QUANTITATIVE SHEAF THEORY

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ABSTRACT. We introduce a notion of complexity of a complex of ℓ-adic sheaves on a quasi-projective variety and prove that the six operations are "continuous", in the sense that the complexity of the output sheaves is bounded solely in terms of the complexity of the input sheaves. A key feature of complexity is that it provides bounds for the sum of Betti numbers that, in many interesting cases, can be made uniform in the characteristic of the base field. As an illustration, we discuss a few simple applications to horizontal equidistribution results for exponential sums over finite fields.

CONTENTS

1. INTRODUCTION

Since its invention, and especially since Deligne's proof [7] of the strongest form of the Riemann Hypothesis over finite fields, étale cohomology has exerted a considerable influence on analytic number theory. Its applications very often rely on estimates for the dimension of various étale cohomology spaces, which appear in "implicit constants" arising from the Grothendieck–Lefschetz trace formula combined with the Riemann Hypothesis and depend on the characteristic p of the finite fields under consideration. This characteristic is typically itself a variable going to infinity, and getting uniform estimates in terms of p turns out to be the crucial difficulty. Except for very simple cases, uniformity of such estimates is *not* a formal feature of étale cohomology (for examples and further discussion, see [25, 11.11]).

In recent years, this issue has been particularly visible in a series of works by Fouvry, Kowalski and Michel (see, e.g., [14] and [15]) that make extensive use of very general sheaves on curves in various problems of analytic number theory. Due to the simpler nature of curves (essentially, the Euler–Poincaré characteristic controls the sum of Betti

numbers, and has an expression in terms of "simple" local invariants), they obtained a satisfactory theory, phrased in terms of a "complexity" invariant of an ℓ-adic sheaf on a curve over a finite field, which they called the "(analytic) conductor". The key feature of this theory is that *most natural operations on sheaves and the analytic resulting estimates depend on the "input" sheaves only through their conductor* (see, e.g., [14, Th. 1.5]).

Another application of a suitable version of complexity for ℓ-adic sheaves on curves is the proof by Deligne [8] of the theorem that, for a lisse $\overline{\mathbf{Q}}_\ell$-adic Weil sheaf on a normal connected scheme of finite type over a finite field such that the traces of Frobenius at all closed points are algebraic numbers, the field generated by these traces is a number field. The complexity that we introduce here could also be used in Deligne's argument.

In this paper, we develop a similar theory for higher-dimensional quasi-projective algebraic varieties over any field. (Being a geometric invariant, the complexity is defined by base change to an algebraic closure of the base field, so most of this paper will only deal with algebraically closed fields.) This leads to very general estimates that solve most of the known problems of estimating Betti numbers in analytic number theory.

We now state somewhat informally the definition of complexity and some of the key statements, focusing for simplicity on sheaves on affine space $\mathbf{A}^n$. Let k be an algebraically closed field and ℓ a prime number different from the characteristic of k. We will define:

- A non-negative integer $c(A)$ for any object A of the bounded derived category of constructible sheaves $D_c^b(\mathbf{A}^n, \overline{\mathbf{Q}}_\ell)$ (Definitions 3.2 and 6.3). Letting u denote the open immersion of $\mathbf{A}^n$ in $\mathbf{P}^n$, the integer $c(A)$ is defined as the maximum over integers $0 \leqslant m \leqslant n$ of the sum of the Betti numbers of the pullback of the extension by zero $u_! A$ to a "generic" linear subspace of dimension m of $\mathbf{P}^n$.
- A non-negative integer $c(f)$ for any morphism $f \colon \mathbf{A}^n \to \mathbf{A}^m$ (Definition 6.6). In general, this is also defined in terms of sums of Betti numbers, but admits in the case at hand a completely explicit bound that only involves n, m and the degrees of the polynomials defining f (Proposition 6.21).

We will then prove the following result (Theorems 6.8 and 6.17). In the statement, $D(A)$ denotes the Verdier dual of A and all functors and operations are considered in the derived sense (so, e.g., we write f_* instead of Rf_*).

Theorem 1.1. *For any* $f \colon \mathbf{A}^n \to \mathbf{A}^m$, *for any objects* A *and* B *of* $D_c^b(\mathbf{A}^n, \overline{\mathbf{Q}}_\ell)$, *and any object* C *of* $D_c^b(\mathbf{A}^m, \overline{\mathbf{Q}}_\ell)$, *the following estimates hold:*

$$c(D(A)) \ll c(A),$$
$$c(A \otimes B) \ll c(A)c(B), \qquad c(\mathscr{H}om(A, B)) \ll c(A)c(B),$$
$$c(f^*C) \ll c(f)c(C), \qquad c(f^!C) \ll c(f)c(C),$$
$$c(f_!A) \ll c(f)c(A), \qquad c(f_*A) \ll c(f)c(A).$$

In all these estimates, the implied constants depend only on (n, m) *and are effective.*

Moreover, let S *be the spectrum of a strictly Henselian discrete valuation ring with special point* σ *and generic point* η, *and let* Ψ *and* Φ *denote the nearby and vanishing cycle functors from* $D_c^b(\mathbf{A}_S^n, \overline{\mathbf{Q}}_\ell)$ *to* $D_c^b(\mathbf{A}_\sigma^n, \overline{\mathbf{Q}}_\ell)$. *For any object* A *of* $D_c^b(\mathbf{A}_S^n, \overline{\mathbf{Q}}_\ell)$, *the following estimates hold:*

$$c(\Psi(A)) \ll c(A_\eta),$$
$$c(\Phi(A)) \ll c(A_\eta) + c(A_\sigma).$$

Over finite fields, the conjunction of the Riemann Hypothesis and the theory of complexity yields the following "quasi-orthogonality" statement (Theorem 7.13):

Theorem 1.2. *Suppose that k is the algebraic closure of a finite field $\mathbf{F}$, and let A and B be irreducible perverse sheaves on $\mathbf{A}^n$ defined over $\mathbf{F}$ that are pure of weight zero, with trace functions t_A and t_B respectively. Then the estimate*

$$\sum_{x \in \mathbf{F}^n} |t_A(x)|^2 = 1 + O(c(A)^2 |\mathbf{F}|^{-1/2})$$

holds, and the estimate

$$\sum_{x \in \mathbf{F}^n} t_A(x) \overline{t_B(x)} \ll c(A) c(B) |\mathbf{F}|^{-1/2}$$

holds if A and B are not geometrically isomorphic. In both estimates, the implied constants depend only on n and are effective.

Remark 1.3. Readers from analytic number theory who are unfamiliar with perverse sheaves may be surprised by the lack of the averaging factor $1/|\mathbf{F}^n|$ in the writing of these sums, in comparison with statements like those in [14]. This is due to the normalization inherent to the definition of weights in this setting: for instance, for a perverse sheaf M on $\mathbf{A}^1$ that is a single lisse sheaf sitting in degree -1, being pure of weight zero means that the eigenvalues of Frobenius at all points have modulus $|\mathbf{F}|^{-1/2}$ (and not 1, as is the case for a lisse sheaf that is pointwise pure of weight zero).

We highlight one first rather simple application (see part (1) of Corollary 7.23), which gives a positive answer to a question of Katz [29, p. 8 and 12.6.6].

Theorem 1.4. *Let $n \geqslant 1$ and $d \geqslant 1$ be integers. Let $D(n,d)$ be the space of Deligne polynomials of degree d in n variables, i.e. those whose homogeneous part of degree d defines a non-singular hypersurface in $\mathbf{P}^{n-1}$. For each $f \in D(n,d)(\mathbf{F}_p)$, set*

$$S(f;p) = \frac{1}{p^{n/2}} \sum_{x \in \mathbf{F}_p^n} e\Big(\frac{f(x)}{p}\Big),$$

where $e(z) = \exp(2i\pi z)$ for $z \in \mathbf{C}$. The families $(S(f,p))_{f \in D(n,d)(\mathbf{F}_p)}$ become equidistributed as $p \to +\infty$ with respect to the image under the trace of the probability Haar measure on the unitary group $\mathrm{U}_{(d-1)^n}(\mathbf{C})$.

We now comment on the approach that we use. Previous Betti number bounds, such as those of Bombieri [4], Adolphson–Sperber [1] and Katz [28], focused primarily on bounding cohomology groups involving certain very explicit sheaves, namely Artin–Schreier and Kummer sheaves. It is possible to apply these bounds to a sheaf cohomology problem involving, say, higher-rank Kloosterman sheaves, but only after unraveling their definition to recast the problem entirely in terms of Artin–Schreier sheaves. When more complicated operations are performed (for example, additive or multiplicative convolution, or Fourier transform), this process of re-interpretation becomes exceedingly cumbersome.

Our approach is instead built around the six functors formalism of étale cohomology, and is closely related to the characteristic classes constructed by T. Saito [39]. We define the "complexity" of an arbitrary bounded complex of constructible ℓ-adic sheaves on a quasi-projective variety, and prove that it satisfies essentially all desired properties suggested by the case of curves and the requirements of applications to analytic number theory. In particular, the complexity of common sheaves such as Artin-Schreier, Kummer and Kloosterman sheaves can be easily calculated, and it turns out to be bounded independently of the characteristic of the underlying field, which is the key uniformity property that we seek.

Remark 1.5. In fact, the complexity of a sheaf on an algebraic variety will also depend on a chosen quasi-projective embedding of the variety; this seems unavoidable to have a theory with good properties, as we explain in Example 6.1.

Remark 1.6. The definition of complexity and the arguments of this paper apply, almost without modification, to the derived category of sheaves with coefficients in $\overline{\mathbf{F}}_\ell$ instead of $\overline{\mathbf{Q}}_\ell$. Neither version is stronger. Although the Betti numbers of a $\overline{\mathbf{Q}}_\ell$-sheaf are bounded by the Betti numbers of the reduction mod ℓ of an integral model of it, this inequality does not help us transfer statements of the form "a bound for the Betti numbers of this sheaf implies a bound for the Betti numbers of that sheaf" in either direction. We have stated and worked out in detail the $\overline{\mathbf{Q}}_\ell$-version as it is the most directly relevant for applications to analytic number theory, but the $\overline{\mathbf{F}}_\ell$-version may also be useful for other purposes.

We believe that this framework has a number of good properties, among which:

(1) Since the deeper aspects of étale cohomology are built primarily around the six functors perspective, rather than the cohomology of varieties with coefficients in some simple explicit sheaves, this framework behaves much better in arguments where sophisticated techniques of étale cohomology are used.

(2) Many applications of exponential sum bounds from étale cohomology revolve around exponential sums that are produced from simpler ones by applying analytic tools like changes of variables, summation over some variables, Fourier transform, etc. Through the "function-sheaf dictionary", each of these usually corresponds to an operation on the sheaf side, which is constructed by means of the six functors (e.g., summation corresponds to direct image with compact support, etc). Since we control the growth of the complexity under the six functors, we obtain automatically a good control of the estimates in such operations.

As we will see, almost all of the bounds for the complexity of the output sheaf of some cohomology operation are linear in the complexity of the input sheaf. This is not always needed for applications, but shows that the theory has good structural properties.

An interpretation. As suggested by Fouvry, Kowalski and Michel in the special case of curves, the "quantitative sheaf theory" that is developed in this paper can be thought of as defining the complexity of ℓ-adic sheaves in such a way that most (if not all) usual operations in étale cohomology are "continuous", in the sense that applying the operation to a sheaf with a given complexity will lead to another one with complexity bounded only in terms of the initial one. Thus, we think of the complexity as being similar to a (semi)-norm on a topological vector space, with functors on categories of sheaves playing the role of (often linear) maps between vector spaces. For instance, the "continuity" of Deligne's ℓ-adic Fourier transform (which was first observed in dimension one in [15, Prop. 8.2]) turns out to be one of the most essential features of applications of étale cohomology to analytic number theory.

Outline of the paper. Although the complexity is defined in terms of sums of Betti numbers, the proof of its main properties deeply relies on T. Saito's construction [39] of the characteristic cycle of ℓ-adic complexes. We survey what we require from this theory in Section 2, and prove a small complement on characteristic cycles of tensor products (Theorem 2.2). In Section 3, we formally define the complexity on projective space, we establish a few simple lemmas concerning "generic" injective linear maps, and most importantly we connect this approach with the characteristic cycle (Proposition 3.17). Section 4 is of technical nature: we define and prove the existence of certain objects called

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"test sheaves" that will ultimately lead to a comparison of the complexity with a norm of the characteristic cycle. Section 5 uses these tools to establish the first fundamental result, namely a bilinear bound for the complexity of the tensor product (Theorem 5.1). Then Section 6 can rather quickly exploit the formalism of étale cohomology to establish the general version of Theorem 1.1, namely Theorems 6.8 and 6.17; later subsections derive various other "continuity" properties. Finally, Section 7 gives some fundamental examples (such as Artin–Schreier and Kummer sheaves) and summarizes a few direct applications (including forms of the Riemann Hypothesis, such as Theorem 1.2, the finiteness statement of Corollary 7.15, and a form of Deligne's equidistribution theorem, from which Theorem 1.4 folllows). In the concluding Section 8, we explain how all the basic estimates can be stated with explicit constants.

Notation and conventions.

Algebraic geometry. We fix throughout a prime number ℓ and we denote by k a field, algebraically closed unless otherwise specified, in which ℓ is invertible.

By an *algebraic variety* over a (not necessarily algebraically closed) field k, we mean a reduced and separated scheme of finite type over the spectrum of k.

By a *geometric generic point* of an irreducible variety X over a separably closed field k', we mean, as is customary in the theory of étale cohomology, a map $\operatorname{Spec} k' \to X$ such that the image of the underlying set-theoretic map consists of the generic point of X.

Let X be a scheme of finite type over k. We denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$ the bounded derived category of constructible complexes of $\overline{\mathbf{Q}}_\ell$-sheaves on X (see, e.g., [32, II.5]). We will usually write distinguished triangles in this category simply as

$$A \longrightarrow B \longrightarrow C.$$

For any object K of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$ and any integer $i \in \mathbf{Z}$, we denote by $\mathrm{H}^i(X, K)$ and $\mathrm{H}^i_c(X, K)$ the étale cohomology and the étale cohomology with compact support groups of X with coefficients in K, and we write

$$h^i(X, K) = \dim \mathrm{H}^i(X, K), \qquad h^i_c(X, K) = \dim \mathrm{H}^i_c(X, K)$$

for the corresponding Betti numbers.

When applied to objects of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$, the symbols $f_!$ and f_* always refer to the derived functors; the tensor product and the hom functor of objects of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$ are also always derived functors. We denote by $\mathrm{D}(A)$ the Verdier dual of an object A of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$.

Given an algebraic variety X over k and objects A and B of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$, the *shriek tensor product* of A and B is the object

$$A \otimes_! B = \Delta^!(A \boxtimes B)$$

of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$, where $\Delta\colon X \to X \times X$ denotes the diagonal embedding. It is related to the usual tensor product by the duality

$$\mathrm{D}(A \otimes_! B) = \mathrm{D}(A) \otimes \mathrm{D}(B).$$

We often use the projection formula in the derived category: for a morphism $f\colon X \to Y$ of algebraic varieties over k, and for objects A of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$ and B of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(Y)$, there is a canonical isomorphism

$$f_!(A \otimes f^*B) \simeq f_!A \otimes B$$

in the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(Y)$ (see, e.g., [18, Th. 7.4.7 (i)]).

We also recall the excision triangle: let $i\colon Z \to X$ be a closed immersion and $j\colon U \to X$ the complementary open immersion, all varieties being defined over k. For any object A of $\mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(X)$ and any morphism $f\colon X \to Y$ over k, there is a distinguished triangle

$$(f \circ j)_! j^* A \longrightarrow f_! A \longrightarrow (f \circ i)_! i^* A$$

in the category $\mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(Y)$ (see, e.g., [18, Th. 7.4.4 (iii)]).

Finite fields. In some sections (e.g., Sections 6.10 and 7.6), we will work over finite fields. We usually denote by $\mathbf{F}$ such a field, which is always assumed to have characteristic different from ℓ. For integers $n \geqslant 1$, we then denote by $\mathbf{F}_n$ the extension of $\mathbf{F}$ of degree n inside some fixed algebraic closure of $\mathbf{F}$ (which often will be the field k).

Let X be an algebraic variety over $\mathbf{F}$. For any object A of $\mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(X)$ and any finite extension $\mathbf{F}_n$ of $\mathbf{F}$, we denote by

$$t_A(\cdot\,; \mathbf{F}_n)\colon X(\mathbf{F}_n) \longrightarrow \overline{\mathbf{Q}}_\ell$$

the *trace function* of A on $\mathbf{F}_n$. We refer the reader to [38, §1] for the basic formalism of trace functions in this generality. We will also write $t_A(x) = t_A(x; \mathbf{F})$ for $x \in X(\mathbf{F})$.

In all arguments involving the formalism of weights (in the sense of Deligne), we will fix an isomorphism $\iota\colon \overline{\mathbf{Q}}_\ell \to \mathbf{C}$ and use it to identify both fields, viewing in particular the trace functions as taking complex values. Weights are then considered to be defined only with respect to ι, e.g. we write "pure of weight zero" instead of "ι-pure of weight zero".

For a real number w and an element $|\mathbf{F}|^{-w}$ in $\overline{\mathbf{Q}}_\ell$ corresponding to a choice of $|\mathbf{F}|^{-w}$ in $\mathbf{C}$ through the isomorphism ι, we denote by $\overline{\mathbf{Q}}_\ell(w)$ the pullback to X of the rank-one ℓ-adic sheaf on $\mathrm{Spec}(\mathbf{F})$ on which Frobenius acts through multiplication by $|\mathbf{F}|^{-w}$. This allows one to define twists $A(w) = A \otimes \overline{\mathbf{Q}}_\ell(w)$ for any object A of $\mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(X)$, and hence to reduce questions about pure sheaves of some weight to those of weight zero.

The Euler–Poincaré characteristic. We recall the Euler–Poincaré characteristic formula for a perverse sheaf on a smooth curve (see, e.g., [38, Th. 2.2.1.2] for the projective case, from which the general case below follows by considering the extension by zero to the compactification). Let k be an algebraically closed field of characteristic $p > 0$, let C be a smooth curve over k, and denote by $\overline{C}$ the smooth projective compactification of C. Given a perverse sheaf A on C, we denote by

$$\mathrm{rank}(A) = \dim \mathscr{H}^{-1}(A)_\eta$$

the *generic rank* of A and, for a closed point x of C, we write

$$\mathrm{drop}_x(A) = \mathrm{rank}(A) - \dim \mathscr{H}^{-1}(A)_x$$
$$\mathrm{jump}_x(A) = \dim \mathscr{H}^0(A)_x.$$

For a closed point x of $\overline{C}$, we denote by $\mathrm{swan}_x(A)$ the *Swan conductor* at x of the cohomology sheaf $\mathscr{H}^{-1}(A)$. We further set

$$\mathrm{loc}(A) = \sum_{x \in C} (\mathrm{drop}_x(A) + \mathrm{jump}_x(A) + \mathrm{swan}_x(A)) + \sum_{x \in \overline{C} - C} \mathrm{swan}_x(A),$$

where both sums run over closed points.

With the above notation, the Grothendieck–Ogg–Shafarevitch formula for the Euler–Poincaré characteristic with compact support of A takes the form

$$\chi_c(C, A) = \mathrm{rank}(A)\chi_c(C, \overline{\mathbf{Q}}_\ell[1]) - \mathrm{loc}(A).$$

The same result holds for the usual Euler–Poincaré characteristic $\chi(C, A)$ since both are in fact equal for any constructible sheaf on any variety by a theorem of Laumon [37].

If $\mathscr{F}$ is a middle-extension sheaf on C (by which we mean that there exists a non-empty open subset $j\colon U \hookrightarrow C$ such that $\mathscr{F}$ is lisse on U and the adjunction morphism is an isomorphism $\mathscr{F} \simeq j_* j^* \mathscr{F}$), then $A = \mathscr{F}[1]$ is a perverse sheaf on C satisfying $\mathscr{H}^{-1}(A) = \mathscr{F}$. We use the notation $\mathrm{rank}(\mathscr{F})$, $\mathrm{drop}_x(\mathscr{F})$, $\mathrm{swan}_x(\mathscr{F})$ and $\mathrm{loc}(\mathscr{F})$ accordingly; note that in this case $\mathrm{jump}_x(A) = 0$ holds for all x.

Asymptotic notation. For complex-valued functions f and g defined on a set (or on objects of a category, in which case the values of f and g are assumed to only depend on their isomorphism classes), the notation $f \ll g$ and $f = O(g)$ are synonymous; they mean that there exists a real number $c \geqslant 0$ such that, for all x in the relevant set (or all objects in the category), the inequality $|f(x)| \leqslant cg(x)$ holds. We call a value of c an "implied constant", and we may point out its (in)dependency on some additional parameters. We also write $f \asymp g$ whenever both $f \ll g$ and $g \ll f$ hold.

Remarks on the text. All the important ideas of this paper are solely due to W. Sawin, and were worked out in 2015 and 2016 while he was an ETH–ITS Junior Fellow. The current text was written by A. Forey, J. Fresán and E. Kowalski, based on the original draft by W. Sawin, in view of the applications to equidistribution results for exponential sums on commutative algebraic groups in their work [11]. The quantitative form of the generic base change theorem is due to A. Forey.

2. Characteristic cycles

In this section, we recall some properties of the characteristic cycles and the characteristic classes of complexes of étale sheaves as defined by Beilinson [2] and T. Saito [39]. We also prove a small complement (Theorem 2.2) regarding the compatibility of characteristic cycles with tensor products.

Let X be a smooth scheme over a perfect field k. In [39], characteristic cycles are defined for complexes of sheaves of Λ-modules on X, where Λ is a finite local ring whose residue characteristic ℓ is invertible in k (e.g., $\Lambda = \mathbf{Z}/\ell^n\mathbf{Z}$). It was observed by Umezaki, Yang and Zhao [42, Section 5] that the whole theory can be readily adapted to the case of $\overline{\mathbf{Q}}_\ell$-coefficients. In what follows, we implicitly rely on [42] in order to apply Saito's results to $\Lambda = \overline{\mathbf{Q}}_\ell$.

Assume X has pure dimension n. The *characteristic cycle* $\mathrm{CC}(A)$ of an object A of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$ is an algebraic cycle on the cotangent bundle $T^* X$ of X of the form

$$\mathrm{CC}(A) = \sum_i m_i C_i,$$

where m_i is an integer and C_i is a closed conical (i.e. stable under the natural action of $\mathbf{G}_m$) subset of $T^* X$ of dimension n. We refer the reader to [39, Def. 5.10] for the definition of the multiplicities m_i, and simply indicate some of the relevant properties of the characteristic cycle.

Another invariant of A is its *singular support* $\mathrm{SS}(A)$, which is a closed conical subset of $T^* X$ of dimension n, containing the support of the characteristic cycle, previously defined by Beilinson in [2]. Unlike what happens for the characteristic cycle, two $\overline{\mathbf{Z}}_\ell$-sheaves that are isomorphic after tensoring with $\overline{\mathbf{Q}}_\ell$ may have distinct singular supports, so before

stating results involving the singular support we will choose a $\overline{\mathbf{Z}}_\ell$-structure $A_{\overline{\mathbf{Z}}_\ell}$ (though the particular choice will matter little), and set

$$\mathrm{SS}(A) = \mathrm{SS}(A_{\overline{\mathbf{Z}}_\ell} \otimes \overline{\mathbf{F}}_\ell).$$

If the object A is perverse, then $\mathrm{CC}(A)$ is effective (i.e., we have $m_i \geqslant 0$ for every i) by [39, Prop. 5.14].

The main property of the characteristic cycle is an index formula ([39, Th. 7.13]) according to which, if X is a smooth projective variety over an algebraically closed field k, the Euler–Poincaré characteristic of A is given by the intersection number

$$\chi(X, A) = \mathrm{CC}(A) \cdot [T_X^* X].$$

In this formula, $T_X^* X$ denotes the zero section of $T^* X$, and the intersection is well-defined since the support of $\mathrm{CC}(A)$ has pure dimension n.

Example 2.1. Let X be a smooth projective curve and $\mathscr{F}$ a lisse sheaf on a dense open subset $j \colon U \to X$. The characteristic cycle of the perverse sheaf $A = j_! \mathscr{F}[1]$ is given by

$$\mathrm{CC}(A) = \mathrm{rank}\, \mathscr{F} \cdot [T_X^* X] + \sum_{x \in X \setminus U} (\mathrm{rank}\, \mathscr{F} + \mathrm{swan}_x A)[T_x^* X],$$

where $T_x^* X$ stands for the conormal bundle of x in $T^* X$. In this situation, the index formula amounts to the Grothendieck–Ogg–Shafarevich formula.

Another deep result is the compatibility of the characteristic cycle with pullbacks. Let $h \colon W \to X$ be a morphism between smooth schemes of pure dimensions m and n respectively and $C \subset T^* X$ a closed conical subset. Let

$$h^* C = W \times_X C \subset W \times_X T^* X$$

be the pullback of C, and denote by $K \subset W \times_X T^* X$ the preimage of the zero section $T_W^* W$ by the map $dh \colon W \times_X T^* X \to T^* W$. Following [39, Def. 3.3 and 7.1], we say that the map h is *properly C-transversal* if the intersection $h^* C \cap K$ is a subset of $W \times_X T_X^* X$ and each irreducible component of $h^* C$ has dimension m. This implies in particular that the restriction of dh to $h^* C$ is finite (by [2, Lem. 1.2. (ii)] or [39, Lem. 3.1]). Hence, given a cycle Z supported on the irreducible components of C, we can define its pullback $h^! Z \subset T^* W$ as $(-1)^{m-n}$ times the push-forward along the finite map $h^* C \to T^* W$ of the preimage of Z in $W \times_X T^* X$ (see [39, Def. 7.1]).

The main theorem of Beilinson and Saito [39, Th. 7.6] regarding pullbacks says that, if A is an object of $\mathrm{D}_\mathrm{c}^\mathrm{b}(X, \overline{\mathbf{Z}}_\ell)$ and $h \colon W \to X$ is a properly $\mathrm{SS}(A)$-transversal morphism, then the following equality holds:

$$\mathrm{CC}(h^* A) = h^! \mathrm{CC}(A).$$

For example, smooth morphisms are properly $\mathrm{SS}(A)$-transversal by [39, Lem. 3.4], and in that case the theorem essentially amounts to the statement that, for smooth varieties X and Y and objects A of $\mathrm{D}_\mathrm{c}^\mathrm{b}(X)$ and B of $\mathrm{D}_\mathrm{c}^\mathrm{b}(Y)$, the equality

$$\mathrm{CC}(A \boxtimes B) = \mathrm{CC}(A) \times \mathrm{CC}(B),$$

holds in $T^*(X \times Y) = T^* X \times T^* Y$, see [40, Th. 2.2] for details.

In what follows, vector bundles on schemes are viewed as schemes in the usual way. A sum of vector bundles $V + W$ is isomorphic to their fiber product over the base, whereas the product $V \times W$ is their product as schemes, and hence is a vector bundle on the product of the bases. As for $T^* X$, a closed subset of a vector bundle is said to be *conical* if it is invariant under the scaling action of $\mathbf{G}_m$. Given a vector bundle V on a variety X,

we denote by $\overline{V} = \mathbf{P}(V + \mathscr{O}_X)$ the projective bundle compactifying V, which admits a decomposition $\overline{V} = V \cup \mathbf{P}(V)$.

The main result of this section is the following theorem concerning characteristic cycles of tensor products of sheaves.

Theorem 2.2. *Let X be a smooth variety over k of pure dimension n. Let A and B be objects of $\mathrm{D}_c^b(X, \overline{\mathbf{Z}}_\ell)$. Consider the summation and the inclusion maps*

$$s\colon T^*X + T^*X \to T^*X, \qquad i\colon T^*X + T^*X \to T^*X \times T^*X.$$

Assume that $\mathrm{SS}(A) \cap \mathrm{SS}(B)$ is supported on the zero section and that each irreducible component of $\mathrm{SS}(A) \times_X \mathrm{SS}(B)$ has dimension at most n. Then:

(1) *Each irreducible component of $\mathrm{SS}(A) \times_X \mathrm{SS}(B)$ has dimension equal to n.*
(2) *The equality $i^*(\mathrm{SS}(A) \times \mathrm{SS}(B)) = \mathrm{SS}(A) \times_X \mathrm{SS}(B)$ holds.*
(3) *The restriction of s to $\mathrm{SS}(A) \times_X \mathrm{SS}(B)$ is a finite map to T^*X.*
(4) *The following equality holds*

$$\mathrm{CC}(A \otimes B) = (-1)^n s_* i^* \left(\mathrm{CC}(A) \times \mathrm{CC}(B) \right),$$

where the pullback and the pushforward are taken in the sense of intersection theory. (By (1) and (2), the inverse image $i^* (\mathrm{SS}(A) \times \mathrm{SS}(B))$ has the expected dimension, so the intersection-theoretic pullback is well-defined.)
(5) *Assume that X is projective. Let $\overline{\mathrm{CC}(A)}$ and $\overline{\mathrm{CC}(B)}$ be the closures of $\mathrm{CC}(A)$ and $\mathrm{CC}(B)$ inside the projective bundle $\overline{T^*X}$. Then the equality*

$$\chi(X, A \otimes B) = (-1)^n \mathrm{CC}(A) \cdot \mathrm{CC}(B) = (-1)^n \overline{\mathrm{CC}(A)} \cdot \overline{\mathrm{CC}(B)}$$

holds, where the dots denote intersection numbers of algebraic cycles.

Note that proving properties (1), (2) and (3) amounts to checking that the diagonal map is properly $\mathrm{SS}(A) \times \mathrm{SS}(B)$-transversal. The proof of the theorem relies on the following lemma:

Lemma 2.3. *Let X be a variety over k and let V be a vector bundle on X. Let $C_1, C_2 \subset V$ be conical subsets and $\overline{C_1}$ and $\overline{C_2}$ their closures inside the projective bundle $\overline{V}$. The following two conditions are equivalent:*

(1) *$C_1 \cap C_2$ is contained in the zero section of V.*
(2) *$\overline{C_1} \cap \overline{C_2}$ does not intersect $\mathbf{P}(V)$ inside $\overline{V}$.*

Proof. The intersection $\overline{C_1} \cap \overline{C_2}$ is proper, as a closed subset of $\overline{V}$. Taking the decomposition $\overline{V} = V \cup \mathbf{P}(V)$ into account, if condition (2) holds, then $\overline{C_1} \cap \overline{C_2}$ is also affine, being a closed subset of V. Therefore, $\overline{C_1} \cap \overline{C_2}$ is a finite conical subset of V, and is hence contained in the zero section. Conversely, (1) implies (2) because if x is a point of $\overline{C_1} \cap \overline{C_2}$ that lies in $\mathbf{P}(V)$, then the line in V corresponding to x is contained in $C_1 \cap C_2$, which therefore is not just the zero section. $\square$

Proof of Theorem 2.2. We first prove statement (1). Since $\mathrm{SS}(A) \times_X \mathrm{SS}(B)$ is the intersection of the inverse images of $\mathrm{SS}(A)$ and $\mathrm{SS}(B)$ inside $T^*X + T^*X$ (i.e. the intersection of two $2n$-dimensional schemes inside a smooth variety of dimension $3n$), each of its irreducible components has dimension at least n, and hence equal to n by assumption.

Statement (2) follows from the fact that pullback by the map i identifies the sum $T^*X + T^*X$ with $T^*X \times_X T^*X$ seen as a closed subset of $T^*X \times T^*X$. In the rest of the proof, we will make this identification.

To prove (3), observe that $s\colon T^*X + T^*X \to T^*X$ is a map of vector bundles on X. Since $\mathrm{SS}(A)$ and $\mathrm{SS}(B)$ are conical and $\mathrm{SS}(A) \cap \mathrm{SS}(B)$ is a subset of the zero section T_X^*X

of T^*X, the intersection of $\mathrm{SS}(A) \times_X \mathrm{SS}(B)$ and the pullback along s of T_X^*X is contained in the zero section of $T^*X + T^*X$. Hence, the restriction of s to $\mathrm{SS}(A) \times_X \mathrm{SS}(B)$ is finite by [39, Lem. 3.1].

We now turn to (4). Recall that the characteristic cycle of the external product $A \boxtimes B$ is equal to $\mathrm{CC}(A) \times \mathrm{CC}(B)$ and that the tensor product $A \otimes B$ is given by $\Delta^*(A \boxtimes B)$, where $\Delta \colon X \to X \times X$ denotes the diagonal map. We will compute $\mathrm{CC}(A \otimes B)$ using the compatibility of characteristic cycles with pullbacks, as recalled above. For this, we need to show that Δ is a properly $\mathrm{SS}(A) \times \mathrm{SS}(B)$-transversal morphism.

From part (2), we know that every irreducible component of $\Delta^*(\mathrm{CC}(A) \times \mathrm{CC}(B))$ has dimension n. For the transversality condition, we need to consider the maps

$$T^*X \xleftarrow{\ s\ } T^*X + T^*X \xrightarrow{\ i\ } T^*(X \times X)$$

defined by the diagonal Δ, and check that the intersection of $i^*(\mathrm{SS}(A) \times \mathrm{SS}(B))$ with the preimage of the zero section T_X^*X by the map s is contained in the zero section of $T^*X + T^*X$. But this is precisely the condition we checked in the proof of (3), and is in fact equivalent to (3) by [39, Lem. 3.1]. Hence, [39, Th. 7.6] applies to Δ and $A \boxtimes B$, which yields the equality

$$\mathrm{CC}(\Delta^*(A \boxtimes B)) = \Delta^! \mathrm{CC}(A \boxtimes B).$$

Combining this with the first part of the proof and the definition of $\Delta^!$, we get

$$\mathrm{CC}(A \otimes B) = \mathrm{CC}(\Delta^*(A \boxtimes B)) = \Delta^! \mathrm{CC}(A \boxtimes B) = (-1)^n s_* i^*(\mathrm{CC}(A) \times \mathrm{CC}(B)).$$

Let us finally prove (5). By the index formula [39, Th. 7.13], the Euler–Poincaré characteristic of $A \otimes B$ is the intersection number of $\mathrm{CC}(A \otimes B)$ with the zero section $[T_X^*X]$. Using part (4), it is hence given by

$$(-1)^n [T_X^*X] \cdot s_* i^*(\mathrm{CC}(A) \times \mathrm{CC}(B)).$$

By the projection formula in intersection theory (see, e.g., [20, Prop. 8.3 (c)]), the equality

$$[T_X^*X] \cdot s_* i^*(\mathrm{CC}(A) \times \mathrm{CC}(B)) = s^*[T_X^*X] \cdot i^*(\mathrm{CC}(A) \times \mathrm{CC}(B))$$

holds, as long as the restriction of s to the support of $i^*(\mathrm{CC}(A) \times \mathrm{CC}(B))$ is a proper map. Taking into account that the support of $i^*(\mathrm{CC}(A) \times \mathrm{CC}(B))$ is $\mathrm{SS}(A) \times_X \mathrm{SS}(B)$, this is indeed the case by part (3). Now, $s^*[T_X^*X]$ is the closed subset in $T^*X + T^*X$ consisting of those elements with zero sum. Being conical, $\mathrm{CC}(B)$ is in particular invariant under multiplication by -1, hence the equality

$$s^*[T_X^*X] \cdot i^*(\mathrm{CC}(A) \times \mathrm{CC}(B)) = d_*[T^*X] \cdot i^*(\mathrm{CC}(A) \times \mathrm{CC}(B)),$$

where $d \colon T^*X \to T^*X + T^*X$ is the diagonal map. Since d is a closed immersion, the projection formula yields

$$d_*[T^*X] \cdot i^*(\mathrm{CC}(A) \times \mathrm{CC}(B)) = [T^*X] \cdot d^* i^*(\mathrm{CC}(A) \times \mathrm{CC}(B)).$$

Furthermore, $i \circ d$ is the diagonal map $\delta \colon T^*X \to T^*X \times T^*X$, so the equality

$$[T^*X] \cdot d^* i^*(\mathrm{CC}(A) \times \mathrm{CC}(B)) = [T^*X] \cdot \delta^*(\mathrm{CC}(A) \times \mathrm{CC}(B))$$
$$= \mathrm{CC}(A) \cdot \mathrm{CC}(B)$$

holds. Combining all these identities we get the first equality of (5). Finally,

$$\overline{\mathrm{SS}(A)} \cap \overline{\mathrm{SS}(B)} = \mathrm{SS}(A) \cap \mathrm{SS}(B)$$

follows from Lemma 2.3, and this implies the equality of intersection numbers

$$\mathrm{CC}(A) \cdot \mathrm{CC}(B) = \overline{\mathrm{CC}(A)} \cdot \overline{\mathrm{CC}(B)},$$

which completes the proof. $\square$

The following definition is taken from Saito [39, Def. 6.7] and relies on the isomorphism

$$\mathrm{CH}_*(\mathbf{P}^n) \longrightarrow \mathrm{CH}_n\left(\overline{T^*\mathbf{P}^n}\right), \quad (a_i) \longmapsto \sum_i p^* a_i h^i,$$

where $p\colon \overline{T^*\mathbf{P}^n} \to \mathbf{P}^n$ is the projection and h is the first Chern class of the dual of the universal sub line bundle of $\overline{T^*\mathbf{P}^n} \times_{\mathbf{P}^n} (T^*\mathbf{P}^n + \mathscr{O}_{\mathbf{P}^n})$ (see [39, (6.12)]).

