FIXED-POINT STATISTICS FROM SPECTRAL MEASURES ON TENSOR ENVELOPE CATEGORIES

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Abstract. We prove old and new convergence statements for fixed-points statistics and characters of symmetric groups using tensor envelope categories, such as the Deligne–Knop category of representations of the "symmetric group" S_t for an indeterminate t. We also speculate on a generalization of Chebotarev's density theorem to pseudopolynomials.

1. Introduction

Spectral measures associated to operators on Hilbert spaces are key tools in functional analysis and its applications, for instance to quantum mechanics and ergodic theory. Recall that a continuous normal linear operator $u\colon \mathrm{E} \to \mathrm{E}$ on a Hilbert space E has a compact spectrum and that, for each vector $x \in \mathrm{E}$, there exists a unique bounded positive Radon measure μ_x on $\mathbf{C}$, supported on the spectrum of u, such that the equality

$$\int_{\mathbf{C}} f \, d\mu_x = \langle x | f(u) x \rangle$$

holds for all continuous functions $f\colon \mathbf{C} \to \mathbf{C}$; see, for instance, [4, IV, p. 190, déf. 2]. This measure is called the *spectral measure of* u *relative to* x. In particular, we have

$$\int_{\mathbf{C}} z^a \bar{z}^b \, d\mu_x(z) = \langle x | u^a (u^*)^b (x) \rangle$$

for all non-negative integers a and b. In this paper, we consider an analogue of this last relation for objects in symmetric monoidal categories.

Definition 1.1 (Spectral measure)**.** Let $\mathscr{C}$ be a symmetric monoidal category with an endofunctor D, and let i be a complex-valued invariant of $\mathscr{C}$, by which we mean a map from the set of isomorphism classes of objects of $\mathscr{C}$ to $\mathbf{C}$. Let M be an object of $\mathscr{C}$. A positive measure μ on $\mathbf{C}$ is called a *spectral measure* of M relative to i if the equality

$$\int_{\mathbf{C}} z^a \, \bar{z}^b \, d\mu(z) = i(\mathrm{M}^{\otimes a} \otimes \mathrm{D}(\mathrm{M})^{\otimes b})$$

holds for all non-negative integers a and b.

We will think of D as a duality functor on $\mathscr{C}$, although no extra condition is required for this general definition. The basic motivation is provided by the following example.

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Example 1.2. Let $r \geqslant 1$ be an integer, and let $\mathrm{G} \subset \mathbf{GL}_r(\mathbf{C})$ be a compact group with probability Haar measure ν. Let $\mathscr{C}$ be the category of finite-dimensional continuous complex representations of G, and D the contragredient endofunctor of $\mathscr{C}$. By representation theory of compact groups, the direct image of ν by the trace $\mathrm{Tr}\colon \mathrm{G} \to \mathbf{C}$ is a spectral measure $\mu = \mathrm{Tr}_*(\nu)$ of the "tautological" object of $\mathscr{C}$ corresponding to the inclusion of G in $\mathbf{GL}_r(\mathbf{C})$, relative to the invariant given on a representation ϱ by

$$i(\varrho) = \dim_{\mathbf{C}}(\varrho^{\mathrm{G}}) = \dim_{\mathbf{C}} \mathrm{Hom}(1, \varrho),$$

where 1 denotes the trivial one-dimensional representation of G. In number theory, measures of this kind are often called *Sato–Tate measures*, the original example being $\mathrm{SU}_2 \subset \mathbf{GL}_2(\mathbf{C})$.

Our first main result is a new proof of a statement which goes back to the very early studies of probability theory through the analysis of card games and the like (see the historical paper of Takács [28] for references). Interestingly, the tensor categories that will arise in the proof are the categories of representations of the "symmetric group" S_t for an indeterminate t of Deligne [10] and Knop [19]. Another, rather different, construction of this category has recently been given by Harman and Snowden [16, § 15].

Theorem 1.3 ("Problème des rencontres"; Montmort [8]; N. Bernoulli I; de Moivre [7])**.** *Let $(\mathrm{X}_n)_{n\geqslant 1}$ be a sequence of random variables with X_n a uniformly chosen random permutation in the symmetric group S_n. The sequence $(|\mathrm{Fix}(\mathrm{X}_n)|)_{n\geqslant 1}$, where $\mathrm{Fix}(\sigma)$ denotes the set of fixed points of $\sigma \in \mathrm{S}_n$, converges in law to a Poisson distribution with parameter* 1.

Recall that the Poisson distribution with parameter a positive real number λ is the measure P_λ supported on non-negative integers given by

$$\mathrm{P}_\lambda(\{r\}) = e^{-\lambda} \frac{\lambda^r}{r!}$$

for all integers $r \geqslant 0$. The meaning of the statement (and how it was originally proved) is therefore that the formula

$$\lim_{n\to+\infty} \frac{1}{n!} |\{\sigma \in \mathrm{S}_n \text{ with } |\mathrm{Fix}(\sigma)| = r\}| = \frac{1}{e}\frac{1}{r!} \tag{1.1}$$

holds for all integers $r \geqslant 0$. Neither categories nor spectral measures appear in the statement; the link comes from the fact that the limit Poisson distribution arises as the spectral measure of a suitable object in the Deligne–Knop category $\mathscr{C}_t = \underline{\mathrm{Rep}}(\mathrm{S}_t)$. This is a $\mathbf{C}(t)$-linear semisimple tensor category that "interpolates" the categories of representations of the symmetric groups S_n. From that point of view, an interesting feature of our proof is that it shows how the Poisson distribution (maybe the most natural measure on non-negative integers) is some kind of analogue of the Sato–Tate measures from Example 1.2. In Remark 3.8, we will see a similar statement for the complex gaussian distribution. By Chebotarev's density theorem, the probability that a random permutation in S_n has r fixed points governs the asymptotic density of the set of primes p such that a generic polynomial of degree n with integer coefficients has r roots modulo p. In Section 5, we will speculate on a generalization of Chebotarev's density theorem that could explain the occurrence of the limit (1.1) in numerical experiments involving certain pseudopolynomials.

Our second main result is the following new theorem.

Theorem 1.4. *Let $m \geqslant 1$ be an integer and let λ be a partition of m with parts*

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots .$$

For $n \geqslant m + \lambda_1$, let $\pi_{\lambda,n}$ be the representation of S_n corresponding to the partition

$$\lambda^{(n)} = (n - m, \lambda_1, \lambda_2, \ldots),$$

and let $\chi_{\lambda,n} \colon \mathrm{S}_n \to \mathbf{C}$ be its character. Then the sequence of measures $(\chi_{\lambda,n}(\mathrm{X}_n))_{n \geqslant m+\lambda_1}$, where as before X_n is a uniformly distributed random permutation on S_n, converges in law as $n \to +\infty$ to a spectral measure of the simple object $x_{\lambda,t}$ of the Deligne–Knop category $\mathscr{C}_t$ associated to the partition λ, relative to the invariant

$$i(\mathrm{M}) = \dim_{\mathbf{C}(t)} \mathrm{Hom}(1_t, \mathrm{M}),$$

where 1_t is the unit objet of $\mathscr{C}_t$.

This second result has a corollary which relates it to the theory of FI-modules of Church, Ellenberg and Farb [6]. An *FI-module* over a field k is a functor V from the category with objects finite sets and morphisms injective maps to the category of k-vector spaces. For each $n \geqslant 0$, we write V_n for the image of the set $\{1, \ldots, n\}$; note that V_n has a natural structure of representation of S_n. An FI-module V is called *finitely generated* if there exists a finite set of elements $x_i \in \mathrm{V}_{n_i}$ that do not "lie" on a proper subfunctor of V. We refer the reader to the introduction of [6] for a list of examples of finitely-generated FI-modules arising in algebra, geometry and topology.

Corollary 1.5. *Let $\mathrm{V} = (\mathrm{V}_n)_{n \geqslant 0}$ be a finitely-generated* FI*-module over* $\mathbf{C}$. *For $n \geqslant 0$, let ξ_n be the character of V_n as an S_n-representation. The sequence of measures $(\xi_n(\mathrm{X}_n))_{n \geqslant 0}$ converges in law as $n \to +\infty$ to a combination of spectral measures of objects of $\mathscr{C}_t$ relative to the invariant i from Theorem 1.4.*

Proof. A fundamental result of the theory of FI-modules [6, Prop. 3.3.3] implies the existence of a polynomial $\mathrm{Q} \in \mathbf{C}[(\mathrm{T}_\lambda)_\lambda]$ (in indeterminates parameterized by all partitions of all integers $m \geqslant 1$) satisfying $\xi_n = \mathrm{Q}((\chi_{\lambda,n})_\lambda)$ for all large enough n, and hence the corollary follows immediately from Theorem 1.4. ∎

This paper is excerpted from a longer work in progress [13], whose status and evolution are however unpredictable. We hope that the simple example of an application of spectral measures to classical problems will motivate the reader's interest in this notion.

Conventions. Let X be a set. A *partition* of X is a set of non-empty subsets of X, pairwise disjoint and with union equal to X. Note that this definition constrasts with that of Bourbaki (E, II, p. 29, déf. 7), where a partition is a *family* of subsets of X, allowing the empty set.

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2. Existence and uniqueness of spectral measures in tensor categories

From now on, we only consider k-linear tensor categories, for some field k, in the sense of Deligne [9, 1.2]. We always use the duality functor of such a category as the endofunctor D in Definition 1.1. An object M is called *self-dual* if there exists an isomorphism $\mathrm{M} \simeq \mathrm{D}(\mathrm{M})$.

