Classical and quantum Lagrangian field theories with boundary

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Outline

Introduction

- Generalize Segal–Atiyah's axioms to perturbative QFTs
  boundaries $\rightsquigarrow$ Hilbert spaces
  manifolds (with boundaries) $\rightsquigarrow$ states/operators
- Do it for general Lagrangian theories (including gauge theories)
- First understand classical picture

Lagrangian Mechanics

- In Lagrangian mechanics $S = \int_{t_0}^{t_1} L\, dt$ as a functional on the path space $N^{[t_0,t_1]}$.
- Usual example: $L = \frac{1}{2}m||v||^2 - V(q)$.
- Newton's equation are recovered as Euler–Lagrange equations (EL), i.e., critical points: $\delta S = 0$.
- A solution is uniquely specified by its initial conditions. Set $C := TN$, the space of **Cauchy data**.
- For this, one sets conditions at t_0 and t_1 (usually by fixing the path endpoints). Otherwise

$$\delta S = \mathrm{EL} + \alpha|_{t_0}^{t_1},$$

$$\alpha = \sum_i \frac{\partial L}{\partial v^i} dq^i \in \Omega^1(C).$$

Here EL denotes the term containing the EL equations. By *EL* we will denote the space of solutions to EL.

Symplectic formulation

$\omega := \mathrm{d}\alpha$ is symplectic iff L is regular. In this case:

- ω is the pullback on $C = TN$ of the canonical symplectic form on T^*N by the Legendre mapping.
- Time evolution is given by a Hamiltonian flow ϕ. In particular,

$$L := \text{graph}\, \phi_{t_0}^{t_1} \in \overline{TN} \times TN$$

  is Lagrangian (**canonical relation**).

Remark

L may also be defined directly as $L = \pi(EL)$ with

$$\begin{array}{rccc} \pi: & N^{[t_0,t_1]} & \to & TN \times TN \\ & \{x(t)\} & \mapsto & ((x(t_0), \dot{x}(t_0)), (x(t_1), \dot{x}(t_1))) \end{array}$$

This picture has to be generalized

Example1: Geodesics

We discuss geodesics on $\mathbb{E}^2$ (Minkowski would be more realistic).

$$L = ||v||,$$

S is defined on $N_0^{[t_0,t_1]} := \{\text{immersed paths}\}$.

- EL = straight lines
- Cauchy data: $C = \mathbb{R}^2 \times \mathbb{R}^2_* = \mathbb{R}^2 \times S^1 \times \mathbb{R}_{>0} \ni (\mathbf{q}, \mathbf{v}, \rho)$.
- $\alpha = \mathbf{v} \cdot \mathrm{d}\mathbf{q}$
- ω degenerate
- $L := \pi(EL) = \{(\mathbf{q}_1, \mathbf{v}, \rho_1), (\mathbf{q}_2, \mathbf{v}, \rho_2)) : \mathbf{q}_1 - \mathbf{q}_2 \,||\, \mathbf{v}\}$ Not a graph!

Geodesics (continued)

However:

- $\omega|_L = 0$, so L is isotropic (actually Lagrangian).
- $\ker \omega(q, v) = \operatorname{span}\left(\mathbf{v} \cdot \frac{\partial}{\partial \mathbf{q}}, \frac{\partial}{\partial \rho}\right) =$ directions parallel to $\mathbf{v}$ and rescalings of velocity, so

$$\varpi\colon C \to \underline{C} := C/\ker \omega = TS^1$$

  with canonical symplectic form (identify T and T^* using the metric).
- $\underline{L} := \varpi(L) = \operatorname{graph} \operatorname{Id}$, so a graph and Lagrangian.
- Actually, no time evolution after reduction (an example of topological theory).

Example 2: Free 2d particle

- $S_M = \int_M \partial_\mu \phi \, \partial^\mu \phi$ on $\mathbb{R}^M$.
- $EL_M = \{\phi \in \mathbb{R}^M : \Delta\phi = 0\}$.
- Cauchy data (for M a cylinder $S^1 \times I$) $C_{S^1} = (\mathbb{R}^{S^1})^2$: field on S^1 together with its normal derivative.
- If ∂M consistst of n circles $\partial_1 M, \ldots, \partial_n M$:

$$\begin{array}{cccc} \pi : & \mathbb{R}^M & \to & C_{S^1}^n \\ & \phi & \mapsto & ((\phi_{\partial_1 M}, \mathbf{n} \cdot \nabla\phi_{\partial_1 M}), \ldots) \end{array}$$

- $L_M := \pi(EL_M)$ is a graph for M a cylinder, otherwise not a graph.
- However, C_{S^1} is symplectic and L_M is Lagrangian in $C_{S^1}^n$.

