UNMOTIVATED ERGODIC AVERAGES

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ABSTRACT. We consider weighted ergodic averages indexed by primes, where the weight depends on the prime, and is a "trace function" coming from algebraic geometry. We obtain extensions of classical results, in both L^2 and topological settings, and raise some further problems.

1. INTRODUCTION

Let (X, μ, f) be a dynamical system: f is a measurable map $f\colon X \to X$ on a probability space (X, μ) and $f_*\mu = \mu$, i.e., μ is an invariant measure.

In this paper, motivated largely by simple curiosity (though see also Remark 1.5 for one arithmetical motivation), we consider weighted ergodic averages of *triangular* form[1], namely averages

$$\frac{1}{p}\sum_{0\leqslant n<p} t_p(n)\ (\varphi \circ f^n), \tag{1.1}$$

for some fixed function $\varphi\colon X \to \mathbf{C}$, where p is a prime and t_p is a function on $\mathbf{Z}$ (depending on p) "of algebraic origin". Precisely, we are interested in the limit of such averages as $p \to +\infty$ when the functions t_p are *trace function* modulo p or short linear combinations of such functions.

We note that this type of triangular ergodic averages are quite natural from the point of view of arithmetic; for instance, they are reminiscent of certain problems in homogeneous dynamics, such as the work of Mozes and Shah [33], which consider limits of measures that are invariant under unipotent subgroups *without enforcing that they arise from a fixed "source"*.

Since the general theory of trace functions (as amplified by Fouvry, Kowalski and Michel in particular) is probably not well-known to most readers, we present right away three basic examples that will indicate the flavor of these averages.

Example 1.1. (1) The function $t_p(n)$ which is the characteristic function of the set of squares modulo p (quadratic residues) is a linear combination

$$\frac{1}{2}\Bigl(1 + \Bigl(\frac{n}{p}\Bigr)\Bigr)$$

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[1] In the sense of the "triangular arrays" of probability theory, e.g., in the Central Limit Theorem.

of two trace functions. Thus the average (1.1) is then the ergodic average where n is restricted to be a square modulo p. Note the clear triangular feature: when the prime p changes, the set of quadratic residues changes also.

(2) Let $q \in \mathbf{Z}[\mathrm{X}]$ be a fixed monic polynomial. Then $t_p(n) = e(q(n)/p)$ is a trace function, where $e(z) = \exp(2i\pi z)$ for any complex number z.

(3) Define $t_p(n) = \mathrm{Kl}_2(n;p)$ where

$$\mathrm{Kl}_2(n;p) = \frac{1}{\sqrt{p}} \sum_{1 \leqslant x \leqslant p-1} e\Bigl(\frac{nx + \bar{x}}{p}\Bigr),$$

where $\bar{x}$ is the inverse of x modulo p. These are the classical *Kloosterman sums*, which are of paramount importance in analytic number theory. The function t_p is then also a trace function.

More generally, we will explain below the definition of two norms $\|\cdot\|_{\mathrm{t}} \leqslant \|\cdot\|_{\mathrm{tf}}$ on the space $\mathscr{C}(\mathbf{F}_p)$ of complex-valued functions on $\mathbf{F}_p = \mathbf{Z}/p\mathbf{Z}$, which we identify with the interval $\{0, \ldots, p-1\}$. For $f\colon \mathbf{F}_p \to \mathbf{C}$, these norms measure the complexity of a decomposition of f into sums of certain trace functions. In the three examples above, we have $\|t_p\|_{\mathrm{tf}} \leqslant c$, where c is independent of p (but depends on the degree of q in the case of Example (2)), and similarly $\|t_p\|_{\mathrm{t}} \leqslant c'$ for some constant c', except in the case of polynomials q of degree 1 in Example (2).

Then, exploiting the remarkable fundamental L^2 properties of trace functions (which are very deep, as they rely on Deligne's most general version of the Riemann Hypothesis over finite fields [11]), we will prove rather easily the following result.

Theorem 1.2 (L^2-ergodic theorems)**.** *Let $(t_p)_p$ be a sequence of functions $t_p\colon \mathbf{F}_p \to \mathbf{C}$, indexed by an infinite subset P of the primes. Let (X, μ, f) be a dynamical system and let*

$$\pi\colon \mathrm{L}^1(\mathrm{X}, \mu) \to \mathrm{L}^1(\mathrm{X}, \mu)$$

be the projection given by the ergodic theorem (see [14, Th. 2.30])*. Assume that there exists a constant $c \geqslant 0$ such that either*

(a) *We have $\|t_p\|_{\mathrm{tf}} \leqslant c$ for $p \in \mathsf{P}$,*

(b) *The system (X, μ, f) is weakly-mixing and $\|t_p\|_{\mathrm{t}} \leqslant c$ for $p \in \mathsf{P}$.*

Let $\varphi \in \mathrm{L}^2(\mathrm{X}, \mu)$. Then the following results hold:

(1) *We have*

$$\frac{1}{p} \sum_{0 \leqslant n < p} t_p(n)\, \varphi \circ f^n - \Bigl(\frac{1}{p} \sum_{0 \leqslant n < p} t_p(n)\Bigr) \pi(f) \to 0$$

in $\mathrm{L}^2(\mathrm{X}, \mu)$ as $p \to +\infty$ along P. Moreover, the convergence is uniform for φ in compact sets of $\mathrm{L}^2(\mathrm{X}, \mu)$.

(2) *Suppose that*

$$\sum_{p \in \mathsf{P}} \frac{(\log p)^2}{p} < +\infty. \tag{1.2}$$

Then for μ-almost all x, we have

$$\frac{1}{p} \sum_{0 \leqslant n < p} t_p(n)\, \varphi(f^n(x)) - \Bigl(\frac{1}{p} \sum_{0 \leqslant n < p} t_p(n)\Bigr) \pi(f)(x) \to 0$$

as $p \to +\infty$ along P.

In addition, we consider the analogue of Sarnak's Möbius randomness conjecture [37] (one of the recent focus points at the intersection of analytic number theory and ergodic theory) for our weighted averages. We can prove a version of this conjecture for certain specific families of trace functions, but since their definition is non-trivial, we only state here some representative examples.

Theorem 1.3 (Topological ergodic theorems)**.** *Let* X *be a locally compact topological space and* $f\colon \mathrm{X} \to \mathrm{X}$ *a continuous map. Assume that either* X *is compact or that* X *is a metric space and* f *uniformly continuous.*

Assume that the topological entropy of f *is zero. Then for all bounded*[2] *continuous functions* $\varphi\colon \mathrm{X} \to \mathbf{C}$ *and all* $x \in \mathrm{X}$*, we have*

$$\lim_{p\to+\infty} \frac{1}{p} \sum_{0\leqslant n<p} \mathrm{Kl}_2(n;p)\varphi(f^n(x)) = 0,$$

$$\lim_{p\to+\infty} \frac{1}{p} \sum_{0\leqslant n<p} \Bigl(\frac{n}{p}\Bigr)\varphi(f^n(x)) = 0.$$

Remark 1.4. (1) Sequences of the form $(\varphi(f^n(x)))_n$, where f has topological entropy 0 and φ is continuous are called *deterministic*. Hence, the result shows that there is no deterministic sequence which can correlate non-trivially with an infinite sequence of Kloosterman sums, or Legendre symbols, modulo primes.

(2) We will show that these families may be replaced by a fairly wide class of trace functions, but not all.

(3) See [8] for Bowen's definition of topological entropy, which applies to uniformly continuous maps between metric spaces, and [1] for the definition of Adler, Konheim and McAndrew which applies to arbitrary compact spaces. It is known that these are equal (when both are defined), see, e.g., [13, Satz 4.8].

(4) The special case of this theorem concerning Kloosterman sums was proved independently by El Abdalaoui, Shparlinski and Steiner [15, Th. 2.8] (when X is compact).

Remark 1.5. From the arithmetic point of view, it is a crucial fact that *there is no systematic rule to construct or constrain the sequences of trace functions that are used in the averages for each prime*. We think that the sequence of Kloosterman sums or Legendre symbols are natural, but the only constraint that we impose in Theorem 1.2 is the boundedness of the trace norms of the functions (as in much previous work). We will see that the situation is very unclear when the system (X, μ, f) is not weakly-mixing.

In general, the search for natural stronger conditions that "bind" a sequence (t_p) of trace functions is, for the author, a very natural arithmetic motivation for the study of our weighted ergodic averages. In other words: is there a natural "coherence" condition for trace functions modulo primes that would naturally distinguish examples like Kloosterman sums or Legendre symbols?

Outline of the paper. We present some concrete "incarnations" of the results in Section 2. Then Section 3 gives the definitions and basic background results concerning trace functions, including defining the "trace norms" $\|\cdot\|_{\mathrm{t}}$ and $\|\cdot\|_{\mathrm{tf}}$. Sections 4 and 5 prove the mean ergodic

[2]Check

theorem, and Section 6 discusses the topological case. We then conclude with a discussion section (including an easy maximal inequality in L^2), and with some further questions that may be of interest in probing further the links between these two subjects.

Notation. For basic references concerning ergodic theory, we will refer to the books of Einsiedler and Ward [14] and of Einsiedler and Schmidt [13] (e.g., for topological entropy, which is not discussed in [14]).

We will summarize in Section 3 the key facts concerning trace functions. More details and examples can be found for instance in the surveys [18, 25] of Fouvry, Kowalski, Michel and Sawin.

We will say that an infinite set P of primes that satisfies (1.2) is *sparse*. In order that P be sparse, it is enough that there exists $\delta > 0$ such that the counting function

$$\pi(x; \mathsf{P}) = \sum_{\substack{p \leqslant x \\ p \in \mathsf{P}}} 1$$

satisfies

$$\pi(x; \mathsf{P}) \ll \frac{x}{(\log x)^{3+\delta}}.$$

Remark on the text. The first draft of these notes was written in 2018/2019. At that time, I put them aside: the absence of applications diminished the interest of the questions, and moreover the results did not seem strong enough (or the proofs conceptually interesting enough) to compensate this fault.

I came back to the text in 2023, first because the appearance of the preprint [15] of El Abdalaoui, Shparlinski and Steiner showed that at least a few other mathematicians did consider similar questions, and then because I decided to talk about this at least once, in the Number Theory Seminar of the University of Turku (where I was present to be the opponent in the PhD defense of O. Järvienemi). Although the defects discussed above still apply,[3] there is (I think) one interesting outcome from working on this topic, namely the diophantine approximation result of Lemma 10.2, which was actually stated without proof in the 2019 draft.

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2. Examples of results

Many of our results may be interpreted as leading to cancellation properties for certain sums involving trace functions. These are often of interest in analytic number theory, and we therefore state in this section a few examples with concrete choices of trace functions and of dynamical systems (X, μ, f). We also present examples which show that some of the assumptions of Theorems 1.2 and 1.3 are needed.

[3] In addition to the fact that there might be lurking mistakes and imprecisions, and that there are significant redundancies in certain arguments.

Example 2.1. We give first some examples related to continued fraction expansions. Let $(]0,1[,\mu,f)$ be the continued fraction dynamical system (see [14, Ch. 3]), in other words

$$\mu = \frac{1}{\log(2)}\frac{dx}{1+x}, \qquad f(x) = \frac{1}{x} - \Big\lfloor \frac{1}{x}\Big\rfloor.$$

This system is ergodic (loc. cit.) and f has positive entropy.

For $x \in [0,1]$, let $(a_n(x))$ be the sequence of partial quotients in the continued fraction expansion of x. We have $a_{n+1}(x) = a_n(f(x))$.

Maybe the simplest result that we can deduce from this work is that for a fixed integer $k \geqslant 0$, and for almost all x, we have

$$\frac{1}{p}\Big|\Big\{1 \leqslant n < p \mid \Big(\frac{n}{p}\Big) = 1 \text{ and } a_n(x) = k\Big\}\Big| \to \frac{1}{2\log 2}\log\Big(\frac{(k+1)^2}{k(k+2)}\Big)$$

as $p \to +\infty$ along a sparse sequence, where (n/p) is the Legendre symbol. This is one half of the density of occurence of $a_n(x) = k$, see [14, Cor. 3.8].

This result follows from Theorem 1.2, (2) when we take

$$t_p(n) = \frac{1}{2}\Big(1 + \Big(\frac{n}{p}\Big)\Big),$$

for p odd, and φ the characteristic function of $a_1(x) = k$, since $\varphi \circ f^n$ is the characteristic function of $a_n(x) = k$, and moreover we have $\|t_p\|_{\mathrm{tf}} \ll 1$ and

$$\frac{1}{p}\sum_{n\in\mathbf{F}_p} t_p(n) = \frac{1}{2}.$$

Example 2.2. For p prime, let t_p be the Kloosterman sum function modulo p (Example 1.1, (3)). We have $\|t_p\|_{\mathrm{tf}} \ll 1$ and

$$\frac{1}{p}\sum_{n\in\mathbf{F}_p} t_p(n) = 0.$$

Define $\mathrm{X} = \mathrm{SL}_2(\mathbf{Z})\backslash \mathrm{SL}_2(\mathbf{R})$ and denote by μ the invariant probability measure on X (induced by a normalized Haar measure on $\mathrm{SL}_2(\mathbf{R})$). Consider the dynamical system with

$$f(g) = g\begin{pmatrix} 2 & 0 \\ 0 & 1/2\end{pmatrix}$$

for $g \in \mathrm{X}$ (a part of the geodesic flow). It is known that (X,μ,f) is ergodic and that f has positive topological entropy (see TODO).

Let $\varphi\colon \mathrm{X} \to \mathbf{C}$ be an L^2-function. Applying Theorem 1.2, (2), we deduce that for almost all $z \in \mathrm{X}$, we have

$$\frac{1}{p}\sum_{1\leqslant n<p} \mathrm{Kl}_2(n;p)\varphi\Big(z\begin{pmatrix} 2^n & 0 \\ 0 & 2^{-n}\end{pmatrix}\Big) \to 0$$

as $p \to +\infty$ along a sparse subsequence.

