MIND THE (MULTIPLICATIVE) GAPS

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Abstract. We consider the gaps between successive elements of the $N \times N$ multiplication table for any positive integer N. We find an explicit list of gaps which occur, and obtain partial results (experimental and otherwise) which suggest that this list might be exhaustive.

1. The gaps in the multiplication table

Given a positive integer $N \geqslant 1$, we denote by $[N]$ the interval $\{1, \ldots, N\}$ of integers. The $N \times N$ *multiplication table* is simply the restriction of multiplication to the finite set $[N] \times [N]$. This is a very simple but fascinating object for analytic number theory, as evidenced by the very subtle behavior of the size of the image of this map (i.e., the number of integers $n \leqslant N^2$ which have a factorization $n = ab$ with $1 \leqslant a, b \leqslant N$).

We are interested here in the values and the distribution of the *gaps* of the multiplication table. Precisely, we want to keep track of multiplicity, both in the original sequence and in the gaps. For this, we let g_N denote the map

$$[N^2] \to [N] \cdot [N]$$

which enumerates in increasing order the multiplication table, taking into account multiplicity (in other words, g_N sorts in increasing order the family $(ab)_{1 \leqslant a,b \leqslant N}$ of all products of elements of $[N]$). We then consider the measure

$$\mu_N = \sum_{1 \leqslant m < N^2} \delta_{g_N(n+1) - g_N(n)},$$

where δ_x is a Dirac mass at x. This is called the *gap measure*, with $\mu_N(m)$ the *multiplicity* of m as a gap. If $\mu_N(m) = 1$, then we say that the gap m is *isolated*.

Below, as a matter of terminology, we will sometimes call "gap" any difference $ab - cd$ of elements of the multiplication table, and speak of "primitive gap" for those differences which really occur between consecutive elements (i.e., those for which the inequalities

$$ab \leqslant ef \leqslant cd$$

with $1 \leqslant e, f \leqslant N$ imply that $ef = ab$ or $ef = cd$).

The following result finds an explicit subset of the set $\mathscr{G}_N$ of primitive gaps in the $N \times N$ multiplication table.

Theorem 1.1. *Let $N \geqslant 2$ be an integer. Let R be the largest integer j such that $\lfloor j^2/4 \rfloor < N$.*

(1) *The set* $\mathscr{G}_{\mathrm{N}}$ *of primitive gaps in the* $\mathrm{N}\times\mathrm{N}$ *multiplication table contains the union of the sets* $\mathscr{S}_{\mathrm{N}}$, $\mathscr{B}_{\mathrm{N}}$ *and* $\mathscr{E}_{\mathrm{N}}$, *where*

$$\mathscr{S}_{\mathrm{N}} = \{0,\ldots,\mathrm{R}-2\},$$
$$\mathscr{B}_{\mathrm{N}} = \{\mathrm{N} - \lfloor j/2\rfloor \lceil j/2\rceil \mid 1 \leqslant j \leqslant \mathrm{R}-2\},$$

and $\mathscr{E}_{\mathrm{N}}$ *is empty unless* N *is of the form* k^2-1, k^2, k^2+1, k^2+k-1, k^2+k *or* k^2+k+1 *for some integer* $k\geqslant 1$, *in which cases we have*

$$\mathscr{E}_{k^2-1} = \{\mathrm{R}-1\} = \{2k-1\}, \qquad \mathscr{E}_{k^2} = \{\mathrm{R}-1,\mathrm{R}\} = \{2k-2, 2k-1\}$$
$$\mathscr{E}_{k^2+1} = \{\mathrm{R}\} = \{2k\}, \qquad \mathscr{E}_{k^2+k-1} = \{\mathrm{R}-1\} = \{2k-1\}$$
$$\mathscr{E}_{k^2+k} = \{\mathrm{R}-1,\mathrm{R}\} = \{2k-1,2k\}, \qquad \mathscr{E}_{k^2+k+1} = \{\mathrm{R}\} = \{2k+1\},$$

with the convention[1] *that* $\mathscr{E}_3 = \{2,3\}$ *and* $\mathscr{E}_5 = \{3,4\}$.

(2) *The sets* $\mathscr{S}_{\mathrm{N}}$, $\mathscr{B}_{\mathrm{N}}$ *and* $\mathscr{E}_{\mathrm{N}}$ *are disjoint, except when* N *is of the form* k^2+1 *or* k^2+k+1, *in which case the only non-trivial intersection is*

$$\mathscr{B}_{\mathrm{N}} \cap \mathscr{E}_{\mathrm{N}} = \{\mathrm{R}\}.$$

Remark 1.2. (1) More explicitly, the set $\mathscr{B}_{\mathrm{N}}$ of "large" gaps is the union of the set of values $\mathrm{N}-j^2/4$ for $2\leqslant j\leqslant \mathrm{R}-2$ even and of the values $\mathrm{N}-(j^2-1)/4$ for $1\leqslant j\leqslant \mathrm{R}-2$ odd.

(2) The integer R in Theorem 1.1 coincides with $\lfloor 2\sqrt{\mathrm{N}}\rfloor$ for all integers which are not squares; for $\mathrm{N}=k^2$, we have $\mathrm{R}=2k-1$ and $\lfloor 2\sqrt{\mathrm{N}}\rfloor = 2k$.

To see this, let $r=\lfloor 2\sqrt{\mathrm{N}}\rfloor$; note the lower bounds

$$\frac{r^2}{4} \geqslant \frac{(2\sqrt{\mathrm{N}}-1)^2}{4} = \mathrm{N}-\sqrt{\mathrm{N}}+\frac{1}{4}, \qquad \frac{r}{2} \geqslant \frac{2\sqrt{\mathrm{N}}-1}{2} = \sqrt{\mathrm{N}}-\frac{1}{2},$$

from which we get

$$\frac{(r+1)^2}{4} = \frac{r^2}{4}+\frac{r}{2}+\frac{1}{4} \geqslant \mathrm{N}-\sqrt{\mathrm{N}}+\frac{1}{2}+\sqrt{\mathrm{N}}-\frac{1}{2} = \mathrm{N},$$

and therefore $\lfloor (r+1)^2/4\rfloor \geqslant \mathrm{N}$, meaning that $\mathrm{R}\leqslant r$. On the other hand, from $\lfloor r^2/4\rfloor \leqslant r^2/4 \leqslant \mathrm{N}$ we deduce that $r\leqslant \mathrm{R}$ (and so $\mathrm{R}=r$) *unless* there is equality $r^2/4=\mathrm{N}$, and N is a square, say $\mathrm{N}=k^2$, in which case $\mathrm{R}=2k-1$ is elementary.

(3) For any integer $j\geqslant 0$, we have $\lfloor j/2\rfloor\lceil j/2\rceil = \lfloor j^2/4\rfloor$ (check for j even and odd separately). See https://oeis.org/A002620 for properties (and other occurences) of this sequence of integers.

Numerical experiments and some theoretical results described below raise the following questions and problems:

Problem 1.3. *Prove or disprove that for all* $\mathrm{N}\geqslant 2$, *the following assertions hold:*

(1) *The set* $\mathscr{G}_{\mathrm{N}}$ *is exactly the union of* $\mathscr{S}_{\mathrm{N}}$, $\mathscr{B}_{\mathrm{N}}$ *and* $\mathscr{E}_{\mathrm{N}}$.

(2) *For* $1\leqslant j\leqslant \mathrm{R}-2$, *the gap* $\mathrm{N}-\lfloor j/2\rfloor\lceil j/2\rceil$ *is an isolated gap, occuring as*

$$\mathrm{N}-\lfloor j/2\rfloor\lceil j/2\rceil = \mathrm{N}(\mathrm{N}-j+1)-(\mathrm{N}-\lfloor j/2\rfloor)(\mathrm{N}-\lceil j/2\rceil).$$

(3) *All the gaps in* $\mathscr{E}_{\mathrm{N}}$ *are isolated.*

[1] Note that $3=1^2+1+1=2^2-1$ and $5=2^2+1=2^2+2-1$.

(4) *All the gaps in $\mathscr{S}_{\mathrm{N}}$ have multiplicity at least* 2.

We will illustrate Theorem 1.1 in the next section. As partial evidence for a positive answer to the questions thus raised, we have the following result:

Proposition 1.4. *There exists a constant $\alpha > 0$ such that for all $n \leqslant \alpha \mathrm{N}^{3/2}/(\log \mathrm{N})^3$ belonging to the $\mathrm{N} \times \mathrm{N}$ multiplication table, the next element n^+ in the table satisfies*

$$n^+ - n \leqslant \mathrm{N}^{1/2}.$$

Remark 1.5. (1) Theorem 1.1 is already a new illustration of the fact that the multiplication table is very far from a "generic" set of integers. Indeed, for a random set $\mathrm{A} \subset [\mathrm{N}^2]$ with relatively large density $\delta = |\mathrm{A}|/\mathrm{N}^2$, one would expect that every interval of length somewhat larger than this should contain some element of A (this is the same basic reasoning behind the predictions of the "Cramér model" for the size of gaps between primes). In particular, it should also be the case that there are at most about $1/\delta$ distinct primitive gaps between elements of A.[2]

In the case of the multiplication table, we have

$$\delta^{-1} \asymp (\log \mathrm{N})^{\alpha} (\log \log \mathrm{N})^{3/2}$$

with $\alpha = 1 - \frac{1+\log(\log(2))}{\log(2)}$ (as proved by Ford [1]), so having (at least) about $2\sqrt{\mathrm{N}}$ distinct gaps is very striking. But the statement of the result already hints that this phenomenon is linked to a certain algebraic rigidity of the larger elements in the multiplication table.

