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The size Ramsey number of a directed path

Ido Ben-Eliezer^{a,1}, Michael Krivelevich^{b,2}, Benny Sudakov^{c,3}

^a School of Computer Science, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel

^b School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel

^c Department of Mathematics, UCLA, Los Angeles, CA 90095, United States

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ABSTRACT

Given a graph H , the size Ramsey number $r_e(H, q)$ is the minimal number m for which there is a graph G with m edges such that every q -coloring of $E(G)$ contains a monochromatic copy of H . We study the size Ramsey number of the directed path of length n in oriented graphs, where no antiparallel edges are allowed. We give nearly tight bounds for every fixed number of colors, showing that for every $q \geq 1$ there are constants $c_1 = c_1(q), c_2$ such that

$$\frac{c_1(q)n^{2q}(\log n)^{1/q}}{(\log \log n)^{(q+2)/q}} \leq r_e(\vec{P}_n, q+1) \leq c_2 n^{2q} (\log n)^2.$$

Our results show that the path size Ramsey number in oriented graphs is asymptotically larger than the path size Ramsey number in general directed graphs. Moreover, the size Ramsey number of a directed path is polynomially dependent on the number of colors, as opposed to the undirected case.

Our approach also gives tight bounds on $r_e(\vec{P}_n, q)$ for general directed graphs with $q \geq 3$, extending previous results.

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1. Introduction

Given an integer $q > 0$, we write $G \rightarrow (H)_q$ if every q -coloring of $E(G)$ contains a monochromatic copy of H .

E-mail addresses: idoebene@post.tau.ac.il (I. Ben-Eliezer), krivelev@post.tau.ac.il (M. Krivelevich), bsudakov@math.ucla.edu (B. Sudakov).

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The study of size Ramsey numbers (initiated in [10]) is concerned with the following questions. Given a graph H , what is the minimum number of edges m for which there is a graph G with m edges such that $G \rightarrow (H)_q$?

Denote by $r_e(H, q)$ the size Ramsey number of H with respect to coloring with q colors. That is,

$$r_e(H, q) = \min\{|E(G)| \mid G \rightarrow (H)_q\}.$$

The study of $r_e(K_n, q)$ is essentially equivalent to the study of the original Ramsey number. Namely, it can be verified that if $r_e(K_n, q) = m$ then a clique with exactly m edges has the desired property. This result is attributed to Chvátal in [10].

Beck proved [8] that for every constant $q > 0$, the size Ramsey number of the undirected path on n vertices is linear in n , answering a question of Erdős. Namely, there is a constant c_q such that

$$r_e(P_n, q) \leq c_q n.$$

Later Alon and Chung [3] provided an explicit construction of graphs with this property. The linearity of the size Ramsey number for bounded degree trees was proved by Friedman and Pippenger [12].

1.1. Directed graphs

In this work we focus on the size Ramsey number in directed graphs, and in particular on the size Ramsey number of a directed path on n vertices.

Raynaud [16] proved that every red–blue coloring of the complete symmetric directed graph (a directed graph in which between every two vertices there are edges in both directions) has a Hamilton cycle which is the union of two monochromatic paths (a simple proof can be found in [14]). In particular, this shows that the size Ramsey number of the path \vec{P}_n for directed graphs with antiparallel (opposite) edges is $O(n^2)$. Reimer [17] proved that every digraph with the property that every red–blue coloring has a path of length n must have $\Omega(n^2)$ edges, therefore proving that Raynaud's bound for non-simple directed graphs is tight up to a constant factor.

In this work we provide nearly tight bounds for the size Ramsey number of directed paths in oriented graphs, where no antiparallel edges are allowed. We show that this number is asymptotically larger than the path size Ramsey number when such edges are allowed. Our approach also generalizes to the case of non-simple directed graphs, and we provide nearly tight bounds for such graphs for every constant $q \geq 3$.

Our first result is a lower bound on the size Ramsey number.

Theorem 1. For every $q \geq 1$ there is a constant $c_1 = c_1(q)$ such that

$$r_e(\vec{P}_n, q + 1) \geq \frac{c_1 n^{2q} (\log n)^{1/q}}{(\log \log n)^{(q+2)/q}}.$$

We also have the following almost matching upper bound.

Theorem 2. There is an absolute constant c_2 such that for every $q \geq 1$

$$r_e(\vec{P}_n, q + 1) \leq c_2 n^{2q} \cdot (\log n)^2.$$

Moreover, a random tournament T_N on $N = \Theta(n^q \log n)$ vertices satisfies $T_N \rightarrow (\vec{P}_n)_{q+1}$ with high probability.

We stress that Theorem 1 proves that the size Ramsey number of a directed path in oriented graphs is asymptotically larger than the size Ramsey number in general directed graphs for any fixed number of colors.

In the course of the proof of Theorem 2, we actually prove the following asymmetric Ramsey property.

Theorem 3. *There is an absolute constant c such that the following holds. There is an oriented graph G with $cn^2 \cdot (\log n)^2$ edges such that every red–blue coloring of $E(G)$ contains a red path of length n or a blue path of length $n \log n$.*

Our approach also gives matching lower and upper bounds for the case of directed non-simple graphs with more than 2 colors, where antiparallel edges are allowed. This generalizes the results of Raynaud [16] and Reimer [17] to any fixed q .

Proposition 4. *The size Ramsey number of a directed path on n vertices for $q + 1$ colors in directed, non-simple graphs is $\Omega(n^{2q})$.*

Proposition 5. *The size Ramsey number of a directed path on n vertices for $q + 1$ colors in directed, non-simple graphs is $O(n^{2q})$.*

Note that here we show that the size Ramsey number (both for simple or non-simple directed graphs) of a directed path is polynomially dependent on the number of colors, in contrast to the undirected case where changing the number of colors changes the size Ramsey number by a constant factor, as follows from the above mentioned result of Beck [8].

