A Numéraire-Independent Version of the Fundamental Theorems of Asset Pricing

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Joint work (in progress) with Travis Fisher and Sergio Pulido

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Executive summary: First FTAP as a bridge between the mathematics and the finance
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- martingale measure
- expectation
- valuation operator
- “absence of arbitrage”
- pricing
- NFLVR for allowable strategies
• We aim to widen the bridge to cover cleanly the case when there are multiple financial assets, any of which may potentially lose all value relative to the others.

• To do this we shift away from having a pre-determined numéraire to a more symmetrical point of view where all assets have equal priority.
• We aim to widen the bridge to cover cleanly the case when there are multiple financial assets, any of which may potentially lose all value relative to the others.
• To do this we shift away from having a pre-determined numéraire to a more symmetrical point of view where all assets have equal priority.
Our own motivation

- Popular model in FX:

\[ S_{1,2}(t) = S_{1,2}(0) + \int_0^t (aS_{1,2}(u)^2 + bS_{1,2}(u) + c) \, dW(u) \]

“Quadratic normal volatility” (stopped when hitting zero)

- Calibration usually yields strict local martingale dynamics.
- Let’s assume a complete market and zero interest rate.
- Superreplication cost of \( S_{1,2}(T) \) is strictly smaller than \( S_{1,2}(0) \) (if we price according to risk-neutral expectations)
- This yields issues with put-call parity, which is a market convention.
- Possible ways out:
  - Use a different model.
  - Change the concept of pricing operator.
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- **Lewis**: “add correction term” to risk-neutral expectation when pricing calls.
- **Madan & Yor**: Exchange expectations and limits.
- **Cox & Hobson**: Restrict class of admissible strategies.
- **Carr & Fisher & Ruf**:
  - Note that a change of numéraire via strict local martingale $S_{1,2}$ yields non-equivalent measure.
  - Then consider the minimal superreplication cost under both measures (the original one and the new one).
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**Issues:**

- Correction term seems non-symmetric in currencies.
- What to do in an incomplete market??
- What to do with more than two currencies??
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Related literature — an incomplete list

- Herdegen (2014)
- Tehranchi (2014): Non-existence of numéraire
- Kardaras (2014): Exchange options
- Carr & Fisher & Ruf (2014)
- Schönbucher’s survival measure (credit risk)
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Outline

1. Underlying objects: scenarios, price processes, ...
2. Valuation operator: maps future random (scenario-dependent) payoffs to present deterministic prices
3. Notions of arbitrage
4. Bringing everything together: Fundamental Theorems of Asset Pricing
5. Aggregation and disaggregation in different currencies

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Relative prices are modelled by an exchange matrix

- \( d \): number of currencies
- \( s_{i,j} \): units of currency \( i \) per unit of currency \( j \)
- values 0 and \( \infty \) for \( s_{i,j} \) are allowed!
- **Exchange matrix**: A \( d \times d \)-dimensional matrix \( s = (s_{i,j})_{i,j} \) taking values in \( [0, \infty]^{d \times d} \) such that
  1. \( s_{i,i} = 1 \)
  2. \( s_{i,j} s_{j,k} = s_{i,k} \), whenever the product is defined.
- Note: there exists always a *strongest* currency \( i^* \) with \( \sum_j s_{i^*,j}(t) \leq d \).
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Underlying objects

- Filtered space $(\Omega, \mathcal{F}, \mathbb{F})$: representing possible scenarios and a flow of information.
- $S_{i,j}(t) \in [0, \infty]$ denotes the price of the $j$:th currency in terms of the $i$:th currency.
- $S(\cdot) = (S_{i,j}(\cdot))_{i,j}$ is an $\mathbb{F}$–progressive, càdlàg process such that $S(t)$ is a $d \times d$ exchange matrix:
  \[ S_{i,j}(t)S_{j,k}(t) = S_{i,k}(t) \quad \text{(whenever defined)} \]
- Define: $\mathcal{A}(t) = \{ i : \sum_j S_{i,j}(t) < \infty \} \neq \emptyset$. 


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Value vector

- A value vector $\nu = (\nu_i)_i$ (with respect to $S(t)$) encodes the price of an asset in terms of the $d$ currencies.
- The $i$:th component describes the price of an asset in terms of the $i$:th currency.
- $\nu$ satisfies consistency condition:
  $$S_{i,j}(t)\nu_j = \nu_i \quad (\text{whenever defined})$$

$$\mathcal{U}^t = \left\{ C : \mathcal{F}(t)-\text{measurable value vector s.t.} \right\}$$
  $$\exists K > 0 \text{ with } C_i \geq -K \sum_j S_{i,j} \text{ for all } i$$

$$\mathcal{D}^t = \mathcal{U}^t \cap (-\mathcal{U}^t).$$
Value vector

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Valuation operator

- A *valuation operator* relates future random prices to present deterministic prices.