Proposition 2.1 (Spectral measures for self-dual objects)**.** *Let $\mathscr{C}$ be a tensor category in which every object is self-dual, and let i be an $\mathbf{R}$-valued* additive *invariant of $\mathscr{C}$. If*

$$2i(\mathrm{M} \otimes \mathrm{N}) \leqslant i(\mathrm{M} \otimes \mathrm{M}) + i(\mathrm{N} \otimes \mathrm{N}) \tag{2.1}$$

holds for all objects M and N of $\mathscr{C}$, then every object of $\mathscr{C}$ admits a spectral measure relative to i which is supported on $\mathbf{R}$.

Proof. By the solution of the Hamburger moment problem (see, e.g., [27, Th. 3.8]), a sequence $(\mu_a)_{a \geqslant 0}$ of real numbers is the sequence of moments of a positive Borel measure μ on $\mathbf{R}$ if and only if the inequality

$$\sum_{1 \leqslant a,b \leqslant \mathrm{A}} \alpha_a \alpha_b \mu_{a+b} \geqslant 0 \tag{2.2}$$

holds for all integers $\mathrm{A} \geqslant 1$ and all real numbers α_a. Therefore, M admits a real spectral measure relative to i if and only if the values $\mu_a = i(\mathrm{M}^{\otimes a})$ satisfy this condition.

We first consider the case where α_a are integers. Setting

$$\mathrm{P} = \bigoplus_{\alpha_a \geqslant 0} \alpha_a \mathrm{M}^{\otimes a} \qquad \text{and} \qquad \mathrm{N} = \bigoplus_{\alpha_a \leqslant -1} (-\alpha_a) \mathrm{M}^{\otimes a},$$

we then get

$$\sum_{1 \leqslant a,b \leqslant \mathrm{A}} \alpha_a \alpha_b \mu_{a+b} = i(\mathrm{P} \otimes \mathrm{P}) + i(\mathrm{N} \otimes \mathrm{N}) - 2i(\mathrm{P} \otimes \mathrm{N}),$$

and hence the assumption (2.1) implies the inequality

$$\sum_{1 \leqslant a,b \leqslant \mathrm{A}} \alpha_a \alpha_b \mu_{a+b} \geqslant 0.$$

This extends to $\mathbf{Q}$ by homogeneity, and to $\mathbf{R}$ by continuity. ☐

If not all objects are self-dual (as often happens), the situation is more subtle, because the moment problem on $\mathbf{C}$ is more challenging than on $\mathbf{R}$: the analogue of the positivity condition above is not sufficient to ensure the existence of a positive measure on $\mathbf{C}$ with given moments. However, under an extra growth condition, one obtains both existence and uniqueness of the spectral measure.

Proposition 2.2 (Spectral measures for general objects)**.** *Let $\mathscr{C}$ be a tensor category. Let i be a $\mathbf{C}$-valued additive invariant of $\mathscr{C}$. Suppose that the inequality*

$$i(\mathrm{M} \otimes \mathrm{D}(\mathrm{N})) + i(\mathrm{D}(\mathrm{M}) \otimes \mathrm{N}) \leqslant i(\mathrm{M} \otimes \mathrm{D}(\mathrm{M})) + i(\mathrm{N} \otimes \mathrm{D}(\mathrm{N})) \tag{2.3}$$

holds for all objects M and N of $\mathscr{C}$. Let M be an object of $\mathscr{C}$ satisfying the Carleman condition

$$\sum_{a \geqslant 1} i((\mathrm{M} \otimes \mathrm{D}(\mathrm{M}))^{\otimes a})^{-1/(2a)} = +\infty. \tag{2.4}$$

Then there exists a unique spectral measure for M relative to i.

Proof. This follows as above (*mutatis mutandis* using complexification) from the fact (due to Nussbaum) that the Carleman condition (2.4) combined with the analogue of (2.2), is a sufficient condition for the existence *and* uniqueness of a measure on $\mathbf{C}$ with given moments; see for instance [27, Th. 15.11]. ☐

Remark 2.3. The Carleman condition holds in particular if there exist $c \geqslant 0$ and $r \geqslant 0$ such that the inequality $i((\mathrm{M} \otimes \mathrm{D}(\mathrm{M}))^{\otimes n}) \leqslant cr^n$ holds for all non-negative integers n. This is a frequent occurrence, but it corresponds to measures with compact support (compare with Deligne's "subexponential growth theorem"; see [12, Th. 9.11.4]).

Definition 2.4 (Positive invariants)**.** An additive invariant i on $\mathscr{C}$ is called a *positive invariant* if it satisfies (2.3) for all objects M and N of $\mathscr{C}$.

The following result gives a usable criterion to check that certain invariants are positive.

Proposition 2.5. *Let $\mathscr{C}$ be an essentially small tensor category. Let $\widehat{\mathscr{C}}$ be a set of objects of $\mathscr{C}$ such that every object of $\mathscr{C}$ is isomorphic to a finite direct sum of objects from $\widehat{\mathscr{C}}$. Let i be an additive invariant of $\mathscr{C}$. Then i is positive if the bilinear form*

$$b(n, m) = \sum_{\mathrm{V},\mathrm{W} \in \widehat{\mathscr{C}}} n_{\mathrm{V}} m_{\mathrm{W}}\ i(\mathrm{V} \otimes \mathrm{D}(\mathrm{W}))$$

on $\mathbf{Z}^{(\widehat{\mathscr{C}})}$ is positive, i.e., $b(n, n) \geqslant 0$ for all functions $n \colon \widehat{\mathscr{C}} \to \mathbf{Z}$ with finite support.

Proof. Let M and N be objects of $\mathscr{C}$, and represent them as direct sums

$$\mathrm{M} = \bigoplus_{\mathrm{V} \in \widehat{\mathscr{C}}} m_{\mathrm{V}} \mathrm{V}, \qquad \mathrm{N} = \bigoplus_{\mathrm{W} \in \widehat{\mathscr{C}}} n_{\mathrm{W}} \mathrm{W}$$

with only finitely many non-zero integers m_{V}, n_{W}. By additivity, we obtain the formulas

$$i(\mathrm{M} \otimes \mathrm{D}(\mathrm{N})) + i(\mathrm{D}(\mathrm{M}) \otimes \mathrm{N}) = \sum_{\mathrm{V},\mathrm{W}} m_{\mathrm{V}} n_{\mathrm{W}}\ i(\mathrm{V} \otimes \mathrm{D}(\mathrm{W})) + \sum_{\mathrm{V},\mathrm{W}} m_{\mathrm{V}} n_{\mathrm{W}}\ i(\mathrm{D}(\mathrm{V}) \otimes \mathrm{W}),$$
$$i(\mathrm{M} \otimes \mathrm{D}(\mathrm{M})) + i(\mathrm{N} \otimes \mathrm{D}(\mathrm{N})) = \sum_{\mathrm{V},\mathrm{W}} m_{\mathrm{V}} m_{\mathrm{W}}\ i(\mathrm{V} \otimes \mathrm{D}(\mathrm{W})) + \sum_{\mathrm{V},\mathrm{W}} n_{\mathrm{V}} n_{\mathrm{W}}\ i(\mathrm{V} \otimes \mathrm{D}(\mathrm{W}))$$

so that we get

$$\Big(i(\mathrm{M} \otimes \mathrm{D}(\mathrm{M})) + i(\mathrm{N} \otimes \mathrm{D}(\mathrm{N}))\Big) - \Big(i(\mathrm{M} \otimes \mathrm{D}(\mathrm{N})) + i(\mathrm{D}(\mathrm{M}) \otimes \mathrm{N})\Big) = b(m - n, m - n),$$

and the result then follows from Proposition 2.2. ☐

As a special case, we deduce:

Corollary 2.6. *Let k be a field. Let $\mathscr{C}$ be any essentially small k-linear semisimple tensor category with unit object $1_{\mathscr{C}}$ in which the* Hom *spaces are finite-dimensional. The formula*

$$i(\mathrm{M}) = \dim_k \mathrm{Hom}(1_{\mathscr{C}}, \mathrm{M})$$

defines a positive invariant on $\mathscr{C}$.

Proof. We apply Proposition 2.5 to the set $\widehat{\mathscr{C}}$ of isomorphism classes of simple objects of $\mathscr{C}$. Then $i(\mathrm{V} \otimes \mathrm{D}(\mathrm{W})) = 0$ for V and W in $\widehat{\mathscr{C}}$, unless V is equal to W, so that the bilinear form b in the statement is diagonal in the canonical basis of $\mathbf{Z}^{(\widehat{\mathscr{C}})}$, with diagonal coefficients equal to $i(\mathrm{V} \otimes \mathrm{D}(\mathrm{V})) = \dim_k \mathrm{Hom}(1_{\mathscr{C}}, \mathrm{V} \otimes \mathrm{D}(\mathrm{V})) \geqslant 0$. ☐

Remark 2.7. (1) We emphasize that it is essential to impose the positivity of the spectral measure in Definition 1.1: it is known by independent work of Boas and Pólya (see, e.g., [3]) that *any* sequence of complex numbers is the sequence of moments of infinitely many complex measures on $\mathbf{R}$.

(2) In general, spectral measures are not uniquely determined given the object of interest, and only their moments are unambiguously known (see the conclusion of Remark 3.8 for a simple example where the spectral measure is not unique).

We conclude this section with a simple observation.