Definition 2.4 (Characteristic class). The *characteristic class* $\mathrm{cc}(A) \in \mathrm{CH}_*(\mathbf{P}^n)$ of an object A of $\mathrm{D}^\mathrm{b}_\mathrm{c}(\mathbf{P}^n)$ is the image of $\overline{\mathrm{CC}(A)} \in \mathrm{CH}_n\left(\overline{T^*\mathbf{P}^n}\right)$ under the inverse of the above isomorphism.

The total Chow group $\mathrm{CH}_*(\mathbf{P}^n)$ is isomorphic to $\mathbf{Z}^{n+1}$, with generators the classes of linear subspaces of dimension 0 to n. Using this, we will view $\mathrm{cc}(A)$ as an element of $\mathbf{Z}^{n+1}$. The lattice $\mathbf{Z}^{n+1}$ inherits the intersection pairing of $\mathrm{CH}_n\left(\overline{T^*\mathbf{P}^n}\right)$, and this pairing is independent of the base field k.

Lemma 2.5. *Let $0 \leqslant m \leqslant n$ be an integer, and let K_m denote the constant $\overline{\mathbf{Q}}_\ell$-sheaf on an m-dimensional linear subspace of $\mathbf{P}^n$, extended by zero and placed in degree $-m$. Then the characteristic cycle $\mathrm{CC}(K_m)$ is the conormal bundle of that m-dimensional subspace, and the characteristic classes $\mathrm{cc}(K_0), \dots, \mathrm{cc}(K_n)$ form a basis of $\mathrm{CH}_*(\mathbf{P}^n)$. In particular, $\mathrm{cc}(K_0), \dots, \mathrm{cc}(K_n)$ form a basis of $\mathbf{Z}^{n+1}$ independent of k and ℓ.*

Proof. The first statement follows from [39, Lem. 5.11(1) and 5.13(2)]. Recall that $\mathrm{CH}_*(\mathbf{P}^n)$ has rank $n+1$, so to show that the characteristic classes $\mathrm{cc}(K_0), \dots, \mathrm{cc}(K_n)$ form a basis, it suffices to prove that they are linearly independent. We will do this by proving that the matrix of intersection numbers $\mathrm{cc}(K_i) \cdot \mathrm{cc}(K_j)$ is invertible. If $i + j < n$, then generic subspaces of dimensions i and j do not intersect, so neither do their conormal bundles, and hence the intersection number is 0. If $i + j = n$, then a generic i-dimensional subspace and a generic j-dimensional subspace intersect transversely at a single point, so their conormal bundles intersect transversely at a single point as well, and hence have intersection number 1. The intersection matrix is thus invertible. The last sentence is just a restatement of this, except for the independence of k and ℓ, which follows from observing that the isomorphism [39, (6.12)], applied to the closure of the conormal bundle of an m-dimensional subspace, can be defined integrally and so is independent of the characteristic. $\square$

3. Complexity and generic linear maps

In this section, we define the complexity of a complex of sheaves on projective space and we establish a few results about generic linear maps between projective spaces and their relationship with characteristic cycles.

Definition 3.1. Let k be a field. Let $0 \leqslant m \leqslant n$ be integers. Let $M_k^{n+1,m+1}$ be the variety of $(n+1) \times (m+1)$ matrices of maximal rank, so that in particular $M_k^{n+1,n+1}$ is equal to $\mathrm{GL}_{n+1,k}$. Given an extension k' of k and a k'-point $\boldsymbol{a} \in M^{n+1,m+1}(k')$, we denote by $l_{\boldsymbol{a}}\colon \mathbf{P}^m_{k'} \to \mathbf{P}^n_{k'}$ the associated linear map.

In the case $m = n$, in which $\boldsymbol{a}$ is an invertible matrix and $l_{\boldsymbol{a}}$ is an automorphism, we may also use $l_{\boldsymbol{a}}$ to refer to the induced map on any scheme depending functorially on $\mathbf{P}^n$ (e.g., its cotangent bundle).

Here is the key definition of this article. Although it is closely related to characteristic cycles, as we will see, we have chosen to define it rather in terms of Betti numbers, so it can be used more directly in situations where some Betti numbers can be computed.

Definition 3.2 (Complexity of a complex of sheaves on projective space)**.** Let k be a field and let $n \geqslant 0$ be an integer. For each $0 \leqslant m \leqslant n$, let $\boldsymbol{a}_m$ be a geometric generic point of $M_k^{n+1,m+1}$ defined over an algebraically closed field k'. The *complexity* $c(A)$ of an object A of $\mathrm{D}_c^b(\mathbf{P}_k^n)$ is the non-negative integer

$$c(A) = \max_{0 \leqslant m \leqslant n} \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}_{k'}^m, l_{\boldsymbol{a}_m}^* A).$$

When working with this definition, we will often use the projection formula to write

$$h^i(\mathbf{P}_{k'}^m, l_{\boldsymbol{a}_m}^* A) = h^i(\mathbf{P}_{k'}^n, A \otimes l_{\boldsymbol{a}_m *} \overline{\mathbf{Q}}_\ell),$$

which gives a slightly different expression for the complexity. Since étale cohomology is invariant under base change of algebraically closed fields, the value of $c(A)$ is independent of the choice of the field of definition k', and we will often drop it from the notation.

Instead of considering geometric generic points of $M^{n+1,m+1}$ in the definition of complexity, one can alternatively consider closed points in a suitable dense open subset of $M^{n+1,m+1}$, as the following lemma shows.

Lemma 3.3. *Let k be a field and $\bar{k}$ an algebraic closure of k. Let $0 \leqslant m \leqslant n$ be an integer, and let A be a complex on $\mathbf{P}_k^n$. There exists a dense open subset $U \subset M_k^{n+1,m+1}$ such that the equality*

$$h^i(\mathbf{P}_{k'}^m, l_{\boldsymbol{a}}^* A) = h^i(\mathbf{P}_{\bar{k}}^m, l_{\boldsymbol{b}}^* A)$$

holds for any geometric generic point $\boldsymbol{a}$ of $M^{n+1,m+1}$ defined over an algebraically closed field k' and for every $\boldsymbol{b} \in U(\bar{k})$ and $i \in \mathbf{Z}$.

Proof. For $\boldsymbol{b} \in M^{m+1,n+1}(\bar{k})$, we view the linear map $l_{\boldsymbol{b}}$ as the composition of the map $x \mapsto (x, \boldsymbol{b})$ from $\mathbf{P}^m$ to $\mathbf{P}^m \times M^{m+1,n+1}$ with the matrix multiplication map

$$\mathrm{mult}\colon \mathbf{P}^m \times M^{m+1,n+1} \to \mathbf{P}^n.$$

Let $p\colon \mathbf{P}^m \times M^{m+1,n+1} \to M^{m+1,n+1}$ be the projection. Let $U \subset M_k^{n+1,m+1}$ be a dense open set such that the complex $p_* \mathrm{mult}^* A$ has lisse cohomology sheaves on U. It follows from the proper base change theorem that the equality

$$h^i(\mathbf{P}_{k'}^m, l_{\boldsymbol{a}}^* A) = h^i(\mathbf{P}_{\bar{k}}^m, l_{\boldsymbol{b}}^* A)$$

holds for every $b \in U(\bar{k})$. $\square$

We also note for later use the inequality $c(l_{\boldsymbol{b}}^* A) \leqslant c(A)$ for all $\boldsymbol{b} \in U(\bar{k})$, which is a straightforward consequence of the previous lemma and the definition of complexity.

Proposition 3.4. *Let A be an object of $\mathrm{D}_c^b(\mathbf{P}_k^n)$. The complexity $c(A)$ vanishes if and only if $A = 0$. More precisely, let d_j be the dimension of the support of the cohomology sheaf $\mathscr{H}^j(A)$ and let r_j be the maximum of the generic ranks of the restrictions of $\mathscr{H}^j(A)$ to the irreducible components of maximal dimension of its support. Setting $d = \max_{j \in \mathbf{Z}} d_j$ and $r = \sum_{\substack{j \in \mathbf{Z} \\ d_j = d}} r_j$, the inequality $c(A) \geqslant r$ holds.*

Proof. Set $m = n - d$ and let $\boldsymbol{a}_m$ be a geometric generic point of $M_k^{n+1,m+1}$ defined over an algebraically closed field k'. The cohomology sheaf $\mathscr{H}^j(l_{\boldsymbol{a}_m}^* A) = l_{\boldsymbol{a}_m}^* \mathscr{H}^j(A)$ vanishes in degrees j such that $d_j < d$, whereas it has finite support S_j for all j such that $d_j = d$. Therefore, the Betti number $h^i(\mathbf{P}_{k'}^n, \mathscr{H}^j(l_{\boldsymbol{a}_m}^* A))$ vanishes for $i \neq 0$ and is equal to the sum of the dimensions of the stalks of $\mathscr{H}^j(l_{\boldsymbol{a}_m}^* A)$ at points of S_j for $i = 0$. By definition, there is a point of S_j at which the stalk has dimension r_j, and hence the inequality $h^i(\mathbf{P}_{k'}^n, \mathscr{H}^0(l_{\boldsymbol{a}_m}^* A)) \geqslant r_j$ holds for all j such that $d_j = d$.

It follows from the vanishing of cohomology in non-zero degrees that the spectral sequence

$$\mathrm{H}^i(\mathbf{P}^n_{k'}, \mathcal{H}^j(l^*_{\boldsymbol{a}_m} A)) \implies \mathrm{H}^{i+j}(\mathbf{P}^n_{k'}, l^*_{\boldsymbol{a}_m} A)$$

degenerates at the first page, hence an isomorphism

$$\mathrm{H}^j(\mathbf{P}^n_{k'}, l^*_{\boldsymbol{a}_m} A) \cong \mathrm{H}^0(\mathbf{P}^n_{k'}, \mathcal{H}^j(l^*_{\boldsymbol{a}_m} A)).$$

By definition of complexity, we then get

$$c(A) \geqslant \sum_{j \in \mathbf{Z}} h^j(\mathbf{P}^n_{k'}, l^*_{\boldsymbol{a}_m} A) = \sum_{j \in \mathbf{Z}} h^0(\mathbf{P}^n_{k'}, \mathcal{H}^j(l^*_{\boldsymbol{a}_m} A)) \geqslant \sum_{\substack{j \in \mathbf{Z} \\ d_j = d}} r_j = r,$$

which concludes the proof, since $r = 0$ if and only if $A = 0$. $\qquad\square$

We will make several arguments involving generic points on different spaces of linear maps. The next two lemmas describe all the properties we will need about relationships between different generic points.

Lemma 3.5. *Let X and Y be geometrically irreducible affine varieties over a field k. Let $x \in X(k')$ and $y \in Y(k')$ be points defined over a field extension k' of k. Let $k(x)$ and $k(y)$ be the fields generated by the coordinates of x and y respectively. If x is a geometric generic point of X_k and y is a geometric generic point of $Y_{k(x)}$, then x is a geometric generic point of $X_{k(y)}$.*

Proof. Letting ξ and η denote the generic points of X and Y respectively, the result follows from the observation that if we identify $X \times \eta$ and $\xi \times Y$ with (irreducible) subschemes of $X \times Y$, then the generic points of $X \times \eta$ and $\xi \times Y$ are the same as the generic point of $X \times Y$. $\qquad\square$

Lemma 3.6. *Let $\boldsymbol{a}$ be an element of $M^{n+1,m+1}(k)$ and let $\boldsymbol{g}$ be a geometric generic point of $\mathrm{GL}_{n+1,k}$. Then $\boldsymbol{g}\boldsymbol{a}$ is a geometric generic point of $M^{n+1,m+1}_k$.*

Proof. Because $\mathrm{GL}_{n+1,k}$ acts transitively on $M^{n+1,m+1}_k$, the map $g \mapsto g\boldsymbol{a}$ from $\mathrm{GL}_{n+1,k}$ to $M^{n+1,m+1}_k$ is surjective, and hence dominant, so the image of $\boldsymbol{g}$ under this map is a geometric generic point of $M^{n+1,m+1}_k$. $\qquad\square$

Lemma 3.7. *Let k be a field and let $0 \leqslant m \leqslant n$ be integers. Let $\boldsymbol{a}$, $\boldsymbol{g}$, and $\boldsymbol{h}$ be points of $M^{n+1,m+1}_k$, $\mathrm{GL}_{n+1,k}$, and $\mathrm{GL}_{m+1,k}$ respectively, all defined over an algebraically closed field k'. Let $\boldsymbol{b} = \boldsymbol{g}\boldsymbol{a}\boldsymbol{h}^{-1}$. Let $k(\boldsymbol{a})$ be the subfield of k' generated by the coordinates of $\boldsymbol{a}$, and similarly for $k(\boldsymbol{g}), k(\boldsymbol{a},\boldsymbol{g})$, and so on. Assume that $\boldsymbol{g}$ is a geometric generic point of $\mathrm{GL}_{n+1,k(\boldsymbol{a})}$ and that $\boldsymbol{h}$ is a geometric generic point of $\mathrm{GL}_{m+1,k(\boldsymbol{a},\boldsymbol{g})}$. Then $\boldsymbol{b}$ is a geometric generic point of $M^{n+1,m+1}_{k(\boldsymbol{a},\boldsymbol{h})}$ and $\boldsymbol{h}$ is a geometric generic point of $\mathrm{GL}_{m+1,k(\boldsymbol{a},\boldsymbol{b})}$.*

Proof. By Lemma 3.5, because $\boldsymbol{g}$ is generic over $k(\boldsymbol{a})$ and $\boldsymbol{h}$ is generic over $k(\boldsymbol{a},\boldsymbol{g})$, $\boldsymbol{g}$ is a geometric generic point of $\mathrm{GL}_{n+1,k(\boldsymbol{a},\boldsymbol{h})}$.

Because $\mathrm{GL}_{n+1,k}$ acts transitively on $M^{n+1,m+1}_k$ by left multiplication, applying a geometric generic point $\boldsymbol{g}$ of $\mathrm{GL}_{n+1,k(\boldsymbol{a},\boldsymbol{h})}$ to a point $\boldsymbol{a}\boldsymbol{h}^{-1}$ of $M^{n+1,m+1}_{k(\boldsymbol{a},\boldsymbol{h})}$ produces a geometric generic point $\boldsymbol{b} = \boldsymbol{g}\boldsymbol{a}\boldsymbol{h}^{-1}$ of $M^{n+1,m+1}_{k(\boldsymbol{a},\boldsymbol{h})}$.

By Lemma 3.5, because $\boldsymbol{h}$ is geometric generic over $k(\boldsymbol{a})$ and $\boldsymbol{b}$ is geometric generic over $k(\boldsymbol{a},\boldsymbol{h})$, it follows that $\boldsymbol{h}$ is geometric generic over $k(\boldsymbol{a},\boldsymbol{b})$. $\qquad\square$

Definition 3.8. For $\lambda_1, \ldots, \lambda_{n+1}$ in $k^\times$, let $\mathrm{Diag}(\lambda_1, \ldots, \lambda_{n+1})$ be the diagonal matrix in $\mathrm{GL}_{n+1}(k)$ whose diagonal entries are $\lambda_1, \ldots, \lambda_{n+1}$.

Lemma 3.9. *Let k be a field and let $n \geqslant 0$ be an integer. Let $\boldsymbol{g}$ be a geometric generic point of $\mathrm{GL}_{n+1,k}$, and let $\lambda_1, \ldots, \lambda_{n+1}$ be independent transcendentals over the field $k(\boldsymbol{g})$ generated by the matrix coefficients of $\boldsymbol{g}$. Then:*

(1) *$\boldsymbol{g}\,\mathrm{Diag}(\lambda_1, \ldots, \lambda_{n+1})\boldsymbol{g}^{-1}$ is a geometric generic point of $\mathrm{GL}_{n+1,k}$.*
(2) *$\boldsymbol{g}\,\mathrm{Diag}(1, \ldots, 1, \lambda_{m+1}, \ldots, \lambda_{n+1})$ is a geometric generic point of $\mathrm{GL}_{n+1,k}$ for all $0 \leqslant m \leqslant n+1$.*

Proof. Because $(\boldsymbol{g}, \lambda_1, \ldots, \lambda_{n+1})$ is a geometric generic point of $\mathrm{GL}_{n+1,k} \times \mathbf{G}_{m,k}^{n+1}$, to check that an element of $\mathrm{GL}_{n+1,k(\boldsymbol{g},\lambda_1,\ldots,\lambda_{n+1})}$, expressed as a function of the tuple $(\boldsymbol{g}, \lambda_1, \ldots, \lambda_{n+1})$ is a geometric generic point of $\mathrm{GL}_{n+1,k}$, it suffices to check that the function in question is a dominant map from $\mathrm{GL}_{n+1,k} \times \mathbf{G}_{m,k}^{n+1}$ to $\mathrm{GL}_{n+1,k}$. For $\boldsymbol{g}\,\mathrm{Diag}(1, \ldots, 1, \lambda_{m+1}, \ldots, \lambda_{n+1})$, dominance is simply the fact that $\mathrm{GL}_{n+1,k}$ acts transitively on itself by left multiplication and for $\boldsymbol{g}\,\mathrm{Diag}(\lambda_1, \ldots, \lambda_{n+1})\boldsymbol{g}^{-1}$ dominance is the well-known fact that generic matrices are diagonalizable. $\qquad\square$

Lemma 3.10. *Let k be a field and let m and m' be integers such that $0 \leqslant m, m' \leqslant n$ and $0 \leqslant m + m' - n$. Let $\boldsymbol{a}$ and $\boldsymbol{a}'$ be independent geometric generic points of $M_k^{n+1,m+1}$ and $M_k^{n+1,m'+1}$ defined over a field k' (i.e. such that $\boldsymbol{a}'$ is a geometric generic of $M_{k(\boldsymbol{a})}^{n+1,m'+1}$). Then there exists a geometric generic point $\boldsymbol{b}$ of $M_k^{n+1,m+m'-n+1}$ such that the image of $l_{\boldsymbol{b}}$ is the intersection of the images of $l_{\boldsymbol{a}}$ and $l_{\boldsymbol{a}'}$ and such that $l_{\boldsymbol{a}*}\overline{\mathbf{Q}}_\ell \otimes l_{\boldsymbol{a}'*}\overline{\mathbf{Q}}_\ell = l_{\boldsymbol{b}*}\overline{\mathbf{Q}}_\ell$.*

Proof. View $\boldsymbol{a}$ and $\boldsymbol{a}'$ as matrices of elements defined over k', and $l_{\boldsymbol{a}}(x) - l_{\boldsymbol{a}'}(y)$ as a linear map from $k'^{m+1+m'+1}$ to k'^{n+1}. Because $\boldsymbol{a}$ and $\boldsymbol{a}'$ are generic and $m + m' - n \geqslant 0$, this map is surjective, with kernel of dimension $m + m' + 1 - n$. We can choose a basis for this kernel by row reduction, and then let the columns of $\boldsymbol{b}$ be $\boldsymbol{a}$ (equivalently $\boldsymbol{a}'$) applied to the vectors in this basis.

We can check that the point $\boldsymbol{b}$ is generic by restricting to the special case where the first $m + m' + 1 - n$ columns of $\boldsymbol{a}$ agree with the first $m + m' + 1 - n$ columns of $\boldsymbol{a}'$, and the other columns are generic. In this case, $\boldsymbol{b}$ will consist of exactly these columns. Since some specialization of generic matrices maps to a generic matrix, generic matrices also map to a generic matrix.

The last claim, on pushforwards, follows from the definition and the calculation of the intersection of the image. $\qquad\square$

Lemma 3.11. *Let A and B be perverse sheaves on $\mathbf{P}_k^n$. For a geometric generic point $\boldsymbol{g} \in \mathrm{GL}_{n+1}(k')$, the object $A \otimes l_{\boldsymbol{g}}^* B[-n]$ is perverse.*

Proof. The external product $A \boxtimes B$ is perverse, and the sheaf $A \otimes l_{\boldsymbol{g}}^* B$ on $\mathbf{P}^n \times \mathrm{GL}_{n+1}$ is the pullback of $A \boxtimes B$ by a smooth morphism of relative dimension $(n+1)^2 - n$, so by [3, 4.2.5] is perverse after shifting by $(n+1)^2 - n$. We then take the pullback to the fiber over a geometric generic point of GL_{n+1}, which preserves perversity after shifting by $-(n+1)^2$ by definition of perversity. $\qquad\square$

Lemma 3.12. *Let $\boldsymbol{a}$ be a geometric generic point of $M_k^{n+1,m+1}$. Let A be an object of $\mathrm{D}_c^b(\mathbf{P}_k^n)$. Then $\mathrm{D}(l_{\boldsymbol{a}}^* A) = l_{\boldsymbol{a}}^* \mathrm{D}(A)[2(m-n)]$, and if A is perverse, then $l_{\boldsymbol{a}}^* A[m-n]$ is perverse.*

Proof. We view the linear map $l_{\boldsymbol{a}}$ as the composition of the map $x \mapsto (x, \boldsymbol{a})$ from $\mathbf{P}^m$ to $\mathbf{P}^m \times M^{m+1,n+1}$ with the matrix multiplication map $\mathbf{P}^m \times M^{m+1,n+1} \to \mathbf{P}^n$. Observe that matrix multiplication is a smooth morphism of relative dimension $(m+1)(n+1) + m - n$ and the pullback to the geometric generic point is an inverse limit of open immersions. As in the previous lemma, the former preserve perversity after shifting by $(m+1)(n+$

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$1) + m - n$, and the latter after shifting by $-(m + 1)(n + 1)$. The statements for duals follow from the equality $f^! = f^*(d)[2d]$ for a smooth morphism f of relative dimension d. $\qquad\square$

Lemma 3.13. *Let $0 \leqslant m \leqslant n - 1$ be an integer, and let A be a perverse sheaf on $\mathbf{P}_k^n$. There exists a dense open subset $U \subset M_k^{n+1,m+1}$ such that, for any geometric generic point $\boldsymbol{a}$ of $M_k^{n+1,m+1}$ defined over an algebraically closed field k' and for any $\boldsymbol{b} \in U(\bar{k})$, the complex $l_{\boldsymbol{b}}^* A[m - n]$ is perverse and satisfies*

$$h^i(\mathbf{P}_{k'}^m, l_{\boldsymbol{a}}^* A) = h^i(\mathbf{P}_{\bar{k}}^m, l_{\boldsymbol{b}}^* A)$$

for every $i \in \mathbf{Z}$.

Proof. Similarly to the proof of Lemma 3.12, we view the linear map $l_{\boldsymbol{b}}$ as the composition of the map $x \mapsto (x, \boldsymbol{b})$ from $\mathbf{P}^m$ to $\mathbf{P}^m \times M^{m+1,n+1}$ with the matrix multiplication map mult $: \mathbf{P}^m \times M^{m+1,n+1} \to \mathbf{P}^n$. Since mult is smooth, the pullback of A along it is perverse up to shift by $(m + 1)(n + 1) - (n - m)$, so it remains to show that, for any perverse $\overline{\mathbf{Q}}_\ell$-sheaf $\mathrm{mult}^* A[(m+1)(n+1)-(n-m)]$ on $\mathbf{P}^m \times M^{m+1,n+1}$, for $\boldsymbol{b}$ in some dense open subset U, the pullback to the fiber over $\boldsymbol{b}$ is perverse up to shift by $(m + 1)(n + 1)$.

To do this, we choose a $\overline{\mathbf{Z}}_\ell$-structure on $\mathrm{mult}^* A[(m + 1)(n + 1) - (n - m)]$, obtaining a perverse $\overline{\mathbf{Z}}_\ell$-sheaf K, and we let U be an open set over which $K \otimes \bar{\mathbf{F}}_\ell$ is locally acyclic. For any point $\boldsymbol{b} \in U(\bar{k})$, after passing to a further open subset we may assume the closure of $\boldsymbol{b}$ is smooth. Then, by [39, Cor. 8.10], the immersion $x \mapsto (x, \boldsymbol{b})$ is K-transversal and thus the pullback along this immersion is a shifted perverse sheaf, as desired.

The second property is granted by Lemma 3.3. $\qquad\square$

We need the following corollary of Theorem 2.2.

Corollary 3.14. *Let A and B be objects of $\mathrm{D}_c^b(\mathbf{P}_k^n)$. For each geometric generic point $\boldsymbol{g}$ of $\mathrm{GL}_{n+1,k}$ over an algebraically closed field k', the following equality holds:*

$$\chi(\mathbf{P}_{k'}^n, A \otimes l_{\boldsymbol{g}}^* B) = (-1)^n \overline{\mathrm{CC}(A)} \cdot \overline{\mathrm{CC}(B)}.$$

Proof. We will prove the statement for objects A and B of $\mathrm{D}_c^b(\mathbf{P}_k^n, \overline{\mathbf{Z}}_\ell)$; the result then follows for A and B in $\mathrm{D}_c^b(\mathbf{P}_k^n, \overline{\mathbf{Q}}_\ell)$ by choosing an integral structure and noting that both the Euler characteristic and the characteristic cycle are preserved by inverting ℓ.

We will check that the singular supports of A and $l_{\boldsymbol{g}}^* B$ fulfil the conditions of Theorem 2.2. From (5) of loc. cit., we will then get the equality

$$\chi(\mathbf{P}_{k'}^n, A \otimes l_{\boldsymbol{g}}^* B) = (-1)^n \overline{\mathrm{CC}(A)} \cdot \overline{\mathrm{CC}(l_{\boldsymbol{g}}^* B)} = (-1)^n \overline{\mathrm{CC}(A)} \cdot \overline{\mathrm{CC}(B)},$$

where the second identity follows from deformation-invariance of intersection numbers, on noting that the algebraic cycles $\overline{\mathrm{CC}(B)}$ and $\overline{\mathrm{CC}(l_{\boldsymbol{g}}^* B)} = l_{\boldsymbol{g}}^* \overline{\mathrm{CC}(B)}$ lie in a family parameterized by $\mathrm{GL}_{n+1,k}$.

The first condition is that $\mathrm{SS}(A) \cap \mathrm{SS}(l_{\boldsymbol{g}}^* B)$ is contained in the zero section. In view of Lemma 2.3, this amounts to showing that $\overline{\mathrm{SS}(A)} \cap \overline{\mathrm{SS}(l_{\boldsymbol{g}}^* B)}$ does not intersect $\mathbf{P}(T^* \mathbf{P}^n)$ inside $\overline{T^* \mathbf{P}^n}$. Note the equality $\overline{\mathrm{SS}(l_{\boldsymbol{g}}^* B)} = l_{\boldsymbol{g}}^* \overline{\mathrm{SS}(B)}$. The space of triples (x, y, g) such that

$$x \in \overline{\mathrm{SS}(A)} \cap \mathbf{P}(T^* \mathbf{P}^n), \quad y \in \overline{\mathrm{SS}(B)} \cap \mathbf{P}(T^* \mathbf{P}^n), \quad g \in \mathrm{GL}_{n+1} \quad \text{and} \quad l_g(y) = x$$

is a bundle over $\overline{\mathrm{SS}(A)} \cap \mathbf{P}(T^* \mathbf{P}^n) \times \overline{\mathrm{SS}(B)} \cap \mathbf{P}(T^* \mathbf{P}^n)$. Since GL_{n+1} acts transitively on $\mathbf{P}(T^* \mathbf{P}^n)$, all its fibers have dimension $(n + 1)^2 - (2n - 1)$, and hence the space itself

has dimension at most

$$\dim\left(\overline{\mathrm{SS}(A)}\cap\mathbf{P}(T^*\mathbf{P}^n)\right)+\dim\left(\overline{\mathrm{SS}(B)}\cap\mathbf{P}(T^*\mathbf{P}^n)\right)+(n+1)^2-(2n-1)$$
$$=(n-1)+(n-1)+(n+1)^2-(2n-1)=(n+1)^2-1.$$

This dimension being less than $(n+1)^2=\dim\mathrm{GL}_{n+1}$, the fiber over generic $\boldsymbol{g}$ is empty, which means that $\overline{\mathrm{SS}(A)}\cap\overline{\mathrm{SS}(l_{\boldsymbol{g}}^*B)}$ does not intersect $\mathbf{P}(T^*\mathbf{P}^n)$.

The second condition is that every irreducible component of

$$\mathrm{SS}(A)\times_{\mathbf{P}^n}\mathrm{SS}(l_{\boldsymbol{g}}^*B)=\mathrm{SS}(A)\times_{\mathbf{P}^n}l_{\boldsymbol{g}}^*\mathrm{SS}(B)$$

has dimension at most n. It is sufficient to prove that every irreducible component of the scheme over GL_{n+1} whose fiber over a point $\boldsymbol{h}$ is $\mathrm{SS}(A)\times_{\mathbf{P}^n}l_{\boldsymbol{h}}^*\mathrm{SS}(B)$ has dimension at most $(n+1)^2+n$, because then the generic fiber has dimension at most n. This scheme maps to $\mathrm{SS}(A)\times\mathrm{SS}(B)$, which has dimension $2n$, and the fiber over any point consists of all elements of GL_{n+1} that send one point in $\mathbf{P}^n$ to another, so the fiber has dimension $(n+1)^2-n$ and the total space has dimension $(n+1)^2+n$, as desired. $\square$

Lemma 3.15. *There exists a unique linear map $f_n\colon\mathbf{Z}^{n+1}\to\mathbf{Z}^n$ such that, for any perverse sheaf A on $\mathbf{P}_k^n$ and any geometric generic point $\boldsymbol{b}$ of $M_k^{n+1,n}$, the equality*

$$\mathrm{cc}(l_{\boldsymbol{b}}^*A)=f_n\left(\mathrm{cc}(A)\right)$$

holds. This linear map does not depend on k and ℓ.

Proof. Let K_m be the perverse sheaf on $\mathbf{P}^{n-1}$ described in Lemma 2.5.

It does not matter what geometric generic point $\boldsymbol{b}$ of $M_k^{n+1,n}$ we take, because any pair of geometric generic points defined over two different fields are isomorphic after a suitable extension of both fields. Let $\boldsymbol{a}$ be a geometric generic point of $M_k^{n+1,n}$, $\boldsymbol{g}$ a geometric generic point of $\mathrm{GL}_{n+1,k(\boldsymbol{a})}$, and $\boldsymbol{h}$ a geometric generic point of $\mathrm{GL}_{n,k(\boldsymbol{a},\boldsymbol{g})}$. Let $\boldsymbol{b}=\boldsymbol{g}\boldsymbol{a}\boldsymbol{h}^{-1}$. Then Lemma 3.7 guarantees that $\boldsymbol{b}$ is a geometric generic point of $M_k^{n+1,n}$ and so we may work with $\boldsymbol{b}$.

Furthermore, $\boldsymbol{h}$ is a geometric generic point of $\mathrm{GL}_{n,k(\boldsymbol{b})}$ by Lemma 3.7, and hence Corollary 3.14 yields the equality

$$\mathrm{cc}(l_{\boldsymbol{b}}^*A)\cdot\mathrm{cc}(K_m)=\chi\left(\mathbf{P}_{k'}^{n-1},l_{\boldsymbol{b}}^*A\otimes l_{\boldsymbol{h}^{-1}}^*K_m\right).$$

Furthermore, since $\boldsymbol{h}$ is generic, the right-hand side is equal to

$$\chi\left(\mathbf{P}_{k'}^{n-1},l_{\boldsymbol{h}}^*l_{\boldsymbol{b}}^*A\otimes K_m\right)=\chi\left(\mathbf{P}_{k'}^{n-1},l_{\boldsymbol{a}}^*l_{\boldsymbol{g}}^*A\otimes K_m\right)$$
$$=\chi\left(\mathbf{P}_{k'}^n,l_{\boldsymbol{g}}^*A\otimes l_{\boldsymbol{a}*}K_m\right)$$
$$=\mathrm{cc}(A)\cdot\mathrm{cc}(l_{\boldsymbol{a}*}K_m)$$

by the projection formula and Corollary 3.14, along with the fact that $\boldsymbol{g}$ is a geometric generic point of $\mathrm{GL}_{n+1,k(\boldsymbol{a})}$. Thus, $\mathrm{cc}(l_{\boldsymbol{b}}^*A)\cdot\mathrm{cc}(K_m)$ is a linear function of $\mathrm{cc}(A)$.

Because the $\mathrm{cc}(K_m)$ form a basis, by Lemma 2.5, the pairings $\mathrm{cc}(l_{\boldsymbol{b}}^*A)\cdot\mathrm{cc}(K_m)$ of $\mathrm{cc}(l_{\boldsymbol{b}}^*A)$ with that basis can be used as coordinates of $\mathrm{cc}(l_{\boldsymbol{b}}^*A)$. Because these pairings are linear functions of $\mathrm{cc}(A)$, the class $\mathrm{cc}(l_{\boldsymbol{b}}^*A)$ is a linear function of A.

By Lemma 2.5, the characteristic classes $\mathrm{cc}(K_m)$ and $\mathrm{cc}(l_{\boldsymbol{a}*}K_m)$ (which is just $\mathrm{cc}(K_m)$ in a different projective space) are independent of ℓ and k, and so the linear map is independent of ℓ and k. $\square$

Definition 3.16. For $n\geqslant 0$, let $f_n\colon\mathbf{Z}^{n+1}\to\mathbf{Z}^n$ be the linear map uniquely determined in Lemma 3.15. We define inductively a bilinear form

$$b_n\colon\mathbf{Z}^{n+1}\times\mathbf{Z}^{n+1}\to\mathbf{Z}$$

by setting $b_0(x, y) = xy$ for $(x, y) \in \mathbf{Z} \times \mathbf{Z}$ and

$$b_n(x, y) = x \cdot y + 4b_{n-1}(f_n(x), f_n(y))$$

for $n \geqslant 1$ and $(x, y) \in \mathbf{Z}^{n+1} \times \mathbf{Z}^{n+1}$, where $x \cdot y$ is the scalar product on $\mathbf{Z}^{n+1}$.

Proposition 3.17. *For any perverse sheaves A and B in $\mathrm{D}^\mathrm{b}_\mathrm{c}(\mathbf{P}^n_k)$, and for any geometric generic point $\boldsymbol{g}$ of $\mathrm{GL}_{n+1,k}$ defined over k', the following holds:*

$$\sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^n_{k'}, A \otimes l^*_{\boldsymbol{g}} B) \leqslant b_n \left(\mathrm{cc}(A), \mathrm{cc}(B) \right).$$

Proof. We prove this proposition by induction on n. The case $n = 0$ is trivial, as then A and B are simply vector spaces, their characteristic classes are their dimensions, and the sum of Betti numbers is the product of the dimensions.

