C_4-FREE SUBGRAPHS WITH LARGE AVERAGE DEGREE

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ABSTRACT

Motivated by a longstanding conjecture of Thomassen, we study how large
the average degree of a graph needs to be to imply that it contains a C_4-free
subgraph with average degree at least t. Kühn and Osthus showed that
an average degree bound which is double exponential in t is sufficient. We
give a short proof of this bound, before reducing it to a single exponential.
That is, we show that any graph G with average degree at least $2^{ct^2 \log t}$
(for some constant $c > 0$) contains a C_4-free subgraph with average degree
at least t. Finally, we give a construction which improves the lower bound
for this problem, showing that this initial average degree must be at least
$t^{3-o(1)}$.

1. Introduction

The girth of a graph G, denoted $g(G)$, is the length of a shortest cycle in G. A celebrated conjecture of Thomassen [13] from 1983 says that, for each t and g, there is some $f(t, g)$ such that every graph G with average degree $d(G) \geq f(t, g)$ contains a subgraph with girth at least g and average degree at least t.

This is straightforward for regular graphs, or, more generally, for graphs whose maximum degree $\Delta(G)$ is at most a constant multiple of $d(G)$. Indeed, given an n-vertex d-regular graph G, take a subgraph H of G by keeping every edge with probability $p = d^{-(g-1)/g}/2$. The expected number of edges in H is $ndp/2 = nd^{1/g}/4$ and the expected number of cycles with length at most g in H is at most $nd^{g-1}p^g = n/2^{g-1}$. Thus, deleting an edge from each short cycle of a typical such H gives a graph with average degree at least $d^{1/g}/4$ and girth larger than g.

This argument can, of course, be used for any graph containing an almost regular subgraph with high average degree. Pyber, Rödl and Szemerédi [11] showed that any graph G whose average degree is at least logarithmic in $\Delta(G)$ has an r-regular subgraph (with r growing together with $d(G)$). This can therefore be used to prove Thomassen's conjecture for such graphs. On the other hand, Pyber, Rödl and Szemerédi [11] also proved that there are graphs G with average degree at least $c \log \log \Delta(G)$ that do not contain even a 3-regular subgraph. This shows that one cannot prove Thomassen's conjecture through reduction to the regular case. However, more progress has been made for graphs where the maximum degree is bounded by a function of the average degree. Indeed, Dellamonica and Rödl [3] proved the conjecture for each graph G with average degree at least $\alpha(\log \log \Delta(G))^\beta$, for some constants α and β. Nevertheless, in general Thomassen's conjecture remains widely open.

For general graphs, as every graph contains a bipartite (and hence odd cycle free) subgraph with at least half of the edges, Thomassen's conjecture is trivial for each $g \leq 4$. The only non-trivial case of Thomassen's conjecture obtained so far was by the breakthrough paper of Kühn and Osthus [7], who proved the case where $g = 6$. More precisely, they showed that, for some constant $c > 0$, every graph G with average degree at least $2^{2^{ct^3}}$ contains a subgraph with average degree at least t and girth at least 6. An alternative proof of this, with a similar double exponential bound, was later given by Dellamonica, Koubek, Martin and Rödl [2]. Using their approach, McCarty [10] recently proved that a bipartite

graph G with large average degree and no copy of $K_{t,t}$ contains an induced C_4-free subgraph with many edges. The assumption that G is bipartite can be further removed, using the result from [8], which says that an H-free graph with average degree d contains an induced bipartite subgraph with average degree at least $\log^{1-o(1)} d$.

With the establishment of an upper bound for the $g = 6$ case of Thomassen's conjecture, it is natural to ask whether this double exponential bound can be improved. In this paper, we study this, starting with a very short proof of the result of Kühn and Osthus. Building on this, we then give the following single exponential bound.

THEOREM 1.1: *There exists $c > 0$ such that, for each $t \in \mathbb{N}$, every graph with average degree at least $2^{ct^2 \log t}$ contains a graph with average degree at least t and girth at least 6.*

On the other hand, it is known that there are graphs with average degree $(1+o(1))t^2$ which contain no subgraphs with average degree at least t and girth bigger than 4. Indeed, this follows easily from well-known bounds on the Turán numbers of the 4-cycle proved by Erdős, Rényi and Sós [6] (see also Problem 10.36(a) in [9]). They showed that the maximum number of edges in any n-vertex C_4-free graph is at most $(1/2 + o(1))n^{3/2}$. Therefore the complete graph on $n = (1 + o(1))t^2$ vertices has average degree $(1 + o(1))t^2$ yet no C_4-free subgraph with average degree at least t. Thus, the bound in Theorem 1.1 must be at least quadratic in t. However, the discussion above on techniques for almost regular graphs suggests we should look to graphs with irregular degree sequences to improve this. By giving a new construction, we show the bound in Theorem 1.1 must be at least roughly cubic in t, as follows.

