RATIONAL APPROXIMATION WITH CHOSEN NUMERATORS

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Abstract. We consider the problem of approaching real numbers with rational numbers with prime denominator and with a single numerator allowed for each denominator. We then present a simple application, related to possible correlations between trace functions and dynamical sequences.

1. Introduction

The following statement is motivated by certain specific applications concerning possible correlations between "trace functions" and "dynamical" sequences (see Section 5 for a concrete statement and the notes [9] for a more general perspective).

Theorem 1.1. *Let $c > 0$ be a real number with $c \leqslant 1/2$. There exist sequences (a_p) indexed by prime numbers, with a_p an integer such that $0 \leqslant a_p < p$ for all p, such that for almost all $x \in [0, 1]$, the set of primes p with*

$$\left| x - \frac{a_p}{p} \right| \leqslant \frac{c}{p}$$

is infinite.

So, informally, we consider a problem of diophantine approximation where, for each denominator, only one numerator is allowed (and with the additional restriction, coming from the original motivation, that the denominators are primes). The approximation is of course then worse than what is allowed by varying the numerator (and of course not every choice of numerator can be successful).

We will give three proofs in Sections 2, 3 and 4. The first is (probably unsurprisingly) very elementary, but has some nice aspects, especially an analogy with sieve. The second proof is more straightforward in principle, but involves more sophisticated ingredients, especially about the distribution of primes. The third proof (suggested by Manuel Hauke) simplifies the second proof by exploiting a lemma of Cassels, which removes the need for serious understanding of the distribution of primes, but involves Lebesgue's density theorem instead.

This simple result suggests some questions:

(1) Can one describe an explicit sequence (a_p) which has the desired property? Note that the second proof will show that it is a generic property (in the sense of the natural probability measure on the space of sequences (a_p), described below).

(2) More specifically, if we define (a_p) using the "greedy" algorithm (taking a_p for successive primes so as to always maximize the measure of the union of the intervals up to that point), does it work? Hauke, in an email, pointed out that a variant of this

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construction is successful for $c = 2$, and yields a deterministic sequence that "works" for all $c > 0$.

(3) For a given value of c, the "exceptional set" always contain rational numbers with denominators $< 1/c$. Are there other elements in this exceptional set? If Yes, can we describe the elements that belong to it, or compute its Hausdorff dimension?

(4) For suitable (a_p) and x, can we estimate asymptotically as $\mathrm{X} \to +\infty$ the number of primes $p \leqslant \mathrm{X}$ such that $|x - a_p/p| \leqslant c/p$? Heuristically, one can hope to have something like

$$2c \sum_{p \leqslant \mathrm{X}} \frac{1}{p} \sim 2c \log \log \mathrm{X}$$

such primes $\leqslant \mathrm{X}$; can this be established for suitable choices of the a_p? The first proof provides some weaker quantitative information, with high probability (with respect to x).

(5) What about multidimensional versions? Variants on manifolds?

Remark 1.2. Although problems of this kind do not seem to be standard in diophantine approximation, one can interpret the question roughly as asking whether there exists a sequence (a_p) such that the set of points a_p/p is "eutaxic" with respect to the radii c/p (see, e.g., the survey of Durand [2, Ch. 8], and also Bugeaud's book [1, Ch. 6]).

A related, but more delicate, question was solved by Shepp [10] (after previous work of Dvoretzky and others), who found a sharp criterion which ensures that a non-increasing sequence $(\ell_n)_{n \geqslant 1}$ of positive real numbers has the property that, almost surely, the union of arcs of length ℓ_n with independent and uniform centers will cover *entirely* a circle of length 1. Shepp proved that this holds if and only if

$$\sum_{n \geqslant 1} \frac{1}{n^2} \exp(\ell_1 + \cdots + \ell_n) = +\infty.$$

As observed by Dvoretzky [3], this is a different condition than asking that the union covers almost all points of the circle, which occurs if and only if the series $\sum \ell_n$ diverges, by an elementary argument as in Section 3.

Notation. We use the Vinogradov notation $f \ll g$ (for complex-valued functions f and g defined on some set X): it means that there exists a real number $c \geqslant 0$ (the "implied constant") such that $|f(x)| \leqslant c g(x)$ for all $x \in \mathrm{X}$.