Proposition 2.8. *Let k be a field. Let $\mathscr{C}$ be a k-linear tensor category with unit object $1_{\mathscr{C}}$ and let i be a positive invariant on $\mathscr{C}$. Let* M *be an object of $\mathscr{C}$. Let μ be a spectral measure for* M *relative to i.*

(1) *For any non-negative integers m and n, the image measure $(z \mapsto z^m \bar{z}^n)_* \mu$ is a spectral measure for $\mathrm{M}^{\otimes a} \otimes \mathrm{D}(\mathrm{M})^{\otimes b}$ relative to i.*
(2) *The image measure $(z \mapsto 2\,\mathrm{Re}(z))_* \mu$ is a spectral measure for $\mathrm{M} \oplus \mathrm{D}(\mathrm{M})$ relative to i.*

Proof. We prove the second statement, the first being similar. The object $\mathrm{N} = \mathrm{M} \oplus \mathrm{D}(\mathrm{M})$ is self-dual, so it suffices to consider $i(\mathrm{N}^{\otimes a})$ for all integers $a \geqslant 0$. From the isomorphism

$$\mathrm{N}^{\otimes a} \simeq \bigoplus_{0 \leqslant b \leqslant a} \binom{a}{b} \mathrm{M}^{\otimes b} \otimes \mathrm{D}(\mathrm{M})^{\otimes (a-b)},$$

and the definition of spectral measures, we get the equality

$$i(\mathrm{N}^{\otimes a}) = \sum_{0 \leqslant b \leqslant a} \binom{a}{b} \int_{\mathbf{C}} z^b \bar{z}^{b-a} d\mu(z) = \int_{\mathbf{C}} (z + \bar{z})^a d\mu(z),$$

which means that $(z \mapsto 2\,\mathrm{Re}(z))_* \mu$ is a spectral measure for N. ☐

Remark 2.9. With obvious conventions, this proposition can be phrased and generalized as follows: for any polynomial $\mathrm{Q} \in \mathbf{N}[z, \bar{z}]$, the measure $\mathrm{Q}_* \mu$ is a spectral measure for the object $\mathrm{Q}(\mathrm{M}, \mathrm{D}(\mathrm{M}))$.

3. Tensor envelopes and fixed-point statistics

Let k be a field of characteristic zero and $t \in k$ an element. Deligne [10, Th. 2.18] defined a rigid k-linear pseudo-abelian symmetric monoidal category $\underline{\mathrm{Rep}}(\mathrm{S}_t, k)$ by generators and relations, relying on some stability properties of representations of symmetric groups. If t is a non-negative integer $n \geqslant 0$, then $\underline{\mathrm{Rep}}(\mathrm{S}_t, k)$ is abelian and semisimple. If $t = n$, then the semisimplication of $\underline{\mathrm{Rep}}(\mathrm{S}_t, k)$ is equivalent to the category $\mathrm{Rep}(\mathrm{S}_n)$ of k-linear representations of the symmetric group S_n. We will mainly deal with the case where $k = \mathbf{C}(t)$ and t is the indeterminate of k, which we simply denote by $\underline{\mathrm{Rep}}(\mathrm{S}_t)$.

Knop [19] discovered an alternative approach to constructing new rigid symmetric monoidal categories which is *a priori* independent of ideas of interpolating other categories; this leads to many more examples, and happens to recover in a special case the categories of Deligne. The input data in Knop's construction is a base category $\mathscr{A}$ satisfying some regularity conditions, a field k and a degree function δ which associates to every surjective morphism e in $\mathscr{A}$ an element $\delta(e)$ of k, again subject to some conditions. The resulting category is denoted $\mathscr{T}(\mathscr{A},\delta)$ by Knop, and is called the *tensor envelope* of $\mathscr{A}$ with respect to δ. For the moment, it is sufficient for us to recall that every object x of $\mathscr{A}$ defines an object $[x]$ of $\mathscr{T}(\mathscr{A},\delta)$, which is always self-dual, and that the k-linear space of morphisms from $[x]$ to $[y]$ admits as a basis the set of all *relations* from x to y, i.e., the set of all subobjects of the product $x \times y$. To give some context, we spell out in Appendix A the construction of $\mathscr{T}(\mathscr{A},\delta)$ in the special case relevant to Theorem 1.3, namely when $\mathscr{A}$ is the *opposite* of the category of finite sets.

3.1. **Proof of Theorem 1.3.** Let P_1 denote the Poisson distribution with parameter 1. By the so-called *Dobiński's formula* (see, e.g., [26]), for each integer $k \geqslant 0$, the k-th moment

$$\mathbf{E}(\mathrm{P}_1^k) = \frac{1}{e}\sum_{r=0}^{\infty}\frac{r^k}{r!}$$

agrees with the k-th Bell number, i.e., the number of partitions of a set with k elements (indeed, both sequences satisfy $a_0 = 0$ and the recurrence relation $a_{k+1} = \sum_{r=0}^{k}\binom{k}{r}a_r$). In particular, $\mathbf{E}(\mathrm{P}_1^k) \leqslant k^k$, so that the Carleman condition holds and P_1 is determined by its moments (as is well-known). Thanks to the method of moments (see, e.g., [2, Th. 30.2]), to prove the convergence in law $|\operatorname{Fix}(\mathrm{X}_n)| \to \mathrm{P}_1$ as $n \to +\infty$, it suffices to prove that, for each integer $k \geqslant 0$, the sequence of moments $(\mathbf{E}(|\operatorname{Fix}(\mathrm{X}_n)|^k))_{n\geqslant 1}$ converges to $\mathbf{E}(\mathrm{P}_1^k)$.

We first observe the equality $|\operatorname{Fix}(\mathrm{X}_n)| = \chi_n(\mathrm{X}_n)$, where χ_n is the character of the "standard" permutation representation Std_n of S_n acting on $\mathbf{C}^n$. By basic representation theory of finite groups, we then get the expression

$$\mathbf{E}(|\operatorname{Fix}(\mathrm{X}_n)|^k) = \frac{1}{n!}\sum_{\sigma\in\mathrm{S}_n}\chi_n(\sigma)^k = \dim_{\mathbf{C}}\operatorname{Hom}_{\mathrm{Rep}(\mathrm{S}_n)}(1_n, \mathrm{Std}_n^{\otimes k}), \tag{3.1}$$

where 1_n is the trivial one-dimensional representation of S_n.

We now appeal to the Deligne–Knop category $\mathscr{C}_t = \underline{\mathrm{Rep}}(\mathrm{S}_t)$, first in the situation where t is the indeterminate in the field $\mathbf{C}(t)$. Before pursuing the proof, we summarize the properties of $\mathscr{C}_t$ that will be useful for us:

(a) Each finite set X defines a self-dual object [X] of $\mathscr{C}_t$, and these objects satisfy

$$\operatorname{Hom}_{\mathscr{C}_t}([\mathrm{X}],[\mathrm{Y}]) = \mathbf{C}(t)\langle\text{partitions of } \mathrm{X}\sqcup\mathrm{Y}\rangle.$$

(b) The tensor product of [X] and [Y] is the object $[\mathrm{X}]\otimes[\mathrm{Y}] = [\mathrm{X}\sqcup\mathrm{Y}]$.

(c) The category $\mathscr{C}_t$ is a semisimple $\mathbf{C}(t)$-linear tensor category.

In particular, $\mathscr{C}_t$ contains objects $1_t = [\emptyset]$ and $\mathrm{Std}_t = [\{1\}]$, the first being the unit object for the tensor product. By Corollary 2.6, the assignment

$$i(\mathrm{M}) = \dim_{\mathbf{C}(t)}\operatorname{Hom}_{\mathscr{C}_t}(1_t, \mathrm{M})$$

defines a positive invariant on $\mathscr{C}_t$.

Lemma 3.1. *The object* Std_t *admits a unique spectral measure with respect to* i*, and this measure is equal to the Poisson distribution* P_1*. In particular,*

$$\mathbf{E}(\mathrm{P}_1^k) = \dim_{\mathbf{C}(t)} \mathrm{Hom}_{\mathscr{C}_t}(1_t, \mathrm{Std}_t^{\otimes k}). \tag{3.2}$$

Proof. Recall that the object Std_t is self-dual. For each integer $k \geqslant 0$, the k-th tensor product $\mathrm{Std}_t^{\otimes k}$ is the object $[\{1,\ldots,k\}]$ of $\mathscr{C}_t$. Hence, $i(\mathrm{Std}_t^{\otimes k}) = \dim_{\mathbf{C}(t)} \mathrm{Hom}_{\mathscr{C}_t}(1_t, \mathrm{Std}_t^{\otimes k})$ is the number of partitions of the set $\{1,\ldots,k\}$, which is also the k-th moment of P_1. ☐

Combining (3.1) and (3.2), the proof of Theorem 1.3 then reduces to showing the equality

$$\lim_{n\to+\infty} \dim_{\mathbf{C}} \mathrm{Hom}_{\mathrm{Rep}(\mathrm{S}_n)}(1_n, \mathrm{Std}_n^{\otimes k}) = \dim_{\mathbf{C}(t)} \mathrm{Hom}_{\underline{\mathrm{Rep}}(\mathrm{S}_t)}(1_t, \mathrm{Std}_t^{\otimes k}).$$

For this, we use the variant $\mathscr{C}_z = \underline{\mathrm{Rep}}(\mathrm{S}_z, \mathbf{C})$ of the Deligne–Knop category obtained by "specializing" the indeterminate t to some fixed complex number z. Properties (a) and (b) from above still hold, and in particular

$$\dim_{\mathbf{C}(t)} \mathrm{Hom}_{\underline{\mathrm{Rep}}(\mathrm{S}_t)}(1_t, \mathrm{Std}_t^{\otimes k}) = \dim_{\mathbf{C}} \mathrm{Hom}_{\mathscr{C}_z}(1_z, \mathrm{Std}_z^{\otimes k})$$

since a basis of both vector spaces is given by the partitions of $\{1,\ldots,k\}$. Unless z is an integer $n \geqslant 0$, the category $\mathscr{C}_z$ is still semisimple.