Thus Theorem 1.1 shows that there are much larger gaps than expected on probabilistic grounds. Nevertheless, one would certainly expect that the bulk of the multiplication table is more random, and this is somewhat confirmed by the proof of Theorem 1.1, where the gaps are located among very large values in $[\mathrm{N}] \cdot [\mathrm{N}]$. This certainly suggests that at least some "asymptotic" version of Problem 1.3 should have a positive answer.

However, I find most intriguing the possibility that the answer is *always* positive, since it is (to me) very unclear how one could so cleanly separate the "random" aspect from the "deterministic" one.

(2) We also note that although the theorem and the problem above suggest that the isolated gaps follow rigid "algebraic" rules, even as far as multiplicity is concerned, the multiplicities of the other gaps are certainly much more arithmetic in nature. Indeed, let

$$d_{\mathrm{N}}(n) = \sum_{\substack{ab=n \\ a,b \leqslant \mathrm{N}}} 1$$

denote the "representation function" for the multiplication table. Then

$$\mu_{\mathrm{N}}(0) = \sum_{n \in \mathscr{M}_{\mathrm{N}}} (d_{\mathrm{N}}(n) - 1) = \mathrm{N}^2 - \mathrm{M}_{\mathrm{N}},$$

which involves the size of the multiplication table. (The inclusion of $m = 0$ as a possible gap is however a matter of convention, and one could decide to dispense with it; only the value of $\mu_{\mathrm{N}}(0)$ would be affected.)

[2] As an aside, Soundararajan suggested that it would be interesting to get a good result on the *number* of distinct primitive gaps between primes $p \leqslant x$.

Similarly, since any gap equal to 1 is primitive, we have

$$\mu_{\mathrm{N}}(1)=\sum_{\substack{1\leqslant a,b,c,d\leqslant \mathrm{N}\\ ab-cd=1}}\frac{1}{d_{\mathrm{N}}(ab)d_{\mathrm{N}}(cd)},$$

and since one knows (see the result [2, Th. 1.1, (3.1)] of Ganguly and Guria) that

$$\sum_{\substack{1\leqslant a,b,c,d\leqslant \mathrm{N}\\ ab-cd=1}}1\sim\frac{2}{\zeta(2)}\mathrm{N}^2$$

as $\mathrm{N}\to+\infty$, this at least shows that the estimates

$$\mathrm{N}^{2-\varepsilon}\ll\mu_{\mathrm{N}}(1)\ll\mathrm{N}^2$$

hold for any $\varepsilon>0$. It would be interesting to determine the asymptotic behavior of $\mu_{\mathrm{N}}(k)$ for k fixed, or $k\ll\mathrm{N}^{1/2-\delta}$ for some $\delta>0$.

The following somewhat surprising corollary was the first experimental observation from which the work on this paper started.

Corollary 1.6. *Let* $\mathrm{N}\geqslant 7$ *be an integer such that Problem 1.3 has a positive answer for the* $[\mathrm{N}]\times[\mathrm{N}]$ *multiplication table. Then the largest integer* $g\geqslant 1$ *such that* $\{0,\ldots,g\}$ *is contained in* $\mathscr{G}_{\mathrm{N}}$ *is equal to the number of isolated gaps in the* $\mathrm{N}\times\mathrm{N}$ *multiplication table.*

Proof. Let R be as in Theorem 1.1. The point is that according to the theorem and the assumption, the interval $\mathscr{S}_{\mathrm{N}}=\{0,\ldots,\mathrm{R}-2\}\subset\mathscr{G}_{\mathrm{N}}$ has length $|\mathscr{B}_{\mathrm{N}}|+1$, and can only be prolonged to a longer interval in $\mathscr{G}_{\mathrm{N}}$ by the elements of $\mathscr{E}_{\mathrm{N}}$; since these, when they exist, are isolated gaps as all the elements of $\mathscr{B}_{\mathrm{N}}$ are, the result follows. ☐

Notation. We use the Vinogradov notation $f\ll g$ (for complex-valued functions f and g defined on some set X): it means that there exists a real number $c\geqslant 0$ (the "implied constant") such that $|f(x)|\leqslant cg(x)$ for all $x\in\mathrm{X}$.

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2. Examples

In the examples, we list the multiplicity distribution of the gaps as the tuple

$$(\mu_{\mathrm{N}}(0),\mu_{\mathrm{N}}(1),\ldots,\mu_{\mathrm{N}}(\mathrm{N})).$$

(1) For $\mathrm{N}=1$, the measure μ_1 is zero. On the other hand, for $2 \leqslant \mathrm{N} \leqslant 6$, every integer between 0 and N occurs as a primitive gap: the multiplicity distributions are given by

$$(1,1,1) \text{ for } \mathrm{N}=2, \qquad (3,3,1,1) \text{ for } \mathrm{N}=3,$$
$$(7,4,2,1,1) \text{ for } \mathrm{N}=4, \qquad (11,8,2,1,1,1) \text{ for } \mathrm{N}=5$$
$$(18,9,4,1,1,1,1) \text{ for } \mathrm{N}=6,$$

and one can check that the assertions of Theorem 1.1 hold (with the stated convention for $\mathrm{N}=3$ or 5, which can be represented in two ways as one of the exceptional forms defining $\mathscr{E}_{\mathrm{N}}$).

These are the only cases where $\mu_{\mathrm{N}}(j) \geqslant 1$ for all integers j with $0 \leqslant j \leqslant \mathrm{N}$.

(2) For $\mathrm{N}=33$, we get the following values of $\mu_{33}(m)$:

$$(717,158,89,43,30,10,12,9,8,3,0,0,0,1,0,0,0,1,0,0,0,1,0,0,1,0,0,1,0,1,0,1,1,1).$$

Here $\mathrm{R}=11$, the set $\mathscr{E}_{33}$ is empty and

$$\mathscr{S}_{33}=\{0,1,\ldots,9\}, \qquad \mathscr{B}_{33}=\{13,17,21,24,27,29,31,32,33\}.$$

(3) For $\mathrm{N}=73=k^2+k+1$ with $k=8$, we get

$$(3678,638,377,227,136,71,63,25,22,23,16,6,10,9,10,2,0,1,0,0,0,0,0,$$
$$0,1,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0,0,1,0,0,$$
$$0,0,1,0,0,0,1,0,0,0,1,0,0,1,0,0,1,0,1,0,1,1,1).$$

We have $\mathrm{R}=17=2k+1$, and

$$\mathscr{B}_{73}=\{17,24,41,37,43,48,53,57,61,63,67,69,71,72,73\}, \qquad \mathscr{E}_{73}=\{17\}.$$

so that 17 is in the intersection of $\mathscr{B}_{73}$ and $\mathscr{E}_{73}$.

(4) For $\mathrm{N}=100$, the values of $\mu_{100}(m)$ are

$$(7094,1039,656,380,265,142,124,64,60,36,40,22,16,10,11,10,6,5,1,1,0,0,0,0,0,$$
$$0,0,0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,0,1,0,0,0,0,0,0,1,0,0,0,0,0,1,$$
$$0,0,0,0,0,1,0,0,0,0,1,0,0,0,0,1,0,0,0,1,0,0,0,1,0,0,1,0,0,1,0,1,0,1,1,1).$$

Here $\mathrm{R}=19$, with $\mu_{100}(\mathrm{R}-2)=\mu_{100}(17)=5$. We see that $\mathscr{E}_{100}=\{18,19\}$, which are both isolated gaps; the set $\mathscr{B}_{100}$ is

$$\{28,36,44,51,58,64,70,75,80,84,88,91,94,96,98,99,100\}.$$

3. Asymptotic multiplication table

The basic idea behind the proof of Theorem 1.1 is simply that "large" elements of the multiplication table are ordered "like polynomials", in the sense that if a and b are suitably small compared to N, then the values $(\mathrm{N}-a)(\mathrm{N}-b)$ are ordered like the values of the polynomials $(\mathrm{X}-a)(\mathrm{X}-b)$ as X tends to infinity. This ordering of polynomials is very explicit and the corresponding gaps can be determined easily. Although we could dispense with this step, we first look at polynomials, since it motivates the shape of set $\mathscr{B}_{\mathrm{N}}$ in Theorem 1.1. We first note that the notion of gaps, and of primitive gaps, makes sense for subsets of any ordered abelian group.