1.2. Organization of the paper

The rest of the paper is organized as follows. In Section 2 we provide some notation and definitions and present well-known theorems that will be used later on. In Section 3 we present the proofs of our lower bounds, proving Theorem 1 and Proposition 4. In Section 3 we prove the upper bounds, showing Theorem 2, Theorem 3 and Proposition 5. Finally, in Section 5 we present some concluding remarks and open problems.

Throughout the paper we assume that the underlying parameter n is large enough. We do not try to optimize constants, and all logarithms are in base 2. We ignore all floor and ceiling signs whenever these are not crucial.

2. Preliminaries

All the graphs we consider are directed. An oriented (or simple) graph is a graph with no antiparallel (or opposite) edges. Given a set of vertices $A \subseteq V$, we denote by $E(A)$ the set of edges in the induced subgraph $G[A]$. The in-degree of a vertex v , denoted by $d^-(v)$, is the number of edges that are directed into v , and the out-degree of v , denoted by $d^+(v)$ is the number of edges directed from v . The degree of v is $d^+(v) + d^-(v)$, and we let $\Delta(G)$ be the maximum degree in a graph G and $\delta(G)$ be the minimum degree in G . For a vertex v we let $N^+(v) = \{u \in V : (v, u) \in E\}$ and call this set the *out-neighbors* of v . We also let $N^-(v) = \{u \in V : (u, v) \in E\}$ and call this set the *in-neighbors* of v . For a set of vertices A we let $N^+(A)$ be $\bigcup_{a \in A} N^+(a)$. A set of vertices is *acyclic* if it does not span a directed cycle. The *length* of a directed path is the number of edges it contains. The *edge density* of a directed graph $G = (V, E)$ is $\frac{|E|}{|V|^2}$.

The complete graph on n vertices, denoted by K_n , is an undirected graph for which every pair of vertices is connected. A *complete symmetric* directed graph is a non-simple directed graph where between every two vertices there are edges in both directions. A tournament is an oriented graph where between every two vertices there is an edge in exactly one of the directions.

A k -coloring of a graph is a mapping of the vertices to $\{1, \dots, k\}$ such that every two adjacent vertices are mapped to distinct values. The chromatic number of a graph G , denoted by $\chi(G)$, is the minimum k such that G is k -colorable. It is well known that every graph of maximum degree d is $(d + 1)$ -colorable. A Hamilton cycle in G is a cycle that visits every vertex in G exactly once.

We will use the following two theorems several times throughout the paper. The first one, also mentioned in the Introduction, is due to Raynaud [16] (see, e.g., [14] for a proof).

Theorem 2.1. *Let G be a complete symmetric directed graph on t vertices. Then every red–blue coloring of $E(G)$ contains a Hamilton cycle which is the union of two monochromatic paths. In particular, it contains a monochromatic path of length $t/2$.*

We also need the following simple theorem that was proved independently by Gallai [13] and Roy [18] (see, e.g., [15, Chapter 9, Problem 9], for a proof).

Theorem 2.2. *Let G be a directed graph with no path longer than t . Then G is $(t + 1)$ -colorable.*

3. Lower bounds

In this section we show that every oriented graph with relatively few edges has an edge coloring without a long monochromatic path. We first prove that every sparse oriented graph has a large acyclic set, and then use it to show that every graph admits a partition into a relatively small number of independent sets and acyclic sets. We conclude by showing that given such a partition we can color the edges with no long monochromatic path.

3.1. Sparse graphs have large acyclic sets

The following lemma is well known and easy.

Lemma 3.1. *Every tournament on n vertices has an acyclic set of size $\lfloor \log n \rfloor + 1$.*

Proof. We may assume that $n = 2^k$ for some integer k , as otherwise we can take any subtournament of size $2^{\lfloor \log n \rfloor}$. We prove it by induction on k , and note that the case $k = 0$ is trivial. Suppose that the induction hypothesis is true for k , and we next prove it for $k + 1$. Indeed, every tournament on 2^{k+1} vertices contains a vertex v with out-degree at least 2^k . By induction hypothesis, $N^+(v)$ contains an acyclic set A of size $k + 1$, and thus $A \cup \{v\}$ is an acyclic set of size $k + 2$, as required. \square

Since every oriented graph is a subgraph of a tournament, we get the following direct consequence.

Corollary 3.2. *Every oriented graph on n vertices has an acyclic set of size $\log n$.*

Our main result in this subsection is that every sparse oriented graph has a large acyclic set. This is a directed version of a lemma by Erdős and Szemerédi [11]. Although it can be derived from their result, we include here a self-contained proof for the sake of completeness.

Lemma 3.3. *There is an absolute constant $c > 0$ such that the following holds. Every oriented graph G with n vertices and at most εn^2 edges, where $\varepsilon \geq \frac{1}{n}$, contains an acyclic set of size $\frac{c \log n}{\varepsilon \log(1/\varepsilon)}$.*

Proof. Let G be an oriented graph with n vertices and εn^2 edges. We can assume that $\varepsilon < 1/4$ as otherwise the lemma follows easily by taking c small enough and applying Corollary 3.2. Let $G' = (V, E)$ be the subgraph of G obtained by removing every vertex with in-degree greater than $2\varepsilon n$. Observe that we removed at most $n/2$ vertices from G and therefore $|V| \geq n/2$.

