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# WEAK FORM OF HOLZAPFEL'S CONJECTURE*

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**Abstract.** Let $\mathbb{B} \subset \mathbb{C}^2$ be the unit ball and $\Gamma$ be a lattice of $\mathrm{SU}(2,1)$. Bearing in mind that all compact Riemann surfaces are discrete quotients of the unit disc $\Delta \subset \mathbb{C}$, Holzapfel conjectures that the discrete ball quotients $\mathbb{B}/\Gamma$ and their compactifications are widely spread among the smooth projective surfaces. There are known ball quotients $\mathbb{B}/\Gamma$ of general type, as well as rational, abelian, K3 and elliptic ones. The present note constructs three non-compact ball quotients, which are birational, respectively, to a hyperelliptic, Enriques or a ruled surface with an elliptic base. As a result, we establish that the ball quotient surfaces have representatives in any of the eight Enriques classification classes of smooth projective surfaces.

## 1. Introduction

In his monograph [4] Rolf-Peter Holzapfel states as a working hypothesis or a philosophy that "... up to birational equivalence and compactifications, all complex algebraic surfaces are ball quotients." By a complex algebraic surface is meant a smooth projective surface over $\mathbb{C}$. These have smooth minimal models, which are classified by Enriques in eight types - rational, ruled of genus $\geq 1$, abelian, hyperelliptic, K3, Enriques, elliptic and of general type. The compact torsion free ball quotients $\mathbb{B}/\Gamma$ are smooth minimal surfaces of general type. Ishida [10], Keum [11, 12] and Dzambic [1] obtain elliptic surfaces, which are minimal resolutions of the isolated cyclic quotient singularities of compact ball quotients. Hirzebruch [2] and then Holzapfel [3], [7], [9] have constructed torsion free ball quotient compactifications with abelian minimal models. In [9] Holzapfel provides a ball quotient compactification, which is birational to the Kummer surface of an abelian surface,

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i.e., to a smooth minimal K3 surface. Rational ball quotient surfaces are explicitly recognized and studied in [6], [8]. The present work constructs smooth ball quotients with a hyperelliptic or, respectively, a ruled model with an elliptic base. It provides also a ball quotient with one double point, which is birational to an Enriques surface. All of them are finite Galois quotients of a non-compact torsion free $\mathbb{B}/\Gamma_{-1}^{(6,8)}$, constructed by Holzapfel in [9] and having abelian minimal model of the toroidal compactification. As a result, we establish the following

**Theorem 1** (Weak Form of Holzapfel's Conjecture). *Any of the eight Enriques classification classes of complex projective surfaces contains a ball quotient surface.*

## 2. Ball Quotient Compactifications with Abelian Minimal Models

Let us recall that the **complex two-ball**

$$\mathbb{B} = \{(z_1, z_2) \in \mathbb{C}^2 \,;\, |z_1|^2 + |z_2|^2 < 1\} = \mathrm{SU}(2,1)/\mathrm{S}(\mathrm{U}(2) \times \mathrm{U}(1))$$

is an irreducible non-compact Hermitian symmetric space. The discrete biholomorphism groups $\Gamma \subset \mathrm{SU}(2,1)$ of $\mathbb{B}$, whose quotients $\mathbb{B}/\Gamma$ have finite $\mathrm{SU}(2,1)$-invariant measure are called ball lattices. The present section studies the image $T$ of the toroidal compactifying divisor $T' = (\mathbb{B}/\Gamma)' \setminus (\mathbb{B}/\Gamma)$ on the minimal model $A$ of $(\mathbb{B}/\Gamma)'$, whenever $A$ is an abelian surface. It establishes that for any subgroup $H \subseteq \mathrm{Aut}(A, T)$ there is a ball quotient $\mathbb{B}/\Gamma_H$, birational to $A/H$.