Definition 3.1. The *asymptotic order* on $\mathbf{R}[\mathrm{X}]$ is the total ordering $\leqslant_{\mathrm{as}}$ determined by $f \leqslant_{\mathrm{as}} g$ if and only if $f(x) \leqslant g(x)$ for all x large enough.

The *asymptotic order* on $\mathbf{R}^d$ is the total ordering $\leqslant_{\mathrm{as}}$ determined by

$$(a_0, \ldots, a_{d-1}) \leqslant (b_0, \ldots, b_{d-1})$$

if and only if

$$\prod_{i=0}^{d-1} (\mathrm{X} - a_i) \leqslant_{\mathrm{as}} \prod_{i=0}^{d-1} (\mathrm{X} - b_i).$$

The asymptotic order is indeed a total order since the difference of two polynomials is a polynomial, hence is either zero, or has the same sign at its leading coefficient for all x large enough. With this ordering, $\mathbf{R}[\mathrm{X}]$ is an ordered abelian group (in fact, it is an ordered ring).

The order induced on $\mathbf{Z}^d$ by the asymptotic order on $\mathbf{R}^d$ is a well-ordering.

Lemma 3.2. *We have $(a, b) \leqslant_{\mathrm{as}} (c, d)$ if and only if either $a + b > c + d$, or $a + b = c + d$ and $ab \leqslant cd$.*

Proof. This is a direct computation from the definition. □

We now determine the primitive gaps between polynomials $(\mathrm{X} - a)(\mathrm{X} - b)$ with a, b non-negative integers, or equivalently between pairs (a, b) for the asymptotic order.

We first note that if $j \geqslant 0$ is an integer, then $\lfloor j/2 \rfloor + \lceil j/2 \rceil = j$ and $\lfloor j/2 \rfloor \lceil j/2 \rceil = \lfloor j^2/4 \rfloor$.

Proposition 3.3. *The asymptotic well-ordering of the set of pairs of non-negative integers (a, b) with $a \leqslant b$ is determined by the following rules.*

(1) *For any integers j, k with $0 \leqslant k < j$, we have*

$$(a, j - a) <_{\mathrm{as}} (b, k - b)$$

for $0 \leqslant a \leqslant j/2$ and $0 \leqslant b \leqslant k/2$.

(2) *For any integer $j \geqslant 0$, we have*

$$(0, j) <_{\mathrm{as}} (1, j - 1) <_{\mathrm{as}} \cdots <_{\mathrm{as}} (\lfloor j/2 \rfloor, \lceil j/2 \rceil).$$

Proof. This is a straightforward consequence of the previous lemma. □

Example 3.4. In decreasing order, we obtain

$$(0,0) >_{\mathrm{as}} (0,1) >_{\mathrm{as}} (1,1) >_{\mathrm{as}} (0,2) >_{\mathrm{as}} (1,2) >_{\mathrm{as}} (0,3) >_{\mathrm{as}} (2,2) >_{\mathrm{as}} (1,3),$$

corresponding to the polynomial comparisons

$$\begin{aligned} \mathrm{X}^2 >_{\mathrm{as}} \mathrm{X}(\mathrm{X}-1) >_{\mathrm{as}} (\mathrm{X}-1)^2 >_{\mathrm{as}} \mathrm{X}(\mathrm{X}-2) >_{\mathrm{as}} (\mathrm{X}-1)(\mathrm{X}-2) \\ >_{\mathrm{as}} \mathrm{X}(\mathrm{X}-3) >_{\mathrm{as}} (\mathrm{X}-2)^2 >_{\mathrm{as}} (\mathrm{X}-1)(\mathrm{X}-3), \end{aligned}$$

valid for large values of the variable. The successive gaps between these polynomials are

$$\mathrm{X}, \quad \mathrm{X}-1, \quad 1, \quad \mathrm{X}-2, \quad 2, \quad \mathrm{X}-4, \quad 1.$$

Corollary 3.5. *The primitive gaps for the asymptotic order between the polynomials of the form* $(\mathrm{X}-a)(\mathrm{X}-b)$ *with* (a,b) *non-negative integers such that* $a \leqslant b$ *are either positive integers or polynomials*

$$\mathrm{X} - \lfloor j/2 \rfloor \lceil j/2 \rceil = \mathrm{X} - \lfloor j^2/4 \rfloor$$

for some $j \geqslant 1$*.*

More precisely, each of the above polynomial is a unique gap, and any positive integer $k \geqslant 1$ *occurs exactly between pairs* $(a, j-a)$ *and* $(a+1, j-a-1)$ *with* $j - 2a - 1 = k$*.*

Proof. By the proposition, the gaps which occur (and which are primitive in view of the well-ordering) between these polynomials are either "large" gaps between successive elements of the form

$$(\lfloor j/2 \rfloor, \lceil j/2 \rceil) <_{\mathrm{as}} (j-1, 0)$$

for $j \geqslant 1$, or "small" gaps between successive elements of the form

$$(a, j-a) <_{\mathrm{as}} (a+1, j-a-1)$$

with $j \geqslant 2$ and $0 \leqslant a \leqslant \lfloor j/2 \rfloor$. These two gaps are, respectively, equal to

$$\mathrm{X}(\mathrm{X}-j+1) - (\mathrm{X} - \lfloor j/2 \rfloor)(\mathrm{X} - \lceil j/2 \rceil) = \mathrm{X} - \lfloor j/2 \rfloor \lceil j/2 \rceil$$

and

$$\begin{aligned}(\mathrm{X}-(a+1))(\mathrm{X}-(j-a-1)) &- (\mathrm{X}-a)(\mathrm{X}-(j-a)) \\ &= (a+1)(j-a-1) - a(j-a) = j - 2a - 1,\end{aligned}$$

which implies the result. ☐

We now go back to the multiplication table of integers, and record a simple fact which explains the relevance of the asymptotic order for the multiplication table.

Lemma 3.6. *Let* N *be an integer. For non-negative integers* (a,b,c,d) *such that* $|ab - cd| < \mathrm{N}$*, we have*

$$(\mathrm{N}-a)(\mathrm{N}-b) > (\mathrm{N}-c)(\mathrm{N}-d)$$

if and only if

$$(a,b) >_{\mathrm{as}} (c,d).$$

Proof. The first condition is equivalent to

$$((c+d) - (a+b))\mathrm{N} > cd - ab,$$

and the second (by the previous lemma) is equivalent to

$$a + b < c + d \quad \text{or} \quad a + b = c + d \text{ and } ab > cd.$$

If $a+b = c+d$, then both conditions are equivalent to $ab > cd$, for all values of (a,b,c,d).

If $a + b < c + d$, then the second condition holds, and using the bounds $|ab - cd| < \mathrm{N}$ and $((c+d) - (a+b)) \geqslant \mathrm{N}$, we see that the first is also true.

If $a + b > c + d$, then the second condition fails, and using the bounds $|ab - cd| < \mathrm{N}$ and $((c+d) - (a+b)) \leqslant -\mathrm{N}$, we see that the first is also false. ☐

The set $\mathscr{D}_{\mathrm{N}}$ of tuples of non-negative integers (a, b, c, d) with $|ab - cd| < \mathrm{N}$ contains the set $[\mathrm{M}]^4$, provided $\mathrm{M}^2 < \mathrm{N}$, since $0 \leqslant ab \leqslant \mathrm{M}^2 < \mathrm{N}$ and $0 \leqslant cd \leqslant \mathrm{M}^2 < \mathrm{N}$ in that case. Thus this lemma suggest that the gaps in the $\mathrm{N} \times \mathrm{N}$ multiplication table will be "the same" as those between polynomials if we restrict (a, b, c, d) to $\mathscr{D}_{\mathrm{N}}$. However, one must be careful that this is only the case if there is no intermediate product

$$(\mathrm{N} - a)(\mathrm{N} - b) < (\mathrm{N} - e)(\mathrm{N} - f) < (\mathrm{N} - c)(\mathrm{N} - d)$$

with either (a, b, e, f) or (e, f, c, d) outside $\mathscr{D}_{\mathrm{N}}$. In addition, the range of values thus allowed does not recover the full range in Theorem 1.1, and its proof will be more involved.