Let U be a maximum acyclic set in G' , and assume that $|U| < \frac{\log n}{10\varepsilon \log(1/\varepsilon)}$. Since $\varepsilon \geq 1/n$, we can also assume that $|U| < n/20$ as otherwise we are done. Since every vertex has in-degree at most $2\varepsilon n$, we have

$$\sum_{v \in U} d^-(v) < 2\varepsilon n|U|.$$

Let R^* be the set of vertices in $V \setminus U$ with at least $5\varepsilon|U|$ out-neighbors in U , then $|R^*| \leq \frac{2n}{5}$, as otherwise the total number of edges directed into U is more than $2\varepsilon n|U|$. Let $R = V \setminus (U \cup R^*)$ be the set of vertices outside U with at most $5\varepsilon|U|$ out-neighbors in U , and we conclude that

$$|R| \geq n/2 - 2n/5 - |U| \geq n/20.$$

For every vertex $v \in R$ we define a set of vertices $S_v \subseteq U$ of size exactly $5\varepsilon|U|$ that contains $N^+(v) \cap U$. Using the inequality $\binom{n}{k} \leq (\frac{en}{k})^k$, we get that the total possible number of subsets of U of this size is

$$\binom{|U|}{5\varepsilon|U|} \leq \left(\frac{e}{5\varepsilon}\right)^{5\varepsilon|U|} \leq 2^{\frac{\log n}{2 \log(1/\varepsilon)} \cdot \log \frac{e}{5\varepsilon}} \leq n^{1/2}.$$

Therefore, by the pigeonhole principle, there is a set $R' \subseteq R$ of size at least $\frac{n}{20n^{1/2}} = \frac{n^{1/2}}{20}$ such that $|N^+(R') \cap U| \leq 5\varepsilon|U|$.

By Corollary 3.2, R' contains an acyclic set $R'' \subseteq R'$ of size at least $\frac{1}{2} \cdot (\log n - 10)$. Note that $R'' \cup (U \setminus N^+(R'))$ is an acyclic set of size at least

$$|R''| + |U| - 5\varepsilon|U| \geq |U| + \frac{1}{2}(\log n - 10) - \frac{\log n}{2 \log(1/\varepsilon)} > |U|,$$

assuming that $\varepsilon < 1/4$ and n is large enough. Therefore, we get a contradiction as U is not a maximum acyclic set, and the lemma follows. \square

3.2. Acyclic colorings and coloring acyclic sets

In this subsection we give two building blocks that will be used later in the proof of Theorem 1. We first consider the case where we are given a coloring of the edges in some induced disjoint sets without a long monochromatic path, and we wish to color the edges between them while keeping the length of a longest monochromatic path relatively small. We then consider the case when we are given an acyclic set, and show how to color its edges with no long monochromatic path.

Given a set of vertices A and a coloring φ of $E(A)$, let $\ell_\varphi(A)$ be the length of a longest monochromatic path of in A with respect to φ .

We prove the following.

Lemma 3.4. *Let A_1, \dots, A_k be disjoint sets of vertices in a directed graph, and let φ be a coloring of $\bigcup_{i=1}^k E(A_i)$ with $(q + 1)$ colors such that for every $1 \leq i \leq k$ we have $\ell_\varphi(A_i) \leq r$. Then φ can be extended to a $(q + 1)$ coloring φ' of $E(\bigcup_{i=1}^k A_i)$ such that*

$$\ell_{\varphi'}\left(\bigcup_{i=1}^k A_i\right) \leq q(r + 1) \cdot \lceil \sqrt[q]{k} \rceil.$$

Proof. Let s be the minimal integer such that $k \leq s^q$. For each $1 \leq i \leq k$, we denote by (i) the representation of i in base- s with exactly q digits. That is, we represent each index as a vector of length q , letting $(i)_y$ be the y th coordinate of i , and for each $1 \leq y \leq q$, $0 \leq (i)_y \leq s - 1$.

We now define an acyclic coloring of the edges between the sets. For two sets A_i and A_j , we color all the edges from A_i to A_j as follows. If there is an index y such that $(i)_y < (j)_y$, we color all the edges in color y (if more than one choice of y is possible, choose one arbitrarily). Otherwise, we color all the edges from A_i to A_j in color $q + 1$.

Note that by the definition of the new coloring, if a monochromatic path leaves a set A_i it will never return to this set. Also, if P is a monochromatic path colored by $1 \leq y \leq q$, then the y th coordinates of all the sets visited by the path are all distinct, and therefore there are at most s sets in the path. For a monochromatic path P that is colored by $(q + 1)$, if A_j is visited by P after it visits A_i then $(i)_y \geq (j)_y$ for every coordinate $1 \leq y \leq q$, and there is a strict inequality in at least

one coordinate. Therefore, the sums of the coordinates of all the sets visited by the path are all distinct, and hence such a path visits at most sq distinct sets.

We conclude that every monochromatic path colored by $1 \leq y \leq q$ visits at most $s \leq sq$ distinct sets, and every monochromatic path colored by $q + 1$ visits at most sq sets.

A longest monochromatic path contains at most r edges from each A_i , plus one edge that leaves A_i . It visits at most $qs = q \lceil \sqrt[q]{k} \rceil$ sets. Therefore, the length of this path is bounded by $q(r + 1) \cdot \lceil \sqrt[q]{k} \rceil$, as claimed. \square

In particular, we get the following immediate corollary, that generalizes an argument of Reimer [17].

Corollary 3.5. *Let G be a k -colorable graph. There is a $(q + 1)$ -coloring of $E(G)$ with no monochromatic path longer than $q \lceil \sqrt[q]{k} \rceil$.*

We next show similarly that one can color the edges of an acyclic graph Z without creating a long monochromatic path, given an upper bound on the length of any directed path in Z .

