A Note on τ-Critical Linear Hypergraphs

Benny Sudakov *

Abstract

J.P. Roudneff as well as R. Aharoni and R. Ziv conjectured that every τ-critical linear hypergraph has at most $\binom{\tau+1}{2}$ edges. In this note we prove this conjecture for $\tau \leq 5$, and obtain a nontrivial upper bound for general τ.

1 Introduction

A *linear hypergraph* H is an ordered pair $H = (V, E)$, where $V = V(H)$ is a finite set of vertices and $E = E(H)$ is a collection of subsets of V, called edges, such that any pair $e_1, e_2 \in E$ satisfies $|e_1 \cap e_2| \leq 1$. A subset $T \subset V$ is called a *cover* of H if it intersects every edge of H. The minimum cardinality of a cover is denoted by $\tau(H)$, and is called the *covering number* of H. A hypergraph is called τ*-critical* if omitting any one of its edges reduces its covering number.

Several problems in Extremal Set Theory deal with the estimation of the size of τ-critical systems of given types (see [2], [3] for various examples). Erdős, Hajnal and Moon [1] proved that any τ-critical *graph* has at most $\binom{\tau+1}{2}$ edges. As a generalization of this result J.P. Roudneff [2], as well as R. Aharoni and R.Ziv [3] conjectured that the same estimate holds for any τ-critical linear hypergraph. In this note we prove this conjecture for $\tau \leq 5$. We also show that for any $\tau \geq 5$ the number of edges in a τ-critical linear hypergraph is at most $\tau^2 - 3\tau + 5$. It is worth noting that an easy upper bound for the number of edges in a τ-critical linear hypergraph is $\tau^2 - \tau + 1$, as it is not too difficult to see that the maximum degree in each such hypergraph is at most τ and there is a set of $\tau - 1$ vertices that covers all the edges except one. But the trivial upper bound proves the conjecture only for $\tau = 2$.

The usual framework for dealing with τ-critical hypergraphs is by the so called set-pair method (see [2], [3] for more details). We start with some definitions.

Definition: *A collection $S = \{(A_i, B_i) | 1 \leq i \leq n\}$ is called an $(a, b)^*$ system iff it satisfies the following conditions:*

$$A_i \cap B_i = \emptyset \textit{ for all } 1 \leq i \leq n$$

*Department of Mathematics, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, Israel. Email: sudakov@math.tau.ac.il. This research forms part of a Ph.D. thesis, written by the author under the supervision of Professor Noga Alon. Mathematics Subject Classification (1991): 05D05, 05D15.

$$A_i \cap B_j \neq \emptyset \textit{ for } i \neq j$$

$$|A_i| = a \textit{ for } 1 \leq i \leq n \textit{ and } |B_i \cap B_j| < b \textit{ for } i \neq j.$$

We denote by $f(a,b)$ the maximum possible size of an $(a,b)^*$ system. From every τ-critical linear hypergraph one can construct a $(\tau-1,2)^*$ system of the same size as follows. Let $H=(V,E)$ be a τ-critical linear hypergraph. By definition, for any edge $e \in E$ one can find a subset $X_e \subset V$ such that $|X_e| = \tau - 1$ and X_e intersects all the edges of H except e. Therefore the family $\{(X_e, e)\}$ forms a $(\tau-1,2)^*$ system of size $|E|$. Thus, the conjecture of Roudneff, Aharoni and Ziv would follow from the following.

Conjecture 1.1 *For all $a \geq 0$: $f(a,2) \leq \binom{a+2}{2}$.*

The conjecture of Aharoni and Ziv was initially formulated in this form. They also conjectured that even more is true:

Conjecture 1.2 *For all pairs a and b: $f(a,b) \leq \binom{a+b}{b}$.*

Note that the bound in Conjecture 1.2 is tight for all pairs a and b as shown by the following example. Let Y be a set of size $a+b$, then $S = \{(A, Y \setminus A) |\ |A| = a,\ A \subset Y\}$ forms an $(a,b)^*$ system of size $\binom{a+b}{b}$.
Here we prove the following theorem.