4. Disjointness

We first quickly handle the second assertion of Theorem 1.1 concerning the intersections of the sets $\mathscr{S}_{\mathrm{N}}$, $\mathscr{B}_{\mathrm{N}}$ and $\mathscr{E}_{\mathrm{N}}$. It is straightforward that $\mathscr{S}_{\mathrm{N}} \cap \mathscr{E}_{\mathrm{N}}$ and $\mathscr{B}_{\mathrm{N}} \cap \mathscr{S}_{\mathrm{N}}$ are empty. To determine the intersection of $\mathscr{E}_{\mathrm{N}}$ and $\mathscr{B}_{\mathrm{N}}$, we note that the smallest element of $\mathscr{B}_{\mathrm{N}}$ is

$$m = \mathrm{N} - \Big\lfloor \frac{(\mathrm{R} - 2)^2}{4} \Big\rfloor,$$

provided $\mathrm{R} \geqslant 3$ (otherwise $\mathscr{B}_{\mathrm{R}}$ is empty), which we assume. We compute

$$\Big\lfloor \frac{(\mathrm{R} - 2)^2}{4} \Big\rfloor \leqslant \frac{(\mathrm{R} - 2)^2}{4} = \frac{\mathrm{R}^2}{4} - \mathrm{R} + 1.$$

We have $\mathrm{R}^2/4 \leqslant \mathrm{N} - 1$: indeed, the definition of R implies that $\lfloor \mathrm{R}^2/4 \rfloor < \mathrm{N}$, so that $\mathrm{R}^2/4 \leqslant \mathrm{N}$, but equality is not possible as it would imply that $\lfloor \mathrm{R}^2/4 \rfloor = \mathrm{N}$. Hence we obtain the lower bound

$$m \geqslant \mathrm{N} - (\mathrm{N} - 1 - \mathrm{R} + 1) = \mathrm{R}.$$

Since $\mathscr{E}_{\mathrm{N}} \subset \{\mathrm{R} - 1, \mathrm{R}\}$ and $\mathrm{R} \in \mathscr{E}_{\mathrm{N}}$ if and only if N is of the form k^2, $k^2 + 1$, $k^2 + k$ or $k^2 + k + 1$ for some $k \geqslant 1$, the intersection of $\mathscr{B}_{\mathrm{N}}$ and $\mathscr{E}_{\mathrm{N}}$ is either empty or equal to $\{\mathrm{R}\}$, and the latter occurs if and only if $m = \mathrm{R}$. We compute the value of m for the different possible forms of N:

$$\mathrm{N} = k^2, \quad \mathrm{R} = 2k - 1, \quad \frac{(\mathrm{R} - 2)^2}{4} = k^2 - 3k + 2 - \frac{1}{4}, \quad m = 3k - 2$$

$$\mathrm{N} = k^2 + 1, \quad \mathrm{R} = 2k, \quad \frac{(\mathrm{R} - 2)^2}{4} = k^2 - 2k + 1, \quad m = 2k$$

$$\mathrm{N} = k^2 + k, \quad \mathrm{R} = 2k, \quad \frac{(\mathrm{R} - 2)^2}{4} = k^2 - 2k + 1, \quad m = 3k - 1$$

$$\mathrm{N} = k^2 + k + 1, \quad \mathrm{R} = 2k + 1, \quad \frac{(\mathrm{R} - 2)^2}{4} = k^2 - k + \frac{1}{4}, \quad m = 2k + 1.$$

In the second and fourth case, we have $m = \mathrm{R}$, so

$$\mathrm{R} \in \mathscr{B}_{k^2+1} \cap \mathscr{E}_{k^2+1}, \qquad \mathrm{R} \in \mathscr{B}_{k^2+k+1} \cap \mathscr{E}_{k^2+k+1}.$$

On the other hand, the only possibility in the first (resp. third) case to have $m = \mathrm{R}$ is if $3k - 2 = 2k - 1$ (resp. $3k - 1 = 2k + 1$). This would mean that $k = 1$, which is impossible in both cases as it would give $k = 1$ and $\mathrm{R} \leqslant 2$.

5. Finding the gaps

In this section, we will prove the first assertion of Theorem 1.1. We fix an integer $\mathrm{N} \geqslant 2$ and we denote by R the integer in Theorem 1.1, i.e., the largest integer $j \geqslant 1$ such that $\lfloor j^2/4 \rfloor < \mathrm{N}$. By "multiplication table", we will always mean the $\mathrm{N} \times \mathrm{N}$ table.

Motivated by the Section 3, we define the integers

$$a_j = (\mathrm{N} - \lfloor j/2 \rfloor)(\mathrm{N} - \lceil j/2 \rceil) = \mathrm{N}^2 - j\mathrm{N} + \lfloor j^2/4 \rfloor,$$
$$b_j = \mathrm{N}(\mathrm{N} - j) = \mathrm{N}^2 - j\mathrm{N}$$

for all integers $j \geqslant 0$. As long as $0 \leqslant j \leqslant \mathrm{N} - 1$, the integers a_j and b_j belong to the $\mathrm{N} \times \mathrm{N}$ multiplication table.

The inequalities

$$b_{\mathrm{R}} < a_{\mathrm{R}} < b_{\mathrm{R}-1} < \cdots < b_2 < a_2 < b_1 = a_1 < b_0 = \mathrm{N}^2 \tag{5.1}$$

are then valid, with (not necessarily primitive) gaps

$$a_j - b_j = \lfloor j^2/4 \rfloor, \qquad b_{j-1} - a_j = \mathrm{N} - \lfloor j^2/4 \rfloor$$

for $1 \leqslant j \leqslant \mathrm{R}$.

We will now investigate which elements of the multiplication table might lie in the intervals defined by the inequalities (5.1). For orientation, we note that b_{R} is close to $\mathrm{N}^2 - 2\mathrm{N}^{3/2}$, so all results here concern the very large elements of the multiplication table.

We will frequently use the following simple lemma.

Lemma 5.1. *Let k be a non-negative integer. As (a, b) vary over pairs of non-negative integers with $a \leqslant b$ and $a + b = k$, we have*

$$ab \leqslant \left\lfloor \frac{k^2}{4} \right\rfloor, \tag{5.2}$$

with equality if and only if $a = \lfloor k/2 \rfloor$ and $b = \lceil k/2 \rceil$ and moreover either $a = 0$ or

$$ab \geqslant k - 1, \tag{5.3}$$

with equality if and only if $a = 1$.

Proof. This is elementary calculus. □

We begin with the interval between b_j and a_j. Given arbitrary integers (a, b) and $j \geqslant 0$ with $\lfloor j^2/4 \rfloor \leqslant \mathrm{N}$, we have

$$b_j \leqslant (\mathrm{N} - a)(\mathrm{N} - b) \leqslant a_j \tag{5.4}$$

if and only if

$$0 \leqslant (j - (a + b))\mathrm{N} + ab \leqslant \lfloor j^2/4 \rfloor. \tag{5.5}$$

Lemma 5.2. *Assume that $\mathrm{N} \geqslant 8$, that $0 \leqslant a \leqslant b \leqslant \mathrm{N}$ and that $\lfloor j^2/4 \rfloor < \mathrm{N}$.*

(1) *If $a + b = j$ then (5.4) holds.*

(2) *If* (5.4) *holds and* $a+b\neq j$*, then* $a+b=j+1$ *and the inequalities*

$$\mathrm{N}\leqslant ab\leqslant \mathrm{N}+\frac{3}{2}\sqrt{\mathrm{N}},$$

$$b_j=\mathrm{N}(\mathrm{N}-j)\leqslant (\mathrm{N}-a)(\mathrm{N}-b)\leqslant b_j+\frac{3}{2}\sqrt{\mathrm{N}}$$

hold.

Proof. The assumption on j implies that $j^2/4\leqslant \mathrm{N}+1$, so $j\leqslant 2\sqrt{\mathrm{N}+1}$.

If $a+b=j$, then it is elementary that (5.5) holds, hence also (5.4). We assume therefore that this is not the case.

Suppose first that $ab<\mathrm{N}$. Then the inequality $b_j\leqslant (\mathrm{N}-a)(\mathrm{N}-b)$ holds if and only if $a+b=j$ or $j>a+b$. In the latter case, we have

$$(j-(a+b))\mathrm{N}+ab\geqslant \mathrm{N}>\lfloor j^2/4\rfloor,$$

so our assumption shows that (5.5) does not hold.

Assume now that $ab\geqslant \mathrm{N}$. We must then have $j<a+b$ since otherwise

$$(j-(a+b))\mathrm{N}+ab\geqslant ab\geqslant \mathrm{N}>\lfloor j^2/4\rfloor,$$

so that (5.5) does not hold. We write $a+b=j+k$ with $k\geqslant 1$.

The left-hand inequality in (5.5) is equivalent to

$$k\mathrm{N}\leqslant ab,$$

and since $ab\leqslant \mathrm{N}^2$, it follows that $k\leqslant \mathrm{N}$. Moreover, from $a\geqslant k\mathrm{N}/b$, we get

$$a+b\geqslant \frac{k\mathrm{N}}{b}+b\geqslant 2\sqrt{k\mathrm{N}},$$

and therefore

$$k=a+b-j\geqslant 2\sqrt{k\mathrm{N}}-2\sqrt{\mathrm{N}+1}.$$

since $j\leqslant 2\sqrt{\mathrm{N}+1}$.