- Concept goes back to Harrison & Pliska (1981); see also Biagini & Cont (2006) and literature on risk measures.

We say that a family of operators $\mathbb{V} = (\mathbb{V}^{r,t})_{0 \leq r \leq t \leq T}$, with

$$\mathbb{V}^{r,t} : \mathcal{D}^{t} \rightarrow \mathcal{D}^{r},$$

is a valuation operator with respect to $S$ if it satisfies:

1. Positivity
2. Linearity
3. Continuity from below
4. Time consistency
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Valuation operator — the conditions

1. (Positivity) If $C \in D^{T}$ and $C \geq 0$ then $V^{0,T}(C) \geq 0$.
2. (Linearity) If $H \in L^{\infty,r}$, and $C, C' \in D^{t}$ then
   \[ V^{r,t}(H1_{\{H \neq 0\}}C + C') = H1_{\{H \neq 0\}}V^{r,t}(C) + V^{r,t}(C'). \]
3. (Continuity from below) If $(C_{n})_{n \in \mathbb{N}} \subset D^{T}$ is a nondecreasing sequence of nonnegative value vectors converging to $C \in D^{T}$, then $V^{0,t}(C_{n})$ converges to $V^{0,t}(C)$.
4. (Time consistency) For $C \in D^{T}$,
   \[ V^{r,t}(V^{t,T}(C)) = V^{r,T}(C). \]
5. (Martingale property) $V^{t,T}(S_{.,i}(T)) = S_{.,i}(t)1_{\{i \in \mathcal{A}(t)\}}$.
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Introduction of a probability measure

- Let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$.
- We say $\mathbb{P}$ satisfies (PSmg) if there exists $(A_i)_i$ with $\bigcup_i A_i = \Omega$ such that for each $i$, $\mathbb{P}(A_i) > 0$ and $S_i$ is a $\mathbb{P}_i$–semimartingale, where $\mathbb{P}_i(\cdot) = \mathbb{P}(\cdot | A_i)$ for each $i$. 
Trading strategies and wealth processes

- Let $\mathbb{P}$ satisfy (PSmg).
- Let $h$ denote a predictable process. Then $V^h$ is a value vector process with $V^h_i(t) = \sum_j h_j S_{i,j}(t)$.
- $h$ is called a $\mathbb{P}$–trading strategy if $h \in L(S_i, \mathbb{P}_i)$ and the self-financing condition holds:
  $$V^h_i - V^h_i(0) = h \cdot \mathbb{P}_i S_i.$$
- $h$ is $\mathbb{P}$–allowable if there exists $\varepsilon > 0$ such that $V_i(t) \geq -\varepsilon \sum_j S_{i,j}(t)$. 
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No-arbitrage condition

Assume that \( \mathbb{P} \) satisfies (PSmg). We say that \( S \) satisfies *NFLVR for \( \mathbb{P} \)-allowable strategies* if for any sequence of \( \mathbb{P} \)-allowable strategies \( (h^n) \) with \( V^{h^n}(0) \leq 0 \) and such that there exist \( (\xi^n) \in L^\infty(\mathbb{R}, \mathbb{P}) \) satisfying

\[
V_i^{h^n}(T) \geq \xi^n \sum_j S_i.j(T),
\]

the following conclusion holds:

\[
\xi = \lim_{n \to \infty} \xi^n \text{ exists and } \mathbb{P}(\xi \geq 0) = 1 \implies \mathbb{P}(\xi = 0) = 1.
\]

Here, the limit is taken in \( L^\infty(\mathbb{R}, \mathbb{P}) \).
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Assume that $\mathbb{P}$ satisfies (PSmg). We say that $S$ satisfies NFLVR for $\mathbb{P}$–allowable strategies if for any sequence of $\mathbb{P}$–allowable strategies $(h^n)$ with $V^{h^n}(0) \leq 0$ and such that there exist $(\xi^n) \in L^\infty(\mathbb{R}, \mathbb{P})$ satisfying

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Here, the limit is taken in $L^\infty(\mathbb{R}, \mathbb{P})$. 
First fundamental theorem

Write $\mathbb{P} \sim \mathbb{V}$ if for a nonnegative $C = (C_i); \in \mathcal{D}^T$, we have $\mathbb{V}^{0,T}(C) = 0$ if and only if $\sum_i 1\{c_i=0\} > 0 \mathbb{P}$–almost surely.

1. If $\mathbb{P}$ satisfies (PSmg) and $S$ satisfies NFLVR for $\mathbb{P}$–allowable strategies then there exists a valuation operator $\mathbb{V} \sim \mathbb{P}$.

