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RICHARDSON-GAUDIN ALGEBRAS AND THE EXACT SOLUTIONS OF THE PROTON-NEUTRON PAIRING

VESSELIN G. GUEORGUIEV and JORGE DUKELSKY[†]

Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences 1784 Sofia, Bulgaria

[†]*Instituto de Estructura de la Materia, CSIC, Serrano 123, 28006 Madrid, Spain*

Abstract. Many exactly solvable models are based on Lie algebras. The pairing interaction is important in nuclear physics and its exact solution for identical particles in non-degenerate single-particle levels was first given by Richardson in 1963. His solution and its generalization to Richardson-Gaudin quasi-exactly solvable models have attracted the attention of many contemporary researchers and resulted in the exact solution of the isovector pn-pairing within the $\mathrm{SO}(5)$ RG-model and the equal strength spin-isospin pn-pairing within the $\mathrm{SO}(8)$ RG-model. Basic properties of the RG-models are summarized and possible applications to nuclear physics are emphasized.

1. Introduction

Symmetry is one of the most important paradigms in modern physics. Any Lie group and its Lie algebra have a naturally defined action on a product of spaces (representations). Thus, they are very suitable for a multi-particle system with an underling symmetry. Usually, this means that the relevant operators are well defined for one particle as well as for any number of particles. This allows one to find exact solutions to a problem, with a given underlining symmetry, by referring to the relevant representation theory of the symmetry group at place. As a result many exactly solvable models are build using Lie algebra representation theory. A well known examples are the theories with $\mathrm{SO}(3)$ and $\mathrm{SU}(2)$ rotational symmetry, the Elliott's $\mathrm{U}(3)$ symmetry model [4–7], the Wigner's $\mathrm{SU}(4)$ spin-isospin symmetry [17], and many more that play a major role in nuclear physics. For example, the $\mathrm{SO}(8)$ and $\mathrm{Sp}(6)$ Ginnocchio models, the Fermion Dynamical Symmetry Models (FDSM), and the three dynamical symmetries of the Interacting Boson Model (IBM).

A nuclear many-body system near equilibrium can be viewed as subject to a mean field **Harmonic Oscillator** (HO) potential

$$H_0 = \frac{\vec{p}^2}{2m} + \frac{1}{2}k^2\vec{x}^2.$$

Since the symmetry group of the three-dimensional HO is $\mathrm{U}(3)$ one can easily see its relevance in the description of nuclei [4]. It is well know that one can understand the magic numbers and the shell structure of nuclei within the three-dimensional HO approximation [11]. Using the HO single-particle states one can write a general Hamiltonian with one- and two-body terms

$$H = \sum_i \varepsilon_i a_i^+ a_i + \frac{1}{4}\sum_{i,j,k,l} V_{ij,kl} a_i^+ a_j^+ a_k a_l.$$

Here, a_i and a_j^+ are fermion **annihilation** and **creation operators**, ε_i single-particle energies, and $V_{ij,kl} = \langle ij|V|kl\rangle$ two-body interaction matrix elements.

The simplest extension of the HO Hamiltonian is to add quadruple interaction terms $Q \cdot Q$ and/or spin-orbit interaction $L.S$. This has been well studied by Elliott and his collaborators [4–7].

Another important interaction is the pairing interaction in nuclei

$$H_P = \sum_i 2\varepsilon_i n_i - g\sum_{i,j} a_{\uparrow,i}^+ a_{\downarrow,i}^+ a_{\uparrow,j} a_{\downarrow,j}. \qquad (1)$$

Here n_i is the **number operator** for pairs. The exact solution of the pairing interaction between identical particles in non-degenerate single particle levels was first given by Richardson [14, 15]. His solution and its generalization to Richardson-Gaudin quasi-exactly solvable models (RG-models) have attracted the attention of various contemporary researchers - resulting in the exact solution of the isovector proton-neutron pairing in nuclei within the $\mathrm{SO}(5)$ RG-model [3] - to be discussed in Section 4.1 and the equal strength spin-isospin proton-neutron pairing within the $\mathrm{SO}(8)$ RG-model [12] to be summarized in in Section 4.2. Basic properties of the integrable RG-models are summarized in Section 3 and their possible applications to variety of nuclear physics models are emphasized in Section 5. In the next section we briefly discuss few dynamical symmetry models of importance to nuclear systems.