Lemma 3.6. *Let Z be an acyclic graph in which a longest directed path has t edges. There is a q -coloring of $E(Z)$ with no monochromatic path longer than $\lceil \sqrt[q]{t + 1} \rceil$.*

Proof. Let s be the minimal number such that $t + 1 \leq s^q$, and we prove that there is a q -coloring with no monochromatic path longer than s . For a vertex v let $\ell_Z(v)$ be the length of a longest path in Z that ends in v , and for each $0 \leq i \leq t$ let

$$Z_i = \{v : \ell_Z(v) = i\}.$$

Note that these sets are well defined since Z is an acyclic graph. Moreover, each Z_i is an independent set and all the edges in the graph are directed from Z_i to Z_j for $j > i$. Let (i) be the encoding of the number i in base s with exactly q digits. Observe that for each $j > i$ there is an index $1 \leq y \leq q$ for which $(j)_y > (i)_y$. We now define the coloring as follows. For $j > i$, we color all the edges from Z_i to Z_j in color y where y is an index such that $(j)_y > (i)_y$. If there is more than one feasible choice of y , we choose one of them arbitrarily.

Again, we get that in every monochromatic path of color y , all the sets Z_i that are visited by the path have distinct y th coordinate, and there is at most one vertex from each Z_i . We conclude that every monochromatic path contains at most s vertices, as required. \square

We will also need the following well-known and simple claim, and we give its proof for completeness.

Claim 3.7. *Let G be an undirected graph with m edges. Then $\chi(G) \leq 2\sqrt{m}$.*

Proof. Every optimal vertex coloring contains an edge between every two color classes, hence $m \geq \binom{\chi(G)}{2}$ and we get that $\chi(G) \leq 2\sqrt{m}$ as required. \square

Proof of Theorem 1. Let $G = (V, E)$ be a directed graph with $\frac{c_1 n^{2q} (\log n)^{1/q}}{(\log \log n)^{(q+2)/q}}$ edges. Define

$$X = \left\{ v \in V : d^+(v) + d^-(v) \leq \left(\frac{n}{2q} \right)^q \right\}.$$

Note that $\Delta(G[X]) \leq \left(\frac{n}{2q} \right)^q$ (when $G[X]$ is considered here as an undirected graph), and therefore, $\chi(G[X]) \leq \left(\frac{n}{2q} \right)^q + 1$. By Corollary 3.5, there is a $(q + 1)$ -coloring of $E(X)$ with no monochromatic path longer than $n/2 + q$.

Let $Y = V \setminus X$, and $|Y| = m$. Since $|E(G)| \leq \frac{c_1 n^{2q} (\log n)^{1/q}}{(\log \log n)^{(q+2)/q}}$ and every vertex in Y has at least $(\frac{n}{2q})^q$ incident edges, we have by double counting

$$m \leq \frac{2|E|}{(\frac{n}{2q})^q} \leq \frac{2c_1(2q)^q n^q (\log n)^{1/q}}{(\log \log n)^{(q+2)/q}}. \tag{3.1}$$

Consider the following procedure that partitions most of the vertices in Y into $\Theta(\log \log n)$ families $Y^{(1)}, Y^{(2)}, \dots$, where each family is composed of just a few acyclic sets.

Roughly speaking, we partition our vertices into acyclic sets as follows. We maintain an index i , and at the i th step we find acyclic sets that cover approximately $m/2^i$ vertices, and group these sets to a family $Y^{(i)}$. To this end, at the beginning of each step we fix a number a_i for which we are guaranteed by Lemma 3.3 that throughout the i th step we can always find an acyclic set of size a_i . By the end of the procedure, most of the vertices are partitioned into families, and each family is composed of acyclic sets of the same size.

We initiate the procedure by taking $i = 1$ and $Y' = Y$, where i represents the step number. At the beginning of step i , we fix $\varepsilon_i = \frac{|E(Y')|}{(m/2^i)^2}$ and $a_i = \frac{c \log(m/2^i)}{\varepsilon_i \log(1/\varepsilon_i)}$ (where c is an absolute constant given by Lemma 3.3), and also set $j = 1$. The step ends when $|Y'| \leq m/2^i$.

At the i th step, we repeatedly find an acyclic set $Z_{i,j}$ in Y' of size a_i , and set $Y' := Y' \setminus Z_{i,j}$ and then increase j by one.

The procedure terminates when $|E(Y')| \leq \frac{n^{2q}}{(16q)^{2q}}$.

We first stress that for every i, j we can find an acyclic set $Z_{i,j}$ as claimed. Observe that during each step, the value ε_i is fixed, the number of edges does not increase, and the number of vertices is at least $m/2^i$. Hence, ε_i is an upper bound on edge density during the step. Therefore, Lemma 3.3 guarantees the existence of an acyclic set $Z_{i,j}$ of size exactly a_i .

Let $Y^{(i)} = \{Z_{i,j}\}$ be the family of acyclic sets that is constructed at step i , and let $k_i = |Y^{(i)}|$. We next bound the size of a longest monochromatic path in $\bigcup_{j=1}^{k_i} Z_{i,j}$ for every i , and start with $i = 1$.