For integer values $z = n$, the semisimplication of $\mathscr{C}_n$ is equivalent, as a tensor category, to the category of finite-dimensional complex representations of S_n, an equivalence being given by a functor that maps an object of the form $[\mathrm{X}]$ to the permutation representation on the space V_{X} of functions $\mathrm{X} \to \mathbf{C}^n$ (see [19, Th. 9.8, Example 1, p. 606]). In particular, such a functor sends the object $[\emptyset]$ to the trivial one-dimensional representation 1_n, and the object $[\{1\}]$ to the standard permutation representation Std_n on $\mathbf{C}^n$. The semisimplification of $\mathscr{C}_n$ is the quotient category $\overline{\mathscr{C}}_n = \mathscr{C}_n/\mathscr{N}_n$, where $\mathscr{N}_n$ denotes the tensor radical of $\mathscr{C}_n$ (see [19, § 4.1]). It is a semisimple abelian tensor category by [19, Th. 6.1].

Lemma 3.2. *Let* $n \geqslant 1$ *be an integer. For each integer* $k \geqslant 0$*, the inequality*

$$\dim_{\mathbf{C}} \mathrm{Hom}_{\mathrm{Rep}(\mathrm{S}_n)}(1_n, \mathrm{Std}_n^{\otimes k}) \leqslant \dim_{\mathbf{C}} \mathrm{Hom}_{\mathscr{C}_n}(1_t, \mathrm{Std}_n^{\otimes k})$$

holds, with equality if and only if $n \geqslant k$*.*

Proof. Let X and Y be finite sets. By definition of the quotient category, the objects of $\overline{\mathscr{C}}_n$ are the same as those of $\mathscr{C}_n$, and the morphisms between the representations V_{X} and V_{Y} of S_n corresponding to $[\mathrm{X}]$ and $[\mathrm{Y}]$ via the equivalence of categories are given by

$$\mathrm{Hom}_{\mathrm{Rep}(\mathrm{S}_n)}(\mathrm{V}_{\mathrm{X}}, \mathrm{V}_{\mathrm{Y}}) = \mathrm{Hom}_{\overline{\mathscr{C}}_n}([\mathrm{X}], [\mathrm{Y}]) = \mathrm{Hom}_{\mathscr{C}_n}([\mathrm{X}], [\mathrm{Y}])/\mathscr{N}_n([\mathrm{X}], [\mathrm{Y}]).$$

Therefore, we obtain an inequality

$$\dim_{\mathbf{C}} \mathrm{Hom}_{\mathrm{Rep}(\mathrm{S}_n)}(\mathrm{V}_{\mathrm{X}}, \mathrm{V}_{\mathrm{Y}}) \leqslant \dim_{\mathbf{C}} \mathrm{Hom}_{\mathscr{C}_n}([\mathrm{X}], [\mathrm{Y}]),$$

with equality if and only if $\mathscr{N}_n([\mathrm{X}], [\mathrm{Y}])$ is reduced to the zero morphism. Taking $\mathrm{X} = \emptyset$ and $\mathrm{Y} = \{1,\ldots,k\}$, so that $\mathrm{V}_{\mathrm{X}} = 1_n$ and $\mathrm{V}_{\mathrm{Y}} = \mathrm{Std}_n^{\otimes k}$, proves the first part of the statement.

It remains to see when $\mathscr{N}_n(1_n, \mathrm{Std}_n^{\otimes k})$ is zero. By a result of Knop [19, Cor. 8.5], this holds if and only if certain invariants ω_e in $\mathbf{C}$ are non-zero for all indecomposable surjective morphisms $e\colon u \to v$ in the category Set^{opp} such that u is a subquotient of $1_t \otimes \mathrm{Std}_t^{\otimes k} = \mathrm{Std}_t^{\otimes k}$. By [19, Ex. 1, p. 596], this invariant is equal to $\omega_e = n - |v|$ for such morphisms; since indecomposable surjective morphisms $u \to v$ in Set^{opp} are injective maps of sets $v \hookrightarrow u$

satisfying $|v| = |u| - 1$, and u is a subquotient of $\mathrm{Std}_n^{\otimes k} = [\{1, \ldots, k\}]$, we have $|v| \leqslant k - 1$, and hence $\omega_e = n - |v| \geqslant 1$ is non-zero for all $n \geqslant k$. $\square$

Remark 3.3. (1) In comparison with other proofs, this abstract argument has the advantage of explaining, to some extent, where the Poisson distribution comes from.

(2) It is natural to ask if similar ideas can be used to reprove other statements in the theory of random permutations, such as the fact that the sequence $(\ell_i(\mathrm{X}_n))_{n\geqslant 1}$, where $\ell_i(\sigma)$ denotes the number of i-cycles in the decomposition of a permutation σ, converges in law to the Poisson distribution $\mathrm{P}_{1/i}$. More ambitiously, one can try to count the number of cycles in a random permutation.

(3) To the best of our knowledge, the fact that the first moments of $|\operatorname{Fix}(\mathrm{X}_n)|$ coincide with those of the Poisson distribution first appears in the work of Diaconis–Shashahani [11, Th. 7].

3.2. **Other fixed-point statistics.** Knop's approach yields many more instances of tensor categories, and the principles above are then applicable. As an example, we recover a result of Fulman (proved in his 1997 unpublished thesis) which appears in a paper of Fulman and Stanton [14, Th. 4.1].

Proposition 3.4 (Fulman)**.** *Let* E *be a finite field and let* $(\mathrm{X}_n)_{n\geqslant 1}$ *be a sequence of random variables with* X_n *uniformly distributed in* $\mathbf{GL}_n(\mathrm{E})$*. The sequence* $(|\operatorname{Fix}(\mathrm{X}_n)|)_{n\geqslant 1}$*, where* $\operatorname{Fix}(g)$ *is the* 1*-eigenspace of* $g \in \mathbf{GL}_n(\mathrm{E})$*, converges in law as* $n \to +\infty$*. For* $k \geqslant 0$*, the* k*-th moment of the limiting distribution is equal to the number of vector subspaces of* E^k*. Moreover, the* k*-th moment of* $|\operatorname{Fix}(\mathrm{X}_n)|$ *is equal to the limiting moment for* $n \geqslant k$*.*

Proof. We argue as in the proof of Theorem 1.3, using instead the base category $\mathsf{Vec}(\mathrm{E})$ of finite-dimensional E-vector spaces and the degree function $\delta(e\colon \mathrm{U} \to \mathrm{V}) = t^{\dim_{\mathrm{E}}(\ker(e))}$ for a surjective E-linear map to construct Knop's category $\mathscr{C}_t$. We use as before the unit object $1_t = [\{0\}]$ and the standard object $\mathrm{Std}_t = [\mathrm{E}]$, which is self-dual.

Specializing to $t = |\mathrm{E}|^n$ for some integer $n \geqslant 1$, the quotient $\overline{\mathscr{C}}_{|\mathrm{E}|^n}$ is naturally equivalent to the category of finite-dimensional complex representations of $\mathbf{GL}_n(\mathrm{E})$ (see [19, Example 5, p. 606]). We obtain

$$\dim_{\mathbf{C}} \operatorname{Hom}_{\mathbf{GL}_n(\mathrm{E})}(1_n, \mathrm{Std}_n^{\otimes k}) \leqslant \dim_{\mathbf{C}(t)} \operatorname{Hom}_{\mathscr{C}_t}(1_t, \mathrm{Std}_t^{\otimes k}),$$

where Std_n is the $|\mathrm{E}|^n$-dimensional permutation representation of $\mathbf{GL}_n(\mathrm{E})$ associated to its natural action on E^n. As before, there is equality if the numerical invariants ω_e are non-zero for indecomposable surjective E-linear maps $e\colon \mathrm{U} \to \mathrm{V}$ where U is a subquotient of $\mathrm{Std}_n^{\otimes k}$ in $\mathscr{C}_{|\mathrm{E}|^n}$. We have $\omega(e) = |\mathrm{E}|^n - |\mathrm{V}|$, and hence there is equality if $n \geqslant k$ (note that in $\mathscr{C}_{|\mathrm{E}|^n}$, the tensor product is defined using the *direct sum* of finite-dimensional E-vector spaces).