2. Some Exactly Solvable Nuclear Models

A quantum system has a dynamical symmetry if the Hamiltonian can be expressed as a function of the Casimir operators of a subgroup chain. A typical example of a rank two dynamical symmetry is the Elliott's $\mathrm{SU}(3)$ model that is used in the description of deformed nuclei [4–7]

$$H_{\text{def}} = \varepsilon N + \chi Q \cdot Q. \tag{2}$$

This Hamiltonian can be rewritten as a linear combinations of the Casimir operator of the $\mathfrak{su}(3)$ Lie algebra $C_2^{\mathfrak{su}(3)} = (Q \cdot Q + 3L^2)/4$ involving the quadrupole-quadrupole interaction $Q \cdot Q$ and the Casimir operator L^2 of the $\mathrm{SO}(3)$ subgroup of the angular momentum. Thus we have a group chain reduction $\mathrm{SO}(3) \subset \mathrm{SU}(3)$ that provides exact solution to our initial Hamiltonian (2)

$$H_{\mathfrak{so}(3)\subset\mathfrak{su}(3)} = \varepsilon N + \frac{1}{2\mathfrak{J}} L^2 + a C_2^{\mathfrak{su}(3)}.$$

Here N counts the number of particles with energy ε of a particular HO shell and L^2 lifts the l-degeneracy of a particular harmonic oscillator shell.

A common feature of the dynamical symmetry nuclear models, is that they are all defined for a degenerate single particle levels. Since single particle energy splitting breaks the dynamical symmetry, it is usually expected that this will prevent the model to be exactly solvable. For example, spin-orbit interaction $l \cdot s$ lifts the total angular momentum degeneracy $j = l + 1/2$ and $j = (l + 1) - 1/2$ and destroys the $\mathrm{SU}(3)$ symmetry [10]. Although, the single particle energy splitting breaks the dynamical symmetry it may still preserve the exact solvability. The pairing model with non-degenerate single particle levels, whose exact solution has been found by Richardson, represents a unique example of an exactly solvable model with these characteristics [14, 15]. The model is exactly-solvable due to the special extension of the relevant dynamical symmetry algebra to a spectral Gaudin algebra.

3. Spectral Lie Algebras

A Gaudin algebra $\mathcal{G}$ ( ) is an infinite dimensional extension of the Lie algebra that associates to any generator $X^{\alpha} \in$ a parameter dependent generator $X^{\alpha}(\lambda) \in \mathcal{G}$ ( ) satisfying the following commutation relations [16]

$$\left[X^{\alpha}(\lambda), X^{\beta}(\mu)\right] = \sum_{\gamma} \Gamma_{\gamma}^{\alpha\beta} \frac{X^{\gamma}(\lambda) - X^{\gamma}(\mu)}{\lambda - \mu}. \tag{3}$$

Here $\Gamma_{\gamma}^{\alpha\beta}$ are the structure constants of the Lie algebra , λ and μ are complex spectral parameters. One can form Hermitian operators by using the dot product defined via the -invariant metric tensor $g^{\alpha\beta} \sim \mathrm{Tr}(\mathrm{ad}(X^{\alpha})\mathrm{ad}(X^{\beta}))$

$$K(\lambda) = X(\lambda) \cdot X(\lambda). \tag{4}$$

$K(\lambda)$ are not Casimir operators because they do not commute with all generators of $\mathcal{G}$ ( ). However, these operators commute among themselves, i.e.,

$$[K(\lambda), K(\mu)] = 0. \tag{5}$$

This implies that the system is integrable and $K(\lambda)$ are the integrals of motion.
There is a unique rational realization[1] of the generators $X^\alpha(\lambda)$ in terms of L copies of the generators of the Lie algebra given by the following expression

$$X^\alpha(\lambda) = \sum_{i=1}^{L} \frac{1}{z_i - \lambda} X_i^\alpha + \rho^\alpha. \tag{6}$$

The z_i are L arbitrary numbers, which will ultimately be related to the single particle energies. Here we deviate from [16] by introducing the set of arbitrary parameters ρ^α. However, most of the expressions related to the Gaudin algebra (3) derived for the case $\rho^\alpha = 0$ are still valid because $X^\alpha(\lambda) \to X^\alpha(\lambda) + \rho^\alpha$ is an algebra isomorphism. The shift of the elements of $\mathcal{G}$ by the ρ parameters is a key to the symmetry breaking in the model.