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2. First proof

We denote by λ the Lebesgue measure. For a sequence $\boldsymbol{a} = (a_p)$ with $0 \leqslant a_p < p$ for all primes p, define

$$\mathrm{A}_{\boldsymbol{a}} = \{x \in [0, 1] \mid \left| x - \frac{a_p}{p} \right| \leqslant \frac{c}{p} \text{ for infinitely many } p\}.$$

We thus want to find $\boldsymbol{a}$ with $\lambda(\mathrm{A}_{\boldsymbol{a}}) = 1$. For real parameters X and Y with $1 \leqslant \mathrm{X} < \mathrm{Y}$, we consider the set

$$\Omega_{\mathrm{X,Y}}(\boldsymbol{a}) = \Big\{ x \in [0,1] \mid \Big| x - \frac{a_p}{p} \Big| > \frac{c}{p} \text{ for } \mathrm{X} < p \leqslant \mathrm{Y} \Big\}.$$

We observe that the set $\mathscr{A}$ of all sequences $\boldsymbol{a}$ is naturally a compact set, as a product of finite sets. In particular, there is a natural product probability measure on this set, where each a_p is uniform over the integers from 0 to $p-1$. We will use the notation $\mathbf{P}(\cdot)$ and $\mathbf{E}(\cdot)$ below to indicate probability and expectation according to this measure.

The crucial lemma is the following. We view it as a kind of sieve statement, on average over $\mathscr{A}$.

Lemma 2.1. *Let*

$$\mathrm{H}_{\mathrm{X,Y}} = \sum_{\mathrm{X} < p \leqslant \mathrm{Y}} \frac{1}{p}.$$

We have

$$\mathbf{E}(\lambda(\Omega_{\mathrm{X,Y}})) \ll \frac{1}{\mathrm{H}_{\mathrm{X,Y}}},$$

where the implied constant depends only on c.

Proof. Let $\varphi_p \colon [0,1] \to \{0,1\}$ denote the characteristic function of the interval $\mathrm{I}_p(a_p) = [a_p/p - c/p, a_p/p + c/p]$, each being viewed as random variables on $\mathscr{A}$ (the φ_p are random functions, the I_p are random intervals). Let

$$\mathrm{N}_{\mathrm{X,Y}} = \sum_{\mathrm{X} < p \leqslant \mathrm{Y}} \varphi_p,$$

again a random variable on $\mathscr{A}$. We denote also

$$\nu_{\mathrm{X,Y}} = \int_0^1 \mathrm{N}_{\mathrm{X,Y}}(x) dx$$

and note that $\nu_{\mathrm{X,Y}} = 2c\mathrm{H}_{\mathrm{X,Y}}$, independently of the value of $\boldsymbol{a}$.

Noting that $\Omega_{\mathrm{X,Y}}$ is the set of those x where $\mathrm{N}_{\mathrm{X,Y}} = 0$, we deduce from Markov's inequality (on $[0,1]$ with the Lebesgue measure) the upper bound

$$\lambda(\Omega_{\mathrm{X,Y}}) \leqslant \lambda\Big(\Big\{ x \in [0,1] \mid |\mathrm{N}_{\mathrm{X,Y}}(x) - \nu_{\mathrm{X,Y}}| \geqslant \nu_{\mathrm{X,Y}} \Big\}\Big) \leqslant \frac{\alpha_{\mathrm{X,Y}}}{\nu_{\mathrm{X,Y}}^2}$$

where

$$\alpha_{\mathrm{X,Y}} = \int_0^1 \Big(\mathrm{N}_{\mathrm{X,Y}}(x) - \nu_{\mathrm{X,Y}} \Big)^2 dx$$

(again a random variable on $\mathscr{A}$).

We have

$$\alpha_{\mathrm{X,Y}} = \int_0^1 \Big(\sum_{\mathrm{X} \leqslant p \leqslant \mathrm{Y}} \Big(\varphi_p(x) - \frac{2c}{p} \Big) \Big)^2 dx = \sum_{\mathrm{X} < p_1, p_2 \leqslant \mathrm{Y}} \int_0^1 \Big(\varphi_{p_1}(x) - \frac{2c}{p} \Big) \Big(\varphi_{p_2}(x) - \frac{2c}{p} \Big) dx.$$

For $p_1 = p_2$, the integral is equal to

$$\int_0^1 \Big(\varphi_{p_1}(x) - \frac{2c}{p_1}\Big)^2 dx = \frac{2c}{p_1}\Big(1 - \frac{2c}{p_1}\Big) \leqslant \frac{2c}{p_1}$$