Assume the inequality holds in dimension $n - 1$ and let A and B be perverse sheaves in $\mathrm{D}^\mathrm{b}_\mathrm{c}(\mathbf{P}^n)$. By Lemma 3.11, the object $A \otimes l^*_{\boldsymbol{g}} B[-n]$ is perverse. Set

$$\sigma = \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^n_{k'}, A \otimes l^*_{\boldsymbol{g}} B),$$

which is the quantity we want to estimate. By the definition of the Euler–Poincaré characteristic, we have

$$h^{-n}(\mathbf{P}^n_{k'}, A \otimes l^*_{\boldsymbol{g}} B) = (-1)^n \chi(\mathbf{P}^n_{k'}, A \otimes l^*_{\boldsymbol{g}} B) + \sum_{i \neq -n} (-1)^{i+n+1} h^i(\mathbf{P}^n_{k'}, A \otimes l^*_{\boldsymbol{g}} B)$$

$$\leqslant (-1)^n \chi(\mathbf{P}^n_{k'}, A \otimes l^*_{\boldsymbol{g}} B) + \sum_{i \neq -n} h^i(\mathbf{P}^n_{k'}, A \otimes l^*_{\boldsymbol{g}} B),$$

hence the inequality

$$\sigma = \sum_{i < -n} h^i(\mathbf{P}^n_{k'}, A \otimes l^*_{\boldsymbol{g}} B) + h^{-n}(\mathbf{P}^n_{k'}, A \otimes l^*_{\boldsymbol{g}} B) + \sum_{i > -n} h^i(\mathbf{P}^n_{k'}, A \otimes l^*_{\boldsymbol{g}} B)$$

$$\leqslant 2 \sum_{i < -n} h^i(\mathbf{P}^n_{k'}, A \otimes l^*_{\boldsymbol{g}} B) + (-1)^n \chi(\mathbf{P}^n_{k'}, A \otimes l^*_{\boldsymbol{g}} B) + 2 \sum_{i > -n} h^i(\mathbf{P}^{n-1}_{k'}, A \otimes l^*_{\boldsymbol{g}} B).$$

Let now $\boldsymbol{a} \in M^{n+1,n}(k')$ be a geometric generic point of $M^{n+1,n}_{k(\boldsymbol{g})}$ defined over an algebraically closed field k'. By excision and the t-exactness of affine morphisms, the canonical map

$$\mathrm{H}^i(\mathbf{P}^n_{k'}, A \otimes l^*_{\boldsymbol{g}} B) \to \mathrm{H}^i(\mathbf{P}^{n-1}_{k'}, l^*_{\boldsymbol{a}}(A \otimes l^*_{\boldsymbol{g}} B))$$

is an isomorphism in degrees less than $-n$. Because duality exchanges $l^*_{\boldsymbol{a}}$ and $l^!_{\boldsymbol{a}}$, and because the dual of $A \otimes l^*_{\boldsymbol{g}} B[-n]$ is also perverse, the canonical map

$$\mathrm{H}^i(\mathbf{P}^{n-1}_{k'}, l^!_{\boldsymbol{a}}(A \otimes l^*_{\boldsymbol{g}} B)) \to \mathrm{H}^i(\mathbf{P}^n_{k'}, A \otimes l^*_{\boldsymbol{g}} B)$$

is an isomorphism in degrees greater than $-n$. We thus obtain

$$\sigma \leqslant 2 \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^{n-1}_{k'}, l^*_{\boldsymbol{a}}(A \otimes l^*_{\boldsymbol{g}} B)) + (-1)^n \chi(\mathbf{P}^n_{k'}, A \otimes l^*_{\boldsymbol{g}} B) + 2 \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^{n-1}_{k'}, l^!_{\boldsymbol{a}}(A \otimes l^*_{\boldsymbol{g}} B)).$$

For a geometric generic point $\boldsymbol{a}$, the functor $l^!_{\boldsymbol{a}}$ coincides with $l^*_{\boldsymbol{a}}$, up to a shift and Tate twist, and hence

$$\sigma \leqslant 4 \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^{n-1}_{k'}, l^*_{\boldsymbol{a}}(A \otimes l^*_{\boldsymbol{g}} B)) + (-1)^n \chi(\mathbf{P}^n_{k'}, A \otimes l^*_{\boldsymbol{g}} B).$$

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Now we observe that $l_{\boldsymbol{a}}^*(A \otimes l_{\boldsymbol{g}}^* B) = l_{\boldsymbol{a}}^* A \otimes l_{\boldsymbol{a}}^* l_{\boldsymbol{g}}^* B$. Take $\boldsymbol{h}$ to be a geometric generic point of $\mathrm{GL}_{n,k(\boldsymbol{a},\boldsymbol{g})}$ and $\boldsymbol{b} = \boldsymbol{g}\boldsymbol{a}\boldsymbol{h}^{-1}$. Then $l_{\boldsymbol{a}}^* l_{\boldsymbol{g}}^* B = l_{\boldsymbol{h}}^* l_{\boldsymbol{b}}^* B$. By Lemma 3.7, $\boldsymbol{h}$ is generic over $k(\boldsymbol{a}, \boldsymbol{b})$. Together with the induction hypothesis and Lemma 3.15, we deduce that

$$\sum_{i \in \mathbf{Z}} h^i(\mathbf{P}_{k'}^{n-1}, l_{\boldsymbol{a}}^* A \otimes l_{\boldsymbol{h}}^* l_{\boldsymbol{b}}^* B) \leqslant b_{n-1}(\mathrm{cc}(l_{\boldsymbol{a}}^* A), \mathrm{cc}(l_{\boldsymbol{b}}^* B)) = b_{n-1}(f_n(\mathrm{cc}(A)), f_n(\mathrm{cc}(B))).$$

Finally, in view of the equality

$$(-1)^n \chi(\mathbf{P}_{k'}^n, A \otimes l_{\boldsymbol{g}}^* B) = \mathrm{cc}(A) \cdot \mathrm{cc}(B)$$

given by Lemma 3.14, we obtain $\sigma \leqslant b_n(\mathrm{cc}(A), \mathrm{cc}(B))$ by definition of b_n. $\qquad\square$

4. Test sheaves

The main result of this section is the construction, achieved in Corollary 4.13, of a specific basis of the group $\mathrm{CH}(\mathbf{P}_k^n)$, which will play an essential role in controlling the complexity of a tensor product in the next section.

Definition 4.1 (Test sheaf). A *test sheaf* on $\mathbf{P}_k^n$ is a perverse sheaf A on $\mathbf{P}_k^n$ such that, for any field extension k' of k, any perverse sheaf B on $\mathbf{P}_{k'}^n$, and any generic point $\boldsymbol{g}$ of $\mathrm{GL}_{n+1,k'}$ defined over an algebraically closed field extension k'' of k', the cohomology group $\mathrm{H}^i(\mathbf{P}_{k'}^n, A \otimes l_{\boldsymbol{g}}^* B)$ vanishes in all degrees $i \neq -n$.

For example, a skyscraper sheaf A on $\mathbf{P}_k^n$ is a test sheaf, since the object $A \otimes l_{\boldsymbol{g}}^* B$ is a skyscraper sheaf sitting in degree $-n$ in that case.

Remark 4.2. The key properties of a test sheaf A are the following:
   (1) the functor defined on perverse sheaves by $B \mapsto \mathrm{H}^{-n}(\mathbf{P}_{k'}^n, A \otimes l_{\boldsymbol{g}}^* B)$ is exact,
   (2) the cohomology groups $\mathrm{H}^i(\mathbf{P}_{k'}^n, A \otimes l_{\boldsymbol{g}}^* B)$ and $\mathrm{H}^{-n}(\mathbf{P}_{k'}^n, A \otimes {}^{\mathfrak{p}}\mathscr{H}^i(B))$ agree, up to renumbering, for all objects B of $\mathrm{D}_\mathrm{c}^\mathrm{b}(\mathbf{P}_k^n)$.

Test sheaves are most useful when they are of the form highlighted in the next definition:

Definition 4.3 (Strong test sheaf). Let $d \geqslant 1$ be an integer. A *strong test sheaf of depth d* on $\mathbf{P}_k^n$ is a test sheaf A that admits a filtration

$$0 = F_0 \subset F_1 \cdots \subset F_d = A$$

such that, for each $1 \leqslant j \leqslant d$, the quotient F_j/F_{j-1} is isomorphic to a perverse sheaf (see Lemma 3.12) of the form $l_{\boldsymbol{a}_j *}\overline{\mathbf{Q}}_\ell[m_j]$ for some $0 \leqslant m_j \leqslant n$ and some $\boldsymbol{a}_j \in M^{n+1,m_j+1}(k)$.

Strong test sheaves have the following crucial property:

Proposition 4.4. *Let A be a strong test sheaf of depth d on $\mathbf{P}_k^n$. The inequality*

$$\sum_{i \in \mathbf{Z}} \left| \mathrm{cc}(A) \cdot \mathrm{cc}({}^{\mathfrak{p}}\mathscr{H}^i(B)) \right| \leqslant dc(B)$$

holds for any field extension k' of k and any object B of $\mathrm{D}_\mathrm{c}^\mathrm{b}(\mathbf{P}_{k'}^n)$.

Proof. Denote by $(F_j)_{1 \leqslant j \leqslant d}$ the terms of the filtration from Definition 4.3.
   Let $\boldsymbol{g}$ be a generic point of $\mathrm{GL}_{n+1,k'}$. By Corollary 3.14, we have:

$$\sum_{i \in \mathbf{Z}} \left| \mathrm{cc}(A) \cdot \mathrm{cc}({}^{\mathfrak{p}}\mathscr{H}^i(B)) \right| = \sum_{i \in \mathbf{Z}} |\chi(\mathbf{P}_{k'}^n, A \otimes l_{\boldsymbol{g}}^{*\,\mathfrak{p}}\mathscr{H}^i(B))|$$

$$\leqslant \sum_{i \in \mathbf{Z}} \sum_{j \in \mathbf{Z}} h^j(\mathbf{P}_{k'}^n, A \otimes l_{\boldsymbol{g}}^{*\,\mathfrak{p}}\mathscr{H}^i(B)).$$

Since A is a test sheaf, the vanishing $\mathrm{H}^j(\mathbf{P}^n_{k'}, A \otimes l^{*\mathfrak{p}}_{\boldsymbol{g}}\mathscr{H}^i(B)) = 0$ holds for all $j \neq -n$, and hence the spectral sequence computing $\mathrm{H}^i(\mathbf{P}^n_{k'}, A \otimes l^*_{\boldsymbol{g}}B)$ via the perverse filtration on B degenerates at the first page. It follows that

$$\sum_{i\in\mathbf{Z}}\sum_{j\in\mathbf{Z}} h^j(\mathbf{P}^n_{k'}, A \otimes l^{*\mathfrak{p}}_{\boldsymbol{g}}\mathscr{H}^i(B)) = \sum_{i\in\mathbf{Z}} h^i(\mathbf{P}^n_{k'}, A \otimes l^*_{\boldsymbol{g}}B).$$

On the other hand, the spectral sequence associated to the filtration (F_j) of A yields the inequality of sums of Betti numbers

$$\sum_{i\in\mathbf{Z}} h^i(\mathbf{P}^n_{k'}, A \otimes l^*_{\boldsymbol{g}}B) \leqslant \sum_{j=1}^{d}\sum_{i\in\mathbf{Z}} h^i(\mathbf{P}^n_{k'}, (F_j/F_{j-1}) \otimes l^*_{\boldsymbol{g}}B).$$

By assumption, there exist isomorphisms $F_j/F_{j-1} = l_{\boldsymbol{a}_j *}\overline{\mathbf{Q}}_\ell[m_j]$ for some integer $0 \leqslant m_j \leqslant n$ and k-point $\boldsymbol{a}_j$ of M^{n+1,m_j+1}, so that the following equalities hold:

$$\sum_{j=1}^{d}\sum_{i\in\mathbf{Z}} h^i(\mathbf{P}^n_{k'}, (F_j/F_{j-1}) \otimes l^*_{\boldsymbol{g}}B) = \sum_{j=1}^{d}\sum_{i\in\mathbf{Z}} h^i(\mathbf{P}^n_{k'}, l_{\boldsymbol{a}_j *}\overline{\mathbf{Q}}_\ell \otimes l^*_{\boldsymbol{g}}B)$$

$$= \sum_{j=1}^{d}\sum_{i\in\mathbf{Z}} h^i(\mathbf{P}^{m_j}_{k'}, l^*_{\boldsymbol{a}_j}l^*_{\boldsymbol{g}}B).$$

Since $\boldsymbol{g}$ is generic over k', the product $\boldsymbol{g}\boldsymbol{a}_j$ is a generic point of $M^{n+1,m+1}_{k'}$, and therefore the definition of $c(B)$ implies the inequality

$$\sum_{j=1}^{d}\sum_{i\in\mathbf{Z}} h^i(\mathbf{P}^{m_j}_{k'}, l^*_{\boldsymbol{a}_j}l^*_{\boldsymbol{g}}B) \leqslant \sum_{j=1}^{d} c(B) = dc(B),$$

which concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We will now construct strong test sheaves forming a basis of $\mathrm{CH}(\mathbf{P}^n_k)$.

Lemma 4.5. *Let A and B be objects of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbf{P}^n_k)$. Let $\boldsymbol{g}$ be a generic point of $\mathrm{GL}_{n,k}$. Then there is a canonical isomorphism*

$$A \otimes l^*_{\boldsymbol{g}}B \cong (A \otimes_! l^*_{\boldsymbol{g}}B)[-2n].$$

Proof. Let $l\colon \mathbf{P}^n \times \mathrm{GL}_{n+1} \to \mathbf{P}^n \times \mathrm{GL}_{n+1}$ be the universal morphism given by

$$l(x,h) = (h \cdot x, h)$$

and let $\pi\colon \mathbf{P}^n \times \mathrm{GL}_{n+1} \to \mathbf{P}^n$ be the projection. Then $A \otimes l^*_{\boldsymbol{g}}B$ and $A \otimes_! l^*_{\boldsymbol{g}}B$ are the generic fibers of $\pi^*A \otimes l^*\pi^*B$ and $(\pi^*A \otimes_! l^*\pi^*B)[-2(n+1)^2]$ respectively, so it suffices to prove that the objects $\pi^*A \otimes l^*\pi^*B$ and $\pi^*A \otimes_! l^*\pi^*B$ are isomorphic up to shift.

For this, consider the morphism $(\pi, \pi \circ l)\colon \mathbf{P}^n \times \mathrm{GL}_{n+1} \to \mathbf{P}^n \times \mathbf{P}^n$, which is smooth because GL_{n+1} acts transitively on $\mathbf{P}^n$. Since $\otimes$ commutes with smooth pullbacks and $\otimes_!$ commutes with smooth pullbacks up to shifting by twice the relative dimension, we obtain isomorphisms

$$\pi^*A \otimes l^*\pi^*B = (\pi, \pi \circ l)^*\left(\mathrm{pr}_1^*A \otimes \mathrm{pr}_2^*B\right),$$

$$\pi^*A \otimes_! l^*\pi^*B = (\pi, \pi \circ l)^*\left(\mathrm{pr}_1^*A \otimes_! \mathrm{pr}_2^*B\right)[2(n+1)^2 - 2n].$$

It is then enough to prove that $\mathrm{pr}_1^* A \otimes \mathrm{pr}_2^* B$ and $\mathrm{pr}_1^* A \otimes_! \mathrm{pr}_2^* B$ are isomorphic up to shift. This follows from the computation

$$
\begin{aligned}
\mathrm{pr}_1^* A \otimes_! \mathrm{pr}_2^* B &= \mathrm{D}(\mathrm{D}(\mathrm{pr}_1^* A) \otimes \mathrm{D}(\mathrm{pr}_2^* B)) \\
&= \mathrm{D}(\mathrm{pr}_1^* \mathrm{D}(A)[2n] \otimes \mathrm{pr}_2^* \mathrm{D}(B)[2n]) \\
&= \mathrm{pr}_1^* \mathrm{D}(\mathrm{D}(A)[2n]) \otimes \mathrm{pr}_2^* \mathrm{D}(\mathrm{D}(B)[2n]) \\
&= (\mathrm{pr}_1^* A \otimes \mathrm{pr}_2^* B)[-4n],
\end{aligned}
$$

which uses the standard properties of Verdier duality and $\mathrm{pr}_i^! = \mathrm{pr}_i^*[2n]$. $\qquad\square$

Lemma 4.6. *Let $n \geqslant 0$ be an integer. Let H_1 and H_2 be hyperplanes in $\mathbf{P}_k^n$ that intersect transversely, and consider the commutative diagram*

$$
\begin{array}{ccc}
\mathbf{P}_k^n - H_1 & \xrightarrow{\;\;a\;\;} & \mathbf{P}_k^n \\
{\scriptstyle b}\big\uparrow & & {\scriptstyle c}\big\uparrow \\
\mathbf{P}_k^n - (H_1 \cup H_2) & \xrightarrow{\;\;d\;\;} & \mathbf{P}_k^n - H_2.
\end{array}
$$

The object $a_* b_! \overline{\mathbf{Q}}_\ell[n]$ is isomorphic to $c_! d_* \overline{\mathbf{Q}}_\ell[n]$ and is a test sheaf.

Proof. By adjunction, there is a natural morphism $c_! d_* \overline{\mathbf{Q}}_\ell \to a_* b_! \overline{\mathbf{Q}}_\ell$ extending the identity on $\mathbf{P}_k^n - (H_1 \cup H_2)$, and it suffices to prove that this map is an isomorphism on stalks at all geometric points of $H_1 \cup H_2$. This is obvious save for points of the intersection $H_1 \cup H_2$. For those, we argue as follows: since H_1 and H_2 intersect transversely, étale locally around each such point the diagram looks like

$$
\begin{array}{ccc}
\mathbf{A}_k^{n-2} \times (\mathbf{A}_k^1 - \{0\}) \times \mathbf{A}_k^1 & \xrightarrow{\;\;a\;\;} & \mathbf{A}_k^n \\
{\scriptstyle b}\big\uparrow & & {\scriptstyle c}\big\uparrow \\
\mathbf{A}_k^{n-2} \times (\mathbf{A}_k^1 - \{0\}) \times (\mathbf{A}_k^1 - \{0\}) & \xrightarrow{\;\;d\;\;} & \mathbf{A}_k^{n-2} \times \mathbf{A}^1 \times (\mathbf{A}_k^1 - \{0\}).
\end{array}
$$

Expressing the constant sheaf on the down left corner as an external product and letting $j \colon \mathbf{A}^1 - \{0\} \hookrightarrow \mathbf{A}^1$ denote the inclusion, the Künneth formula then implies that the map $c_! d_* \overline{\mathbf{Q}}_\ell \to a_* b_! \overline{\mathbf{Q}}_\ell$ is locally given by the identity on $\overline{\mathbf{Q}}_\ell \boxtimes j_* \overline{\mathbf{Q}}_\ell \boxtimes j_! \overline{\mathbf{Q}}_\ell$.

We now proceed to the proof that $a_* b_! \overline{\mathbf{Q}}_\ell[n] \simeq c_! d_* \overline{\mathbf{Q}}_\ell[n]$ is a test sheaf. First of all, we check that $a_* b_! \overline{\mathbf{Q}}_\ell[n]$ is perverse. Indeed, all maps in the diagram are affine open immersions, and the direct image and exceptional direct image by those preserve perversity. Next, let B be a perverse sheaf on $\mathbf{P}_k^n$ and let $i \in \mathbf{Z}$. Using the projection formula, we get the equalities

$$
\begin{aligned}
\mathrm{H}^i(\mathbf{P}_{k'}^n, c_! d_* \overline{\mathbf{Q}}_\ell[n] \otimes l_{\boldsymbol{g}}^* B) &= \mathrm{H}^{i+n}(\mathbf{P}_{k'}^n, c_!(d_* \overline{\mathbf{Q}}_\ell \otimes c^* l_{\boldsymbol{g}}^* B)) \\
&= \mathrm{H}_c^{i+n}(\mathbf{P}_{k'}^n - H_2, d_* \overline{\mathbf{Q}}_\ell \otimes c^* l_{\boldsymbol{g}}^* B) \\
&= \mathrm{H}_c^{i+n}(\mathbf{P}_{k'}^n - H_2, c^*(c_! d_* \overline{\mathbf{Q}}_\ell \otimes l_{\boldsymbol{g}}^* B)).
\end{aligned}
$$

The object $c_! d_* \overline{\mathbf{Q}}_\ell \otimes l_{\boldsymbol{g}}^* B$ is perverse by Lemma 3.11. Since pullbacks by open immersions preserve perversity, the object $c^*(c_! d_* \overline{\mathbf{Q}}_\ell \otimes c^* l_{\boldsymbol{g}}^* B)$ is a perverse sheaf on the affine variety $\mathbf{P}_{k'}^n - H_2$, and hence its compactly supported cohomology vanishes when $i + n < 0$ by Artin's vanishing theorem.

Dually, using Lemma 4.5, we get

$$\mathrm{H}^i(\mathbf{P}^n_{k'}, a_*b_!\overline{\mathbf{Q}}_\ell[n] \otimes l^*_{\boldsymbol{g}}B) = \mathrm{H}^{i-n}(\mathbf{P}^n_{k'}, a_*b_!\overline{\mathbf{Q}}_\ell \otimes_! l^*_{\boldsymbol{g}}B)$$
$$= \mathrm{H}^{i-n}(\mathbf{P}^n_{k'}, a_*(b_!\overline{\mathbf{Q}}_\ell \otimes_! a^*l^*_{\boldsymbol{g}}B))$$
$$= \mathrm{H}^{i-n}(\mathbf{P}^n_{k'} - H_1, b_!\overline{\mathbf{Q}}_\ell \otimes_! a^*l^*_{\boldsymbol{g}}B)$$
$$= \mathrm{H}^{i+n}(\mathbf{P}^n_{k'} - H_1, a^*(a_*b_!\overline{\mathbf{Q}}_\ell \otimes_! l^*_{\boldsymbol{g}}B)[-2n]).$$

Arguing as above using Lemma 4.5, we see that $a^*(a_*b_!\overline{\mathbf{Q}}_\ell \otimes_! l^*_{\boldsymbol{g}}B)[-2n]$ is a perverse sheaf on the affine variety $\mathbf{P}^n_{k'} - H_1$, so that its cohomology vanishes for $i + n > 0$ by Artin's vanishing theorem. This completes the proof. $\qquad \square$

Lemma 4.7. *Let* $\boldsymbol{a} \in M^{n+1,m+1}_k(k)$. *If* A *is a test sheaf on* $\mathbf{P}^m_k$, *then* $l_{\boldsymbol{a}*}A$ *is a test sheaf on* $\mathbf{P}^n_k$.

Proof. Let B be a perverse sheaf on $\mathbf{P}^n_{k'}$ and let $\boldsymbol{g}$ be a generic point in $\mathrm{GL}_{n+1,k'}$. By the projection formula, we have

$$\mathrm{H}^i(\mathbf{P}^n_{k'}, l_{\boldsymbol{a}*}A \otimes l^*_{\boldsymbol{g}}B) = \mathrm{H}^i(\mathbf{P}^m_{k'}, A \otimes l^*_{\boldsymbol{a}}l^*_{\boldsymbol{g}}B).$$

Let $\boldsymbol{h}$ be a generic point of $\mathrm{GL}_{m+1,k'(\boldsymbol{a},\boldsymbol{g})}$. Set $\boldsymbol{b} = \boldsymbol{g}\boldsymbol{a}\boldsymbol{h}^{-1}$ and $k'' = k'(\boldsymbol{a},\boldsymbol{g},\boldsymbol{h})$. By Lemma 3.7, the point $\boldsymbol{b}$ is a generic linear embedding and $\boldsymbol{h}$ is generic over the field of definition of $k'(\boldsymbol{b})$, so that

$$\mathrm{H}^i(\mathbf{P}^m_{k''}, A \otimes l^*_{\boldsymbol{a}}l^*_{\boldsymbol{g}}B) = \mathrm{H}^i(\mathbf{P}^m_{k''}, A \otimes l^*_{\boldsymbol{h}}l^*_{\boldsymbol{b}}B) = \mathrm{H}^{i+n-m}(\mathbf{P}^m_{k''}, A \otimes l^*_{\boldsymbol{h}}l^*_{\boldsymbol{b}}B[m-n]).$$

Because $\boldsymbol{b}$ is generic, the object $l^*_{\boldsymbol{b}}B[m-n]$ is perverse by Lemma 3.12, and therefore this cohomology group vanishes for $i + n - m \neq -m$, i.e., for $i \neq -n$. $\qquad \square$

Definition 4.8 (Standard test sheaf). For each integer $0 \leqslant m \leqslant n$, pick a point $\boldsymbol{a}_m \in M^{n+1,m+1}(k)$ and pick transversal hyperplanes H_1 and H_2 in $\mathbf{P}^m_k$. As in Lemma 4.6, let $c \colon \mathbf{P}^m_k - H_2 \to \mathbf{P}^m_k$ and $d \colon \mathbf{P}^m_k - (H_1 \cup H_2) \to \mathbf{P}^m_k - H_2$ be the corresponding immersions. Define the m-th *standard test sheaf* A_m as

$$A_m = l_{\boldsymbol{a}_m*}c_!d_*\overline{\mathbf{Q}}_\ell[m].$$

Remark 4.9. In the case $m = 0$, choosing $\boldsymbol{a}_0$ amounts to choosing a k-point of $\mathbf{P}^n_k$, the only possible choices for H_1 and H_2 are the empty hyperplanes in $\mathbf{P}^0_k = \mathrm{Spec}\,k$, and A_0 is a skyscraper sheaf supported at the chosen point.

Corollary 4.10. *For each* $0 \leqslant m \leqslant n$, *the object* A_m *is a strong test sheaf on* $\mathbf{P}^n_k$ *of depth at most* 4.

Proof. The fact that A_m is a test sheaf follows from applying Lemma 4.6 and Lemma 4.7. Because $l_{\boldsymbol{a}*}$ preserves constant sheaves on linear subspaces, it preserves any filtration into constant sheaves on linear subspaces of the type required for a strong test sheaf, and thus it suffices to find such a filtration for $c_!d_*\overline{\mathbf{Q}}_\ell$.

We use the notation of Lemma 4.6. Let h be the closed immersion of H_2 into $\mathbf{P}^m_k$. Since c is an open immersion, there is a base change isomorphism $d_*\overline{\mathbf{Q}}_\ell = c^*a_*\overline{\mathbf{Q}}_\ell$. By adjunction, there is hence a morphism $c_!d_*\overline{\mathbf{Q}}_\ell = c_!c^*a_*\overline{\mathbf{Q}}_\ell \to a_*\overline{\mathbf{Q}}_\ell$, which is an isomorphism away from H_2. Because $c_!d_*\overline{\mathbf{Q}}_\ell$ vanishes on H_2, the morphism $c_!d_*\overline{\mathbf{Q}}_\ell \to a_*\overline{\mathbf{Q}}_\ell$ has mapping cone $h^*a_*\overline{\mathbf{Q}}_\ell$. The complex $a_*\overline{\mathbf{Q}}_\ell$ has a filtration whose associated quotients are the constant sheaf and a shift of the constant sheaf on H_1, so $h^*a_*\overline{\mathbf{Q}}_\ell$ has a filtration whose associated quotients are the constant sheaf on H_2 and a shift of the constant sheaf on the intersection $H_1 \cap H_2$. The mapping cone triangle gives the desired filtration of $c_!d_*\overline{\mathbf{Q}}_\ell$. $\qquad \square$

Remark 4.11. (1) One can also give a proof "by pure thought" of Corollary 4.10, based on the fact that A_m is equivariant for the subgroup of PGL_{m+1} acting on $\mathbf{P}_k^m$ and preserving the hyperplanes H_1 and H_2. Because this group action has finitely many orbits, and its stabilizers are connected, one can show that the only irreducible elements of the category of perverse sheaves invariant under this group are the intersection cohomology complexes of the closures of the orbits, which in this case are simply the constant sheaves on $\mathbf{P}^m$, H_1, H_2, and $H_1 \cap H_2$. An analogous method is used in the theory of the geometric Satake isomorphism to classify perverse sheaves on the affine Grassmanian that are equivariant for the left action of the formal arc group (see, e.g., [21, proof of Prop. 1]).

(2) The standard test sheaves depend on the choices in Definition 4.8; whenever we use them, we assume implicitly these choices have been made for all m, and that they are the same in the remainder of the arguments.

Lemma 4.12. *Let $(A_j)_{0 \leqslant j \leqslant n}$ be standard test sheaves. The following holds:*

$$\mathrm{cc}(A_{m_1}) \cdot \mathrm{cc}(A_{m_2}) = \begin{cases} 0 & \text{if } m_1 + m_2 < n, \\ 1 & \text{if } m_1 + m_2 = n. \end{cases}$$

Proof. This is a straightforward consequence of Corollary 3.14, which identifies the intersection number $\mathrm{cc}(A_{m_1}) \cdot \mathrm{cc}(A_{m_2})$ with $(-1)^n \chi(\mathbf{P}_{k'}^n, A_{m_1} \otimes l_g^* A_{m_2})$. Indeed, the support of A_m is of dimension m. If $m_1 + m_2 < n$, it follows that the support of A_{m_1} does not intersect that of A_{m_2} after generic translation, and consequently their tensor product vanishes, from which $\mathrm{cc}(A_{m_1}) \cdot \mathrm{cc}(A_{m_2}) = 0$ follows. Similarly, if $m_1 + m_2 = n$, the support of of A_{m_1} intersects that of A_{m_2} after generic translation at a single point. At this point, both sheaves have stalk $\overline{\mathbf{Q}}_\ell$, so their tensor product is a skyscraper sheaf and has Euler–Poincaré characteristic 1. $\qquad\square$

Corollary 4.13. *Let $(A_j)_{0 \leqslant j \leqslant n}$ be standard test sheaves on $\mathbf{P}_k^n$. Then the characteristic classes $(\mathrm{cc}(A_j))_{0 \leqslant j \leqslant n}$ form a basis of $\mathrm{CH}(\mathbf{P}_k^n) \simeq \mathbf{Z}^{n+1}$.*

Proof. Indeed, by Lemma 4.12, the matrix of intersection pairings between the characteristic classes $\mathrm{cc}(A_i)$ is invertible. $\qquad\square$

5. Complexity of a tensor product

In this section, we prove the crucial theorem showing that the complexity controls the sum of Betti numbers of a tensor product of complexes of ℓ-adic sheaves, namely:

Theorem 5.1. *Let A and B be objects of $\mathrm{D}_c^b(\mathbf{P}_k^n)$. Then the estimate*

$$\sum_{i \in \mathbf{Z}} h^i(\mathbf{P}_k^n, A \otimes B) \ll c(A)c(B)$$

holds, with an implied constant that only depends on n.

Remark 5.2. In Theorem 8.1, we state and sketch a proof of a version of this result with an explicit constant.

Before starting the proof, we show how this implies an important corollary.

Corollary 5.3. *Let A and B be objects of $\mathrm{D}_c^b(\mathbf{P}_k^n)$. Then the estimate*

$$c(A \otimes B) \ll c(A)c(B)$$

holds, with an implied constant that only depends on n.

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Proof. Let $\boldsymbol{g}$ be a geometric generic point of $\mathrm{GL}_{n+1,k}$. For $0 \leqslant m \leqslant n$, let $\boldsymbol{a}_m$ be a geometric generic point of $M_k^{n+1,m+1}$. From the definition of complexity and Theorem 5.1, we get the estimate

$$c(A \otimes B) = \sup_{0 \leqslant m \leqslant n} \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}_k^n, A \otimes B \otimes l_{\boldsymbol{a}_m *}\overline{\mathbf{Q}}_\ell)$$

$$\ll c(A) \sup_{0 \leqslant m \leqslant n} c(B \otimes l_{\boldsymbol{a}_m *}\overline{\mathbf{Q}}_\ell).$$

For $0 \leqslant m' \leqslant n$, let $\boldsymbol{a}'_{m'}$ be a geometric generic point of $M_{k(\boldsymbol{a}_m)}^{n+1,m'+1}$. Again by definition, the complexity above is given by

$$c(B \otimes l_{\boldsymbol{a}_m *}\overline{\mathbf{Q}}_\ell) = \sup_{0 \leqslant m' \leqslant n} \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}_k^n, B \otimes l_{\boldsymbol{a}_m *}\overline{\mathbf{Q}}_\ell \otimes l_{\boldsymbol{a}'_{m'} *}\overline{\mathbf{Q}}_\ell)$$

for any $m \leqslant n$. By Lemma 3.10, the equality $l_{\boldsymbol{a}_m *}\overline{\mathbf{Q}}_\ell \otimes l_{\boldsymbol{a}'_{m'} *}\overline{\mathbf{Q}}_\ell = l_{\boldsymbol{b}*}\overline{\mathbf{Q}}_\ell$ holds for a geometric generic point $\boldsymbol{b}$ of $M_k^{n+1,m+m'-n+1}$. This implies the inequality

$$\sum_{i \in \mathbf{Z}} h^i(\mathbf{P}_k^n, B \otimes l_{\boldsymbol{a}_m *}\overline{\mathbf{Q}}_\ell \otimes l_{\boldsymbol{a}'_{m'} *}\overline{\mathbf{Q}}_\ell) \leqslant c(B)$$

for any m and m', from which the result follows. $\qquad\square$

The strategy for proving Theorem 5.1 consists in first establishing a "generic" version of it, and then deducing from this the precise statement. In what follows, we denote by $(A_m)_{0 \leqslant m \leqslant n}$ the family of standard test sheaves from Definition 4.8. For $x \in \mathbf{R}^{n+1} = \mathrm{CH}_*(\mathbf{P}^n) \otimes \mathbf{R}$, we set

$$\|x\| = \sum_{m=0}^{n} |x \cdot \mathrm{cc}(A_m)|$$

Because $(\mathrm{cc}(A_m))_{0 \leqslant m \leqslant n}$ is a basis of $\mathrm{CH}_*(\mathbf{P}^n)$, by Corollary 4.13, and the intersection form is non-degenerate, this is simply the ℓ^1-norm in the dual basis, and in particular defines a norm on $\mathbf{R}^{n+1}$.

Proposition 5.4. *For any object B of $\mathrm{D}_c^b(\mathbf{P}_k^n)$, the following inequality holds:*

$$\sum_{i \in \mathbf{Z}} \| \mathrm{cc}(^{\mathfrak{p}}\mathscr{H}^i(B))\| \leqslant 4(n+1)c(B).$$

Proof. Let $0 \leqslant m \leqslant n$. Because A_m is a strong test sheaf of depth at most 4, Proposition 4.4 yields the inequality

$$\sum_{p \in \mathbf{Z}} |\mathrm{cc}(^{\mathfrak{p}}\mathscr{H}^p(B)) \cdot \mathrm{cc}(A_m)| \leqslant 4c(B).$$

Thus, we have

$$\sum_{p \in \mathbf{Z}} \| \mathrm{cc}(^{\mathfrak{p}}\mathscr{H}^p(B))\| = \sum_{m=0}^{n} \sum_{p \in \mathbf{Z}} |\mathrm{cc}(^{\mathfrak{p}}\mathscr{H}^p(B) \cdot \mathrm{cc}(A_m)| \leqslant 4(n+1)c(B). \qquad\square$$

The second part of the next corollary shows that the complexity of a complex of sheaves is, up to constants, the sum of the norms of the characteristic cycles of its perverse cohomology sheaves.

Corollary 5.5. *Let A and B be objects of $\mathrm{D}_c^b(\mathbf{P}_k^n)$.*

(1) *For a generic point $\boldsymbol{g}$ of $\mathrm{GL}_{n+1,k}$ defined over k', the estimate*

$$\sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^n_{k'}, A \otimes l^*_{\boldsymbol{g}} B) \ll c(A)c(B)$$

holds, with an implied constant that only depends on n.

(2) *We have*

$$c(B) \asymp \sum_{i \in \mathbf{Z}} \| \operatorname{cc}(^{\mathfrak{p}}\mathscr{H}^i(B)) \|,$$

where the implied constants depend only on n.