THEOREM 1.2: *There is a constant $c > 0$ such that, for all t, there exists a graph with average degree at least t^3 yet no C_4-free subgraph with average degree at least $ct \log t$.*

NOTATION. Given a graph G we denote by $\Delta(G)$ the maximum degree of G, by $e(G)$ the number of edges of G and by $d(G)$ the average degree of G. For a vertex x we denote by $N_G(x)$ the set of neighbours of x and by $N_G^2(x)$ the set of vertices in G of distance two from x. For any pair of vertices $x, y \in V(G)$, the codegree $d_G(x, y)$ is the number of common neighbours of x and y in G. Given $A \subset V(G)$ and $x \in V(G)$ we denote by $d(x, A)$ the number of neighbours of x

in A. For a pair of disjoint vertex subsets A, B in G we denote by $e_G(A, B)$ the number of edges of G between A and B and by $G[A, B]$ the induced bipartite subgraph of G with all such edges. All logarithms are to the base 2.

2. Double exponential upper bound

In this section, we give a short proof of the following slightly stronger form of the result of Kühn and Osthus [7].

THEOREM 2.1: *There exists $c > 0$ such that every graph with average degree at least $2^{2^{ct^2}}$ contains a graph with average degree at least t and girth at least 6.*

Since, as is well known, every graph contains a bipartite subgraph with at least half of the edges, we can assume that G is bipartite. That is, we can assume the initial graph G has girth at least four and no odd cycles. In order to obtain a subgraph of G with girth at least 6, we need only find a dense subgraph which has no 4-cycles, that is, which is C_4-free. The main idea of the proof (inspired by that of Kühn and Osthus) is to find either a dense subgraph of G with small codegrees, and hence few 4-cycles, or a large complete bipartite subgraph of G. In the first case a further random subgraph is likely to be still dense but have no 4-cycles, and in the second case we use the following well known construction of Reiman for Zarankiewicz's problem (see Problem 10.15(a) in [9]). It is based on a projective plane and gives a dense C_4-free subgraph of the complete bipartite graph.

LEMMA 2.2: *If $s = 4k^2$, then $K_{s,s}$ has a C_4-free subgraph with average degree at least k.*

The next lemma, which is one of the main steps in the proof, finds either a dense subgraph with small codegrees or can be used to build a large complete bipartite subgraph, one vertex at a time.

LEMMA 2.3: *Let G be a bipartite graph with vertex classes A and B and let $\lambda \geq 1$. Then, either*

  (1) *there is some vertex $v \in B$ and sets $A' \subset N(v)$ and $B' \subset B \setminus \{v\}$ such that $d(G[A', B']) \geq \lambda$, or*
  (2) *there is a spanning subgraph $H \subset G$ with $d(H) \geq d(G)/(\lambda + 1)$ and $d_H(x, y) \leq \lambda$, for all distinct $x, y \in B$.*

Proof. Let $G = G_0$, $n = |B|$ and label $B = \{v_1, \ldots, v_n\}$. For each $i = 1, \ldots, n$, repeat the following. Given a spanning subgraph G_{i-1}, let V_i be the set of all vertices in $\{v_{i+1}, \ldots, v_n\}$ which have at least λ neighbours in the set $N_{G_{i-1}}(v_i)$. By definition, there are at least $\lambda|V_i|$ edges of G_{i-1} from V_i to $N_{G_{i-1}}(v_i)$. If in addition $e_{G_{i-1}}(V_i, N_{G_{i-1}}(v_i)) \geq \lambda d_{G_{i-1}}(v_i)$, then the induced subgraph $G_{i-1}[V_i, N_{G_{i-1}}(v_i)]$ has average degree at least λ. Therefore, (1) holds with $v = v_i$, $A' = N_{G_{i-1}}(v_i)$ and $B' = V_i$. Otherwise, if $e_{G_{i-1}}(V_i, N_{G_{i-1}}(v_i)) < \lambda d_{G_{i-1}}(v_i)$, remove all the edges of G_{i-1} between V_i and $N_{G_{i-1}}(v_i)$ to form G_i. Note that $v_i \notin V_i$, so that no edges adjacent to v_i are removed in this operation.

Suppose that (1) never holds, so that the process terminates with

$$H := G_n \subset G_{n-1} \subset \cdots \subset G_0 = G.$$

Now, for every $i < j$, by construction of G_i we have $d_{G_i}(v_i, v_j) < \lambda$. Thus $d_H(v_i, v_j) < \lambda$ for each $i \neq j$. Most importantly, edges incident to v_i in G_{i-1} are never subsequently removed. Indeed, in step i no edges incident to v_i are removed (as $v_i \notin V_i$), and, after step i, v_i has codegrees less than λ with all $v_j, j > i$ and we only remove edges incident to pairs with high codegree. Therefore, $d_{G_i}(v_i) = d_{G_{i-1}}(v_i)$.

Since we removed fewer than $\lambda d_{G_{i-1}}(v_i)$ edges from G_{i-1} to get G_i we have $d_H(v_i) = d_{G_{i-1}}(v_i) > (e(G_{i-1}) - e(G_i))/\lambda$. Therefore,

$$e(H) = \sum_{i \in [n]} d_H(v_i) > \sum_{i \in [n]} \frac{e(G_{i-1}) - e(G_i)}{\lambda} = \frac{e(G) - e(H)}{\lambda}.$$

Hence,

$$e(H) > e(G)/(\lambda + 1),$$

and thus $d(H) > d(G)/(\lambda + 1)$, namely, H satisfies (2). ∎

If a graph has small codegrees yet many edges, a typical random subgraph will be C_4-free with large average degree, as follows.