On the other hand, let

$$\widetilde{f}(g) = g\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}$$

for $g \in \mathrm{X}$ (part of the horocycle flow). Then $(\mathrm{X}, \mu, \widetilde{f})$ is ergodic and has zero entropy (note that X is not compact, but $\widetilde{f}$ is uniformly continuous, so Bowen's definition of entropy applies). Thus we have

$$\frac{1}{p}\sum_{1\leqslant n<p} \mathrm{Kl}_2(n;p)\varphi\Big(z\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix} z\Big) \to 0$$

for any bounded continuous function φ on X and any $z \in \mathrm{X}$ by Theorem 1.3.

Example 2.3. It is not surprising that pointwise convergence may fail in full generality, since this means considering arbitrary sequences a_n instead of $\varphi(f^n(x))$ (using the shift on $[-1, 1]$ on the space of bounded sequences). As a simple example, consider again $t_p(n) = (n/p)$ (the Legendre symbol modulo p).

Let (p_k) be an increasing sequence of primes with $p_{k+1}/p_k \to +\infty$; the set of primes thus defined is of course sparse. Define a sequence a_n by

$$a_n = \begin{cases} 1 \text{ if } n \text{ is a square modulo } p_{k+1}\\ 0 \text{ if } n \text{ is not a square modulo } p_{k+1},\end{cases}$$

where $p_k \leqslant n < p_{k+1}$. Then

$$\frac{1}{p_k}\sum_{0\leqslant n<p_k} t_{p_k}(n)a_n = 1 + \mathrm{O}\Big(\frac{p_{k-1}}{p_k}\Big) \to 1.$$

This example can, for instance, be embedded in the continued fraction setting, and can be adapted to pretty arbitrary sequences of trace functions.

Example 2.4. Let p be a prime and $t_p(n) = e(a_p n/p)$ for some $a_p \in \mathbf{F}_p$. These are trace functions, but we will show that Theorem 1.3 does not hold with $\mathrm{Kl}_2(n; p)$ replaced by $t_p(n)$, at least if (a_p) is chosen in a suitable manner.

Pick $\theta \in \mathbf{R}/\mathbf{Z}$ which is irrational. There exists $\delta > 0$ such that there are infinitely many approximations a_p/p by rational numbers with prime denominators with

$$\Big|\theta - \frac{a_p}{p}\Big| \leqslant \frac{1}{p^{1+\delta}}. \tag{2.1}$$

Indeed, this was proved by Vinogradov for arbitrary $\delta < 1/5$, and the best-known result by Matomäki [31] applies for any $\delta < 1/3$. The irrational translation $f(x) = x + \theta$ on $\mathbf{R}/\mathbf{Z}$ has entropy zero; pick the starting point $x = 0$ and the continuous function $\varphi(\alpha) = e(\alpha)$ on $\mathbf{R}/\mathbf{Z}$. Then, for primes p for which (2.1) holds, we get

$$\frac{1}{p}\sum_{0\leqslant n<p} e\Big(-\frac{na_p}{p}\Big)e(n\theta) = \frac{1}{p}\frac{1 - e(p(\theta - a_p/p))}{1 - e(\theta - a_p/p)} \to 1$$

as $p \to +\infty$ along this sequence.

We note in passing that Theorem 1.2 does *not* apply here (the system is not weakly-mixing, and the norms $\|t_p\|_{\mathrm{tf}}$ are not bounded).

(Also, we note that one could obtain easier examples using the fact that (2.1) holds with $\delta = 1$ for almost all $\theta \in [0, 1]$, which goes back at least to Duffin and Schaeffer [12].)

Example 2.5. Here are some additional standard examples of functions on $\mathbf{Z}$ that arise as trace functions modulo p with bounded conductor, and which moreover are "geometrically irreducible" (an important property which means essentially that their mean-square average modulo p is close to 1). More examples are found, e.g., in [18]. This should give an idea of the variety of ergodic averages that we are considering.

(1) For any a modulo p, the additive character $n \mapsto e(an/p)$ is a trace function of a so-called Artin-Schreier sheaf; it has conductor uniformly bounded.

(2) For any non-trivial multiplicative character χ modulo p, extended by 0 to $\mathbf{Z}/p\mathbf{Z}$, the corresponding Dirichlet character is a trace function of a so-called Kummer sheaf; it has conductor uniformly bounded.

(3) More generally, let $f \in \mathbf{Z}[\mathrm{X}]$ be a non-constant polynomials. The functions $n \mapsto e(f(n)/p)$ and $n \mapsto \chi(f(n))$ are trace functions with conductor bounded in terms of the degree of f only. Similarly if f is a non-constant rational function, with the trace function having value 0 at poles of f, and with conductor depending on the degrees of the numerator and denominators of f.

(4) If t_p is a geometrically irreducible trace function modulo p, and is not proportional to an additive character, then its normalized Fourier transform

$$\widehat{t_p}(n) = \frac{1}{\sqrt{p}} \sum_{0 \leqslant m < p} e\Big(\frac{mn}{p}\Big) t_p(m)$$

is also a trace function with conductor bounded only in terms of that of t_p (see [19, Prop. 8.2]).

So for instance, the fact that the Kloosterman sums used above (see Example 1.1), namely

$$t_p(n) = \frac{1}{\sqrt{p}} \frac{1}{\sqrt{p}} \sum_{1 \leqslant x \leqslant p-1} e\Big(\frac{nx + \bar{x}}{p}\Big),$$

define a geometrically irreducible trace function modulo p with bounded conductor follows from this principle applied to the trace function $n \mapsto e(\bar{n}/p)$ (extended by 0 for $n = 0$), which is a special case of Example (3).

As a final remark, we emphasize that trace functions behave in many ways like random functions (e.g., they often have Gowers norms that are as small as those of random functions, as shown by Fouvry, Kowalski and Michel in [21]), and one can think of them in these terms in a first reading.

3. Properties of trace functions

We summarize here the properties of trace functions that we will use. These are essentially related to the Fourier transform (which was already mentioned in Example 2.5, (4), as an operation preserving trace functions).

First, we fix throughout the paper a prime number ℓ, and impose that all other prime numbers we consider below are different from ℓ (one can take $\ell = 2$ and only consider odd primes). We fix an isomorphism $\iota \colon \overline{\mathbf{Q}}_\ell \to \mathbf{C}$. We first clarify our terminology and conventions concerning sheaves:

Definition 3.1 (Sheaves and uniform sheaves)**.** Let $p \neq \ell$ be a prime.

(1) A *sheaf* $\mathscr{F}$ modulo p is a middle extension $\overline{\mathbf{Q}}_\ell$-sheaf on $\mathbf{A}^1_{\mathbf{F}_p}$, pure of weight 0. A *Fourier sheaf* modulo p is a sheaf modulo p that is of Fourier type in the sense of Katz [27, 7.3.4],

in other words, none of its geometrically irreducible components is geometrically isomorphic to an Artin-Schreier sheaf.

(2) The *trace function* $t_{\mathscr{F}}$ of a sheaf $\mathscr{F}$ modulo p is the complex-valued function on $\mathbf{Z}$ defined by

$$t_{\mathscr{F}}(x) = \iota(\mathrm{Tr}(\mathrm{Fr}_{x,\mathbf{F}_p} \,|\mathscr{F}_{\bar{x}}))$$

where $\mathrm{Fr}_{x,\mathbf{F}_p}$ is the Frobenius at $x \in \mathbf{F}_p$, and $\bar{x}$ is a geometric point above x.

(3) Let $\mathscr{F}$ be a sheaf modulo p and let $k \geqslant 0$ be an integer. We say that $\mathscr{F}$ is *k-uniform* if no geometrically irreducible component of $\mathscr{F}$ is geometrically isomorphic to a sheaf of the type $\mathscr{L}_{\psi(\mathrm{P})}$ where ψ is a non-trivial additive character of $\mathbf{F}_p$ and $\mathrm{P} \in \mathbf{F}_p[\mathrm{X}]$ is a polynomial of degree $\leqslant k$.

(4) We say that $\mathscr{F}$ is almost k-uniform with average μ if $\mathscr{F} \simeq \mu\overline{\mathbf{Q}}_\ell \oplus \mathscr{G}$ where $\mathscr{G}$ is k-uniform.

Note that speaking of geometrically irreducible components of a sheaf $\mathscr{F}$ modulo p is legitimate, since such sheaves (being pure of some weight) are geometrically semisimple by work of Deligne.

Example 3.2. To say that $\mathscr{F}$ is 0-uniform (resp. 1-uniform) means that $\mathscr{F}$ has no trivial geometrically irreducible component (resp. is of Fourier type in the sense of Katz [27, 7.3.5]).

Let $\mathscr{F}$ be a sheaf modulo p. Fouvry, Kowalski and Michel defined its *conductor* $\mathbf{c}(\mathscr{F})$ in [19, Def. 1.13]; it is a positive integer which vanishes if and only if $\mathscr{F}$ is zero.

The conductor measures quantitatively the complexity of a sheaf in many estimates.. One essential property is a bound on the size of the trace function: for any sheaf $\mathscr{F}$ modulo p, and any $x \in \mathbf{Z}$, we have

$$|t_{\mathscr{F}}(x)| \leqslant \mathbf{c}(\mathscr{F}). \tag{3.1}$$

Using the conductor, we define the trace norms as follows.

Definition 3.3 (Trace norms)**.** Let p be a prime different from ℓ. Let $\mathscr{C}(\mathbf{F}_p)$ denote the vector space of $\mathbf{C}$-valued functions on $\mathbf{F}_p$.

For $f \in \mathscr{C}(\mathbf{F}_p)$, we define

$$\|f\|_{\mathrm{t}} = \inf\Bigl\{\sum_i \mathbf{c}(\mathscr{F}_i)|a_i| \;\mid\; f = \sum_i a_i t_{\mathscr{F}_i},\; \mathscr{F}_i \text{ geometrically irreducible}\Bigr\},$$

and

$$\|f\|_{\mathrm{tf}} = \inf\Bigl\{\sum_i \mathbf{c}(\mathscr{F}_i)|a_i| \;\mid\; f = \sum_i a_i t_{\mathscr{F}_i},\; \mathscr{F}_i \text{ geometrically irreducible Fourier}\Bigr\}.$$

In both cases, the infimum runs over decompositions of f in linear combinations of trace functions of sheaves of the indicated type.

It is straightforward that both of these are norms, and clear that $\|f\|_{\mathrm{t}} \leqslant \|f\|_{\mathrm{tf}}$.

Remark 3.4. Although we mentioned that trace functions can be thought of as "random" functions, one should note that for most simple models of random functions $f\colon \mathbf{F}_p \to \mathbf{C}$ (e.g., taking all $f(n)$ to be independent and uniform over the unit disc), the norm $\|f\|_{\mathrm{t}}$ will in fact be very large, as explained in a paper of Fouvry, Kowalski and Michel (see [22, Th. 5.1]).

We now state some of the fundamental analytic properties of trace functions, starting with the general form of the "completion method" for short sums of trace functions.

Proposition 3.5 (Completion method)**.** *Let $\mathscr{F}$ be a Fourier sheaf modulo p and $t\colon \mathbf{Z} \to \mathbf{C}$ its trace function. For any interval I in $\mathbf{Z}$ of length $\leqslant p$, we have*

$$\sum_{n \in \mathrm{I}} t(n) \ll \sqrt{p}(\log p)$$

where the implied constant depends only on the conductor of $\mathscr{F}$.

See [24, §1.1, §2.2] for the argument, which is straightforward, granted the very deep fact (a case of Deligne's Riemann Hypothesis in its strongest form) that the normalized discrete Fourier transform of t is the trace function of a sheaf $\mathrm{FT}(\mathscr{F})$, which is also a middle-extension of weight 0, with conductor $\leqslant 10\,\mathbf{c}(\mathscr{F})^2$ (this last important estimate is proved by Fouvry, Kowalski and Michel in [19, Prop. 8.2]).

Proposition 3.6. *Let $\mathscr{F}$ and $\mathscr{G}$ be middle-extension ℓ-adic sheaves of weight 0 modulo p.*

(1) *The additive middle convolution $\mathscr{F} *_! \mathscr{G}$ is a middle-extension ℓ-adic sheaf of weights $\leqslant 0$, and it has conductor bounded in terms of the conductors of $\mathscr{F}$ and $\mathscr{G}$.*

(2) *Suppose that $\mathscr{F}$ is a Fourier sheaf. The additive middle convolution $\mathscr{F} *_! \mathrm{D}(\mathscr{F})$ contains no Artin-Schreier sheaf as geometrically irreducible component.*

Proof. For (1), the first assertion follows from the definition of the middle convolution and from Deligne's Riemann Hypothesis. To estimate the conductor, it is simplest here to appply the Fourier transform, which is an exact functor transforming middle-convolution into tensor product, so that

$$\mathscr{F} *_! \mathscr{G} = \overline{\mathrm{FT}}(\mathrm{FT}(\mathscr{F}) \otimes \mathrm{FT}(\mathscr{G})).$$

We can then apply the estimate [19, Prop. 8.2] for the conductor of a Fourier transform.

For (2), applying the Fourier transform, a hypothetical injection $\mathscr{L}_{\psi(ax)} \hookrightarrow \mathscr{F} *_! \mathrm{D}(\mathscr{F})$ would imply the existence of an injection

$$\delta_a \hookrightarrow \mathrm{FT}_\psi(\mathscr{F}) \otimes \mathrm{D}(\mathrm{FT}_\psi(\mathscr{F}))$$

of a punctual skyscraper sheaf into $\mathrm{FT}_\psi(\mathscr{F}) \otimes \mathrm{D}(\mathrm{FT}_\psi(\mathscr{F}))$. However, since both $\mathrm{FT}_\psi(\mathscr{F})$ and its dual are middle-extension sheaves when $\mathscr{F}$ is a middle-extension, their tensor product has no punctual part. $\square$

The following definition will be convenient in some places.

Definition 3.7. A family $(\mathscr{F}_p)_p$ of sheaves modulo p indexed by (a subset of) the primes $\neq \ell$ is an *almost Fourier family* if the conductor of $\mathscr{F}_p$ is bounded independently of p, and if there exists an integer $r \geqslant 0$ such that $\mathscr{F}_p = r\overline{\mathbf{Q}}_\ell \oplus \widetilde{\mathscr{F}_p}$ for all p, where $\widetilde{\mathscr{F}_p}$ is a Fourier sheaf modulo p. We say that r is the mean of the family.