(variance of a Bernoulli random variable with probability of success $2c/p_1$), again independently of $\boldsymbol{a}$. Thus

$$\sum_{\mathrm{X}<p_1\leqslant \mathrm{Y}} \mathbf{E}\Big(\int_0^1 \Big(\varphi_{p_1}(x) - \frac{2c}{p_1}\Big)^2 dx\Big) \leqslant 2c\mathrm{H}_{\mathrm{X,Y}}.$$

We now suppose that $p_1 \neq p_2$. We then have

$$\int_0^1 \Big(\varphi_{p_1}(x) - \frac{2c}{p_1}\Big)\Big(\varphi_{p_2}(x) - \frac{2c}{p_2}\Big) dx = \lambda(\mathrm{I}_{p_1} \cap \mathrm{I}_{p_2}) - \frac{4c^2}{p_1p_2}, \tag{2.1}$$

where the first term depends on $\boldsymbol{a}$.

We next estimate the expectation

$$\mathbf{E}\Big(\lambda(\mathrm{I}_{p_1} \cap \mathrm{I}_{p_2})\Big)$$

over $\boldsymbol{a}$. For this purpose, we may (and do) assume that $p_1 < p_2$. We then have the formula

$$\mathbf{E}\Big(\lambda(\mathrm{I}_{p_1} \cap \mathrm{I}_{p_2})\Big) = \frac{1}{p_1p_2} \sum_{0\leqslant a<p_1} \lambda\Big(\mathrm{I}_{p_1}(a) \cap \bigcup_{0\leqslant b<p_2} \Big[\frac{b}{p_2} - \frac{c}{p_2}, \frac{b}{p_2} + \frac{c}{p_2}\Big]\Big).$$

Drawing a picture if need be, we get

$$\mathbf{E}\Big(\lambda(\mathrm{I}_{p_1} \cap \mathrm{I}_{p_2})\Big) = \frac{1}{p_1p_2} \times p_1 \times \Big(\frac{4c^2}{p_1} + \mathrm{O}\Big(\frac{1}{p_2}\Big)\Big) = \frac{4c^2}{p_1p_2} + \mathrm{O}\Big(\frac{1}{p_2^2}\Big).$$

Combined with (2.1), this leads to

$$\sum_{\mathrm{X}\leqslant p_1<p_2\leqslant \mathrm{Y}} \mathbf{E}\Big(\int_0^1 \Big(\varphi_{p_1}(x) - \frac{2c}{p_1}\Big)\Big(\varphi_{p_2}(x) - \frac{2c}{p_2}\Big) dx\Big) \ll \mathrm{H}_{\mathrm{X,Y}}.$$

Multiplying by two to account for the case $p_1 > p_2$ and adding the contribution where $p_1 = p_2$, we conclude that $\mathbf{E}(\alpha_{\mathrm{X,Y}}) \ll \mathrm{H}_{\mathrm{X,Y}}$, and hence

$$\mathbf{E}(\lambda(\Omega_{\mathrm{X,Y}})) \ll \mathrm{H}_{\mathrm{X,Y}}^{-1},$$

as claimed. □

Remark 2.2. The start of the argument is essentially of form of sieve inequality, especially similar to those used in certain geometric group theory works by Lubotzky and Meiri, see for instance the account in [8, Th. 5.3.1].

More generally, sieve methods in analytic number theory lead to bounds (also sometimes lower bounds) for the sizes of sets of the form

$$\{n \leqslant \mathrm{N} \mid n \,(\mathrm{mod}\, p) \notin \mathrm{I}_p \text{ for } p \leqslant \mathrm{X}\}$$

for suitable choices of subsets $\mathrm{I}_p \subset \mathbf{Z}/p\mathbf{Z}$ and of parameters N and X. It was pointed out in [7] that in fact some of the basic techniques (e.g., the so-called "large sieve") can be extended to much more general settings than the integers, and the lemma above provides another illustration.