General case (after V. Fock)

- Let $S_M = \int_M L$ be a class of local actions determined by a Lagrangian L. Here M is a d-manifold.
- S_M is defined on a space of fields F_M (e.g., maps from M to another manifold, connections on $M, \ldots$)
- $EL_M :=$ solutions to $\delta S_M = 0$ modulo boundary terms.
- Cauchy data: Let Σ be a $(d-1)$-dimensional manifold. $C_\Sigma :=$ fields on Σ that determine a unique solution to EL_M for $M = \Sigma \times [0, \epsilon]$, ϵ small.

By restricting the fields on the boundary, we have

$$\pi \colon F_M \to C_{\partial M}$$

The variation is now

$$\delta S_M = \mathrm{EL}_M + \pi^* \alpha_{\partial M} \tag{1}$$

where $\alpha_{\partial M}$ is determined by the boundary contributions.
Actually, working on $\Sigma \times [0, \epsilon]$, we have for every $(d-1)$-manifold Σ

$$\alpha_\Sigma \in \Omega^1(C_\Sigma)$$

Boundary structure

- $\omega_\Sigma := \mathrm{d}\alpha_\Sigma$ is a (pre)symplectic structure on C_Σ (symplectic iff L is regular).
- $L_M := \pi(EL_M)$ is isotropic in $C_{\partial M} \Longleftarrow$ (1) (in general not a graph) i.e., $\omega_{\partial M}|_{L_{\partial M}} = 0$(in all relevant examples $L_{\partial M}$ is Lagrangian)

Remark (Composition)

If $M = M_1 \cup_\Sigma M_2$, where Σ is (part of) the boundary of M_1 and of M_2,

$$L_M = L_{M_1} \circ L_{M_2} \subset C_{(\partial M_1 \setminus \Sigma) \coprod (\partial M_2 \setminus \Sigma)},$$

where $\circ$ denotes the composition of relations.

Definition

We call $L_{\partial M}$ the **evolution relation**. (More precisely, we split $\partial M = \partial_{\text{in}} M \coprod \partial_{\text{out}} M$ and regard L_M as a relation in $\overline{C_{(\partial_{\text{in}} M)^{\text{opp}}}} \times C_{\partial_{\text{out}} M}$.)

For a regular theory on a cylinder $M = \Sigma \times I$, L_M is a graph and the composition of cylinders yields the usual composition of maps.

Boundary structure (continued)

Remark (EL)

By definition the fiber of EL_M over L_M is just one point if M is a short cylinder, but in general it may be much bigger.
So it makes sense to remember it and think of $EL_M \to C_{\partial_M}$ as a correspondence, the **evolution correspondence**.

Remark (Reduction)

If ω_Σ is degenerate, we may consider symplectic reduction

$$\varpi \colon C_\Sigma \to \underline{C_\Sigma}$$

and also consider reduced evolution relations

$$\underline{L_{\partial M}} := \varpi(L_{\partial M}) \subset \underline{C_{\partial M}}.$$

They are automatically isotropic.
In all known examples, they are Lagrangian. Maybe a theorem.

Axiomatics

We may then think of a classical Lagrangian field theory in d dimensions as the following data:

- A space of field F_M for every d-manifold M
- A presymplectic space C_Σ for every $(d-1)$-manifold Σ
- An isotropic correspondence $\pi\colon EL_M \to C_{\partial M}$ for every M such that $\pi(EL_M)$ is Lagrangian after reduction.
- $(F_\bullet, C_\bullet)$ should be thought as a functor.

Remark

In the reduced picture (in case of trivial fibers), the target "category" is that of (singular) symplectic manifolds and canonical relations.
Notice that the reduced evolution relation for a (short) cylinder is a graph, actually a flow. In particular,

$$\text{“}\lim_{\epsilon\to 0}\text{”}\,\underline{L_{\Sigma\times[0,\epsilon]}} = \mathrm{graph}(\mathrm{Id}_{\underline{C_\Sigma}}).$$

For regular L, ω_Σ is nondegenerate for every Σ, so no reduction is needed.