For an almost Fourier family, the trace functions t_p of $\mathscr{F}_p$ satisfy

$$t_p(x) = r + \widetilde{t}_p(x)$$

where $\widetilde{t}_p$ is the trace function of $\widetilde{\mathscr{F}_p}$.

The following proposition will be only be used for polynomials P of degree 1, but since it is of independent interest, we state and prove it in general (see [15, Th. 2.7] for a special case).

Proposition 3.8. *Let $k \geqslant 1$ be an integer and define $\gamma_k = 2^{-k}$. Let p be a prime and let $\mathscr{F}$ be a k-uniform ℓ-adic sheaf modulo p with trace function $t(n)$. Let $\mathrm{P} \in \mathbf{R}[\mathrm{X}]$ be a polynomial of degree $\leqslant k$. Let I be an interval in $\mathbf{Z}$ of length $|\mathrm{I}| \geqslant 1$. We have*

$$\sum_{n\in\mathrm{I}} t(n)e(\mathrm{P}(n)) \ll \mathbf{c}(\mathscr{F})^2\Big(|\mathrm{I}|^{1-2\gamma_k}p^{\gamma_k} + |\mathrm{I}|p^{-\gamma_k}\Big)(\log p)^{2\gamma_k} \tag{3.2}$$

where the implied constant is absolute.

For $k \geqslant 2$, the proof will use the following lemma; readers only interested in main results of this paper may skip this in a first reading.

Lemma 3.9. *Let $k \geqslant 1$ be an integer and p a prime. Let $\mathscr{F}$ be a geometrically isotypic k-uniform ℓ-adic sheaf modulo p for some integer $k \geqslant 1$. Let $h \in \mathbf{F}_p$ be such that the set of singularities of $[+h]^*\mathscr{F}$ and $\mathrm{D}(\mathscr{F})$ are disjoint. If $p > k$ and $\mathbf{c}(\mathscr{F}) < p$, and if $h \neq 0$, then $[+h]^*\mathscr{F} \otimes \mathrm{D}(\mathscr{F})$ is a $(k-1)$-uniform ℓ-adic sheaf modulo p with conductor $\ll \mathbf{c}(\mathscr{F})^2$.*

Proof. This is implicit in the work of Fouvry, Kowalski and Michel in [21, §5]. Precisely, under the assumption on h, the tensor product $[+h]^*\mathscr{F} \otimes \mathrm{D}(\mathscr{F})$ is an ℓ-adic sheaf modulo p (the key point is that it is a middle-extension, see [21, Lemma 2.2]). If the conclusion does not hold, we deduce from the definition of $(k-1)$-uniform sheaf that there exists a polynomial P of degree $\leqslant k-1$ such that

$$\mathscr{F} \simeq [+h]^*\mathscr{F} \otimes \mathscr{L}_{\psi(\mathrm{P})}$$

(see [21, Lemma 5.3 (2)]). From this, we see first that $\mathbf{c}(\mathscr{F}) \geqslant p$, if $\mathscr{F}$ is not lisse on $\mathbf{A}^1_{\mathbf{F}_p}$ (because the orbit of a singularity under $x \mapsto x+h$ is contained in the set of singularities, so there are at least p of them, each of which contributes at least 1 to the sum of drops of $\mathscr{F}$). Otherwise, since $p > k$, by [21, Lemma 5.4 (2)], it follows that either $\mathbf{c}(\mathscr{F}) \geqslant p$ (because of the contribution of the Swan conductor at ∞) or $\mathscr{F}$ is isomorphic to $\mathscr{L}_{\psi(\mathrm{Q})}$ for some polynomial of degree $\leqslant k$. The lemma follows, by contraposition. □

Proof. We first consider the case $|\mathrm{I}| \leqslant p$. We then need to show that

$$\sum_{n\in\mathrm{I}} t(n)e(\mathrm{P}(n)) \ll \mathbf{c}(\mathscr{F})^2|\mathrm{I}|^{1-2\gamma_k}p^{\gamma_k}(\log p)^{2\gamma_k}, \tag{3.3}$$

and we may assume (by additive change of variable) that I is contained in $\{0,\ldots,p-1\}$.

We assume (as we may) that $\mathrm{P}(0) = 0$. If we decompose the arithmetic semisimplification of $\mathscr{F}$ in arithmetically irreducible components, say $\mathscr{F}_i$, then one of the following is true (see [21, Lemma 5.3]):

(1) For some $n \geqslant 2$, the sheaf $\mathscr{F}_i$ is induced from some irreducible sheaf on $\mathrm{Spec}(\mathbf{F}_{p^n})$ by pushforward along the map $\mathrm{Spec}(\mathbf{F}_{p^n}) \to \mathrm{Spec}(\mathbf{F}_p)$; in this case, the trace function t_i of $\mathscr{F}_i$ is identically 0 (see [21, Lemma 5.3] or [19, Proof of Prop. 8.3]), so that the estimate (3.2) is trivial.

(2) The sheaf $\mathscr{F}_i$ is geometrically isotypic.

Since the estimate (3.2) is linear in $\mathscr{F}$, we see that we may reduce the proof to the case where $\mathscr{F}$ is geometrically isotypic.

We now proceed by induction on k. The key tool is Weyl differencing. Assume first that $k = 1$ and that $\mathrm{P}(n) = \theta n$ (here we do not need to assume that $\mathscr{F}$ is isotypic). By discrete

Fourier inversion, we obtain

$$\sum_{n\in \mathrm{I}} t(n)e(\theta n) = \sum_{0\leqslant h<p} \widehat{t}(h)\alpha_p(h,\theta)$$

where

$$\alpha_p(h,\theta) = \frac{1}{\sqrt{p}}\sum_{n\in\mathrm{I}} e\Bigl(n\Bigl(\frac{h}{p}+\theta\Bigr)\Bigr), \qquad \widehat{t}(h) = \frac{1}{\sqrt{p}}\sum_{0\leqslant n<p} t(n)e\Bigl(\frac{nh}{p}\Bigr).$$

Since $\mathscr{F}$ is 1-uniform, it is a Fourier sheaf, and we have $|\widehat{t}(h)| \leqslant \mathbf{c}(\mathrm{FT}(\mathscr{F})) \ll \mathbf{c}(\mathscr{F})^2$ (by [19, Prop. 8.2]). On the other hand, by summing the geometric sum, we have

$$|\alpha_p(h,\theta)| \leqslant \min\Bigl(\frac{|\mathrm{I}|}{\sqrt{p}}, \frac{1}{\sqrt{p}}\frac{1}{\|\frac{h}{p}+\theta\|}\Bigr),$$

where $\|\cdot\|$ on the right-hand side is the distance to the nearest integer. We use the first bound for that value h_0 of h where $|h_0/p+\theta| \leqslant 1/p$, and the other values of $\alpha_p(h,\theta)$ are then bounded by

$$\frac{\sqrt{p}}{2}, \quad \ldots, \quad \frac{\sqrt{p}}{p},$$

so that

$$\sum_{0\leqslant h<p} |\alpha_p(h,\theta)| \ll \sqrt{p}(\log p),$$

with an absolute implied constant. Combining these results we obtain

$$\Bigl|\sum_{0\leqslant h<p} \widehat{t}(h)\alpha_p(h,\theta)\Bigr| \ll \mathbf{c}(\mathscr{F})\sqrt{p}\log p,$$

with an absolute implied constant, which implies the bound (3.3) for $k=1$.

Now assume that $\deg(\mathrm{P}) = k \geqslant 2$ and that the proposition is true for polynomials of degree $k-1$; assume (as we saw that we may) that $\mathscr{F}$ is geometrically isotypic. We write

$$\begin{aligned}\Bigl|\sum_{n\in\mathrm{I}} t(n)e(\mathrm{P}(n))\Bigr|^2 &= \sum_{n,m\in\mathrm{I}} t(n)\overline{t(m)}e(\mathrm{P}(n)-\mathrm{P}(m))\\ &= \sum_{h\in\mathrm{I}-\mathrm{I}}\sum_{m\in\mathrm{I}_h} t(m+h)\overline{t(m)}e(\mathrm{P}(m+h)-\mathrm{P}(m))\\ &= \sum_{h}\sum_{m\in\mathrm{I}_h} t(m+h)\overline{t(m)}e(\mathrm{Q}_h(m))\end{aligned}$$

where $\mathrm{Q}_h = \mathrm{P}(\mathrm{X}+h)-\mathrm{P}(\mathrm{X})$ is a polynomial of degree $\leqslant k-1$ and I_h is an interval, depending on h, of length $|\mathrm{I}_h| \leqslant |\mathrm{I}|$.

For $h\in \mathrm{I}-\mathrm{I}$ such that the set of singularities of $[+h]^*\mathscr{F}$ and $\mathrm{D}(\mathscr{F})$ are not disjoint, we use the trivial bound

$$\Bigl|\sum_{m\in\mathrm{I}_h} t(m+h)\overline{t(m)}e(\mathrm{Q}_h(m))\Bigr| \leqslant \mathbf{c}(\mathscr{F})^2|\mathrm{I}|.$$

Note that there are at most n^2 values of h with this property, where $n\leqslant \mathbf{c}(\mathscr{F})$ is the number of singularities of $\mathscr{F}$.

Now suppose that the set of singularities of $[+h]^*\mathscr{F}$ and $\mathrm{D}(\mathscr{F})$ are disjoint. The function

$$m \mapsto t(m+h)\overline{t(m)}$$

is the trace function of the sheaf $[+h]^*\mathscr{F} \otimes \mathrm{D}(\mathscr{F})$, which is $(k-1)$-uniform by Lemma 3.9. Hence, by induction, we have

$$\sum_{m\in \mathrm{I}_h} t(m+h)\overline{t(m)}e(\mathrm{Q}_h(m)) \ll \mathbf{c}(\mathscr{F})^2 |\mathrm{I}_h|^{1-2\gamma_{k-1}} p^{\gamma_{k-1}} (\log p)^{2\gamma_{k-1}}$$

where the implied constant is absolute. Finally, gathering the estimates together, since $|\mathrm{I}-\mathrm{I}| \leqslant 2|\mathrm{I}|$ and $|\mathrm{I}_h| \leqslant |\mathrm{I}|$, we obtain

$$\Bigl|\sum_{n\in\mathrm{I}} t(n)e(\mathrm{P}(n))\Bigr|^2 \ll \mathbf{c}(\mathscr{F})^4|\mathrm{I}| + \mathbf{c}(\mathscr{F})^2|\mathrm{I}|^{2-2\gamma_{k-1}} p^{\gamma_{k-1}} (\log p)^{2\gamma_{k-1}},$$

and (3.3) follows for degree k by taking the square root since $\gamma_{k-1} = 2\gamma_k$.

We now assume that $|\mathrm{I}| > p$. We can decompose the interval I into $\lfloor |\mathrm{I}|/p \rfloor$ intervals of length p and one remaining interval J of length $|\mathrm{J}| \leqslant p$. Using shifts, each of these sums is of the type above for a shifted sheaf, with the same conductor, and an interval of length $\leqslant p$. The previous case therefore implies

$$\sum_{n\in\mathrm{I}} t(n)e(\mathrm{P}(n)) \ll \mathbf{c}(\mathscr{F})^2 \frac{|\mathrm{I}|}{p} \times p^{1-2\gamma_k} p^{\gamma_k} (\log p)^{2\gamma_k}$$

(since the implied constant is independent of the coefficients of P), and this is of the desired shape for $|\mathrm{I}| > p$. ☐

Corollary 3.10. *Let $\mathscr{F}$ be a Fourier sheaf modulo p with trace function t_p, and $\theta \in \mathbf{R}/\mathbf{Z}$. We have*

$$\sum_{0\leqslant n<p} t_p(n)e(-\theta n) \ll \sqrt{p}\log p,$$

where the implied constant depends only on the conductor of $\mathscr{F}$.

Remark 3.11. (1) The estimate of Proposition 3.8 cannot be improved without some additional assumption, since $t(n) = e(n^k/p)$ is the trace function of a sheaf that is $(k-1)$-uniform but not k-uniform.

(2) Estimates similar to that of Proposition 3.8 have been proved by a number of authors when $t(n) = \chi(n)$ is a multiplicative character, beginning with Enflo [16]; more recent works include those of Chang [10], Heath-Brown and Pierce [26] and Pierce [35]. In that special case, rather stronger results hold; as far as the size of I is concerned, they are comparable to the Burgess bound for short character sums, i.e., non-trivial provided that I is a bit larger than $p^{1/4}$.

(3) If $\theta = a/p$ for some integer a, then the estimate of Corollary 3.10 holds without the factor $\log p$, by the existence of Deligne's Fourier transform. It would be interesting to know if this factor is really needed in general.

4. The mean ergodic theorem in the Fourier case

This section considers the mean ergodic theorem in L^2. As can be expected from the good L^2 properties of trace functions, a very satisfactory theory exists, and it is reasonably easy to derive. Roughly speaking, we will see that non-trivial interactions arise only from the Artin-Schreier components (on the side of trace functions) and from the Kronecker factor (on the dynamical side). So if either the Artin-Schreier component or the Kronecker factor is trivial (the latter means that the dynamical system is weakly-mixing), then the statements are particularly clear.

We fix a measurable dynamical system (X, μ, f). We denote by

$$u_f \colon \mathrm{L}^2(\mathrm{X}, \mu) \to \mathrm{L}^2(\mathrm{X}, \mu)$$

the associated unitary operator, defined by $u_f(\varphi) = \varphi \circ f$ for all φ.

We also fix a family $(\mathscr{F}_p)_p$ of sheaves modulo p with bounded conductor, indexed by an infinite set of primes P. We denote by t_p the trace function of $\mathscr{F}_p$, viewed as a function on $\mathbf{Z}$. Finally, we denote by

$$v_p = \frac{1}{p} \sum_{0 \leqslant n < p} t_p(n)\ u_f^n$$

the ergodic averaging operator with weight t_p acting on $\mathrm{L}^2(\mathrm{X}, \mu)$.