Theorem 1.3 *Conjecture 1.1 is true for all $a \leq 4$. Moreover, for any $a \geq 4$: $f(a,2) \leq a^2 - a + 3$.*

The proof of Theorem 1.3 is given in Section 2. In Section 3 we show how to apply this result and derive a simple upper bound for $f(a,b)$ in general.

2 The Proof of Theorem 1.3

In this section we prove our main result. Let $S = \{(A_i, B_i) | 1 \leq i \leq n\}$ be an $(a,2)^*$ system. Put $X = \cup_{i=1}^n A_i$. We may assume that $B_i \subset X$ for all i. Otherwise replace each B_i by $B_i \cap X$, and note that the resulting family remains an $(a,2)^*$ system. We claim that no family of $a+2$ of the sets B_i share a common point. Assume this is false, and renumber these sets as $B_1, \ldots, B_{a+2}$. Suppose $\{y\} = \cap_{i=1}^{a+2} B_i$. By definition $A_{a+2} \cap B_{a+2} = \emptyset$, thus $y \notin A_{a+2}$. Therefore $A_{a+2} \cap (B_i \setminus \{y\}) \neq \emptyset$ for all $1 \leq i \leq a+1$. Since the sets $B_i \setminus \{y\}$ are pairwise disjoint for $i \leq a+1$, we conclude that $|A_{a+2}| \geq a+1$. This contradiction proves the claim.

The following lemma may be interesting in its own right.

Lemma 2.1 *Let $S = \{(A_i, B_i) | 1 \leq i \leq n\}$ be an $(a,2)^*$ system, which contains a family of $a+1$ of the sets B_i with a nonempty intersection. Then*

$$|S| \leq a + 1 + f(a-1, 2).$$

Proof: Suppose $B_1, \dots, B_{a+1}$ have a nonempty intersection. Let $\{y\} = \cap_{i=1}^{a+1} B_i$. Consider the new system $S' = \{(A_j \setminus \{y\}, B_j) | a+2 \leq j \leq n\}$. It is easy to see that $y \in A_j$ for $j \geq a+2$, since otherwise $A_j \cap (B_i \setminus \{y\}) \neq \emptyset$ for $1 \leq i \leq a+1$. But the sets $B_i \setminus \{y\}$ are pairwise disjoint for $1 \leq i \leq a+1$. Hence A_j has size at least $a+1$, contradiction. Also $y \notin B_j$ for $j \geq a+2$, since any $a+2$ of the sets B_j have an empty intersection. Thus S' is an $(a-1,2)^*$ system, implying that

$$|S| = a + 1 + |S'| \leq a + 1 + f(a-1, 2). \quad \square$$

Now we can prove Conjecture 1.1 for $a \leq 3$.

Proof: The assertion is trivial for $a = 0$. Assuming $a > 0$, let $S = \{(A_i, B_i) | 1 \leq i \leq n\}$ be an $(a,2)^*$ system and suppose $a \leq 3$. If S contains a family of $a+1$ of the sets B_i with a nonempty intersection, then we proceed by induction using Lemma 2.1. Otherwise, since $A_1 \cap B_i \neq \emptyset$ for $2 \leq i \leq n$ there is a point in A_1, which is contained in at least $\frac{n-1}{a}$ of the sets B_i. Thus $\frac{n-1}{a} \leq a$. Therefore $n \leq a^2 + 1$. This proves Conjecture 1.1 for $a \leq 3$. $\square$

Before proving Theorem 1.3 we need the following additional lemma.

Lemma 2.2 *Let $S = \{(A_i, B_i) | 1 \leq i \leq n\}$ be an $(a,2)^*$ system, such that all $A_i, 1 \leq i \leq n$ are pairwise disjoint. Then $|S| \leq 2a + 1$.*