If $k\geqslant 5$ and $\mathrm{N}>4$, this leads to the contradiction $k>\mathrm{N}$ (since then $2\sqrt{k\mathrm{N}}-2\sqrt{\mathrm{N}+1}>\sqrt{k\mathrm{N}}$). If $2\leqslant k\leqslant 4$, the inequality $2\sqrt{k\mathrm{N}}-2\sqrt{\mathrm{N}+1}\leqslant k$ only happens if N is bounded, indeed if $\mathrm{N}\leqslant 8$ by inspection (achieved for $k=2$).

There only remains the possibility that $k=1$. We then have

$$2\sqrt{\mathrm{N}}\leqslant a+b=j+1\leqslant 2\sqrt{\mathrm{N}+1}+1\leqslant 2\sqrt{\mathrm{N}}+2$$

(for $\mathrm{N}\geqslant 1$). We write

$$a=\sqrt{\mathrm{N}}-h,\qquad b=\sqrt{\mathrm{N}}+h',$$

where $h\in\mathbf{R}$ and $h'\geqslant 0$. The previous result implies

$$2\sqrt{\mathrm{N}}\leqslant 2\sqrt{\mathrm{N}}+h'-h\leqslant 2\sqrt{\mathrm{N}}+2$$

hence $0\leqslant h'-h\leqslant 2$. But also

$$\mathrm{N}>\lfloor j^2/4\rfloor\geqslant \frac{(a+b-1)^2}{4}-1=\mathrm{N}+(h'-h-1)\sqrt{\mathrm{N}}+\frac{(h'-h-1)^2}{4}-1,$$

which implies that $h'-h-1\leqslant 3/2$.[3]

[3] For $\mathrm{N}\geqslant 8$, this can be improved.

We now get

$$\mathrm{N} \leqslant ab = \mathrm{N} + (h' - h)\sqrt{\mathrm{N}} - hh' \leqslant \mathrm{N} + \frac{3}{2}\sqrt{\mathrm{N}},$$

and

$$(\mathrm{N} - a)(\mathrm{N} - b) = \mathrm{N}^2 - (a + b)\mathrm{N} + ab = \mathrm{N}(\mathrm{N} - j) + (ab - \mathrm{N}),$$

hence

$$\mathrm{N}(\mathrm{N} - j) \leqslant (\mathrm{N} - a)(\mathrm{N} - b) \leqslant \mathrm{N}(\mathrm{N} - j) + \frac{3}{2}\sqrt{\mathrm{N}},$$

which gives the desired statement. □

We next look at the interval between a_j and b_{j-1}. Given again arbitrary (a, b) and $j \geqslant 0$ with $\lfloor j^2/4 \rfloor \leqslant \mathrm{N}$, we have

$$a_j \leqslant (\mathrm{N} - a)(\mathrm{N} - b) \leqslant b_{j-1} \tag{5.6}$$

if and only if

$$\lfloor j^2/4 \rfloor \leqslant (j - (a + b))\mathrm{N} + ab \leqslant \mathrm{N}. \tag{5.7}$$

Lemma 5.3. *Assume that $0 \leqslant a \leqslant b \leqslant \mathrm{N}$ and that $\lfloor j^2/4 \rfloor < \mathrm{N}$ and $\lceil j/2 \rceil < \mathrm{N}$ so that $a_j \geqslant 1$. Then (5.6) holds if and only if*

$$(a, b) = (0, j - 1) \text{ or } (a, b) = (\lfloor j/2 \rfloor, \lceil j/2 \rceil).$$

Proof. Suppose first that $ab < \mathrm{N}$. The inequality $(\mathrm{N} - a)(\mathrm{N} - b) \leqslant b_{j-1}$ is equivalent with $(a, b) \leqslant_{\mathrm{as}} (0, j - 1)$, i.e. to $a + b > j - 1$ or $a + b = j - 1$ and $ab \leqslant 0$. The second case means that $a = 0$, in which case $(a, b) = (0, j - 1)$. Otherwise, we get $a + b \geqslant j$. On the other hand, if $a + b \geqslant j + 1$, then $(j - (a + b))\mathrm{N} + ab \leqslant -\mathrm{N} + an < 0$, which contradicts (5.7), so only $a + b = j$ remains possible. The inequality

$$\lfloor j/2 \rfloor \lceil j/2 \rceil \leqslant ab$$

only occurs when $(a, b) = (\lfloor j/2 \rfloor, \lceil j/2 \rceil)$. So we obtain the two allowed cases.

We suppose now that $ab \geqslant \mathrm{N}$. This implies also that $b \geqslant \sqrt{\mathrm{N}}$. The inequality (5.7) implies then that $a + b > j$ (the case $a + b = j$ is impossible, since then $ab \leqslant \lfloor j^2/4 \rfloor < \mathrm{N}$). We write $a + b = j + k$ with $k \geqslant 1$. The left-hand side of (5.7) becomes

$$\lfloor (a + b - k)^2/4 \rfloor \leqslant -k\mathrm{N} + ab,$$

hence implies

$$\frac{(a + b)^2}{4} - \frac{k(a + b)}{2} + \frac{k^2}{4} \leqslant -k\mathrm{N} + ab + 1.$$

Since $(a + b)^2/4 - ab = (a - b)^2/4$, this gives

$$-\frac{k(a + b)}{2} + \frac{k^2}{4} \leqslant \frac{(a - b)^2}{4} - \frac{k(a + b)}{2} + \frac{k^2}{4} \leqslant -k\mathrm{N} + 1$$

and this implies that

$$b \geqslant \frac{a + b}{2} \geqslant \mathrm{N} + \frac{k}{4} - \frac{1}{k}.$$

If $k \geqslant 3$, this is a contradiction since $b \leqslant \mathrm{N}$; if $1 \leqslant k \leqslant 2$, then it implies that $b \geqslant \mathrm{N} - 3/4$, so that $b = \mathrm{N}$, which is incompatible with (5.6) since $a_j \geqslant 1$. □

We now can show that all the integers in Theorem 1.1 occur as primitive gaps in the multiplication table.

Proposition 5.4. *Let* $\mathrm{N} \geqslant 2$ *and let* R *be the largest non-negative integer such that* $\lfloor \mathrm{R}^2/4 \rfloor < \mathrm{N}$*. The following properties hold.*

(1) *For all integers* j *such that* $1 \leqslant j \leqslant \mathrm{R}$*, the integer* $\mathrm{N} - \lfloor j^2/4 \rfloor$ *is a primitive gap in the* $\mathrm{N} \times \mathrm{N}$ *multiplication table.*
(2) *For all integers* j *such that* $1 \leqslant j \leqslant \mathrm{R} - 2$*, the integer* j *is a primitive gap in the* $\mathrm{N} \times \mathrm{N}$ *multiplication table.*
(3) *If* R *is of the form* k^2*,* $k^2 - 1$*,* $k^2 + k$*,* $k^2 + k - 1$ *for some integer* $k \geqslant 1$*, then* $\mathrm{R} - 1$ *is a primitive gap in the* $\mathrm{N} \times \mathrm{N}$ *multiplication table.*
(4) *If* R *is of the form* k^2*,* $k^2 + 1$*,* $k^2 + k$ *or* $k^2 + k + 1$ *for some integer* $k \geqslant 1$*, then* R *is a primitive gap in* $\mathrm{N} \times \mathrm{N}$ *multiplication table.*

Proof. We can easily check numerically the statement for small values of N, and so we will assume that (say) $\mathrm{N} \geqslant 100$.

From Lemma 5.3, we see that the $\mathrm{N} \times \mathrm{N}$ multiplication table contains the primitive gaps

$$\mathrm{N} - \lfloor j^2/4 \rfloor$$

for $1 \leqslant j \leqslant \mathrm{R}$.

From Lemma 5.2 and the fact that

$$\mathrm{N}(\mathrm{N} - j) \leqslant (\mathrm{N} - 1)(\mathrm{N} - (j - 1)) \leqslant \cdots \leqslant (\mathrm{N} - \lfloor j/2 \rfloor)(\mathrm{N} - \lceil j/2 \rceil)$$

with successive gaps

$$(\mathrm{N} - (a + 1))(\mathrm{N} - (j - a - 1)) - (\mathrm{N} - a)(\mathrm{N} - (j - a)) = j - 2a - 1$$

for $0 \leqslant a \leqslant \lfloor j/2 \rfloor - 1$, we see that the $\mathrm{N} \times \mathrm{N}$ multiplication table contains as primitive gaps the integers

$$j - 2a - 1$$

where, for $1 \leqslant j \leqslant \mathrm{R}$, we let a run over integers such that $0 \leqslant a \leqslant \lfloor j/2 \rfloor - 1$, *provided*

$$a(j - a) > \frac{3}{2}\sqrt{\mathrm{N}}.$$

Indeed, this last condition implies that

$$(\mathrm{N} - a)(\mathrm{N} - (j - a)) = \mathrm{N}^2 - j\mathrm{N} + a(j - a) > \mathrm{N}(\mathrm{N} - j) + \frac{3}{2}\sqrt{\mathrm{N}},$$

so that all the products given by the second part of Lemma 5.2 are at most $(\mathrm{N}-a)(\mathrm{N}-(j-a))$, implying that the gap $j - 2a - 1$ in the interval

$$(\mathrm{N} - a)(\mathrm{N} - (j - a)) < (\mathrm{N} - (a + 1))(\mathrm{N} - (j - a - 1))$$

(which lies between b_j and a_j is primitive).

