

MAXIMUM UNION-FREE SUBFAMILIES

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ABSTRACT

An old problem of Moser asks: what is the size of the largest union-free subfamily that one can guarantee in every family of m sets? A family of sets is called **union-free** if there are no three distinct sets in the family such that the union of two of the sets is equal to the third set. We show that every family of m sets contains a union-free subfamily of size at least $\lfloor \sqrt{4m+1} \rfloor - 1$ and that this bound is tight. This solves Moser's problem and proves a conjecture of Erdős and Shelah from 1972.

More generally, a family of sets is **a -union-free** if there are no $a+1$ distinct sets in the family such that one of them is equal to the union of a others. We determine up to an absolute multiplicative constant factor the size of the largest guaranteed a -union-free subfamily of a family of m sets. Our result verifies in a strong form a conjecture of Barat, Füredi, Kantor, Kim and Patkos.

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1. Introduction

A set A of integers is **sum-free** if there are no $x, y, z \in A$ such that $x + y = z$. Erdős [8] in 1965 proved that every set of n nonzero integers contains a sum-free subset of size at least $n/3$. The proof is an influential application of the probabilistic method in extremal number theory. This result was rediscovered by Alon and Kleitman [3], who showed how to find a sum-free subset of size at least $(n + 1)/3$. Finally, Bourgain [5] using harmonic analysis improved the lower bound to $(n + 2)/3$. This result is the current state of the art for this problem. It is not even known if the constant factor $1/3$ is best possible.

The analogous problem in extremal set theory has also been studied for a long time. A family of sets is called **union-free** if there are no three distinct sets X, Y, Z in the family such that $X \cup Y = Z$. An old problem of Moser asks: what is the size of the largest union-free subfamily that one can guarantee in every family of m sets? Denote this number by $f(m)$. The study of $f(m)$ has attracted considerable interest. Riddell observed that $f(m) \geq \sqrt{m}$ (this follows immediately from Dilworth’s theorem, see below). Erdős and Komlós [9] determined the correct order of magnitude of $f(m)$ by proving that $f(m) \leq 2\sqrt{2m} + 4$. They conjectured that $f(m) = (c - o(1))\sqrt{m}$ for some constant c , without specifying the right value of c . In 1972, Erdős and Shelah [10] improved both the upper and lower bounds by showing that $\sqrt{2m} - 1 < f(m) < 2\sqrt{m} + 1$ (the lower bound was also obtained independently by Kleitman [14]). Erdős and Shelah conjectured that their upper bound is asymptotically tight.

CONJECTURE 1.1: $f(m) = (2 - o(1))\sqrt{m}$.

We verify this conjecture and solve Moser’s problem.

THEOREM 1.2: *For all m , we have*

$$f(m) = \lfloor \sqrt{4m + 1} \rfloor - 1.$$

Let $a \geq 2$ be an integer. A family of sets is called **a -union-free** if there are no $a + 1$ distinct sets X_1, \dots, X_{a+1} such that $X_1 \cup \dots \cup X_a = X_{a+1}$. Let $g(m, a)$ be the minimum over all families of m sets of the size of the largest a -union-free subfamily. In particular, $g(m, 2) = f(m)$. The same proof which shows $f(m) > \sqrt{2m} - 1$ also shows that $g(m, a) > \sqrt{2m} - 1$. Recently, Barat, Füredi, Kantor, Kim and Patkos [4] proved that $g(m, a) \leq c(a + a^{1/4}\sqrt{m})$ for

some absolute constant c and made the following conjecture on the growth of $g(m, a)$.

CONJECTURE 1.3: $\lim_{a \rightarrow \infty} \liminf_{m \rightarrow \infty} \frac{g(m, a)}{\sqrt{m}} = \infty$.

We prove this conjecture in the following strong form, which further gives the correct order of magnitude for $g(m, a)$.

THEOREM 1.4: *For all $m \geq a \geq 2$, we have $g(m, a) \geq \max\{a, a^{1/4}\sqrt{m}/3\}$.*

The lower bound in Theorem 1.4 is tight apart from an absolute multiplicative constant factor by the above-mentioned upper bound from [4]. Of course, if $m \leq a$, we have trivially $g(m, a) = m$.