LEMMA 2.4: *There is some d_0 such that the following holds for each $d \geq d_0$. Let G be a bipartite graph with vertex classes A and B, with $|A| \geq |B|$, $d_G(x, y) \leq d^{1/5}$ for each $x, y \in B$ with $x \neq y$, $d(x) \leq d$ for each $x \in A$ and $d(G) \geq d^{3/4}$. Then, G contains a C_4-free subgraph with average degree at least $d^{1/4}$.*

Proof. Let $n = |A|$ and note that $n \leq |G| \leq 2n$. For each $v \in A$, there are at most d^2 distinct pairs of vertices $x, y \in N(v)$, and hence at most $d^{1/5} \cdot d^2 = d^{11/5}$ copies of C_4 in G containing v. Thus, G contains at most $nd^{11/5}$ copies of C_4. Let H be a random subgraph of G formed by including each edge independently at random with probability $p = 3d^{-1/2}$. Let X be the number of copies of C_4 in H. As $e(G) = n \cdot d(G)/2 \geq nd^{3/4}/2$, we have

$$\mathbb{E}(e(H) - X) \geq p \cdot e(G) - nd^{11/5}p^4 \geq 3nd^{1/4}/2 - 81nd^{1/5} \geq nd^{1/4}.$$

Thus, there is some subgraph $H \subset G$ with $e(H) - X \geq nd^{1/4}$. As $|G| \leq 2n$, removing an edge from each C_4 in H thus gives a C_4-free subgraph with average degree at least $d^{1/4}$.   ∎

We apply this lemma through the following corollary.

COROLLARY 2.5: *There is some d_0 such that the following holds for each $d \geq d_0$. Let G be a bipartite graph with average degree $d \geq t^4$ which contains no C_4-free graph with average degree at least t. Then, there is some vertex $v \in V(G)$ and sets $A' \subset N(v)$ and $B' \subset V(G) \setminus \{v\}$ such that*

$$d(G[A', B']) \geq d^{1/5}.$$

Proof. Remove from G one by one vertices of degree less than $d/2$. This does not decrease its average degree, and produces a subgraph $G_0 \subset G$ with $d(G_0) \geq d(G)$ and $\delta(G_0) \geq d/2$. Suppose this (bipartite) subgraph has vertex classes A and B with $|A| \geq |B|$. For each $v \in A$, select $\lceil d/2 \rceil$ incident edges and add them to G_1. Then, $e(G_1) \geq d|A|/2$ and $d_{G_1}(v) \leq d$, for each $v \in A$. By Lemma 2.3 with $\lambda = d^{1/5}$, if G_1 does not satisfy the assertion of the corollary, then G_1 contains a spanning subgraph $H \subset G_1$ with $d(H) \geq d(G_1)/(d^{1/5} + 1) \geq d^{3/4}$ and $d_H(x, y) \leq d^{1/5}$, for all distinct x and $y \in B$. Then, by Lemma 2.4, H contains a C_4-free subgraph with average degree at least $d^{1/4} \geq t$, a contradiction.   ∎

Applying this corollary iteratively, we can now prove Theorem 2.1.

Proof of Theorem 2.1. Suppose t is large enough that Corollary 2.5 holds for each $d \geq t^{5^{t^2}}$. Suppose G is bipartite, with average degree at least $t^{5^{9t^2}}$, and let $G_0 = G$. Suppose, for contradiction, that G_0 contains no C_4-free subgraph with average degree at least t, and note that $d(G_0) \geq t^4$.

For each $i = 1, \ldots, 8t^2$, by Corollary 2.5, we can find a vertex v_i and disjoint sets $A_i \subset N(v_i)$ and $B_i \subset V(G_{i-1}) \setminus \{v_i\}$ and a graph $G_i = G_{i-1}[A_i, B_i]$ with average degree at least $t^{5^{9t^2-i}}$. Moreover, note that, for each $j < i$, $A_i \subset N(v_j)$ or $B_i \subset N(v_j)$.

Let $s = 4t^2$. Thus, we have vertices $v_1, \ldots, v_{2s}$ and a graph G_{2s} with average degree at least $t^{5^{t^2}} > s$ and vertex sets A_{2s} and B_{2s} such that, for each $i \in [s]$, either $A_{2s} \subset N(v_i)$ or $B_{2s} \subset N(v_i)$. Relabelling, we can assume that we have vertices $v_1, \ldots, v_s$ with $A_{2s} \subset N(v_i)$ for each $i \in [s]$. As $d(G_{2s}) \geq s$, $|A_{2s}| \geq s$, and therefore G_{2s}, and hence G, contains a copy of $K_{s,s}$. Then, by Lemma 2.2, G contains a C_4-free subgraph with average degree at least t, a contradiction. ∎