**Lemma 1.** *If a ball quotient $\mathbb{B}/\Gamma$ is birational to an abelian surface $A$ then $\mathbb{B}/\Gamma$ is smooth and non-compact.*

**Proof:** Assume that $\mathbb{B}/\Gamma$ is singular. For a compact $\mathbb{B}/\Gamma$ set $U = \mathbb{B}/\Gamma$. If $\mathbb{B}/\Gamma$ is non-compact, let $U = (\mathbb{B}/\Gamma)'$ be the toroidal compactification of $\mathbb{B}/\Gamma$. In either case $U$ is a compact surface with isolated cyclic quotient singularities. Consider the minimal resolution $\varphi : Y \to U$ of $p_i \in U^{\mathrm{sing}}$ by Hirzebruch-Jung strings $E_i = \sum_{t=1}^{\nu_i} E_i^t$. The irreducible components $E_i^t$ of $E_i$ are smooth rational curves of self-intersection $(E_i^t)^2 \leq -2$. The birational morphism $Y \dashrightarrow A$ transforms $E_i^t$ onto rational curves on $A$. It suffices to observe that an abelian surface $A$ does not support rational curves $C$, in order to conclude that $\mathbb{B}/\Gamma$ is smooth. The compact smooth ball quotients are known to be of general type, so that $\mathbb{B}/\Gamma$ is to be non-compact.

Assume that there is a rational curve $C \subset A$. Its desingularization $f : \widetilde{C} \to C$ can be viewed as a holomorphic map $F : \widetilde{C} \to A$. Homotopy lifting property applies to $F$ and provides a holomorphic immersion $\widetilde{F} : \widetilde{C} \to \widetilde{A} = \mathbb{C}^2$ in the universal cover $\widetilde{A}$ of $A$, due to simply connectedness of the smooth rational curve $\widetilde{C}$. Its image

$\widetilde{F}(\widetilde{C})$ is a compact complex-analytic subvariety of $\mathbb{C}^2$, which maps to compact complex-analytic subvarieties $\mathrm{pr}_i(\widetilde{F}(\widetilde{C})) \subset \mathbb{C}$ by the canonical projections $\mathrm{pr}_i\colon \mathbb{C}^2 \to \mathbb{C}$, $1 \leq i \leq 2$. Thus, $\mathrm{pr}_i(\widetilde{F}(\widetilde{C}))$ and, therefore, $\widetilde{F}(\widetilde{C})$ are finite. The contradiction justifies the non-existence of rational curves on $A$.

□

The next lemma lists some immediate properties of the image $T$ of the toroidal compactifying divisor $T'$ of $A' = (\mathbb{B}/\Gamma)'$ on its abelian minimal model $A$.

**Lemma 2.** *Let $A' = (\mathbb{B}/\Gamma)'$ be a smooth toroidal ball quotient compactification, $\xi : A' \to A$ be the blow-down of the $(-1)$-curves $L = \sum_{j=1}^{s} L_j$ on $A'$ to an abelian surface $A$ and $T_i'$, $1 \leq i \leq h$ be the disjoint smooth elliptic irreducible components of the toroidal compactifying divisor $T' = (\mathbb{B}/\Gamma)' \setminus (\mathbb{B}/\Gamma)$. Then*

i) *$T_i = \xi(T_i')$ are smooth irreducible elliptic curves on $A$*

ii) $T^{\mathrm{sing}} = \sum_{1 \leq i < j \leq h} T_i \cap T_j = \xi(L)$

iii) *$T_i \cap T^{\mathrm{sing}} \neq \emptyset$ and the restrictions $\xi : T_i' \to T_i$ are bijective for all $1 \leq i \leq h$.*

**Proof:** i) According to the birational invariance of the genus, the curves $T_i = \xi(T_i')$ have smooth elliptic desingularizations. It suffices to show that any curve $C \subset A$ of genus one is smooth. If $C$ is singular then its desingularization $\widetilde{C}$ is a smooth elliptic curve. Therefore, the composition $\widetilde{C} \to C \hookrightarrow A$ of the desingularization map with the identical inclusion of $C$ is a morphism of abelian varieties. In particular, it is unramified, which is not the case for $\widetilde{C} \to C$. Therefore any curve $C \subset A$ of genus one is smooth.