3.1. Richardson-Gaudin Operators

For the realization (6) of the Gaudin algebra, the integrals of motion are

$$K(\lambda) = \rho \cdot \rho + \sum_i \frac{C_2^{(i)}}{(z_i - \lambda)^2} + 2 \sum_i \frac{R_i}{z_i - \lambda}.$$

Here $C_2^{(i)}$ is the **second Casimir operator** of the i-th copy of . Thus the first two terms are constants for states built on a tensor product of irreducible representations of . The R_i are the **Richardson-Gaudin operators** [2, 9] and are one-half of the residue of $K(\lambda)$ at $\lambda = z_i$

$$R_i = \sum_{j(\neq i)} \frac{X_i \cdot X_j}{z_i - z_j} + \xi_i, \qquad \xi_i = \rho \cdot X_i. \tag{7}$$

By taking the residues of (5) at $\lambda = z_i$ and $\mu = z_j$ one can see that the R_i operators commute among themselves. Therefore, they define a new set of integrals of motion. Thus any function of the R_i operators can be used as a model Hamiltonian for this integrable system. In particular, any linear combination of the R_i operators is at most quadratic in the generators.

For a singular semi-simple algebras, the eigenvalues of the R_i operators can be obtained from the eigenvalues $k(\lambda)$ of the $K(\lambda)$ operator given in [16] by taking the appropriate residue (1/2 $\mathrm{Res}(k(\lambda), \lambda = z_i)$)

$$r_i = \Lambda_i \cdot \rho + \sum_{j(\neq i)} \frac{\Lambda_i \cdot \Lambda_j}{z_i - z_j} + \sum_{a=1}^{r} \sum_{\alpha=1}^{M^a} \frac{\Lambda_i \cdot \pi^a}{z_i - e_{a,\alpha}}. \tag{8}$$

[1]There are non-rational Gaudin algebra realizations but we will not consider them here.

Here $e_{a,\alpha}$ are solutions of the **generalized Richardson equations** [1, 13, 16]

$$\sum_{b=1}^{r}\sum_{\beta=1}^{M^b \, '} \frac{\pi^b \cdot \pi^a}{e_{b,\beta} - e_{a,\alpha}} - \sum_{i=1}^{L} \frac{\Lambda_i \cdot \pi^a}{z_i - e_{a,\alpha}} = \xi \cdot \pi^a \tag{9}$$

and L is the number of copies of the algebra and Λ_i is the weight for the i-th copy. The $\Lambda_i \cdot \pi^a$ are actually the eigenvalues of the generators $H^a = \pi^a_s h^s$ at the lowest/highest weigh state of the i-th copy of . The π^a_s are the components of the positive simple roots[2] π^a of the Lie algebra in the Cartan-Weyl basis ($[h_s, E^a] = \pi^a_s E^a$), and $\xi \cdot \pi^a = \pi^a_s \rho^s$, where ρ^s are the components of the symmetry breaking one-body operator $\xi = \rho^s h_s$ along the Cartan generators h_s. The rank of is r and M^a are positive numbers related to the eigenvalues m^a of H^a at the desired eigenstate ($M^a = \sum_i \Lambda^a_i - m^a_i$).