Note that the values of a (in the indicated range) with $a(j - a) > \frac{3}{2}\sqrt{\mathrm{N}}$ is an interval (if the conditions holds for a, it also does for $a + 1$, if $a + 1$ is still in the allowed range).

Taking $j = \mathrm{R}$, we get all gaps $\mathrm{R} - 2a - 1$ for $1 \leqslant a \leqslant \lfloor \mathrm{R}/2 \rfloor - 1$, since $\mathrm{R} - 1 > 3\sqrt{\mathrm{N}}/2 + 1$ for $\mathrm{N} > 37$; taking $j = \mathrm{R} - 1$, we get all gaps $\mathrm{R} - 2a - 2$ for $1 \leqslant a \leqslant \lfloor (\mathrm{R} - 1)/2 \rfloor - 1$, since $\mathrm{R} - 2 > 3\sqrt{\mathrm{N}}/2 + 1$ for $\mathrm{N} > 64$. All together, this gives all gaps between 1 and $\mathrm{R} - 3$. (If R is odd, one checks that the "boundaries" are 2 and 1, respectively, whereas they are 1 and 2 if R is even.)

It remains to handle the cases asserting the existence of R, $\mathrm{R}-1$ and $\mathrm{R}-2$ as gaps.

Case 1. We claim that as soon as $\mathrm{N} \geqslant 8$, the inequality

$$\mathrm{N}(\mathrm{N}-\mathrm{R}+1) < (\mathrm{N}-1)(\mathrm{N}-(\mathrm{R}-2))$$

gives a primitive gap of size $\mathrm{R}-2$ in the multiplication table. Indeed, an inequality

$$\mathrm{N}(\mathrm{N}-\mathrm{R}+1) < (\mathrm{N}-a)(\mathrm{N}-b) < (\mathrm{N}-1)(\mathrm{N}-(\mathrm{R}-2))$$

with $0 \leqslant a \leqslant b$ is equivalent to

$$0 < \mathrm{N}(\mathrm{R}-(a+b+1)) + ab < \mathrm{R}-2.$$

Proceeding as in the proof of Lemma 5.2, we find that this implies that $a+b = (\mathrm{R}-1)+1 = \mathrm{R}$ and $\mathrm{N} \leqslant ab \leqslant \mathrm{N}+\mathrm{R}-2$. However, for $a+b=\mathrm{R}$, we have $ab \leqslant (\mathrm{R}/2)^2 \leqslant \lfloor \mathrm{R}^2/4 \rfloor < \mathrm{N}$, so this is impossible.

Case 2. We claim that if N is of one of the forms k^2, k^2-1, k^2+k or k^2+k-1 for $k \geqslant 4$, then the integer $\mathrm{R}-1$ is a primitive gap. Note that the value of R is then, respectively, equal to $2k-1$, $2k-1$, $2k$, $2k$. It suffices then to show that the following two inequalities define primitive gaps of size respectively $2k-2$ and $2k-1$ in the indicated multiplication tables:

$$\begin{aligned}(k(k-1))^2 = k^2(k^2-2k+1) &< (k^2-1)(k^2-2k+2), \quad k^2\text{-table}\\ (k^2+k)(k^2-k) = k^2(k^2-1) &< (k^2+k-1)(k^2-k+1), \quad (k^2+k)\text{-table.}\end{aligned}$$

Indeed, the first one takes care of the cases $\mathrm{N} = k^2$ or k^2-1 (because the elements on each side of the inequality belong to the (k^2-1)-table), and the second one of the cases $\mathrm{N} = k^2+k$ or k^2+k-1.

We can handle both inequalities together, since they are the same as

$$\mathrm{N}(\mathrm{N}-\mathrm{R}) < (\mathrm{N}-1)(\mathrm{N}-\mathrm{R}+1).$$

As before, an inequality

$$\mathrm{N}(\mathrm{N}-\mathrm{R}) < (\mathrm{N}-a)(\mathrm{N}-b) < (\mathrm{N}-1)(\mathrm{N}-\mathrm{R}+1)$$

with $0 \leqslant a \leqslant b$ is equivalent to

$$0 < \mathrm{N}(\mathrm{R}-(a+b)\mathrm{N}) + ab < \mathrm{R}-1,$$

and is only possible if $a+b = \mathrm{R}+1$. But then $ab \leqslant \lfloor (\mathrm{R}+1)^2/4 \rfloor$, and by inspection we have $\lfloor (\mathrm{R}+1)^2/4 \rfloor = \mathrm{N}$ in the present case (we have $(\mathrm{R}+1)^2/4 = k^2$ if $\mathrm{N} = k^2$ and $(\mathrm{R}+1)^2/4 = k^2+k+1/4$ if $\mathrm{N} = k^2+k$).

Case 3. We claim that if N is of one of the forms k^2, k^2+1, k^2+k or k^2+k+1 for $k \geqslant 4$, then the integer R is a primitive gap. Note that the value of R is then, respectively, equal to $2k-1$, $2k$, $2k$, $2k+1$. It suffices then to show that the following four inequalities define primitive gaps of size, respectively, $2k-1$, $2k$, $2k$ and $2k+1$ in the indicated tables:

$$\begin{aligned}
(k^2-k+1)^2 &< k^2(k^2-2k+3), && k^2\text{-table},\\
(k^2-k+2)^2 &< (k^2+1)(k^2-2k+4), && (k^2+1)\text{-table},\\
k^2(k^2+1) &< (k^2+k)(k^2-k+2), && (k^2+k)\text{-table},\\
(k^2+1)(k^2+2) &< (k^2+k+1)(k^2-k+3), && (k^2+k+1)\text{-table}.
\end{aligned}$$

If we transcribe in terms of N and R these inequalities, and consider when a product $(\mathrm{N}-a)(\mathrm{N}-b)$ lies strictly between these quantities, we find that this is equivalent to

$$\begin{aligned}
-\mathrm{R} &< \mathrm{N}(\mathrm{R}-(a+b+2))+ab<0\\
-\mathrm{R} &< \mathrm{N}(\mathrm{R}-(a+b+3))+ab<0\\
-\mathrm{R} &< \mathrm{N}(\mathrm{R}-(a+b+2))+ab<0\\
-\mathrm{R} &< \mathrm{N}(\mathrm{R}-(a+b+3))+ab<0,
\end{aligned}$$

respectively. (Each of these depends on the precise relation between N and R; for instance, for the first of these, we use the fact that

$$\mathrm{N}+\Bigl(\frac{\mathrm{R}-1}{2}\Bigr)^2=2\mathrm{N}-\mathrm{R}$$

when $\mathrm{N}=k^2$ and $\mathrm{R}=2k-1$.) In particular, there are really only two cases to consider.

Consider the first and third inequalities. Arguing as previously,[4] we find that $\mathrm{R}-(a+b+2)=-1$, hence $a+b=\mathrm{R}-1$, and the inequalities are equivalent to $\mathrm{N}-\mathrm{R}<ab<\mathrm{N}$. But for $a+b=\mathrm{R}-1$, we get

$$ab\leqslant\Bigl\lfloor\frac{(\mathrm{R}-1)^2}{4}\Bigr\rfloor,$$

which in both cases here is equal to $\mathrm{N}-\mathrm{R}$. (For instance, if $\mathrm{N}=k^2$, $\mathrm{R}=2k-1$, we get $(\mathrm{R}-1)^2/4=(k-1)^2=\mathrm{N}-\mathrm{R}$.)

A similar reasoning implies that the second and fourth inequalities have no solutions. ☐

6. Exhaustion, I

In the remaining of this paper, we present some evidence for a positive answer to Problem 1.3. In the present section, this is through the proof of Proposition 1.4.

Let $\mathrm{N}\geqslant 2$ be an integer. Let R denote the integer in Theorem 1.1. In the (generic) case where $\mathscr{E}_{\mathrm{N}}$ is empty, we would ideally like to show that there is no primitive gap $\geqslant \mathrm{R}-1$ among elements ab of the multiplication table with $ab\leqslant b_{\mathrm{R}-2}=\mathrm{N}(\mathrm{N}-\mathrm{R}+2)$, which is of size about $\mathrm{N}^2-2\mathrm{N}^{3/2}$. Recalling the definition

$$d_{\mathrm{N}}(n)=\sum_{\substack{ab=n\\ a,b\leqslant \mathrm{N}}}1,$$

this would mean proving that

$$\sum_{\mathrm{X}<n\leqslant \mathrm{X}+\mathrm{Y}} d_{\mathrm{N}}(n)>0$$

[4] It is helpful to note here that $\sqrt{k\mathrm{N}-\mathrm{R}+2}\geqslant\sqrt{k\mathrm{N}}$ as soon as $\mathrm{N}\geqslant 10$, for instance.

if $\mathrm{R}^2 \leqslant \mathrm{X} \leqslant \mathrm{N}^2 - \mathrm{NR}$ and $\mathrm{Y} \geqslant \mathrm{R}$.