3. Proof of the main result

In this section we prove Theorem 1.1. We still use an iterative procedure which finds either a dense C_4-free subgraph or makes progress towards a complete bipartite graph (cf. Corollary 2.5). However, now the average degree will decrease much less on each iteration. Instead of passing from average degree d to average degree at least $d^{1/5}$ (which led to our double exponential bound), the average degree d decreases to only average degree at least $d/50t$ at each iteration, where t is the average degree we are aiming for. To do this we wish to apply Lemma 2.3 to a graph G with $\lambda = d(G)/50t$ instead of $\lambda = d(G)^{1/5}$. If (1) in Lemma 2.3 holds, we can iterate as before. If (2) holds, we need to do more work. In general, the conditions in (2) are not strong enough for our techniques to find the required dense C_4-free subgraph, but we can do this if, in addition, the subgraph is a very unbalanced bipartite graph (see Lemma 3.5). Therefore, as this subgraph is always spanning, we ensure this by only applying Lemma 2.3 to very unbalanced bipartite graphs. Fortunately, in each dense bipartite graph we can find either a very unbalanced dense bipartite subgraph, or a dense bipartite subgraph whose maximum degree is at most polynomial in the average degree (see Lemma 3.1). This latter case can be solved using the techniques for nearly regular graphs discussed in the introduction (see also Lemma 3.3). Therefore, by applying Lemma 3.1 before applying Lemma 2.3 in each iteration, we gain the additional property that the graph is very unbalanced, which we use when (2) in Lemma 2.3 holds.

We start with Lemma 3.1, which finds a dense subgraph that is either very unbalanced or has low maximum degree.

LEMMA 3.1: *Every bipartite graph G with $d(G) \geq k \geq 2$ contains a subgraph H with vertex classes A and B such that $d(H) \geq k/4$, $d_H(v) \leq k$ for each $v \in A$, and either*

    (1) $|A| \geq k^6|B|$, *or*
    (2) $\Delta(H) \leq k^7$.

Proof. Remove from G one by one vertices of degree less than $k/2$. This does not decrease its average degree, and produces a subgraph $G_0 \subset G$ with $d(G_0) \geq d(G)$ and $\delta(G_0) \geq k/2$. Suppose this (bipartite) subgraph has vertex classes A and B_0 with $|A| \geq |B_0|$. For each $v \in A$, select $\lceil k/2 \rceil$ incident edges and add them to G_1. Then, $e(G_1) \geq k|A|/2$ and $d_{G_1}(v) \leq k$, for each $v \in A$.

Let $B_1 \subset B_0$ be the set of vertices with degree at least k^7 in G_1, and let $B_2 = B \setminus B_0$. Note that

$$d(G_1[B_1, A]) + d(G_1[B_2, A]) = \frac{2e(G_1[B_1, A])}{|B_1| + |A|} + \frac{2e(G_1[B_2, A])}{|B_2| + |A|}$$
$$\geq \frac{2e(G_1[B_1, A]) + 2e(G_1[B_2, A])}{|A| + |B_0|} \geq \frac{2e(G_1)}{2|A|} \geq \frac{k}{2}.$$

Therefore, either

$$d(G_1[B_1, A]) \geq k/4 \quad \text{or} \quad d(G_1[B_2, A]) \geq k/4.$$

If $d(G_1[B_2, A]) \geq k/4$, then, letting $H = G_1[B_2, A]$, we have that $\Delta(H) \leq k^7$, and so (2) is satisfied. On the other hand, if $d(G_1[B_1, A]) \geq k/4$, then, letting $H = G_1[B_1, A]$, we have that $|B_1|k^7 \leq \lceil k/2 \rceil \cdot |A|$, so that (1) is satisfied. ∎

For graphs satisfying (2) in Lemma 3.1, we show that with a small reduction in average degree we can reduce the maximum degree bound even further.

LEMMA 3.2: *Let G be a bipartite graph with vertex classes A and B satisfying $\Delta(G) \leq k^7$, $d(G) \geq k/4$, and $d_G(v) \leq k$ for each $v \in A$. If k is sufficiently large, then G contains a subgraph with maximum degree at most k and average degree at least $k/400 \log k$.*

Proof. Note that we can assume that G has no isolated vertices. Let $r = 10 \log k$, and, for each $i \in [r]$, let

$$B_i = \{v \in B : 2^{i-1} \leq d_G(v) < 2^i\},$$

noting that these sets partition B as $2^r \geq k^7$. For each $i \in [r]$, let $G_i = G[A, B_i]$. As $e(G) \geq k|A|/8$, there must be some i with $e(G_i) \geq k|A|/80 \log k$.

Let $d = 2^{i-1}$, so that, for each $v \in B_i$, $d \leq d_{G_i}(v) \leq 2d$. If $2d \leq k$, then G_i satisfies the lemma, so assume that $2d > k$. Thus, $p := k/4d < 1/2$. Now, let $A' \subset A$ be chosen by including each vertex in A independently at random with probability p. Let B' be the set of vertices in B_i with at most k neighbours in A' and let $H = G_i[B' \cup A']$. Then, by definition, $\Delta(H) \leq k$.

For each $v \in B_i$, its degree in A' is binomially distributed with expectation $\mathbb{E}(d(v, A'))$ satisfying $k/4 = pd \leq \mathbb{E}(d(v, A')) \leq 2pd = k/2$. Therefore, using Chernoff's bound (see, e.g, Appendix A of [1]), we have

$$\mathbb{P}(d(v, A') \geq k) \leq \mathbb{P}(d(v, A') \geq 2 \cdot \mathbb{E}(d(v, A')))$$
$$\leq 2 \exp(-\mathbb{E}(d(v, A'))/3) \leq 2 \exp(-k/12).$$

Thus, $\mathbb{E}(|B_i \setminus B'|) \leq 2 \exp(-k/12)|B_i|$. Note that, as $d|B_i| \leq k|A|$, we have $|B_i| \leq 4p|A|$, and, therefore, $\mathbb{E}(|B_i \setminus B'|) \leq 8 \exp(-k/12)p|A|$.