Proposition 4.1. *Suppose that $\mathscr{F}_p$ is a Fourier sheaf for all p. The endomorphisms $(v_p)_p$ of $\mathrm{L}^2(\mathrm{X}, \mu)$ converge to 0 as $p \to +\infty$ with respect to the operator norm. In fact, we have*

$$\|v_p\| \ll p^{-1/2}(\log p), \tag{4.1}$$

where the implied constant depends only on $\mathbf{c}(\mathscr{F}_p)$.

Although the proof may seem rather trivial, it relies on the Riemann Hypothesis over finite fields.

Proof. Let $\varphi \in \mathrm{L}^2(\mathrm{X}, \mu)$ have norm 1. Let ν be the spectral measure of the unitary operator u_f relative to the unit vector φ, i.e., the Borel probability measure on $\mathbf{R}/\mathbf{Z}$ such that

$$\int_{\mathbf{R}/\mathbf{Z}} \varrho(e(\theta)) d\nu(\theta) = \langle \varrho(u_f)\varphi | \varphi \rangle$$

for any continuous function ϱ on $\mathbf{S}^1$ (see, e.g., [4, Déf. 4, p. 268]). We obtain in particular

$$\|v_p(\varphi)\|^2 = \int_0^1 \Big| \frac{1}{p} \sum_{0 \leqslant n < p} t_p(n) e(n\theta) \Big|^2 d\nu(\theta).$$

Applying Corollary 3.10, we get

$$\|v_p(\varphi)\|^2 \ll \frac{(\log p)^2}{p}$$

where the implied constant depends only on the conductor of $\mathscr{F}_p$. This concludes the proof. $\square$

This implies the first part of Theorem 1.2, in the case of Fourier sheaves (with uniform convergence over bounded sets), because

$$\frac{1}{p}\sum_{0\leqslant n<p} t_p(n) \ll \frac{1}{\sqrt{p}} \to 0$$

in that case.

For arbitrary functions $t_p\colon \mathbf{F}_p \to \mathbf{C}$, provided they satisfy Assumption (a), namely $\|t_p\|_{\mathrm{tf}} \ll 1$, we can represent t_p as a finite combination (with coefficients bounded in ℓ_1) of trace functions of Fourier sheaves, and obtain the same result by linearity.

Moreover, this also implies the second part, still in the case of Fourier sheaves, by a standard trick: if p ranges over a sparse set of primes P, then for any fixed $\varphi \in \mathrm{L}^2(\mathrm{X},\mu)$, the series

$$\sum_p \|v_p(\varphi)\|^2$$

converges (by (4.1) and the definition by sparseness), and this implies that the function

$$x \mapsto \sum_p |v_p(\varphi)(x)|^2$$

is finite almost surely, hence that $v_p(\varphi)(x)$ converges to 0 for almost all x. Once again, this gives the second part of Theorem 1.2 under Assumption (a) by linearity.

Remark 4.2. For the sake of variety, here is an argument which provides a proof of the weaker result

$$\|v_p\| \ll p^{-1/4}(\log p)^{1/2},$$

without using the spectral theorem. Let $\varphi \in \mathrm{L}^2(\mathrm{X},\mu)$ and

$$\psi_p = v_p(\varphi) = \frac{1}{p}\sum_{0\leqslant n<p} t_p(n)\ u_f^n(\varphi).$$

We compute

$$\begin{aligned}\|\psi_p\|^2 &= \frac{1}{p^2}\sum_{\substack{0\leqslant n<p\\ 0\leqslant m<p}} t_p(n)\overline{t_p(m)}\langle u_f^n(\varphi)|u_f^m(\varphi)\rangle\\ &= \frac{1}{p^2}\sum_{|h|<p}\langle u_f^h(\varphi)|\varphi\rangle \sum_{\substack{0\leqslant n,m<p\\ n-m=h}} t_p(n)\overline{t_p(m)}.\end{aligned}$$

The contribution coming from $h=0$ is

$$\frac{\|\varphi\|^2}{p^2}\sum_{x\in\mathbf{F}_p}|t_p(x)|^2 \leqslant \mathbf{c}(\mathscr{F}_p)^2\|\varphi\|^2 p^{-1}$$

by (3.1). Now fix h with $1\leqslant |h|<p$. The corresponding summand is $\langle u_f^h(\varphi)|\varphi\rangle\sigma_h$, where

$$\sigma_h = \sum_{\substack{0\leqslant n,m<p\\ n-m=h}} t_p(n)\overline{t_p(m)} = \sum_{\max(0,h)\leqslant n<\min(p,p+h)} t_p(n)\overline{t_p(n-h)}.$$

By completion and by the properties of the additive convolution of trace functions of Fourier sheaves (see Proposition 3.5 and Proposition 3.6), we have

$$\sigma_h \ll \sqrt{p}(\log p)$$

for all $h \neq 0$, where the implied constant depends only on $\mathbf{c}(\mathscr{F}_p)$. Therefore we derive

$$\|\psi_p\|^2 \ll \|\varphi\|^2 p^{-1} + \|\varphi\|^2 p^{-1/2}(\log p)$$

where the implied constant depends only on $\mathbf{c}(\mathscr{F}_p)$. This gives the result.

We can immediately extend the mean-ergodic theorem for Fourier sheaves to L^r when $1 \leqslant r \leqslant 2$. For $r > 2$, see Section 7.

Corollary 4.3. *Suppose that $\mathscr{F}_p$ is a Fourier sheaf for all p. Let $r \in [1, 2]$. The endomorphisms*

$$\widetilde{v}_p = \frac{1}{p} \sum_{0 \leqslant n < p} t_p(n)\ u_f^n$$

of $\mathrm{L}^r(\mathrm{X}, \mu)$ converge to 0 as $p \to +\infty$ in the norm topology.

Proof. Suppose first that φ is bounded. Since $r \leqslant 2$ an μ is a probability measure, we have

$$\|\widetilde{v}_p(\varphi)\|_r^r = \int_{\mathrm{X}} \Big| \frac{1}{p} \sum_{0 \leqslant n < p} t_p(n)\, \varphi(f^n(x)) \Big|^r d\mu(x) \leqslant \Big(\int_{\mathrm{X}} \Big| \frac{1}{p} \sum_{0 \leqslant n < p} t_p(n)\, \varphi(f^n(x)) \Big|^2 d\mu(x) \Big)^{r/2},$$

hence $\|\widetilde{v}_p\| \leqslant \|v_p\|$, which tends to 0. ☐

Recall that we denote by π the ergodic projection $\mathrm{L}^1(\mathrm{X}, \mu) \to \mathrm{L}^1(\mathrm{X}, \mu)$. It restricts to the orthogonal projection on the 1-eigenspace of $\mathrm{L}^2(\mathrm{X}, \mu)$. The standard mean-ergodic theorem in L^2 implies that

$$\frac{1}{p} \sum_{0 \leqslant n < p} u_f^n \to \pi$$

in the space of endomorphisms of $\mathrm{L}^2(\mathrm{X}, \mu)$ with the topology of pointwise convergence (see, e.g., [14, Th. 2.21]).

Recall further that almost Fourier families are defined in Definition 3.7.

Corollary 4.4. *Assume that the family $(\mathscr{F}_p)$ is almost Fourier with mean $r \geqslant 0$. Then the sequence of endomorphisms (v_p) of $\mathrm{L}^2(\mathrm{X}, \mu)$ converges to $r\pi$ as $p \to +\infty$ with respect to the topology of uniform convergence on compact subsets of $\mathrm{L}^2(\mathrm{X}, \mu)$.*

Proof. The assumption implies that $t_p = r + \widetilde{t}_p$, where $\widetilde{t}_p$ is the trace function of a Fourier sheaf with conductor $\leqslant \mathbf{c}(\mathscr{F}_p)$, and we may combine Proposition 4.1, applied to $\widetilde{t}_p$, with the usual mean-ergodic theorem to derive the convergence of v_p to $r\pi$ in the topology of pointwise convergence. Moreover, since $\|v_p\| \leqslant \mathbf{c}(\mathscr{F}_p)$ for all p, the family (v_p) is equicontinuous, and hence the convergence holds in fact uniformly over compact subsets of $\mathrm{L}^2(\mathrm{X}, \mu)$ (see [3, p. 16, th. 1]). ☐

Example 4.5. Let S_p be the set of quadratic residues modulo p. Assume that f is μ-ergodic, so that the 1-eigenspace is spanned by the constant function 1 and $\pi(\varphi) = \int_{\mathrm{X}} \varphi d\mu$ for all

$\varphi \in \mathrm{L}^2(\mathrm{X}, \mu)$. We then have

$$\frac{1}{p} \sum_{\substack{0 \leqslant n < p \\ n \in \mathrm{S}_p}} \varphi \circ f^n \to \frac{1}{2} \int_{\mathrm{X}} \varphi \, d\mu$$

uniformly for $\varphi \in \mathrm{L}^2(\mathrm{X}, \mu)$ in compact subsets of $\mathrm{L}^2(\mathrm{X}, \mu)$. Indeed, the characteristic function of S_p, for $p \geqslant 3$, is

$$\frac{1}{2}(1 + \chi_p)$$

where χ_p is the Legendre character modulo p, and the latter is the trace function of a rank 1 non-trivial Kummer sheaf.

Using [20, §6.2], one can extend straightforwardly this result by replacing S_p with the set $\mathrm{S}_{q,p} = q(\mathbf{F}_p)$ of the values modulo p of a fixed polynomial $q \in \mathbf{Z}[\mathrm{X}]$ (except that the leading constant $1/2$ might be replaced by a value depending on p).

5. Weakly-mixing systems

It remains to prove Theorem 1.2 under Assumption (b). By linearity, it suffices to treat the case of trace functions of (geometrically irreducible) sheaves modulo primes $p \in \mathsf{P}$ with bounded conductor.

We keep the notation of the previous section concerning the dynamical system and the family $(\mathscr{F}_p)$ as well as the operator u_f. We write

$$\alpha_p = \frac{1}{p} \sum_{0 \leqslant n < p} t_p(n).$$

We use a suitable decomposition of the trace function t_p. We write $t_p = t_p^{\mathrm{AS}} + \widetilde{t}_p$, where t_p^{AS} is the Artin-Schreier component and $\widetilde{t}_p$ is the trace function of a Fourier sheaf $\widetilde{\mathscr{F}_p}$ with bounded conductor. Using the Riemann Hypothesis, we can express further

$$t_p^{\mathrm{AS}} = \alpha_p + \widetilde{t}_p^{\mathrm{AS}} + \mathrm{O}(p^{-1/2}),$$

for all p, where $\widetilde{t}_p^{\mathrm{AS}}$ is the trace function of an Artin-Schreier sheaf $\mathscr{A}_p$ with no trivial geometrically irreducible component and with bounded conductor, and where the implied constant depends only on $\mathbf{c}(\mathscr{F}_p)$.

Proposition 5.1. *Suppose that the system (X, μ, f) is ergodic and that the Kronecker factor of (X, μ, f) is trivial, or in other words that (X, μ, f) is weakly mixing.*

The endomorphisms

$$v_p - \alpha_p \pi = \frac{1}{p} \sum_{0 \leqslant n < p} t_p(n)\ u_f^n - \alpha_p \pi$$

of $\mathrm{L}^2(\mathrm{X}, \mu)$ converge to 0 in the topology of uniform convergence on compact subsets, and

$$\frac{1}{p} \sum_{0 \leqslant n < p} t_p(n) \varphi(f^n(x)) - \alpha_p \to 0$$

for almost all x.

Proof. Using the decomposition

$$t_p = \alpha_p + \widetilde{t}_p + \widetilde{t}_p^{\mathrm{AS}},$$

we have

$$\frac{1}{p}\sum_{0\leqslant n<p} \widetilde{t}_p(n)\ u_f^n(\varphi) \to 0$$

by Proposition 4.1 applied to the sheaves $\widetilde{\mathscr{F}_p}$, and

$$\frac{1}{p}\sum_{0\leqslant n<p} \alpha_p\ u_f^n(\varphi) - \alpha_p\pi(\varphi) \to 0$$

by the classical mean-ergodic theorem [14, Th. 2.21].

Similarly, the pointwise convergence holds almost surely for these two components by Theorem 1.2 and the classical pointwise ergodic theorem (see, e.g., [14, Th. 2.30]).

We now use the assumption that the dynamical system is weakly mixing: a result of Bourgain (the uniform Wiener–Wintner Theorem, see the proof by Assani [2, Th. 6]) then implies that

$$\frac{1}{p}\sum_{0\leqslant n<p} e(n\theta)\varphi(f^n(x)) \to 0$$

for almost all x, uniformly for $\theta \in [0,1]$. Since the trace function of $\widetilde{t}_p^{\mathrm{AS}}$ is a finite linear, combination with coefficients of size 1, of additive characters $n \mapsto e(an/p)$, it follows that

$$\frac{1}{p}\sum_{0\leqslant n<p} \widetilde{t}_p^{\mathrm{AS}}(n)\varphi(f^n(x)) \to 0$$

almost surely (although that the number of such additive characters may depend on p, this doesn't affect this argument).

This concludes the proof of the pointwise part of Theorem 1.2 for weakly mixing systems. The mean-ergodic convergence follows by the dominated convergence theorem. ☐

Besides this proof, we now give an alternative argument for the mean-ergodic theorem in this case, which does not use the uniform Wiener–Wintner Theorem. This can be skipped (we include it since these are informal notes, and the arguments were elaborated before we were aware of this result of Bourgain).

We will need the following definition to state the basic technical fact.

Definition 5.2. Let $\theta \in \mathbf{R}/\mathbf{Z}$ and let $(a_p)_p$ be a sequence of integers, indexed by an infinite set of primes. We say that a_p/p *converges emphatically* to θ if

$$\limsup_{p\to+\infty} p\Big|\frac{a_p}{p} - \theta\Big| < +\infty,$$

and if moreover no subsequence of $(p|\frac{a_p}{p} - \theta|)_p$ converges to a positive integer.

Remark 5.3. If a_p/p converges emphatically to θ, then a_p/p converges to θ. If $\theta = 0$, then one sees that the condition means that $a_p = 0$ for all but finitely many p.