On the one hand, for all $n \geqslant 1$, the function $g \mapsto |\operatorname{Fix}(g)|$ is the character of the standard representation, and on the other hand, by Knop's construction, the dimension

$$\dim_{\mathbf{C}(t)} \operatorname{Hom}_{\mathscr{C}_t}(1_t, \mathrm{Std}_t^{\otimes k})$$

is the number of subspaces of E^k. Thus, $\mathbf{E}(|\operatorname{Fix}(\mathrm{X}_n)|^k)$ converges to this number. To conclude, we need however to apply Lemma 3.5 below, since in this case the size of the moments do not satisfy the Carleman condition, but it is known that

$$|\{\text{subspaces of } \mathrm{E}^k\}| \ll |\mathrm{E}|^{k(k+1)/4}. \qquad \square$$

Lemma 3.5 (Heath–Brown)**.** *Let $q \geqslant 1$ be an integer, and let $(m_k)_{k\geqslant 0}$ be a sequence of real numbers such that $m_k \ll q^{k(k+1)/4}$ for $k \geqslant 0$. Let $(\mathrm{Z}_n)_{n\geqslant 1}$ be a sequence of random variables such that*

(1) *For all n, the support of Z_n is contained in the set of powers q^r for $r \geqslant 0$.*
(2) *For all $k \geqslant 0$, we have $\mathbf{E}(\mathrm{Z}_n^k) \to m_k$.*

Then (Z_n) converges in law to a random variable Z supported on powers of q with moments m_k for all $k \geqslant 0$.

Proof. This is implicit in [17, Lemmas 17 and 18]. More precisely, it follows from standard results in the method of moments that the second assumption implies that any subsequence of $(\mathrm{Z}_n)_{n\geqslant 0}$ which converges in law has a limit with moments m_k, and it is elementary from the first assumption that all such limits are supported on powers of q. Heath–Brown's result (proved in [17] in the case $q = 4$, but with immediate generalization) is that there is a unique probability measure on $\mathbf{R}$ with these two properties. Since moreover the convergence of moments implies uniform integrability (or tightness), this means that the sequence $(\mathrm{Z}_n)_{n\geqslant 0}$ is relatively compact and has a unique limit point, and hence converges. The stated properties of the limit are then clear. $\square$

Remark 3.6. A result of Christiansen [5] (also cited by Fulman and Stanton) shows that the limiting measure of Proposition 3.4, as a measure on $\mathbf{R}$, is not characterized by its moments. Thus, some extra condition is necessary to ensure uniqueness, and this is provided by the assumption that the support is restricted to powers of q.

Considering another example of Knop leads by the same method to a similar result which is new, to the best of our knowledge.

Proposition 3.7. *Let E be a finite field and let $(\mathrm{X}_n)_{n\geqslant 1}$ be a sequence of random variables with X_n uniformly distributed in the affine-linear group $\mathbf{Aff}_n(\mathrm{E})$ of E^n.*

The sequence $(|\operatorname{Fix}(\mathrm{X}_n)|)_{n\geqslant 1}$, where $\operatorname{Fix}(g)$ is the set of fixed points of $g \in \mathbf{Aff}_n(\mathrm{E})$, converges in law as $n \to +\infty$. For $k \geqslant 0$, the k-th moment of the limiting distribution is equal to the number of affine subspaces of E^{k-1}.

Moreover, the k-th moment of $|\operatorname{Fix}(\mathrm{X}_n)|$ is equal to the limiting moment for $n \geqslant k$.

Proof. We argue as above with the base category $\mathscr{A}$ of (non-empty) affine spaces over E (see [19, p. 597, Ex. 6; p. 607, Ex. 7]). $\square$

Remark 3.8. It is also natural to consider the category $\underline{\mathrm{Rep}}(\mathrm{GL}_t)$ of Deligne and Milne (see [10, § 10, Déf. 10.2]), interpolating the categories of representations of $\mathbf{GL}_n(\mathbf{C})$. Indeed, the argument applies rather similarly, and leads to the analogue of Theorem 1.3 in this context: the direct image under the trace $\mathrm{Tr}\colon \mathrm{U}_n \to \mathbf{C}$ of the probability Haar measure on the unitary group U_n converges as $n \to +\infty$ to a standard complex gaussian. This was first proved by Diaconis and Shashahani [11]; see also Larsen's paper [24] for the case of the symplectic groups and real gaussians.

First, by Corollary 2.6, the assignment $i(\mathrm{M}) = \dim_{\mathbf{C}(t)} \operatorname{Hom}_{\underline{\mathrm{Rep}}(\mathrm{GL}_t)}(1_t, \mathrm{M})$ defines a positive invariant on $\underline{\mathrm{Rep}}(\mathrm{GL}_t)$. One can then show that there exists an object Std_t, which for

$t = n$ corresponds to the standard representation of $\mathbf{GL}_n(\mathbf{C})$ through the equivalence from the semisimplification of $\underline{\mathrm{Rep}}(\mathrm{GL}_t)$ to the category of representations of $\mathbf{GL}_n(\mathbf{C})$, satisfying

$$i(\mathrm{Std}_t^{\otimes a} \otimes \mathrm{D}(\mathrm{Std}_t)^{\otimes b}) = \dim_{\mathbf{C}(t)} \mathrm{Hom}(1_t, \mathrm{Std}_t^{\otimes a} \otimes \mathrm{D}(\mathrm{Std}_t)^{\otimes b}) = \begin{cases} 0 & \text{if } a \neq b, \\ a! & \text{if } a = b. \end{cases}$$

More precisely, with the notation of [10, Déf. 10.2], the object Std_t corresponds to the pair of finite sets $(\{1\}, \emptyset)$ and is denoted by $\mathrm{X}_0^{\otimes\{1\}}$. Thus, $\mathrm{Std}_t^{\otimes a} \otimes \mathrm{D}(\mathrm{Std}_t)^{\otimes b}$ corresponds to the pair $(\{1, \ldots, a\}, \{1, \ldots, b\})$ and the value of $i(\mathrm{Std}_t^{\otimes a} \otimes \mathrm{D}(\mathrm{Std}_t)^{\otimes b})$ is the dimension of the space

$$\mathrm{Hom}((\emptyset, \emptyset), (\{1, \ldots, a\}, \{1, \ldots, b\})),$$

which is by definition the number of bijections $\{1, \ldots, b\} \to \{1, \ldots, a\}$. These values are known to be equal to the moments

$$\frac{1}{\pi} \int_{\mathbf{C}} z^a \bar{z}^b e^{-|z|^2} dz$$

of a standard complex gaussian random variable, which is therefore the spectral measure associated to Std_t. Using a stabilization property of the corresponding invariants for $\mathbf{GL}_n(\mathbf{C})$ when $n > a + b$, one gets convergence as before (see [10, Prop. 10.6]).

This proof is not as satisfactory as that of Theorem 1.3, because Deligne and Milne's definition of $\underline{\mathrm{Rep}}(\mathrm{GL}_t)$ involves some *a priori* knowledge of stability properties of representations and linear invariants of $\mathbf{GL}_n(\mathbf{C})$. The argument does show, however, that the convergence to the gaussian can be interpreted in terms of spectral measures, and that the standard gaussian can also be interpreted as a "generalized" Sato–Tate measure. Moreover, it suggests the question: what are the spectral measures for other objects of $\underline{\mathrm{Rep}}(\mathrm{GL}_t)$?

We note also that since Berg [1] proved that the third power of a real gaussian random variable is not determined by its moments, the third tensor power of $\mathrm{Std}_t \oplus \mathrm{D}(\mathrm{Std}_t)$ (which, thanks to Proposition 2.8, has spectral measure the cube of a real gaussian), gives an example of an object of $\underline{\mathrm{Rep}}(\mathrm{GL}_t)$ whose spectral measure is not unique.

4. Proof of Theorem 1.4

Let $m \geqslant 1$ be an integer and let λ be a partition of m with parts $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$. Recall that the statement concerns the limiting behavior of the measures $(\chi_{\lambda,n}(\mathrm{X}_n))_{n \geqslant m+\lambda_1}$, where X_n is a uniformly distributed random permutation in S_n and $\chi_{\lambda,n} \colon \mathrm{S}_n \to \mathbf{C}$ is the character of the representation of S_n corresponding to the partition $(n - m, \lambda_1, \lambda_2, \ldots)$.

The argument will consist of two stages:

- We prove *a priori* that the sequence of measures $(\chi_{\lambda,n}(\mathrm{X}_n))_{n \geqslant m+\lambda_1}$ converges in law as $n \to +\infty$ to some measure μ_λ.
- We compute the moments of the limiting measure and show that they coincide with those of a spectral measure of the object $x_{\lambda,t}$ of the Deligne–Knop category.

Lemma 4.1. *The sequence $(\chi_{\lambda,n}(\mathrm{X}_n))_{n \geqslant m+\lambda_1}$ converges in law to a measure μ_λ as $n \to +\infty$.*

Proof. For each $i \geqslant 1$, let $\ell_i(\sigma)$ denote the number of i-cycles (fixed points if $i = 1$) in the representation of σ as a product of cycles with disjoint support. It is known from the theory of symmetric functions that there exists a so-called character polynomial $q_\lambda \in \mathbf{Q}[(\mathrm{L}_i)_{i\geqslant 1}]$ such that, for all large enough n, the equality

$$\chi_{\lambda,n}(\sigma) = q_\lambda(\ell_1(\sigma), \ldots, \ell_i(\sigma), \ldots)$$

holds for all $\sigma \in \mathrm{S}_n$ (see, for instance, [25, Ex. I.7.14]). Since the sequences $(\ell_i(\mathrm{X}_n))_{i\geqslant 1}$ are also known to converge in law as $n \to +\infty$ to a sequence $(\mathrm{P}_{1/i})_{i\geqslant 1}$ of independent Poisson random variables with parameters $1/i$ (see, e.g., [11, Th. 7]), the sequence $(\chi_{\lambda,n}(\mathrm{X}_n))_{n\geqslant m+\lambda_1}$ converges in law to $\mu_\lambda = q_\lambda(\mathrm{P}_1, \ldots, \mathrm{P}_{1/i}, \ldots)$. ☐

Remark 4.2. In the spirit of Remark 3.3 (2), it would be interesting to prove Lemma 4.1 without using character polynomials.