Let $k_0 = k/800 \log k$, so that $e(G_i) = e(B_i, A) \geq 10k_0|A|$. Note also that every vertex of $B_i \setminus B'$ has degree at most $\Delta(G) \leq k^7$ in A'. Hence

$$e(G_i[B' \cup A']) \geq e(G_i[B_i, A']) - k^7|B_i \setminus B'|.$$

Since $\mathbb{E}(e(G_i[B_i \cup A'])) = p \cdot e(G_i[B_i, A])$ and $\mathbb{E}(|A'|) = p|A|$, we have

$$\mathbb{E}(e(G_i[B' \cup A']) - k_0(|A'| + |B'|))$$
$$\geq \mathbb{E}(e(G_i[B_i, A']) - k^7|B_i \setminus B'| - k_0|A'| - k_0|B_i|)$$
$$\geq p \cdot e(G_i[B_i, A]) - k^7 \cdot 8 \exp(-k/12)p|A| - k_0 \cdot p|A| - k_0 \cdot 4p|A|$$
$$\geq p(10k_0|A| - k^7 \cdot 8 \exp(-k/12)|A| - 5k_0|A|)$$
$$= p|A|(5k_0 - k^7 \cdot 8 \exp(-k/12)) \geq 0.$$

Thus, there is some choice of A' for which $e(H) - k_0(|A'| + |B'|) \geq 0$. Then $d(H) \geq 2k_0 = k/400 \log k$ and $\Delta(H) \leq k$ (as explained above), completing the proof. ∎

Graphs produced by Lemma 3.2 have high enough average degree compared to their maximum degree that taking a random subgraph is likely, with a small alteration, to find a dense C_4-graph, as follows.

LEMMA 3.3: *Let G be a graph with maximum degree Δ and average degree $d \geq \Delta^{3/4}$. Then, G contains a C_4-free subgraph with average degree at least $d \cdot \Delta^{-3/4}/4$.*

Proof. Let $n = |G|$. For every $v \in V(G)$, there are clearly at most Δ^3 paths of length three starting at v and hence at most Δ^3 copies of C_4 containing v. Thus, G contains at most $n\Delta^3$ copies of C_4. Let H be a random subgraph of G formed by including each edge independently at random with probability $p = \Delta^{-3/4}/2$. Let X be the number of copies of C_4 in H. As $e(G) = n \cdot d(G)/2 = nd/2$, we have

$$\mathbb{E}(e(H) - X) \geq p \cdot e(G) - n\Delta^3 p^4 \geq nd\Delta^{-3/4}/4 - n/16 \geq nd\Delta^{-3/4}/8.$$

Thus, there is some subgraph $H \subset G$ with $e(H) - X \geq nd\Delta^{-3/4}/8$. Removing an edge from each C_4 in H gives a C_4-free subgraph with average degree at least $d\Delta^{-3/4}/4$. ∎

By combining Lemma 3.2 and Lemma 3.3, we can now find a dense C_4-free subgraph in graphs satisfying (2) in Lemma 3.1. For convenience, we record this as follows.

COROLLARY 3.4: *The following holds for sufficiently large k, and $t \leq k^{1/5}$. Every bipartite graph G with $d(G) \geq k$ contains either*

 (1) *a subgraph H with vertex classes A and B such that $d(H) \geq k/4$, $d_H(v) \leq k$ for each $v \in A$, and $|A| \geq k^6|B|$, or*
 (2) *a C_4-free subgraph with average degree at least t.*

Proof. By Lemma 3.1, if (1) does not hold, then G contains a subgraph H which is bipartite with vertex classes A and B such that $d(H) \geq k/4$, $d_H(v) \leq k$ for every $v \in A$ and $\Delta(H) \leq k^7$. By Lemma 3.2, H contains a subgaph H' with maximum degree at most k and average degree at least $k/400\log k$. Then, letting $\Delta = k$, for sufficiently large k, by Lemma 3.3 H' contains a C_4-free subgraph with average degree at least $(k/400\log k) \cdot k^{-3/4}/4 \geq k^{1/5} \geq t$, so that (2) holds. ∎

We now find dense C_4-free subgraphs in graphs which satisfy (2) in Lemma 2.3 and are very unbalanced.

LEMMA 3.5: *Let $d \geq 2$. Suppose a bipartite graph G with vertex classes A and B satisfies*

 - *$|A| \geq d^6|B|$,*
 - *$d(x,y) \leq d$, for all distinct $x, y \in B$, and*
 - *$d_G(v) \leq d$, for every $v \in A$.*

Then, G contains a subgraph G' which is C_4-free and has average degree at least $d(G)/5$.