Let ε be the edge density in Y . Then $\varepsilon = \varepsilon_1/4$. It is easy to verify that $\frac{\log m - 1}{\log(1/\varepsilon) - 2} \geq \frac{\log m}{\log(1/\varepsilon)}$ (since clearly $m^2 \geq 1/\varepsilon$) and hence the size of each acyclic set is at least

$$\frac{c \log(m/2)}{\varepsilon_1 \log(1/\varepsilon_1)} = \frac{c(\log m - 1)}{4\varepsilon(\log(1/\varepsilon) - 2)} \geq \frac{c \log m}{4\varepsilon \log(1/\varepsilon)}.$$

As we complete the first step just after we cover at least $m/2$ vertices with acyclic sets, we get that the number of acyclic sets satisfies $k_1 \leq \frac{(m/2) \cdot 4\varepsilon \log(1/\varepsilon)}{c \log m} + 1$. Moreover, since all acyclic sets are of the same size, the size of each $Z_{1,j}$ is bounded by $\frac{m/2}{k_1 - 1}$.

By Lemma 3.6, we get that the edges of each acyclic set $E(Z_{1,j})$, for every $1 \leq j \leq k_1$, can be colored with $(q + 1)$ colors with no monochromatic path longer than $q^{+1} \sqrt{\frac{m/2}{k_1 - 1}} + 1$. We then can apply Lemma 3.4 to color the edges between Z_{1,j_1} and Z_{1,j_2} for each j_1, j_2 . The number of sets is k_1 , and the size of a longest monochromatic path in each of them is bounded by $q^{+1} \sqrt{\frac{m/2}{k_1 - 1}} + 1$. We conclude that the edges of $\bigcup Z_{1,j}$ admit a $(q + 1)$ -coloring with no monochromatic path longer than

$$q(\sqrt[q]{k_1} + 1) \cdot \left(\sqrt[q+1]{\frac{m/2}{k_1 - 1}} + 2 \right) \leq 2q \left(\frac{m^q k_1}{2^q} \right)^{\frac{1}{q(q+1)}}.$$

Note that the edge density ε in Y satisfies

$$\varepsilon \leq \frac{|E|}{m^2} = \frac{c_1 n^{2q} (\log n)^{1/q}}{m^2 (\log \log n)^{(q+2)/q}}. \tag{3.2}$$

Moreover, since we assume that the number of edges is at least $\frac{n^{2q}}{(16q)^{2q}}$ (as otherwise the procedure terminates before this step begins), we get that $\frac{1}{\varepsilon} = \frac{m^2}{|E(Y)|} \leq \frac{m^2(16q)^{2q}}{n^{2q}}$ and therefore by (3.1)

$$\log(1/\varepsilon) \leq \log\left(\frac{(16q)^{2q}}{n^{2q}} \cdot \frac{4c_1^2(2q)^{2q}n^{2q}(\log n)^{2/q}}{(\log \log n)^{(2q+4)/q}}\right) \leq 2 \log \log n, \tag{3.3}$$

assuming that n is large enough.

Therefore, we have the following bound:

$$\begin{aligned} \frac{m^q k_1}{2^q} &\leq \frac{m^{q+1} \varepsilon \log(1/\varepsilon)}{2^{q-2} c \log m} \\ &\leq \frac{m^{q-1} c_1 n^{2q} (\log n)^{1/q} \log \log n}{2^{q-3} c \log m \cdot (\log \log n)^{(q+2)/q}} \\ &\leq \frac{8(2q)^{q(q-1)} c_1^q n^{q(q+1)} \log n \log \log n}{(\log \log n)^{q+2} c \log m} \\ &\leq \frac{8(2q)^{q(q-1)} c_1^q n^{q(q+1)}}{c(\log \log n)^{q+1}}, \end{aligned}$$

where the first inequality follows from the bound on k_1 , the second one from substituting ε and $\log 1/\varepsilon$ and using (3.2) and (3.3), and the third one from applying estimate (3.1) on m . The last inequality follows from the bound $\log n \leq \log m$ that clearly holds.

Taking $c_1 < \frac{c^{1/q}}{8(2q)^q \cdot (16q^2)^{q+1}}$ (we do not try to optimize the dependence on q here), we get that

$$2q \left(\frac{m^q k_1}{2^q}\right)^{\frac{1}{q(q+1)}} \leq \frac{1}{8q} \cdot \frac{n}{\sqrt[q]{\log \log n}}.$$

It is not difficult to verify that taking smaller values of m only decreases the last expression. Hence, by repeating the same coloring method, we get that for each i , there is a $(q + 1)$ -coloring of the edges spanned by the union of the sets in $Y^{(i)}$ with no monochromatic path longer than $\frac{1}{8q} \cdot \frac{n}{\sqrt[q]{\log \log n}}$. Note that the number of vertices in Y' is decreased by a factor of at least 2 in each step. Moreover, the procedure terminates when $|E(Y')| \leq \frac{n^{2q}}{(16q)^{2q}}$, and thus it essentially terminates if $|Y'| \leq \frac{n^q}{(16q)^q}$. Hence by applying (3.1) we get that the number of families is bounded by

$$\log \frac{(16q)^q m}{n^q} \leq \log\left(\frac{(16q)^q}{n^q} \cdot \frac{2c_1(2q)^q n^q (\log n)^{1/q}}{(\log \log n)^{(q+2)/q}}\right) \leq \log \log n,$$

assuming again that n is large enough.