Proof. For the proof of (1), we apply the spectral sequences associated to the perverse filtrations of A and B to get

$$\sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^n_{k'}, A \otimes l^*_{\boldsymbol{g}} B) \leqslant \sum_{p \in \mathbf{Z}} \sum_{q \in \mathbf{Z}} \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^n_{k'}, {}^{\mathfrak{p}}\mathscr{H}^p(A) \otimes l^*_{\boldsymbol{g}} {}^{\mathfrak{p}}\mathscr{H}^q(B)).$$

Applying Proposition 3.17, and denoting by M_n the norm of the bilinear form b_n, so that the inequality $b_n(\alpha, \beta) \leqslant M_n \|\alpha\| \, \|\beta\|$ holds for any α and β, we get

$$\sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^n_{k'}, A \otimes l^*_{\boldsymbol{g}} B) \leqslant M_n \sum_{p \in \mathbf{Z}} \sum_{q \in \mathbf{Z}} \| \operatorname{cc}(^{\mathfrak{p}}\mathscr{H}^p(A)) \| \, \| \operatorname{cc}(^{\mathfrak{p}}\mathscr{H}^q(B)) \|,$$

which is at most $16(n+1)^2 M_n c(A)c(B)$ by Proposition 5.4.

For (2), one bound follows from Proposition 5.4. In the other direction, we begin as above (with the same $\boldsymbol{g}$) until we reach the bound

$$\sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^n_{k'}, A \otimes l^*_{\boldsymbol{g}} B) \ll \sum_{p \in \mathbf{Z}} \| \operatorname{cc}(^{\mathfrak{p}}\mathscr{H}^p(A)) \| \sum_{q \in \mathbf{Z}} \| \operatorname{cc}(^{\mathfrak{p}}\mathscr{H}^q(B)) \|.$$

Taking $A = l_{\boldsymbol{a_m}*}\overline{\mathbf{Q}}_\ell$, where $\boldsymbol{a_m}$ is an arbitrary element of $M^{n+1,m+1}(k)$, this gives

$$\sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^m_{k'}, l^*_{\boldsymbol{a_m}} l^*_{\boldsymbol{g}} B) \ll \sum_{q \in \mathbf{Z}} \| \operatorname{cc}(^{\mathfrak{p}}\mathscr{H}^q(B)) \|$$

for $0 \leqslant m \leqslant n$. This implies the desired bound, since $\boldsymbol{g}\boldsymbol{a_m}$ is a generic point of $M^{n+1,m+1}_k$ by Lemma 3.6. $\qquad\square$

Our main theorem in this section is the same as Corollary 5.5, but without the generic pullback $l^*_{\boldsymbol{g}}$. To remove it, we will use a double induction strategy. We view a generic point $\boldsymbol{g}$ of GL_{n+1} as a generic diagonal matrix conjugated by a generic invertible matrix. We will show that the theorem remains true if some of the entries of the diagonal matrix are set to 1, by induction on the number of entries. Once all the diagonal entries are set to 1, any conjugate is the identity matrix, and so we obtain our desired statement.

At each step, we have a family of cohomology groups parameterized by $\mathbf{G}_m$, i.e. a complex of sheaves on $\mathbf{G}_m$, and we want to bound the stalk at the identity using the stalk at the generic point. In general, the stalk at the generic point could be very small and the stalk at the identity could still be very large. However, we will now show that this cannot happen as long as we also control the cohomology. In our case, it turns out that controling the cohomology corresponds to a lower-dimensional version of the problem, and we may use induction on the dimension to achieve our goal.

Lemma 5.6. *For each object A of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbf{G}_{m,k})$, the inequality*

(1) $$\sum_{i \in \mathbf{Z}} \dim \mathscr{H}^i(A)_1 \leqslant \sum_{i \in \mathbf{Z}} \dim \mathscr{H}^i(A)_\eta + \sum_{i \in \mathbf{Z}} h^i_c(\mathbf{G}_{m,k}, A)$$

holds, where η denotes the generic point of $\mathbf{G}_{m,k}$.

Proof. We first reduce the proof to the case when A is perverse. On the one hand, the spectral sequence associated to the perverse filtration of A gives the inequality

$$\sum_{i \in \mathbf{Z}} \dim \mathscr{H}^i(A)_1 \leqslant \sum_{j \in \mathbf{Z}} \sum_{i \in \mathbf{Z}} \dim \mathscr{H}^i(^{\mathfrak{p}}\mathscr{H}^j(A))_1.$$

On the other hand, because perverse sheaves are supported in a single degree at the generic point, we have

$$\sum_{i \in \mathbf{Z}} \dim \mathscr{H}^i(A)_\eta = \sum_{j \in \mathbf{Z}} \sum_{i \in \mathbf{Z}} \dim \mathscr{H}^i(^{\mathfrak{p}}\mathscr{H}^j(A))_\eta.$$

Because the compactly supported cohomology of an affine curve with coefficients in a perverse sheaf is concentrated in degrees 0 and 1, the spectral sequence for the perverse filtration of A degenerates and

$$\sum_{i \in \mathbf{Z}} h_c^i(\mathbf{G}_{m,k}, A) = \sum_{j \in \mathbf{Z}} \sum_{i \in \mathbf{Z}} h_c^i(\mathbf{G}_{m,k}, {}^{\mathfrak{p}}\mathscr{H}^j(A)).$$

Therefore, it is sufficient to prove the inequality (1) when A is a perverse sheaf.

If A is perverse, most of the terms in the sums in (1) vanish. Removing all the terms that are known to vanish, the desired inequality can be stated as

$$\dim \mathscr{H}^0(A)_1 + \dim \mathscr{H}^{-1}(A)_1 \leqslant \dim \mathscr{H}^{-1}(A)_\eta + h_c^0(\mathbf{G}_{m,k}, A) + h_c^1(\mathbf{G}_{m,k}, A).$$

Since $\dim \mathscr{H}^{-1}(A)_1 \leqslant \dim \mathscr{H}^{-1}(A_\eta)$, it suffices to show the inequality

$$\dim \mathscr{H}^0(A)_1 \leqslant h_c^0(\mathbf{G}_{m,k}, A) + h_c^1(\mathbf{G}_{m,k}, A),$$

or even

$$\chi_c(\mathbf{G}_{m,k}, A) \geqslant \dim \mathscr{H}^0(A)_1.$$

This follows from the Euler–Poincaré characteristic formula for perverse sheaves on smooth curves, where the local term at the point 1 is at least $\dim \mathscr{H}^0(A)_1$, all other local terms are non-negative, and the global term is a multiple of the Euler–Poincaré characteristic of $\mathbf{G}_m$ and hence vanishes. $\qquad\square$

Let $n \geqslant 1$ be an integer. For each point $\boldsymbol{g}$ of $GL_{n+1,k}$ valued in an algebraically closed field, the fixed points of the linear map $l_{\boldsymbol{g}}$ are the projectivizations of the eigenspaces of $\boldsymbol{g}$. In particular, for any $\lambda \neq 1$, the fixed points of $l_{\mathrm{Diag}(\lambda,1,\dots,1)}$ on $\mathbf{P}^n$ consist of the isolated point $x_0 = [1 : 0 : \cdots : 0]$ and the hyperplane $H_0 = \{[0 : a_1 : \cdots : a_n]\}$. We identify H_0 with $\mathbf{P}^{n-1}$.

In the next two lemmas, we will denote by $U \subseteq \mathbf{P}^n$ the complement of $\{x_0\} \cup H_0$, by $j \colon U \to \mathbf{P}^n$ the corresponding open immersion, and by $\pi \colon U \to H_0 \simeq \mathbf{P}^{n-1}$ the projection map from U to the hyperplane H_0. We also denote by $p \colon \{x_0\} \to \mathbf{P}^n$ the closed immersion of the isolated fixed point and by $h \colon H_0 = \mathbf{P}^{n-1} \to \mathbf{P}^n$ the closed immersion of the hyperplane.

Lemma 5.7. *Let $n \geqslant 1$ be an integer. Let A and B be objects of $\mathrm{D}_c^b(\mathbf{P}_k^n)$. Let λ be a geometric generic point of $\mathbf{G}_{m,k}$ defined over an algebraically closed field k'. With notation*

as above, the following inequality holds:

$$\sum_{i\in\mathbf{Z}} h^i(\mathbf{P}^n_k, A\otimes B) \leqslant \sum_{i\in\mathbf{Z}} h^i(\mathbf{P}^n_{k'}, A\otimes l^*_{\mathrm{Diag}(\lambda,1,\ldots,1)}B)$$

$$+ 2\sum_{i\in\mathbf{Z}} h^i(\mathrm{Spec}(k), p^*A\otimes p^*B) + 2\sum_{i\in\mathbf{Z}} h^i(\mathbf{P}^{n-1}_k, h^*A\otimes h^*B)$$

$$+ \sum_{i\in\mathbf{Z}} h^i(\mathbf{P}^{n-1}_k, \pi_!j^*A\otimes \pi_!j^*B).$$

Proof. Let K be the pushforward of $A\otimes l^*_{\mathrm{Diag}(\mu,1,\ldots,1)}B$ from $\mathbf{P}^n\times\mathbf{G}_m$ to $\mathbf{G}_m$, where $\mathbf{G}_m$ has coordinate μ. Proper base change yields the equalities

$$\sum_{i\in\mathbf{Z}} \dim\mathscr{H}^i(K)_1 = \sum_{i\in\mathbf{Z}} h^i(\mathbf{P}^n_k, A\otimes B),$$

$$\sum_{i\in\mathbf{Z}} \mathscr{H}^i(K)_\eta = \sum_{i\in\mathbf{Z}} h^i(\mathbf{P}^n_{k'}, A\otimes l^*_{\mathrm{Diag}(\lambda,1,\ldots,1)}B).$$

Hence, by Lemma 5.6, it suffices to prove the inequality

$$\sum_{i\in\mathbf{Z}} h^i_c(\mathbf{G}_{m,k}, K) \leqslant 2\sum_{i\in\mathbf{Z}} h^i(\mathrm{Spec}(k), p^*A\otimes p^*B)$$

$$+ 2\sum_{i\in\mathbf{Z}} h^i(\mathbf{P}^{n-1}_k, h^*A\otimes h^*B) + \sum_{i\in\mathbf{Z}} h^i(\mathbf{P}^{n-1}_k, (\pi_!j^*A)\otimes(\pi_!j^*B)).$$

By the Leray spectral sequence, the equality

$$(2) \qquad \sum_{i\in\mathbf{Z}} h^i_c(\mathbf{G}_{m,k}, K) = \sum_{i\in\mathbf{Z}} h^i_c\left(\mathbf{P}^n\times\mathbf{G}_m, A\otimes l^*_{\mathrm{Diag}(\mu,1,\ldots,1)}B\right).$$

holds. To compute the right-hand side, we partition $\mathbf{P}^n\times\mathbf{G}_m = X_0\cup X_1\cup X_2$ as

$$X_0 = \{x_0\}\times\mathbf{G}_m, \quad X_1 = H_0\times\mathbf{G}_m, \quad X_2 = U\times\mathbf{G}_m.$$

Note that X_0 and X_1 are closed, and X_2 is open. By excision, the right-hand side of (2) is bounded by

$$\sum_{j=0}^{2}\sum_{i\in\mathbf{Z}} h^i_c(X_j, A\otimes l^*_{\mathrm{Diag}(\mu,1,\ldots,1)}B).$$

Since X_0 and X_1 are fixed points of $l^*_{\mathrm{Diag}(\lambda,1,\ldots,1)}$ for all λ, the restriction of the complex $A\otimes l^*_{\mathrm{Diag}(\lambda,1,\ldots,1)}B$ to either of them is the same as that of $A\otimes B$. The Künneth formula gives

$$\mathrm{H}^*_c(X_0, A\otimes B) \simeq \mathrm{H}^*(\{x_0\}, A\otimes B)\otimes\mathrm{H}^*_c(\mathbf{G}_m, \overline{\mathbf{Q}}_\ell)$$

$$\mathrm{H}^*_c(X_1, A\otimes B) \simeq \mathrm{H}^*(H_0, A\otimes B)\otimes\mathrm{H}^*_c(\mathbf{G}_m, \overline{\mathbf{Q}}_\ell),$$

and hence (since the sum of Betti numbers of $\mathbf{G}_m$ is equal to 2) we get

$$\sum_{i\in\mathbf{Z}} h^i_c(X_0, A\otimes B) = 2\sum_{i\in\mathbf{Z}} h^i(\{x_0\}, A\otimes B) = 2\sum_{i\in\mathbf{Z}} h^i(\mathrm{Spec}(k), p^*A\otimes p^*B)$$

$$\sum_{i\in\mathbf{Z}} h^i_c(X_1, A\otimes B) = 2\sum_{i\in\mathbf{Z}} h^i(H_0, A\otimes B) = 2\sum_{i\in\mathbf{Z}} h^i(\mathbf{P}^{n-1}, h^*A\otimes h^*B).$$

Finally, note that the $\mathbf{G}_m$-action on U is free, so that there is an isomorphism between X_2 and the space of pairs of points (x,y) in $U\times U$ such that x and y lie in the same orbit of the $\mathbf{G}_m$-action, which identifies the projection to U and the pullback along $l_{\mathrm{Diag}(\lambda,1,\ldots,1)}$ with the first and the second projections. The space of such pairs is isomorphic to

the fiber product of U with itself over the quotient $U/\mathbf{G}_m$. There is an isomorphism $U/\mathbf{G}_m \simeq H_0 \simeq \mathbf{P}^{n-1}$ induced by π, hence the equality

$$\sum_{i \in \mathbf{Z}} h_c^i(X_2, A \otimes l^*_{\mathrm{Diag}(\lambda,1,\ldots,1)}B) = \sum_{i \in \mathbf{Z}} h_c^i(U \times_{\mathbf{P}^{n-1}} U, j^*A \boxtimes j^*B).$$

By the Künneth formula (or the projection formula applied twice), the right-hand side is also equal to

$$\sum_{i \in \mathbf{Z}} h_c^i(U, A \otimes \pi_! j^*B) = \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}_k^{n-1}, \pi_! j^*A \otimes \pi_! j^*B).$$

Gathering all these computations, we obtain the result. $\qquad\square$

Lemma 5.8. *Let $n \geqslant 1$. Let A be an object of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbf{P}^n_k)$ and let $\boldsymbol{g}$ be a geometric generic point of $\mathrm{GL}_{n+1,k}$ defined over an algebraically closed field k'. Then the following inequalities hold:*

$$c(h^*l^*_{\boldsymbol{g}}A) \leqslant c(A),$$
$$c(\pi_! j^*l^*_{\boldsymbol{g}}A) \leqslant 3c(A),$$
$$\sum_{i \in \mathbf{Z}} h^i(\mathrm{Spec}(k'), p^*l^*_{\boldsymbol{g}}A) \leqslant c(A).$$

Proof. For each $0 \leqslant m \leqslant n-1$, let $\boldsymbol{a}_m$ be a geometric generic point of $M^{n,m+1}_{k(\boldsymbol{g})}$. By definition,

$$c(h^*l^*_{\boldsymbol{g}}A) = \sup_m \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^m, l^*_{\boldsymbol{a}_m} h^* l^*_{\boldsymbol{g}}A).$$

The composition $l_{\boldsymbol{g}} \circ h \circ l_{\boldsymbol{a}_m}$ is a generic linear embedding from $\mathbf{P}^m$ to $\mathbf{P}^n$, and hence this quantity is $\leqslant c(A)$ by definition.

For any $i \in \mathbf{Z}$, by proper base change, there is an isomorphism

$$(3) \qquad \mathrm{H}^i(\mathbf{P}^m, l^*_{\boldsymbol{a}_m} \pi_! j^* l^*_{\boldsymbol{g}}A) = \mathrm{H}^i(\pi^{-1} l_{\boldsymbol{a}_m}(\mathbf{P}^m), l^*_{\boldsymbol{g}}A).$$

The inverse image $\pi^{-1} l_{\boldsymbol{a}_m}(\mathbf{P}^m)$ is of the form

$$L - \{x_0\} - (H_0 \cap L),$$

where L is a general $(m+1)$-dimensional subspace of $\mathbf{P}^n$ that contains x_0. By excision, the sum of the Betti numbers of (3) is at most

$$\sum_{i \in \mathbf{Z}} h^i(L, l^*_{\boldsymbol{g}}A) + \sum_{i \in \mathbf{Z}} h^i(\{x_0\}, l^*_{\boldsymbol{g}}A) + \sum_{i \in \mathbf{Z}} h^i(H_0 \cap L, l^*_{\boldsymbol{g}}A).$$

Each of these is bounded by $c(A)$ (since $\boldsymbol{a}_m$ and $l_{\boldsymbol{g}}$ are generic, see Lemmas 3.5 and 3.6), hence $c(\pi_! j^* l^*_{\boldsymbol{g}}A) \leqslant 3c(A)$.

Finally, $l_{\boldsymbol{g}} \circ p$ is the inclusion map of a generic point, so the inequality

$$\sum_{i \in \mathbf{Z}} h^i(\mathrm{Spec}(k'), p^*l^*_{\boldsymbol{g}}A) \leqslant c(A)$$

holds by the definition of $c(A)$. $\qquad\square$

We are finally ready to prove the main theorem of this section.

Proof of Theorem 5.1. We prove the theorem by induction on $n \geqslant 0$. The theorem holds for $n = 0$, because A and B are then simply complexes of vector spaces so

$$c(A) = \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^0, A),$$

and hence
$$\sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^0, A \otimes B) = \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^0, A) \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^0, B).$$

Now assume that $n \geqslant 1$ and that Theorem 5.1 holds for $\mathbf{P}^{n-1}$.

Let $\boldsymbol{g}$ be a generic point of $\mathrm{GL}_{n+1,k}$ and let $\lambda_1, \ldots, \lambda_{n+1}$ be independent transcendentals over $k(\boldsymbol{g})$. Let k' be an algebraically closed extension of $k(\boldsymbol{g})$ containing $\lambda_1, \ldots, \lambda_{n+1}$. For $1 \leqslant m \leqslant n+1$, we denote
$$\delta_m = \mathrm{Diag}(1, \ldots, 1, \lambda_m, \ldots, \lambda_{n+1}).$$

Then
$$\sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^n_k, A \otimes B) = \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^n_{k'}, l^*_{\boldsymbol{g}} A \otimes l^*_{\boldsymbol{g}} B) = \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^n_{k'}, l^*_{\mathrm{Diag}(1,\ldots,1)} l^*_{\boldsymbol{g}} A \otimes l^*_{\boldsymbol{g}} B).$$

On the other hand, we have
$$\sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^n_{k'}, l^*_{\delta_{n+1}} l^*_{\boldsymbol{g}} A \otimes l^*_{\boldsymbol{g}} B) = \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^n_{k'}, A \otimes l^*_{\boldsymbol{g} \delta_{n+1} \boldsymbol{g}^{-1}} B)$$
$$\ll c(A) c(B)$$

by Corollary 5.5 (1) and Lemma 3.9 (1).

Therefore, using a telescoping sum, we get
$$\sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^n_k, A \otimes B)$$
$$= \sum_{m=1}^{n+1} \left(\sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^n_{k'}, l^*_{\boldsymbol{g}} A \otimes l^*_{\delta_{m+1}} l^*_{\boldsymbol{g}} B) - \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^n_{k'}, l^*_{\boldsymbol{g}} A \otimes l^*_{\delta_m} l^*_{\boldsymbol{g}} B) \right)$$
$$+ O(c(A) c(B)).$$

By Lemma 5.7, for a fixed m, the term between parentheses is
$$\leqslant 2 \sum_{i \in \mathbf{Z}} h^i(\mathrm{Spec}\, k', p^* l^*_{\boldsymbol{g}} A \otimes p^* l^*_{\delta_{m+1}} l^*_{\boldsymbol{g}} B)$$
$$+ 2 \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^{n-1}_{k'}, h^* l^*_{\boldsymbol{g}} A \otimes h^* l^*_{\delta_{m+1}} l^*_{\boldsymbol{g}} B)$$
$$+ \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^{n-1}_{k'}, (\pi_! j^* l^*_{\boldsymbol{g}} A) \otimes (\pi_! j^* l^*_{\delta_{m+1}} l^*_{\boldsymbol{g}} B)).$$

Applying the induction hypothesis, this is
$$\ll 2 c(p^* l^*_{\boldsymbol{g}} A) c(p^* l^*_{\delta_{m+1}} l^*_{\boldsymbol{g}} B) + 2 c(h^* l^*_{\boldsymbol{g}} A) c(h^* l^*_{\delta_{m+1}} l^*_{\boldsymbol{g}} B) + c(\pi_! j^* l^*_{\boldsymbol{g}} A) c(\pi_! j^* l^*_{\delta_{m+1}} l^*_{\boldsymbol{g}} B)$$

and finally, applying Lemma 5.8 and Lemma 3.9 (2), this is
$$\leqslant 2 c(A) c(B) + 2 c(A) c(B) + 9 c(A) c(B) = 13 c(A) c(B).$$

This concludes the proof. $\qquad\qquad\square$

6. Quantitative sheaf theory on quasi-projective varieties

The goal of this section is to define the complexity of objects of the derived category for any quasi-projective algebraic variety over a field, or rather the pair consisting of such a variety and a given quasi-projective embedding.

6.1. **Definition of complexity and continuity of the six operations.** It turns out that it is not really possible to define a complexity invariant of sheaves on an algebraic variety, with its expected properties, that only depends on the algebraic variety. We give two examples, related to two desirable properties of a complexity function (the second was suggested by an anonymous referee).

Example 6.1. (1) A hypothetical canonical complexity invariant $c(A)$ for objects A of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$ should of course be invariant under automorphisms of X. Another basic requirement for applications of a complexity invariant should be that, if $\mathbf{F}$ is a finite field and k an algebraic closure of $\mathbf{F}$, then there should be, up to isomorphism over k, only finitely many irreducible perverse sheaves of bounded complexity (see Corollary 7.15). However, there are examples of algebraic varieties X over $\mathbf{F}$ with infinitely many $\mathbf{F}$-automorphisms and perverse sheaves A on X such that there are infinitely many pairwise non-isomorphic perverse sheaves among the $\sigma^* A$ for such automorphisms σ (e.g., let X be the affine plane, and let A be a suitable shift of the constant sheaf $\overline{\mathbf{Q}}_\ell$ on an irreducible plane curve).

(2) Consider $X = \mathbf{A}^2$, with coordinates (x, y). For any polynomial $f \in k[x, y]$, the map $u_f \colon (x, y) \mapsto (x, y - f(x))$ is an automorphism of X. Let $i \colon \mathbf{A}^1 \to X$ be the closed immersion $x \mapsto (x, 0)$. For any complexity function c on $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$ that satisfies the "bilinearity" property of Corollary 5.3 and is invariant under automorphisms, the following estimate would hold:

$$c(i_* \overline{\mathbf{Q}}_\ell[1] \otimes u_f^*(i_* \overline{\mathbf{Q}}_\ell[1])) \ll 1.$$

This is impossible since the support of $i_* \overline{\mathbf{Q}}_\ell[1] \otimes u_f^*(i_* \overline{\mathbf{Q}}_\ell[1])$ is the intersection of the line $y = 0$ and its image under u_f, which can be an arbitrarily large finite set. A similar projective example arises from $X = E \times E$ for an elliptic curve E, with the immersion $E \simeq E \times \{0\} \to E \times E$ and the automorphisms $(x, y) \mapsto (x, y + nx)$ for $n \in \mathbf{Z}$.

We will work instead with pairs (X, u) consisting of an algebraic variety X over k and a locally closed immersion $u \colon X \to \mathbf{P}^n_k$ for some integer $n \geqslant 0$, called the *embedding dimension* of (X, u).

Remark 6.2. (1) Recall that a locally closed immersion u is a morphism that can be factored as $u = i \circ j$, where j is an open immersion and i is a closed immersion.

(2) We will often simply denote by X the pair (X, u), when no confusion is possible, and call this simply a *quasi-projective variety*. We will also sometimes do this over a more general base S than the spectrum of a field.

Definition 6.3 (Complexity of a complex of sheaves on a quasi-projective variety)**.** Let (X, u) be a quasi-projective variety over an algebraically closed field k. The *complexity relative to u* of an object A of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$ is the non-negative integer

$$c_u(A) = c(u_! A).$$

When the embedding is clearly understood, we will simply speak of the complexity of A.

Remark 6.4. The complexity $c_u(A)$ vanishes if and only if $A = 0$ (Proposition 3.4).

The first main objective is to prove that the complexity is under control when performing all usual operations, starting with Grothendieck's six functors (i.e., in the language used in the introduction, these functors are "continuous"). For this purpose, we will also define a complexity invariant for morphisms.

Definition 6.5 (Complexity of a locally closed immersion)**.** Let (X, u) be a quasi-projective variety over k. We define the complexity of u to be

$$c(u) = c_u(\overline{\mathbf{Q}}_\ell) = c(u_! \overline{\mathbf{Q}}_\ell).$$

Definition 6.6 (Complexity of a morphism). Let $f \colon (X, u) \to (Y, v)$ be a morphism of quasi-projective varieties over k. Let m and n be the embedding dimension of X and Y respectively. For integers $0 \leqslant p \leqslant m$ and $0 \leqslant q \leqslant n$, let $\boldsymbol{a}_p$ and $\boldsymbol{b}_q$ be geometric generic points of $M_k^{m+1,p+1}$ and $M_k^{n+1,q+1}$ respectively, all defined over a common algebraically closed field k'.

The *complexity of the morphism f relative to (u, v)* is defined as

$$c_{u,v}(f) = \max_{0 \leqslant p \leqslant m} \max_{0 \leqslant q \leqslant n} \sum_{i \in \mathbf{Z}} h_c^i(X_{k'}, u^* l_{\boldsymbol{a}_p *} \overline{\mathbf{Q}}_\ell \otimes f^* v^* l_{\boldsymbol{b}_q *} \overline{\mathbf{Q}}_\ell).$$

In the vein of Lemma 3.3, the definition of $c_{u,v}(f)$ can be phrased in terms of closed points over a dense open subset $W \subset M_k^{m+1,p+1} \times M_k^{n+1,q+1}$ rather than geometric generic points. Namely, letting $\Gamma \subset \mathbf{P}^m \times \mathbf{P}^n$ denote the graph of f relative to the locally closed immersions u and v, there exist such W with the property that the above Betti numbers are equal to

$$h_c^j(\Gamma \cap (\ker(l_{\boldsymbol{a}_p}) \times \ker(l_{\boldsymbol{b}_q})), \overline{\mathbf{Q}}_\ell)$$

for all points $(\boldsymbol{a}_p, \boldsymbol{b}_q) \in W(\bar{k})$.

Remark 6.7. These definitions are compatible in the sense that, for a locally closed immersion $u \colon X \to \mathbf{P}^n$, using Lemma 3.10, we have $c_{u,\mathrm{Id}}(u) = c(u)$.

In a similar vein, if $i \colon X \to Y$ is an immersion of quasi-projective varieties (X, u) and (Y, v), such that $u = v \circ i$, then $c_{u,v}(i) = c(u)$, which can be seen readily from the definition, the projection formula and Lemma 3.10.

The fundamental result is the following:

Theorem 6.8 (Continuity of the six operations). *Let (X, u) and (Y, v) be quasi-projective varieties over k, and let $f \colon X \to Y$ be a morphism. For objects A and B of $\mathrm{D}_c^b(X)$ and C of $\mathrm{D}_c^b(Y)$, the following estimates hold:*

(4) $$c_u(\mathrm{D}(A)) \ll c(u) c_u(A)$$

(5) $$c_u(A \otimes B) \ll c_u(A) c_u(B)$$

(6) $$c_u(A \otimes_! B) \ll c(u)^3 c_u(A) c_u(B)$$

(7) $$c_u(\mathscr{H}om(A, B)) \ll c(u) c_u(A) c_u(B)$$

(8) $$c_u(f^* C) \ll c_{u,v}(f) c_v(C)$$

(9) $$c_v(f_! A) \ll c_{u,v}(f) c_u(A)$$

(10) $$c_u(f^! C) \ll c(u) c(v) c_{u,v}(f) c_v(C)$$

(11) $$c_v(f_* A) \ll c(u) c(v) c_{u,v}(f) c_u(A).$$

In all these estimates, the implied constants only depend on the embedding dimensions of (X, u) and (Y, v).

In the next subsection, we will prove Theorem 6.8. In later subsections, we will then handle similarly a number of additional operations, e.g., decomposing a complex into irreducible perverse sheaves.

Remark 6.9. Let k_0 be a field of characteristic coprime to ℓ. For any quasi-projective variety (X_0, u_0) over k_0 and any object A_0 of $\mathrm{D}_c^b(X_0)$, we set

$$c_{u_0}(A_0) = c_u(A),$$

where (X, u) is the base change of (X_0, u_0) to an algebraically closed field extension k of k_0 and A is the base change of A_0. Similarly, we define the complexity of a morphism $f_0 \colon (X_0, u_0) \to (Y_0, v_0)$ as

$$c_{u_0, v_0}(f_0) = c_{u,v}(f),$$

where f is the base change of f_0 to an algebraically closed field. Since the complexity is independent of the chosen algebraically closed field, Theorem 6.8 also holds for the complexity over non-algebraically closed fields.

Remark 6.10. Theorem 1.1 from the introduction follows from Theorem 6.8 applied to $X = \mathbf{A}^n$ and $Y = \mathbf{A}^m$ with u and v the standard open immersions to $\mathbf{P}^n$ and $\mathbf{P}^m$ respectively.

6.2. **Proof of the continuity of the six functors.** We begin with a lemma concerning complexes on projective space.

Lemma 6.11. *For each object A of $\mathrm{D}_c^b(\mathbf{P}_k^n)$, the equality $c(A) = c(\mathrm{D}(A))$ holds.*

Proof. Let $\boldsymbol{a}$ be a geometric generic point of of $M_k^{n+1, m+1}$. Using Verdier duality and Lemma 3.12, the computation

$$\sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^m, l_{\boldsymbol{a}}^* A) = \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^m, \mathrm{D}(l_{\boldsymbol{a}}^* A)) = \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^m, l_{\boldsymbol{a}}^* \mathrm{D}(A))$$

yields $c(A) = c(\mathrm{D}(A))$ by definition of complexity. $\qquad\square$

Proof of (4). The equalities

$$c_u(\mathrm{D}(A)) = c(u_! \mathrm{D}(A)) = c(u_* \mathrm{D}(A) \otimes u_! \overline{\mathbf{Q}}_\ell)$$

hold by definition, and hence we get the estimate

$$c_u(\mathrm{D}(A)) = c(\mathrm{D}(u_! A) \otimes u_! \overline{\mathbf{Q}}_\ell) \ll c(\mathrm{D}(u_! A)) c(u_! \overline{\mathbf{Q}}_\ell) = c(u_! A) c(u_! \overline{\mathbf{Q}}_\ell) = c_u(A) c(u)$$

by combining Corollary 5.3 and Lemma 6.11. $\qquad\square$

Proof of (5) *and* (6). By definition, we have

$$c_u(A \otimes B) = c(u_!(A \otimes B)).$$

Then by Corollary 5.3, we obtain

$$c_u(A \otimes B) = c(u_!(A \otimes B)) = c(u_! A \otimes u_! B) \ll c(u_! A) c(u_! B) = c_u(A) c_u(B).$$

Combining this with (4), we obtain (6). $\qquad\square$

Proof of (7). We have by definition of internal Hom

$$c_u(\mathscr{H}\mathrm{om}(A, B)) = c_u(\mathrm{D}(A) \otimes B) \ll c_u(\mathrm{D}(A)) c_u(B) \ll c(u) c_u(A) c_u(B),$$

where the estimates follow by applying (5) and (4). $\qquad\square$

Proof of (8) *and* (9). Let m be the embedding dimension of X and n that of Y. By definition, we have

$$c_u(f^* C) = c(u_! f^* C) = \max_{0 \leqslant p \leqslant m} \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}_{k'}^p, l_{\boldsymbol{a}_p}^* u_! f^* C).$$

Applying three times the projection formula, we have for any p and $i \in \mathbf{Z}$

$$h^i(\mathbf{P}_{k'}^p, l_{\boldsymbol{a}_p}^* u_! f^* C) = h_c^i(Y_{k'}, f_! u^* l_{\boldsymbol{a}_p!} \overline{\mathbf{Q}}_\ell \otimes C).$$

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Applying the Leray spectral sequence for étale cohomology with compact support to the morphism $v\colon Y \to \mathbf{P}^n$, we derive

$$\sum_{i \in \mathbf{Z}} h_c^i(Y_{k'}, f_! u^* l_{\boldsymbol{a}_p!}\overline{\mathbf{Q}}_\ell \otimes C) = \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}_{k'}^n, v_! f_! u^* l_{\boldsymbol{a}_p!}\overline{\mathbf{Q}}_\ell \otimes v_! C).$$

Then, by Theorem 5.1 and the definition, we get

$$\sum_{i \in \mathbf{Z}} h^i(\mathbf{P}_{k'}^n, v_! f_! u^* l_{\boldsymbol{a}_p!}\overline{\mathbf{Q}}_\ell \otimes v_! C) \ll c_v(f_! u^* l_{\boldsymbol{a}_p!}\overline{\mathbf{Q}}_\ell) c_v(C).$$

Thus,

$$c_u(f^* C) \ll c_v(C) \max_{0 \leqslant p \leqslant m} c_v(f_! u^* l_{\boldsymbol{a}_p!}\overline{\mathbf{Q}}_\ell).$$

Applying again the definition and the projection formula three times, we have

$$c_v(f_! u^* l_{\boldsymbol{a}_p!}\overline{\mathbf{Q}}_\ell) = \max_{0 \leqslant q \leqslant n} \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^q, l_{\boldsymbol{b}_q}^* v_! f_! u^* l_{\boldsymbol{a}_p!}\overline{\mathbf{Q}}_\ell)$$

$$= \max_{0 \leqslant q \leqslant n} \sum_{i \in \mathbf{Z}} h_c^i(X_{k'}, f^* v^* l_{\boldsymbol{b}_q!}\overline{\mathbf{Q}}_\ell \otimes u^* l_{\boldsymbol{a}_p!}\overline{\mathbf{Q}}_\ell)$$

for any p. This concludes the proof of (8) by definition of $c_{u,v}(f)$, and the proof of (9) is similar. $\qquad\square$

Proof of (10) *and* (11). From $f^! C = \mathrm{D}(f^* \mathrm{D}(C))$ we get the estimates

$$c_u(f^! C) \ll c(u) c_u(f^* \mathrm{D}(C)) \ll c(u) c_{u,v}(f) c_v(\mathrm{D}(C)) \ll c(u) c(v) c_{u,v}(f) c_v(C)$$

by (4) and (8). This establishes (10), and (11) is proved similarly using (9) instead of (8). $\qquad\square$

We can also bound the complexity, with respect to the Segre embedding, of the external product of complexes.

Proposition 6.12. *Let (X, u) and (Y, v) be quasi-projective algebraic varieties over k with embedding dimensions n and m respectively, and consider their product $(X \times Y, u \boxtimes v)$, where $u \boxtimes v = s \circ (u \times v)$ is the composition of $u \times v$ with the Segre embedding*

$$s\colon \mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^{(n+1)(m+1)-1}.$$

Let $p_1\colon X \times Y \to X$ and $p_2\colon X \times Y \to Y$ denote the projections. Then the estimate

$$c_{u \boxtimes v}(A \boxtimes B) \ll c_{u \boxtimes v, u}(p_1) c_{u \boxtimes v, v}(p_2) c_u(A) c_v(B)$$

holds for all objects A of $\mathrm{D}_c^b(X)$ and B of $\mathrm{D}_c^b(Y)$, with an implied constant that only depends on (n, m).