Proof: Since all A_i are pairwise disjoint and $B_i \cap A_j \neq \emptyset$ for $i \neq j$, it follows that $|B_i| \geq n - 1$ for all i. By definition of an $(a,2)^*$ system, $|B_i \cap B_j| \leq 1$. Therefore, since $B_i \subset \cup_{j=1}^{n} A_j$ for all i,

$$| \cup_{i=1}^{n} A_i| = an \geq | \cup_{i=1}^{n} B_i| \geq \sum_{i=1}^{n} |B_i| - \sum_{i<j} |B_i \cap B_j|$$

$$\geq n(n-1) - \frac{n(n-1)}{2} = \frac{n(n-1)}{2}.$$

Therefore $n \leq 2a + 1$, as needed. $\square$

Proof of Theorem 1.3: Let $S = \{(A_i, B_i) | 1 \leq i \leq n\}$ be any $(a,2)^*$ system where $a \geq 4$. We prove that $|S| \leq a^2 - a + 3$. This implies the assertion of Conjecture 1.1 for $a = 4$. We assume that no family of $a+1$ of the sets B_i share a common point, since if such a family exists, than we can proceed by induction using Lemma 2.1. The proof is by contradiction. Suppose $n = a^2 - a + 4$ (if n is greater than that, simply consider only $a^2 - a + 4$ of the pairs). We show that in this case all the sets A_i are pairwise disjoint and obtain a contradiction to Lemma 2.2.

Without loss of generality it is enough to prove that A_1 is disjoint from A_i for $i \geq 2$. We use the following greedy procedure. Consider a point in A_1 that belongs to the maximum number of the sets B_i. Denote this point by x_1. Let $\mathcal{F}_1 = \{(A_j^{(1)}, B_j^{(1)})\}$ be the family of all pairs (A_i, B_i) in our system for which $x_1 \in B_i$. Remove these pairs from the family and continue the process. Let x_2 be the point in A_1 contained in the maximum number of the remaining sets B_i. Let $\mathcal{F}_2 = \{(A_j^{(2)}, B_j^{(2)})\}$ be the family of all remaining pairs (A_i, B_i) for which $x_2 \in B_i$. Remove the family $\mathcal{F}_2$ from our system. We continue the process until it stops after a stages, since $|A_1| = a$. As a result the system $S \setminus (A_1, B_1)$ has been partitioned into families $\mathcal{F}_i = \{(A_j^{(i)}, B_j^{(i)})\}, 1 \leq i \leq a$, with the following properties:

$$|\mathcal{F}_1| \geq \ldots \geq |\mathcal{F}_a|, \tag{1}$$

$$\mathcal{F}_i \cap \mathcal{F}_k = \emptyset \text{ for } i \neq k, \tag{2}$$

$$A_1 = \{x_1, \ldots, x_a\},\ x_i \in B_j^{(i)} \text{ for all } j \text{ and also } x_i \notin B_j^{(k)} \text{ for } k > i. \tag{3}$$

Let $|\mathcal{F}_a| = a - t$. By our assumptions we have that $|\mathcal{F}_i| \leq a$ and $\sum_1^a |\mathcal{F}_i| = a^2 - a + 3$. Therefore $t \geq 1$ and at least $t + 2$ families, namely $\mathcal{F}_1, \ldots, \mathcal{F}_{t+2}$ have size a precisely.

We give a detailed proof that $A_1^{(1)} \cap A_1 = \emptyset$. The same reasoning can be used to show that all other sets $A_j^{(i)}$ do not intersect A_1.

Claim 1: $x_i \notin A_1^{(1)}$ for $i \leq t + 2$.

Proof: Since $x_1 \in B_1^{(1)}$, by definition $x_1 \notin A_1^{(1)}$. Assume, first, that $x_i \in A_1^{(1)}$ for some $2 \leq i \leq t+2$. We claim in this case that all x_i, $2 \leq i \leq t + 2$, belong to $A_1^{(1)}$. Without loss of generality assume that x_2 belongs to $A_1^{(1)}$ but x_j does not for some $j > 2$. By definition $A_1^{(1)}$ intersects $B_i^{(j)}$ for all i. Then $A_1^{(1)}$ has also a nonempty intersection with $B_i^{(j)} \setminus \{x_j\}$ for all i. Since any $a + 1$ of the sets B_i are disjoint we have that $x_2 \notin B_i^{(j)}$. Note that $\cap_i B_i^{(j)} = \{x_j\}$, thus $B_i^{(j)} \setminus \{x_j\}$ are pairwise disjoint. Therefore $A_1^{(1)}$ has size at least $a + 1$, contradiction. Therefore all x_i, $2 \leq i \leq t + 2$, belong to $A_1^{(1)}$. Since any $a + 1$ of the sets B_i are disjoint, $x_j \notin B_i^{(1)}$ for all i and $2 \leq j \leq t + 2$. By definition $A_1^{(1)}$ intersects $B_i^{(1)}$ for $i \geq 2$. Therefore we conclude that $|A_1^{(1)}| \geq (t + 1) + (a - 1) = a + t \geq a + 1$, contradiction. Hence no x_i, $i \leq t + 2$, belongs to $A_1^{(1)}$. $\square$