In the next two sections, we prove Theorem 1.2 and Theorem 1.4, respectively. For the proofs of these theorems, it is helpful to study the structure of the partial order on sets given by inclusion. Recall that a chain (antichain) in a poset is a collection of pairwise comparable (incomparable) elements. Dilworth’s theorem [6] implies that any poset with m elements contains a chain or antichain of size at least \sqrt{m} . Notice that a chain or antichain of sets is a -union-free for all a , but the lower bound $g(m, a) \geq \sqrt{m}$ we get from this simple argument is not strong enough. For the proof of Theorem 1.4, we find considerably larger structures in posets which imply that the subfamily is a -union-free. The existence of such large structures in posets may be of independent interest.

2. Erdős–Shelah conjecture

In this section we prove that every family of m sets contains a union-free subfamily of size at least $\lfloor \sqrt{4m+1} \rfloor - 1$. This verifies in a strong form the Erdős–Shelah conjecture. We finish the section with a modification of the construction of Erdős and Shelah which shows that this bound is tight.

Let \mathcal{F} be a family of m sets and assume that the maximum union-free subfamily has size α . Let t be the length of the longest chain in \mathcal{F} . Let \mathcal{F}_t be the family of maximal (by inclusion) sets in \mathcal{F} , and for $i = t - 1, \dots, 1$, inductively define \mathcal{F}_i as the family of maximal sets in $\mathcal{F} \setminus (\bigcup_{j=i+1}^t \mathcal{F}_j)$. We call \mathcal{F}_i the **i -th level of \mathcal{F}** . Note that since there exists a chain of length t , \mathcal{F}_i is non-empty for all i and $\mathcal{F} = \bigcup_{i=1}^t \mathcal{F}_i$, and for every set in \mathcal{F}_i there is a chain of length $t - i + 1$ starting at that set which hits every level above \mathcal{F}_i exactly once. Furthermore,

if $X \subset Y$, then X lies in a level below Y , and thus each level is an antichain. For a set $X \in \mathcal{F}_i$, let i be the **rank** of X .

We define an auxiliary graph $H = H(\mathcal{F})$ with vertex set $V(H) = \{1, \dots, t\}$ as follows. The pair (i, i') with $i < i'$ is an edge of H if there exist two disjoint chains \mathcal{X} and \mathcal{Y} which both start at \mathcal{F}_i and end at $\mathcal{F}_{i'-1}$ such that

- (i) \mathcal{X}, \mathcal{Y} both have length $i' - i$, and
- (ii) at most one set in $\{X \cup Y \mid X \in \mathcal{X}, Y \in \mathcal{Y}\}$ has rank i' and all the rest of the sets of this form have rank greater than i' or do not belong to family \mathcal{F} .

Thus, by definition, we can see that if (i, i') is an edge of H and $i < i'' < i'$, then (i, i'') and (i'', i') are also edges of H , i.e., we have **monotonicity** of H .

We first prove that if there are many pairs of sets $X, Y \in \mathcal{F}$ such that $X \cup Y \in \mathcal{F}$, then there are many edges of H .

LEMMA 2.1: *Let $i < i'$ and X, Y be two sets of rank at most i such that $X \cup Y \in \mathcal{F}_{i'}$. Then (i, i') is an edge of H .*

Proof. Let X, Y be sets which have rank a, b respectively ($a, b \leq i$), and whose union lies in $\mathcal{F}_{i'}$. Let $\mathcal{X} = \{X_a = X, X_{a+1}, \dots, X_{i'-1}\}$ be a chain starting at X and ending in $\mathcal{F}_{i'-1}$. Similarly define a chain $\mathcal{Y} = \{Y_b = Y, Y_{b+1}, \dots, Y_{i'-1}\}$ starting at Y and ending in $\mathcal{F}_{i'-1}$. If there exists an element Z which lies in both \mathcal{X} and \mathcal{Y} , then $X_a, Y_b \subset Z$ and thus the rank of $X \cup Y = X_a \cup Y_b$ is at most the rank of Z which is at most $i' - 1$. This contradicts our assumption that $X \cup Y \in \mathcal{F}_{i'}$. Therefore \mathcal{X} and \mathcal{Y} are disjoint. Moreover, for every $X_c \in \mathcal{X}$ and $Y_d \in \mathcal{Y}$, we have $X \cup Y \subset X_c \cup Y_d$ and since $X \cup Y$ has rank i' , we know that $X_c \cup Y_d$ either is equal to $X \cup Y$, has rank larger than i' , or is not in \mathcal{F} . By taking a subchain $\{X_i, \dots, X_{i'-1}\}$ of \mathcal{X} and a subchain $\{Y_i, \dots, Y_{i'-1}\}$ of \mathcal{Y} , we can see that (i, i') is an edge of H . ■