Proof. The estimate

$$c_{u \boxtimes v}(A \boxtimes B) \ll c_{u \boxtimes v}(p_1^* A) c_{u \boxtimes v}(p_2^* B)$$

holds by (5). Since the first factor on the right-hand side satisfies

$$c_{u \boxtimes v}(p_1^* A) \ll c_{u \boxtimes v, u}(p_1) c_u(A)$$

by (8), and similarly for $c_{u \boxtimes v}(p_2^* B)$, the result follows. $\qquad\square$

Remark 6.13. It should be possible to estimate the complexities $c_{u \boxtimes v, u}(p_1)$ and $c_{u \boxtimes v, v}(p_2)$ in terms of $c(u)$ and $c(v)$. Indeed, this amounts to estimating the cohomology of $u_! \overline{\mathbf{Q}}_\ell \boxtimes v_! \overline{\mathbf{Q}}_\ell$ on $\mathbf{P}^n \times \mathbf{P}^m$ restricted to the intersection of a general linear subspace with the inverse image of a general linear subspace under the Segre embedding. One can bound this by

$$c_s(u_! \overline{\mathbf{Q}}_\ell \boxtimes v_! \overline{\mathbf{Q}}_\ell) c_s(\mathscr{F}),$$

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where $\mathscr{F}$ is the constant sheaf on the intersection of a general linear subspace with the inverse image of a general linear subspace under the Segre embedding, and then attempt to estimate the two factors separately.

6.3. Linear operations.

Proposition 6.14. *Let (X, u) be a quasi-projective variety over k.*

(1) *For any distinguished triangle $A_1 \to A_2 \to A_3$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$, the following holds:*

$$c_u(A_2) \leqslant c_u(A_1) + c_u(A_3),$$
$$c_u(A_1) \leqslant c_u(A_2) + c_u(A_3),$$
$$c_u(A_3) \leqslant c_u(A_1) + c_u(A_2).$$

(2) *For any objects A and B of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$, the following inequalities hold:*

$$\max(c_u(A), c_u(B)) \leqslant c_u(A \oplus B) \leqslant c_u(A) + c_u(B).$$

(3) *Let A be an object of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$. For any $h \in \mathbf{Z}$, the following equality holds:*

$$c_u(A[h]) = c_u(A).$$

Proof. The first bound in (1), as well as the (in)equalities in (2) and (3), follow immediately from the definition $c_u(A) = c(u_! A)$ and the expression of the complexity in terms of Betti numbers. The second and third inequalities are then deduced from the first (and from (3)) using the distinguished triangles $A_2 \to A_3 \to A_1[1]$ and $A_3[-1] \to A_1 \to A_2$. □

In general, it is not obvious how sharp the first inequality $c(A_2) \leqslant c(A_1) + c(A_3)$ is. However, in the important special case of the decomposition of a perverse sheaf into its irreducible constituents, there is a more satisfactory estimate.

Theorem 6.15. *Let (X, u) be a quasi-projective variety over k. Let A be an object of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$. For any $i \in \mathbf{Z}$, let n_i be the length of ${}^{\mathfrak{p}}\mathscr{H}^i(A)$ in the abelian category of perverse sheaves and let $(A_{i,j})_{1 \leqslant j \leqslant n_i}$ be the family of its Jordan-Hölder factors, repeated with multiplicity. Then the estimates*

$$c_u(A) \leqslant \sum_{i \in \mathbf{Z}} \sum_{j=1}^{n_i} c_u(A_{i,j}) \ll c(u) c_u(A)$$

hold, with an implied constant that only depends on the embedding dimension of (X, u).

Proof. The first inequality follows immediately from the previous proposition. We prove the second one.

For $i \in \mathbf{Z}$, let m_i be the length of ${}^{\mathfrak{p}}\mathscr{H}^i(u_! A)$ in the category of perverse sheaves on the projective space target of u, and let $(B_{i,j})_{1 \leqslant j \leqslant m_i}$ be the Jordan-Hölder factors (repeated with multiplicity) of these perverse sheaves.

Let $\overline{X}$ be the closure of the image of u. The $B_{i,j}$ are irreducible perverse sheaves with support contained in $\overline{X}$, and hence each $u^* B_{i,j}$ is either zero or is an irreducible perverse sheaf on X.

Because $u_! A$ is supported on $\overline{X}$, its perverse filtration is stable under pullback to $\overline{X}$. The perverse filtration is always stable under open immersions; since u is a locally closed immersion, it follows that it is stable under u^*.

Therefore, we have ${}^{\mathfrak{p}}\mathscr{H}^i(A) = {}^{\mathfrak{p}}\mathscr{H}^i(u^* u_! A) = u^{*\,\mathfrak{p}}\mathscr{H}^i(u_! A)$. This perverse sheaf is an iterated extension of $u^* B_{i,j}, \ldots, u^* B_{i,m_i}$; by the uniqueness of the Jordan-Hölder factors, the perverse sheaves $u^* B_{i,1}, \ldots, u^* B_{i,m_i}$ that do not vanish coincide with the $A_{i,1}, \ldots, A_{i,n_i}$,

including multiplicity. We may therefore assume that the equality $u^*B_{i,j} = A_{i,j}$ holds for $1 \leqslant j \leqslant n_i$ and $u^*B_{i,j} = 0$ for $j > n_i$.

Then we have

$$\sum_{i \in \mathbf{Z}} \sum_{j=1}^{n_i} c_u(A_{i,j}) = \sum_{i \in \mathbf{Z}} \sum_{j=1}^{n_i} c_u(u^*B_{i,j}) \ll c(u) \sum_{i \in \mathbf{Z}} \sum_{j=1}^{m_i} c(B_{i,j})$$

$$\ll c(u) \sum_{i \in \mathbf{Z}} \sum_{j=1}^{m_i} \| \operatorname{cc}(B_{i,j}) \|,$$

by using (8) and Corollary 5.5 (2).

On the other hand, we have

$$\operatorname{cc}({}^{\mathfrak{p}}\mathscr{H}^i(u_!A)) = \sum_{j=1}^{m_j} \operatorname{cc}(B_{i,j}).$$

Since the $B_{i,j}$ are perverse sheaves, their characteristic cycles are effective. The norm on $\mathbf{R}^{n+1}$ is additive on the cone of effective cycles, hence the equality

$$\| \operatorname{cc}({}^{\mathfrak{p}}\mathscr{H}^i(u_!A)) \| = \sum_{j=1}^{m_i} \| \operatorname{cc}(B_{i,j}) \|.$$

The estimate

$$\sum_{i \in \mathbf{Z}} \sum_{j=1}^{n_i} c_u(A_{i,j}) \ll c(u) \sum_{i \in \mathbf{Z}} \| \operatorname{cc}({}^{\mathfrak{p}}\mathscr{H}^i(u_!A)) \| \ll c(u)c(u_!A) = c(u)c_u(A)$$

then follows by Proposition 5.4. $\qquad\square$

Corollary 6.16. *Let (X, u) be a quasi-projective variety over k and $w \colon W \to X$ the embedding of a locally closed subvariety. For each perverse sheaf A on W, the middle extension perverse sheaf $w_{!*}A$ on X satisfies*

$$c_{u \circ w}(A) \ll c(u \circ w)c_u(w_{!*}A),$$
$$c_u(w_{!*}A) \ll (c(u)c(u \circ w))^2 c_{u \circ w}(A),$$

where the implied constants only depend on the embedding dimension of (X, u).

Proof. We recall that $w_{!*}A$ is the image of the canonical morphism

$${}^{\mathfrak{p}}\mathscr{H}^0(w_!A) \to {}^{\mathfrak{p}}\mathscr{H}^0(w_*A)$$

of perverse sheaves. It satisfies $A = w^*w_{!*}A$, and hence the estimates

$$c_{u \circ w}(A) = c_{u \circ w}(w^*w_{!*}A) \ll c_{u \circ w, u}(w)c_u(w_{!*}A) = c(u \circ w)c_u(w_{!*}A)$$

follow from (8) and Remark 6.7. Conversely, Proposition 6.14 gives the bound

$$c_u(w_{!*}A) \leqslant c_u({}^{\mathfrak{p}}\mathscr{H}^0(w_!A)) + c_u({}^{\mathfrak{p}}\mathscr{H}^0(w_*A)).$$

Then Theorem 6.15, combined with (9) and (11) and Remark 6.7 again, gives the estimate

$$c_u(w_{!*}A) \ll c(u)(c_u(w_!A) + c_u(w_*A)) \ll c(u)(c(u \circ w) + c(u \circ w)^2 c(u))c_{u \circ w}(A),$$

as we wanted to show. $\qquad\square$

6.4. **Nearby and vanishing cycles.** In this section, we prove the continuity of the functors of nearby and vanishing cycles. Let S be the spectrum of a strictly Henselian discrete valuation ring, with special point σ and generic point η. Let $\bar\eta$ be a geometric generic point above η. Let $f\colon X \to S$ be a quasi-projective morphism and $u\colon X \to \mathbf{P}^n_S$ a locally closed embedding. We denote by X_σ and X_η the fibers of f over σ and η respectively. We also consider the induced embeddings $u_\sigma\colon X_\sigma \to \mathbf{P}^n$ and $u_\eta\colon X_\eta \to \mathbf{P}^n$. For an object A of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$, we denote by A_σ and A_η the restrictions of A to X_σ and X_η respectively. We denote by Ψ_f (resp. Φ_f) the functor of nearby cycles (resp. of vanishing cycles) from $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$ to $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X_\sigma)$.

Theorem 6.17. *Let X, S and f be as above. For each object $A \in \mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$, the following estimates hold:*

$$c_{u_\sigma}(\Psi_f(A)) \ll c(u_\sigma)c_{u_\eta}(A_\eta),$$
$$c_{u_\sigma}(\Phi_f(A)) \ll c(u_\sigma)c_{u_\eta}(A_\eta) + c_{u_\sigma}(A_\sigma).$$

Before starting the proof, recall the following compatibilities of the nearby cycle functor with pushforwards and pullbacks. Let $h : X' \to X$ be an S-morphism, and $f' = f \circ h$. Recall from [10, Exp. XIII (2.1.7.1)] that, if h is proper, there is a canonical isomorphism

$$\Psi_f h_* \to h_* \Psi_{f'} \tag{12}$$

by proper base change. Moreover, if h is smooth then by [10, Exp. XIII (2.1.7.2)], there is a canonical isomorphism

$$h^* \Psi_f \to \Psi_{f'} h^*. \tag{13}$$

Proof. By definition of the vanishing cycles, there is a distinguished triangle

$$A_\sigma \to \Psi_f(A) \to \Phi_f(A),$$

and hence the inequality $c_{u_\sigma}(\Phi_f(A)) \leqslant c_{u_\sigma}(A_\sigma) + c_{u_\sigma}(\Psi_f(A))$ holds by Proposition 6.14. It is thus enough to prove the first inequality.

We first assume that f is proper, so that u is a closed immersion. Using (12) with $h = u$, we can replace X by $\mathbf{P}^m_S$ and A by $u_!(A)$.

From (12), for every $i \in \mathbf{Z}$ we get the equality

$$h^i(X_{\bar\eta}, A_{\bar\eta}) = h^i(X_\sigma, \Psi_f(A)). \tag{14}$$

By definition, we have

$$c_{u_\sigma}(\Psi_f(A)) = \max_{0 \leqslant m \leqslant n} \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^n_{k'}, l^*_{\boldsymbol{a}_m}(A_\eta)),$$

where $\boldsymbol{a}_m$ is a geometric generic point of $M^{n+1,m+1}_{k(\sigma)}$ defined over an algebraically closed field k', and similarly for $c_{u_\eta}(A_\eta)$. As in the proof of Lemma 3.12, we view $l_{\boldsymbol{a}_m}$ as the composition of the map $s\colon x \mapsto (x, \boldsymbol{a}_m)$ from $\mathbf{P}^m$ to $\mathbf{P}^m \times M^{m+1,n+1}$ with the matrix multiplication map $p\colon \mathbf{P}^m \times M^{m+1,n+1} \to \mathbf{P}^n$. We still denote by p the smooth map $\mathbf{P}^m_S \times_S M^{m+1,n+1}_S \to \mathbf{P}^n_S$. By (13), there is an isomorphism

$$p^* \Psi_f \to \Psi_f p^*, \tag{15}$$

where we still denote by f the natural morphism $\mathbf{P}^m_S \times_S M^{m+1,n+1}_S \to S$.

Let S' denote the strict localization of $M^{m+1,n+1}_S$ at the point $\boldsymbol{a}_m$ and let $s\colon S' \to M^{m+1,n+1}_S$ be the localization map. The scheme S' is the spectrum of a strictly Henselian discrete valuation ring, with special point σ' and generic point η', which are mapped to σ and η respectively by the canonical morphism $s'\colon S' \to S$. The situation is similar

to the discussion preceding [6, Lem. 3.3]. Set $\tilde{k} = k(\bar{\eta}) \times_{k(\eta)} k(\eta')$, which is the fraction field of the strict localization of $M_{\bar{S}}^{m+1,n+1}$, where $\bar{S}$ is the normalization of S in $\bar{\eta}$. Then $\mathrm{Gal}(\bar{\eta}/\eta) = \mathrm{Gal}(\tilde{k}/\eta')$ holds. Let $\bar{\eta}'$ the spectrum of an algebraic closure of $\tilde{k}$ and $P = \mathrm{Gal}(\bar{\eta}'/\tilde{k})$, which is a pro-$p$ group for p the characteristic exponent of k.

Let $f' \colon \mathbf{P}_{S'}^n \to S'$ be the strict localization at $\boldsymbol{a}_m$ of the morphism

$$\mathbf{P}_S^m \times_S M_S^{m+1,n+1} \longrightarrow M_S^{m+1,n+1}.$$

We have the following cartesian diagram:

$$
\begin{array}{ccc}
\mathbf{P}_{S'}^n & \xrightarrow{\;\;f'\;\;} & S' \\
\downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} \quad \searrow{\scriptstyle s'} \\
\mathbf{P}_S^n \times_S M_S^{m+1,n+1} & \longrightarrow & M_S^{m+1,n+1} \longrightarrow S.
\end{array}
$$

with f the composite to S.

As recalled in (13), there is an isomorphism

(16) $$s^* \Psi_f \to \Psi_{s' \circ f'} s^*.$$

By [6, Th. Finitude, Lem. 3.3], for every object B of $\mathrm{D}_c^b(\mathbf{P}_{S'}^n)$ and $i \in \mathbf{Z}$, the equality

(17) $$\Psi_{s' \circ f'}(B) = \Psi_{f'}(B)^P$$

holds. Since P is a pro-p group, taking the P-fixed part is an exact functor, hence the inequality

(18) $$h^i(\mathbf{P}_{k'}^n, \Psi_{f'}(B)^P) \leqslant h^i(\mathbf{P}_{k'}^n, \Psi_{f'}(B)).$$

Combining equations (15), (16) and (17), we find

(19) $$l_{\boldsymbol{a}_m}^* \Psi_f(A) = s^* p^* \Psi_f(A) = s^* \Psi_f(p^* A) = \Psi_{s' \circ f'}(s^* p^* A) = \Psi_{f'}(l_{\boldsymbol{a}_m}^* A)^P.$$

Hence, by (18) applied to $B = s^* p^* A$, we get

(20) $$h^i(\mathbf{P}_{k'}^n, l_{\boldsymbol{a}_m}^* \Psi_f(A)) \leqslant h^i(\mathbf{P}_{k'}^n, \Psi_{f'}(l_{\boldsymbol{a}_m}^* A)).$$

By (14) applied to f', we have

(21) $$h^i(\mathbf{P}_{k'}^n, \Psi_{f'}(l_{\boldsymbol{a}_m}^* A)) = h^i(\mathbf{P}_{\bar{\eta}'}^n, (l_{\boldsymbol{a}_m}^* A)_\eta) = h^i(\mathbf{P}_{\bar{\eta}'}^n, l_{\boldsymbol{a}_m}^*(A_{\bar{\eta}'})).$$

Combining (20) and (21), we find the sought-after inequality

$$c_{u_\sigma}(\Psi_f(A)) \leqslant c_{u_\eta}(A_\eta).$$

We now come back to the general case, i.e. we do not assume that f is proper. We factor the immersion $u \colon X \to \mathbf{P}_S^n$ as $i \circ j$, where i is a closed immersion and j is an open immersion.

By the first step, applied to $j_! A$ and the morphism $\bar{f}$ given by the composition of i and the projection $\mathbf{P}_S^n \to S$, we get $c_{i_\eta}((j_! A)_\eta) \leqslant c_{i_\sigma}(\Psi_{\bar{f}} j_! A)$. By (13), the object $j_\sigma^* \Psi_{\bar{f}} j_! A$ is isomorphic to $\Psi_f(A)$, and hence we get

$$
\begin{aligned}
c_{u_\sigma}(\Psi_f(A)) &\ll c_{u_\sigma, i_\sigma}(j_\sigma) c_{i_\sigma}(\Psi_{\bar{f}} j_! A) \\
&\leqslant c_{u_\sigma, i_\sigma}(j_\sigma) c_{i_\eta}((j_! A)_\eta) \\
&= c_{u_\sigma, i_\sigma}(j_\sigma) c_{u_\eta}(A_\eta) = c(u_\sigma) c_{u_\eta}(A_\eta)
\end{aligned}
$$

by (8) and Remark 6.7. This finishes the proof. $\qquad\square$

Corollary 6.18. *Let k be an algebraically closed field, $f : (X, u) \to (S, v)$ a flat morphism of quasi-projective varieties over k, with S smooth and irreducible of dimension 1. Let σ be a closed point of S. Let $S_{(\sigma)}$ be the strict localization of S at σ. Denote by Ψ_f and Φ_f the nearby and vanishing cycles functors of the base change of f to $S_{(\sigma)}$, which we view as functors from $D^b_c(X)$ to $D^b_c(X_\sigma)$. For any $A \in D^b_c(X)$, the following estimates hold:*

$$c_{u_\sigma}(\Psi_f(A)) \ll c_{u,v}(f)^2 c_u(A),$$
$$c_{u_\sigma}(\Phi_f(A)) \ll c_{u,v}(f)^2 c_u(A).$$

Proof. Let $\Gamma \subset X \times_k S$ be the graph of f. We denote by $h : X \to \Gamma$ the canonical S-morphism, and by $f' : \Gamma \to S$ the projection. Abusively, we will also write $u : \Gamma \to \mathbf{P}^n_S$ for the canonical embedding induced by $u : X \to \mathbf{P}^n_k$.

Since h is an isomorphism, the base change morphism $\Psi_{f'} h_{\eta*} \to h_{\sigma*} \Psi_f$ is an isomorphism as well. Moreover, h_η and h_σ are isomorphisms and the equality $c_{u_\sigma}(B) = c_{u_\sigma}(h_{\sigma*}B)$ holds for any object B of $D^b_c(X_\sigma)$. By Theorem 6.17 applied to the strict localization at σ of $f' : \Gamma \to S$ and h_*A, the following estimates hold:

$$c_{u_\sigma}(\Psi_{f'}(h_*A)) \ll c(u_\sigma) c_{u_\eta}(h_*A_\eta),$$
$$c_{u_\sigma}(\Phi_{f'}(h_*A)) \ll c(u_\sigma) c_{u_\eta}(h_*A_\eta) + c_{u_\sigma}(h_*A_\sigma).$$

Using the base change isomorphism previously quoted, this implies that

$$c_{u_\sigma}(\Psi_f(A)) \ll c(u_\sigma) c_{u_\eta}(A_\eta),$$
$$c_{u_\sigma}(\Phi_f(A)) \ll c(u_\sigma) c_{u_\eta}(A_\eta) + c_{u_\sigma}(A_\sigma).$$

We will now prove the estimates

$$c_{u_\sigma}(A_\sigma) \ll c_{u,v}(f) c_u(A),$$
$$c(u_\sigma) \ll c_{u,v}(f),$$
$$c_{u_\eta}(A_\eta) \ll c_{u,v}(f) c_u(A),$$

which will conclude the proof.

First, let $\delta_\sigma \in D^b_c(S)$ be the rank-one skyscraper sheaf supported at $\sigma \in S$. We have $c_v(\delta_\sigma) = 1$ and $f^* \delta_\sigma$ is the constant sheaf supported on X_σ, so that by Theorem 6.8, we obtain $c_u(f^* \delta_\sigma) \ll c_{u,v}(f) c_v(\delta_\sigma)$. Hence, the estimate

$$c(u_\sigma) = c_u(f^* \delta_\sigma) \ll c_{u,v}(f)$$

holds. This proves the second inequality.

Moreover, we have $A_\sigma = A \otimes \overline{\mathbf{Q}}_{\ell|X_\sigma}$, and hence the estimate

$$c_{u_\sigma}(A_\sigma) \ll c_u(A) c(u_\sigma)$$

holds by Theorem 6.8. Combined with the second inequality, this yields the first one.

For the third inequality, let $\boldsymbol{b}$ be a geometric generic point of M^{n_S, n_S+1} such that the intersection of $v(S)$ and the image of $l_{\boldsymbol{b}}$ in $\mathbf{P}^{n_S}_{k'}$ is finite. This intersection consists of finitely many geometric generic points $\bar{\eta}_1, \ldots, \bar{\eta}_n$ of S. The complex $f^* v^* l_{\boldsymbol{b}*} \overline{\mathbf{Q}}_\ell$ is then the constant sheaf supported on $f^{-1}(\{\bar{\eta}_1, \ldots, \bar{\eta}_n\})$, hence the estimate

$$c_{u_\eta}(A_\eta) \leqslant c_u(A \otimes f^* v^* l_{\boldsymbol{b}*} \overline{\mathbf{Q}}_\ell).$$

From Theorem 6.8 and the definition of $c_{u,v}(f)$, we deduce

$$c_u(A \otimes f^* v^* l_{\boldsymbol{b}*} \overline{\mathbf{Q}}_\ell) \ll c_u(A) c_u(f^* v^* l_{\boldsymbol{b}*} \overline{\mathbf{Q}}_\ell) \leqslant c_u(A) c_{u,v}(f),$$

which ends the proof. $\qquad\square$

6.5. **Uniformity in families.** In Theorem 6.8, the factors $c(u)$ or $c_{u,v}(f)$ appear. The next proposition ensures that these are bounded in algebraic families; for all practical purposes, they will thus behave as constants in applications.

Proposition 6.19. *Let S be a noetherian scheme in which ℓ is invertible. Let X be a scheme of finite type over S, and let $u\colon X \to \mathbf{P}_S^n$ be a locally closed immersion of schemes over S, so that for each geometric points s of S, the pair (X_s, u_s) is a quasi-projective variety over the algebraically closed field $k(s)$. Then:*

   (1) *There exists a constant M_1, depending only on (X, S, u), such that the inequality $c(u_s) \leqslant M_1$ holds for all s.*

   (2) *For any object A of $\mathrm{D}_c^b(X)$, there exists a constant M_2, depending only on (X, S, u, A), such that the inequality $c_{u_s}(A_s) \leqslant M_2$ holds for all s.*

   (3) *Let Y be a scheme over S and let $v\colon Y \to \mathbf{P}_S^{n'}$ be a locally closed immersion of schemes over S. Let $f\colon X \to Y$ be a morphism of schemes over S. Then there exists a constant M_3, depending only on (X, S, u, Y, v, f) such that the inequality $c_{u_s, v_s}(f_s) \leqslant M_3$ holds for all s.*

Proof. By definition of $c(u_s)$, part (1) will follow from (2) applied to $A = \overline{\mathbf{Q}}_\ell$.

We now prove (2). Let $0 \leqslant m \leqslant n$ be an integer. Let $r\colon M_S^{n+1,m+1} \times_S \mathbf{P}_S^m \to \mathbf{P}_S^n$ be the morphism defined by $r(\boldsymbol{a}, x) = l_{\boldsymbol{a}}(x)$. Consider the diagram

$$X \xrightarrow{\ u\ } \mathbf{P}_S^n \xleftarrow{\ r\ } M_S^{n+1,m+1} \times_S \mathbf{P}_S^m$$
$$\downarrow{\scriptstyle \mathrm{pr}_1}$$
$$M_S^{n+1,m+1}.$$

Let s be a geometric point of S. Let $\boldsymbol{a}_m$ be a geometric generic point of the fiber of $M_S^{n+1,m+1}$ over s. Let $i \in \mathbf{Z}$. By the proper base change theorem, the equality

$$\sum_{i \in \mathbf{Z}} h^i(\mathbf{P}_{k'}^m, l_{\boldsymbol{a}_m}^* u_{s!} A_s) = \sum_{i \in \mathbf{Z}} \dim \mathscr{H}^i(\mathrm{pr}_{1*} r^* u_! A)_{\boldsymbol{a}_m}$$

holds. Since u is a locally closed immersion and r and pr_1 are morphisms of finite type, the complex $\mathrm{pr}_{1*} r^* u_! A$ is an object of $\mathrm{D}_c^b(M_S^{n+1,m+1})$, so by constructibility, the sum above is bounded as s varies.

For (3), we argue in a similar way with $M_S^{n+1,m+1} \times_S M_S^{n'+1,m'+1}$. $\qquad\square$

Remark 6.20. For instance, if $f\colon \mathbf{A}^n \to \mathbf{A}^m$ is a morphism given by m polynomials in n variables of degree at most d, then using the universal family of such polynomials it follows that there exists a constant $b(n, m, d)$ such that the inequality $c_{u,v}(f) \leqslant b(n, m, d)$ holds for all f (with the standard embeddings $u\colon \mathbf{A}^n \to \mathbf{P}^n$ and $v\colon \mathbf{A}^m \to \mathbf{P}^m$). We can in fact make some cases fully explicit by appealing to results of Katz [28]. Since this can be useful in some applications, we spell out one such case in the next proposition.

For non-negative integers N, r and d, set

$$B(N, r, d) = 6 \cdot 2^r \cdot (3 + rd)^{N+1}.$$

Proposition 6.21. *Let r, d, n_1 and n_2 be non-negative integers. Let X be a closed subvariety of $\mathbf{A}^{n_1}$ over k defined by r equations of degree at most d. Let Y be any subvariety of $\mathbf{A}_k^{n_2}$. Let u and v be the natural immersions of $\mathbf{A}^{n_1}$ and $\mathbf{A}^{n_2}$ in $\mathbf{P}^{n_1}$ and $\mathbf{P}^{n_2}$ respectively. For any morphism $f\colon X \to Y$ defined by n_2 polynomials of degree at most d, the following inequality holds:*

$$c_{u,v}(f) \leqslant B(n_1, n_2 + r, d).$$

Proof. By definition, the complexity $c_{u,v}(f)$ is the maximum over all integers $m_1 \leqslant n_1$ and $m_2 \leqslant n_2$ of the sum of the compactly supported Betti numbers (with $\overline{\mathbf{Q}}_\ell$-coefficients) of the intersection of X with a generic linear subspace of dimension m_1 in $\mathbf{P}^{n_1}$ and with the pullback by f of a generic linear subspace of dimension m_2 in $\mathbf{P}^{n_2}$. The intersection of the first linear subspace with $\mathbf{A}^{n_1}$ is an affine space of dimension n_1. Its intersection with X is defined by r equations of degree at most d, and the pullback of a linear subspace is defined by $n_2 - m_2$ linear combinations of the polynomials defining f, which are hence also of degree at most d. Thanks to [28, Cor. of Th. 1, p. 34], we deduce the inequalities

$$\sum_{i \in \mathbf{Z}} h_c^i(X, u^* l_{\boldsymbol{a}_{m_1}}^* \overline{\mathbf{Q}}_\ell \otimes f^* v^* l_{\boldsymbol{b}_{m_2}}^* \overline{\mathbf{Q}}_\ell) \leqslant B(m_1, n_2 - m_2 + r, d) \leqslant B(n_1, n_2 + r, d)$$

(with notation as in Definition 6.6) for all m_1 and m_2, and the result follows. $\qquad\square$

Remark 6.22. In some cases, Katz's remark in [28, p. 43] leads to better estimates.

6.6. **The open set of lissity of a complex.** Another invariant governed by the complexity is the degree of the complement of a dense open set where an object of the derived category is lisse (i.e., where all its cohomology sheaves are lisse). Below, by *degree* of a subvariety Z of projective space, we mean the sum of the degrees of its irreducible components (which may have different dimensions).

Theorem 6.23. *Let (X, u) be an irreducible quasi-projective variety over k. Let A be an object of $\mathrm{D}_c^b(X)$. Let Z be the complement of the maximal open subset where X is smooth and A is lisse. Then the estimate*

$$\deg(u(Z)) \ll (3 + s)c(u)c_u(A)$$

holds, where the degrees are computed in the projective space target of u, and s is the degree of the codimension 1 part of the singular locus of X.

Proof. We first assume that A is an irreducible perverse object. Let n be the embedding dimension of (X, u). Let S be the support of A.

If S has codimension $m \geqslant 1$, then A is lisse (being zero) outside S; we will show that the inequality $c_u(A) \geqslant \deg(u(S))$ holds. Indeed, let $\boldsymbol{a}_m$ be a geometric generic point of $M_k^{m+1,n+1}$, so that $l_{\boldsymbol{a}_m} : \mathbf{P}^m \to \mathbf{P}^n$ is a generic linear embedding that intersects S in $\deg(u(S))$ general points. The complex $l_{\boldsymbol{a}_m}^* u_! A$ is then supported on $\deg(u(S))$ points and non-zero on each of them, so the sum of its Betti numbers is $\geqslant \deg(u(S))$, which gives the stated bound.

Now assume that S is equal to X. In case A is lisse on the non-singular locus of X, the subset Z is the singular locus of X, and the statement trivially holds since the left-hand side of the estimate is independent of A. Otherwise, Z has codimension 1 in X. Indeed, were its codimension bigger, A would be lisse on the non-singular locus of X by purity of the branching locus [22, Exp. X, Cor. 3.3]. Let d be the degree of $u(Z)$. Set $m = 1 + \mathrm{codim}(u(X))$, and let $\boldsymbol{a}_m$ be a geometric generic point of $M_k^{m+1,n+1}$, so that $l_{\boldsymbol{a}_m} : \mathbf{P}^m \to \mathbf{P}^n$ is a generic linear embedding that intersects the support of A in a curve C, of which d points lie in the singular locus Z. Let j be the embedding of C in X. Moreover, let $\tilde{C}$ be the normalization of C and let $\pi : \tilde{C} \to C$ be the canonical morphism.

We can view the pullback $j^* A$ of A to C as obtained first by pullback along a smooth morphism $M^{m+1,n+1} \times \mathbf{P}^m \to \mathbf{P}^n$, followed by pullback to the geometric generic fiber. Thus any property of A that is preserved by smooth pullback and by restriction to the geometric generic fiber will be preserved by pullback to C. In particular, by Lemma 3.12 a shift of $j^* A$ is perverse on C, and because an irreducible perverse sheaf on a curve is a shift of a middle-extension sheaf, the pullback $j^* A$ is a shift of a middle extension

sheaf $\mathscr{F}$. In addition, $\mathscr{F}$ is not lisse on at least d points, since neither smooth pullback nor restriction to the geometric generic fiber can make a singular point disappear.

Let $\tilde{\mathscr{F}}$ be the middle extension of $\pi^*\mathscr{F}$ from the maximal open set where it is lisse; then $\mathscr{F}$ is canonically isomorphic to $\pi_*\tilde{\mathscr{F}}$, since both are middle-extension sheaves on C that are isomorphic on a dense open set. Let d' be the number of points where $\mathscr{F}$ is not lisse and that are not contained in the singular support of C. We have $d \leqslant d' + s$ and $\tilde{\mathscr{F}}$ is not lisse in at least d' points.

From the equalities

$$\sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^n, l_{\boldsymbol{a}_m}^* u_! A) = \sum_{i \in \mathbf{Z}} h_c^i(C, j^*A) = \sum_{i \in \mathbf{Z}} h_c^i(C, \mathscr{F}) = \sum_{i \in \mathbf{Z}} h_c^i(\tilde{C}, \tilde{\mathscr{F}}),$$

we get

$$-\chi_c(\tilde{C}, \tilde{\mathscr{F}}) \leqslant \sum_{i \in \mathbf{Z}} h_c^i(\mathbf{P}^n, l_{\boldsymbol{a}_m}^* u_! A) \leqslant c_u(A).$$

Let r be the "generic rank of A" in the sense of Proposition 3.4. By Proposition 3.4, the inequality $r \leqslant c_u(A)$ holds. Since $l_{\boldsymbol{a}_m}$ is a generic linear embedding, r is also the "generic rank of j^*A" in the sense of Proposition 3.4. Since j^*A is a shifted middle-extension sheaf $\mathscr{F}$, r is also the generic rank of $\mathscr{F}$, i.e. the rank of a lisse sheaf of which $\mathscr{F}$ is the middle extension. Since $\tilde{\mathscr{F}}$ be the middle extension of $\pi^*\mathscr{F}$ from the maximal open set where it is lisse, r is also the generic rank of $\tilde{\mathscr{F}}$. Using the Grothendieck-Ogg-Shafarevich formula, we get $\chi_c(\tilde{C}, \tilde{\mathscr{F}}) \leqslant d' - 2r$. From this we derive the inequalities

$$d' \leqslant 2r + c_u(A) \leqslant 3c_u(A).$$

We finally get

$$d \leqslant d' + s \leqslant (3 + s)c_u(A),$$

since $c_u(A) \geqslant 1$.

Now consider the general case. For $i \in \mathbf{Z}$, let n_i be the length of ${}^{\mathfrak{p}}\mathscr{H}^u(A)$ and let $(A_{i,j})_{1 \leqslant j \leqslant n_i}$ be the Jordan–Hölder factors of ${}^{\mathfrak{p}}\mathscr{H}^i(A)$, repeated with multiplicity. The object A is lisse on the intersection over i and j of the maximal open sets $U_{i,j}$ where $A_{i,j}$ is lisse. (Note that all the irreducible perverse factors of a lisse sheaf are lisse, so that if $A_{i,j}$ is not lisse at a point, then neither is A).

By the case of perverse sheaves, the complement of $U_{i,j}$ is the union of subvarieties with total degree $\leqslant (3 + s)c_u(A_{i,j})$. Thus, we obtain subvarieties Z_i with

$$\sum_{i=1}^{m} \deg(Z_i) \leqslant (3 + s) \sum_{i \in \mathbf{Z}} \sum_{j=1}^{n_i} c_u(A_{i,j}) \ll (3 + s)c(u)c_u(A)$$

by Theorem 6.15. $\qquad\square$

6.7. Complexity of the cohomology sheaves of a complex.

In this section, we consider an analogue of Theorem 6.15 where the Jordan–Hölder components of the perverse cohomology sheaves are replaced by the ordinary cohomology sheaves. This is a case where we will only be able to prove "continuity" abstractly, without explicit estimates. Precisely, we have:

Proposition 6.24. *The exist a function* $N\colon \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$ *with the following property: for any quasi-projective algebraic variety* (X, u) *over* k *with embedding dimension* n, *and for any object* A *of* $\mathrm{D}_c^b(X)$, *the following inequality holds:*

$$\sum_{i \in \mathbf{Z}} c_u(\mathscr{H}^i(A)) \leqslant N(n, c_u(A)).$$

We begin with two lemmas.

Lemma 6.25. *For any non-negative integers d and n, there exists $C(d, n) \geqslant 0$ such that, for any algebraically closed field k, for any prime ℓ invertible in k, and for any closed immersion $u \colon X \to \mathbf{P}^n$ whose image is a union of irreducible subvarieties of total degree d, the inequalities*

$$c(u) \leqslant C(d, n)$$
$$\deg(X_s) \leqslant C(d, n)$$

hold, where X_s is the singular locus of X.

Proof. The theory of Chow varieties (see, e.g., [33]) provides a quasi-projective scheme $\mathrm{Chow}_{d,n}$ over k whose points "are" the closed immersions $i \colon X \to \mathbf{P}^n$ whose image is a union of irreducible subvarieties of total degree d. The first inequality then follows from Proposition 6.19 (1), and the second is a consequence of the constructibility of the function that maps a point of $\mathrm{Chow}_{d,n}$ to the degree of the singular locus of the corresponding quasi-projective variety. $\qquad\square$

Lemma 6.26. *For any non-negative integers d and n, there exists $C(d, n) \geqslant 0$ with the following property: for any algebraically closed field k, for any prime ℓ invertible in k, for any quasi-projective variety (X, u) over k with embedding dimension n, and for any object A of $\mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(X)$ with $c_u(A) \leqslant d$, there exists a stratification*

$$X_{n+1} \subseteq X_n \subseteq X_{n-1} \subseteq \ldots X_1 \subseteq X$$

of X such that

- *The subvariety X_1 is the support of A.*
- *For all i, the subvariety $X_i - X_{i-1}$ is smooth.*
- *For all i, the object A is lisse on $X_i - X_{i-1}$.*
- *For all i, the subvariety X_i is a union of at most $C(d, n)$ subvarieties of degree $\leqslant C(d, n)$.*
- *For all i, the inequality $c(u_i) \leqslant C(d, n)$ holds, where $u_i \colon X_i - X_{i+1} \to \mathbf{P}^n$ is the natural immersion.*

Proof. The equality $c_u(A) = c(u_! A)$ holds by definition, so we may replace X with the projective space $\mathbf{P}^n$ (and u with the identity), and A with $u_! A$, provided we describe the subvarieties X_i as forming a stratification of the support of A.