ii) The inclusion $T^{\mathrm{sing}} \subseteq \sum_{1 \leq i < j \leq h} T_i \cap T_j$ follows from i). For the opposite inclusion, note that $\xi|_{A' \setminus L} = \mathrm{Id}_{(A' \setminus L)} : A' \setminus L \to A \setminus \xi(L)$ guarantees $T_i = \xi(T_i') \neq \xi(T_j') = T_j$ and different elliptic curves on an abelian surface intersect transversally at any of their intersection points. Thus, $T^{\mathrm{sing}} = \sum_{1 \leq i < j \leq h} T_i \cap T_j$. The disjointness of $T_i'$ yields $\sum_{1 \leq i < j \leq h} T_i \cap T_j \subseteq \xi(L)$. Conversely, the Kobayashi hyperbolicity of $\mathbb{B}/\Gamma$ requires $\mathrm{card}(L_j \cap T') \geq 2$ for all $1 \leq j \leq s$. However, $\mathrm{card}(L_j \cap T_i') \leq 1$ by the smoothness of $T_i = \xi(T_i')$, so that there exist at least two $T_i' \neq T_k'$ with $\mathrm{card}(L_j \cap T_i') = \mathrm{card}(L_j \cap T_k') = 1$. In other words, the point $\xi(L_j) \in T_i \cap T_k$. That verifies the inclusion $\xi(L) \subseteq \sum_{1 \leq i < j \leq h} T_i \cap T_j$, whereas the coincidence $\xi(L) = \sum_{1 \leq i < j \leq h} T_i \cap T_j$.

iii) If $T_i \cap \xi(L) = \emptyset$ then the intersection numbers $(T_i')^2 = T_i^2$ coincide. By the Adjunction Formula

$$0 = -e(T_i) = T_i^2 + K_A.T_i = T_i^2 + \mathcal{O}_A.T_i = T_i^2$$

so that $(T_i')^2 = 0$. That contradicts the contractibility of $T_i'$ to the corresponding cusp of $\mathbb{B}/\Gamma$ and justifies $T_i \cap T^{\mathrm{sing}} \neq \emptyset$ for all $1 \leq i \leq h$.

Note that $\xi|_{T_i' \setminus L} = \mathrm{Id}\,|_{T_i' \setminus L} : T_i' \setminus L \to T_i \setminus \xi(L)$ is bijective. In order to define $\xi^{-1} : T_i \cap \xi(L) \to T_i' \cap L$, let us recall that for any $p \in \xi(L)$ the smooth rational curve $\xi^{-1}(p)$ has $\mathrm{card}(\xi^{-1}(p) \cap T_i') \leq 1$. More precisely, $\mathrm{card}(\xi^{-1}(p) \cap T_i') = 1$ if and only if $p \in T_i$, so that for any $p \in T_i \cap \xi(L)$ there is a unique point $\{q(p)\} = T_i' \cap \xi^{-1}(p)$. That provides a regular morphism $\xi^{-1}(p) = q(p)$ for all $p \in T_i \cap \xi(L)$.

□

According to Lemma 2, the image $T = \xi(T')$ of the toroidal compactifying divisor $T' = (\mathbb{B}/\Gamma)' \setminus (\mathbb{B}/\Gamma)$ under the blow-down $\xi : (\mathbb{B}/\Gamma)' \to A$ of the $(-1)$-curves is a multi-elliptic divisor, i.e., $T = \sum_{i=1}^{h} T_i$ has smooth elliptic irreducible components $T_i$, which intersect transversally. Note also that $(A, T)$ determines uniquely $(\mathbb{B}/\Gamma)'$ as the blow-up of $A$ at $T^{\mathrm{sing}}$.

**Definition 2.** *A pair $(A, T)$ of an abelian surface $A$ and a divisor $T \subset A$ is an abelian ball quotient model if there exists a torsion free toroidal ball quotient compactification $(\mathbb{B}/\Gamma)'$, such that the blow-down $\xi : (\mathbb{B}/\Gamma)' \to A$ of the $(-1)$-curves on $(\mathbb{B}/\Gamma)'$ maps the pair $\big((\mathbb{B}/\Gamma)', T' = (\mathbb{B}/\Gamma)' \setminus (\mathbb{B}/\Gamma)\big)$ onto $(A, T)$.*

The next lemma explains the construction of non-compact ball quotients, which are finite Galois quotients of torsion free non-compact $\mathbb{B}/\Gamma$, birational to abelian surfaces.