3.2. Symmetry Breaking

Although one can use any function of R_i as a Hamiltonian, there is a particular liner combination of the Gaudin operators R_i that results in a simple expression which is linear in the spectral parameters $e_{a,\alpha}$. From this expression one can see that the breaking of the -symmetry is due to the ξ terms

$$H = \sum_i z_i R_i = \sum_i z_i \xi_i - \frac{1}{2}\left(C_2 - \sum_i C_2^{(i)}\right). \tag{10}$$

In general $z_i \neq z_j$, thus the term $\sum_i z_i \xi_i$ would mix different irreps because it will not commute with the total Casimir operator C_2 that is built from the generators $X = \sum_i X_i$. The final symmetry of this Hamiltonian is determined by the set of generators that commute with ξ. In order to see that the eigenvalues of (10) are linear in the spectral parameters $e_{a,\alpha}$ one has to multiply the generalized Richardson equations (9) by $e_{a,\alpha}$ and sum over all the indexes. After some manipulations one would observe that the last term in (8) appears in the relevant expressions.

As we already discussed in the previous section, common feature of the dynamical symmetry nuclear models is that they are all defined for degenerate single particle levels. Single particle energy splitting breaks the dynamical symmetry but may still preserve the integrability. The pairing model with non-degenerate single particle levels, whose exact solution has been found by Richardson, along with the above discussed Richardson-Gaudin constructions are examples of exactly solvable models with such characteristics [14].

Models based on fermion realization of the generators of the type $a^+_\alpha a^+_\beta$, such as the s $(2n)$ algebras, are naturally suitable for non-degenerate single particle systems.

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[2] E^a is a simple root vector if it cannot be witten as a commutator of any other two positive root vectors.

This is related to the fact that the fermion realizations of the corresponding Cartan generators are related to the fermion number operators. This is an easy observation if one looks at the commutator $[X^+_{\alpha\beta}, X^-_{\alpha\beta}] = n_\alpha + n_\beta - 1$ where the non-Cartan generators are in the form $X^+_{\alpha\beta} = a^+_\alpha a^+_\beta$ and $(a_\alpha)^+ = a^+_\alpha$. Note that n_α and n_β enter on an equal footing. In contrast models that are built on generators of the form $X^+_{\alpha\beta} = a^+_\alpha a_\beta$, such as $\mathfrak{su}(n)$ algebras, result in Cartan generators of the form $n_\alpha - n_\beta$.

4. Pairing in Nuclei

Now we will direct our discussion towards the applications of the generalized Richardson-Gaudin models in nuclear physics. For this reason we shall specify z_i to be related to the single particle energies ε_i by the expression $z_i = 2\varepsilon_i$. ρ^α to be non-zero only for X^α that are elements of the Cartan algebra of , that is $\rho^a \neq 0$ and $a = 1...r$. Then a "pairing" like Hamiltonian H_P is obtained from (10) by considering $H_P = gH$ and its form is

$$H_P = \sum_i 2\varepsilon_i \delta \cdot h_i - g \sum_{\beta \in \Omega_+} \sum_{i \neq j} Y_i^{-\beta} Y_{\beta,j}$$

$$E_P = \sum_{a=1}^{r} \sum_{\alpha=1}^{M^a} e_{a,\alpha} \delta_a, \quad g = \prod_{\rho^a \neq 0} \frac{1}{\rho^a}, \qquad \delta_a = g\rho_a.$$

Here the E_P are the eigenvalues of H_P and Y_β are the positive root vectors with respect to the chosen Cartan algebra $\{h^a\}$ of . Now g plays the row of a coupling constant for the two-body interaction term $Y^{-\beta} Y_\beta$.

This is particularly clear in the case of the pairing model where $=\mathfrak{su}(2)$ with generators $Y_+ = a^+_\uparrow a^+_\downarrow$, $Y_- = a_\uparrow a_\downarrow$ and $h = a^+_\uparrow a_\uparrow + a^+_\downarrow a_\downarrow$. Since there is only one Cartan generator h, there is only one $\rho = \frac{1}{g}$, and $\delta = 1$, and E_P results in the usual expression of a sum over the pair energies e_i, $E_P = \sum_{i=1}^{M} e_i$ for the standard pairing Hamiltonian (1).