Indeed, we define X_1 to be the support of A, and then inductively X_{i+1} to be the complement in X_i of the maximal smooth open subset of X_i on which A is lisse. On noting the inequality $\dim(X_{i+1}) < \dim(X_i)$, this provides a stratification of the support of A with at most $n+1$ non-empty subvarieties. We denote by v_i the immersion $X_i \to \mathbf{P}^n$.

We now prove by induction on i with $1 \leqslant i \leqslant n + 1$ that X_i is a union of varieties whose total degree is bounded only in terms of (d, n), and that $c(v_i)$ is bounded only in terms of (d, n).

Since X_1 is the support of A, either X_1 is equal to $\mathbf{P}^n$ or it is the complement of the maximum open subset on which A is lisse (in fact, zero). In the second case, Lemma 6.23 shows that that X_1 is a union of varieties of degree at most $3c_u(A)$. In the first case, the same inequality holds trivially. Then by the first inequality in Lemma 6.25, the complexity $c(v_1)$ has a bound in terms of $c_u(A)$ and n, and hence in terms of (d, n). This establishes the base case of the induction.

Assume that the induction assumption holds for some i with $1 \leqslant i \leqslant n$. Then, by (8), we obtain $c_{v_i}(v_i^* A) \leqslant c_{v_i, \mathrm{Id}}(v_i) c(A) = c(v_i) c(A)$. By the definition of X_{i+1}, applying Lemma 6.23 and the second inequality of Lemma 6.25, we deduce that X_{i+1} is a

union of varieties of total degree bounded only in terms of (d, n). The first inequality of Lemma 6.25 applied once more shows that $c(v_{i+1})$ is bounded only in terms of (d, n). This completes the induction.

It only remains to bound the complexity of the immersions u_i. The excision triangle gives

$$u_{i!}\overline{\mathbf{Q}}_\ell \longrightarrow v_{i!}\overline{\mathbf{Q}}_\ell \longrightarrow v_{(i+1)!}\overline{\mathbf{Q}}_\ell$$

in $\mathrm{D}_c^b(\mathbf{P}^n)$, and hence the inequality

$$c(u_i) = c(u_{i!}\overline{\mathbf{Q}}_\ell) \leqslant c(v_{i!}\overline{\mathbf{Q}}_\ell) + c(v_{(i+1)!}\overline{\mathbf{Q}}_\ell) = c(v_i) + c(v_{i+1})$$

follows by Proposition 6.14 (1). $\qquad\square$

Proof of Proposition 6.24. We apply the previous lemma with $d = c_u(A)$, and we let $C(d, n) \geqslant 0$ denote the corresponding number and

$$X_{n+1} \subseteq X_n \subseteq X_{n-1} \subseteq \ldots X_1 \subseteq X$$

a stratification of X with the properties of the lemma.

Applying excision repeatedly and Proposition 6.14 (1), we get

$$\sum_{i \in \mathbf{Z}} c_u(\mathscr{H}^j(A)) = \sum_{i \in \mathbf{Z}} c(u_! \mathscr{H}^j(A))$$

$$\leqslant \sum_{i \in \mathbf{Z}} \sum_{j=1}^{n+1} c(u_{j!} u_j^* u_! \mathscr{H}^i(A)) = \sum_{j=1}^{n+1} \sum_{i \in \mathbf{Z}} c(u_{j!} u_j^* u_! \mathscr{H}^i(A)).$$

Applying (9) to $f = u_j$ viewed as a morphism from $(X_j - X_{j+1}, u_j)$ to $(\mathbf{P}^n, \mathrm{Id})$, we have

$$c(u_{j!} u_j^* u_! \mathscr{H}^i(A)) \ll c_{u_j, \mathrm{Id}}(u_j) c_{u_j}(u_j^* u_! \mathscr{H}^i(A)) = c(u_j) c_{u_j}(u_j^* u_! \mathscr{H}^i(A))$$

for all i and j. By construction, $u_j^* u_{j!} \mathscr{H}^i(A)$ is lisse on the smooth scheme $X_j - X_{j+1}$; since the perverse homology sheaves and the usual homology sheaves of a lisse sheaf on a smooth scheme agree, the estimates

$$\sum_{i \in \mathbf{Z}} c(u_{j!} u_j^* u_! \mathscr{H}^i(A)) \ll c(u_j) \sum_{i \in \mathbf{Z}} c_{u_j}({}^{\mathfrak{p}}\mathscr{H}^i(u_j^* u_! A)) \ll c(u_j)^2 c_{u_j}(u_j^* u_! A)$$

hold by Theorem 6.15. Applying (8), we get

$$c_{u_j}(u_j^* u_! A) \ll c(u_j) c(u_! A) = c(u_j) c_u(A),$$

and therefore

$$\sum_{i \in \mathbf{Z}} c_u(\mathscr{H}^j(A)) \ll c_u(A) \sum_{j=1}^{n+1} c(u_j)^3$$

is bounded only in terms of (n, d), since $c(u_j) \leqslant C(d, n)$ for all j. $\qquad\square$

6.8. **Generic base change.** We now prove an effective version of Deligne's generic base change theorem [6, Th. finitude, 1.9]. This argument is due to A. Forey.

Theorem 6.27. *Let (X, u), (Y, v) and (S, w) be quasi-projective algebraic varieties over k. Let $f: X \to Y$ and $g: Y \to S$ be morphisms.*

For any object A of $\mathrm{D}_c^b(X)$, there exists an integer $N \geqslant 0$, depending only on $c_u(A)$ and (f, g, u, v, w), and a dense open set $U \subset S$ such that
   (i) *The image of the complement of U has degree $\leqslant N$.*
   (ii) *The object $f_* A$ is of formation compatible with any base change $S' \to U \subset S$.*

Remark 6.28. The original generic base change theorem is stated for a constructible sheaf of R-modules, where R is a Noetherian ring satisfying $nR = 0$ for some integer n that is invertible in S, and S is not supposed to be defined over a field k. An additional statement of the theorem is the constructibility of f_*A on Y_U. In the setting of the above statement, it is already known that f_*A is constructible, precisely by applying [6, Th. finitude, 1.9] to $f: X \to Y$ over k. However, in order to understand the complexity of the complement of the open set U over which f_*A is of formation compatible with base change, we would need to redo the whole proof, following closely Deligne's argument (see also [18, Th. 9.3.1]).

In this section, we will often write simply $c(X)$ for the complexity $c(u)$, where (X, u) is a quasi-projective variety. Recall also from Remark 6.7 that if $i: Z \to X$ is an immersion, then $c_{u \circ i, u}(i) = c(Z)$.

Before starting the proof, we state a useful lemma.

Lemma 6.29. *Let (X, u) be a quasi-projective variety over k of embedding dimension n. There exists a finite open cover (U_i) of X into at most $\dim(X) + 1$ affine subvarieties with open immersions $u_i: U_i \to X$ such that $c_{u \circ u_i, u}(u_i) = c(U_i)$ is bounded in terms of $c(u) = c(X)$ and n.*

Proof. The image of u in $\mathbf{P}^n$ can be written as $Z - W$, where Z and W are closed subvarieties of $\mathbf{P}^n$. As in the proof of Lemma 6.26, by Theorem 6.23 the degrees of Z and W are bounded in terms of $c(u)$ and n. Let H be a hypersurface of $\mathbf{P}^n$ such that $W \subset H$ but $Z \notin W$. One can choose W of degree at most the degree of W. Then set $U_1 = Z - H$, which is an affine open subset of $Z - W$, of complexity bounded by the degrees of Z and H. Since $\dim(Z \cap H) < \dim(Z)$, one can conclude by induction: the affine open subsets of $Z \cap H$ that are obtained are restrictions of affine open subsets of $\mathbf{P}^n$, and hence their restrictions to Z produce affine open subsets of $Z - W$, which, together with U_1, cover $Z - W$. $\square$

During the proof of Theorem 6.27, we will repeatedly consider subschemes of X, Y and S. These will always implicitly be considered with the locally closed embedding inherited from X, Y or S. All complexities will be computed with respect to those implicit embeddings, and hence we will simplify the notation by dropping the embeding from the complexity. Moreover, whenever we say that we can shrink S, we mean that we can replace S by a dense open subset whose complement has bounded degree in terms of $c(A)$.

Proof of Theorem 6.27. Working successively with each irreducible component of S, there is no loss of generality in assuming, as we do from now on, that S is irreducible.

Step 1. We first consider the particular case where $S = Y$, the variety X is smooth of pure relative dimension d over S, and A is a lisse sheaf such that $R^i f_! A'$ is lisse for each $i \in \mathbf{Z}$. Here, A' denotes the "naive" dual $A' = \mathscr{H}om(A, \overline{\mathbf{Q}}_\ell)$, i.e. non-derived sheaf homomorphisms. Then Deligne [6, Th. finitude, 2.1] proves the result with $U = S$ (this is essentially a consequence of the proper base change theorem and Poincaré duality).

Step 2. We now assume that $S = Y$, X is smooth over S of pure relative dimension d, and that A is a lisse sheaf. Let again $A' = \mathscr{H}om(A, \overline{\mathbf{Q}}_\ell)$; by (9) and Theorem 6.23, there exists an open set $U \subset S$ whose complement has degree bounded in terms of $c(A')$, and hence in terms of $c(A)$, such that $f_! A'$ is lisse on U. Over the dense open set U, we are in the situation of Step 1, and hence the result holds in this case too.

Step 3. We now perform some reductions for the proof of the general case. We first observe that the problem is local on Y, so that we may assume that Y is affine. Using

a finite affine cover of X and excision, we may also assume that X is affine. Note that the complexity of the restriction of A to those affine open subsets is bounded in terms of $c(A)$ by (8).

Up to shrinking X again, we can now factor f into an open immersion followed by a proper morphism g (this is a form of Nagata's compactification theorem).

By proper base change, the result holds for the morphism g with $U = S$, and hence it is enough to prove the result when f is an open immersion with dense image. We will then prove the result by induction on the relative dimension $n \geqslant 0$ of X over S.

For $n = 0$, since X and Y are of dimension 0 over S, up to shrinking S to a dense open subset (whose complement is of degree bounded in terms of X and Y), we have $X = Y$. Hence the result holds.

We now consider $n \geqslant 1$, and assume that the result holds for relative dimension up to $n - 1$.

We will prove below the following sublemma:

Lemma 6.30. *With notation and assumption as before, up to shrinking S, there exists a dense open subset $Y' \subset Y$ such that the result holds over Y' and such that the complement of Y' in Y is finite of bounded degree over S.*

Step 4. Under the conditions of Step 3, we further assume that X is smooth over S and that A is a lisse sheaf. We can replace Y by its closure in the image of v, and hence assume that Y is projective over S. Using Lemma 6.30, we can assume that there exists $Y' \subset Y$ such that the result holds for Y' and that $Y - Y'$ is finite over S of bounded degree. We then have the following commutative diagram:

$$
\begin{array}{ccc}
 & Y' & \\
 & \downarrow{\scriptstyle j} & \\
X \xrightarrow{\;f\;} & Y & \xleftarrow{\;\;i\;\;} Y - Y' \\
\searrow{\scriptstyle a}\; \downarrow{\scriptstyle b} & & \swarrow{\scriptstyle c} \\
 & S &
\end{array}
$$

By Step 2, up to shrinking S, the object a_*A is of formation compatible with any base change. By the choice of Y' (after again shrinking S) the object j^*f_*A is of formation compatible with any base change. By the proper base change theorem, the same holds for $b_*j_!j^*f_*A$. Now we have a distinguished triangle

$$j_!j^*f_*A \longrightarrow f_*A \longrightarrow i_*i^*f_*A$$

Applying b_* to it, we get the distinguished triangle

$$b_*j_!j^*f_*A \longrightarrow a_*A \longrightarrow c_*i^*f_*A$$

Since the first two complexes in this triangle are of formation compatible with any base change, the same holds for the third one, namely $c_*i^*f_*A$. Since c is finite, this implies that i^*f_*A also has the same property. Hence in the first triangle, the first and third complexes are of formation compatible with any base change, and hence the middle one, namely f_*A, also has this property. This finishes the proof in this case.

Step 5. We now show how to reduce the situation (after the reduction in Step 3) to that of Step 4. We will also prove below the following additional sublemma:

Lemma 6.31. *Up to shrinking S, and performing a base change along a finite surjective radicial morphism and reducing X, in an effective way, we can find an open dense subset V of X that is smooth over S and has complement of bounded degree.*

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By Proposition 6.24, the complexity of the cohomology sheaves of A are bounded in terms of $c(A)$; hence, up to replacing A by each of its cohomology sheaves in turn, we reduce to the case where A is a sheaf. If the support of A is not dense in X, then the support has relative dimension $\leqslant n-1$, and we are done by induction. If the support of A is dense in X, then by Theorem 6.23, up to shrinking V, we may assume that the restriction of A to V is a lisse sheaf. Applying induction to the restriction of A to $X - V$ and excision, we can assume that A is a lisse sheaf supported on V.

Let j be the open immersion $V \to X$. By Step 4, the result holds for j, and hence up to shrinking S, we may assume that $j_* j^* A$ is of formation compatible with any base change. Choose a cone C of the canonical adjunction morphism $A \to j_* j^* A$. By definition, we have a distinguished triangle

$$A \longrightarrow j_* j^* A \longrightarrow C$$

and the cohomology of C is supported on $Z = X - V$. Since the first two complexes are of formation compatible with any base change, the same is true for C. By Theorem 6.8 and Proposition 6.14, the complexity of the restriction of C to Z is bounded in terms of the complexity of A. Since the relative dimension of Z is $< n$, the induction hypothesis applies to the closed immersion $Z \to X$ and to the restriction of C to Z. Hence, up to shrinking S, we can assume that $f_* C$ is of formation compatible with any base change.

We then apply f_* to the previous distinguished triangle and obtain

$$f_* A \to (fj)_* j^* A \to f_* C$$

By Step 4 again, the result holds for the open immersion $fj \colon V \to Y$, so that after shrinking S, the objet $(fj)_* j^* A$ is of formation compatible with any base change. Since we have seen that this also holds for $f_* C$, we deduce that $f_* A$ is of formation compatible with any base change, as desired. $\qquad\square$

We now prove the sublemmas above.

Proof of Lemma 6.30. The problem is local on Y, so by Lemma 6.29, we can also assume that Y is affine over S. We hence assume that $Y \subset \mathbf{A}_S^r$, where the choice of coordinates is induced by v.

For $1 \leqslant i \leqslant r$, consider the i-th coordinate projection $p_i \colon Y \to \mathbf{A}_S^1$. We view X and Y as $\mathbf{A}_S^1$-schemes using p_i as structure morphism. The generic relative dimensions of X and Y are then $\leqslant n-1$, and hence we can apply the induction hypothesis to this situation. We find a dense open subset $U_i \subset \mathbf{A}_S^1$ such that the complement of U_i in $\mathbf{A}_S^1$ is of degree bounded in terms of $c(A)$, and such that $f_* A$ is of formation compatible with any base change $S' \to U_i \subset \mathbf{A}_S^1$. Define

$$Y' = \bigcup_{1 \leqslant i \leqslant r} p_r^{-1}(U_i).$$

Since the complexity of the U_i is bounded in terms of $c(A)$, the same holds for Y'. From the definition of U_i, the result holds for the restriction of f to $X' = f^{-1}(Y')$. Since the morphism

$$Y - Y' = (\mathbf{A}_S^1 - U_1) \times_S \cdots \times_S (\mathbf{A}_S^1 - U_r) \longrightarrow S$$

is generically finite, there exists a dense open $S' \subset S$ over which this morphism is finite. The degree of $S - S'$ is bounded in terms of the degrees of $\mathbf{A}_S^1 - U_i$, for $i = 1, \dots, r$, and hence is itself bounded in terms of $c(A)$, which ends the proof of the sublemma. $\qquad\square$

Proof of Lemma 6.31. Recall that X is assumed to be affine, with a fixed closed embedding into $\mathbf{A}_S^r$ for some integer $r \geqslant 0$. By the effective version of generic flatness, see, e.g., [43, Th. 2.45], we find a dense open subset S' of S over which X is flat and such that

the complement of S' in S is of bounded degree. We may therefore assume that X is flat over S. To obtain smoothness, we use the Jacobian criterion. We want to define U to be the complement in X of the vanishing locus of the Jacobian ideal, which is indeed of bounded degree. However, for this to define a dense open subset of X, we need to perform first a base change along a finite surjective radicial morphism $S' \to S$ (which does not change étale cohomology), then replace X and Y by their reductions. The degree of the finite surjective radicial morphism is determined by the degrees (according to the affine embedding of S) of the coefficients of the polynomials defining X in $\mathbf{A}_S^r$, and hence is bounded. Once the result is known for $X_{S'}$ over S', we conclude the proof by considering the image U of the dense open subset of S' by the morphism $S' \to S$, since the degree of the complement of U is still bounded. $\qquad\square$

6.9. **Tannakian operations.** Let (X, u) be a connected quasi-projective algebraic variety over k, and let $\mathscr{F}$ be a lisse ℓ-adic sheaf on X. We shall view $\mathscr{F}$ as a continuous representation π of the étale fundamental group of X on a finite-dimensional $\overline{\mathbf{Q}}_\ell$-vector space. The *monodromy group* of $\mathscr{F}$ is then defined as the Zariski closure of its image, which is an algebraic group G over $\overline{\mathbf{Q}}_\ell$ equipped with a distinguished faithful representation. Given a further algebraic representation $\rho \colon G \to \mathrm{GL}_N(\overline{\mathbf{Q}}_\ell)$, we denote by $\rho(\mathscr{F})$ the lisse ℓ-adic sheaf corresponding to the representation $\rho \circ \pi$.

The following lemma is very useful in deriving properties of $\rho(\mathscr{F})$ from those of $\mathscr{F}$. By a reductive group over a field of characteristic zero, we mean a group G all whose finite-dimensional representations are completely reducible (that is, G is not assumed to be connected).

Lemma 6.32. *Let $G \subset \mathrm{GL}_N$ be a reductive group over an algebraically closed field of characteristic zero and denote by Std its tautological N-dimensional representation. For any algebraic irreducible representation $\rho \colon G \to \mathrm{GL}(V)$, there exist non-negative integers (a, b) such that ρ is a subrepresentation of $\mathrm{Std}^{\otimes a} \otimes \mathrm{D}(\mathrm{Std})^{\otimes b}$. If the determinant of Std has finite order, there exists such a pair with $b = 0$.*

Proof. For the first part of the statement, see, e.g., [5, Th. III.4.4] in the case of compact Lie groups, and use the equivalence of categories between reductive groups and compact Lie groups.

When the determinant has finite order, it suffices to prove that the contragredient of the tautological representation is a subrepresentation of a tensor power of the standard representation. But, using brackets to denote multiplicity, we have

$$\langle \mathrm{D}(\mathrm{Std}), \mathrm{Std}^{\otimes a} \rangle = \langle \mathbf{1}, \mathrm{Std}^{\otimes(a+1)} \rangle$$

for all $a \geqslant 0$, and if $\det(\mathrm{Std})^m = \mathbf{1}$ for some $m \geqslant 1$, we have $\langle \mathbf{1}, \mathrm{Std}^{\otimes mN} \rangle \geqslant 1$. Indeed, as the determinant is the highest exterior power, there is an inclusion $\det \subset \mathrm{Std}^{\otimes N}$, and this induces $\mathbf{1} = \det^{\otimes m} \subset \mathrm{Std}^{\otimes mN}$. $\qquad\square$

Proposition 6.33. *With notation as above, assume that the group G is reductive with finite center. There exists an integer $a \geqslant 0$, depending only on ρ, such that*

$$c_u(\rho(\mathscr{F})) \ll c_u(\mathscr{F})^a,$$

where the implied constant depends only on the embedding dimension of u.

Proof. Let Std denote the standard faithful representation of G corresponding to the sheaf $\mathscr{F}$. The identity component G^0 of G is reductive, so the restriction of the one-dimensional character $\det(\mathrm{Std})$ of G to G^0 is non-trivial if and only if its restriction to the central torus of G^0 is non-trivial. The central torus of G^0 is a torus with an action

of the (finite) component group of G by conjugation which does not fix any non-trivial cocharacter. Hence, it does not fix any non-trivial character, so the restriction of $\det(\mathrm{Std})$ to G^0 is trivial, and thus $\det(\mathrm{Std})$ has finite order. By Lemma 6.32, there exists a non-negative integer a such that ρ is a direct summand of $\mathrm{Std}^{\otimes a}$. Thus, the estimates

$$c_u(\rho(\mathscr{F})) \leqslant c_u(\mathscr{F}^{\otimes a}) \ll c_u(\mathscr{F})^a$$

hold by Proposition 6.14 (2) and parts (4) and (5) of Theorem 6.8. $\qquad\square$

Remark 6.34. If X is a curve, the estimate can be strengthened considerably to

$$c_u(\rho(\mathscr{F})) \ll \dim(\rho) c_u(\mathscr{F})$$

by using the "local" formula of Theorem 7.3 below (see [26, Ch. 3] for arguments of this kind). It would be very interesting to know whether such a strong bound holds in higher dimension.

In some applications, it is natural to try to improve on this bound by estimating a effectively, or in other words, to make Lemma 6.32 effective. We record here one such estimate, since this might be of interest for other applications.

Let $N \geqslant 1$ be an integer and let $G \subset \mathrm{GL}_N$ be a reductive algebraic group over an algebraically closed field of characteristic zero. As above, we denote by Std the tautological faithful representation of G in GL_N.

Definition 6.35. For any irreducible algebraic representation ρ of G, we define

$$w(\rho) = \min\left\{ a + b \mid \rho \text{ is a subrepresentation of } \mathrm{Std}^{\otimes a} \otimes \mathrm{D}(\mathrm{Std})^{\otimes b} \right\}.$$

Lemma 6.32 means that $w(\rho)$ is finite for all irreducible algebraic representations ρ of G.

Proposition 6.36. *Let Λ be the weight lattice of the identity component G_0 of G. Fix a norm $\|\cdot\|$ on Λ. There exist real numbers α, $\beta \geqslant 0$ such that, for any irreducible algebraic representation ρ of G, the inequality*

$$w(\rho) \leqslant \alpha \max_i \|\lambda_i\| + \beta$$

holds, where the λ_i are the highest weights of the irreducible components of the restriction of ρ to G_0.

Proof. In this proof, we will say that a pair (a, b) is a *spot* for a representation ρ of G if ρ appears as a subrepresentation of $\mathrm{Std}^{\otimes a} \otimes \mathrm{D}(\mathrm{Std})^{\otimes b}$.

Let n be the semi-simple rank of G. For a positive dominant weight λ in Λ, we denote by π_λ the irreducible representation of G^0 with highest weight λ.

We first assume that G is a connected semisimple group. In this case, the representation ρ is associated to a single highest weight λ. Moreover, we can choose a basis (e_i) of the root lattice tensored with $\mathbf{R}$ such that a Weyl chamber can be identified with the cone of vectors with coordinates $\geqslant 0$.

For integers i with $1 \leqslant i \leqslant n$, we denote by λ_i the weight in the line spanned by e_i with minimal i-th coordinate, and we write $\lambda_i = x_i e_i$ for some integer $x_i \geqslant 1$. Further, for each i, we fix spots (a_i, b_i) of π_{λ_i} (which exist by Lemma 6.32).

We denote by A the finite set of positive dominant weights μ such that each coordinate of μ is $\leqslant x_i$ for all i. For each $\mu \in A$, we fix spots (a_μ, b_μ) of π_μ.

By subtracting from a weight λ of G suitable multiples of $\lambda_1, \ldots, \lambda_n$, until all coordinates are $\leqslant x_i$, we see that we may write

$$\lambda = \mu + \sum_{i=1}^r n_i \lambda_i$$

where $\mu \in A$ and $0 \leqslant n_i \leqslant \ell_i(\lambda)$ for some linear maps $\ell_i \colon \Lambda \to \mathbf{Z}$.

From highest weight theory, the representation ρ with highest weight λ is a summand of

$$\pi_\mu \otimes \bigotimes_{1 \leqslant i \leqslant r} \pi_{\lambda_i}^{\otimes n_i}.$$

Hence, by definition, the pair (a, b) given by

$$a = a_\mu + \sum_{i=1}^r n_i a_i, \quad b = b_\mu + \sum_{i=1}^r n_i b_i,$$

is a spot of ρ. We have then

$$w(\rho) \leqslant a + b \leqslant \alpha + \beta \|\lambda\|$$

where

$$\alpha = \max_{\mu \in A} (a_\mu + b_\mu), \quad \beta = \max_{1 \leqslant i \leqslant r} \|\ell_i\| \times \sum_{i=1}^r (a_i + b_i)$$

(with the usual norm for linear maps $\Lambda \to \mathbf{R}$ with respect to the given norm on Λ).

Assume now that G is a connected reductive group. We construct the weights λ_i and A of the derived group of G as before, then lift them (keeping the notation) to the whole group. Let further $(\gamma_j)_{1 \leqslant j \leqslant k}$ be a basis of the subspace of the weights whose associated representations are trivial on the derived group.

For any highest weight λ of $G = G^0$, the representation π_λ can then be expressed as a summand of a representation of the form

$$\pi_\mu \otimes \bigotimes_{1 \leqslant i \leqslant r} \pi_{\lambda_i}^{\otimes n_i} \otimes \bigotimes_{1 \leqslant j \leqslant k} \pi_{\gamma_j}^{\otimes m_j}$$

where, as before, the n_i and m_j are bounded by linear functions of λ. This leads to a bound of the desired type as in the semisimple case.

Finally, we consider the general case. Let Std_0 be the tautological representation of G^0. Note that for any non-negative integers (a, b), the representation

(22) $$\mathrm{Ind}_{G^0}^G (\mathrm{Std}_0^{\otimes a} \otimes \mathrm{D}(\mathrm{Std}_0)^{\otimes b})$$

contains $\mathrm{Std}^{\otimes a} \otimes \mathrm{D}(\mathrm{Std})^{\otimes b}$ as a subrepresentation. Moreover, by Frobenius reciprocity, the multiplicity of ρ in (22) is the dimension of the space of G^0-morphisms from the restriction ρ_0 of ρ to G^0 to $\mathrm{Std}_0^{\otimes a} \otimes \mathrm{D}(\mathrm{Std}_0)^{\otimes b}$. For an irreducible component ρ' of ρ_0, this space is non-zero for some (a, b) with $a + b = w(\rho')$. The bound then follows from the case of connected groups established in the first part. $\square$

6.10. **Independence of ℓ.** In this section and the next one, we work over a finite field $\mathbf{F}$ and k is an algebraic closure of $\mathbf{F}$. We use the notation for trace functions from the introduction. We first prove that the complexities of a compatible system of ℓ-adic sheaves are largely independent of ℓ, following ideas from Katz [28].

We fix a quasi-projective variety (X, u) over $\mathbf{F}$ and a field K of characteristic zero. In order to vary the prime ℓ, we make the following definitions. Let Λ be a non-empty set and let $\mathscr{S}$ be a family $(\ell_\lambda, \iota_\lambda)_{\lambda \in \Lambda}$ of pairs where each ℓ_λ is a prime number invertible in $\mathbf{F}$ and ι_λ is a field embedding of the given field K in $\overline{\mathbf{Q}}_{\ell_\lambda}$.

Definition 6.37. An $\mathscr{S}$-compatible system of complexes on X is a family $(A_\lambda)_{\lambda \in \Lambda}$ consisting of objects A_λ of $\mathrm{D}_c^b(X, \overline{\mathbf{Q}}_{\ell_\lambda})$ such that the following conditions hold:

- for any $\lambda \in \Lambda$, any finite extension $\mathbf{F}_n$ of $\mathbf{F}$, and any $x \in X(\mathbf{F}_n)$, the trace $t_{A_\lambda}(x; \mathbf{F}_n)$ takes values in the subfield $\iota_\lambda(K)$ of $\overline{\mathbf{Q}}_{\ell_\lambda}$;

- for any λ, μ in Λ, any finite extension $\mathbf{F}_n$ of $\mathbf{F}$, and any $x \in X(\mathbf{F}_n)$, we have

$$\iota_\lambda^{-1} t_{A_\lambda}(x; \mathbf{F}_n) = \iota_\mu^{-1} t_{A_\mu}(x; \mathbf{F}_n).$$

Lemma 6.38. *Let $(A_\lambda)_{\lambda \in \Lambda}$ be an $\mathscr{S}$-compatible system on X. The Euler–Poincaré characteristic $\chi_c(X \times k, A_\lambda)$ of A_λ is independent of λ.*

Proof. For any λ, it follows from the definition of $\mathscr{S}$-compatible system that the L-function of A_λ is independent of λ. From the expression for this L-function given by the Grothendieck–Lefschetz formula, we know that $\chi_c(X \times k, A_\lambda)$ is the degree of this rational function (degree of the numerator minus degree of the denominator), and hence is independent of λ. □

We then have the following result:

Theorem 6.39. *Let $\mathscr{S}$ be as above.*

(1) Let $n \geqslant 0$ be an integer and let $(A_\lambda)_\lambda$ be an $\mathscr{S}$-compatible system of perverse sheaves on $\mathbf{P}^n_{\mathbf{F}}$. Then for λ and μ in Λ, we have

$$c(A_\lambda) \asymp c(A_\mu),$$

where the implied constants only depend on n.

(2) Let (X, u) be a smooth quasi-projective variety over $\mathbf{F}$ and let $(\mathscr{F}_\lambda)_\lambda$ be an $\mathscr{S}$-compatible system of lisse sheaves on X. Then for all λ and μ in Λ, we have

$$c_u(\mathscr{F}_\lambda) \asymp c_u(\mathscr{F}_\mu),$$

where the implied constants only depend on the embedding dimension of (X, u).

Proof. (1) We argue by induction on n, following the strategy of Katz [28, Th. 7]. It is enough to prove the estimate

$$c(A_1) \ll c(A_2)$$

for two compatible perverse sheaves A_1 in $D_c^b(\mathbf{P}^n, \overline{\mathbf{Q}}_{\ell_1})$ and A_2 in $D_c^b(\mathbf{P}^n, \overline{\mathbf{Q}}_{\ell_2})$, with an implied constant that only depends on n.

If $n = 0$, then A_1 and A_2 are vector spaces of the same dimension and their complexity $c(A_1) = c(A_2)$ is this common dimension.

We now fix $n \geqslant 1$ and assume that the result holds on $\mathbf{P}^m$ for all $0 \leqslant m \leqslant n - 1$. For each $0 \leqslant m \leqslant n - 1$, we pick a geometric generic point $\boldsymbol{a}_m \in M^{n+1, m+1}(k')$ for some algebraically closed field k'.

Let $0 \leqslant m \leqslant n - 1$. By applying Lemma 3.13 to A_1 and A_2, we find open dense subsets $U_{m,1}$ and $U_{m,2}$ of $M_k^{n+1, m+1}$ satisfying the properties of Lemma 3.13. Up to replacing $\mathbf{F}$ by a finite extension, which we may do, we can assume that $(U_{m,1} \cap U_{m,2})(\mathbf{F})$ is non-empty for all m, and we then pick $\boldsymbol{b}_m \in (U_{m,1} \cap U_{m,2})(\mathbf{F})$, again for all m. The complexes $l_{\boldsymbol{b}_m}^* A_1[m-n]$ and $l_{\boldsymbol{b}_m}^* A_2[m-n]$ on $\mathbf{P}^m$ over $\mathbf{F}$ are perverse and form a compatible system; hence, by the induction hypothesis, the equivalence

$$(23) \qquad\qquad c(l_{\boldsymbol{b}_m}^* A_1) \asymp c(l_{\boldsymbol{b}_m}^* A_2)$$

holds, with implied constants that only depend on n. We have

$$\max_{0 \leqslant m \leqslant n-1} \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}_{k'}^m, l_{\boldsymbol{a}_m}^* A_1) = \max_{0 \leqslant m \leqslant n-1} \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}_{k'}^m, l_{\boldsymbol{b}_m}^* A_1),$$

again by Lemma 3.13. Hence

$$(24) \qquad\qquad \max_{0 \leqslant m \leqslant n-1} \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}_{k'}^m, l_{\boldsymbol{a}_m}^* A_1) = c(l_{\boldsymbol{b}_m}^* A_1),$$

by the definition of the complexity and Lemma 3.7.

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Together with (23), it follows that

$$c(A_1) = \max_{0 \leqslant m \leqslant n} \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^m_{k'}, l^*_{\boldsymbol{a}_m} A_1)$$

$$\leqslant \max_{0 \leqslant m \leqslant n-1} \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^m_{k'}, l^*_{\boldsymbol{a}_m} A_1) + \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^n_{k'}, l^*_{\boldsymbol{a}_n} A_1)$$

$$\ll c(l^*_{\boldsymbol{b}_m} A_2) + \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^n_{k'}, l^*_{\boldsymbol{a}_n} A_1)$$

$$\ll c(A_2) + \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^n_{k'}, l^*_{\boldsymbol{a}_n} A_1).$$

Since the geometric generic point $\boldsymbol{a}_n \in M^{n+1,n+1}_{k'}$ induces an isomorphism $l_{\boldsymbol{a}_n}$ of $\mathbf{P}^n_{k'}$, it is enough to prove the estimate

$$\sigma_1 = \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^n_{k'}, A_1) \ll c(A_2).$$

Arguing as in the proof of Proposition 3.17, we have

$$h^0(\mathbf{P}^n_{k'}, A_1) = \chi(\mathbf{P}^n_{k'}, A_1) + \sum_{i<0}(-1)^{i+1} h^i(\mathbf{P}^n_{k'}, A_1) + \sum_{i>0}(-1)^{i+1} h^i(\mathbf{P}^n_{k'}, A_1),$$

and therefore

$$\sigma_1 = \sum_{i<0} h^i(\mathbf{P}^n_{k'}, A_1) + h^0(\mathbf{P}^n_{k'}, A_1) + \sum_{i>0} h^i(\mathbf{P}^n_{k'}, A_1)$$

$$\leqslant 2 \sum_{i<0} h^i(\mathbf{P}^n_{k'}, A_1) + \chi(\mathbf{P}^n_{k'}, A_1) + 2 \sum_{i>0} h^i(\mathbf{P}^n_{k'}, A_1).$$

By excision and Artin's vanishing theorem for affine varieties, and the fact that A_1 is perverse, the canonical map

$$\mathrm{H}^i(\mathbf{P}^n_{k'}, A_1) \longrightarrow \mathrm{H}^i(\mathbf{P}^{n-1}_{k'}, l^*_{\boldsymbol{a}_{n-1}} A_1)$$

is an isomorphism for $i < 0$. Similarly, because the dual of a perverse sheaf is perverse and duality exchanges $l^*_{\boldsymbol{a}_{n-1}}$ and $l^!_{\boldsymbol{a}_{n-1}}$, the canonical map

$$\mathrm{H}^i(\mathbf{P}^{n-1}_{k'}, l^!_{\boldsymbol{a}_{n-1}} A_1) \longrightarrow \mathrm{H}^i(\mathbf{P}^n_{k'}, A_1)$$

is an isomorphism for $i > 0$. Since $l_{\boldsymbol{a}_{n-1}}$ is a closed immersion, the functors $l^*_{\boldsymbol{a}_{n-1}}$ and $l^!_{\boldsymbol{a}_{n-1}}$ are equal up to a shift and a Tate twist. Hence,

$$\sigma_1 = 2 \sum_{i<0} h^i(\mathbf{P}^n_{k'}, l^*_{\boldsymbol{a}_{n-1}} A_1) + \chi(\mathbf{P}^n_{k'}, A_1) + 2 \sum_{i>0} h^i(\mathbf{P}^n_{k'}, l^!_{\boldsymbol{a}_{n-1}} A_1)$$

$$\leqslant 4 \sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^{n-1}, l^*_{\boldsymbol{a}_{n-1}} A_1) + \chi(\mathbf{P}^n_{k'}, A_2)$$

using Lemma 6.38. Hence,

$$\sigma_1 \ll c(A_2) + \chi(\mathbf{P}^n_{k'}, A_2) \ll c(A_2),$$

by (23) and (24), and the elementary Proposition 7.1 below.