Proof. Let $\ell = d(G)$ and let $A' \subset A$ be a subset of vertices chosen independently at random with probability $p = 1/d^6$ and let $G' = G[A' \cup B]$. Each vertex $v \in B$ has at most $d \cdot d(v)$ paths of length two starting at it. Since $d(v, u) \leq d$ for all $u \in N^2(v)$, v is contained in at most $d^2 \cdot d(v)$ copies of C_4. Thus, there are at most

$$\sum_{v \in B} d^2 \cdot d(v) \leq d^2 \cdot e(G)$$

copies of C_4 in G. Let X be the number of copies of C_4 in G', and note that

$$\mathbb{E}(X) \leq p^2 d^2 e(G) \leq p e(G)/2.$$

Also note that $|B| \leq p|A|$, $\mathbb{E}|A'| = p|A|$ and $\ell|A| \leq 2e(G)$. Hence

$$\mathbb{E}(e(G') - X - (|A'| + |B|)\ell/10) \geq p e(G) - p e(G)/2 - 2p|A| \cdot \ell/10$$
$$= p(e(G)/2 - \ell|A|/5) \geq 0.$$

Thus, there is some subgraph $G' \subset G$ with $e(G') - X - (|A'| + |B|)\ell/10 \geq 0$. Taking G' and removing one edge from each C_4 gives a C_4-free graph with average degree at least $\ell/5$. ∎

In our proof, at every iteration (if required), we wish to apply Corollary 3.4, then Lemma 2.3, then Lemma 3.5. For convenience we combine these steps in the following corollary. The improvement we have made can be seen by comparing this to Corollary 2.5.

COROLLARY 3.6: *For sufficiently large t, let G be a bipartite graph with $d(G) \geq d \geq t^5$. Then, G contains either*

(1) *a C_4-free subgraph with average degree at least t, or*
(2) *a vertex v and sets $A \subset N_G(v)$ and $B \subset V(G) \setminus (A \cup \{v\})$ with $d(G[A, B]) \geq d/50t$.*

Proof. By Corollary 3.4, if the required C_4-free subgraph does not exist, then G contains a subgraph H with vertex classes A and B such that $d(H) \geq d/4$, $d_H(v) \leq d$ for each $v \in A$, and $|A| \geq d^6|B|$. Apply Lemma 2.3 to H with $\lambda = d/50t$. If case (1) in Lemma 2.3 holds, then (2) holds here. Therefore, assume that there is some spanning subgraph $H' \subset H$ with $d(H') \geq d(H)/(\lambda + 1) \geq 6t$ and, $d_{H'}(x, y) \leq \lambda \leq d$ for all distinct $x, y \in B$. By Lemma 3.5, H' contains a C_4-free subgraph with average degree at least t, as required. ∎

Proof of Theorem 1.1.   Suppose G is bipartite, with average degree at least $(50t)^{9t^2}$ and suppose, for contradiction, that G contains no C_4-free subgraph with average degree at least t. For each $i = 1, \ldots, 4t^2$, by Corollary 3.6, we can find a vertex v_i and sets $A_i \subset N(v_i)$ and $B_i \subset B_{i-1} \backslash \{v_i\}$ or $B_i \subset A_{i-1} \backslash \{v_i\}$ and a graph $G_i = G_{i-1}[A_i, B_i]$ with average degree at least $(50t)^{9t^2-i}$. Moreover, note that, for each $j < i$, $A_i \subset N(v_j)$ or $B_i \subset N(v_j)$.

Let $s = 4t^2$. Thus we have vertices $v_1, \ldots, v_{2s}$ and a graph G_{2s} with average degree at least $(50t)^{t^2} \geq s$ and vertex sets A_{2s} and B_{2s} such that, for each $i \in [s]$, either $A_{2s} \subset N(v_i)$ or $B_{2s} \subset N(v_i)$. Relabelling, we can assume that we have vertices $v_1, \ldots, v_s$ with $A_s \subset N(v_i)$ for each $i \in [s]$. As $d(G_{2s}) \geq s$, $|A_{2s}| \geq s$, and therefore G_{2s}, and hence G, contains a copy of $K_{s,s}$. Hence, by Proposition 2.2, G contains a C_4-free subgraph with average degree at least t, a contradiction.   ∎

4. Lower bound

In this section we prove Theorem 1.2. Recall, from the introduction, the following bound of Erdős, Rényi and Sós [6] on the Turán number of C_4. It says that every n-vertex C_4-free graph G has $e(G) \leq n^{3/2}/2 + n/4$. In particular, then, the complete bipartite graph $K_{n,n}$ has average degree n but no C_4-free subgraph with average degree larger than $2n^{1/2} + 1/2$. As noted already in the beginning of the paper, such regular graphs are unlikely to provide good lower bounds. Instead, we base our construction on (highly irregular) graphs without regular subgraphs, constructed by Pyber, Rödl and Szemerédi [11], as follows.

THEOREM 4.1: *For all d, there exists a graph with $d(G) \geq d$ which contains no 3-regular subgraph.*

To prove our lower bound, we take (essentially) the Pyber–Rödl–Szemerédi graph H with parameter $4d$ and blow up every vertex into a set of d^2 vertices, replacing each edge by a copy of K_{d^2,d^2}. Intuitively, this graph should not have a C_4-free subgraph H' with average degree much bigger than d. Indeed, firstly, the above estimate for the Turán number of C_4 prevents the edges in H' between any two blown-up vertices from having C_4-free subgraphs with average degree $\geq d$. Moreover, the subgraph of H whose edges correspond to pairs of blown-up vertices with large average degree between them in H', can be shown to have maximum degree bounded by d^4. Then, from the properties of the

Pyber–Rödl–Szemerédi graph, it follows that such a graph has few edges. To prove this rigorously, we first modify the Pyber–Rödl–Szemerédi graph slightly, using another result from [11].