4.1. T=1 Proton-Neutron Pairing as $\mathrm{SO}(5)$ RG-model

To better illustrate the current framework we will briefly discuss the $T = 1$ pairing in proton-neutron systems which is related to an $\mathfrak{s}$ (5) Lie algebra [13]. In this case the one-level system is constructed from various proton-proton, neutron-neutron, and proton-neutron pairs. By choosing the Cartan generators to be the total particle number operator $h_2 = 1 - (\hat{N}_p + \hat{N}_n)/2$ and the third projection of the isospin $h_1 = T_0 = (\hat{N}_p - \hat{N}_n)/2$. We find the positive root vectors[3] of the algebra to be

[3]Positiveness of a root vector is determined by the positiveness of the corresponding eigenvalues: first with respect to h_r, if zero then one has to look at h_{r-1} and so on.

the hard boson annihilation operators $\{b(\mu) : \mu = 1,2,3\} = \{n_\uparrow^- n_\downarrow^-, (p_\uparrow^- n_\downarrow^- + n_\uparrow^- p_\downarrow^-)/\sqrt{2}, p_\uparrow^- p_\downarrow^-\}$ plus the isospin rising operator $T_+ = (p_\downarrow^+ n_\downarrow^- + p_\uparrow^+ n_\uparrow^-)/\sqrt{2}$. The simple root vectors are $\{b(3), T_+\}$ and $b(3) = p_\uparrow^- p_\downarrow^-$ is the singular root vector [16]. In the chosen basis the corresponding commutation relations are

$$\begin{array}{lll} [h_2, h_1] = 0, & [b(\mu), b(\nu)] = 0, & \mu, \nu = 1,2,3 \\ [h_2, b(\mu)] = b(\mu), & [h_2, T_+] = 0, & [h_1, T_+] = T_+ \\ [h_1, b(1)] = b(1), & [h_1, b(2)] = 0, & [h_1, b(3)] = -b(3) \\ [b(3), T_+] = b(2), & [T_+, b(2)] = b(1), & [T_+, b(1)] = 0. \end{array}$$

By including the conjugated operators one closes the algebra $\mathfrak{so}(5)$. The isospin sub-algebra $\mathfrak{su}_T(2) \subset \mathfrak{so}(5)$ is generated by the $\mathfrak{so}(5)$ generators that are not related to the singular root vector $b(3) = p_\uparrow^- p_\downarrow^-$. That is, $\mathfrak{su}_T(2)$ is generated by T_+, $T_- = (T_+)^+$, and $[T_+, T_-] = T_0 = h_1$. Even more, all the given $\mathfrak{so}(5)$ generators can be recognized as $\mathfrak{su}_T(2)$ tensor operators.

Since $\mathfrak{so}(5)$ is a rank two algebra, we have two types of spectral parameters: $e_{1,\alpha}$ and $e_{2,\beta}$ which we will denote by w_α and v_β in the following discussion. The upper bounds M^1 and M^2, for the indices α and β, are related to the isospin T and the total number of pairs M via the expressions $M^1 = M - T$ and $M^2 = M$. The scalar products of the simple roots are $(\pi_2, \pi_2) = 2, (\pi_1, \pi_1) = 1, (\pi_2, \pi_1) = -1$. If we consider now the spherical shell model, where protons and neutrons can occupy single particle states with quantum numbers (j, m_j) then the sub-index $\uparrow$ corresponds to $m_j > 0$ and $\downarrow$ to $m_j < 0$ and the single particle index i labels the states $(j, |m_j|)$. Due to the rotational symmetry, we can use the angular momentum j instead of i but have to take into account the corresponding degeneracy $\Omega_j = (2j+1)/2$. Finally the weights Λ_i^a are the same for any i and correspond to the fundamental representation of $\mathfrak{so}(5)$, that is, $\Lambda^1 = 0$, and $\Lambda^2 = 1$. Putting all this together with the choice $\rho^1 = 0, \rho^2 = -1/g$, and $z_i = 2\varepsilon_i$ in the generalized Richardson equating (9) we obtain the equations for the proton-neutron $T = 1$ pairing that were given also by Links et al [13] and Asorey *et al* [1]

$$\begin{gathered} \frac{1}{g} = \sum_{i=1}^{L} \frac{\Omega_i}{2\varepsilon_i - v_\alpha} + \sum_{\beta \neq \alpha}^{M} \frac{2}{v_\alpha - v_\beta} + \sum_{\gamma=1}^{M-T} \frac{1}{w_\gamma - v_\alpha} \\ 0 = \sum_{\alpha=1}^{M} \frac{1}{v_\alpha - w_\gamma} + \sum_{\delta \neq \gamma}^{M-T} \frac{1}{w_\gamma - w_\delta}, \quad E = \sum_{\alpha=1}^{M} v_\alpha. \end{gathered} \tag{11}$$

The spectral parameters v_α have the same meaning as pair energies. If one allows for isospin breaking then one has to set $\rho^1 = \Delta \neq 0$ [3].