(2) The proof is similar, using the obvious adaptation of Lemma 3.13 to lisse sheaves, and the fact that the dual of a lisse sheaf on a smooth scheme is lisse. Alternatively, but at the cost of adding a dependency of the implied constants on $c(u)$, we may use [19, Th. 3] to reduce to (1); indeed, this shows that the middle extensions $u_{!*} A_\lambda$ of the components of a compatible system of lisse sheaves is a compatible system of perverse sheaves on the

projective space target of u, and one can apply Corollary 6.16 to bound the complexity of the middle extensions. $\square$

6.11. **Complex conjugation.** We keep working over a finite field $\mathbf{F}$ with algebraic closure k. In applications to analytic number theory, taking the complex conjugate of the trace function t_A is a natural operation, but this has no canonical counterpart at the level of $\mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(X)$. Nevertheless, we have the following result:

Proposition 6.40. *Let* (X, u) *be a quasi-projective algebraic variety over* $\mathbf{F}$*. Given any mixed object* A *of* $\mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(X)$*, there exists a mixed object* A' *of* $\mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(X)$ *such that*

$$t_{A'}(\cdot; \mathbf{F}_n) = \overline{t_A(\cdot; \mathbf{F}_n)} \quad \text{for any } n \quad \text{and} \quad c_u(A') \ll c(u)c_u(A),$$

where the implied constant depends only on the embedding dimension of (X, u)*.*

Proof. For each $n \geqslant 1$, the trace function $t_A(\cdot; \mathbf{F}_n)$ is the restriction to the set $X(\mathbf{F}_n)$ of $\mathbf{F}_n$-rational points of X of the trace function of the complex $u_! A$ on $\mathbf{P}^n$, which is again mixed if so is A. Assume that we have found a mixed object $\widetilde{A}$ of $\mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(\mathbf{P}^n)$ that "works" for $u_! A$, then $A' = u^* \widetilde{A}$ works for A, and

$$c_u(A') \ll c(u)c(\widetilde{A}) \ll c(u)c(u_! A) = c(u)c_u(A),$$

where the first bound is part (8) of Theorem 6.8. Thus, we may assume $X = \mathbf{P}^n$ (with the identity embedding).

We first assume that A is a perverse sheaf. If A is pure of weight zero, then A is geometrically semisimple by [3, 5.3.8]. As explained in [29, Lem. 1.8.1. (1)], a result of Gabber on the independence of ℓ of traces of intermediate extensions [19, Th. 3] then implies that we can take $A' = \mathrm{D}(A)$. The perverse sheaf A' is also pure of weight zero and has complexity $c(A') = c(A)$ by Lemma 6.11. If A is pure of weight w, then we can simply take $A' = \mathrm{D}(A(w))(-w)$, which is pure of weight w.

Suppose now that A is a mixed perverse sheaf. Let (F_j) be the canonical weight filtration on A, with quotients F_{j+1}/F_j pure of weight w_j (see [3, Th. 5.3.5]). We first claim that the estimate

$$\sum_j c(F_{j+1}/F_j) \ll c(A)$$

holds. Indeed, since the quotients are perverse sheaves, we have

$$\sum_j c(F_{j+1}/F_j) \ll \sum_j \| \mathrm{cc}(F_{j+1}/F_j) \| = \left\| \sum_j \mathrm{cc}(F_{j+1}/F_j) \right\| = \| \mathrm{cc}(A) \| \ll c(A)$$

by Corollary 5.5 and the fact that the characteristic cycles of perverse sheaves are effective (compare with the end of the proof of Theorem 6.15). In view of the equality

$$t_A(\cdot; \mathbf{F}_n) = \sum_j t_{F_{j+1}/F_j}(\cdot; \mathbf{F}_n)$$

for all $n \geqslant 1$, we can take

$$A' = \bigoplus_j \mathrm{D}((F_{j+1}/F_j)(w_i))(-w_i),$$

which is a mixed perverse sheaf with complexity

$$c(A') = \sum_j c(\mathrm{D}(F_{j+1}/F_j)) = \sum_j c(F_{j+1}/F_j) \ll c(A),$$

by Lemma 6.11 and the above.

Finally, in the general case, if A is a mixed complex of weights $\leqslant w$, then its perverse cohomology sheaves $^{\mathfrak{p}}\mathcal{H}^i(A)$ are mixed of weights $\leqslant w + i$ for all $\in \mathbf{Z}$ by [3, Th. 5.4.1] and we have

$$t_A(\cdot; \mathbf{F}_n) = \sum_{i \in \mathbf{Z}} (-1)^i t_{^{\mathfrak{p}}\mathcal{H}^i(A)}(\cdot; \mathbf{F}_n)$$

for all $n \geqslant 1$. For all i, let A_i' be a mixed perverse sheaf satisfying the conditions of the proposition for $^{\mathfrak{p}}\mathcal{H}^i(A)$. We can define

$$A' = \bigoplus_{i \in \mathbf{Z}} A_i',$$

which satisfies

$$c(A') = \sum_{i \in \mathbf{Z}} c(A_i') \ll \sum_{i \in \mathbf{Z}} c(^{\mathfrak{p}}\mathcal{H}^i(A)) \ll c(A)$$

by the previous case. $\qquad\square$

Remark 6.41. In practice, the conjugate of the trace functions that occur in concrete applications are often clearly trace functions with essentially the same complexity (e.g., the conjugate of the trace function of an Artin–Schreier sheaf $\mathcal{L}_{\psi(f)}$ is simply $\mathcal{L}_{\psi(-f)}$). Nevertheless, Proposition 6.40 might be useful for certain theoretical arguments.

Remark 6.42. In fact, the assumption that A is a mixed object is always satisfied thanks to the deep theorem of L. Lafforgue that an irreducible lisse sheaf with determinant of finite order on a normal variety X is pure of weight zero (see [36, Prop. VII.7 (i)]). We briefly sketch how to deduce from this that A is mixed. By definition, it suffices to treat the case where A is a constructible sheaf and, by induction on the dimension of X, we can even suppose that A is lisse. The successive quotients of a Jordan–Hölder filtration of A are then irreducible lisse sheaves; up to a twist by some $\overline{\mathbf{Q}}_\ell(w)$, their determinant is of finite order by geometric class field theory (see [7, (1.3.6)]), and hence they are pure thanks to Lafforgue's theorem.

7. Examples and applications

7.1. **Sums of Betti numbers.** One of the most useful properties of the complexity is that it controls Betti numbers, as the following proposition shows.

Proposition 7.1. *Let (X, u) be a quasi-projective variety over k. For any A in $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$, the following holds:*

$$|\chi_c(X, A)| \leqslant \sum_{i \in \mathbf{Z}} h^i_c(X, A) \leqslant c_u(A),$$

$$|\chi(X, A)| \leqslant \sum_{i \in \mathbf{Z}} h^i(X, A) \ll c(u)c_u(A).$$

Moreover, the implied constant in the second estimate only depends on the embedding dimension of (X, u).

Proof. Let n be the embedding dimension of (X, u). For each object A of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(X)$, the equality $h^i_c(X, A) = h^i(\mathbf{P}^n_{\bar{k}}, u_! A)$ holds. Since étale cohomology is invariant under extension of scalars to an algebraically closed field k' and $l_{\boldsymbol{a}_n} : \mathbf{P}^n_{k'} \to \mathbf{P}^n_{k'}$ is an isomorphism, the first bound is a straightforward consequence of the definition of the complexity $c_u(A)$. On noting that Verdier duality implies the equality of sums of Betti numbers $\sum_{i \in \mathbf{Z}} h^i(X, A) = \sum_{i \in \mathbf{Z}} h^i_c(X, \mathrm{D}(A))$, the second estimate then follows from the first one and the estimate $c_u(\mathrm{D}(A)) \ll c(u)c_u(A)$ from (4), with an implied constant that only depends on n. $\qquad\square$

Remark 7.2. The converse estimate

$$c_u(A) \ll \sum_{i \in \mathbf{Z}} h_c^i(X, A)$$

does not hold in general, since the right-hand side is independent of u, whereas we have seen in Example 6.1 that there is no reasonable intrinsic notion of complexity of an object of $\mathrm{D}_c^b(X)$. This can also be seen concretely, e.g. in the case of curves.

7.2. Complexity of sheaves on curves. In the case of curves, we can write down an explicit formula for the complexity.

Theorem 7.3. *Let (C, u) be a smooth connected quasi-projective curve over k. Let $\overline{C}$ be the smooth compactification of C, and denote by g the genus of $\overline{C}$, by n the number of points of $\overline{C} - C$, and by d the degree of $u(C)$.*

(1) *The complexity of an object A of $\mathrm{D}_c^b(C)$ is given by*

$$(25) \qquad c_u(A) = \max\Big(d \sum_{i \in \mathbf{Z}} \dim \mathscr{H}^i(A)_\eta, \sum_{i \in \mathbf{Z}} h_c^i(C, A)\Big),$$

where η stands for the generic point of C.

(2) *If A is a perverse sheaf on C, the following inequalities hold:*

$$\max(d, 2g + n - 2)\operatorname{rank}(A) + \operatorname{loc}(A) \leqslant c_u(A) \leqslant \max(d, 2g + n + 2)\operatorname{rank}(A) + \operatorname{loc}(A).$$

(3) *If $\mathscr{F}$ is a middle-extension sheaf on C, the following inequalities hold:*

$$\max(d, 2g + n - 2)\operatorname{rank}(\mathscr{F}) + \operatorname{loc}(\mathscr{F}) \leqslant c_u(\mathscr{F}) \leqslant \max(d, 2g + n + 2)\operatorname{rank}(\mathscr{F}) + \operatorname{loc}(\mathscr{F}).$$

Proof. Part (1) is simply a translation of the definition of $c_u(A)$.

To deduce (2), we first note that a perverse sheaf A satisfies

$$\sum_{i \in \mathbf{Z}} \dim \mathscr{H}^i(A)_\eta = \dim \mathscr{H}^{-1}(A) = \operatorname{rank}(A).$$

Writing the sum of Betti numbers as

$$\sum_{i \in \mathbf{Z}} h_c^i(C, A) = -\chi_c(C, A) + 2h^0(C, A) + 2h^{-2}(C, A),$$

we then have

$$-\chi_c(C, A) \leqslant \sum_{i \in \mathbf{Z}} h_c^i(C, A) \leqslant -\chi_c(C, A) + 4\operatorname{rank}(A).$$

Taking the equality $\chi_c(C, \overline{\mathbf{Q}}_\ell[1]) = 2g - 2 + n$ into account, the statement then follows from the Grothendieck–Ogg–Shafarevitch formula.

Finally, (3) is a special case of (2), for the perverse sheaf $A = \mathscr{F}[1]$. □

As a corollary, we can now compare the complexity defined in this paper with the "analytic conductor" of Fouvry, Kowalski and Michel for the affine line over the algebraic closure of a finite field. Precisely, for a middle-extension sheaf $\mathscr{F}$ on $\mathbf{A}^1$ over such a field, the latter is defined in [15, Def. 1.13] as

$$c_{\mathrm{fkm}}(\mathscr{F}) = \operatorname{rank}(\mathscr{F}) + \operatorname{loc}_{\mathrm{fkm}}(\mathscr{F})$$

$$\operatorname{loc}_{\mathrm{fkm}}(\mathscr{F}) = (\text{number of singular points of } \mathscr{F} \text{ in } \mathbf{P}^1) + \sum_{x \in \mathbf{P}^1} \operatorname{swan}_x(\mathscr{F}).$$

Corollary 7.4. *For a middle-extension sheaf $\mathscr{F}$ on $\mathbf{A}^1$ over a finite field, the inequalities*

$$c_{\mathrm{fkm}}(\mathscr{F}) \leqslant c_u(\mathscr{F}) \leqslant 3c_{\mathrm{fkm}}^2(\mathscr{F})$$

hold, with u the obvious embedding $\mathbf{A}^1 \to \mathbf{P}^1$ of degree 1.

Proof. We apply Theorem 7.3 with $g = 0$, $n = 1$, and $d = 1$. On noting the inequality $\mathrm{drop}_x(\mathscr{F}) \leqslant \mathrm{rank}(\mathscr{F})$, the upper-bound in (3) leads to

$$c_u(\mathscr{F}) \leqslant 3\,\mathrm{rank}(\mathscr{F}) + \mathrm{loc}(\mathscr{F}) \leqslant 3\,\mathrm{rank}(\mathscr{F}) + \mathrm{rank}(\mathscr{F})\,\mathrm{loc}_{\mathrm{fkm}}(\mathscr{F})$$
$$\leqslant 3\,\mathrm{rank}(\mathscr{F})c_{\mathrm{fkm}}(\mathscr{F}) \leqslant 3c_{\mathrm{fkm}}(\mathscr{F})^2.$$

In addition, the inequality $\mathrm{loc}_{\mathrm{fkm}}(\mathscr{F}) \leqslant \mathrm{loc}(\mathscr{F})$ holds, hence the inequalities

$$c_{\mathrm{fkm}}(\mathscr{F}) \leqslant \mathrm{rank}(\mathscr{F}) + \mathrm{loc}(\mathscr{F}) \leqslant c_u(\mathscr{F})$$

by the lower-bound in loc. cit. $\qquad\square$

7.3. Artin–Schreier and Kummer sheaves.

In this section, k is the algebraic closure of a finite field $\mathbf{F}$. Given an ℓ-adic additive character $\psi\colon \mathbf{F} \to \overline{\mathbf{Q}}_\ell^{\times}$ (resp. an ℓ-adic multiplicative character $\chi\colon \mathbf{F}^\times \to \overline{\mathbf{Q}}_\ell^{\times}$), we denote by $\mathscr{L}_\psi$ (resp. $\mathscr{L}_\chi$) the corresponding *Artin–Schreier sheaf* on $\mathbf{A}^1$ over $\mathbf{F}$ (resp. the *Kummer sheaf* on $\mathbf{G}_m$ over $\mathbf{F}$), as defined, e.g., in [6, Sommes exp.]. These are lisse sheaves of rank one; the Artin–Schreier sheaf is wildly ramified at ∞ if ψ is non-trivial, whereas the Kummer sheaf is tamely ramified at 0 and ∞. We also denote by $\mathscr{L}_\psi$ and $\mathscr{L}_\chi$ the middle extensions of these sheaves to $\mathbf{P}^1$.

Proposition 7.5. *Let $C = \mathbf{P}^1$ over k and $u = \mathrm{Id}$.*
  (1) *For any non-trivial additive character ψ of $\mathbf{F}$, we have $c_u(\mathscr{L}_\psi) = 1$.*
  (2) *For any non-trivial multiplicative character χ of $\mathbf{F}^\times$, we have $c_u(\mathscr{L}_\chi) = 1$.*

Proof. These follow from Theorem 7.3 (1) by the standard knowledge of the relevant Betti numbers. $\qquad\square$

We can now easily estimate the complexity of more general Artin–Schreier and Kummer sheaves, which are building blocks of many of the sheaves used in applications to analytic number theory.

Definition 7.6. Let X be a scheme over $\mathbf{F}$ and U a dense open subset of X. Let j be the open immersion $U \to X$. For each morphism $f\colon U \to \mathbf{A}^1$, we define the *Artin–Schreier sheaf $\mathscr{L}_{\psi(f)}$* on X as $\mathscr{L}_{\psi(f)} = j_! f^* \mathscr{L}_\psi$. For each morphism $g\colon U \to \mathbf{G}_m$, we define the *Kummer sheaf $\mathscr{L}_{\chi(g)}$* on X as $\mathscr{L}_{\chi(g)} = j_! g^* \mathscr{L}_\chi$.

Let (X, u) be a quasi-projective variety over k, and let $v\colon \mathbf{A}^1 \to \mathbf{P}^1$ be the obvious embedding (or its restriction to $\mathbf{G}_m$). Combined with (8) and (9), Proposition 7.5 gives the estimates

$$c_u(\mathscr{L}_{\psi(f)}) \ll c_{u\circ j, u}(j)c_{u\circ j, v}(f)$$
$$c_u(\mathscr{L}_{\chi(g)}) \ll c_{u\circ j, u}(j)c_{u\circ j, v}(g),$$

(with the same notation as above, and when ψ and χ are non-trivial), where the implied constants only depend on the embedding dimension of u. According to Remark 6.20, these quantities are uniformly bounded if f or g varies among morphisms with "bounded degree". One can sometimes be more explicit, using tools like Proposition 6.21. Here is a simple example:

Corollary 7.7. *Let f_1 and f_2 be polynomials in n variables with coefficients in k, with f_2 non-zero. Let $U \subset \mathbf{A}^n$ be the open set where f_2 does not vanish. Set $f = f_1/f_2\colon U \to \mathbf{A}^1$. Then the following estimate holds:*

$$c_u(\mathscr{L}_{\psi(f)}) \ll (4 + \deg(f_1) + \deg(f_2))^{n+2}$$

where $u\colon \mathbf{A}^n \to \mathbf{P}^n$ is the standard embedding and the implied constant only depends on n.

Proof. Apply the bound above combined with Proposition 6.21, viewing U as the subscheme of $\mathbf{A}^{n+1}$ defined by the equation $f_2(x)y = 1$, and $f \colon U \to \mathbf{A}^1$ as the morphism $(x, y) \mapsto f_1(x)y$. $\qquad\square$

7.4. **Further examples.** We collect here a few more examples of "standard" sheaves whose complexity is easily estimated. Again, k is the algebraic closure of a finite field $\mathbf{F}$.

Proposition 7.8.

(1) *Let $\mathscr{K}$ be any hypergeometric sheaf on $\mathbf{G}_m$ in the sense of Katz[1]. We have*

$$c_u(\mathscr{K}) \ll \mathrm{rank}(\mathscr{K})$$

where the implied constant is absolute and $u \colon \mathbf{G}_m \to \mathbf{P}^1$ is the standard embedding.

(2) *Let (E, u) be an elliptic curve over $\mathbf{F}$ embedded in $\mathbf{P}^2$ as a cubic curve. Let θ be a non-trivial character $E(\mathbf{F}) \to \overline{\mathbf{Q}}_\ell^\times$, and let $\mathscr{L}_\theta$ be the associated character sheaf defined by the Lang isogeny. Then $c_u(\mathscr{L}_\theta) = 3$.*

(3) *Let $\{x_1, \ldots, x_4\}$ be four distinct points in $\mathbf{P}^1(\mathbf{F})$. Let $\mathscr{F}$ be an irreducible middle-extension sheaf of rank 2 on $\mathbf{P}^1$ over $\mathbf{F}$ with unipotent local monodromy at those four points.[2] Then $c(\mathscr{F}) = 2$.*

Proof. All these statements follow straightforwardly from calculating Betti numbers and Theorem 7.3 (1).

In the case of a hypergeometric sheaf, the cohomology is one-dimensional.

In case (2), we apply (25) with $(n, g, d) = (0, 1, 3)$. The rank of $\mathscr{L}_\theta$ is 1, so the first term in the maximum is equal to 3. The second term is 0, since $h^i(E, \mathscr{L}_\theta) = 0$ for all i by the assumption that θ is non-trivial.

In case (3), we apply the formula with $(n, g, d) = (0, 0, 1)$. The first term in the maximum is 2 since the rank of $\mathscr{F}$ is 2. In the second term, we have

$$h^0(\mathbf{P}^1, \mathscr{F}) = h^2(\mathbf{P}^1, \mathscr{F}) = 0,$$

since $\mathscr{F}$ is an irreducible middle-extension sheaf of rank 2. Thus, the sum of Betti numbers is equal to $-\chi(\mathbf{P}^1, \mathscr{F})$. The Euler–Poincaré characteristic formula for $\mathscr{F}$ gives

$$\chi(\mathbf{P}^1, \mathscr{F}) = 2\chi(\mathbf{P}^1, \overline{\mathbf{Q}}_\ell) - 4 \times 1 = 0$$

Hence, the sum of the Betti numbers vanishes and $c(\mathscr{F}) = 2$. $\qquad\square$

7.5. **Tame ramification.** When a sheaf naturally appears in higher dimension without being obtained from simpler sheaves by means of Grothendieck's six functors, one may still be able to estimate its complexity, using either the direct definition of complexity, or sometimes the uniformity statement of Proposition 6.19. We illustrate this here with some results involving tame ramification.

Proposition 7.9. *Let X be a smooth scheme over k, and let $j \colon X \to \overline{X}$ be an embedding into a smooth scheme $\overline{X}$ such that the complement $D = \overline{X} \setminus X$ is a divisor with normal crossings. Let $i \colon \overline{X} \to \mathbf{P}_k^n$ be a closed immersion, and let $\mathscr{F}$ be a lisse sheaf on X that is tamely ramified along D. Then the following holds:*

$$c_{i \circ j}(\mathscr{F}) \ll \mathrm{rank}(\mathscr{F})c(i \circ j).$$

---

[1]See [27, §8.2] for the definition and properties of hypergeometric sheaves.

[2] Deligne and Flicker [9, Prop. 7.1] have proved that there are $|\mathbf{F}|$ such sheaves up to isomorphism and $\mathbf{F}$-twisting.

Proof. Set $d = \dim(X)$. The object $\mathscr{F}[d]$ is perverse since $\mathscr{F}$ is a lisse sheaf on a smooth scheme of dimension d. Since j is an affine open immersion, the object $j_! \mathscr{F}[d]$ is perverse on $\overline{X}$, and since i is a closed immersion, the object $A = i_! j_! \mathscr{F}[d]$ is also perverse (see [3, Cor. 4.1.3]). Since A is perverse, Corollary 5.5 (2) and Lemma 2.5 yield

$$c_{i \circ j}(\mathscr{F}) = c(A) \ll \| \operatorname{CC}(A) \| \ll \sup_{0 \leqslant m \leqslant n} | \operatorname{CC}(A) \cdot \operatorname{CC}(K_m) |,$$

where K_m is the constant sheaf on a generic m-dimensional subspace of $\mathbf{P}^n$.

By Theorem 2.2, $| \operatorname{CC}(A) \cdot \operatorname{CC}(K_m) |$ is the absolute value of the Euler–Poincaré characteristic of the pullback of A to a generic m-dimensional subspace, or equivalently the Euler–Poincaré characteristic of the restriction of $\mathscr{F}$ to the intersection of a generic m-dimensional subspace with X.

For a generic m-dimensional subspace H, the intersection $H \cap \overline{X}$ is smooth, the intersection with $\overline{D}$ has normal crossings, and the restriction of $\mathscr{F}$ has tame ramification around the intersection with D. This implies the equality

$$\chi(X \cap H, \mathscr{F}) = \operatorname{rank}(\mathscr{F}) \chi(X \cap H),$$

and since the linear subspace H is generic, the Euler–Poincaré characteristic of $X \cap H$ is bounded by $c(i \circ j)$. $\qquad\square$

Corollary 7.10. *Let (X, u) and (Y, v) be quasi-projective algebraic varieties over k, and let $\pi \colon Y \to X$ be a finite étale covering. Assume that v is the composition of an embedding into a smooth normal crossings compactification $\overline{Y}$ with a closed immersion. Let $\mathscr{F}$ be a lisse sheaf on X such that $\pi^* \mathscr{F}$ is tamely ramified along $\overline{Y} - Y$. Then the estimate*

$$c_u(\mathscr{F}) \ll \operatorname{rank}(\mathscr{F}) c_{v,u}(\pi) c(v)$$

holds, with an implied constant that only depends on the embedding dimensions of (X, u) and (Y, v).

Proof. The composition of the adjunction maps

$$\mathscr{F} \to \pi_* \pi^* \mathscr{F} = \pi_! \pi^* \mathscr{F} \to \mathscr{F}$$

is the multiplication by the degree of π. The sheaf $\mathscr{F}$ is hence a direct factor of $\pi_! \pi^* \mathscr{F}$, and the estimates

$$c_u(\mathscr{F}) \leqslant c_u(\pi_! \pi^* \mathscr{F}) \ll c_{v,u}(\pi) c_v(\pi^* \mathscr{F}) \ll \operatorname{rank}(\mathscr{F}) c_{v,u}(\pi) c(v),$$

hold by Proposition 6.14, part (9) of Theorem 6.8, and Proposition 7.9 applied to $\pi^* \mathscr{F}$ on Y. $\qquad\square$

7.6. **The Riemann Hypothesis.** In this section, we record the general "quasi-orthogonality" version of the Riemann Hypothesis over finite fields arising from Deligne's work and the theory of complexity developed here.

We work over a finite field $\mathbf{F}$, and assume that k is an algebraic closure of $\mathbf{F}$. We fix an isomorphism ι from $\overline{\mathbf{Q}}_\ell$ to $\mathbf{C}$ to define weights, as explained in the notation section. The notation for trace functions is also recalled there.

We first explain the type of bounds for values of trace functions of perverse sheaves that can be obtained using the complexity.

Proposition 7.11. *Let (X, u) be a quasi-projective variety over $\mathbf{F}$ of pure dimension d. Let M be a perverse sheaf on X that is pure of weight zero.*

(1) *For any $x \in X(\mathbf{F})$, the estimate $t_M(x) \ll c_u(M)$ holds.*

(2) *If $d \geqslant 1$ and M is geometrically irreducible with support equal to X, then the following estimate holds:*

$$t_M(x) \ll c_u(M)|\mathbf{F}|^{-1/2}.$$

(3) *There exists a stratification of X, defined over $\mathbf{F}$, into locally closed strata S_j that are irreducible and smooth of pure dimension d_j such that the degree of $u(S_j)$ is bounded in terms of $c_u(M)$ and the estimate*

$$t_M(x) \ll c_u(M)|\mathbf{F}|^{\frac{\max\{-d,-d_j-1\}}{2}}$$

holds for all j and all $x \in S_j(\mathbf{F})$.

In all these estimates, the implied constants only depend on the embedding dimension of (X, u).

Proof. (1) The perversity of M implies the vanishing of the cohomology sheaves $\mathscr{H}^i(M)$ in all degrees $i > 0$ and $i < -d$. By the definition of weights for perverse sheaves, if M is pure of weight zero, then $\mathscr{H}^i(M)$ is pointwise mixed of weight $\leqslant i$, which means that the eigenvalues of Frobenius acting on the stalks $\mathscr{H}^i(M)_x$ all have modulus $\leqslant |\mathbf{F}|^{i/2} \leqslant 1$. Therefore, the inequality

$$|t_M(x)| \leqslant \sum_{-d \leqslant i \leqslant 0} |t_{\mathscr{H}^i(M)}(x)| \leqslant \sum_{-d \leqslant i \leqslant 0} \dim \mathscr{H}^i(M)_x$$

holds, and it suffices to estimate the dimensions of the stalks of $\mathscr{H}^i(M)$. From (8) applied to the morphism $x \colon \operatorname{Spec}(\mathbf{F}) \to X$, we derive the estimates

$$(26) \qquad \dim \mathscr{H}^i(M)_x \leqslant c_{u \circ x}(x^*M) \ll c_{u \circ x, u}(x)c_u(M) = c(u \circ x)c_u(M) = c_u(M),$$

and hence the estimate $t_M(x) \ll c_u(M)$ as claimed.

(2) If we assume, moreover, that M is a geometrically irreducible perverse sheaf with support X, by [3, Th. 4.3.1 (ii)] there exists a dense open subset $j \colon U \to X$ and a lisse sheaf $\mathscr{F}$ on U such that M is the middle extension $j_{!*}\mathscr{F}[d]$. Then $\mathscr{H}^0(M)$ vanishes by [3, Prop. 2.2.4], so that the eigenvalues of Frobenius have modulus $\leqslant |\mathbf{F}|^{-1/2}$ and the same argument as in (1) yields the stronger estimate

$$|t_M(x)| \ll c_u(M)|\mathbf{F}|^{-1/2}.$$

(3) By Theorem 6.23, there exists a stratification of X, defined over $\mathbf{F}$, into locally closed strata S_j such that the restriction of $\mathscr{H}^i(M)$ to each S_j is lisse and the degree of $u(S_j)$ is bounded in terms of $c_u(M)$. By [3, Prop. 2.2.4], there is an inequality $\dim \operatorname{Supp}(\mathscr{H}^i(M)) < -i - 1$ for all $-i < d$. Since $\mathscr{H}^i(M)$ restricts to a lisse sheaf on S_j, we must have $d_j < -i$ whenever $\mathscr{H}^i(M)$ is non-zero and $d_j \neq d$. $\qquad\square$

Remark 7.12. The following example shows that the estimates in Proposition 7.11 are best possible in general. Assume that the finite field satisfies $|\mathbf{F}| \equiv 1 \pmod 3$. Let χ be a non-trivial multiplicative character of order three of $\mathbf{F}$ and let $\mathscr{L}_\chi$ be the associated Kummer sheaf on $\mathbf{G}_m$. Denote by $j \colon U \to \mathbf{A}^n$ the inclusion of the complement of the hypersurface $F \subset \mathbf{A}^n$ given by the equation

$$x_1^3 + \cdots + x_n^3 = 0.$$

Let $\mathscr{F}$ be the lisse sheaf $\mathscr{L}_{\chi(x_1^3 + \cdots + x_n^3)}(-n/2)$ on U and let $M = j_{!*}(\mathscr{F}[n])$ be its middle extension to $\mathbf{A}^n$, shifted to make it perverse. Then M is a geometrically irreducible perverse sheaf of weight zero and the equality $|t_M(x)| = |\mathbf{F}|^{-n/2}$ holds for all $x \in U(\mathbf{F})$. We claim that, at the origin, the trace function of M satisfies

$$|t_M(0)| \gg |\mathbf{F}|^{-1/2}.$$

To see this, let $h\colon X' \to \mathbf{A}^n$ denote the blow-up of $\mathbf{A}^n$ at the origin, $E \simeq \mathbf{P}^{n-1}$ the exceptional divisor, F' the strict transform of F, and $j'\colon U' \to X'$ the inclusion of the inverse image $U' = h^{-1}(U)$ of U. We define $\mathscr{F}' = h^*\mathscr{F}$ and $M' = j'_{!*}(\mathscr{F}'[n])$. Since χ is a character of order three, a local computation shows that $\mathscr{F}'$ extends to a lisse sheaf outside F' (for example, in the affine chart given by $y_1 = 1$ and $y_i = x_i/x_1$ for $i = 2,\dots,n$, this follows from the equality $\mathscr{L}_{\chi(x_1^3 + \cdots + x_n^3)} = \mathscr{L}_{\chi(x_1^3(1+y_2^3+\cdots+y_n^3))} = \mathscr{L}_{\chi(1+y_2^3+\cdots+y_n^3)}$). Since F' is a smooth divisor, the perverse sheaf M' is the extension by zero of the shift of this lisse sheaf to X'. Hence, its trace function at a point $y = [y_1\colon \cdots \colon y_n] \in E(\mathbf{F})$ equals

$$t_{M'}(y) = \begin{cases} \chi(y_1^3 + \cdots + y_n^3) & \text{if } y_1^3 + \cdots + y_n^3 \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

By the decomposition theorem [3], the perverse sheaf M is a direct factor of $h_!(M')$, and since h is an isomorphism outside the origin, the other irreducible components of $h_!(M')$ are skyscraper sheaves supported at zero. By the Grothendieck–Lefschetz trace formula, the trace function of M at 0 is equal to

$$\frac{1}{|\mathbf{F}|^{n/2}} \sum_{\substack{[y_1\colon\cdots\colon y_n]\in\mathbf{P}^{n-1}(\mathbf{F}) \\ y_1^3+\cdots+y_n^3\neq 0}} \chi(y_1^3 + \cdots + y_n^3),$$

which up to the normalizing factor $|\mathbf{F}|^{-n/2}$ is a sum of Jacobi sums of weight $n-1$ by a classical computation (see [24, Ch. 8, Th. 5]). Weyl's equidistribution theorem implies that this sum does not cancel, at least for some extensions of any given finite field.

Theorem 7.13 (Quasi-orthogonality). *Let $\mathbf{F}$ be a finite field of characteristic different from ℓ. Let (X, u) be a geometrically irreducible quasi-projective algebraic variety of dimension d over $\mathbf{F}$.*

(1) Assume that X is smooth. Let A and B be ℓ-adic constructible sheaves on X. Suppose that A and B are mixed of weights $\leqslant 0$ and that there exists a dense open subset U of X on which A and B are lisse sheaves, pure of weight zero, and geometrically irreducible. Then there exists a complex number α of modulus 1 such that the estimate

$$\left| \frac{1}{|\mathbf{F}|^d} \sum_{x\in X(\mathbf{F})} t_A(x)\overline{t_B(x)} - \alpha\delta(A, B) \right| \ll c(u)c_u(A)c_u(B)|\mathbf{F}|^{-1/2}$$

$$+ c(u)(c_u(A) + c_u(B))|\mathbf{F}|^{-1}$$

holds, with $\delta(A, B) = 1$ if A and B are geometrically isomorphic over U and $\delta(A, B) = 0$ otherwise.

(2) Let A and B be geometrically irreducible ℓ-adic perverse sheaves on X. Suppose that A and B are pure of weight zero. Then there exists a complex number α of modulus 1 such that the estimate

$$\left| \sum_{x\in X(\mathbf{F})} t_A(x)\overline{t_B(x)} - \alpha\delta(A, B) \right| \ll c(u)c_u(A)c_u(B)|\mathbf{F}|^{-1/2}$$

holds, with $\delta(A, B) = 1$ if A and B are geometrically isomorphic and $\delta(A, B) = 0$ otherwise.

In both estimates, the implied constant only depends on the embedding dimension of (X, u).

Proof. We start with (1). Since B restricts (up to a shift and a Tate twist) to a perverse sheaf of weight zero on U, the proof of Proposition 6.40 shows that $\overline{t_B}$ coincides with the trace function of $\mathrm{D}(B)$ on $U(\mathbf{F})$. Since A and B are geometrically irreducible, the

object $A \otimes \mathrm{D}(B)$ has non-trivial cohomology with compact support in top degree $2d$ if and only if A and B are geometrically isomorphic, and in that case this cohomology is one-dimensional. By [29, Lem. 1.8.1], if A and B are geometrically isomorphic, there is an isomorphism $B \simeq A \otimes \alpha^{\deg}$ for a unique $\alpha \in \overline{\mathbf{Q}}_\ell$, which has modulus 1 when viewed as a complex number. Arguing as in the proof of Katz's orthogonality theorem [29, Th. 1.7.2], the combination of the Grothendieck–Lefschetz trace formula and Deligne's Riemann Hypothesis thus gives the estimate

$$\left| \frac{1}{|\mathbf{F}|^d} \sum_{x \in U(\mathbf{F})} t_A(x)\overline{t_B(x)} - \alpha\delta(A, B) \right| \leqslant \sigma|\mathbf{F}|^{-1/2},$$

where σ denotes the sum of Betti numbers

$$\sigma = \sum_{i \in \mathbf{Z}} h_c^i(U_{\overline{\mathbf{F}}}, A \otimes \mathrm{D}(B)).$$

By Proposition 7.1 and (5), this quantity is bounded by

$$\sigma \ll c(A \otimes \mathrm{D}(B)) \ll c_u(A)c_u(\mathrm{D}(B)) \ll c(u)c_u(A)c_u(B).$$

On the other hand, on the complement $X - U$, the functions t_A and $\overline{t_B}$ are bounded by the generic rank of A and B respectively, which are in turn bounded by their complexity taking (26) into account. This gives the estimate

$$\left| \frac{1}{|\mathbf{F}|^d} \sum_{x \in (X-U)(\mathbf{F})} t_A(x)\overline{t_B(x)} \right| \ll c_u(A)c_u(B)|(X - U)(\mathbf{F})||\mathbf{F}|^{-d}.$$

We then note that U is contained in the intersection of the maximal open set U_1 where A is lisse and the maximal open set U_2 where B is lisse, and hence the estimates

$$\deg(X - U) \ll \deg(X - U_1) + \deg(X - U_2) \ll c(u)(c_u(A) + c_u(B))$$

hold by Theorem 6.23 (recall that X is assumed to be smooth). We conclude using the classical bound $|(X - U)(\mathbf{F})| \leqslant \deg(X - U)|\mathbf{F}|^{\dim(X-U)}$.