THEOREM 4.2: *There is a constant $c > 0$ such that the following holds. Let G be a graph with $\Delta(G) = \Delta > 1$ and $d(G) \geq c \log \Delta$. Then G has a 3-regular subgraph.*

COROLLARY 4.3: *There is a constant $c > 0$ such that for all $d \geq 2$ there exists a balanced bipartite graph H with bipartition classes A, B having $d(a) = d$ for all $a \in A$, and which contains no subgraph Γ with $d(\Gamma) \geq c \log d$ and $\Delta(\Gamma) \leq d^4$.*

Proof. Let c be such that Theorem 4.2 holds with the constant $c/4$. Using Theorem 4.1, let G be a graph with $d(G) \geq 8d$ and no 3-regular subgraph. By Theorem 4.2, G then must have no subgraph Γ with $d(\Gamma) \geq (c \log \Delta(\Gamma))/4$. Thus, G has no subgraph Γ with $d(\Gamma) \geq c \log d$ and $\Delta(\Gamma) \leq d^4$.

Now, to deduce the corollary, we need only show that G has a balanced bipartite subgraph with all vertices in one class having degree d. First choose a bipartite subgraph G_1 of G with $d(G_1) \geq d(G)/2$ (a random bipartition has this in expectation). Then, by iteratively deleting vertices with degree at most $2d$ and using that $d(G_1) \geq 4d$, we find a subgraph G_2 of G_1 with $\delta(G_2) \geq 2d$. Say G_2 has bipartition classes A and B, with $|A| \geq |B|$. For each vertex $a \in A$, delete all but d edges adjacent to a. Then, delete $|A| - |B|$ vertices from A. Note that the final graph, H say, is bipartite, balanced, and every vertex in the class from A has degree d. ∎

As we already explained, our example will be a blow up of the graph from the previous corollary. The following lemma will be used to show that certain C_4-free subgraphs of this example have to be sparse. The sets A_i and B_j here represent the blown-up vertices.

LEMMA 4.4: *Let $k \geq 4$ and $d \geq 2$. Let G be a C_4-free bipartite graph whose vertex classes are A and B. Let A have partition $A_1 \cup \cdots \cup A_r$ and B have partition $B_1 \cup \cdots \cup B_s$, with $|A_i|, |B_j| \leq d^2$ for all $i \in [r]$ and $j \in [s]$. Define a graph H with the vertex set $\{a_1, \ldots, a_r, b_1, \ldots, b_s\}$ and $a_i b_j$ an edge exactly if $e(A_i, B_j) > 0$. Suppose H is k-degenerate. Then, $d(G) \leq 13kd$.*

To show this, we use the following lemma.

LEMMA 4.5: *Let $d \geq 2$. Let G be a C_4-free bipartite graph whose vertex classes are A and B. Suppose A has a partition $A_1 \cup \cdots \cup A_m$ with $|A_i| \leq d^2$ for each $i \in [m]$. Suppose that for all $b \in B$ and $i \in [m]$, we have $|N(b) \cap A_i| = 0$ or $|N(b) \cap A_i| \geq d$. Then*

$$d(G) \leq 18d.$$

Proof. Fix an arbitrary $i \in [m]$. Let B' be the set of vertices $b \in B$ with $|N(b) \cap A_i| \geq d$. As G is C_4-free, every pair of vertices in A_i has at most 1 common neighbour in B. Thus, as each vertex in B' is adjacent to at least $\binom{d}{2}$ different pairs of vertices in A_i, we have $\binom{|A_i|}{2} \geq |B'|\binom{d}{2}$. Combining this with $|A_i| \leq d^2$ gives

$$|B'| \leq \frac{d}{d-1}(|A_i| - 1) \leq 2|A_i|.$$

Since G is C_4-free, by the bound of Erdős, Rényi and Sós [4], we have

$$e(A_i, B') \leq \frac{(|A_i| + |B'|)^{3/2}}{2} + \frac{|A_i| + |B'|}{4} \leq (3|A_i|)^{3/2}.$$

Thus, using $|A_i| \leq d^2$, we get

$$e(A_i, B) = e(A_i, B') \leq (3|A_i|)^{3/2} \leq 9d|A_i|.$$

Summing this for all $i \in [m]$, we get

$$e(G) = \sum_{i=1}^{m} e(A_i, B) \leq 9d|A|$$

and hence $d(G) \leq 2e(G)/|A| \leq 18d$. ∎

Using this lemma, we can now prove Lemma 4.4.

Proof of Lemma 4.4. For an edge ab with $a \in A_i, b \in B_j$, we say that ab is Type 1 if $|N(a) \cap B_j| \geq d$, Type 2 if $|N(b) \cap A_i| \geq d$, and Type 3 if neither of these occur. (Note ab may be both Type 1 and Type 2.) Let G_1, G_2, and G_3 be the subgraphs of G with vertex set $V(G)$ consisting of Type $1, 2$, and 3 edges respectively.