For any $\theta_0 \in \mathbf{R}/\mathbf{Z}$, there is a sequence (a_p) indexed by primes such that (a_p) converges emphatically to θ_0, by taking a_p/p the closest to θ_0, so that $p|\frac{a_p}{p} - \theta_0| < 1$.

Proposition 5.4. *Let (a_p) be a sequence of integers indexed by an infinite subset of primes. Assume that a_p/p converges to θ_0 in $\mathbf{R}/\mathbf{Z}$. Let $\varphi \in \mathrm{L}^2(\mathrm{X},\mu)$ of norm 1 and define*

$$\psi_p = \frac{1}{p}\sum_{0\leqslant n<p} e\Big(-\frac{na_p}{p}\Big)\ u_f^n(\varphi).$$

(1) *Suppose that $\theta_0 \neq 0$ in $\mathbf{R}/\mathbf{Z}$. If the sequence $(\|\psi_p\|)$ converges to a non-zero number, then the sequence (a_p/p) converges emphatically to θ_0, and $e(\theta_0)$ is an eigenvalue of u_f.*

(2) *Suppose that $\theta_0 = 0$ and $p \nmid a_p$ for all p. Then $\psi_p \to 0$.*

Proof. As before, let ν be the spectral measure of the unitary operator u_f relative to the unit vector φ. We obtain

$$\|\psi_p\|^2 = \int_{\mathbf{R}/\mathbf{Z}}\Big|\frac{1}{p}\sum_{0\leqslant n<p} e\Big(n\Big(\theta-\frac{a_p}{p}\Big)\Big)\Big|^2 d\nu(\theta) = \int_{\mathbf{R}/\mathbf{Z}}\frac{1}{p}\mathrm{F}_p\Big(\theta-\frac{a_p}{p}\Big)d\nu(\theta),$$

where F_p is the Fejér kernel: $\mathrm{F}_p(0)=p$ and

$$\mathrm{F}_p(\theta) = \frac{1}{p}\Big(\frac{\sin(\pi p\theta)}{\sin(\pi\theta)}\Big)^2$$

for $\theta\neq 0$.

Recall that $0\leqslant \mathrm{F}_p\leqslant p$, so $p^{-1}|\mathrm{F}_p|\leqslant 1$. Moreover, $\mathrm{F}_p(\theta)\to 0$ uniformly on the complement of any neighborhood of 0 in $\mathbf{R}/\mathbf{Z}$. Thus, using the limit assumption $a_p/p\to\theta_0$, we have

$$\mathrm{F}_p\Big(\theta-\frac{a_p}{p}\Big)\to 0$$

as $p\to+\infty$ for any fixed $\theta\neq\theta_0$, and a fortiori we have the same limit after dividing the left-hand side by p.

We first prove (2), and thus assume that $\theta_0=0$ and $p\nmid a_p$. Then

$$\mathrm{F}_p(\theta_0-\tfrac{a_p}{p}) = \mathrm{F}_p(-\tfrac{a_p}{p}) = 0,$$

for all p, hence we obtain $\|\psi_p\|\to 0$ by the dominated convergence theorem.

Now we prove (1), and assume that $\theta_0\neq 0$ and that $\|\psi_p\|$ converges to a non-zero number. If the sequence $(p|\frac{a_p}{p}-\theta_0|)$ is unbounded, then using the assumption $\theta_0\neq 0$ and the formula defining F_p, we see that there is a subsequence of primes such that

$$\frac{1}{p}\mathrm{F}_p\Big(\theta_0-\frac{a_p}{p}\Big) \ll \frac{1}{p^2\big|\theta_0-\frac{a_p}{p}\big|^2}\to 0$$

as $p\to+\infty$. We conclude using the dominated convergence theorem that $\|\psi_p\|^2\to 0$ along this subsequence, contrary to the assumption.

Thus we have

$$\sup_{p\to+\infty} p\Big|\frac{a_p}{p}-\theta_0\Big| = \mathrm{C} < +\infty.$$

Consider any subsequence of primes where the sequence $(p|\frac{a_p}{p}-\theta_0|)_p$ converges to some real number $c\geqslant 0$. Then

$$\frac{1}{p}\mathrm{F}_p\Big(\theta_0-\frac{a_p}{p}\Big)\to\Big(\frac{\sin(\pi c)}{\pi c}\Big)^2,$$

hence, along this subsequence, the dominated convergence theorem gives

$$\lim_{p\to+\infty} \|\psi_p\|^2 = \Big(\frac{\sin(\pi c)}{\pi c}\Big)^2 \nu(\{\theta_0\}).$$

Since we assumed that the left-hand side exists and is non-zero, we conclude that θ_0 is an atom of ν. As is well-known, this implies that $e(\theta_0)$ is an eigenvalue of u_f (because it implies that the spectral projector relative to $\{e(\theta_0)\}$ is non-zero; see, e.g., [4, p. 279, Cor.]). ☐

Corollary 5.5. *Suppose that the system* (X, μ, f) *is ergodic and that the Kronecker factor of* (X, μ, f) *is trivial, or in other words that* (X, μ, f) *is weakly mixing. Let*

$$\alpha_p = \frac{1}{p} \sum_{0\leqslant n<p} t_p(n).$$

Then the endomorphisms

$$v_p - \alpha_p\pi = \frac{1}{p} \sum_{0\leqslant n<p} t_p(n)\ u_f^n - \alpha_p\pi$$

of $\mathrm{L}^2(\mathrm{X}, \mu)$ *converge to* 0 *in the topology of uniform convergence on compact subsets.*

Proof. Since $|\alpha_p| \leqslant \mathbf{c}(\mathscr{F}_p)$, the family of endomorphisms $v_p - \alpha_p\pi$ is equicontinuous, hence it suffices to prove pointwise convergence to 0 for all $\varphi \in \mathrm{L}^2(\mathrm{X}, \mu)$. We may further assume that φ has norm 1.

We write $t_p = t_p^{\mathrm{AS}} + \widetilde{t}_p$, where t_p^{AS} is the Artin-Schreier component and $\widetilde{t}_p$ is the trace function of a Fourier sheaf $\widetilde{\mathscr{F}_p}$ with bounded conductor. Using the Riemann Hypothesis, we can express further

$$t_p^{\mathrm{AS}} = \alpha_p + \widetilde{t}_p^{\mathrm{AS}} + \mathrm{O}(p^{-1/2}),$$

for all p, where $\widetilde{t}_p^{\mathrm{AS}}$ is the trace function of an Artin-Schreier sheaf $\mathscr{A}_p$ with no trivial geometrically irreducible component and with bounded conductor, and where the implied constant depends only on $\mathbf{c}(\mathscr{F}_p)$. Then we have

$$\frac{1}{p} \sum_{0\leqslant n<p} \widetilde{t}_p(n)\ u_f^n(\varphi) \to 0$$

by Proposition 4.1 applied to the sheaves $\widetilde{\mathscr{F}_p}$, and

$$\frac{1}{p} \sum_{0\leqslant n<p} \alpha_p\ u_f^n(\varphi) - \alpha_p\pi(\varphi) \to 0$$

by the classical mean-ergodic theorem [14, Th. 2.21].

We are now done unless $\mathscr{A}_p$ has rank $\geqslant 1$ for an infinite sequence of primes. We now assume this and consider only such primes. Let

$$\psi_p = \frac{1}{p} \sum_{0\leqslant n<p} \widetilde{t}_p^{\mathrm{AS}}(n)\ u_f^n(\varphi).$$

The sequence $(\|\psi_p\|)_p$ is bounded by the maximum of the ranks of the sheaves $\mathscr{A}_p$. Let $c \geqslant 0$ be a limiting value, obtained for a subsequence of primes which we omit from the notation. Assume that $c>0$. By passing to a further subsequence, we may assume that the rank of $\mathscr{A}_p$ is a constant $r\geqslant 1$. We have geometric isomorphisms

$$\mathscr{A}_p \simeq \bigoplus_{j=1}^{r} \mathscr{L}_{\psi(-a_{p,j}x)}$$

for some integers $0<a_{p,j}<p$. There must exist some fixed j such that the norm of

$$\widetilde{\psi}_p = \frac{1}{p}\sum_{0\leqslant n<p} e\Bigl(-\frac{a_{p,j}n}{p}\Bigr)\, u_f^n(\varphi)$$

does not converge to 0, as otherwise we would obtain $c=0$. We may then assume, again by passing to a subsequence, that $-a_{p,j}/p$ converges to some $\theta_0\in\mathbf{R}/\mathbf{Z}$. Since $\widetilde{\psi}_p$ does not converge to 0, we have $\theta_0\neq 0$ by Proposition 5.4, (2).

Now, by definition (see [14, Th. 2.36 or §6.4]), the assumption on (X,μ,f) means that u_f has no eigenvalue different from 1 (and that 1 is an eigenvalue of multiplicity one). We have then a contradiction to Proposition 5.4, (1). This means that all limit points of the bounded sequence $(\|\psi_p\|)_p$ are equal to 0, hence it converges to 0. ☐

Using linearity, this corollary implies Theorem 1.2, (1) under Assumption (b). It remains to deal with the pointwise ergodic theorem in this case. (TODO)

Example 5.6. Examples of weakly mixing systems (X,μ,f) are Bernoulli shifts, ergodic automorphisms of compact abelian groups (e.g., elements of $\mathrm{SL}_d(\mathbf{Z})$ acting on $(\mathbf{R}/\mathbf{Z})^d$ which have no root of unity as an eigenvalue) or the Gauss map in the theory of continued fractions [34].

Another important class arises in homogeneous dynamics. Let G be a locally compact group, Γ a lattice in G and consider the action of G on $\mathrm{X}=\Gamma\backslash\mathrm{G}$. Denote by μ_{X} the G-invariant probability measure on X. Assume that the action is mixing [14, § 8.1]. Let $x\in\mathrm{G}$ be such that $x^n\to+\infty$ in G as $n\to+\infty$. Then defining $f(\Gamma y)=\Gamma yx$, we obtain a system $(\mathrm{X},\mu_{\mathrm{X}},f)$ that is mixing by definition, hence weakly mixing. This applies for instance to $\mathrm{G}=\mathrm{SL}_2(\mathbf{R})$ and x a non-trivial unipotent element.

6. The topological case

In this section, we prove Theorem 1.3. Thus let X be a compact topological space and $f\colon \mathrm{X}\to\mathrm{X}$ a continuous map, such that the topological entropy $h(f)$ is zero (see, e.g, [13, §4] for an introduction to topological entropy). Let $\varphi\colon\mathrm{X}\to\mathbf{C}$ be continuous and $x\in\mathrm{X}$. The goal is to find conditions on a family $(\mathscr{F}_p)$ of sheaves modulo p with bounded conductor which imply that

$$\lim_{p\to+\infty}\frac{1}{p}\sum_{0\leqslant n<p} t_p(n)\varphi(f^n(x))=0,$$

with no exceptions or sparseness assumption. The claim of Theorem 1.3 is that this is the case when the family consists of Kloosterman sheaves or Kummer sheaves associated to real characters, for which $t_p(n)=\mathrm{Kl}_2(n;p)$ or $t_p(n)=(\frac{n}{p})$, respectively.

The proof is in fact a straightforward adaptation of the combinatorial argument that shows that decay of multiple correlations of the Möbius function (what is called the Chowla conjecture) implies Sarnak's conjecture, as presented e.g. on Tao's blog [38], and extends to a certain class of sheaves introduced in [23] under the name of "bountiful sheaves" ([23, Def. 1.2]). For a clearer perspective, we make the following definition:

Definition 6.1. Let $(\mathscr{F}_p)$ be a family of sheaves modulo p with bounded conductor. We say that it has *positive monodromy-entropy* if for any integers $k \geqslant 1$ and $\mathrm{H} \geqslant 1$, the number $\mathrm{N}_p(k, \mathrm{H})$ of tuples of non-negative integers $(h_1, \dots, h_k, h'_1, \dots, h'_k)$ with h_i, $h_j \leqslant \mathrm{H}$ such that

$$\bigotimes_{i=1}^{k} [+h_i]^* \mathscr{F}_p \otimes \bigotimes_{i=1}^{k} [+h'_i]^* \,\mathrm{D}(\mathscr{F}_p)$$

contains a geometrically trivial irreducible component satisfies

$$\mathrm{N}_p(k, \mathrm{H}) \ll (2k)^k \mathrm{H}^k.$$

A key point in this definition is that the number $\mathrm{N}_p(k, \mathrm{H})$ is bounded independently of p, but it is also important that the exponent of H is no larger than k.

Here is our general statement:

Proposition 6.2. *Let $(\mathscr{F}_p)_p$ be a family of sheaves modulo p with positive monodromy-entropy and bounded conductor.*

Let X be a locally compact topological space and $f\colon \mathrm{X} \to \mathrm{X}$ a continuous map. Assume that either X is compact or that X is a metric space and f uniformly continuous.

Assume that the topological entropy of f is zero. Then for all bounded[4] *continuous functions $\varphi\colon \mathrm{X} \to \mathbf{C}$ and all $x \in \mathrm{X}$, we have*

$$\lim_{p \to +\infty} \frac{1}{p} \sum_{0 \leqslant n < p} t_p(n) \varphi(f^n(x)) = 0. \tag{6.1}$$

This implies Theorem 1.3 in view of the following lemma:

Lemma 6.3. (1) *If $(\mathscr{F}_p)_p$ is a family of bountiful sheaves, then it has positive monodromy-entropy.*

(2) *If $(\mathscr{F}_p)_p$ is a family such that $\mathscr{F}_p$ is a non-trivial Kummer sheaf for all p, then it has positive monodromy-entropy.*

Proof. In case (1), this follows immediately from [23, Def. 1.2,Th. 1.5] and elementary combinatorics, taking into account the definitions of normal and r-normal tuples (see [23, Def. 1.3]).