We now conclude the proof of the theorem. Since

$$\lim_{Y\to+\infty} H_{X,Y} = \sum_{p>X} \frac{1}{p} = +\infty$$

for any $X \geqslant 1$ (one of the most elementary quantitative forms of the infinitude of primes, already known to Euler), Lemma 2.1 implies that for any $X \geqslant 2$ and any $\varepsilon > 0$, we can find $(a_p)_{X<p\leqslant Y}$ (with $0 \leqslant a_p < p$) such that

$$\lambda\Bigl(\Bigl\{x \in [0,1] \mid \Bigl|x - \frac{a_p}{p}\Bigr| \leqslant \frac{c}{p} \text{ for some prime } p \text{ with } X < p \leqslant Y\Bigr\}\Bigr) \geqslant 1 - \varepsilon.$$

Let $X_1 = 1$. Apply the previous remark first with (say) $X = 1$ and $\varepsilon = 1/2$, and denote X_2 a suitable value of Y. Then apply the assumption with $X = X_2$ and $\varepsilon = 1/4$, calling X_3 the value of Y; repeating, we obtain a strictly increasing sequence $(X_n)_{n\geqslant 1}$ of integers and a sequence $(a_p) \in \mathscr{A}$ such that the set

$$B_n = \Bigl\{x \in [0,1] \mid \Bigl|x - \frac{a_p}{p}\Bigr| \leqslant \frac{c}{p} \text{ for some prime } p \text{ with } X_n < p \leqslant X_{n+1}\Bigr\}$$

satisfies $\lambda(B_n) \geqslant 1 - 2^{-n}$ for any $n \geqslant 1$.

If $x \in [0,1]$ belongs to infinitely sets B_n, then $x \in A_{\boldsymbol{a}}$. On the other hand, since

$$\sum_{n\geqslant 1} \lambda([0,1] - B_n) < +\infty,$$

the easy Borel–Cantelli Lemma shows that almost every $x \in [0,1]$ belongs at most to finitely many sets $[0,1] - B_n$.

3. Second proof

We now give the second proof. This is based on an application of Fubini's Theorem (which is a standard approach, as in the first few lines of Dvoretzky's paper [3]).

We write again $I_p(\boldsymbol{a}) = [a_p/p - c/p, a_p/p + c/p]$, viewed as random intervals on the probability space $\mathscr{A}$ to which $\mathbf{P}(\cdot)$ and $\mathbf{E}(\cdot)$ refer. Let $x \in [0,1]$. We then have

$$\mathbf{P}(x \in I_p) = \frac{1}{p} \sum_{\substack{0\leqslant a<p \\ |x-a/p|<c/p}} 1$$

and hence $\mathbf{P}(x \in I_p)$ is either 0 or $1/p$, depending on whether there exists an integer a such that the fractional part of xp is $< c$, or not.

It is a non-trivial fact from the distribution of primes that, if x is irrational, then we have

$$\sum_{\{xp\}<c} \frac{1}{p} = +\infty \tag{3.1}$$

(precisely, this follows by summation by parts from the much more precise results of Vinogradov [11, Ch. XI] which give an asymptotic formula for the number of primes $p \leqslant X$

satisfying $\{xp\} < c$; we note in passing that this result has been improved since then, notably by Vaughan). Thus, since the events $\{x \in \mathrm{I}_p\}$ are independent by construction, the non-trivial direction of the Borel–Cantelli Lemma implies

$$\mathbf{P}(x \in \mathrm{I}_p \text{ for infinitely many } p) = 1$$

for any irrational x.

Now by Fubini's Theorem, we obtain

$$\begin{aligned} \mathbf{E}(\lambda(\mathrm{A}_{\boldsymbol{a}})) &= \mathbf{E}\Big(\int_0^1 1_{\{x \in \mathrm{I}_p \text{ for infinitely many } p\}}\, dx\Big) \\ &= \int_0^1 \mathbf{P}(x \in \mathrm{I}_p \text{ for infinitely many } p) dx = 1, \end{aligned}$$

and since $\lambda(\mathrm{A}_{\boldsymbol{a}}) \leqslant 1$, this means that $\mathrm{A}_{\boldsymbol{a}}$ has measure 1 for almost all sequences (a_p).

Remark 3.1. We do not require the full force of Vinogradov's theorem, but in any case, the formula (3.1) for an arbitrary irrational number x seems to be comparable to the similar divergence of the sum of inverses of primes in an arithmetic progression.

It would also be enough to know that the divergence of the series (3.1) holds for almost all x (instead of all irrationals), and it is quite likely that this can be proved more easily.

4. Third proof

The key ingredient is the third proof is the following result of Cassels, which was pointed out by Hauke.