The second step will rely on Deligne's construction of the category of representations of S_t, which enjoys some functoriality properties that have not been explicitly established by Knop. Since Theorem 1.4 is new, the fact that Deligne's definition involves some a priori knowledge of representations of the symmetric groups is not an instance of circular reasoning.

Let A be a commutative ring and $t \in \mathrm{A}$. We use the A-linear category $\underline{\mathrm{Rep}}(\mathrm{S}_t, \mathrm{A})$ of Deligne [10, Déf. 2.17], keeping the notation $\underline{\mathrm{Rep}}(\mathrm{S}_t)$ for $\mathrm{A} = \mathbf{C}(t)$ and the indeterminate t. The basic objects of this category are associated to finite sets U and denoted by[1] [U]; their Hom spaces are introduced in [10, Déf. 2.12]. For each integer $\mathrm{N} \geqslant 0$, we consider the full subcategory $\underline{\mathrm{Rep}}(\mathrm{S}_t, \mathrm{A})^{(\mathrm{N})}$ whose objects are the direct factors of sums of [U] for U of cardinal $\leqslant \mathrm{N}$. Deligne [10, Prop. 5.1] proved that $\underline{\mathrm{Rep}}(\mathrm{S}_t)^{(\mathrm{N})}$ is a semisimple abelian category if t is not an integer between 0 and $2\mathrm{N}-2$. Moreover, under the assumptions

$$t - k \in \mathrm{A}^\times \text{ for } 0 \leqslant k \leqslant 2\mathrm{N}-2 \quad \text{and} \quad \mathrm{N}! \in \mathrm{A}^\times, \tag{4.1}$$

he associated to any pair (y, ϱ) consisting of a finite set y with $|y| \leqslant 2\mathrm{N}$ and an irreducible representation ϱ of the symmetric group S_y, an object $\boldsymbol{x}_{y,\varrho,\mathrm{A}}$ of $\underline{\mathrm{Rep}}(\mathrm{S}_t, \mathrm{A})^{(\mathrm{N})}$; see [10, Prop. 5.1 and Rem. 5.6]. (This object is independent, up to isomorphism, of the choice of N, provided (4.1) holds, and hence the value of N is omitted from the notation.)

The objects $\boldsymbol{x}_{y,\varrho,\mathrm{A}}$ are functorial with respect to A under the natural base-change functor

$$\mathrm{T}_{\mathrm{A},\mathrm{B}} \colon \underline{\mathrm{Rep}}(\mathrm{S}_t, \mathrm{A}) \longrightarrow \underline{\mathrm{Rep}}(\mathrm{S}_t, \mathrm{B})$$

when B is an A-algebra (see [10, Déf. 2.17]), i.e., there are isomorphisms

$$\boldsymbol{x}_{y,\varrho,\mathrm{B}} \simeq \mathrm{T}_{\mathrm{A},\mathrm{B}}(\boldsymbol{x}_{y,\varrho,\mathrm{A}}).$$

If B is a field of characteristic zero and the image of t is not a non-negative integer, then the full category $\underline{\mathrm{Rep}}(\mathrm{S}_t, \mathrm{B})$ is a semisimple abelian category, and its simple objects are precisely those of the form $\boldsymbol{x}_{y,\varrho,\mathrm{B}}$, for a unique pair (y, ϱ), up to isomorphism.

From now on, we fix an integer $\mathrm{N} \geqslant 1$ and consider the ring

$$\mathrm{A} = \mathbf{C}[t]\Big[\Big(\frac{1}{t-k}\Big)_{0\leqslant k\leqslant 2\mathrm{N}-2}\Big],$$

which is a principal ideal domain (being a localization of the principal ideal domain $\mathbf{C}[t]$) and satisfies the assumption (4.1).

[1] Deligne's basic generators [U] are not the same as the basic objects in Knop's definition, but the precise relation between them is explained by Knop in [20, Rem. 1.2].

Let $m \geqslant 1$ be an integer and λ a partition of m. We then set $\boldsymbol{x}_{\lambda,\mathrm{A}} = \boldsymbol{x}_{y,\varrho,\mathrm{A}}$, where $y = \{1,\dots,m\}$ and ϱ is the irreducible representation of S_m associated to the partition λ. We denote by $x_{\lambda,t}$ the base change of $\boldsymbol{x}_{\lambda,\mathrm{A}}$ to $\underline{\mathrm{Rep}}(\mathrm{S}_t)$ under the natural inclusion $\mathrm{A} \hookrightarrow \mathbf{C}(t)$. Furthermore, if $n > 2\mathrm{N}-2$, then we denote by $x_{\lambda,n}$ the base change of $\boldsymbol{x}_{\lambda,\mathrm{A}}$ to $\underline{\mathrm{Rep}}(\mathrm{S}_n)$ under the morphism $\mathrm{A} \to \mathbf{C}(t)$ that maps t to n.

We begin with a lemma generalizing the first step of the proof of Lemma 3.2 (in the sense that it shows that certain Hom spaces have the same dimension in all Deligne–Knop categories $\underline{\mathrm{Rep}}(\mathrm{S}_t)$, even when t is a non-negative integer, provided it is "large enough").

Lemma 4.3. *Let λ be a partition of an integer $m \geqslant 1$ and let $a \geqslant 0$ be an integer. For any integer $n \geqslant 4am-1$, the following equality holds:*

$$\dim_{\mathbf{C}(t)} \mathrm{Hom}_{\underline{\mathrm{Rep}}(\mathrm{S}_t)}(1_t, x_{\lambda,t}^{\otimes a}) = \dim_{\mathbf{C}} \mathrm{Hom}_{\underline{\mathrm{Rep}}(\mathrm{S}_n)}(1_n, x_{\lambda,n}^{\otimes a}).$$

Proof. Let $\mathrm{N} \geqslant 1$ be an integer such that $\mathrm{N} \geqslant 2am$. Then both $\boldsymbol{x}_{\lambda,\mathrm{A}}$ and $\boldsymbol{x}_{\lambda,\mathrm{A}}^{\otimes a}$ are objects of $\underline{\mathrm{Rep}}(\mathrm{S}_t,\mathrm{A})^{(\mathrm{N})}$ (this follows from the fact that the tensor product of two basic objects $[\mathrm{U}]$ and $[\mathrm{V}]$ is a direct sum of objects $[\mathrm{W}]$ with $|\mathrm{W}| \leqslant |\mathrm{U}| + |\mathrm{V}|$; see [10, § 5.10]). Consequently, by [10, Rem. 5.6] and the fact that A is principal, there is a direct sum decomposition

$$\boldsymbol{x}_{\lambda,\mathrm{A}}^{\otimes a} \simeq \bigoplus_{|\mu| \leqslant \mathrm{N}} v(\mu) \boldsymbol{x}_{\mu,\mathrm{A}} \tag{4.2}$$

for some non-negative integers $v(\mu)$, where the sum is over partitions of integers $\leqslant \mathrm{N}$. Assume $n > 2\mathrm{N}-2$. Applying base-change to $\mathbf{C}(t)$ and to $\mathbf{C}$ as above $t \mapsto n$, we derive from (4.2) direct sum decompositions

$$x_{\lambda,t}^{\otimes a} \simeq \bigoplus_{|\mu| \leqslant \mathrm{N}} v(\mu) x_{\mu,t} \qquad x_{\lambda,n}^{\otimes a} \simeq \bigoplus_{|\mu| \leqslant \mathrm{N}} v(\mu) x_{\mu,n}.$$

By [10, Rem. 5.6], the objects $\boldsymbol{x}_{\mu,\mathrm{A}}$ have the property that

$$\mathrm{Hom}(\boldsymbol{x}_{\mu,\mathrm{A}}, \boldsymbol{x}_{\nu,\mathrm{A}}) = \begin{cases} 0 & \text{if } \mu \neq \nu, \\ \mathrm{A} & \text{if } \mu \neq \nu. \end{cases}$$

Since the unit objects of $\underline{\mathrm{Rep}}(\mathrm{S}_t)$ and $\underline{\mathrm{Rep}}(\mathrm{S}_n)$ are $x_{\mu,t}$ and $x_{\mu,n}$, respectively, for the partition $\mu = (m)$ corresponding to the trivial representation of S_m, we therefore deduce from these decompositions that the equalities

$$\dim_{\mathbf{C}(t)} \mathrm{Hom}_{\underline{\mathrm{Rep}}(\mathrm{S}_t)}(1_t, x_{\lambda,t}^{\otimes a}) = v((m)) = \dim_{\mathbf{C}} \mathrm{Hom}_{\underline{\mathrm{Rep}}(\mathrm{S}_n)}(1_n, x_{\lambda,n}^{\otimes a})$$

hold for all $n \geqslant 4am-1$, which concludes the proof. ☐

Remark 4.4. A combinatorial formula for

$$\dim_{\mathbf{C}(t)} \mathrm{Hom}_{\underline{\mathrm{Rep}}(\mathrm{S}_t)}(1_t, x_{\lambda,t}^{\otimes a})$$

has been obtained (in the generality of tensor envelopes) by Knop [21, Cor. 5.4, Ex. 5.6].