Notice that Lemma 4.5 applies to G_1 and G_2, so that $d(G_1), d(G_2) \leq 18d$. We claim that G_3 is kd-degenerate. That is, every subgraph of G_3 has a vertex with degree at most kd, and thus, by an easy induction, $d(G_3) \leq 2kd$. Therefore, if we can show G_3 is kd-degenerate, we will have

$$d(G) \leq d(G_1) + d(G_2) + d(G_3) \leq 36d + 2kd \leq 13kd.$$

To show G_3 is kd-degenerate, let G' be an arbitrary subgraph of G_3. Let H' be the subgraph of H with $a_i b_j$ an edge exactly if $e_{G'}(A_i, B_j) > 0$. Since H is k-degenerate, there is a vertex of degree $\leq k$ in H', say a_i. Let $v \in A_i \cap V(G')$. Since $d_{H'}(a_i) \leq k$, there are at most k sets B_j with $N_{G'}(v) \cap B_j \neq \emptyset$. By the definition of G_3, $|N_G(v) \cap B_j| \leq d$ for all j. Thus, $d'_G(v) \leq kd$. That is, G' has a vertex with degree at most kd, as required. ∎

We now prove the main result of this section.

Proof of Theorem 1.2. Note that it is sufficient to find, for all $d \geq 4$, a graph with average degree at least d^3 and no C_4-free subgraph with average degree greater than $15cd \log d$, where c is some fixed constant.

Let c be such that, by Corollary 4.3, we can find a graph H satisfying the conditions in that corollary. Let the bipartition classes of H be $\{a_1, \ldots, a_m\}$ and $\{b_1, \ldots, b_m\}$. Let G be the graph formed from H by replacing each a_i and b_j by a set A_i and B_j of d^2 vertices, respectively, and by replacing each edge $a_i b_j$ of H by a complete bipartite graph K_{d^2, d^2} between A_i and B_j. Let $A = A_1 \cup \cdots \cup A_m$ and $B = B_1 \cup \cdots \cup B_m$, and note that these sets have size $d^2 m$. We have $e(G) = d^4 \cdot e(H) = d^5 m$, and hence $d(G) = 2e(G)/(2d^2 m) = d^3$.

Now, let G' be a C_4-free subgraph of G. We will show that $d(G') \leq 15cd \log d$, completing the proof. For this, let H_1 be the subgraph of H formed by including the edge $a_i b_j$ exactly if the subgraph $G'[(A_i \cup B_j) \cap V(G')]$ has a vertex in A_i with at least two neighbours in B_j. Let $H_2 = H \setminus H_1$. Let G_1 and G_2 be the subgraphs of G' on vertex set $V(G')$ formed from the union of the pairs $G'[(A_i \cup B_j) \cap V(G')]$ corresponding to edges $a_i b_j$ in H_1 and H_2 respectively.

Similarly to the simple argument in the proof of Lemma 4.5, since G' is C_4-free, for each j there are at most $\binom{|B_j|}{2} \leq d^4$ vertices $a \in A$ with ≥ 2 neighbours in B_j. Therefore, $d_{H_1}(b_j) \leq d^4$. From the properties of H from Corollary 4.3, we have $d_H(a_i) = d$ for all $i \in [m]$, and therefore $\Delta(H_1) \leq d^4$. Again, by the properties of H, we have that $d(H_1) < c \log d$. This holds also for any subgraph of H_1, since it has maximum degree at most d^4 as well. Hence H_1 is $(c \log d)$-degenerate. By Lemma 4.4, therefore, $d(G_1) \leq 13cd \log d$ (for this application notice that, by the definition of G_1, the only pairs with $e_{G_1}(A_i, B_j) > 0$ are those with $a_i b_j \in E(H_1)$). Notice that, by the definition of G_2, each $a \in A_i$ has $d_{G_2}(a) \leq d_H(a_i) = d$. Indeed, by definition, a has in G_2 at most one neighbour in every set B_j with $a_i b_j \in E(H)$. Therefore, $d(G_2) \leq 2d$, and hence $d(G') = d(G_1) + d(G_2) \leq 13cd \log d + 2d \leq 15cd \log d$, as required. ∎

5. Concluding remarks

- Denote by $f_6(t)$ the smallest number such that every graph G with average degree at least $f_6(t)$ contains a subgraph with girth at least 6 and average degree at least t. In this paper, we studied the growth rate of $f_6(t)$ and proved that it is at most exponential in $t^{2+o(1)}$. The best lower bound that we obtained is roughly cubic. It would be very interesting to close this gap further and, in particular, to decide whether $f_6(t)$ is polynomial in t.

- The next open case of Thomassen's conjecture remains a fascinating, and seemingly very difficult problem. That is, do graphs with large average degree always contain subgraphs with large average degree and girth at least 8?

- A related question, posed by Erdős and Hajnal [4, 5] in the 1960's, asks whether, for every k and g, there is a function $\chi(k, g)$ such that any graph with chromatic number at least $\chi(k, g)$ contains a subgraph with chromatic number at least k and girth greater than g. Unlike Thomassen's conjecture, here the case $g = 3$ is already highly nontrivial, though it was solved by Rödl [12] using a very elegant argument. All other cases remain open.

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