We will show that if α is small, then the total number of sets cannot be m , and this will give us a contradiction. Thus we need some tools which allow us to bound the size of the levels. The next lemma shows that by collecting the levels into groups whose indices form independent sets in H , we can obtain a good bound on the size of the levels.

LEMMA 2.2: *Let $a_1 < \dots < a_k$ be the vertices of an independent set in H . Then $\sum_{j=1}^k |\mathcal{F}_{a_j}| \leq \alpha$. Further, if (a_1, b) is an edge of H with $b < a_1$, then*

$$\sum_{j=1}^k |\mathcal{F}_{a_j}| \leq \alpha - 2(a_1 - b) + 1.$$

Proof. The first assertion is a straightforward corollary of Lemma 2.1, as the lemma implies that $\bigcup_{j=1}^k \mathcal{F}_{a_j}$ is a union-free subfamily.

Let \mathcal{X} and \mathcal{Y} be two chains guaranteed by the fact (b, a_1) is an edge of H . Let Z be the unique set (if it exists) in $\{X \cup Y \mid X \in \mathcal{X}, Y \in \mathcal{Y}\}$ which has rank a_1 . We claim that $\mathcal{W} = \mathcal{X} \cup \mathcal{Y} \cup (\bigcup_{j=1}^k \mathcal{F}_{a_j}) \setminus \{Z\}$ is a union-free subfamily. Assume that this is not the case and let $W_1, W_2, W_3 \in \mathcal{W}$ be three distinct sets in \mathcal{W} such that $W_1 \cup W_2 = W_3$. If $W_3 \in \mathcal{F}_{a_i}$ for some $i \geq 2$, then since each level is an antichain we know that both W_1 and W_2 have rank at most a_{i-1} . However, by Lemma 2.1 this would contradict the assumption that a_{i-1} and a_i are not adjacent in H . We may therefore assume that $W_3 \in \mathcal{X} \cup \mathcal{Y} \cup \mathcal{F}_{a_1}$. If $W_3 \in \mathcal{F}_{a_1}$, then W_1, W_2 must have smaller rank than a_1 and thus lie in $\mathcal{X} \cup \mathcal{Y}$. As W_1, W_2, W_3 are distinct, $W_3 = W_1 \cup W_2$, and \mathcal{X} and \mathcal{Y} are chains, the sets W_1, W_2 cannot both lie in \mathcal{X} or both lie in \mathcal{Y} . Recall that (by condition (ii) above) Z was the only set in $\{X \cup Y \mid X \in \mathcal{X}, Y \in \mathcal{Y}\}$ with rank a_1 . Since $W_3 \neq Z$ and $W_3 = W_1 \cup W_2$, the rank of W_3 is not a_1 . If $W_3 \in \mathcal{X}$, then since W_1, W_2, W_3 are distinct and \mathcal{X} is a chain, one of the sets W_1 or W_2 must lie in \mathcal{Y} . Without loss of generality assume that $W_1 \in \mathcal{Y}$. This implies that $W_1 \cup W_3 = W_3$, which is impossible again by condition (ii) since all the sets in $\{X \cup Y \mid X \in \mathcal{X}, Y \in \mathcal{Y}\}$ have rank at least a_1 and the rank of W_3 is smaller than a_1 . Similarly, $W_3 \notin \mathcal{Y}$. Therefore, there cannot exist such sets W_1, W_2, W_3 , and \mathcal{W} is a union-free subfamily whose size satisfies

$$\sum_{j=1}^k |\mathcal{F}_{a_j}| + 2(a_1 - b) - 1 \leq |\mathcal{W}| \leq \alpha,$$

which implies our claimed inequality. ■

Now we are prepared to prove Theorem 1.2. The main idea is to properly color the vertices of H using as few colors as possible (i.e., partition the vertex set of H into as few independent sets as possible) in order to maximize the power of Lemma 2.2. Consider the following greedy algorithm for finding such a partition into independent sets. The first independent set I_1 contains 1. We