The proof of (2) is the same, except that the term $|\mathbf{F}|^{-d}$ does not appear due to the normalization of weights for perverse sheaves (see Remark 1.3) and that we do not need to treat the sum over points of $X - U$. $\qquad\square$

Taking $X = \mathbf{A}^n$, the second statement immediately implies Theorem 1.2.

Remark 7.14. In the setting of part (2) of Theorem 7.13, suppose that one knows that there is $\beta \geqslant 0$ such that

$$\mathrm{H}_c^i(X_{\overline{\mathbf{F}}}, A \otimes \mathrm{D}(B)) = 0$$

for $i \geqslant -\beta$ (the case $\beta = 0$ corresponds to the assumption that A and B are not geometrically isomorphic). Then by the same argument and the Riemann Hypothesis, we obtain the stronger estimate

$$\sum_{x \in X(\mathbf{F})} t_A(x)\overline{t_B(x)} \ll c(u)c_u(A)c_u(B)|\mathbf{F}|^{-(\beta+1)/2}.$$

The case where $\beta = \dim(X) - 1$ corresponds to full square-root cancellation for the sum over $X(\mathbf{F})$.

A basic finiteness statement follows from this result:

Corollary 7.15. *Let* $\mathbf{F}$ *be a finite field of characteristic different from* ℓ. *Let* (X, u) *be a geometrically irreducible quasi-projective algebraic variety over* $\mathbf{F}$.

For any $c \geqslant 1$, *there are, up to geometric isomorphism, only finitely many* ℓ-*adic perverse sheaves* A *on* X *of complexity* $c_u(A) \leqslant c$ *that are geometrically irreducible and pure of weight zero.*

Proof. By Theorem 7.13, applied to finite extensions of $\mathbf{F}$, we first see that there exists a finite extension $\mathbf{F}_n$ of $\mathbf{F}$ (depending on c) such that the equality of trace functions $t_A(\cdot; \mathbf{F}_n) = t_B(\cdot; \mathbf{F}_n)$ implies that A and B are geometrically isomorphic, for irreducible perverse sheaves A and B of weight zero on X with $c_u(A) \leqslant c$ and $c_u(B) \leqslant c$. Moreover, we can also ensure that $X(\mathbf{F}_n)$ is not empty and that $t_A(\cdot; \mathbf{F}_n) \neq 0$ when $c_u(A) \leqslant c$.

Replacing $\mathbf{F}$ by $\mathbf{F}_n$, it is then enough to prove that there are only finitely many functions t_A for irreducible perverse sheaves of weight zero with $c_u(A) \leqslant c$, up to geometric isomorphism. Let $C(X(\mathbf{F}_n))$ be the finite-dimensional Hilbert space of functions $X(\mathbf{F}_n) \to \mathbf{C}$ with norm

$$\|f\| = \sum_{x \in X(\mathbf{F}_n)} |f(x)|^2.$$

Theorem 7.13 again implies that for A not geometrically isomorphic to B with $c_u(A) \leqslant c$ and $c_u(B) \leqslant c$, the functions $t_A/\|t_A\|$ and $t_B/\|t_B\|$, viewed as elements of the unit sphere of $C(X(\mathbf{F}_n))$, make an angle $\geqslant \theta$ for some $\theta > 0$ independent of A and B. It is well-known that there can only be finitely many such vectors. $\qquad\square$

Remark 7.16. This argument can be made quantitative, although it is probably far from sharp; see [12] for the case of curves.

7.7. **Examples.** We collect here for ease of reference some immediate corollaries of the formalism of Section 6. These contain and generalize all the basic "continuity" estimates of Fouvry, Kowalski and Michel in the case of curves.

Example 7.17 (Fourier Transform)**.** Let $\mathbf{F}$ be a finite field. Let $n \geqslant 1$ be an integer and let $X = \mathbf{A}^n$ viewed as a commutative algebraic group over $\mathbf{F}$, with the obvious embedding in $\mathbf{P}^n$ to define the complexity. Let ψ be a fixed non-trivial ℓ-adic additive character of $\mathbf{F}$ and consider the Artin–Schreier sheaf $\mathscr{L} = \mathscr{L}_{\psi(x \cdot y)}$ of rank 1 on $\mathbf{A}^n \times \mathbf{A}^n$, where $x \cdot y$ is the standard scalar product. Deligne defined the ℓ-adic Fourier transform FT_ψ as the functor $\mathrm{D}_c^b(\mathbf{A}^n) \to \mathrm{D}_c^b(\mathbf{A}^n)$ given by

$$\mathrm{FT}_\psi(A) = p_{2,!}(p_1^* A \otimes \mathscr{L}) = p_{2,*}(p_1^* A \otimes \mathscr{L})$$

(see [38, 1.2.1.1]; the equality of the two expressions is a highly non-trivial fact, often referred to as "the miracle of the Fourier transform").

From the results of Section 6 and Corollary 7.7, we therefore deduce:

Proposition 7.18. *There exists an integer* $N \geqslant 0$, *depending only on* n, *such that*

$$c(\mathrm{FT}_\psi(A)) \leqslant N c(A)$$

holds for any object A *of* $\mathrm{D}_c^b(\mathbf{A}^n)$.

The results of Section 8, together with Propositions 6.21 and 7.5, lead to explicit estimates for N. These are growing at least as fast as $n!$, and it might be interesting to have a better estimate for this "norm" of the ℓ-adic Fourier transform.

For $n = 1$, Fouvry, Kowalski and Michel proved the inequality

$$c_{\mathrm{fkm}}(\mathrm{FT}_\psi(A)) \leqslant 10 c_{\mathrm{fkm}}(A)^2$$

(for middle-extension Fourier sheaves) in [15, Prop. 8.2]; this estimate plays an essential role in many analytic applications, and one can expect a similar use of Proposition 7.18.

Example 7.19 (Other cohomological transforms). Let (X, u) and (Y, v) be quasi-projective varieties over k and K an object of $\mathrm{D}_c^b(X \times Y)$. Let $T_K \colon \mathrm{D}_c^b(X) \to \mathrm{D}_c^b(Y)$ denote the "cohomological transform with kernel K", i.e., the functor such that

$$T_K(A) = p_{2,!}(p_1^* A \otimes K),$$

where $p_1 \colon X \times Y \to X$ and $p_2 \colon X \times Y \to Y$ are the projections.

Applying again the formalism, there exists a constant N_K, depending on K, such that the inequality

$$c_v(T_K(A)) \leqslant N_K c_u(A)$$

holds for any object A of $\mathrm{D}_c^b(X)$. Precisely, this holds with

$$N_K = c_{u \boxtimes v, v}(p_2) c_{u \boxtimes v, v}(p_1) c_{u \boxtimes v}(K)$$

(where $u \boxtimes v$ is used to denote the composition of $u \times v$ with the Segre embedding of the product of the projective spaces target of u and v; as in the case of the Fourier transform, other embeddings of $X \times Y$ might be possible).

In the very special case where $X = Y = \mathbf{A}^1$ (over finite fields) and K is a rank 1 Artin–Schreier or Kummer sheaf on $\mathbf{A}^2$, we can apply Corollary 7.7 to estimate the complexity of the kernel sheaf; a weaker form of the resulting estimate was proved by Fouvry, Kowalski and Michel [16, Th. 2.3].

Other special cases that have been already considered (when $n = 1$) for the varieties $X = Y = \mathbf{A}^1$ or $X = Y = \mathbf{G}_m$ are additive convolution and multiplicative convolution.

More generally, let G be a commutative quasi-projective algebraic group over k, with a given locally closed embedding u. Let $\sigma \colon G \times G \to G$ be the addition morphism. We have two convolution functors

$$* \colon \mathrm{D}_c^b(G) \times \mathrm{D}_c^b(G) \to \mathrm{D}_c^b(G), \qquad *_! \colon \mathrm{D}_c^b(G) \times \mathrm{D}_c^b(G) \to \mathrm{D}_c^b(G)$$

defined by

$$A * B = \sigma_*(A \boxtimes B), \qquad A *_! B = \sigma_!(A \boxtimes B).$$

Use the composition of the Segre embedding with $u \times u$ to embed $G \times G$ in projective space.[3] Then by (11) and (9) and Proposition 6.12, there exists an integer N such that the inequalities

$$c_u(A * B) \leqslant N c_u(A) c_u(B), \qquad c_u(A *_! B) \leqslant N c_u(A) c_u(B)$$

hold for all objects A and B of $\mathrm{D}_c^b(G)$.

Example 7.20 (Gowers uniformity sheaves). Let (G, u) be a commutative quasi-projective algebraic group over a field k_0, with the group law written additively. For each integer $d \geqslant 1$, we denote by $\pi_d \colon G^{d+1} \to G$ the projection $(x, h_1, \ldots, h_d) \mapsto x$. Given a complex A of $\mathrm{D}_c^b(G)$, we define the *uniformity object* $U_d(A)$ as

$$U_d(A) = \pi_{d,!}\Big(\bigotimes_I \mathrm{D}^{\circ |I|}(\sigma_I^* A) \Big),$$

where the tensor product runs over all subsets $I \subset \{1, \ldots, d\}$. For each of them, $\mathrm{D}^{\circ |I|}$ denotes the Verdier duality functor if $|I|$ is odd and the identity functor if $|I|$ is even, and the morphism $\sigma_I \colon G^{d+1} \to G$ is given by

$$\sigma_I(x, h_1, \ldots, h_d) = x + \sum_{i \in I} h_i.$$

---

[3] For certain groups, such as affine groups, one can also use other embeddings.

The point of this construction is that, in the case where k_0 is a finite field $\mathbf{F}$ and A is a perverse sheaf pure of weight zero, the trace function of $U_d(A)$ satisfies

$$\sum_{x \in \mathbf{F}} t_{U_d(A)}(x) = \|t_A\|_{U_d}^{2^d},$$

where $\|\cdot\|_{U_d}$ is the d-Gowers norm for a complex-valued function on the finite group $G(\mathbf{F})$, see [41, Def. 11.2]. For $G = \mathbf{G}_a$, this construction is implicitly used in [13] to obtain an "inverse theorem" for Gowers norms of one-variable trace functions, and various results from loc. cit. can actually be interpreted as bounds for the complexity of $U_d(A)$ in terms of that of A. Thanks to Theorem 6.8, we can generalize them to any group: there exists an integer $N_d \geqslant 0$ such that the inequality

$$c_u(U_d(A)) \leqslant N_d c_u(A)^{2^d}$$

holds for all objects A of $\mathrm{D}_c^b(G)$.

Example 7.21 (Sum-product sheaves)**.** In [34] and [35], special cases of the following construction are applied to Kloosterman sheaves: given an input object A in $\mathrm{D}_c^b(\mathbf{A}^1)$ and an integer $l \geqslant 1$, one constructs a "sum-product" object

$$B_l = \bigotimes_{1 \leqslant i \leqslant l} f_i^* A \otimes f_{i+l}^* \mathrm{D}(A)$$

on $\mathbf{A}^{2+2l}$, with coordinates $(r, s, \boldsymbol{b})$, by means of the functions $f_i(r, s, \boldsymbol{b}) = r(s + b_i)$. It follows from the general formalism that, performing this construction with *any* input object A in $\mathrm{D}_c^b(\mathbf{A}^1)$, we have

$$c_v(B_l) \leqslant N_l c_u(A)^{2l}$$

for some constant N_l depending only on l (the embeddings are the standard embeddings $u \colon \mathbf{A}^1 \to \mathbf{P}^1$ and $v \colon \mathbf{A}^{2l+1} \to \mathbf{P}^{2l+1}$).

7.8. **Equidistribution.** Using the theory of complexity developed in this paper, the "horizontal" version of Deligne's Equidistribution Theorem formulated by Katz in the case of curves [26, Ch. 3] admits a straightforward extension to higher-dimensional varieties. As before, we fix an isomorphism $\iota \colon \overline{\mathbf{Q}}_\ell \to \mathbf{C}$ through which both fields are identified.

Theorem 7.22. *Let $N \geqslant 1$ be an integer and let (X, u) be a smooth and connected quasi-projective scheme over $\mathrm{Spec}(\mathbf{Z}[1/N])$ with geometrically irreducible generic fiber. For each prime p not dividing N, let $(X_{\mathbf{F}_p}, u_p)$ denote the reduction of (X, u) modulo p. Let $\mathscr{P}$ be an infinite set of primes not dividing $N\ell$, and assume that we are given, for each $p \in \mathscr{P}$, a lisse ℓ-adic sheaf $\mathscr{F}_p$ on $X_{\mathbf{F}_p}$ that is pure of weight zero and whose complexity satisfies $c_{u_p}(\mathscr{F}_p) \ll 1$.*

Assume that, for each $p \in \mathscr{P}$, the geometric and arithmetic monodromy groups of $\mathscr{F}_p$ coincide and are isomorphic to a fixed semisimple (i.e., the connected component of the identity is semisimple) algebraic group G. Let K be a maximal compact subgroup of $G(\mathbf{C})$.

For $p \in \mathscr{P}$ and $x \in X(\mathbf{F}_p)$, let $\theta_p(x)$ be the unique conjugacy class in K that intersects the conjugacy class of the semisimplification of the image in G of the Frobenius at x relative to $\mathbf{F}_p$, so that the equality

$$\mathrm{Tr}(\theta_p(x)) = t_{\mathscr{F}_p}(x)$$

holds. Then the families $(\theta_p(x))_{x \in X(\mathbf{F}_p)}$ become equidistributed as $p \to +\infty$ in the space of conjugacy classes of K, with respect to the Haar probability measure.

Proof. Let d be the relative dimension of X over $\mathrm{Spec}(\mathbf{Z}[1/N])$. Shrinking $\mathscr{P}$ and X if necessary, we may assume that $X_{\mathbf{F}_p}$ is a smooth and geometrically connected variety of dimension d and that $X(\mathbf{F}_p)$ is not empty for all primes p in $\mathscr{P}$. In fact, the number of $\mathbf{F}_p$-points behaves asymptotically like $|X(\mathbf{F}_p)| \sim p^d$ as $p \to +\infty$ by the Lang–Weil estimate.

Each lisse ℓ-adic sheaf $\mathscr{F}_p$ corresponds to a representation of the fundamental group $\pi_1(X_{\mathbf{F}_p})$ of $X_{\mathbf{F}_p}$. According to the Weyl Criterion, equidistribution follows from the vanishing

$$(27) \qquad \lim_{p \to +\infty} \frac{1}{|X(\mathbf{F}_p)|} \sum_{x \in X(\mathbf{F}_p)} \mathrm{Tr}(\rho(\theta_p(x))) = 0$$

for any non-trivial irreducible representation ρ of K. By the correspondance between semisimple algebraic groups and compact Lie groups, such a representation ρ corresponds uniquely to an algebraic irreducible representation of the algebraic group G, which is still denoted by ρ. The lisse sheaf $\rho(\mathscr{F}_p)$ on $X_{\mathbf{F}_p}$ (see Section 6.9) satisfies

$$\mathrm{Tr}(\rho(\theta_p(x))) = t_{\rho(\mathscr{F}_p)}(x)$$

for all $x \in X(\mathbf{F}_p)$. Moreover, this sheaf is of weight zero (the group G being semisimple, it is a subsheaf of some tensor power $\mathscr{F}^{\otimes a}$ by Lemma 6.32), geometrically irreducible (since its arithmetic and geometric monodromy groups are equal), and non-trivial. By the Riemann Hypothesis (Theorem 7.13 (1)), applied with $B = \overline{\mathbf{Q}}_\ell$, the estimate

$$\frac{1}{p^d} \sum_{x \in X(\mathbf{F}_p)} \mathrm{Tr}(\rho(\theta_p(x))) \ll c_{u_p}(\rho(\mathscr{F}_p))p^{-1/2}$$

holds for all $p \in \mathscr{P}$. By Proposition 6.33, there exists an integer $a \geqslant 0$, depending only on ρ, such that

$$c_{u_p}(\rho(\mathscr{F}_p)) \ll c_{u_p}(\mathscr{F})^a.$$

Since we assumed that $c_{u_p}(\mathscr{F}_p)$ is bounded, this estimate implies (27). $\qquad \square$

In practice, some multi-variable cases of this theorem could be proved by applying the Deligne–Katz equidistribution theorems to families of curves covering X.

Katz–Sarnak [31, §9.6] and Katz [29, Ch. 12] have proved earlier statements of the same kind, but assuming that $\mathscr{F}_p$ is the base change to $\mathbf{F}_p$ of a "common" sheaf or perverse sheaf on X over $\mathbf{Z}[1/N]$.

In fact, we now show that Theorem 7.22 gives a positive answer to the question of Katz [29, p. 8 and 12.6.6] concerning equidistribution of certain higher-dimensional families of additive character sums. This illustrates that the theory of complexity can, to a certain extent, obviate the lack of a theory of exponential sums over $\mathbf{Z}$ (the second part implies Theorem 1.4 from the introduction).

Let $n \geqslant 1$ and $d \geqslant 1$ be integers. Let $P(n,d)$ be the space of polynomials of degree d in n variables, and $P(n,d,\mathrm{odd}) \subset P(n,d)$ the subspace consisting of odd polynomials, by which we mean that only monomials of odd degree have non-zero coefficients. For a prime number p and $f \in P(n,d)(\mathbf{F}_p)$, set

$$S(f;p) = \frac{1}{p^{w(f)/2}} \sum_{x \in \mathbf{F}_p^n} e\Big(\frac{f(x)}{p}\Big),$$

where $w(f)$ is the smallest integer such that the vanishing

$$\mathrm{H}_c^i(\mathbf{A}_{\overline{\mathbf{F}}_p}^n, \mathscr{L}_{\psi(f)}) = 0$$

holds for all $i > w(f)$, and we recall the notation $e(z) = \exp(2i\pi z)$.

For an even integer N, we denote by
$$\mathrm{USp}_N(\mathbf{C}) = \mathrm{U}_N(\mathbf{C}) \cap \mathrm{Sp}_N(\mathbf{C})$$
the group of unitary symplectic matrices of size N, which is a maximal compact subgroup of the symplectic group $\mathrm{Sp}_N(\mathbf{C})$.

Corollary 7.23. *Let $n \geqslant 1$ and $d \geqslant 1$ be integers. Set $K_n = \mathrm{U}_{(d-1)^n}(\mathbf{C})$ and, for odd d,*
$$K_{n,odd} = \begin{cases} \mathrm{USp}_{(d-1)^n}(\mathbf{C}) & \text{if } n \text{ is odd,} \\ \mathrm{O}_{(d-1)^n}(\mathbf{C}) & \text{if } n \text{ is even.} \end{cases}$$

(1) *The families $(S(f;p))_{f \in P(n,d)(\mathbf{F}_p)}$ become equidistributed as $p \to +\infty$ with respect to the measure which is the image under the trace of the probability Haar measure on K_n.*

(2) *Suppose that d is odd. The families $(S(f;p))_{f \in P(n,d,odd)(\mathbf{F}_p)}$ become equidistributed as $p \to +\infty$ with respect to the measure which is the image under the trace of the probability Haar measure on $K_{n,odd}$.*

Proof. Note that $P(n,d)$ is a dense open subset of the affine space of polynomials of degree $\leqslant d$ in n variables. Let $D(n,d) \subset P(n,d)$ denote the dense open subset of Deligne polynomials (namely, those for which the homogeneous part of highest degree defines a smooth hypersurface in $\mathbf{P}^{n-1}$) and let $D(n,d,\mathrm{odd}) \subset P(n,d,\mathrm{odd})$ be the subset of odd Deligne polynomials. Both $D(n,d)$ and $D(n,d,\mathrm{odd})$ are smooth schemes over $\mathbf{Z}$.

Because (by the Riemann Hypothesis) the estimate
$$|S(f;p)| \leqslant \sum_{i \leqslant w(f)} h_c^i(\mathbf{A}_{\overline{\mathbf{F}}_p}^n, \mathscr{L}_{\psi(f)}) \ll c(\mathscr{L}_{\psi(f)}) \ll 1$$
holds for all p and $f \in P(n,d)(\mathbf{F}_p)$, it is enough to prove the equidistribution of the sums $S(f;p)$ for $f \in D(n,d)(\mathbf{F}_p)$ (resp. $f \in D(n,d,\mathrm{odd})(\mathbf{F}_p)$).

We first handle separately the cases $d = 1$ and $d = 2$. If $d = 1$, the sum $S(f;p)$ vanishes for any non-zero linear polynomial f. Since $K_0 = \mathrm{U}_0(\mathbf{C})$ is the trivial group, whose only element has trace zero, the equidistribution holds in that case.

If $d = 2$, then $K_2 = \mathrm{U}_1(\mathbf{C})$ is the unit circle. Write a polynomial $f \in D(n,2)(\mathbf{F}_p)$ as
$$f(x) = Q(x) + \lambda(x) + \mu,$$
where Q is a non-degenerate quadratic form, λ a linear form and μ a constant. For each prime $p \geqslant 3$, it is an elementary consequence of the fact that normalized Gauss sums have modulus one that $|S(f;p)| = 1$. Let $h \geqslant 1$ be an integer. Then
$$\sum_{f \in D(n,2)(\mathbf{F}_p)} S(f;p)^h = \frac{1}{p^{nh/2}} \sum_{x_1,\dots,x_h \in \mathbf{F}_p^n} \sum_{Q,\lambda} e\Big(\frac{1}{p}\Big(\sum_i (Q(x_i) + \lambda(x_i))\Big)\Big) \sum_{\mu \in \mathbf{F}_p} e\Big(\frac{h\mu}{p}\Big).$$
This vanishes as soon as $p > h$. If $h \leqslant -1$, we obtain the same conclusion after noting that $S(f;p)^{-1} = \overline{S(f;p)} = S(-f;p)$. Thus,
$$\lim_{p \to +\infty} \frac{1}{|D(n,2)(\mathbf{F}_p)|} \sum_{f \in D(n,2)(\mathbf{F}_p)} S(f;p)^h = 0$$
for any non-zero integer h, which proves equidistribution in $\mathrm{U}_1(\mathbf{C}) = K_2$.

We now assume that $d \geqslant 3$. For a prime p, let ψ be an ℓ-adic character with trace function $x \mapsto e(x/p)$, under the identification of $\overline{\mathbf{Q}}_\ell$ and $\mathbf{C}$. On $\mathbf{A}^n \times D(n,d)$ over $\mathbf{F}_p$ with coordinates (x,f) we have the Kummer sheaf $\mathscr{L}_{\psi(f(x))}$. Define
$$\mathscr{D}_p = p_{2,!}\mathscr{L}_{\psi(f(x))}[n](n/2),$$

where $p_2 \colon \mathbf{A}^n \times D(n, d) \to D(n, d)$ denotes the projection to the second factor, so that the equality

$$S(f; p) = (-1)^n t_{\mathscr{D}_p}(f)$$

holds by the trace formula. This is *a priori* an object of the derived category of $D(n, d)$ over $\mathbf{F}_p$, but Deligne has shown (see [29, Cor. 3.5.11, Cor. 3.5.12]) that $\mathscr{D}_p$ is in fact a lisse sheaf of rank $N = (d - 1)^n$ and pure of weight zero for each prime p not dividing d.

We first prove (2), which fits exactly the statement of Theorem 7.22. The relevant monodromy groups have been computed by Katz [29, Th. 12.6.3]: if $p \geqslant 7$ and $p \nmid d$, the geometric and arithmetic monodromy groups of the restriction $\mathscr{F}_p$ of $\mathscr{D}_p$ to $D(n, d, \mathrm{odd})$ coincide and are equal to Sp_N if n is odd and to O_N if n is even (precisely, these references show that the geometric monodromy group is as stated, but by [29, Th. 3.10.6], these sheaves are arithmetically self-dual, so their arithmetic monodromy groups cannot be bigger). Hence, $K_{n, \mathrm{odd}}$ is a maximal compact subgroup of the complex points of the geometric monodromy group for any p. Moreover, denoting by i the closed immersion $D(n, d, \mathrm{odd}) \to D(n, d)$, the estimate

$$c(\mathscr{F}_p) = c(i^* p_{2,!} \mathscr{L}(\psi(f(x)))) \ll c(i)c(p_2)c(\mathscr{L}_{\psi(f(x))})$$

holds by (8) and (9) for any prime p, where the complexities are computed using the embeddings of $\mathbf{A}^n$, $D(n, d)$ and their product and subschemes induced by the natural embeddings of affine spaces in the projective space of the same dimension.

By the bound in Section 7.3 and by Proposition 6.19, we deduce that $c(\mathscr{F}_p) \ll 1$ for all p, where the implied constant depends only on (d, n). Thus, the equidistribution in (2) follows from Theorem 7.22.

We now come back to (1). Here we note that, as above, we have $c(\mathscr{D}_p) \ll 1$ for all p, but the setting is not exactly that of Theorem 7.22. Indeed, Katz proved in [29, Th. 6.8.34] that the geometric monodromy group of $\mathscr{D}_p$, for $p \geqslant 7$ that does not divide $d(d - 1)$, is the group

$$G_{2p} = \{g \in \mathrm{GL}_N \mid \det(g)^{2p} = 1\},$$

which depends on p. We argue then by repeating the use of the Weyl Criterion in the proof of Theorem 7.22.

For $f \in D(n, d)(\mathbf{F}_p)$, we denote by $\theta_p(f)$ the conjugacy class in G_{2p} corresponding to the Frobenius at f, whose trace is equal to $S(f; p)$. Let ρ be a non-trivial irreducible representation of $\mathrm{U}_N(\mathbf{C})$, which can also be viewed as a representation of GL_N. The lisse sheaf $\rho(\mathscr{D}_p)$ satisfies $c(\rho(\mathscr{D}_p)) \ll 1$ for all p (as in loc. cit.). The restriction of ρ to G_{2p} is a direct sum of a bounded number of irreducible representations. We claim that if p is large enough, depending on ρ, then this restriction does not contain the trivial representation of G_{2p}. Indeed, by Frobenius reciprocity, the multiplicity of the trivial representation is equal to the sum over $h \in \mathbf{Z}$ of the multiplicity of the character $\det(\cdot)^{2ph}$ in ρ, i.e., it is equal to one if $\rho = \det(\cdot)^{2ph}$ for some non-zero h (because ρ is non-trivial), and zero otherwise. The first case cannot occur if p is large enough, hence the claim.

Now applying the Riemann Hypothesis as above to each irreducible subrepresentation of $\rho(\mathscr{D}_p)$, we obtain

$$\frac{1}{|D(n, d)(\mathbf{F}_p)|} \sum_{f \in D(n, d)(\mathbf{F}_p)} \mathrm{Tr}(\rho(\theta_p(f))) \ll p^{-1/2}$$

for all p, where the implied constant depends only on (n, d, ρ). This implies the equidistribution in (2). $\qquad\square$

In [11], the three last-named authors use the theory of complexity, among other tools, to study the equidistribution of families of exponential sums arising as discrete Fourier–Mellin transforms of trace functions on commutative algebraic groups, generalizing the equidistribution theorems of Deligne [7] (for powers of $\mathbf{G}_a$, through the Fourier transform) and Katz [30] (for $\mathbf{G}_m$), see [17] for a survey. Among other things, this has applications, also discussed in [11], to the study of the variance of arithmetic functions over function fields in arithmetic progressions, improving results of Hall, Keating and Roddity-Gershon [23]. We state here (a form of) the basic result, in the "vertical" direction (see [11, Th. 2]).

Theorem 7.24. *Let (G, u) be a connected commutative algebraic group over a finite field $\mathbf{F}$ with a given quasi-projective embedding. Denote by $\mathbf{F}_n$ the extension of $\mathbf{F}$ of degree n in an algebraic closure of $\mathbf{F}$ and by $\widehat{G}(\mathbf{F}_n)$ the group of characters of $G(\mathbf{F}_n)$.*

Let A be a geometrically irreducible perverse sheaf on G which is pure of weight zero. There exists a complex reductive algebraic group G_A with a maximal compact subgroup K_A such that the sums

$$S(A, \chi) = \sum_{x \in G(\mathbf{F}_n)} \chi(x) t_A(x; \mathbf{F}_n),$$

defined for $\chi \in \widehat{G}(\mathbf{F}_n)$, become equidistributed on average in $\mathbf{C}$ with respect to the image under the trace map of the Haar probability measure μ on K_A, i.e. the equality

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{1 \leqslant n \leqslant N} \frac{1}{|G(\mathbf{F}_n)|} \sum_{\chi \in \widehat{G}(\mathbf{F}_n)} f(S(A, \chi)) = \int_{K_A} f(\mathrm{Tr}(g)) d\mu(g)$$

holds for any continuous and bounded function $f \colon \mathbf{C} \to \mathbf{C}$.

8. Effective bounds

It is clear that the implied constants in our bounds for all of Grothendieck's six functors can be made effective as long as that for the Betti numbers of a tensor product of complexes in Theorem 5.1 can be made effective. We state here such an effective bound and sketch the proof.

Theorem 8.1. *For all objects A and B of $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\mathbf{P}^n_k)$, the following estimate holds:*

$$\sum_{i \in \mathbf{Z}} h^i(\mathbf{P}^n_k, A \otimes B) \leqslant \frac{2^{16}}{3^4} e^{4/13} 13^n (n+2)! \ c(A) c(B).$$

Sketch of proof. The primary issue is to control the bilinear form of Defintion 3.16. It is convenient to calculate with this bilinear form in a basis generated by constant sheaves (identical to that of Lemma 2.5 up to a sign). Let e_m be the class of $\overline{\mathrm{CC}(K_m)}$ for K_m the constant sheaf on an m-dimensional subspace, so that $e_0, \ldots, e_n$ form a basis of the vector space $\mathrm{CH}_n\left(\overline{T^*\mathbf{P}^n}\right) \otimes \mathbf{Q}$. By Theorem 2.2, the intersection pairing

$$e_{m_1} \cdot e_{m_2} = \overline{\mathrm{CC}(K_{m_1})} \cdot \overline{\mathrm{CC}(K_{m_2})}$$

is equal to $(-1)^n$ times the Euler–Poincaré characteristic of the intersection of a general m_1-dimensional subspace and a general m_2-dimensional subspace. This intersection is a projective space of dimension $m_1 + m_2 - n$, and hence has Euler–Poincaré characteristic $\min(m_1 + m_2 + 1 - n, 0)$, so the intersection number is equal to $(-1)^n \min(m_1 + m_2 + 1 - n, 0)$.

By construction, the function of Lemma 3.15 is easy to calculate with respect to this basis: it sends e_m to e_{m-1}. Therefore, the bilinear form of Lemma 5.5 satisfies

$$f(e_{m_1}, e_{m_2}) = g(m_1 + m_2 + 1 - n) \text{ with}$$

$$g(x) = \sum_{k=0}^{x} 4^k(x - k) = \frac{4^{x+2} - 3x - 4}{9}$$

for $x \geqslant 0$ and $g(x) = 0$ for $x \leqslant 0$. For simplicity, we will upper-bound it by the simpler bilinear form f such that $f(e_{m_1}, e_{m_2}) = 4^{m_1+m_2+3-n}/9$.

Set $e_i' = \sum_{j \leqslant i} 2^{i-j} e_j$. Then $e_0', \ldots, e_n'$ also form a basis. Fix a norm $\|\cdot\|$ on the vector space $\mathrm{CH}_n\left(\overline{T^*\mathbf{P}^n}\right) \otimes \mathbf{R}$ which is the ℓ^∞ norm in this basis. This basis is convenient because the intersection of e_i' with the characteristic cycle of a test sheaf supported on $\mathbf{P}^m$ is 1 if $m + i = n$ and 0 otherwise. To check this, it suffices to check that the intersection number of e_i with the characteristic cycle of a test sheaf supported on $\mathbf{P}^m$ is equal to 1 if $i + m = n$, to -2 if $i + m - n = 1$, and to 0 otherwise. This follows from the fact that the tensor product of K_i with the test sheaf supported on $\mathbf{P}^m$ is the test sheaf supported on $\mathbf{P}^{i+m-n}$, which has Euler–Poincaré characteristic equal to 1 if $i + 2 - n = 0$, to -2 if $i + m - n = 1$, and to 0 otherwise.

Because the intersection number of e_i' with the characteristic cycle of a test sheaf is equal to 1 on $\mathbf{P}^m$ if $m + i = n$ and to 0 otherwise, Proposition 5.4 then shows that for a perverse sheaf A, the norm $\|\overline{\mathrm{CC}(A)}\|$ is bounded by $4(n+1)c(A)$.

Rewriting the bilinear form f in the basis e_i' gives

$$f(e_{i_1}', e_{i_2}') = \sum_{j_1 \leqslant i_1} \sum_{j_2 \leqslant i_2} \frac{2^{i_1-j_1} 2^{i_2-j_2} 4^{j_1+j_2+3-n}}{9}$$

$$= \frac{2^{i_1+i_2} 4^{3-n}}{9} \left(\sum_{j_1 \leqslant i_1} 2^{j_1}\right)\left(\sum_{j_2 \leqslant i_2} 2^{j_2}\right)$$

$$\leqslant \frac{2^{i_1+i_2} 4^{4-n}}{9} 2^{i_1+1} 2^{i_2+1} = \frac{4^{i_1+i_2+4-n}}{9}.$$

Hence, the total norm of the bilinear form is at most

$$\sum_{i_1=0}^{n} \sum_{i_2=0}^{n} \frac{4^{i_1+i_2 4-n}}{9} = \frac{4^{4-n}}{9} \left(\sum_{i=0}^{n} 4^i\right)^n = \frac{4^{4-n}}{9}\left(\frac{4^{n+1}}{3}\right)^2 = \frac{4^{6+n}}{3^4}$$

and the constant of Corollary 5.5 involves an extra factor of $16(n+1)^2$ coming from the constant of Proposition 5.4, for a total of $(n+1)^2 4^{8+n}/3^4$.

Let b_n be the constant in Theorem 5.1 in dimension n. Then we can see from the induction argument that there are n terms (it may appear to be $n+1$, but we may set $\lambda_1 = 1$ at the start by scaling the whole matrix, which does not affect the automorphism of $\mathbf{P}^m$), each of size $13b_{n-1}c(A)c(B)$, plus one term coming from Corollary 5.5 of size $4^{8+n}(n+1)^2 c(A)c(B)/3^4$, so we obtain:

$$b_n = 13n b_{n-1} + \frac{4^{8+n}}{3^4}(n+1)^2 = \sum_{k=0}^{n} 13^{n-k} \frac{n!}{k!} \frac{4^{8+k}(k+1)^2}{3^4}$$

$$= n! 13^n \sum_{k=0}^{n} \frac{4^{8+k}(k+1)^2}{3^4 13^k k!} \leqslant (n+1)^2 n! 13^n \frac{2^{16}}{3^4} e^{4/13} \leqslant (n+2)! 13^n \frac{2^{16}}{3^4} e^{4/13},$$

as we wanted to show. $\qquad\square$

Remark 8.2. Being more careful in the numerical arguments would lead to a significantly improved constant (the dominant terms in the sum defining b_n are those for small k, and for such values our bounds could see significant improvement, e.g. we could use the

constant 1 in Corollary 5.5 for $k = 0$ instead of $2^{16}/3^4$), and some minor adjustments to the algebraic geometry can lower the base of the exponent 13^n, but we do not know how to improve on the factorial growth, except in characteristic zero where a completely different argument offers exponential growth.

From this statement, it is completely straightforward to make explicit the inequalities of Section 6, because all the implicit constants in those inequalities come from repeated applications of Theorem 5.1, which can be replaced with this effective version.

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