4.2. Spin-Isospin pn-pairing as $\mathrm{SO}(8)$ RG-model

In the previous section on our discussion of the T=1 proton-neutron pairing as $\mathrm{SO}(5)$ RG-model, we considered the intrinsic symmetry space to be the isospin $\mathfrak{su}_T(2)$ symmetry while the extrinsic spaces labeled by i were related to the total spin states (j, m_j). In particular, the pairs operators $\{b(\mu), b^+(\mu)\}$ and the $\mathfrak{su}_T(2)$ algebra generators $\{T_0, T_\pm\}$ were time-reversal invariant. For the study of the spin-isospin pn-pairing it is appropriate to consider the Wigner's $\mathfrak{su}_{ST}(4) = \mathfrak{su}_S(2) \times \mathfrak{su}_T(2)$ as intrinsic symmetry of the system [17] and the orbital angular momentum (l, m_l) as the extrinsic space labeled with i in the appropriate sums. This way the protons and neutrons are described by the operators: $a_{l_i,m;s,\sigma;t,\tau}$ where $s = t = \frac{1}{2}$ or briefly $a_{l_i,m,\sigma,\tau}$. The pair operators are defined as isovector and spinvector tensor operators

$$P_{\tau i} = \sqrt{\Omega_{l_i}}\, [a_{l_i} a_{l_i}]^{001}_{00\tau}\,, \qquad D_{\sigma i} = \sqrt{\Omega_{l_i}}\, [a_{l_i} a_{l_i}]^{010}_{0\sigma 0}\,.$$

The $\Omega_{l_i} = (2l_i + 1)/2)$ appears here due to the structure of the Clebsch-Gordan coefficients $<lm, l-m|00>$. The Wigner's $\mathfrak{su}(4)$ along with the $\mathfrak{u}(1)$ number operator $N = N_{n_\uparrow} + N_{n_\downarrow} + N_{p_\uparrow} + N_{p_\downarrow} = 2n$ which is double of the pair number operator n, form $\mathfrak{u}(4)$ algebra with the following 16 generators

$$X_{\tau_1\sigma_1\tau_2\sigma_2 i} = \sum_m a^+_{l_i m,\tau_1\sigma_1} a_{l_i m,\tau_2\sigma_2}$$

that are $\mathfrak{u}(1)$, $\mathfrak{su}_S(2)$, and $\mathfrak{su}_T(2)$ tensors up to a factor $\sqrt{\Omega_{l_i}}$

$$N_i \sim \left[a^+_{l_i} a_{l_i}\right]^{000}_{000}, \quad S_{\sigma i} \sim \left[a^+_{l_i} a_{l_i}\right]^{010}_{0\sigma 0}, \quad T_{\tau i} \sim \left[a^+_{l_i} a_{l_i}\right]^{001}_{00\tau}, \quad Y_{\sigma\tau i} \sim \left[a^+_{l_i} a_{l_i}\right]^{011}_{0\sigma\tau}.$$

The relevant hamiltonian has equal spin and isospin pairing strength

$$H_P = \sum_i^L 2\varepsilon_i n_i - g \sum_{ij,\mu} (P^\dagger_{\mu i} P_{\mu j} + D^\dagger_{\mu i} D_{\mu j})$$

with eigenvalues: $E = \sum_\alpha^{M_1} e_\alpha$ where each M_1 pair contributes pair energies e_α determined by the four equations [12]

$$\frac{1}{g}=\sum_{i}^{L}\frac{\Omega_i}{2\epsilon_i-e_\alpha}-\sum_{\alpha'(\neq\alpha)}^{M_1}\frac{2}{e_{\alpha'}-e_\alpha}+\sum_{\alpha'}^{M_2}\frac{1}{\omega_{\alpha'}-e_\alpha}$$