Claim 2: $x_k \notin A_1^{(1)}$ for all $k \geq t + 3$.

Proof: Since $x_i \notin A_1^{(1)}$ for $1 \leq i \leq t + 2$ we have that

$$A_1^{(1)} \cap B_j^{(i)} \setminus \{x_i\} \neq \emptyset, \text{ for } 1 \leq j \leq a \text{ and for } 2 \leq i \leq t + 2.$$

We also have that $|A_1^{(1)}| = a$. Therefore each point in $A_1^{(1)}$ belongs to $t + 1$ of the sets B_i, one from each of the families $\mathcal{F}_2, \ldots, \mathcal{F}_{t+2}$. Suppose x_k belongs to $A_1^{(1)}$ for some $k \geq t + 3$. Since $\mathcal{F}_k$ is disjoint from $\mathcal{F}_i$ for $1 \leq i \leq t + 2$ we conclude that x_k belongs to $t + 1 + |\mathcal{F}_k| \geq t + 1 + a - t = a + 1$ of the sets B_i, contradiction. $\square$

Claim 1 together with Claim 2 prove that $A_1^{(1)}$ is disjoint from A_1, since $A_1 = \{x_1, \ldots, x_a\}$.

Similarly, each $A_j^{(i)}$ is disjoint from A_1. Therefore all the sets A_i are pairwise disjoint. Thus by Lemma 2.2

$$|S| \ \leq 2a + 1 \ < a^2 - a + 4,$$

contradiction. This completes the proof of Theorem 1.3. $\square$

3 Concluding remarks

It is not clear how to adapt our combinatorial method to get a good upper bound for $f(a, b)$ in general. Perhaps some algebraic tools will be useful here. Note however that the following simple observation, together with Theorem 1.3 gives some upper bound for the size of general $(a, b)^*$ systems.

Proposition 3.1 *For all pairs a and b: $f(a, b+1) \leq af(a, b) + 1$.*

Proof: Let $S = \{(A_i, B_i)|1 \leq i \leq n\}$ be an $(a, b)^*$ system. Suppose that $A_1 = \{x_1, \ldots, x_a\}$. For every x_i consider the family $\mathcal{F}_i = \{(A_j^{(i)}, B_j^{(i)})\}$ of pairs in S where $\{B_j^{(i)}\}$ are all the sets B_i which contain the point x_i. It is easy to see that $\{(A_j^{(i)}, B_j^{(i)}) \setminus \{x_i\}\}$ forms an $(a, b-1)^*$ system. Therefore $|\mathcal{F}_i| \leq f(a, b-1)$. Since $S = \cup_{i=1}^{a} \mathcal{F}_i \cup \{(A_1, B_1)\}$ then,

$$|S| \leq \sum_{i=1}^{a} |\mathcal{F}_i| + 1 \leq af(a, b-1) + 1. \quad \Box$$

Acknowledgement: I am grateful to Noga Alon for his suggestions and comments on an early version of this paper.

References

[1] P. Erdős, A. Hajnal and J. Moon, *A problem in graph theory*, Amer. Math. Monthly 71 (1964), 1107-1110.

[2] Z. Füredi, *Matchings and covers in hypergraphs*, Graphs and Combinatorics 4 (1988), 115-206.

[3] Z. Tuza, *Applications of the set-pair method in extremal hypergraph theory*, Extremal problems for finite sets, Bolyai Society Mathematical Studies 3, Budapest, 1994, 479-514.