find the least $a_{\frac{1}{2}}$ which is not adjacent to 1 and add it to I_1 . After finding the j th element a_j^1 of I_1 , we find the least $a_{j+1}^1 > a_j^1$ which is not adjacent to a_j^1 and add it to I_1 . We continue this procedure until we cannot add any more vertices. Note that by the monotonicity condition satisfied by graph H , the set I_1 is indeed an independent set. Assume that we finished constructing I_i . The independent set I_{i+1} contains the least a_1^{i+1} not in any of the previous independent sets. After finding the j th element a_j^{i+1} of I_{i+1} , we find the least $a_{j+1}^{i+1} > a_j^{i+1}$ which is not in any of the previously chosen independent sets and is not adjacent to a_j^{i+1} and add it to I_{i+1} . We continue this procedure until we cannot add any more vertices. Note that by the monotonicity condition satisfied by graph H , the set I_{i+1} is indeed an independent set. We continue picking out independent sets until there are no remaining vertices of H .

By the first part of Lemma 2.2, the sum of the sizes of the levels with index in I_1 satisfies $\sum_{j \in I_1} |\mathcal{F}_j| \leq \alpha$. For the other independent sets we can use the second part of Lemma 2.2 to obtain a better bound.

LEMMA 2.3: For all $i > 1$, $\sum_{j \in I_i} |\mathcal{F}_j| \leq \alpha - 2i + 3$.

Proof. Let a be the least element in I_i . For each $j < i$, let a_j be the largest vertex in I_j with $a_j < a$. Since $a \notin I_j$ by the greedy algorithm, we know that a_j is adjacent to a in graph H . Since there are $i - 1$ independent sets I_j with $j < i$, there is a vertex b adjacent to a in graph H such that $a - b \geq i - 1$. Thus, by the second part of Lemma 2.2, we have

$$\sum_{j \in I_i} |\mathcal{F}_j| \leq \alpha - 2(i - 1) + 1 = \alpha - 2i + 3. \quad \blacksquare$$

Lemma 2.3 in particular implies that the total number of independent sets is at most $\lfloor \frac{\alpha+3}{2} \rfloor$.

Proof of Theorem 1.2. We first prove the lower bound on $f(m)$. The proof splits into two cases. Suppose first that $n^2 \leq m < n^2 + n$ for some integer n so that $\lfloor \sqrt{4m + 1} \rfloor = 2n$. Assume for contradiction that the size α of the largest union-free subfamily satisfies $\alpha \leq 2n - 2$.

This implies with Lemma 2.3 that

$$m = \sum_i \left| \bigcup_{j \in I_i} \mathcal{F}_j \right| \leq (2n - 2) + \sum_{i=2}^n (2n - 2i + 1) = (2n - 2) + (n - 1)^2 = n^2 - 1,$$

which contradicts $m \geq n^2$. Hence, in this case we must have $\alpha \geq 2n - 1 = \lfloor \sqrt{4m + 1} \rfloor - 1$.

We may therefore assume that $n^2 + n \leq m < (n + 1)^2$ for some integer n so that $\lfloor \sqrt{4m + 1} \rfloor = 2n + 1$. Assume for contradiction that $\alpha \leq 2n - 1$. This implies with Lemma 2.3 that

$$m = \sum_i \left| \bigcup_{j \in I_i} \mathcal{F}_j \right| \leq (2n - 1) + \sum_{i=2}^{n+1} (2n - 2i + 2) = (2n - 1) + n^2 - n = n^2 + n - 1,$$

which contradicts $m \geq n^2 + n$ and we must have $\alpha \geq 2n = \lfloor \sqrt{4m + 1} \rfloor - 1$. This completes the proof of the lower bound in Theorem 1.2.

We use the following construction given by Erdős and Shelah [10] to show that the lower bound on $f(m)$ is tight.