In case (2), if $\mathscr{F}_p = \mathscr{L}_\chi$, where χ has order $d \mid p-1$, with $d \geqslant 2$, then note that

$$\bigotimes_{i=1}^{k} [+h_i]^* \mathscr{F}_p \otimes \bigotimes_{i=1}^{k} [+h'_i]^* \,\mathrm{D}(\mathscr{F}_p) = \mathscr{L}_{\chi(\mathrm{G}/\mathrm{H})},$$

where G and H are the polynomials

$$\mathrm{G} = \prod_{i=1}^{k} (\mathrm{X} + h_i), \qquad \mathrm{H} = \prod_{j=1}^{k} (\mathrm{X} + h'_j).$$

This contains a geometrically trivial component if and only if G/H is a d-th power of a rational function. The bound on $\mathrm{N}_p(k, \mathrm{H})$ is therefore clear (the worse case is when $d = 2$, and then the estimate is the same as that for normal tuples, as in [23, Def. 1.5, (1)]). ∎

[4] Check

We now prove Proposition 6.2, following closely [38]. The next statement, which provides the analogue of decay of multiple correlations of the Möbius function, could also be derived from the work of Perret-Gentil [32] in most cases of interest.

Proposition 6.4. *Let $(\mathscr{F}_p)_p$ be a family of sheaves modulo p with positive monodromy-entropy and bounded conductor. Let $(\alpha_n)_{n\geqslant 0}$ be a sequence of complex numbers bounded by* 1.

Fix an integer $m \geqslant 1$. There exists a absolute constant $\mathrm{C} > 0$ such that, for any $\varepsilon > 0$, we have

$$\frac{1}{p}\Big|\{0 \leqslant n < p \mid \Big|\sum_{0\leqslant i<m} t_p(n+i)\alpha_i\Big| \geqslant \varepsilon m\}\Big| \leqslant \mathrm{C}\exp\Big(-\frac{\varepsilon^2 m}{\mathrm{C}}\Big) + \mathrm{O}(\varepsilon^{-\varepsilon^2 m} p^{-1/2}),$$

where the implied constant depends only on the conductor of $(\mathscr{F}_p)$.

Proof. Let $k \geqslant 1$ be an integer to be chosen later. We have

$$\frac{1}{p}|\{0 \leqslant n < p \mid \Big|\sum_{0\leqslant i<m} t_p(n+i)\alpha_i\Big| \geqslant \varepsilon m\}| \leqslant \frac{1}{(\varepsilon m)^{2k}}\frac{1}{p}\sum_{0\leqslant n<p}\Big|\sum_{0\leqslant i<m} t_p(n+i)\alpha_i\Big|^{2k}.$$

Since $|\alpha_i| \leqslant 1$, if we expand the right-hand side, we obtain the upper bound

$$\frac{1}{(\varepsilon m)^{2k}}\sum_{\substack{0\leqslant a_1,\dots,a_k<m\\ 0\leqslant b_1,\dots,b_k<m}}\sum \frac{1}{p}\sum_{0\leqslant n<p} t_p(n+a_1)\cdots t_p(n+a_k)\overline{t_p(n+b_1)\cdots t_p(n+b_k)}.$$

Because of the monodromy-entropy assumption and the Riemann Hypothesis (see [23, Prop. 1.1]), the inner sum is $\ll p^{-1/2}$, with implied constant depending only on k and $\mathbf{c}(\mathscr{F}_p)$, for all but $\leqslant (2k)^k m^k$ tuples (a_i, b_j).

It follows that

$$\frac{1}{p}|\{0 \leqslant n < p \mid \Big|\sum_{0\leqslant i<m} t_p(n+i)\alpha_i\Big| \geqslant \varepsilon m\}| \leqslant \frac{1}{(\varepsilon m)^{2k}}\Big((2k)^k m^k + \mathrm{O}\Big(\frac{m^{2k}}{\sqrt{p}}\Big)\Big)$$

where the implied constant depends only on k and $\mathbf{c}(\mathscr{F}_p)$. Taking k to be the closest integer $\leqslant \varepsilon^2 m/10$, the result follows. ∎

The crucial feature of this estimate is the fact that the first term decays exponentially with respect to m. We sketch the argument for completeness. We may assume that φ is real-valued and bounded by 1. Let $0 < \varepsilon < 1$ be fixed, and let $(\varphi_\varepsilon(n))$ be a sequence with values in $\mathbf{Z}\varepsilon$, such that $|\varphi_\varepsilon(n)| \leqslant 1$ and $|\varphi(f^n(x)) - \varphi_\varepsilon(n)| \leqslant \varepsilon$ for all $n \geqslant 0$. Fix an integer $m \geqslant 1$. Define $\kappa_\varepsilon(m)$ so that the tuples

$$(\varphi_\varepsilon(n), \dots, \varphi_\varepsilon(n+m-1)) \in (\mathbf{Z}\varepsilon \cap [-1,1])^m \tag{6.2}$$

take $\exp(\kappa_\varepsilon(m))$ values as n ranges over the non-negative integers. The fact that the topological entropy of f is zero (i.e, the sequence $(\varphi(f^n(x)))_n$ is deterministic) implies that

$$\lim_{m\to+\infty}\frac{\kappa_\varepsilon(m)}{m} = 0.$$

Let p be a large prime. For any tuple (6.2), say $(\alpha_0,\ldots,\alpha_{m-1})$, Proposition 6.4 shows that we have

$$\frac{1}{p}\Big|\{0\leqslant n<p\ \mid\ \Big|\sum_{0\leqslant i<m}t_p(n+i)\alpha_i\Big|\geqslant \varepsilon m\}\Big|\leqslant \mathrm{C}\exp\Big(-\frac{\varepsilon^2 m}{\mathrm{C}}\Big)+\mathrm{O}(\varepsilon^{-\varepsilon^2 m}p^{-1/2}),$$

and hence

$$\frac{1}{p}\Big|\{0\leqslant n<p\ \mid\ \Big|\sum_{0\leqslant i<m}t_p(n+i)\varphi_\varepsilon(n+i)\Big|\geqslant \varepsilon m\}\Big|$$
$$\leqslant \mathrm{C}\exp\Big(-\frac{\varepsilon^2 m}{\mathrm{C}}+\kappa_\varepsilon(m)\Big)+\mathrm{O}(\varepsilon^{-\varepsilon^2 m}e^{\kappa_\varepsilon(m)}p^{-1/2}).$$

Since $\kappa_\varepsilon(m)/m\to 0$, we may take m large enough (depending on ε) so that this implies

$$\frac{1}{p}\Big|\{0\leqslant n<p\ \mid\ \Big|\sum_{0\leqslant i<m}t_p(n+i)\varphi_\varepsilon(n+i)\Big|\geqslant \varepsilon m\}\Big|\leqslant \varepsilon+o(1)$$

as $p\to+\infty$. But then we deduce that

$$\Big|\frac{1}{p}\sum_{0\leqslant n<p}\frac{1}{m}\sum_{0\leqslant i<m}t_p(n+i)\varphi_\varepsilon(n+i)\Big|\leqslant 2\varepsilon+o(1)$$

because $|\varphi_\varepsilon|\leqslant 1$ (write the average as the sum of a term where it is $>\varepsilon$, handled by the above inequality, and one where it is $\leqslant\varepsilon$, which has a contribution $\leqslant\varepsilon$).

Now notice that for $0\leqslant i<m$, we have

$$\frac{1}{p}\sum_{0\leqslant n<p}t_p(n+i)\varphi_\varepsilon(n+i)=\frac{1}{p}\sum_{0\leqslant n<p}t_p(n)\varphi_\varepsilon(n)+\mathrm{O}\Big(\frac{m\,\mathbf{c}(\mathscr{F}_p)}{p}\Big)$$

with an absolute implied constant, so we get

$$\Big|\frac{1}{p}\sum_{0\leqslant n<p}t_p(n)\varphi_\varepsilon(n)\Big|\leqslant 2\varepsilon+o(1),$$

hence

$$\Big|\frac{1}{p}\sum_{0\leqslant n<p}t_p(n)\varphi(f^n(x))\Big|\leqslant 3\varepsilon+o(1).$$

The limit (6.1) follows.

7. Mean-ergodic theorems in L^r

This section may be skipped in a first reading. Our goal is to extend the mean-ergodic theorem to the spaces $\mathrm{L}^r(\mathrm{X},\mu)$ when $r>2$. We will achieve this goal, however, only for sheaves satisfying an extra condition.

Proposition 7.1. *Suppose that $(\mathscr{F}_p)$ is a family of sheaves with positive monodromy-entropy. Let $r>2$ be fixed. The endomorphisms*

$$\widetilde{v}_p=\frac{1}{p}\sum_{0\leqslant n<p}t_p(n)\ u_f^n$$

of $\mathrm{L}^r(\mathrm{X},\mu)$ converge to 0 as $p\to+\infty$.

Proof. Using monotonicity, as in Corollary 4.3, it is enough to prove this when $r = 2k$ for some integer $k \geqslant 2$ to deduce it for $r \leqslant 2k$.

Let $\varphi \in \mathrm{L}^{2k}(\mathrm{X}, \mu)$ and denote $\psi_p = \widetilde{v}_p(\varphi)$. We have

$$\|\psi_p\|_{2k}^{2k} = \frac{1}{p^{2k}} \sum_{\substack{n_1,\dots,n_k \\ 0\leqslant n_i<p}} \sum_{\substack{m_1,\dots,m_k \\ 0\leqslant m_j<p}} t_p(n_1)\cdots t_p(n_k)\overline{t_p(m_1)\cdots t_p(m_k)} \langle u_f^{n_1}(\varphi)\cdots u_f^{n_k}(\varphi), u_f^{m_1}(\varphi)\cdots u_f^{m_k}(\varphi)\rangle.$$

Since u_f is isometric, we have

$$\langle u_f^{n_1}(\varphi)\cdots u_f^{n_k}(\varphi), u_f^{m_1}(\varphi)\cdots u_f^{m_k}(\varphi)\rangle = \langle \varphi\cdots u_f^{n_k-n_1}(\varphi), u_f^{m_1-n_1}(\varphi)\cdots u_f^{m_k-n_1}(\varphi)\rangle.$$

Hence, we may sum over $h = n_1$ first, obtaining

$$\|\psi_p\|_{2k}^{2k} = \frac{1}{p^{2k}} \sum_{n_2,\dots,n_k} \sum_{m_1,\dots,m_k} \langle \varphi\ u_f^{n_2}(\varphi)\cdots u_f^{n_k}(\varphi), u_f^{m_1}(\varphi)\cdots u_f^{m_k}(\varphi)\rangle \sum_h t_p(h)t_p(h+n_1)\cdots t_p(h+n_k)\overline{t_p(h+m_1)\cdots t_p(h+m_k)},$$

where the sum is over integers $0 \leqslant h < p$ such that

$$0 \leqslant h + n_i < p, \qquad 0 \leqslant h + m_j < p$$

for $2 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant k$, respectively. This is a sum over an interval of length $< p$. The assumption on $\mathscr{F}_p$ then implies that

$$\sum_h t_p(h)t_p(h+n_1)\cdots t_p(h+n_k)\overline{t_p(h+m_1)\cdots t_p(h+m_k)} \ll p^{1/2}(\log p),$$

where the implied constant depends only on $\mathbf{c}(\mathscr{F}_p)$ and k, unless $(0, n_2, \dots, n_k)$ is a permutation of $(m_1, \dots, m_k)$ (see [23, Th. 1.5]). This occurs for $\ll p^{k-1}$ tuples $(n_2, \dots, m_k)$, and for these we have a bound $\ll p$ for the sum, where the implied constant depends only on $\mathbf{c}(\mathscr{F}_p)$ and k. Thus we derive

$$\|\psi_p\|_{2k}^{2k} \ll p^{-1/2}(\log p) + p^{-k-1}.$$

This shows that $\|\widetilde{v}_p\| \to 0$ in the space of endomorphisms of $\mathrm{L}^{2k}(\mathrm{X}, \mu)$, and concludes the proof. $\square$

Remark 7.2. Analyzing the proof of the proposition further, we can reach a stronger conclusion and indeed derive a slightly stronger pointwise statement than the one in Theorem 1.2, although under assumptions that are reasonable in principle, but difficult to check.

We take the case $k = 2$ of the proposition, and rewrite the first steps above: for $\varphi \in \mathrm{L}^4(\mathrm{X}, \mu)$, we have

$$\|\psi_p\|_4^4 = \frac{1}{p^4} \sum_b \sum_{c,d} \langle \varphi\ u_f^b(\varphi), u_f^c(\varphi)u_f^d(\varphi)\rangle \sum_a t_p(a)t_p(a+b)\overline{t_p(a+c)t_p(a+d)}.$$

We rewrite the sum in the form

$$\|\psi_p\|_4^4 = \frac{1}{p^{7/2}} \sum_{c,d} \langle \varphi\ u_f^b(\varphi), u_f^c(\varphi)u_f^d(\varphi)\rangle \tau_{c,d}(b)$$

where

$$\tau_{c,d}(b) = \frac{1}{\sqrt{p}} \sum_a t_p(a) t_p(a+b) \overline{t_p(a+c) t_p(a+d)},$$

hence

$$\|\psi_p\|_4^4 = \frac{1}{p^{5/2}} \sum_{c,d} \langle w_{p,c,d}(\varphi), \bar{\varphi}\ u_f^c(\varphi) u_f^d(\varphi)\rangle$$

where

$$w_{p,c,d}(\varphi) = \frac{1}{p} \sum_{0\leqslant b<p} \tau_{c,d}(b)\, \varphi \circ f^b.$$

Now assume that $\varphi \in \mathrm{L}^6(\mathrm{X},\mu)$, which implies that $\bar{\varphi}\ u_f^c(\varphi)u_f^d(\varphi)$ belongs to $\mathrm{L}^2(\mathrm{X},\mu)$ and has norm $\ll 1$. Assume moreover that the family $(\mathscr{F}_p)$ satisfies the condition that *for most* (c,d)*, with* $\ll p$ *exceptions, the function* $\tau_{c,d}$ *is a trace function of a Fourier sheaf, with weights* $\leqslant 0$. By Quantitative Sheaf Theory (see [36, Th. 1.1, Cor. 7.4]), the conductor of $\tau_{c,d}$ is $\ll 1$.