- If ω_Σ is degenerate, we say that S defines a gauge theory.
- Notice that L_M is not a graph, even if M is a cylinder. In particular,

$$R_\Sigma := \text{“}\lim_{\epsilon\to 0}\text{''}\, L_{\Sigma\times[0,\epsilon]} \subset \overline{C_\Sigma} \times C_\Sigma$$

  is not a graph.
  It is an equivalence relation (gauge transformation) in C_Σ and

$$\underline{C_\Sigma} = C_\Sigma / R_\Sigma .$$

A topological field theory is a Lagrangian field theory that is invariant under diffeomorphisms.
So, in particular, it is a gauge theory and moreover

$$\underline{L_{\Sigma\times I}} = \mathrm{graph}(\mathrm{Id}_{\underline{C_\Sigma}})$$

for every interval I (no evolution).
One usually also requires all $\underline{C_\Sigma}$s to be finite dimensional (sometimes even compact).

Quantization of regular Lagrangian field theories

- In a regular theory, C_Σ is symplectic;
  geometric quantization: Hilbert space H_Σ
- To the canonical relation $L_M \subset C_{\partial M}$ associate a state $\psi_M \in H_{\partial M}$.
  Asymptotically,
  $$\psi_M = \int \mathrm{e}^{\frac{i}{\hbar} S_M} \in H_{\partial M}$$
  We integrate over bulk fields perturbing boundary fields
  (belonging to a section of the chosen polarization of $C_{\partial M}$).
- If $\partial M = \partial_{\text{in}} M \coprod \partial_{\text{out}} M$, then $\psi_M \in H^*_{\partial_{\text{in}} M} \otimes H_{\partial_{\text{out}} M}$.
  Hence, operator $H_{\partial_{\text{in}} M} \to H_{\partial_{\text{out}} M}$.
  Composition of relations goes to composition of operators.
- Cfr. Segal's axiomatization of CFT and Atiyah's axiomatization of TFT.

Coisotropic submanifolds

If the Lagrangian is not regular, $(C_\Sigma, \omega_\Sigma)$ is not symplectic. It is better to think of $C_\Sigma \subset F^\partial_\Sigma$ with

- F^∂_Σ a symplectic space of fields (for every Σ)
- C_Σ is a coisotropic submanifold of F^∂_Σ i.e., $(T_{C_\Sigma} F^\partial_\Sigma)^\perp \subset TC_\Sigma$

We will call F^∂_Σ the **space of boundary fields**.

Remark

By a Theorem of Gotay every presymplectic manifold may be embedded in a symplectic manifold as a coisotropic submanifold. So we are going to assume that every C_Σ is a presymplectic manifold (i.e., smooth manifold and $\ker \omega_\Sigma$ a smooth subbundle of TC_Σ). This symplectic extension is locally unique.

We assume that π extends to $F_M \to F^\partial_{\partial M}$ as a surjective submersion. The reduction $\underline{C_\Sigma}$ is usually singular, so it is better to work in terms of resolutions.

The BFV construction

Let C be a coisotropic submanifold of a symplectic manifold (F, ω).
Denote by $C^\infty(C)^{\text{invt}}$ the Poisson algebra of functions invariant under the distribution $\ker \omega|_C$.
(If the reduction $\underline{C}$ is smooth, then $C^\infty(C)^{\text{invt}} = C^\infty(\underline{C})$.)
The Koszul–Tate resolution can be given the following form:

Theorem (Batalin–Fradkin–Vilkovisky, Stasheff, Schätz)

One can embed F in a graded symplectic manifold $\mathcal{F}$ and find a function S of degree 1 satisfying $\{S, S\} = 0$ s.t. $C^\infty(C)^{invt}$ is isomorphic, as a Poisson algebra, to the degree-zero cohomology of $C^\infty(\mathcal{F})$ with differential $Q = \{S, \ \}$.

This requires some assumptions, e.g., the finite dimensionality of C.
Under some assumptions the construction works on spaces of fields and preserves locality.
It is better not to go to cohomology.

Keep working with the complex $(C^\infty(\mathcal{F}), Q)$

Example

Suppose C is codimension one: $C =$ zeros of a function ϕ.
Let $X := \{\phi, \}$. Then

$$C^\infty(C)^{\text{invt}} = (C^\infty(F)/ < \phi >)^X.$$

- First add a new odd coordinate b (degree -1) and set $Qb = \phi$. Hence the degree zero cohomology is $C^\infty(C)$.
- Then add another odd coordinate c (degree $+1$) and set $Qf = c\,X(f)$, $f \in C^\infty(F)$. Now the degree zero cohomology is what we want.
- Extend the symplectic form by the term $\mathrm{d}b\,\mathrm{d}c$ and define $S := c\phi$.