**Lemma 3.** *Let $A' = (\mathbb{B}/\Gamma)' = (\mathbb{B}/\Gamma) \cup T'$ be a torsion free ball quotient compactification by a toroidal divisor $T'$, $\xi : A' \to A$ be the blow-down of the $(-1)$-curves on $A'$ to the abelian minimal model $A$ and $T = \xi(T')$. Then*

i) $\mathrm{Aut}(A, T) = \mathrm{Aut}(A', T')$ *is a finite group*

ii) *any subgroup $H \subseteq \mathrm{Aut}(A, T)$ lifts to a ball lattice $\Gamma_H$, such that $\Gamma$ is a normal subgroup of $\Gamma_H$ with quotient group $\Gamma_H/\Gamma = H$ and $\mathbb{B}/\Gamma_H$ is a non-compact ball quotient, birational to $X = A/H$.*

*Moreover, if $X = A/H$ is a smooth surface then $\mathbb{B}/\Gamma_H$ is a smooth ball quotient.*

**Proof:** i) If $G = \mathrm{Aut}(A, T)$, then Lemma 2 ii) implies the $G$-invariance of $\xi(L)$. By the means of an arbitrary automorphism of the smooth projective line $\mathbb{P}^1$, one extends the $G$-action to $L$ and, therefore, to

$$A' = (A' \setminus L) \cup L = (A \setminus \xi(L)) \cup L.$$

The $G$-invariance of $T' = \sum_{i=1}^{h} T'_i$ follows from Lemma 2 iii). That justifies the inclusion $G \subseteq \mathrm{Aut}(A', T')$. For the opposite inclusion, note that the union $L$ of the $(-1)$-curves is invariant under an arbitrary automorphism of $A'$. As a result, there arises a $G$-action on $\xi(L)$ and $A = (A \setminus \xi(L)) \cup \xi(L) = (A' \setminus L) \cup \xi(L)$. The multi-elliptic divisor $T = \sum_{i=1}^{h} T_i$ is $G$-invariant according to Lemma 2 iii). Consequently, $\mathrm{Aut}(A', T') \subseteq G$, whereas $G = \mathrm{Aut}(A', T')$.

In order to show that $G$ is finite, let us consider the natural representation

$$\varphi : G \longrightarrow \mathrm{Sym}(T_1, \ldots, T_h) \simeq \mathrm{Sym}_h$$

in the permutation group of the irreducible components $T_i$ of $T$. It suffices to prove that the kernel $\ker\varphi$ is finite, in order to assert that $G$ is finite. For any $g = \tau_p g_o \in \ker \varphi \subset \mathrm{Aut}(A)$ with linear part $g_o \in \mathrm{GL}_2(\mathbb{C})$ and translation part $\tau_p$, $p \in A$, we show that $g_o$ and $\tau_p$ take finitely many values. Note that the identical inclusions $T_i \subset A$ are morphisms of abelian varieties. Thus, for any choice of an origin $\check{o}_A \in T_i$ there is a $\mathbb{C}$-linear embedding $\mathcal{E}_i : \widetilde{T_i} = \mathbb{C} \hookrightarrow \mathbb{C}^2 = \widetilde{A}$ of the corresponding universal covers. If $\mathcal{E}_i(1) = (a_i, b_i)$ then

$$T_i = E_{a_i,b_i} = \{(a_i t, b_i t) (\mathrm{mod}\, \pi_1(A)) \; ; \; t \in \mathbb{C}\} \subset A.$$