Under these conditions, by Proposition 4.1, we obtain

$$\|w_{p,c,d}(\varphi)\|_2^2 \ll p^{-1/2} \log p,$$

for most (c,d), and hence conclude that

$$\|\psi_p\|_4^4 \ll p^{-1} \log p.$$

If we assume that the family $(\mathscr{F}_p)$ is indexed by a set of primes P such that

$$\sum_{p\in\mathsf{P}} \frac{\log p}{p} < +\infty,$$

then this result means that

$$\sum_p \|\psi_p\|_4^4 < +\infty,$$

or in other words that the non-negative function

$$\sum_p |\psi_p|^4$$

is integrable on X. This imples that $\psi_p = \widetilde{v}_p(\varphi)$ converges almost everywhere to 0, a pointwise theorem. This is a bit stronger than the pointwise part of Theorem 1.2, but the latter does not require any extra condition, and hence we do not pursue the verification that the assumption above holds in reasonable situations.

8. Maximal inequalities in L^2

We now consider maximal inequalities in L^2, i.e., we endeavor to estimate functions like

$$\mathrm{M}\varphi\colon x \mapsto \sup_p \Big|\frac{1}{p} \sum_{0\leqslant n<p} t_p(n)\ \varphi(f^n(x))\Big|$$

in L^2-norm, where we have fixed the dynamical system (X, μ, f) and the family of sheaves $(\mathscr{F}_p)$ with bounded conductor, and with trace functions t_p. In fact, we will need to restrict the supremum to sparse subsets of the primes, and so we use the notation

$$\mathrm{M}_{\mathsf{P}}(\varphi)(x) = \sup_{p \in \mathsf{P}} \Big| \frac{1}{p} \sum_{0 \leqslant n < p} t_p(n)\ \varphi(f^n(x)) \Big|$$

for any set P of primes. We write

$$s(\mathsf{P}) = \sum_{p \in \mathsf{P}} \frac{(\log p)^2}{p},$$

which is finite if and only if P is sparse.

Proposition 8.1. *Suppose that $(\mathscr{F}_p)_p$ is an almost Fourier family* (Definition 3.7) *with mean $r \geqslant 0$. Suppose further that P is sparse. Let $\varphi \in \mathrm{L}^2(\mathrm{X}, \mu)$. We have*

$$\|\mathrm{M}_{\mathsf{P}} \varphi\|_2 \leqslant \mathrm{C}_2 \|\varphi\|_2$$

for some constant C_2 depending only on the conductor of $(\mathscr{F}_p)$ and on $s(\mathsf{P})$.

The method that we use is a direct adaptation of that of Bourgain [5, §2, §3] (it is in fact much simpler). In the remainder of this section, we fix the sparse set P, and we will omit it from the notation unless it is required for context.

The first step is to transfer the problem to $\mathbf{Z}$. For any bounded function ϖ on $\mathbf{Z}$, we define $\widetilde{\mathrm{M}}(\varpi) \colon \mathbf{Z} \to \mathbf{C}$ by

$$\widetilde{\mathrm{M}}(\varpi)(k) = \sup_{p \in \mathsf{P}} \Big| \frac{1}{p} \sum_{0 \leqslant n < p} t_p(n)\ \varpi(k+n) \Big|.$$

Lemma 8.2. *Suppose that there exists $\mathrm{C}_3 \geqslant 0$, depending only on the conductor of $(\mathscr{F}_p)$ and on $s(\mathsf{F})$, such that*

$$\|\widetilde{\mathrm{M}} \varpi\|_2 \leqslant \mathrm{C}_3 \|\varpi\|_2$$

for all ϖ bounded on $\mathbf{Z}$. Then Proposition 8.1 holds with $\mathrm{C}_2 = \mathrm{C}_3$.

Proof. We use the classical method of transfer to $\mathbf{Z}$. It suffices to prove that for all $\mathrm{P} \geqslant 2$ and all $\varphi \in \mathrm{L}^\infty(\mathrm{X}, \mu)$, we have

$$\|\mathrm{M}^{\mathrm{P}} \varphi\|_2 \leqslant 2\mathrm{C}_3 \|\varphi\|_2$$

where

$$\mathrm{M}^{\mathrm{P}}(\varphi) = \sup_{p \leqslant \mathrm{P}} \Big| \frac{1}{p} \sum_{0 \leqslant n < p} t_p(n)\ (\varphi \circ f^n) \Big| \in \mathrm{L}^2(\mathrm{X}, \mu).$$

Fix such a P and φ bounded and measurable on X. Let $\lambda > 1$ be a parameter and $\mathrm{Q} = \lambda \mathrm{P}$. Let $x \in \mathrm{X}$. Define $\widetilde{\varphi} \colon \mathbf{Z} \to \mathbf{C}$ by

$$\widetilde{\varphi}(n) = \begin{cases} \varphi(f^n(x)) & \text{if } 0 \leqslant n < \mathrm{Q} \\ 0 & \text{otherwise.} \end{cases}$$

Note that for any prime $p \leqslant \mathrm{P}$ and n, k such that $0 \leqslant n+k < \mathrm{Q}$, we have

$$\widetilde{\varphi}(n+k) = \varphi(f^{n+k}(x)) = \varphi(f^n(f^k(x)))$$

so that for $0 \leqslant k < \mathrm{Q} - \mathrm{P}$, we get

$$\begin{aligned} \mathrm{M}^{\mathrm{P}}(\varphi)(f^k(x)) &= \sup_{p \leqslant \mathrm{P}} \Big| \frac{1}{p} \sum_{0 \leqslant n < p} t_p(n)\ (\varphi(f^n(f^k(x)))) \Big| \\ &= \sup_{p \leqslant \mathrm{P}} \Big| \frac{1}{p} \sum_{0 \leqslant n < p} t_p(n)\ \widetilde{\varphi}(k+n) \Big| = \widetilde{\mathrm{M}}^{\mathrm{P}}(\widetilde{\varphi})(k), \end{aligned} \tag{8.1}$$

say. By assumption, we have $\|\widetilde{\mathrm{M}}^{\mathrm{P}}(\widetilde{\varphi})\|_2 \leqslant \|\widetilde{\mathrm{M}}(\widetilde{\varphi})\|_2 \leqslant \mathrm{C}_3 \|\widetilde{\varphi}\|_2$. This means that

$$\sum_{k \in \mathbf{Z}} |\widetilde{\mathrm{M}}^{\mathrm{P}}(\widetilde{\varphi})(k)|^2 \leqslant \mathrm{C}_3^2 \sum_{n \in \mathbf{Z}} |\widetilde{\varphi}(n)|^2 = \mathrm{C}_3^2 \sum_{0 \leqslant n < \mathrm{Q}} |\varphi(f^n(x))|^2,$$

hence by (8.1), we obtain

$$\sum_{0 \leqslant k < \mathrm{Q}-\mathrm{P}} |\mathrm{M}^{\mathrm{P}}(\varphi)(f^k(x))|^2 \leqslant \mathrm{C}_3^2 \sum_{0 \leqslant n < \mathrm{Q}} |\varphi(f^n(x))|^2.$$

This inequality is valid for all $x \in \mathrm{X}$. After integrating over X, we get

$$\sum_{0 \leqslant k < \mathrm{Q}-\mathrm{P}} \|\mathrm{M}^{\mathrm{P}}(\varphi) \circ f^k\|_2^2 \leqslant \mathrm{C}_3^2 \sum_{0 \leqslant n < \mathrm{Q}} \|\varphi \circ f^n\|_2^2.$$

But μ is f-invariant, and therefore both sums are sums of equal terms, which means that

$$(\lambda - 1)\mathrm{P}\|\mathrm{M}^{\mathrm{P}}(\varphi)\|^2 \leqslant \mathrm{C}_3^2 \lambda \mathrm{P} \|\varphi\|^2.$$

The result follows by taking $\lambda \to +\infty$. ∎

Proof of Proposition 8.1. We will prove Lemma 8.2. Since $(\mathscr{F}_p)_p$ is an almost Fourier family of mean r, we have

$$t_p(n) = r + \tau_p(n)$$

where τ_p is the trace function of Fourier sheaves with bounded conductor.

Let ϖ be a function on $\mathbf{Z}$ with finite support. We denote by

$$\widehat{\varpi}(\theta) = \sum_{k \in \mathbf{Z}} \varpi(k) e(-k\theta),$$

and

$$\widehat{v}_p(\theta) = \frac{1}{p} \sum_{0 \leqslant n < p} \tau_p(n) e(n\theta)$$

the Fourier transforms on $\mathbf{R}/\mathbf{Z}$ of the function ϖ and of the discrete measures corresponding to the average $\tau_p(n)$. We have

$$\frac{1}{p} \sum_{0 \leqslant n < p} \tau_p(n) \varpi(n+k) = \int_{\mathbf{R}/\mathbf{Z}} \widehat{\varpi}(\theta) \widehat{v}_p(\theta) e(k\theta) d\theta \tag{8.2}$$

for all $k \in \mathbf{Z}$.

For any $k \in \mathbf{Z}$, we have

$$\sup_p \frac{1}{p} \Big| \sum_{0 \leqslant n < p} t_p(n) \varpi(n+k) \Big| \leqslant \sup_p \frac{1}{p} \Big| \sum_{0 \leqslant n < p} \varpi(n+k) \Big| + \Big(\sum_p \Big| \frac{1}{p} \sum_{0 \leqslant n < p} \tau_p(n) \varpi(n+k) \Big|^2 \Big)^{1/2}$$

(where p always ranges over P). Hence

$$\|\widetilde{\mathrm{M}}(\varpi)\|_2^2 = \sum_{k\in\mathbf{Z}} \sup_p \frac{1}{p}\Big|\sum_{0\leqslant n<p} t_p(n)\varpi(n+k)\Big|^2$$
$$\leqslant 2\sum_{k\in\mathbf{Z}} \sup_p \frac{1}{p}\Big|\sum_{0\leqslant n<p} \varpi(n+k)\Big|^2 + 2\sum_{k\in\mathbf{Z}}\sum_p \Big|\frac{1}{p}\sum_{0\leqslant n<p} \tau_p(n)\varpi(n+k)\Big|^2.$$

The first expression is $\leqslant \mathrm{C}_3'\|\varpi\|_2^2$ by the classical maximal ergodic theorem in L^2 for functions on $\mathbf{Z}$ (see [14, § 2.6]). By (8.2) and the Plancherel formula, we estimate the second one as follows:

$$\sum_{k\in\mathbf{Z}}\sum_p \Big|\frac{1}{p}\sum_{0\leqslant n<p} \tau_p(n)\varpi(n+k)\Big|^2 = \sum_{k\in\mathbf{Z}}\sum_p \Big|\int_{\mathbf{R}/\mathbf{Z}} \widehat{\varpi}(\theta)\widehat{v}_p(\theta)e(k\theta)d\theta\Big|^2$$
$$= \sum_p \int_{\mathbf{R}/\mathbf{Z}} |\widehat{\varpi}(\theta)\widehat{v}_p(\theta)|^2 d\theta \leqslant \Big(\sum_p \|\widehat{v}_p(\theta)\|_\infty^2\Big)\|\varpi\|_2^2.$$

Applying Corollary 3.10, we have

$$\sum_p \|\widehat{v}_p(\theta)\|_\infty^2 \ll \sum_{p\in\mathsf{P}} \frac{(\log p)^2}{p} = s(\mathsf{P}),$$

where the implied constant depends only on the conductor of $(\mathscr{F}_p)$, and the result follows. □

9. Pointwise ergodic theorem

We give in this section a second proof of Theorem 1.2, (2), arguing using a transfer principle as in the previous section. This is obviously more complicated than our first proof, but it is interesting that the sparseness condition turns out to be the same in both arguments.

We consider a dynamical system (X, μ, f) and a family of sheaves $(\mathscr{F}_p)$ with bounded conductor as in the previous section, with trace functions t_p, defined for p in a sparse set P. We assume that the family is almost Fourier (Definition 3.7) of mean $r \geqslant 0$.

Proposition 9.1. *Let $\varphi \in \mathrm{L}^2(\mathrm{X}, \mu)$. Then*

$$\frac{1}{p}\sum_{0\leqslant n<p} t_p(n)\ \varphi(f^n(x))$$

converges for μ-almost all $x \in \mathrm{X}$. If $r = 0$, or in other words, if all sheaves $\mathscr{F}_p$ are Fourier sheaves, or if (X, μ, f) is weakly mixing, then the limit is zero.

For the proof, we reduce to the shift by means of an intermediate inequality. For a function ϖ on $\mathbf{Z}$, we write as before

$$u_p(\varpi)(k) = \frac{1}{p}\sum_{0\leqslant n<p} t_p(n)\varpi(n+k).$$

Lemma 9.2. *Assume that for any infinite subset $\mathsf{Q} \subset \mathsf{P}$, there exists a constant C_4 such that, for any function ϖ on $\mathbf{Z}$ with bounded support, we have*

$$\sum_{\ell\in\mathsf{Q}} \Big\|\sup_{\ell<p<\ell^+} |u_p(\varpi) - u_{\ell^+}(\varpi)|\Big\|_2^2 \leqslant \mathrm{C}_4\|\varpi\|^2,$$

where ℓ^+ is the element following ℓ in the subset Q, and p ranges over elements in P. Then Proposition 9.1 holds.

Proof. This has two steps. First, in the same manner that Lemma 8.2 is proved, the statement, if it holds, implies the corresponding bound

$$\sum_{\ell\in\mathsf{Q}}\Big\|\sup_{\ell<p<\ell^+}|u_p(\varphi)-u_{\ell^+}(\varphi)|\Big\|_2^2\leqslant \mathrm{C}_4\|\varphi\|^2,$$

for any $\varphi\in\mathrm{L}^2(\mathrm{X},\mu)$, for any infinite subset Q.

Next, one argues by contradiction that this last set of bounds, for a given φ, implies that $u_p(\varphi)$ converges μ-almost everywhere. ∎

Finally, we prove the auxiliary bounds.