The general case is treated similarly as a starting point. The symplectic form and S are then constructed iteratively in powers of the bs using cohomological perturbation theory [BFV, Stasheff]. It is eventually possible to globalize the construction [Schätz].
In field theory, one may arrange things so that S keeps being a local functional (often at the expense of introducing new coordinates of higher degree).

Quantization

Working in geometric quantization:

- First assume that $\mathcal{F}$ can be quantized to a graded Hilbert space $\mathcal{H}$.
- Then assume that S can be quantized to an operator Ω (of degree 1) satisfying
$$\Omega^2 = 0$$
Notice that the classical condition $\{S, S\} = 0$ implies the quantum condition only up $\hbar^2$.
- Take the degree zero cohomology of the complex $(\mathcal{H}, \Omega)$ as the Hilbert space quantizing $\underline{C}$.

Again one does not have to go to cohomology

Back to our boundary case

Using the BFV construction, we replace the boundary presymplectic manifold C_Σ by the data

$$(\mathcal{F}^\partial_\Sigma, \omega^\partial_\Sigma = \mathrm{d}\alpha^\partial_\Sigma, S^\partial_\Sigma, Q^\partial_\Sigma)$$

of an exact BFV manifold, where

1. ω^∂_Σ symplectic form of degree zero
2. S^∂_Σ of degree 1 satisfying $\{S^\partial_\Sigma, S^\partial_\Sigma\} = 0$
3. $Q^\partial_\Sigma = \{S^\partial_\Sigma, \ \}$ is the Hamiltonian vector field of S^∂_Σ and hence has degree 1 and satisfies $[Q^\partial_\Sigma, Q^\partial_\Sigma] = 0$ (cohomological vector field). Other notation:
$$\iota_{Q^\partial_\Sigma}\omega^\partial_\Sigma = \mathrm{d}S^\partial_\Sigma$$

One can also show that $\mathcal{C}^\partial_\Sigma := \{\text{zeros of } Q^\partial_\Sigma\}$ is coisotropic in $\mathcal{F}^\partial_\Sigma$ and $C^\infty(\mathcal{C}^\partial_\Sigma) = H^\bullet_{Q^\partial_\Sigma}$.

The BV construction

In the bulk we have the problem that EL_M might also be singular. Moreover, if there are symmetries, we wish to consider the quotient $\underline{EL_M}$ as a starting point for perturbation theory in the functional integral.

- If M has no boundary, the Batalin–Vilkovisky (BV) construction yields a BV manifold

  $$(\mathcal{F}_M, \omega_M, S_M, Q_M)$$

  satisfying the same equations as in BFV but
  1. ω_M has degree -1 and S_M has degree zero.
  2. F_M is a submanifold of $\mathcal{F}_M$ and S_M is an extension of the classical action.
  3. $EL_M = \{\text{zeros of } Q_M\}_{\text{degree zero}}$ and $C^\infty(EL_M)^{\text{invt}} =$ degree zero cohomology of $(C^\infty(\mathcal{F}_M), Q_M)$.
- Explictly, $\mathcal{F}_M = T^*[-1](F_M \times \{\text{generators of symmetries}\})$ and $S_M = S_M^{\text{cl}} + \sum x^+ X_c + \cdots$.
  $Q_M = \sum \frac{\partial S_M^{\text{cl}}}{\partial x} \frac{\partial}{\partial x^+} + \cdots$. (EL appears in the first term.)

The case with boundary

The equation

$$\iota_{Q_M}\omega_M = \mathrm{d}S_M$$

no longer holds if M has boundary. We have to deal with the boundary terms as in the first part of this talk.
Putting BV+BFV+Fock together, we get the following axiomatics [C, Mnëv, Reshetikhin]:

- To each $(d-1)$-manifold Σ we associate a BFV-manifold $\mathcal{F}^{\partial}_{\Sigma}, \omega^{\partial}_{\Sigma} = \mathrm{d}\alpha^{\partial}_{\Sigma}, S^{\partial}_{\Sigma}, Q^{\partial}_{\Sigma})$.
- To each d-manifold M we associate the data $(\mathcal{F}_M, \omega_M, S_M, Q_M)$ together with a surjective submersion $\pi\colon \mathcal{F}_M \to \mathcal{F}^{\partial}_{\partial M}$ satisfying:
  1. $Q^{\partial}_{\partial M} = \mathrm{d}\pi Q_M$;
  2. $\iota_{Q_M}\omega_M = \mathrm{d}S_M + \pi^*\alpha^{\partial}_{\partial M}$.
- Plus functoriality and some regularity assumptions.