We define

$$\mathrm{S}_{\mathrm{N}}(\mathrm{X}) = \sum_{n \leqslant \mathrm{X}} d_{\mathrm{N}}(n)$$

for $\mathrm{N} \geqslant 1$ and $\mathrm{X} \geqslant 1$.

We also define as usual the function ψ on $\mathbf{R}$ by $\psi(x) = x - \lfloor x \rfloor - 1/2$.

Proposition 6.1. *Assume that* $\mathrm{N} \leqslant \mathrm{X} \leqslant \mathrm{N}^2$. *We have*

$$\mathrm{S}_{\mathrm{N}}(\mathrm{X}) = 2\mathrm{N}\Big\lfloor \frac{\mathrm{X}}{\mathrm{N}} \Big\rfloor + 2\mathrm{X} \sum_{\mathrm{X}/\mathrm{N} < a \leqslant \sqrt{\mathrm{X}}} \frac{1}{a} - \Big(\sqrt{\mathrm{X}} - \frac{\mathrm{X}}{\mathrm{N}} + \mathrm{O}(1)\Big) - \lfloor \mathrm{X}^{1/2} \rfloor - 2\mathrm{W}_{\mathrm{N}}(\mathrm{X})$$

where

$$\mathrm{W}_{\mathrm{N}}(\mathrm{X}) = \sum_{\mathrm{X}/\mathrm{N} \leqslant a \leqslant \sqrt{\mathrm{X}}} \psi\Big(\frac{\mathrm{X}}{a}\Big).$$

Proof. From the definition, it follows that

$$\mathrm{S}_{\mathrm{N}}(\mathrm{X}) = 2 \sum_{\substack{ab \leqslant \mathrm{X} \\ a \leqslant b \leqslant \mathrm{N}}} 1 - \lfloor \sqrt{\mathrm{X}} \rfloor.$$

Furthermore,

$$\sum_{\substack{ab \leqslant \mathrm{X} \\ a \leqslant b \leqslant \mathrm{N}}} 1 = \sum_{a \leqslant \sqrt{\mathrm{X}}} \sum_{\substack{b \leqslant \mathrm{N} \\ b \leqslant \mathrm{X}/a}} 1 = \sum_{\substack{a \leqslant \sqrt{\mathrm{X}} \\ \mathrm{N} \leqslant \mathrm{X}/a}} \sum_{b \leqslant \mathrm{N}} 1 + \sum_{\substack{a \leqslant \sqrt{\mathrm{X}} \\ \mathrm{X}/a < \mathrm{N}}} \sum_{b \leqslant \mathrm{X}/a} 1.$$

We observe that the assumption $\mathrm{X} \leqslant \mathrm{N}^2$ implies that $\mathrm{X}/\mathrm{N} \leqslant \sqrt{\mathrm{X}}$, so that the first sum in this last expression is

$$\mathrm{N} \sum_{a \leqslant \mathrm{X}/\mathrm{N}} 1 = \mathrm{N}\Big\lfloor \frac{\mathrm{X}}{\mathrm{N}} \Big\rfloor.$$

For the same reason, the second term is

$$\sum_{\mathrm{X}/\mathrm{N} < a \leqslant \sqrt{\mathrm{X}}} \Big\lfloor \frac{\mathrm{X}}{a} \Big\rfloor = \sum_{\mathrm{X}/\mathrm{N} \leqslant a \leqslant \sqrt{\mathrm{X}}} \Big(\frac{\mathrm{X}}{a} - \frac{1}{2} - \psi\Big(\frac{\mathrm{X}}{a}\Big)\Big) = \sum_{\mathrm{X}/\mathrm{N} \leqslant a \leqslant \sqrt{\mathrm{X}}} \Big(\frac{\mathrm{X}}{a} - \frac{1}{2}\Big) - \mathrm{W}_{\mathrm{N}}(\mathrm{X}).$$

Gathering these identities, the proposition follows. $\square$

To estimate $\mathrm{W}_{\mathrm{N}}(\mathrm{X})$, we use standard exponential sums methods, refering to the book of Graham and Kolesnik [3]. Let (k, l) be an exponent pair in the sense of [3, p. 31]. We recall that $0 \leqslant k \leqslant 1/2 \leqslant l \leqslant 1$.

Proposition 6.2. *Assume that* $\mathrm{N} \leqslant \mathrm{X} \leqslant \mathrm{N}^2$. *We have*

$$\mathrm{W}_{\mathrm{N}}(\mathrm{X}) \ll \mathrm{X}^{k/(k+1)} \mathrm{N}^{(l-k)/(k+1)} \log \mathrm{X},$$

where the factor $\log \mathrm{X}$ *can be omitted if* $(k, l) \neq (1/2, 1/2)$.

Proof. This is the same as the bound for the error term in the divisor problem (see [3, Th. 4.6]), but we sketch the details: if I is an interval contained in $]x, 2x]$, the upper bound

$$\sum_{n\in \mathrm{I}} \psi(y/n) \ll y^{k/(k+1)}x^{(l-k)/(k+1)} + y^{-1}x^2$$

holds by [3, Lemma 4.3] (applied with $f(t) = y/t$, with $s = 2$), with implied constant depending only on the exponent pair. Decomposing the sum in $\mathrm{W_N(X)}$ in dyadic intervals $2^{-j-1}\mathrm{X}^{1/2} \leqslant a \leqslant 2^{-j}\mathrm{X}^{1/2}$, and using the bound $\mathrm{X}^{1/2} \leqslant \mathrm{N}$, we deduce

$$\mathrm{W_N(X)} \ll \sum_j \mathrm{X}^{k/(k+1)}(2^{-j}\mathrm{N})^{(l-k)/(k+1)} + \sum_j 2^{-2j}$$

where j runs over non-negative integers for which the indicated dyadic interval intersects the interval $\mathrm{X/N} \leqslant a \leqslant \sqrt{\mathrm{X}}$. The result follows, noting that $l - k > 0$ if $(l, k) \neq (1/2, 1/2)$ (see the definition in [3, p. 31] again). ☐

Corollary 6.3. *Assume that* $\mathrm{N} \leqslant \mathrm{X} \leqslant \mathrm{N}^2$ *and that* $1 \leqslant \mathrm{Y} \leqslant 2\sqrt{\mathrm{X}}$ *and* $\sqrt{\mathrm{X}} + 2 \leqslant \mathrm{N}$.

There exists a constant $c < 0$ *such that if* N *is large enough and*

$$(\mathrm{X+Y})^{1/2} \leqslant c\mathrm{N}, \qquad \mathrm{X}^{k/(k+1)}\mathrm{N}^{(l-k)/(k+1)} \log \mathrm{X} \leqslant \mathrm{Y},$$

then

$$\mathrm{S_N(X+Y) - S_N(X)} > 0.$$

Proof. We note that under the stated assumptions, we have $(\mathrm{X+Y})^{1/2} - \mathrm{X}^{1/2} \ll 1$. Thus, according to Propositions 6.1 and 6.2, we have

$$\mathrm{S_N(X+Y) - S_N(X)} \geqslant \mathrm{S_1 + S_2 + S_3 + O(1)}$$

where

$$\begin{aligned}
\mathrm{S_1} &= 2(\mathrm{X+Y}) \sum_{(\mathrm{X+Y})/\mathrm{N} < a \leqslant (\mathrm{X+Y})^{1/2}} \frac{1}{a} - 2\mathrm{X} \sum_{\mathrm{X/N} < a \leqslant \mathrm{X}^{1/2}} \frac{1}{a},\\
\mathrm{S_2} &= 2\mathrm{N}\Big(\Big\lfloor \frac{\mathrm{X+Y}}{\mathrm{N}} \Big\rfloor - \Big\lfloor \frac{\mathrm{X}}{\mathrm{N}} \Big\rfloor\Big),\\
\mathrm{S_3} &\ll \mathrm{X}^{k/(k+1)}\mathrm{N}^{(l-k)/(k+1)} \log \mathrm{X}.
\end{aligned}$$

We have $\mathrm{S_2} \geqslant 0$. On the other hand, we have

$$\frac{\mathrm{X+Y}}{\mathrm{N}} \leqslant \frac{\mathrm{X} + 2\sqrt{\mathrm{X}}}{\mathrm{N}} \leqslant \sqrt{\mathrm{X}}$$

by our assumptions, hence

$$\mathrm{S_1} = 2\mathrm{Y} \sum_{(\mathrm{X+Y})/\mathrm{N} < a \leqslant (\mathrm{X+Y})^{1/2}} \frac{1}{a} + 2\mathrm{X}\Big(\sum_{\mathrm{X}^{1/2} < a \leqslant (\mathrm{X+Y})^{1/2}} \frac{1}{a} - \sum_{\mathrm{X/N} < a \leqslant (\mathrm{X+Y})/\mathrm{N}} \frac{1}{a}\Big).$$