Proposition 4.1. *Let $(\mathrm{I}_n)_{n \geqslant 1}$ be a sequence of intervals in $\mathbf{R}/\mathbf{Z}$ with length $\lambda(\mathrm{I}_n)$ converging to 0 as $n \to +\infty$. Assume that almost every $x \in \mathbf{R}/\mathbf{Z}$ is contained in infinitely many I_n's. Fix some positive real number δ. Let $\mathrm{X}_n \subset \mathrm{I}_n$ be arbitrary measurable subsets such that $\lambda(\mathrm{X}_n) \geqslant \delta\lambda(\mathrm{I}_n)$ for all $n \geqslant 1$. Then almost every $x \in \mathbf{R}/\mathbf{Z}$ is contained in infinitely many X_n's.*

See, e.g., the accounts by Gallagher [4, Lemma 2] or Harman [5, Lemma 2.1] for the proof. This relies in an essential way on the existence of density points (in the sense of Lebesgue) for sets of positive measure.

Assuming the result, we see that it is enough to prove Theorem 1.1 for $c = 2$ (or even for larger values of c). We can then implement the second proof, where we will have

$$\mathbf{P}(x \in \mathrm{I}_p) = \frac{1}{p} \sum_{\substack{0 \leqslant a < p \\ |x - a/p| < 2/p}} 1 = \frac{1}{p}$$

for all p. Thus the Borel–Cantelli and Fubini steps follow using only the fact that the sum of $1/p$ over primes diverges, as in the first proof.

Remark 4.2. More generally, the lemma of Cassels implies that the set of sequences (a_p) for which Theorem 1.1 applies is *independent* of c. In particular, any "deterministic" construction which applies for one value of c will also apply for any other.

5. Application

Let $\mathrm{X} = (\mathbf{R}/\mathbf{Z})^2$ and μ the Lebesgue measure on X. Further, let $f\colon \mathrm{X} \to \mathrm{X}$ be the map defined by $f(x,y) = (x+y,y)$. We have $f_*\mu = \mu$. Define $\varphi\colon \mathrm{X} \to \mathbf{C}$ by $\varphi(x,y) = e(x)$.

We chose a sequence (a_p) as in Theorem 1.1 with $c = 1/2$. For p prime and $n \in \mathbf{Z}$, we define $t_p(n) = e(-na_p/p)$. (This is a trace function modulo p, but this aspect is not important here.)

Proposition 5.1. *For p prime, define $s_p\colon \mathrm{X} \to \mathbf{C}$ by*

$$s_p(x,y) = \frac{1}{p} \sum_{0 \leqslant n < p} t_p(n)\varphi(f^n(x,y)).$$

The following properties hold:

(1) *The sequence (s_p) does not converge almost everywhere as $p \to +\infty$.*

(2) *If P is an infinite set of primes such that*

$$\sum_{p \in \mathsf{P}} \frac{\log p}{p} < +\infty,$$

then the sequence $(s_p)_{p \in \mathsf{P}}$ converges almost everywhere to 0.

Proof. Since $f^n(x,y) = (x+ny,y)$ for all integers $n \in \mathbf{Z}$, we can compute s_n by summing a finite geometric progression, and we obtain

$$s_p(x,y) = \frac{e(x)}{p} \frac{\sin(\pi p(y - a_p/p))}{\sin(\pi(y - a_p/p))} e\Big(\frac{(p-1)}{2}(y - a_p/p)\Big).$$

It follows that $s_p(x,y) \to 0$ along any infinite set P of primes such that

$$\lim_{\substack{p \to +\infty \\ p \in \mathsf{P}}} p\Big|y - \frac{a_p}{p}\Big| = +\infty.$$

Thus (2) follows because the assumption there implies that almost all (x,y) satisfy

$$\Big|y - \frac{a_p}{p}\Big| \geqslant \frac{\log p}{p}$$

for all but finitely many $p \in \mathsf{P}$, by the easy Borel–Cantelli lemma.

We now prove (1). Note that if (s_p) converges almost everywhere, the limit must be zero according to (2). But the formula for s_p and the defining property of (a_p) imply that for all x and almost all $y \in \mathbf{R}/\mathbf{Z}$, we have $|s_p(x,y)| \gg 1$ for infinitely many primes p. Thus there is almost surely a subsequence which does not converge to 0. ∎

Remark 5.2. The condition in (2) can be replaced by

$$\sum_{p \in \mathsf{P}} \frac{\psi_p}{p} < +\infty,$$

where $(\psi_p)_p$ is an arbitrary sequence of non-negative real numbers such that $\psi_p \to +\infty$.

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