$$0=-\sum_{\alpha'}^{M_1}\frac{1}{e_{\alpha'}-\omega_\alpha}+\sum_{\alpha'(\neq\alpha)}^{M_2}\frac{2}{\omega_{\alpha'}-\omega_\alpha}-\sum_{\alpha'}^{M_3}\frac{1}{\eta_{\alpha'}-\omega_\alpha}-\sum_{\alpha'}^{M_4}\frac{1}{\gamma_{\alpha'}-\omega_\alpha}$$

$$0=-\sum_{\alpha'}^{M_2}\frac{1}{\omega_{\alpha'}-\eta_\alpha}+\sum_{\alpha'(\neq\alpha)}^{M_3}\frac{2}{\eta_{\alpha'}-\eta_\alpha}$$

$$0=-\sum_{\alpha'}^{M_2}\frac{1}{\omega_{\alpha'}-\gamma_\alpha}+\sum_{\alpha('\neq\alpha)}^{M_4}\frac{2}{\gamma_{\alpha'}-\gamma_\alpha}.$$

Since this is $\mathrm{SO}(8)$ RG model of rank is 4 there are 4 sets of spectral parameters. The number of spectral parameters in each set is determined by the relevant $\mathfrak{u}(4)$ Wigner multiplet. For a given number of pairs M, these $\mathfrak{u}(4)$ multiplets can be classified using Young tableaux. Each multiplet is defined by a partition of M in 4 numbers, $[\lambda_1\lambda_2\lambda_3\lambda_4]$, constrained by: $\sum_i \Omega_i \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq 0$. The labels λ_i are related to the number of pairs in the lowest/highest weight state. Thus $M_1 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$ is the number of pairs M, $M_2 = \lambda_2 + \lambda_3 + \lambda_4$, $M_3 = \lambda_3 + \lambda_4$, and $M_4 = \lambda_4$.

5. Conclusions and Discussions

Using the Cartan classification of the semi-simple Lie algebras one can see that many physics models can be generalized within the above framework. Table 1 presents the semi-simple Lie algebras up to rank four and the corresponding main fermion models. R&R denotes the model developed by G. Rosensteel and D. Rowe and TCF stands for Trapped Cold Fermions [8].

Table 1. Lie algebra structures associated with important nuclear physics models. Pairing models are listed in boldface. The symbol $\sim$ indicates the isomorphisms of the corresponding Lie algebras.

rank n	A_n $\mathfrak{su}(n+1)$	B_n $\mathfrak{so}(2n+1)$	C_n $\mathfrak{sp}(2n)$	D_n $\mathfrak{so}(2n)$
1	$\boldsymbol{\mathfrak{su}(2)}$ **pairing**	$\mathfrak{so}(3)\sim\mathfrak{su}(2)$	$\mathfrak{sp}(2)\sim\mathfrak{su}(2)$	$\mathfrak{so}(2)\sim\mathfrak{u}(1)$
2	$\mathfrak{su}(3)$ Elliott	$\boldsymbol{\mathfrak{so}(5)}$ **T=1 pairing**	$\mathfrak{sp}(4)\sim\mathrm{SO}(5)$	$\mathfrak{so}(4)\sim\mathfrak{su}(2)\oplus\mathfrak{su}(2)$
3	$\mathfrak{su}(4)$ Wigner	$\mathfrak{so}(7)\subset\mathfrak{so}(8)$ FDSM	$\mathfrak{sp}(6)$ R&R	$\boldsymbol{\mathfrak{so}(6)\sim\mathfrak{su}(4)}$ **TCF**
4	$\mathfrak{su}(5)$	$\mathfrak{so}(9)$	$\mathfrak{sp}(8)$	$\boldsymbol{\mathfrak{so}(8)}$ **Evans, FDSM**

The RG-models are a new important mathematical tool to study the behavior of physical systems. They are finding applications to the such fields of studies as super-conducting grains, atomic nuclei, and trapped fermion atoms.

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