We begin with the case $m = n^2$ is a perfect square. Consider the collection \mathcal{F} of n^2 sets $X_{ij} = \{x \in \mathbb{N} \mid n + 1 - i \leq x \leq n + j\}$ where $1 \leq i, j \leq n$. We may assume that each set X_{ij} is placed on the (i, j) position of the $n \times n$ grid. Let \mathcal{F}' be a union-free subfamily of \mathcal{F} . Note that \mathcal{F}' cannot contain a triple $X_{i'j}, X_{ij'}, X_{ij}$ with $i' < i$ and $j' < j$ since $X_{i'j} \cup X_{ij'} = X_{ij}$. Delete from each column in the grid the bottommost set in \mathcal{F}' . Note that we removed at most n sets and, after removing these sets, there will be no set of \mathcal{F}' remaining in the lowest row of the grid. Then remove from each row in the grid the leftmost remaining set in \mathcal{F}' . This removes at most $n - 1$ additional sets. Now there cannot be any remaining set as otherwise \mathcal{F}' will contain some triple $X_{i'j}, X_{ij'}, X_{ij}$ with $i' < i$ and $j' < j$. Since we removed at most $n + n - 1 = 2n - 1$ sets, this implies that \mathcal{F}' has size at most $2n - 1$. Thus our theorem is tight for $m = n^2$. We can modify this construction by taking \mathcal{F} to be the collection $X_{ij} = \{x \in \mathbb{N} \mid n + 1 - i \leq x \leq n + j\}$ with $1 \leq i \leq n + 1$ and $1 \leq j \leq n$. In this way we will have a total of $m = n^2 + n$ sets, the corresponding grid will have size $(n + 1) \times n$, and the maximum union-free subfamily will have size at most $2n$. This again shows that our theorem is tight for $m = n^2 + n$. Note that, in this construction, X_{11} is less than all other sets, and hence every maximal union-free subfamily of \mathcal{F} contains X_{11} . Therefore, deleting X_{11} from \mathcal{F} decreases the number of sets in it by one, but also decreases by one the size of the maximum union-free subfamily. This shows that $f(n^2 - 1) \leq 2n - 2$ and $f(n^2 + n - 1) \leq 2n - 1$. Of course, $f(m)$ is a monotone increasing function, and this gives the upper bound on $f(m)$ for all other values of m . ■

3. *a*-union-free subfamilies

In this section we prove Theorem 1.4, which says that for $m \geq a$, every family of m sets contains an a -union-free subfamily of size at least $\max\{a, a^{1/4}\sqrt{m}/3\}$. We will in fact prove a stronger theorem for general posets which implies Theorem 1.4. In a poset P , a (ℓ, α) -**ladder** consists of a chain $X_1 < \dots < X_\ell$, and ℓ antichains $\mathcal{Y}_1, \dots, \mathcal{Y}_\ell$ each of order α such that each $Y \in \mathcal{Y}_i$ satisfies $Y \leq X_i$, and for each i and each $Y \in \mathcal{Y}_{i+1}$, $Y \not\leq X_i$. For $\alpha \geq 1$, the **size** of an (ℓ, α) -ladder is the number of elements in $\mathcal{Y}_1 \cup \dots \cup \mathcal{Y}_\ell$, which is $\ell\alpha$. The size of an $(\ell, 0)$ -ladder we define to be ℓ . Notice that a chain of length ℓ forms a $(\ell, 0)$ -ladder.

LEMMA 3.1: *An (l, α) -ladder in a family of sets contains an a -union-free subfamily of size at least the size of the ladder, which is $l \cdot \max\{\alpha, 1\}$.*

Proof. If $\alpha \leq 1$, then we can just take the chain as the a -union-free subfamily. Otherwise, we claim that the collection $\mathcal{Y} = \bigcup_{i=1}^l \mathcal{Y}_i$ is an a -union-free family and this proves the lemma. Consider a linear extension of the partial order on the sets given by inclusion. Let Y_1, \dots, Y_a be a distinct sets in \mathcal{Y} and assume without loss of generality that $Y_a \in \mathcal{Y}_t$ is the largest of these a sets in this linear extension. By the definition of a ladder, each of these sets is a subset of X_t and therefore the union $Y_{a+1} := Y_1 \cup \dots \cup Y_a$ of these a sets is a subset of X_t . For $j > t$, we have $Y_{a+1} \notin \mathcal{Y}_j$ as no set in \mathcal{Y}_j is a subset of X_t . Since $Y_a \subset Y_{a+1}$ and \mathcal{Y}_t is an antichain, $Y_{a+1} \notin \mathcal{Y}_t$. If $Y_{a+1} \in \mathcal{Y}_j$ with $j < t$, then $Y_a \subset Y_{a+1} \subset X_{t-1}$, which contradicts $Y_a \in \mathcal{Y}_t$ and no set in \mathcal{Y}_t is a subset of X_{t-1} . Thus $Y_{a+1} \notin \mathcal{Y}$ and hence \mathcal{Y} is a -union-free. ■

In a poset P , a set S of elements is **a -degenerate** if no element of S is larger than a other elements of S . It is easy to check that if P is a family of sets with partial order defined by inclusion, then any a -degenerate subset is a -union-free. Thus in order to prove Theorem 1.4, it is enough to establish the following theorem.