We can now conclude the proof of Theorem 1.4. Let $a \geqslant 0$ be an integer. Thanks to Lemma 4.3, the equalities

$$i(x_{\lambda,t}^{\otimes a}) = \dim_{\mathbf{C}(t)} \mathrm{Hom}_{\underline{\mathrm{Rep}}(\mathrm{S}_t)}(1_t, x_{\lambda,t}^{\otimes a}) = \dim_{\mathbf{C}} \mathrm{Hom}_{\underline{\mathrm{Rep}}(\mathrm{S}_n)}(1_n, x_{\lambda,n}^{\otimes a})$$

hold for all large enough integers n. Besides, Deligne [10, Prop. 6.4] has shown that, provided $n > 2m$, the semisimplification functor

$$\underline{\mathrm{Rep}}(\mathrm{S}_n) \to \underline{\mathrm{Rep}}(\mathrm{S}_n)/\mathscr{N}_n = \mathrm{Rep}(\mathrm{S}_n)$$

maps the object $x_{\lambda,n}$ to the representation $\pi_{\lambda,n}$ of S_n associated to the partition $\lambda^{(n)}$. Thus, we obtain the lower bound

$$i(x_{\lambda,t}^{\otimes a}) = \dim_{\mathbf{C}} \mathrm{Hom}_{\underline{\mathrm{Rep}}(\mathrm{S}_n)}(1_n, x_{\lambda,n}^{\otimes a}) \geqslant \dim_{\mathbf{C}} \mathrm{Hom}_{\mathrm{Rep}(\mathrm{S}_n)}(1_n, \pi_{\lambda,n}^{\otimes a}),$$

with equality if and only if $\mathscr{N}(1_n, x_{n,\lambda}^{\otimes a}) = 0$. For all large enough (depending on a and λ) integers n, we have $\mathscr{N}(1_n, x_{\lambda,n}^{\otimes a}) = 0$ (e.g., by Knop's criterion), and hence for such n, we get

$$\int_{\mathbf{R}} x^a \mu_\lambda(x) = i(x_{\lambda,t}^{\otimes a}) = \dim_{\mathbf{C}} \mathrm{Hom}_{\mathrm{Rep}(\mathrm{S}_n)}(1_n, \pi_{\lambda,n}^{\otimes a}) = \frac{1}{n!} \sum_{\sigma \in \mathrm{S}_n} \chi_{\lambda,n}(\sigma)^a,$$

as we wanted to show. □

5. Arithmetic speculations

The distribution of the number of fixed points of random permutations in S_n for a given integer $n \geqslant 1$ occurs naturally in number theory as a limiting distribution for the number of zeros modulo a prime number p of a fixed polynomial with integer coefficients $f \in \mathbf{Z}[\mathrm{T}]$ of degree n and Galois group S_n. Indeed, let $\varrho_f(p)$ be this number. A special case of Chebotarev's density theorem states in that case[2] that the limit formula

$$\lim_{x \to +\infty} \frac{1}{\pi(x)} \big|\{p \leqslant x \mid \varrho_f(p) = r\}\big| = \frac{1}{n!} |\{\sigma \in \mathrm{S}_n \text{ with } |\operatorname{Fix}(\sigma)| = r\}|$$

holds for all integers $r \geqslant 0$, where $\pi(x)$ denotes the number of primes $p \leqslant x$ (this was already observed by Kronecker [23] in 1880, who also pointed out the limiting behavior as $n \to +\infty$).

One may ask if a similar framework can give rise to the Poisson distribution, viewed as the number of fixed points of a "random element" of S_t for an indeterminate t. Some work of Kowalski and Soundararajan [22, § 2.4] involving *pseudopolynomials* might be related. Indeed, they have formulated the following conjecture:

Conjecture 5.1 (Kowalski–Soundararajan)**.** *Let* $\mathrm{F}(n) = \sum_{k=0}^{n} n!/k!$ *for integers* $n \geqslant 0$. *For any prime number* p, *let* $\varrho_{\mathrm{F}}(p)$ *be the number of integers* x *satisfying* $0 \leqslant x \leqslant p-1$ *and* $\mathrm{F}(x) \equiv 0 \pmod p$. *Then, for each integer* $r \geqslant 0$, *the following limit formula holds:*

$$\lim_{x \to +\infty} \frac{1}{\pi(x)} \big|\{p \leqslant x \mid \varrho_{\mathrm{F}}(p) = r\}\big| = \frac{1}{e}\frac{1}{r!}.$$

A *pseudopolynomial* in the sense of Hall [15] is a sequence $(a_n)_{n\geqslant 0}$ of integers such that $m-n$ divides $a_m - a_n$ for all $m > n$. Setting $\mathrm{G}(n) = a_n$, this condition guarantees that the value $\mathrm{G}(x) \pmod p$ is well-defined for $x \in \mathbf{Z}/p\mathbf{Z}$, independently of the choice of a representative to compute it. Besides the sequences $(f(n))_n$ of values of a polynomial with integer coefficients $f \in \mathbf{Z}[\mathrm{X}]$, a standard example is $\mathrm{F}(n)$ as in Conjecture 5.1. This function can

[2] For an arbitrary irreducible polynomial $f \in \mathbf{Z}[\mathrm{T}]$, the corresponding limit would be the probability that a uniformly distributed random element of the Galois group of the splitting field of f, viewed as a permutation of the complex roots of f, has r fixed points.

also be written as $e\int_1^\infty x^n e^{-x}dx$ for all $n \geqslant 0$ (an incomplete gamma function), or $\lfloor en!\rfloor$ for $n \geqslant 1$.

Numerical evidence is favour of Conjecture 5.1 is quite convincing [22, § 2.4]. We speculate that, if true, this limiting behaviour might be explained by appealing to the properties of S_t and some avatar of Chebotarev's density theorem.

Another tantalizing experimental parallel observation is the following. It results from Deligne's equidistribution theorem and the work of Katz (see [18, Th. 7.10.6]) that, given a polynomial $f \in \mathbf{Z}[\mathrm{X}]$ of degree $n \geqslant 6$ whose derivative f' has Galois group S_{n-1}, the exponential sums

$$\mathrm{W}_f(a;p) = \frac{1}{\sqrt{p}} \sum_{x \,(\mathrm{mod}\, p)} \exp\Bigl(2\pi i \frac{af(x)}{p}\Bigr)$$

for $a \in (\mathbf{Z}/p\mathbf{Z})^\times$ become equidistributed as $p \to +\infty$ like the traces of random matrices in a compact group $\mathrm{K} \subset \mathrm{U}_n$ which contains SU_n.

By analogy and comparison with the results of Diaconis–Shashahani and Larsen, we are then led to expect the following:

Conjecture 5.2. *Let* $\mathrm{F}(n) = \sum_{k=0}^{n} n!/k!$ *for integers* $n \geqslant 0$. *For a prime number* p *and* $a \in (\mathbf{Z}/p\mathbf{Z})^\times$, *set*

$$\mathrm{W}_\mathrm{F}(a;p) = \frac{1}{\sqrt{p}} \sum_{x \,(\mathrm{mod}\, p)} \exp\Bigl(2\pi i \frac{a\mathrm{F}(x)}{p}\Bigr).$$

Then the values $(\mathrm{W}_\mathrm{F}(a;p))_{a\in(\mathbf{Z}/p\mathbf{Z})^\times}$ *become equidistributed as* $p \to +\infty$ *like a standard complex gaussian, i.e., for any continuous bounded function* $\varphi\colon \mathbf{C} \to \mathbf{C}$, *the following holds:*

$$\lim_{p\to+\infty} \frac{1}{p-1} \sum_{a\in(\mathbf{Z}/p\mathbf{Z})^\times} \varphi(\mathrm{W}_\mathrm{F}(a;p)) = \frac{1}{\pi}\int_{\mathbf{C}} \varphi(z) e^{-|z|^2} dz.$$

Numerical evidence is again very convincing here. A potential link suggests itself with the representations of $\mathbf{GL}_t$, and even more tantalizing is the suggestion of a form of Schur–Weyl duality relating the categories of representations of S_t and $\mathbf{GL}_t$.

Appendix A. Knop's construction of the category $\underline{\mathrm{Rep}}(\mathrm{S}_t)$

In this section, we recall the steps of Knop's construction of tensor envelopes, specialized to the case of the opposite of the category of finite sets which leads to Deligne's category of "representations" of S_t.

Given sets X, Y and Z with maps $f\colon \mathrm{Y} \to \mathrm{X}$ and $g\colon \mathrm{Y} \to \mathrm{Z}$, we define the gluing $\mathrm{X} \sqcup_\mathrm{Y} \mathrm{Z}$ as the quotient of the disjoint union $\mathrm{X} \sqcup \mathrm{Z}$ by the smallest equivalence relation that identifies $f(y) \in \mathrm{X}$ with $g(y) \in \mathrm{Z}$ for all $y \in \mathrm{Y}$.

Recall that a partition of a set X is a set of non-empty subsets of X, pairwise disjoint and with union equal to X; we will identify partitions with equivalence relations on X.

Given sets X, Y and Z, and partitions α of $\mathrm{X} \sqcup \mathrm{Y}$ and β of $\mathrm{Y} \sqcup \mathrm{Z}$, one defines a partition $\beta \odot \alpha$ of $\mathrm{X} \sqcup \mathrm{Z}$ as follows:

– the equivalence class of an element $x \in \mathrm{X}$ is the union of the α-equivalence class of x and of the set of $z \in \mathrm{Z}$ such that there exists $y \in \mathrm{Y}$ which is α-equivalent to x and β-equivalent to z;
– the equivalence class of an element $z \in \mathrm{Z}$ is the union of the β-equivalence class of z and of the set of $x \in \mathrm{X}$ such that there exists $y \in \mathrm{Y}$ which is α-equivalent to x and β-equivalent to z.