Let $\mathcal{W} = \bigcup_{i,j} Z_{i,j}$ be the set of vertices that were covered throughout the procedure. We color all the edges in $E(\mathcal{W})$ whose endpoints belong to sets from distinct $Y^{(i)}$'s with $(q + 1)$ colors according to Lemma 3.4. Since the length of each monochromatic path spanned by the sets of each family is bounded by $\frac{1}{8q} \cdot \frac{n}{\sqrt[q]{\log \log n}}$, and the number of families is bounded by $\log \log n$, we get that the induced subgraph of all vertices that are contained in sets from $\bigcup Y^{(i)}$ admits a $(q + 1)$ -coloring with no path longer than

$$q \cdot \left(\frac{1}{8q} \cdot \frac{n}{\sqrt[q]{\log \log n}} + 1\right) \cdot (\sqrt[q]{\log \log n} + 1) \leq n/4.$$

Finally, after the procedure terminates, we are left with a set Y' that satisfies $|E(Y')| \leq \frac{n^{2q}}{(16q)^{2q}}$. By Claim 3.7, we get that $G[Y']$ is $\frac{2n^q}{(16q)^q}$ -colorable. Hence Corollary 3.5 implies that $E(Y')$ admits a $(q + 1)$ -coloring with no monochromatic path longer than $n/8 + 1$.

We therefore found a partition of $V(G)$ into three sets X, Y', \mathcal{W} . The set $E(X)$ admits a $(q + 1)$ -coloring with no monochromatic path longer than $n/2 + q$. The set $E(Y')$ admits a coloring with no monochromatic path longer than $n/8$, and $E(\mathcal{W})$ admits a coloring with no monochromatic path longer than $n/4$. We color all the edges that are either from X to Y' or from X to \mathcal{W} or from Y' to \mathcal{W} by the first color, and all the edges that are either from Y' to X or from \mathcal{W} to X or from \mathcal{W} to Y' by the second color. Every monochromatic path in $E(G)$ that leaves one of these three sets does not return to this set. Therefore, the length of a longest monochromatic path is bounded by the sum of the lengths of longest monochromatic paths in X, Y', \mathcal{W} , plus at most two edges between them. Thus $E(G)$ admits a coloring with no monochromatic path longer than $n/2 + q + n/8 + 1 + n/4 + 2 < n$, and Theorem 1 follows. \square

Proof of Proposition 4. Let G be a non-simple directed graph with $(\frac{n}{3q})^{2q}$ edges. By Claim 3.7, G is $2(\frac{n}{3q})^q$ -colorable, and hence by Corollary 3.5 it admits a $(q + 1)$ -coloring with no monochromatic path longer than n , and the proposition follows. \square

4. Upper bounds

In this section we provide an oriented graph for which every q -coloring of its edges contains a long monochromatic path. We start with the case of two colors. In Subsection 4.1 we define the notion of a k -pseudorandom digraph, and show that a random tournament on n vertices is $\Theta(\log n)$ -pseudorandom with high probability. We next show that every red–blue coloring of a k -pseudorandom digraph on n vertices contains a directed red path of length $\Omega(\frac{n}{k})$ or a directed blue path of length $\Omega(n)$. This proves Theorem 3. We conclude this section by showing how to reduce the case of any fixed number of colors to the case of two colors, proving Theorem 2 and Proposition 5. For notational convenience we use n in this section to denote the number of vertices in a Ramsey digraph G , rather than the length of a target path P_n as in Theorem 2, Theorem 3 and Proposition 5.

4.1. Pseudorandom digraphs

We start with the following natural definition for k -pseudorandomness of directed graphs.

Definition 4.1. We say that a directed graph G is k -pseudorandom if for every two disjoint sets A, B such that $|A|, |B| \geq k$, there is at least one edge of G from A to B .

We first show the existence of a $\Theta(\log n)$ -pseudorandom digraph by showing that a random tournament satisfies this property with high probability.

Claim 4.2. A random tournament on n vertices is $2 \log n$ -pseudorandom with high probability.

Proof. Let T_n be a random tournament and fix two disjoint sets A, B of size k in T_n . Since every edge is oriented in each way uniformly and independently of the other edges, the probability that all the edges are directed from B to A is exactly 2^{-k^2} . Since clearly $(k!)^2 \geq 2^{2k-2}$ there are at most $\binom{n}{k}^2 \leq \frac{n^{2k}}{(k!)^2} \leq 2^{2k \cdot \log n - 2k + 2}$ choices of ordered pairs of sets of size k , and thus by the union bound the probability that there are two sets of size k for which all the edges are oriented in one direction is bounded by

$$2^{2k \cdot \log n - k^2 - 2k + 2} = o(1),$$

for $k = 2 \log n$. \square

We also need the following property of k -pseudorandom digraphs.

Claim 4.3. Let G be a k -pseudorandom directed graph, and let A_1, A_2, \dots, A_t be disjoint sets, each of size at least $2k$. Then there is a directed path $v_1 v_2 \dots v_t$, where for each $1 \leq i \leq t$, $v_i \in A_i$.

Proof. We say that a vertex $u_j \in A_j$ is good if there is a directed path $u_j u_{j+1} \dots u_t$ such that $u_s \in A_s$, $s = j, \dots, t$. Clearly, our goal is to prove that there is a good vertex in A_1 . Denote by A_j^* the set of good vertices in A_j . By definition, every vertex in A_t is good, and thus $|A_t^*| \geq 2k$. Also, if $u_{j+1} \in A_{j+1}^*$ and there is an edge from u_j to u_{j+1} , then $u_j \in A_j^*$. Using a reverse induction, assume that we know that for some $j \leq t$, $|A_j^*| \geq k$, then since G is k -pseudorandom all but at most k of the vertices in A_{j-1} have an edge to A_j^* . Therefore, all but at most k of the vertices in A_{j-1} are actually in A_{j-1}^* , and thus $|A_{j-1}^*| \geq k$. We conclude that $|A_1^*| \geq k$ and thus there is a path as required. \square

The following lemma shows that every k -pseudorandom graph contains a long path. The proof follows ideas from [9,7].

Lemma 4.4. Let G be a k -pseudorandom oriented graph on n vertices. Then G contains a directed path of length $n - 2k + 1$.