THEOREM 3.2: *Every poset P on $m \geq a$ elements contains a subset of size at least $\max\{a, a^{1/4}\sqrt{m}/3\}$ which is a -degenerate or forms a ladder.*

The bounds in Theorems 1.4 and 3.2 are tight up to an absolute constant factor for all m and a by a construction of Barat et al. [4] which we will give at the end of this section.

Proof. Let $c = 1/3$. If $m \leq c^{-2}a^{3/2}$, then we can take any a elements as an a -degenerate set. Since $a \geq ca^{1/4}\sqrt{m}$ we have nothing to prove in this range. Thus we will assume that $m > c^{-2}a^{3/2}$. We will show that there exists a subset of size at least $ca^{1/4}\sqrt{m}$ which is a -degenerate or forms a ladder. Since every poset on m elements contains a chain or an antichain of size at least \sqrt{m} , and a chain forms a ladder and an antichain is a -degenerate, we can also assume that $a > 81$.

We may assume that the height of poset P is less than $ca^{1/4}\sqrt{m}$ as otherwise we can take the longest chain as the ladder. In each step, we will either find the desired subset, or we will delete some elements and the height of the remaining poset will drop significantly compared to the drop in the number of remaining elements.

Let $P_0 = P$, $h_0 = ca^{1/4}\sqrt{m}$, and $m_0 = m$. Suppose that after i steps the remaining poset P_i has at least m_i elements and height at most h_i where $m_i = 2^{-i}m_0$ and $h_i = 2^{-2i}h_0$. Notice that this condition is satisfied for $i = 0$. For technical reason we need additional parameters x_i . Let $x_0 = 2^T$ for some integer T so that $2^T = \beta c^2 a^{1/2}$ with $\frac{9}{4} \leq \beta < \frac{9}{2}$ and let $x_i = 2^{-i}x_0$. Note that such T indeed exists since $\frac{9}{2}c^2 a^{1/2} > 1$, and x_i is an integer for all $i = 0, \dots, T$.

Suppose we have just finished step $i \leq T - 1$. Partition P_i into levels $\mathcal{F}_1 \cup \dots \cup \mathcal{F}_{\lfloor h_i \rfloor}$, where \mathcal{F}_j is the minimal elements of the subposet of P_i formed by deleting all \mathcal{F}_k with $k < j$ (some of the levels in the end can be empty). Partition the poset P_i into sets S_k each consisting of x_i consecutive levels of P_i (except possibly one set which consists of less than x_i consecutive levels). Call an element $p \in S_k$ **nice** if it is larger than a other elements of S_k , and **bad** otherwise. For each k , the bad elements within S_k form an a -degenerate set, and hence we may assume S_k contains less than $ca^{1/4}\sqrt{m}$ bad elements. Thus, the total number of bad elements is at most the number of sets S_k times $ca^{1/4}\sqrt{m}$, which is

$$\begin{aligned} \left\lceil \frac{h_i}{x_i} \right\rceil ca^{1/4}\sqrt{m} &= \left\lceil \frac{2^{-2i}h_0}{2^{-i}x_0} \right\rceil ca^{1/4}\sqrt{m} \leq \left(\frac{ca^{1/4}\sqrt{m}}{2^i x_0} + 1 \right) ca^{1/4}\sqrt{m} \\ &= \frac{m}{\beta \cdot 2^i} + ca^{1/4}\sqrt{m}. \end{aligned}$$

By $m \geq c^{-2}a^{3/2}$ and $i \leq T - 1$, we have

$$ca^{1/4}\sqrt{m} = ca^{1/4} \frac{m}{\sqrt{m}} \leq c^2 a^{-1/2} m = \beta c^4 \frac{m}{2^T} \leq \frac{\beta c^4}{2} \frac{m}{2^i}.$$

Since $c = 1/3$ and $\frac{9}{4} \leq \beta \leq \frac{9}{2}$, one can easily check $\frac{1}{\beta} + \frac{\beta c^4}{2} \leq 4/9 + 1/36 < 1/2$, so the number of bad elements in P_i is at most $m_i/2$.