2. If there exists a valuation operator $\mathbb{V}$ then there exists a probability measure $\mathbb{P} \sim \mathbb{V}$ that satisfies (PSmg) and such that $S$ satisfies NFLVR for $\mathbb{P}$–allowable strategies.
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Second fundamental theorem

Suppose that there exists a valuation operator $\tilde{V}$ with respect to $S$. Then, the market is complete if and only if $\tilde{V}$ is the unique valuation operator equivalent to $\tilde{V}$.

Moreover, if a valuation operator exists, then

$$\inf\{\tilde{V}^h(0) : h \text{ super-replicates } C\}$$

$$= \sup\{\tilde{V}^0, T(C) : \tilde{V} \sim \tilde{V} \text{ is a valuation operator}\},$$

Furthermore, the infimum is obtained if the above expression is finite.
Second fundamental theorem

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Furthermore, the infimum is obtained if the above expression is finite.
Disaggregation and aggregation

A family \((\mathbb{Q}_i)_i\) of probability measures such that \(S_i\) a \(\mathbb{Q}_i\)–supermartingale is called consistent if the following change-of-numéraire formula holds

\[
S_{j,i}(r)\mathbb{E}_{r,i}^\mathbb{Q}_i [S_{i,j}(t)X] = \mathbb{E}_{r}^{\mathbb{Q}_j} [X1_{\{S_{j,i}(t) > 0\}}],
\]

where \(X\) is a bounded, nonnegative random variable.

Given a valuation operator \(\mathbb{V}\) there exist a consistent family of supermartingale measures \((\mathbb{Q}_i)_i\) such that

\[
\mathbb{V}^{r,t}_{j,i}(C) = \sum_i S_{j,i}(r)\mathbb{E}_{r}^\mathbb{Q}_i \left[ \frac{C_i}{\mathbb{A}(t)} \right]
\]

(1)

for all \(r \leq t, j \in \mathbb{A}(r), C \in \mathbb{D}^t\).

Conversely, given a consistent family of supermartingale measures \((\mathbb{Q}_i)_i\), (1) defines a valuation operator \(\mathbb{V} \sim \sum_i \mathbb{Q}_i\).
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Given a valuation operator \(\mathbb{V}\) there exist a consistent family of supermartingale measures \((\mathbb{Q}_i)_i\) such that

\[ \mathbb{V}_{r,t}^j (C) = \sum_i S_{j,i}(r) \mathbb{E}_{\mathbb{Q}_i}^r \left[ \frac{C_i}{\mathcal{A}(t)} \right] \quad (1) \]

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Conversely, given a consistent family of supermartingale measures \((\mathbb{Q}_i)\), (1) defines a valuation operator \(\mathbb{V} \sim \sum_i \mathbb{Q}_i\).
The appearance of strict local martingales

Consistent family \((\mathbb{Q}_i)_{i};\)

\[
S_{j,i}(r)\mathbb{E}^{\mathbb{Q}_i}_{r}[S_{i,j}(t)X] = \mathbb{E}^{\mathbb{Q}_j}_{r}[X1\{S_{j,i}(t)>0\}],
\]

- \(S_{i,j}\) is a \(\mathbb{Q}_i\)–martingale if and only if \(\mathbb{Q}_j(S_{j,i}(T) = 0) = 0\).
- \(S_{i,j}\) is a \(\mathbb{Q}_i\)–local martingale if and only if \(S_{j,i}(T)\) does not jump to zero under \(\mathbb{Q}_j\).
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The case of two assets

d = 2, with \( C = (C_1, C_2)^T \)
E.g., \( C = ((S_{1,2}(T) - K)^+, (1 - KS_{2,1}(T))^+)^T \)

\[
\nabla_j^{r,t}(C) = S_{j,1}(r)E_{\eta_1}^{Q_1}\left[\frac{C_1}{\mathcal{A}(t)}\right] + S_{j,2}(r)E_{\eta_2}^{Q_2}\left[\frac{C_2}{\mathcal{A}(t)}\right] \\
= S_{j,1}(r)E_{\eta_1}^{Q_1}[C_1] + S_{j,2}(r)E_{\eta_2}^{Q_2}[C_2 \mathbf{1}_{\{s_{1,2}(t)=\infty\}}]
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E.g., \( C = ((S_{1,2}(T) - K)^+, (1 - KS_{2,1}(T))^+)^T \)

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\mathbb{V}^{r,t}_j(C) = S_{j,1}(r)\mathbb{E}_r^{Q_1} \left[ \frac{C_1}{\mathcal{L}(t)} \right] + S_{j,2}(r)\mathbb{E}_r^{Q_2} \left[ \frac{C_2}{\mathcal{L}(t)} \right] \\
= S_{j,1}(r)\mathbb{E}_r^{Q_1}[C_1] + S_{j,2}(r)\mathbb{E}_r^{Q_2}[C_21\{s_{1,2}(t)=\infty\}] 
\]
The concept of “no obvious devaluations”