If the origin $\check{o}_A \not\in T_i$, then for any point $(P_i, Q_i) \in T_i$ the elliptic curve $T_i = E_{a_i,b_i} + (P_i, Q_i)$. In either case, all $v_i = (a_i, b_i)$ are eigenvectors of the linear part $g_o$ of $g = \tau_p g_o \in \ker\varphi$. We claim that there are at least three pairwise non-proportional $v_i$. Indeed, if all $v_i$ were parallel, then $T^{\mathrm{sing}} = \emptyset$, which contradicts $T_i \cap T^{\mathrm{sing}} \neq \emptyset$ for $1 \leq i \leq h$ by Lemma 2 iii). Suppose that among $v_1, \ldots, v_h$ there are two non-parallel and all other $v_i$ are proportional to one of them. Then after an eventual permutation there is $1 \leq k \leq h-1$, such that $v_1, v_k$ are linearly independent, $v_i = \mu_i v_1$ for $\mu_i \in \mathbb{C}, 2 \leq i \leq k$ and $v_i = \mu_i v_{k+1}$ for $\mu_i \in \mathbb{C}, k+2 \leq i \leq h$. Holzapfel has proved in [9] that any abelian ball quotient model $(A, T)$ is subject to $\sum_{i=1}^{h} \mathrm{card}(T_i \cap T^{\mathrm{sing}}) = 4\,\mathrm{card}(T^{\mathrm{sing}})$. In the case under consideration

$$\mathrm{card}(T^{\mathrm{sing}}) = \sum_{i=1}^{k} \sum_{j=k+1}^{h} \mathrm{card}(T_i \cap T_j)$$

$$\mathrm{card}(T_i \cap T^{\mathrm{sing}}) = \sum_{j=k+1}^{h} \mathrm{card}(T_i \cap T_j) \quad \text{for} \quad 1 \leq i \leq k \quad \text{and}$$

$$\mathrm{card}(T_j \cap T^{\mathrm{sing}}) = \sum_{i=1}^{k} \mathrm{card}(T_i \cap T_j) \quad \text{for} \quad k+1 \leq j \leq h.$$

Therefore $\sum\limits_{i=1}^{h} \mathrm{card}(T_i \cap T^{\mathrm{sing}}) = 2\mathrm{card}(T^{\mathrm{sing}}) \neq 4\mathrm{card}(T^{\mathrm{sing}})$ and there are at least three pairwise non-proportional eigenvectors $v_1, v_2, v_3$ of $g_o$. Let $\lambda_i$ be the corresponding eigenvalues of $v_i$ and $v_3 = \rho_1 v_1 + \rho_2 v_2$ for some $\rho_1, \rho_2 \in \mathbb{C}^*$. Then $\lambda_3 v_3 = g_o(v_3) = \rho_1 \lambda_1 v_1 + \rho_2 \lambda_2 v_2$ implies that $\lambda_1 = \lambda_3 = \lambda_2$ and $g_o = \lambda_o I_2$ is a scalar matrix. On the other hand, $g(T_i) = g_o(T_i) + p = T_i$ for all $1 \leq i \leq h$, so that $g_o$ permutes among themselves the parallel elliptic curves among $T_1, \ldots, T_h$. Since $T_i$ are finitely many, there is a natural number $m$, such that $g_o^m \in \ker\varphi$. Therefore, $\lambda_o^m \in \mathrm{End}(T_i)$ and $\lambda_o^{-m} \in \mathrm{End}(T_i)$ for all $1 \leq i \leq h$, due to $(g_o^m)^{-1} = g_o^{-m} \in \ker\varphi$. Recall that the units group $\mathrm{End}^*(T_i) = \mathbb{Z}^* = \{\pm 1\}$ for $T_i$ without a complex multiplication. If the elliptic curve $T_i$ has complex multiplication by an imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$, $d \in \mathbb{N}$, then $\mathrm{End}(T_i)$ is a subring of the integers ring $\mathcal{O}_{-d}$ of $\mathbb{Q}(\sqrt{-d})$. The units groups $\mathcal{O}_{-1}^* = \langle \mathrm{i} \rangle$, $\mathcal{O}_{-3}^* = \langle e^{\frac{2\pi i}{6}} \rangle$, and $\mathcal{O}_{-d}^* = \langle -1 \rangle$ for all $d \neq 1, 3$ are finite cyclic groups. As a subgroup of $\mathcal{O}_{-d}^*$, the units group $\mathrm{End}^*(T_i)$ is a finite cyclic group. Therefore $\lambda_o^m \in \mathrm{End}^*(T_i)$ and $g_o = \lambda_o I_2$ take finitely many values.