Using the quotient maps

$$\mathrm{Y} \to (\mathrm{X} \sqcup \mathrm{Y})/\alpha, \qquad \mathrm{Y} \to (\mathrm{Y} \sqcup \mathrm{Z})/\beta,$$

we define the gluing $(\mathrm{X} \sqcup \mathrm{Y})/\alpha \ \sqcup_{\mathrm{Y}} \ (\mathrm{Y} \sqcup \mathrm{Z})/\beta$ as above. There is an injective map

$$j\colon (\mathrm{X} \sqcup \mathrm{Z})/\beta \odot \alpha \to (\mathrm{X} \sqcup \mathrm{Y})/\alpha \ \sqcup_{\mathrm{Y}} \ (\mathrm{Y} \sqcup \mathrm{Z})/\beta,$$

and we define $\gamma(\alpha, \beta)$ as the cardinality of the complement of the image of j. Concretely, this is the number of equivalence classes of elements of Y which are not α-equivalent to an element of X neither β-equivalent to an element of Z.

We fix a ring k and an element t of k. The category

$$\mathscr{C}_t = \underline{\mathrm{Rep}}(\mathrm{S}_t)$$

is constructed in three steps. One first defines a k-linear category $\mathscr{C}_t^0$: its objects are finite sets, and the morphism space $\mathrm{Hom}_{\mathscr{C}_t^0}(\mathrm{X}, \mathrm{Y})$ is the free k-module generated by partitions of the finite set $\mathrm{X} \sqcup \mathrm{Y}$. The composition maps are the k-bilinear maps given by

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}_t^0}(\mathrm{Y}, \mathrm{Z}) \times \mathrm{Hom}_{\mathscr{C}_t^0}(\mathrm{X}, \mathrm{Y}) &\longrightarrow \mathrm{Hom}_{\mathscr{C}_t^0}(\mathrm{X}, \mathrm{Z}) \\ (\beta, \alpha) &\longmapsto \beta \circ \alpha = t^{\gamma(\alpha,\beta)} \beta \odot \alpha. \end{aligned}$$

Associativity is not obvious, and relates to basic properties of the function γ.

If $f\colon \mathrm{X} \to \mathrm{Y}$ is a map of finite sets, then there is an associated morphism $\mathrm{Y} \to \mathrm{X}$ in $\mathscr{C}_t^0$ given by the smallest equivalence relation α_f on $\mathrm{Y} \sqcup \mathrm{X}$ that identifies $x \in \mathrm{X}$ with $f(x) \in \mathrm{Y}$ for all $x \in \mathrm{X}$. This construction gives rise to a contravariant functor from the category of finite sets to the category $\mathscr{C}_t^0$ (because it is elementary that $\gamma(\beta, \alpha) = 0$ whenever α and β are equivalence relations associated to maps, and hence $\alpha_{g \circ f} = \alpha_g \circ \alpha_f$ holds for composable maps f and g); this functor is faithful.

From $\mathscr{C}_t^0$, a category $\mathscr{C}_t'$ is constructed as the category of formal finite direct sums of objects of $\mathscr{C}_t^0$, with morphisms given by matrices in the obvious way. Finally, Knop's tensor envelope category $\mathscr{C}_t$ is defined by "adding images of projectors": an object is a pair (X, p) of an object X of $\mathscr{C}_t'$ and an endomorphism p of X such that $p \circ p = p$, and

$$\mathrm{Hom}_{\mathscr{C}_t}((\mathrm{X}, p), (\mathrm{Y}, q)) = q \circ \mathrm{Hom}_{\mathscr{C}_t'}(\mathrm{X}, \mathrm{Y}) \circ p \subset \mathrm{Hom}_{\mathscr{C}_t'}(\mathrm{X}, \mathrm{Y}).$$

The category $\mathscr{C}_t^0$ admits a monoidal structure in the sense of [12, Def. 2.1.1]. The tensor product bifunctor is defined on objects as $\mathrm{X} \otimes \mathrm{Y} = \mathrm{X} \sqcup \mathrm{Y}$ for finite sets X and Y. As for morphisms, the tensor product $\alpha \otimes \beta \in \mathrm{Hom}_{\mathscr{C}_t^0}(\mathrm{X} \otimes \mathrm{Y}, \mathrm{X}' \otimes \mathrm{Y}')$ of $\alpha \in \mathrm{Hom}_{\mathscr{C}_t^0}(\mathrm{X}, \mathrm{X}')$ and $\beta \in \mathrm{Hom}_{\mathscr{C}_t^0}(\mathrm{Y}, \mathrm{Y}')$ is the equivalence relation on

$$(\mathrm{X} \otimes \mathrm{Y}) \sqcup (\mathrm{X}' \otimes \mathrm{Y}') = (\mathrm{X} \sqcup \mathrm{Y}) \sqcup (\mathrm{X}' \sqcup \mathrm{Y}')$$

which "coincides" with α on $\mathrm{X} \sqcup \mathrm{X}'$ and with β on $\mathrm{Y} \sqcup \mathrm{Y}'$. The commutativity constraint $\mathrm{X} \otimes \mathrm{Y} \xrightarrow{\sim} \mathrm{Y} \otimes \mathrm{X}$ and the associativity constraint $(\mathrm{X} \otimes \mathrm{Y}) \otimes \mathrm{Z} \xrightarrow{\sim} \mathrm{X} \otimes (\mathrm{Y} \otimes \mathrm{Z})$ are given, respectively, by the morphisms associated to the obvious identifications $X \sqcup Y = Y \sqcup X$ and $(X \sqcup Y) \sqcup Z = X \sqcup (Y \sqcup Z)$. The unit object $\mathbf{1}$ is the empty set, along with the unique morphism $\mathbf{1} \otimes \mathbf{1} \to \mathbf{1}$.

It is then elementary that if p and q are projectors, then $p \otimes q$ is also one, so the rules

$$(X, p) \otimes (Y, q) = (X \otimes Y, p \otimes q)$$

and bilinearity define a symmetric monoidal structure on $\mathscr{C}_t$.

The monoidal category $\mathscr{C}_t^0$ is rigid ([12, Def. 2.10.1]). Indeed, the dual of a finite set X is defined to be $D(X) = X$ itself, and the evaluation and coevaluation morphisms

$$\mathrm{ev}_X \colon D(X) \otimes X \to \mathbf{1}, \qquad \mathrm{coev}_X \colon \mathbf{1} \to X \otimes D(X)$$

are both identified with the equivalence relation on $X \sqcup X$ associated to the identity map on X. For a morphism $\alpha \in \mathrm{Hom}_{\mathscr{C}_t^0}(X, Y)$, the transpose ${}^{t}\alpha \in \mathrm{Hom}_{\mathscr{C}_t^0}(Y, X)$ is defined as the composition

$$\begin{aligned} D(Y) = Y = Y \otimes \mathbf{1} &\xrightarrow{\mathrm{id} \otimes \mathrm{coev}_X} Y \otimes (X \otimes X) \simeq (Y \otimes X) \otimes X \\ &\xrightarrow{(\mathrm{id} \otimes \alpha) \otimes \mathrm{id}} (D(Y) \otimes Y) \otimes X \xrightarrow{\mathrm{ev}_Y \otimes \mathrm{id}} \mathbf{1} \otimes X = X = D(X), \end{aligned}$$

and corresponds to the obvious equivalence relation on $X \sqcup Y$ which is "the same" as α on $Y \sqcup X$.

The duality functor thus defined extends by linearity to $\mathscr{C}_t'$, and finally to $\mathscr{C}_t$: we have

$$D(X, p) = (D(X), \mathrm{Id}_{D(X)} - {}^{t}p)$$

for an object (X, p) of $\mathscr{C}_t$, and ${}^{t}(q \circ \alpha \circ p) = {}^{t}p \circ {}^{t}\alpha \circ {}^{t}q$ for $q \circ \alpha \circ p \in \mathrm{Hom}_{\mathscr{C}_t}((X, p), (Y, q))$.

Thus, $\mathscr{C}_t$ has the structure of a rigid symmetric monoidal k-linear category.

Suppose that the ring k is a field of characteristic 0 and t is not a non-negative integer. Then Knop proved that the category $\mathscr{C}_t$ is semisimple [19, Th. 6.1 along with Ex. 1, p. 596].

Let A be a fixed finite set and $G = \mathrm{Aut}(A)$ the corresponding symmetric group. An element $g \in G$ acts by precomposition with g^{-1} on the sets of maps from A to any finite set X. The contravariant functor

$$h_A(X) = \mathrm{Hom}_{\mathsf{Set}}(A, X)$$

from the category of finite sets to the category Set_G of finite sets with a G-action can be extended to a tensor functor $T_A \colon \mathscr{C}_t \to \mathsf{Rep}_k(G)$ of finite-dimensional k-linear representations of G so that the diagram

$$\begin{tikzcd} \mathsf{Set}^{opp} \arrow[r, "h_A"] \arrow[d] & \mathsf{Set}_G \arrow[d] \\ \mathscr{C}_t \arrow[r, "T_A"] & \mathsf{Rep}_k(G) \end{tikzcd}$$

commutes [19, proof of Th. 9.4, (9.23)], where the functor $\mathsf{Set}_G \to \mathsf{Rep}_k(G)$ associates to a finite set Y with a G-action the permutation representation of G on the free k-module with basis Y.

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