We say that a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F}(T))$ satisfies “No Obvious Devaluations” (NOD) if

$$\mathbb{P}(i \in \mathcal{A}(T) | \mathcal{F}(\tau)) > 0 \text{ on } \{\tau < \infty\} \cap \{i \in \mathcal{A}(\tau)\}$$

for all $i$ and stopping times $\tau$. 
Aggregation without consistency

Let \((Q_i)_i\) be so that \(S_i\) is a \(Q_i\)-local martingale. Then there exists a martingale valuation operator \(\mathbb{V} \sim (\sum_i Q_i)\) if one of the following two conditions is satisfied:

1. \(S_i\) is a \(Q_i\)-martingale.

2. The following three conditions hold:
   2.1 \(\sum_i Q_i\) satisfies (NOD).
   2.2 \(Q_k|_{\mathcal{F} \cap \{\sum_j S_{k,j}(T) < \infty\}} \sim \left(\sum_i Q_i\right)|_{\mathcal{F} \cap \{\sum_j S_{k,j}(T) < \infty\}}\).
   2.3 There exist \(\epsilon > 0\), \(N \in \mathbb{N}\) and predictable times \(T_1 \leq T_2 \leq \cdots \leq T_N\) such that

\[
\left\{(t, \omega) : \sum_j S_{k,j} \text{ jumps to } \infty \right\} \cap \left\{(t, \omega) : \sum_j S_{k,j} \leq d + \epsilon \right\} \subset \bigcup_{n=1}^{N} [T_n].
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Let \((\mathbb{Q}_i)_i\) be so that \(S_i\) is a \(\mathbb{Q}_i\)-local martingale. Then there exists a martingale valuation operator \(V \sim (\sum_i \mathbb{Q}_i)\) if one of the following two conditions is satisfied:

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An example for lack of aggregation

- $d = 2$; probability measure $\mathbb{P}$
- $R$: a three-dimensional Bessel process:

$$R(t) = 1 + \int_0^t \frac{1}{R(s)}ds + W(t)$$

- Stopping time $\tau$ with $\mathbb{P}(\tau = \infty) > 0$
- $S_{1,2}(t) = 1$ for all $t < \tau$ and $S_{1,2}(t) = 1 + R(t) - R(\tau)$ for all $t \geq \tau$.
- $\mathbb{Q}_1(\cdot) = \mathbb{P}(\cdot|\tau = \infty)$ and $\mathbb{Q}_2 = \mathbb{P}$.
- Then $\mathbb{Q}_1(S_{1,2} \equiv 1) = 1$, $S_{2,1}$ is a $\mathbb{Q}_2$-local martingale, and $(\mathbb{Q}_1 + \mathbb{Q}_2) \sim \mathbb{P}$. 
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An example for lack of aggregation (cont’d)

• Obviously, no complete evaluations occur, thus (2.1) and (2.3) hold.

• We do not have

\[ Q_k \bigg| \mathcal{F} \cap \{ \sum_j S_{k,j}(T) < \infty \} \sim \left( \sum_i Q_i \right) \bigg| \mathcal{F} \cap \{ \sum_j S_{k,j}(T) < \infty \} \]

• Indeed, no martingale valuation operator \( \mathbb{V} \sim (Q_1 + Q_2) \) exists.
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Conclusion

- We consider an exchange economy with $d$ currencies, where each currency has the possibility to complete devaluate against any other currency.
- In such an economy, we introduce the concept of a valuation operator and link it to a no-arbitrage condition.
- We interpret the lack of martingale property of an asset price as a reflection of the possibility that the numéraire currency may devalue completely.
- We study conditions under which not necessarily equivalent measures, corresponding to different numéraires, may be aggregated to obtain a numéraire-independent valuation operator.
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Many thanks for your attention!
Digression: Hyperinflation

- **Hyperinflation**: complete devaluation of the corresponding domestic numéraire and an explosion of the exchange rate with respect to any other currency.

- **Examples:**
  - The price of one Dollar, measured in units of the respective domestic currency, went up by a factor of over 4500 in Austria from January 1919 to August 1922 and by a factor of over $10^{10}$ from January 1922 to December 1923 in Germany.
  - Hungary, August 1945 to July 1946. Prices soared by a factor of over $10^{27}$ in that 12-month period to which the month of July contributed a staggering raise of $4 \times 10^{16}$ percent of prices.
  - Bolivia, August 1984 to August 1985: Price levels increased by 20,000 percent.
  - Zimbabwe, July 2009: for instance, prices increased by an annualized inflation rate of over $2 \times 10^8$ percent.
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