Proposition 9.3. *Let $\mathsf{Q}\subset\mathsf{P}$ be an infinite subset. There exists a constant C_4 such that, for any function ϖ on $\mathbf{Z}$ with bounded support, we have*

$$\sum_{\ell\in\mathsf{Q}}\Big\|\sup_{\ell<p<\ell^+}|u_p(\varpi)-u_{\ell^+}(\varpi)|\Big\|_2^2\leqslant \mathrm{C}_4\|\varpi\|^2.$$

Proof. Writing $t_p(n)=r+\tau_p(n)$, where $\tau_p(n)$ is the trace function of a Fourier sheaf of bounded conductor, and applying the known behavior from the standard pointwise ergodic theory to the first term, we are reduced to showing that

$$\sum_{\ell\in\mathsf{Q}}\Big\|\sup_{\ell<p<\ell^+}|\nu_p(\varpi)-\nu_{\ell^+}(\varpi)|\Big\|_2^2\leqslant \mathrm{C}_5\|\varpi\|^2.$$

for some constant C_5, where ν_p is the averaging operator for the trace function τ_p. The left-hand side of the inequality is equal to

$$\sum_{\ell\in\mathsf{Q}}\sum_{k\in\mathbf{Z}}\Big(\sup_{\ell<p<\ell^+}|\nu_p(\varpi)(k)-\nu_{\ell^+}(\varpi)(k)|\Big)^2\ll\sum_{\ell\in\mathsf{Q}}\sum_{k\in\mathbf{Z}}\sum_{\ell<p<\ell^+}|\nu_p(\varpi)(k)|^2+\sum_{\ell\in\mathsf{Q}}\sum_{k\in\mathbf{Z}}|\nu_{\ell^+}(\varpi)(k)|^2$$

where the implied constant is absolute. The first sum here is larger than the second, and it is at most

$$\sum_{p\in\mathsf{P}}\sum_{k\in\mathbf{Z}}|\nu_p(\varpi)(k)|^2=\sum_{p\in\mathsf{P}}\int_{\mathbf{R}/\mathbf{Z}}|\widehat{\varpi}(\theta)\widehat{\nu}_p(\theta)|^2d\theta$$

by the Plancherel formula and (8.2). Using Corollary 3.10, we obtain the desired bound. ∎

10. Is sparseness necessary?

It is now natural to ask whether the restriction to sparse sets of primes necessary in the maximal and pointwise ergodic theorems, or not.

The first remark is that, for a classical (even weighted) sequence of ergodic averages

$$u_{\mathrm{N}}(x)=\frac{1}{\mathrm{N}}\sum_{0\leqslant n<\mathrm{N}}w(n)\varphi(f^n(x)),$$

convergence along sparse sequences of N implies convergence of the whole sequence. For instance, assume that there is convergence to 0 for N growing at least like a geometric progression with ratio $1+\delta > 0$, and assume that w and φ are bounded. For an arbitrary $\mathrm{N} \geqslant 1$, pick $\mathrm{M} \geqslant 1$ such that $\mathrm{M} \leqslant \mathrm{N} < (1+\delta)\mathrm{M}$. We obtain an obvious upper bound

$$|u_{\mathrm{N}}| \leqslant |u_{\mathrm{M}}| + \frac{\mathrm{C}\delta\mathrm{M}}{\mathrm{N}} \leqslant |u_{\mathrm{M}}| + \delta\mathrm{C}$$

for some constant $\mathrm{C} \geqslant 0$, so that

$$\limsup_{\mathrm{N}\to+\infty} |u_{\mathrm{N}}| \leqslant \delta\mathrm{C},$$

and if this holds for any $\delta > 0$, we obtain $u_{\mathrm{N}} \to 0$. Here the key point is that the restriction of the weight $w(n)$ to a shorter interval is the same as the weight used for the average over that interval – this property fails for "triangular" averages like those appearing in our situation.

Here is an abstract example which could be a guide to an example where almost everywhere convergence is *not true* in our setting.[5] Let X be the product over primes ℓ of copies of $\mathbf{R}/\mathbf{Z}$, viewed as a compact topological group and as a probability space with its Haar measure μ. For ℓ prime, fix an arbitrary measurable subset $\mathrm{A}_\ell \subset \mathbf{R}/\mathbf{Z}$ with measure $(\log \ell)^2/\ell$ (in $\mathbf{R}/\mathbf{Z}$).

Now, for p prime, let φ_p be the characteristic function of the set $\mathrm{Y}_p \subset \mathrm{X}$ of all $(\theta_\ell) \in \mathrm{X}$ such that the p-component θ_p belongs to A_p. Thus $\mu(\mathrm{Y}_p) = (\log p)^2/p$.

We claim that:

(1) the sequence (φ_p) does *not* converge almost everywhere;
(2) but, for any *sparse* set of primes P, the subsequence $(\varphi_p)_{p\in\mathsf{P}}$ converges almost everywhere to 0.

Indeed, the first assertion results from the independence of the functions φ_p (in probabilistic terms, they are independent random variables on X) and from the non-trivial direction of the Borel-Cantelli lemma, since

$$\sum_p \mu(\mathrm{Y}_p) = \sum_p \frac{(\log p)^2}{p} = +\infty, \qquad \sum_p \mu(\mathrm{X} - \mathrm{Y}_p) = \sum_p \Bigl(1 - \frac{(\log p)^2}{p}\Bigr) = +\infty,$$

which shows that for almost all $\theta = (\theta_\ell)$ in X, we have $\theta \in \mathrm{Y}_p$ (resp. $\theta \notin \mathrm{Y}_p$) for infinitely many p, so both $\varphi_p(\theta) = 0$ and $\varphi_p(\theta) = 1$ occur infinitely often.

The second assertion results from the easy direction of the Borel-Cantelli lemma, which implies that if P is a sparse set of primes, then μ-almost every element $\theta = (\theta_\ell) \in \mathrm{X}$ belongs only to finitely many Y_p for $p \in \mathsf{P}$, so that $\varphi_p(\theta) = 0$ for all p large enough in P.

The question is now whether such a model situation can arise in ergodic averages with trace functions (of sheaves with bounded conductor). Roughly speaking, this would amount to having a dynamical system (X, μ, f) and a function φ on X such that

$$\frac{1}{p} \sum_{0\leqslant n<p} t_p(n)\, \varphi(f^n(x)) \to \begin{cases} 1 & \text{with probability } (\log p)^2/p, \\ 0 & \text{with probability } 1 - (\log p)^2/p, \end{cases}$$

and the respective sets of x where these limits hold should be asymptotically independent enough to apply the Borel-Cantelli lemma (exact independence is not necessary, e.g., a sufficient amount of pairwise independence suffices, as in the Erdős-Rényi version of the Borel-Cantelli theorem, see [17, § 1]). (Moreover, the limits could obviously be different, it

[5] This is related to the well-known fact that convergence almost everywhere is not convergence with respect to any topology.

is enough that the two possibilities be separated enough that both occuring infinitely often excludes convergence).

It does not seem impossible to have such a configuration, especially since the trace function is *a priori* ours to select, with the condition that the conductors remain bounded, which might make it possible to exploit the frequent rough independence of primes.

Remark 10.1. (1) We would also show that convergence does not hold almost surely if the ergodic average converges to 1 with probability $1/p$ (instead of $(\log p)^2/p$), which would allow for convergence over all sets of primes with

$$\sum_{p\in \mathsf{P}} \frac{1}{p} < +\infty.$$

This configuration is maybe more likely to be possible.

(2) If we have a system where the ergodic averages converge *everywhere* for all sparse subsets of the primes, then they converge everywhere. (Indeed, the limit ψ would have to be independent of the sparse subset, since the union of two sparse sets is sparse, and then by contraposition, if the sequence was not convergent to ψ, some subsequence would avoid a fixed neighborhood of ψ, and some further subsequence would be sparse.)

The following is currently the closest example that we know. It doesn't quite address the main question, since it involves non-Fourier sheaves and systems with non-trivial Kronecker factors.

Let $\mathrm{X} = (\mathbf{R}/\mathbf{Z})^2$ (viewed as column vectors) with the Haar measure μ. Let $f(x, y) = (x + y, y)$, so that f is the action of an $\mathrm{SL}_2(\mathbf{Z})$-matrix, and therefore preserves μ. For $(x, y) \in \mathrm{X}$, we have

$$f^n(x, y) = (x + ny, y).$$

Define $\varphi\colon \mathrm{X} \to \mathbf{C}$ by $\varphi(x, y) = e(x)$. The ergodic averages are therefore

$$\frac{1}{p}\sum_{0\leqslant n<p} t_p(n)\varphi(f^n(x, y)) = \frac{e(x)}{p}\frac{\sin(\pi p(y - a_p/p))}{\sin(\pi(y - a_p/p))}e\Big(\frac{(p-1)}{2}(y - a_p/p)\Big).$$

Lemma 10.2. *There exists a sequence $(a_p)_p$ of integers such that $0 \leqslant a_p < p$ for all primes p, with the following property: for almost all $\theta \in \mathbf{R}/\mathbf{Z}$, there exist infinitely many p such that $|\theta - a_p/p| \leqslant 1/(100p)$.*

Proof. Here is one quick proof using fairly standard (but non-trivial) facts about the distribution of primes. Another more elementary argument is explained in the note [29] for the simple proof, which also has some more discussion of this somewhat unusual diophantine approximation statement.

Let $\mathscr{A}$ be the product over primes of the sets $\{0, \ldots, p-1\}$; it is a probability space with the product of the uniform probability measures.

Let $c = 1/100$ (any other positive constant would work). For any prime p and $\boldsymbol{a} \in \mathscr{A}$, we write $\mathrm{I}_p(\boldsymbol{a}) = [a_p/p - c/p, a_p/p + c/p]$, viewed as random intervals on $\mathscr{A}$. Let $x \in [0, 1]$. We then have

$$\mathbf{P}(x \in \mathrm{I}_p) = \frac{1}{p}\sum_{\substack{0\leqslant a<p\\ |x-a/p|<c/p}} 1$$

and hence $\mathbf{P}(x \in \mathrm{I}_p)$ is either 0 or $1/p$, depending on whether there exists an integer a such that the fractional part of xp is $< c$, or not.

It is known that if x is irrational, then we have

$$\sum_{\{xp\}<c} \frac{1}{p} = +\infty \tag{10.1}$$

(precisely, this follows by summation by parts from the more precise results, first proved by Vinogradov, which give an asymptotic formula with main term $c\pi(\mathrm{X})$ for the number of primes $p \leqslant \mathrm{X}$ satisfying $\{xp\} < c$, as $\mathrm{X} \to +\infty$; see [39, Ch. XI], and note that this result has been improved and simplified since then). Thus, since the events $\{x \in \mathrm{I}_p\}$ are independent by construction, the Borel–Cantelli Lemma implies

$$\mathbf{P}(x \in \mathrm{I}_p \text{ for infinitely many } p) = 1$$

for any irrational x.

Now by Fubini's Theorem, we obtain

$$\begin{aligned}\mathbf{E}(\lambda(\mathrm{A}_{\boldsymbol{a}})) &= \mathbf{E}\Big(\int_0^1 1_{\{x\in \mathrm{I}_p \text{ for infinitely many } p\}}\, dx\Big)\\ &= \int_0^1 \mathbf{P}(x \in \mathrm{I}_p \text{ for infinitely many } p) dx = 1,\end{aligned}$$

and since $\lambda(\mathrm{A}_{\boldsymbol{a}}) \leqslant 1$, this means that $\mathrm{A}_{\boldsymbol{a}}$ has measure 1 for almost all sequences (a_p). $\square$

Now fix a sequence (a_p) as given by that lemma and define $t_p(n) = e(-a_p n/p)$. These are trace functions of Artin-Schreier sheaves with bounded conductor. Let P be any set of primes with

$$\sum_{p\in\mathsf{P}} \frac{\log p}{p} < +\infty.$$

Then, for almost all (x, y), we have

$$\Big|y - \frac{a_p}{p}\Big| \geqslant \frac{\log p}{p}$$

for all but finitely many $p \in \mathsf{P}$, by the easy Borel–Cantelli lemma, hence

$$\frac{1}{p}\sum_{0\leqslant n<p} t_p(n)\varphi(f^n(x, y)) \to 0$$

almost surely along P. (And note that sparseness could be measured with $\log p$ replaced by any function tending to infinity with p.)

On the other hand, for almost all $(x, y) \in \mathrm{X}$, the properties of the sequence (a_p) prove that there exists a subsequence of primes for which

$$\frac{1}{p}\sum_{0\leqslant n<p} t_p(n)e(n\theta) \gg 1,$$

hence for which

$$\frac{1}{p}\sum_{0\leqslant n<p} t_p(n)\varphi(f^n(x, y))$$

does *not* converge to 0 along the primes. Since the result for sparse sequences mean that this sequence could only converge to 0 almost surely, we conclude that the ergodic averages

$$\frac{1}{p}\sum_{0\leqslant n<p} t_p(n)\varphi\circ f^n$$

do not converge almost surely.

11. Questions

The following further natural questions arise from this note:

(1) Are there maximal and pointwise ergodic theorems with trace functions for $\varphi \in \mathrm{L}^p$ where $p \neq 2$, especially for $p = 1$? For $p > 1$, one can certainly expect to be able to prove theorems in L^p by adapting the ideas of Bourgain [7]. The case $p = 1$ might well be the most interesting; we recall here that Buczolich and Mauldin [9] have proved that there is no maximal or pointwise ergodic theorem in L^1 for averages along the squares (see also LaVictoire's generalization of this fact [30], which relies on non-trivial arithmetic information).

(2) Are there similar results for "classical non-conventional averages" with trace functions, such as

$$\frac{1}{p}\sum_{0\leqslant n<p} t_p(n)\ (\varphi\circ f^n)\ (\varphi\circ f^{2n})\cdots(\varphi\circ f^{kn})$$

(where k is fixed; these occur without weights in Furstenberg's approach to Szemerédi's Theorem, see [14, Ch. 7]) or

$$\frac{1}{p}\sum_{0\leqslant n<p} t_p(n)\ \varphi\circ f^{n^2},$$

and other polynomials in place of n^2? The versions without weights are parts of Bourgain's celebrated work [6, 7, 5].

The first type of averages is intriguing, if only because trace functions are known to satisfy a very strong from of the inverse theorem for Gowers norms (by work of Fouvry, Kowalski and Michel [21]).

(3) Maybe most important: are there interesting applications of such ergodic averages?

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