Proof. Recall that the DFS (Depth First Search) is a graph algorithm that visits all the vertices of a (directed or undirected) graph G as follows. It maintains three sets of vertices, letting S be the set of vertices which we have completed exploring them, T be the set of unvisited vertices, and $U = V(G) \setminus (S \cup T)$, where the vertices of U are kept in a stack (a last in, first out data structure). The DFS starts with $S = U = \emptyset$ and $T = V(G)$.

While there is a vertex in $V(G) \setminus S$, if U is non-empty, let v be the last vertex that was added to U . If v has a neighbor $u \in T$, the algorithm inserts u to U and repeats this step. If v does not have a neighbor in T then v is popped out from U and is inserted to S . If U is empty, the algorithm chooses an arbitrary vertex from T and pushes it to U .

We are now proceed to the proof of the lemma. We execute the DFS algorithm. We let again S, T, U be three sets of vertices as defined above. At the beginning of the algorithm, all the vertices are in T , and at each step a single vertex either moves from T to U or from U to S . At the end of the algorithm, all the vertices are in S . Therefore, at some point we have $|S| = |T|$. Observe crucially that all the vertices in U form a directed path, and that there are no edges from S to T . We conclude that $|S| \leq k - 1$, and therefore $|U| \geq n - 2k + 2$, so there is a directed path with $n - 2k + 1$ edges in U , as required. \square

4.2. The case of two colors

In this subsection we prove that every k -pseudorandom directed graph on n vertices has a monochromatic red path of length $\Omega(\frac{n}{k})$ or a monochromatic blue path of length $\Omega(n)$, and this will prove Theorem 3. We prove the following main lemma.

Lemma 4.5. Let G be a k -pseudorandom directed graph on n vertices. Then every red–blue coloring of its edges yields a red directed path of length $\frac{n}{28k}$ or a blue directed path of length $n/28$.

Proof. Fix a red–blue coloring of $E(G)$. Let G_R be the red graph, that contains only the red edges.

If G_R contains a directed path of length $\frac{n}{14k}$, we are done. Otherwise, by the Gallai–Roy theorem (Theorem 2.2), G_R is $\frac{n}{14k}$ -colorable and therefore has a partition into $\frac{n}{14k}$ independent sets. We partition these independent sets into sets of size exactly $7k$, rounding down the remaining vertices (in particular, we remove every independent set smaller than $7k$). Since we remove at most $7k \cdot \frac{n}{14k} = n/2$ vertices, we remain with $t \geq \frac{n}{14k}$ independent sets B_1, B_2, \dots, B_t , each containing exactly $7k$ vertices.

Note that each B_i , $1 \leq i \leq t$, spans a k -pseudorandom graph and contains only blue edges. Therefore, by Lemma 4.4, each B_i contains a blue path of length at least $5k$. Since there is a directed edge from the last k vertices of this path to the first k vertices in the path, we conclude that each B_i contains a directed blue cycle C_i of length at least $3k$.

We next construct an auxiliary complete symmetric directed auxiliary graph H on t vertices in which each vertex corresponds to a cycle C_i , and with a slight abuse of notation we denote these vertices by C_1, C_2, \dots, C_t . We color the edge from C_i to C_j by blue if there are at least k vertices in C_i that have blue edges directed towards C_j , and red otherwise. Since H is complete and symmetric, by Raynaud’s theorem (Theorem 2.1), it has a monochromatic path of length $t/2 \geq \frac{n}{28k}$. Let $C_{i_1}, C_{i_2}, \dots, C_{i_{t/2}}$ be the vertices in H along the path, each such vertex represents a cycle. We complete the proof by considering the following two cases.

H contains a red path of length $t/2$. For each C_{i_j} there is a set R_{i_j} for which only red edges are going towards $C_{i_{j+1}}$. Moreover, for every $1 \leq j \leq t/2$ we have $|R_{i_j}| \geq 2k$. Observe that all the edges from R_{i_j} to $R_{i_{j+1}}$ are red, and therefore by Claim 4.3, there is a red path of length $t/2 \geq \frac{n}{28k}$ with exactly one vertex from each R_{i_j} , as required.

H contains a blue path of length $t/2$. Call a vertex in C_{i_j} an *endpoint* if it has a blue edge towards $C_{i_{j+1}}$. By the assumption, there are at least k endpoints in C_{i_j} for every $1 \leq j \leq t/2$, and therefore from each vertex in C_{i_j} there is a path of length at least $k - 1$ that ends at some endpoint, in which we can travel along one additional edge towards $C_{i_{j+1}}$. We construct a blue path of length $\frac{n}{28}$ by taking an arbitrary path of length $k - 1$ that ends at some endpoint in C_{i_1} , moving through the endpoint to C_{i_2} , walking through such a path to an endpoint of C_{i_2} and so on till we arrive at $C_{i_{t/2}}$, where we can again walk along a path of length at least $3k - 1$ (that visits all the vertices in $C_{i_{t/2}}$). We conclude that there is a blue path of length at least $k \times \frac{n}{28k} = \frac{n}{28}$, as claimed.

Lemma 4.5 and Theorem 3 follow. \square

4.2.1. Explicit constructions

Given an explicit construction of a k -pseudorandom tournament on n vertices, our approach shows that every red–blue coloring of such a tournament has a monochromatic path of length $\Omega(\frac{n}{k})$. A simple construction of a k -pseudorandom tournament is given by Quadratic Residue tournaments, defined as follows (see [4, Chapter 9]). Let $p \equiv 3 \pmod 4$ be a prime number. The vertices of the tournament T_p are all the elements in the finite field \mathbb{Z}_p . For two vertices i and j , there is an edge from i to j if and only if $i - j$ is a quadratic residue. It can be shown that since $p \equiv 3 \pmod 4$, -1 is a quadratic nonresidue and therefore for each i, j either there is an edge from i to j or an edge from j to i but not both. This construction gives $k = \Theta(\sqrt{n})$ (see [2] and [4]).