Let P_{i+1} be the superset of P_i consisting of the nice elements in P_i . Since there are at most $m_i/2$ bad elements in P_i , the total number of elements in P_{i+1} is at least $m_i - m_i/2 = m_{i+1}$. If the height of this poset is at least h_{i+1} , then by definition, we can find a chain of length at least h_{i+1} . Let $X_1 < \dots < X_\ell$ be a subchain of this chain such that each X_j belongs to different sets S_k . Since S_k is a union of x_i levels, we can guarantee such a subchain of length at least $\ell \geq \lceil h_{i+1}/x_i \rceil$. By the definition of nice elements, each element $X_j \in S_k$ is greater than a other elements in S_k . Therefore, X_j is greater than $\alpha = \lceil a/x_i \rceil$ elements which all belong to a single level of S_k . By the construction of levels, these elements form an antichain. Denote this antichain by \mathcal{Y}_j . We have thus constructed an (ℓ, α) -ladder in P . The size of this ladder is

$$\ell\alpha \geq \left\lceil \frac{h_{i+1}}{x_i} \right\rceil \cdot \left\lceil \frac{a}{x_i} \right\rceil \geq \frac{2^{-2i-2}h_0 \cdot a}{2^{-2i}x_0^2} = \frac{ca^{1/4}\sqrt{m}}{4 \cdot \beta^2 c^4} \geq ca^{1/4}\sqrt{m},$$

where we used that $x_0 = 2^T = \beta c^2 a^{1/2}$, $\beta < 9/2$, $h_0 = ca^{1/4}\sqrt{m}$ and $c = 1/3$. Thus we find a sufficiently large ladder to complete this case.

Otherwise the height of the poset P_{i+1} is less than h_{i+1} , and since it contains at least m_{i+1} elements and has height at most h_{i+1} , we can move on to the next step. Assume that the process continues until $i = T$. Then we have a poset P_T which has at least m_T elements and has height at most h_T . In this case, there exists an antichain of size at least

$$\left\lceil \frac{m_T}{h_T} \right\rceil \geq \frac{2^{-T}m_0}{2^{-2T}h_0} = \frac{2^T m}{ca^{1/4}\sqrt{m}} \geq \frac{c^2 a^{1/2} m}{ca^{1/4}\sqrt{m}} = ca^{1/4}\sqrt{m}.$$

An antichain is an a -degenerate set, so this concludes the proof. ■

We end this section with the construction given in [4] of a family of sets which does not contain a large a -union-free subfamily. Let $k = \lceil \sqrt{a} \rceil$. Let \mathcal{F}_1 be the family of sets given in the end of the previous section, i.e., it contains sets $X_{ij} = \{x \in \mathbb{N} \mid n + 1 - i \leq x \leq n + j\}$ for all $1 \leq i, j \leq n$. A similar argument as there shows that the largest a -union-free subfamily of \mathcal{F}_1 has size at most $2kn$. Indeed we again assume that each set X_{ij} is placed on the (i, j) position of the $n \times n$ grid. Let \mathcal{F}'_1 be an a -union-free subfamily of \mathcal{F}_1 . Delete from each column in the grid the bottommost k sets in \mathcal{F}'_1 . Note that we removed at most kn sets. Then remove from each row in the grid the k leftmost remaining

sets in \mathcal{F}'_1 . This removes at most kn additional sets. Now there cannot be any remaining set. Otherwise \mathcal{F}'_1 will contain sets $X_{i_{1j}}, \dots, X_{i_{kj}}$ and X_{ij} with $i_\ell < i$ for all $1 \leq \ell \leq k$, such that X_{ij} and each set $X_{i_\ell j}$ have at least k other sets from \mathcal{F}'_1 in their column below them. This gives at least $k^2 \geq a$ sets (all properly contained in X_{ij}) from which one can easily choose a sets whose union equals X_{ij} . Since we removed at most $kn + kn = 2kn$ sets, this implies that \mathcal{F}'_1 has size at most $2kn$.