Several examples:

YM, *BF*, CS, PSM (actually, all AKSZ theories)

Example: Electromagnetism

- Maxwell's equations: $\mathrm{d}^*\mathrm{d}A = 0$, A connection 1-form.
- First-order formalism: $S_M^{\mathrm{cl}} = \int_M B\,\mathrm{d}A + \frac{1}{2}B * B$
  B a $(d-2)$-form. Then $EL = \{*B = \mathrm{d}A,\ \mathrm{d}B = 0\}$.
- BV: $S_M = \int_M B\,\mathrm{d}A + \frac{1}{2}B * B + A^+\,\mathrm{d}c$
  A^+: $(d-1)$-form, ghost number -1; c: 0-form, ghost number 1.
  $\omega_M = \int_M \delta A\,\delta A^+ + \delta B\,\delta B^+ + \delta c\,\delta c^+$,
  B^+ and c^+ do not show up in the action.
  $QA = \mathrm{d}c,\ QA^+ = \mathrm{d}B,\ QB^+ = *B + \mathrm{d}A,\ Qc^+ = \mathrm{d}A^+$.
- Boundary fields: A, B, A^+, c,
  $S_\Sigma^\partial = \int_\Sigma c\,\mathrm{d}B$,
  $\alpha_\Sigma^\partial = \int_\Sigma B\,\delta A + A^+\,\delta c$,
  $Q^\partial A^+ = \mathrm{d}B$, $Q^\partial A = \mathrm{d}c$.
  Interpretation:
  $A =$ vector potential, up to gauge transformations $A \mapsto A + \mathrm{d}c$
  $B =$ electric field constrained by Gauss law $\mathrm{d}B = 0$.

Properties

The fundamental equation

$$\iota_{Q_M}\omega_M = \mathrm{d}S_M + \pi^*\alpha^\partial_{\partial M} \tag{2}$$

has several consequences:

1. $L_{Q_M}\omega_M = \pi^*\omega^\partial_{\partial M}$ (Q_M not symplectic).
2. $Q_M(S_M) = 2S^\partial_{\partial M} - \pi^*(\iota_{Q^\partial_{\partial M}}\alpha^\partial_{\partial M})$ (modified CME).
3. $\mathcal{EL}_M := \{\text{zeros of } Q_M\}$ coisotropic,
$$\mathcal{L}_M := \pi(\mathcal{EL}_M) \overset{\text{isotropic/Lagrangian}}{\subset} \mathcal{C}^\partial_{\partial M} \overset{\text{coisotropic}}{\subset} \mathcal{F}^\partial_{\partial M}.$$
4. For every $\ell \in \mathcal{L}_M$, let
$\mathcal{E}_\ell := \pi^{-1}$(orbit through ℓ of coisotropic foliation).
Then $\mathcal{E}_\ell$ presymplectic and we have a fibration $\underline{\mathcal{EL}_M} \to \underline{\mathcal{L}_M}$ with finite dimensional odd symplectic fiber $\underline{\mathcal{E}_\ell}$ over $\underline{\ell}$.
BV canonical correspondence

Example EM:
$\underline{\mathcal{E}_\ell} = H^1(M, \partial M) \oplus H^{n-1}(M)[-1] \oplus H^0(M, \partial M)[1] \oplus H^n(M)[-2]$

Boundaries of boundaries

Sometimes it is possible to push this construction to even lower dimension.
For example in EM:

- Boundary fields: A, B, A^+, c, $\quad S^\partial_\Sigma = \int_\Sigma c\,\mathrm{d}B$, $\alpha^\partial_\Sigma = \int_\Sigma B\,\delta A + A^+\,\delta c$, $\quad Q^\partial A^+ = \mathrm{d}B$, $Q^\partial A = \mathrm{d}c$.
- Boundary of boundary: $\gamma = (d-2)$-manifold
  BB fields: B, c, $\quad \alpha^{\partial\partial}_\gamma = \int_\gamma B\,\delta c$, of degree $+1$
  $S^{\partial\partial}_\gamma = 0$, $\quad Q^{\partial\partial}_\gamma = 0$.
- Again we have $\iota_{Q^\partial_\Sigma}\omega^\partial_\Sigma = \mathrm{d}S^\partial_\Sigma + \pi^*\alpha^{\partial\partial}_{\partial\Sigma}$
- $$\underline{\mathcal{EL}_\Sigma} = \Omega^1(\Sigma)/_{\text{exact}} \oplus \Omega^{d-2}_{\text{closed}}(\Sigma, \partial\Sigma) \oplus H^0(\Sigma, \partial\Sigma)[1] \oplus H^{d-1}(\Sigma)[-1].$$

  For $d = 2$ this space is finite dimensional.