We have also the lower bound

$$2\mathrm{X}\Big(\sum_{\mathrm{X}^{1/2} < a \leqslant (\mathrm{X+Y})^{1/2}} \frac{1}{a} - \sum_{\mathrm{X/N} < a \leqslant (\mathrm{X+Y})/\mathrm{N}} \frac{1}{a}\Big) \geqslant -2\mathrm{X} \cdot \frac{\mathrm{Y}}{\mathrm{N}} \cdot \frac{\mathrm{N}}{\mathrm{X}} \geqslant -2\mathrm{X}.$$

Moreover, from

$$\log(n) \leqslant \sum_{i=1}^{n} \frac{1}{i} \leqslant \log(n) + 1$$

for $n \geqslant 1$, we deduce that

$$2\mathrm{Y} \sum_{(\mathrm{X}+\mathrm{Y})/\mathrm{N} < a \leqslant (\mathrm{X}+\mathrm{Y})^{1/2}} \frac{1}{a} \geqslant 2\mathrm{Y}\log\Bigl(\frac{\mathrm{N}}{\sqrt{\mathrm{X}+\mathrm{Y}}}\Bigr) - 2\mathrm{Y}.$$

These bounds imply the lower bound

$$\mathrm{S_N}(\mathrm{X}+\mathrm{Y}) - \mathrm{S_N}(\mathrm{X}) \geqslant 2\mathrm{Y}\log(\mathrm{N}/(\mathrm{X}+\mathrm{Y})^{1/2}) - 4\mathrm{Y} + \mathrm{O}(\mathrm{X}^{k/(k+1)}\mathrm{N}^{(l-k)/(k+1)}\log \mathrm{X}).$$

The result follows if we take c small enough so that

$$2\log(1/c) \geqslant 4 + c',$$

where c' is the implied constant in the last upper bound (i.e., in Proposition 6.2). ☐

We can now prove Proposition 1.4. We recall the statement.

Corollary 6.4. *There exists a real number $\alpha > 0$ such that if*

$$n \leqslant \frac{\alpha \mathrm{N}^{3/2}}{(\log \mathrm{N})^3}$$

belongs to the $\mathrm{N}\times\mathrm{N}$ multiplication table, then $n^+ - n \leqslant \mathrm{N}^{1/2}$, where for $n = ab$ in the $\mathrm{N}\times\mathrm{N}$ multiplication table, we write n^+ for the next element, if $n \neq \mathrm{N}^2$.

Proof. If $n < \mathrm{N}/4$, we observe that writing $n = ab$, with $a \leqslant b \leqslant \mathrm{N}$, we have $a^2 \leqslant n \leqslant \mathrm{N}/4$, so $a \leqslant \frac{1}{2}\sqrt{\mathrm{N}}$. Moreover since $a < \mathrm{N}$, we have certainly $n^+ \leqslant (a+1)b = n+a$, so $n^+ - n \leqslant \frac{1}{2}\sqrt{\mathrm{N}}$.

We may thus assume that $n \geqslant \mathrm{N}/4$ and that N is sufficiently large to apply the previous corollary with $\mathrm{X} = n$, $\mathrm{Y} = \sqrt{\mathrm{N}}$ (so that $1 \leqslant \mathrm{Y} \leqslant 2\sqrt{\mathrm{X}}$), and with the "standard" exponent pair $(1/2, 1/2)$ (in the usual notation of [3], this is the pair $\mathrm{B}(0,1)$, and $(0,1)$ is trivially an exponent pair; see [3, Th. 3.10]).

We deduce that $\mathrm{S_N}(\mathrm{X}+\mathrm{Y}) - \mathrm{S_N}(\mathrm{X}) > 0$ provided

$$\sqrt{n} + 2 \leqslant \mathrm{N}, \qquad (n+\sqrt{\mathrm{N}})^{1/2} \leqslant c\mathrm{N}, \qquad n^{1/3}(\log n) \leqslant \sqrt{\mathrm{N}},$$

where $c > 0$ is given by the corollary. The last condition implies the other two for N large enough, and is satisfied as soon as

$$n \leqslant 2^{-3}\mathrm{N}^{3/2}/(\log \mathrm{N})^3$$

(since $n \leqslant \mathrm{N}^2$). Adjusting the constant to account for all values of N, we obtain the result. ☐

Remark 6.5. (1) The exponent $3/2$ can be improved immediately by using any of the exponent pairs which lead to better results in the divisor problem.

(2) The Bombieri–Iwaniec–Huxley exponent pair $(9/56 + \varepsilon, 37/56 + \varepsilon)$ (see [3, Th. 7.1]) gives (omitting epsilons) the condition

$$\mathrm{X}^{9/65}\mathrm{N}^{28/65} \ll \mathrm{Y},$$

which for Y of size $\mathrm{N}^{1/2}$ translates cleanly to $\mathrm{X} \ll \mathrm{N}^{1/2}$. This is a trivial bound.

(3) The optimal (very much conjectural) exponent pair would be $(k,l)=(0,1/2)$ (compare with [4, § 8.4], noting the different conventions: the pair (p,q) of loc. cit. corresponds to $(k,l)=(p,q+1/2)$ in the notation of [3]) and in that case the condition above would be $\mathrm{N}^{1/2}\ll \mathrm{Y}$ which is what we want – except that the constant is not necessarily right...

In fact, the only *potential* exponent pair which gives the optimal condition for Y of size $\mathrm{N}^{1/2}$ is $(0,1/2)$. Indeed, if $k=0$, we get the condition

$$\mathrm{N}^{l}\ll \mathrm{N}^{1/2},$$

hence the result since $l\geqslant 1/2$. If $k\neq 0$, then the condition is

$$\mathrm{X}\ll \mathrm{N}^{3/2-l/k+1/(2k)}.$$

and would only give the optimal result, allowing X of size very close to N^2, if $(1-2l)/k=1$. Since $l\geqslant 1/2$ again, this condition is never satisfied.

7. Exhaustion, II

We conclude with some very simple observations, which give some strong restrictions (though not yet full control) of the gaps of size about $3\mathrm{N}/4$ or $\mathrm{N}/2$. We haven't however yet been able to deduce from this that these gaps are all accounted for by Theorem 1.1.

Proposition 7.1. *Let $0\leqslant \alpha\leqslant \mathrm{N}-1$ be an integer. Suppose that $\mathrm{N}-\alpha$ is a primitive gap. Let*

$$ab-(\mathrm{N}-\alpha)=cd$$

where $1\leqslant c\leqslant d\leqslant \mathrm{N}$ and $1\leqslant a\leqslant b\leqslant \mathrm{N}$.

(1) *If $\alpha<(\mathrm{N}-1)/2$, then $b=\mathrm{N}$.*
(2) *If $\alpha\leqslant \mathrm{N}/4$, then $d=c$ or $d=c+1$.*

Proof. Since $a(b-1)=ab-a$ belongs to the multiplication table, we must have $a\geqslant \mathrm{N}-\alpha$.

Assume first that $b<\mathrm{N}$. Then $(a-1)(b+1)=ab-(b-a+1)$ belongs to the multiplication table, and hence $b-a+1\geqslant \mathrm{N}-\alpha$. Consequently we get $b=a+(b-a)\geqslant 2\mathrm{N}-2\alpha-1$, and since $b\leqslant \mathrm{N}$, this implies that $\alpha\geqslant(\mathrm{N}-1)/2$. This implies (1) by contraposition.

If $d=\mathrm{N}$, then the gap is necessarily $\mathrm{N}(\mathrm{N}-1)<\mathrm{N}^2$ (since α is divisible by N, it is equal to N).

If $d<\mathrm{N}$, then from $c(d+1)=cd+c$, we also deduce that $c\geqslant \mathrm{N}-\alpha$, and from the fact that $(c+1)(d-1)=cd+d-c-1$ is in the multiplication table, we deduce that *either* $d\in\{c,c+1\}$ (so the right-hand side is $\leqslant cd$), *or*

$$d-c-1\geqslant \mathrm{N}-\alpha.$$

However, in that case, we get $d=c+(d-c)\geqslant 2\mathrm{N}-2\alpha+1$, hence

$$\mathrm{N}^2\geqslant cd\geqslant 2(\mathrm{N}-\alpha)^2=2\mathrm{N}^2-4\alpha\mathrm{N}+\alpha^2,$$

and in particular $\alpha>\mathrm{N}/4$. This implies (2). ∎

So the first question that this suggests is the following:

Question 7.2. *For $1\leqslant a<\mathrm{N}$, what is the smallest element cd in the $\mathrm{N}\times\mathrm{N}$-table which is larger than $a\mathrm{N}$, and what is the difference $cd-a\mathrm{N}$?*

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