Note that set X_{nn} contains all the other sets in \mathcal{F}_1 . Let \mathcal{G}_2 be a family of sets constructed in the same manner but over a different (disjoint) universe of elements from \mathcal{F}_1 , and let \mathcal{F}_2 be the family $\{G \cup X_{nn} \mid G \in \mathcal{G}_2\}$. Thus every set in \mathcal{F}_2 contains all the sets in \mathcal{F}_1 and the a -union-free subfamily of \mathcal{F}_2 also has size at most $2kn$. Repeat this process and construct families $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_k$ such that for all ℓ , every set in $\mathcal{F}_{\ell+1}$ contains all the sets in \mathcal{F}_ℓ . Let $\mathcal{F} = \bigcup_{\ell=1}^k \mathcal{F}_\ell$ be our family.

Then the number of sets in \mathcal{F} is kn^2 . We will use this family to obtain a bound on $g(m, a)$ for all values of m such that $k(n - 1)^2 < m \leq kn^2$. We can bound the size of the largest a -union-free subfamily \mathcal{F}' of \mathcal{F} as follows. Assume that ℓ is the first index such that there are more than a sets of \mathcal{F}' in $\mathcal{F}_1 \cup \dots \cup \mathcal{F}_\ell$. Then in \mathcal{F}' there are less than a sets in the levels up to $\mathcal{F}_{\ell-1}$, at most $2kn$ sets in \mathcal{F}_ℓ , and it is easy to see for each $t > \ell$ that the sets in \mathcal{F}' from \mathcal{F}_t form a 2-union-free subfamily. Thus there are at most $2n$ such sets. Therefore the number of sets in \mathcal{F}' is at most

$$a + 2kn + (k - i) \cdot 2n \leq a + 4k + 4k(n - 1) \leq 5a + 4(a^{1/4} + 1)\sqrt{m},$$

where we used the inequalities $m > k(n - 1)^2$ and $a + 4\lceil \sqrt{a} \rceil \leq 5a$ for all integers $a \geq 1$. This shows that the bound in Theorem 1.4 is tight up to the constant factor for all a and m .

4. Concluding remarks

In this paper, we solved Moser’s problem by determining the largest $f(m)$ for which every family of m sets has a union-free subfamily of size $f(m)$. We do not know whether or not the family of sets given by Erdős and Shelah [10] is essentially the only extremal family which shows that this bound is tight. It would be interesting to further study this problem of classifying the extremal

families. Specifically, can one classify the extremal families which achieve the bound $f(m) = \lfloor \sqrt{4m+1} \rfloor - 1$?

We also determined up to an absolute constant factor the largest $g(m, a)$ for which every family of m sets has an a -union-free subfamily of size $g(m, a)$. We did not try to optimize constants in the proof of Theorem 1.4 for the sake of clarity of presentation. Although the bound in this theorem can be further improved, some new ideas are needed to determine the asymptotic behavior of $g(m, a)$ for $a \geq 3$.

The proofs of Theorems 1.2 and 1.4 giving lower bounds for $f(m)$ and $g(m, a)$ can easily be made algorithmic. That is, we can find a union-free or a -union-free subfamily of a family of m with size guaranteed by these theorems in polynomial time.

There are other directions that have been studied concerning union-free subfamilies. For example, Abbott and Hanson [1] proposed studying the minimum number of colors necessary to color the subsets of an n -element set so that each color class is union-free. This problem was further studied by Aigner, Duffus and Kleitman [2]. A related result of Kleitman [13], improving on an earlier paper [12], solved a conjecture of Erdős [7] by showing that the largest union-free family of subsets of an n -element set has cardinality at most $(1 + \frac{c}{\sqrt{n}}) \binom{n}{\lfloor n/2 \rfloor}$, where c is an absolute constant. We can also replace the union-free condition by other conditions. For example, Gunderson, Rödl and Sidorenko [11] studied the maximum cardinality of a family of subsets of an n -element set which does not contain a d -dimensional Boolean algebra. A related problem of Erdős and Shelah was solved by Barat, Füredi, Kantor, Kim and Patkos [4], who estimated the size of a maximum subfamily of a family of m sets which does not contain a d -dimensional Boolean algebra.

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