In CS, BF, and all AKSZ theories, one can go down up to zero dimensions!

Quantization

1. Fix a polarization on $\mathcal{F}^{\partial}_{\partial M}$ such the quantization $\Omega_{\partial M}$ of $S^{\partial}_{\partial M}$ squares to zero.
2. For simplicity, assume we have a transversal $\mathcal{L}'$ to the polarization. So $\mathcal{H}_{\partial M}$ = functions on $\mathcal{L}'$.
3. Define
$$\psi_M = \int e^{\frac{i}{\hbar} S_M} \in \mathcal{H}_{\partial M}$$
where the integral is over a Lagrangian submanifold of the fiber over a boundary field in $\mathcal{L}'$.
4. By standard techniques in BV, one may prove that
$$\Omega_{\partial M} \psi_M = 0.$$
Moreover, changing gauge fixing modifies ψ_M by an $\Omega_{\partial M}$-exact term. Thus,

ψ_M defines a class in the physical Hilbert space $H_{\Omega_{\partial M}} 0(\mathcal{H}_{\partial M})$.

Perturbative quantization

Usually, the only way of computing the functional integral is to perturb around a Gaussian theory.
Let S^0 be the Gaussian theory and denote by $\mathcal{Z}_M^0$ the space of functions on the fiber of $\underline{\mathcal{EL}}_M^0$ (“vacua"). Then

1. We get
$$\psi_M = \int \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_M} \in \mathcal{H}_{\partial M} \otimes \mathcal{Z}_M^0$$
2. Because of the odd symplectic structure on these fibers, $\mathcal{Z}_M^0$ has a BV structure. The modified CME is quantized as
$$\Delta_{\mathcal{Z}_M^0} \psi_M + \Omega_{\partial M} \psi_M = 0$$
Setting $\psi_M = \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_{\text{eff}}}$, we get the modified QME
$$\{S_{\text{eff}}, S_{\text{eff}}\} - \mathrm{i}\hbar \Delta_{\mathcal{Z}_M^0} S_{\text{eff}} + (\mathrm{i}\hbar)^2 \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} S_{\text{eff}}} \Omega_M \mathrm{e}^{\frac{\mathrm{i}}{\hbar} S_{\text{eff}}} = 0.$$

Axiomatics

- To each $(d-1)$-manifold Σ we associate a complex $(\mathcal{H}_\Sigma, \Omega_\Sigma)$ of Hilbert spaces.
- To each d-manifold we as associate a f.d. BV manifold $\underline{\mathcal{EL}}_M$ ("moduli space of vacua"), the BV algebra $\mathcal{Z}_M$ of functions on $\underline{\mathcal{EL}}_M$ (endowed with a BV operator Δ), and an element ψ_M of $\mathcal{H}_{\partial M} \otimes \mathcal{Z}_M$ satisfying the modified QME.
- Plus functorial properties.

Eventually, we may integrate over a Lagrangian submanifold of $\underline{\mathcal{EL}}_M$ and go to the Ω_Σ-cohomology getting just a state in the physical Hilbert space.

Remark

The full power of this approach is that we may cut the original manifold M into simple, or tiny, pieces; do the perturbative quantization there; and eventually glue and reduce.
This could provide some new insight for physical theories.
In TFTs it yields a perturbative version of Atiyah's axioms. We expect to be able to compute, e.g., perturbative CS invariants.

Example: *BF* theory

$$S = \int_M \left\langle B, \mathrm{d}A + \tfrac{1}{2}[A, A] \right\rangle,\ A \in \Omega(M, \mathfrak{g}),\ B \in \Omega(M, \mathfrak{g}^*)$$

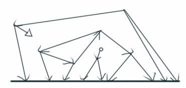

Figure: $\frac{\delta}{\delta B}$-foliation

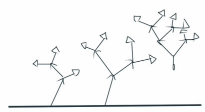

Figure: $\frac{\delta}{\delta A}$-foliation