A much better construction can be based on explicit constructions of Ramsey type bipartite graphs that are provided in [5,6]. In particular, Barak et al. [6] provided an explicit construction of an N by N zero-one matrix such that every K by K sub-matrix is neither an all-ones matrix nor an all-zeros matrix, where $K = 2^{\log^{o(1)} N}$. Observe that this implies the existence of a $2K$ -pseudorandom tournament on N vertices. Indeed, given such a matrix M , define a tournament $T = ([n], E)$ as follows. For every $1 \leq i < j \leq n$, direct the edge from i to j if $M_{i,j} = 1$ and direct it from j to i if $M_{i,j} = 0$. Note that T depends only on the entries above the diagonal of M . Consider two disjoint sets of vertices A, B in T of size $2K$. Without loss of generality we can assume that there are a set $A' \subset A$ and a set $B' \subseteq B$, each of size K , such that all the indices of vertices in A' precede all the indices of vertices in B' . Therefore, the set of edges between A' and B' corresponds to some K by K sub-matrix above the diagonal, which is not a constant matrix. We conclude that there is at least one edge from A' to B' , and at least one edge from B' to A' . This provides an explicit construction of a $2^{\log^{o(1)} N}$ -pseudorandom tournament on N vertices.

4.3. The general case

Here we prove by induction on q that for every k -pseudorandom directed graph G on $28kn^q$ vertices we have $G \rightarrow (\overrightarrow{P_n})_{q+1}$. Theorem 2 will clearly follow by the fact that a random tournament is $\Theta(\log n)$ -pseudorandom with high probability (Claim 4.2).

The base case of the induction ($q = 1$) follows directly from Lemma 4.5. Suppose that the result holds for q colors and we next prove it for $q + 1$ colors. Indeed, let G be a k -pseudorandom graph on $28kn^q$ vertices. Fix a coloring of $E(G)$ with the colors $1, 2, \dots, q + 1$. Denote by $G_{q+1} \subseteq G$ the subgraph with all edges that are colored $q + 1$. If G_{q+1} contains a monochromatic path of length n we

are done. Otherwise, by the Gallai–Roy theorem (Theorem 2.2), we get that G_{q+1} is n -colorable and hence contains an independent set A of size $\frac{28kn^q}{n} = 28kn^{q-1}$.

Note that A spans a subgraph of G and hence $G[A]$ is k -pseudorandom. Also, the colors of all the edges of $G[A]$ are among $\{1, 2, \dots, q\}$. Therefore, by the induction hypothesis $G[A] \rightarrow (\vec{P}_n)_q$, and we conclude that G contains a monochromatic path of length n , as desired. \square

Proof of Proposition 5. The proof for non-simple directed graphs follows similar lines. Note that the Gallai–Roy theorem is valid for non-simple directed graphs as well. Therefore, we can use the same induction on q . The base case ($q = 1$) follows from Raynaud’s theorem (Theorem 2.1). Suppose that the result holds for q colors, the correctness for $q + 1$ colors follows by taking a complete symmetric graph on n^q vertices, and considering the subgraph with all edges colored by the $(q + 1)$ th color. If this graph contains a directed path of length n we are done, otherwise we find an induced subgraph of order n^{q-1} in which all edges are colored $1, 2, \dots, q$, and applying the induction hypothesis, Proposition 5 follows. \square

5. Concluding remarks and open problems

We proved nearly tight bounds for the size Ramsey number of a directed path for oriented graphs. We proved that every red–blue coloring of the edges of a k -pseudorandom graph on n vertices contains a red path of length $\Omega(\frac{n}{k})$ or a blue path of length $\Omega(n)$, but it might be the case that this approach can also give better symmetric Ramsey bounds. An interesting question is whether every red–blue coloring of a k -pseudorandom graph contains a monochromatic path of length $\Omega(\frac{n}{\sqrt{k}})$. Clearly every progress in this direction will improve our upper bounds.

Another related question is about the asymptotic behavior of the maximum length of a monochromatic path in every red–blue coloring of a random tournament. Here we proved that every tournament T has a coloring with no monochromatic path longer than $O(\frac{n}{\sqrt{\log n}})$, and also that with high probability a random tournament T_n has a monochromatic path of length $\Omega(\frac{n}{\log n})$ in every red–blue coloring. It would be very interesting to close the gap between these bounds.

When proving the lower bound on the size Ramsey number, we study the minimal number k for which a certain graph can be partitioned into k acyclic sets. This parameter was studied, e.g., in [1], and it was conjectured that in every oriented graph $G = (V, E)$ there is an acyclic set of size $(1 + o(1)) \frac{|V|^2}{|E|} \cdot \log \frac{|E|}{|V|}$. If this conjecture is true, then our lower bound can be slightly improved.

It is easy to verify that there is no k -pseudorandom oriented graph on n vertices for $k \leq \frac{\log n}{2}$, as every such graph has an acyclic set of size $\log n$ and therefore has two sets of size $\frac{\log n}{2}$ with no edges in one of the directions. On the other hand we proved that for $k = 2 \log n$ such graphs do exist. Hence, it will be interesting to determine the minimum k for which there is a k -pseudorandom oriented graph on n vertices. Another appealing question is to provide better explicit constructions of k -pseudorandom oriented graphs.

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