Concerning the translation part $\tau_p$ of $g \in \ker\varphi$, one can always move the origin $\check{o}_A$ of $A$ at one of the singular points of $T$. Due to the $G$-invariance of $T^{\mathrm{sing}}$, there follows $g(\check{o}_A) = \tau_p g_o(\check{o}_a) = \tau_p(\check{o}_A) = p \in T^{\mathrm{sing}}$. Therefore $p$ takes finitely many values and $\ker\varphi$ is finite.

ii) Since $\Gamma \subset \mathrm{SU}(2,1)$ is a torsion free lattice, any subgroup $H$ of

$$G = \mathrm{Aut}(A', T') \subseteq \mathrm{Aut}(A' \setminus T') = \mathrm{Aut}\,(\mathbb{B}/\Gamma)$$

lifts to a subgroup $\Gamma_H \subset \mathrm{Aut}(\mathbb{B}) = \mathrm{SU}(2,1)$, which normalizes $\Gamma$ and has quotient $\Gamma_H/\Gamma = H$. We claim that $\Gamma_H$ is discrete. Indeed, $\Gamma_H = \cup_{i=1}^{k} \gamma_i \Gamma$ is a finite disjoint union of cosets, relative to $\Gamma$. Suppose that $\Gamma_H$ is not discrete and there is a sequence $\{\nu_n\}_{n=1}^{\infty} \subset \Gamma_H$ with a limit point $\nu_o \in \gamma_{i_o}\Gamma$. Then pass to a subsequence $\{\nu_{m_n}\}_{n=1}^{\infty} \subset \gamma_{i_o}\Gamma$, converging to $\nu_o$. As a result $\{\gamma_{i_o}^{-1}\nu_{m_n}\}_{n=1}^{\infty} \subset \Gamma$ converges to $\gamma_{i_o}^{-1}\nu_o \in \Gamma$ and contradicts the discreteness of $\Gamma$. Thus, $\Gamma_H \supseteq \Gamma$ is discrete and, therefore, a ball lattice. Straightforwardly,

$$A'/H = [(\mathbb{B}/\Gamma) \,/\, (\Gamma_H/\Gamma)] \cup \left(T'/H\right) = (\mathbb{B}/\Gamma_H) \cup \left(T'/H\right) = \overline{(\mathbb{B}/\Gamma_H)}$$

is the compactification of the ball quotient $\mathbb{B}/\Gamma_H$ by the divisor $T'/H$. The $H$-Galois covers $\zeta_H : A \to A/H$ and $\zeta'_H : A' \to \overline{(\mathbb{B}/\Gamma_H)}$ fit in a commutative diagram

$$\begin{array}{ccc} A & \xleftarrow{\ \xi\ } & A' \\ {\scriptstyle \zeta_H}\downarrow & & \downarrow{\scriptstyle \zeta'_H} \\ A/H & \xleftarrow{\ \xi_H\ } & \overline{(\mathbb{B}/\Gamma_H)} \end{array}$$

with the contraction $\xi_H$ of $L/H$ to $\xi(L)/H$.

Note that $X = A/H$ is smooth exactly when $H$ has no isolated fixed points on $A$. The blow-up $\xi : A' \to A$ replaces an arbitrary $p_j = \xi(L_j)$ with stabilizer $\mathrm{Stab}_H(p_j)$ by a smooth rational curve $L_j$ with $\mathrm{Stab}_H(q) = \mathrm{Stab}_H(p_j)$ for all $q \in L_j$. Therefore the blow-up $\xi$ does not create isolated $H$-fixed points on $A'$ and $A'/H = \overline{(\mathbb{B}/\Gamma_H)}$ is a smooth compactification. Its open subset $\mathbb{B}/\Gamma_H$ is smooth.

□

## 3. Explicit Constructions

The present section applies Lemma 3 to a specific abelian ball quotient model over the Gauss numbers $\mathbb{Q}(\mathrm{i})$, in order to provide ball quotient compactifications, which are birational to a hyperelliptic, Enriques or a ruled surface with an elliptic base.

**Theorem 3** (Holzapfel [9]). *Let us consider the elliptic curve* $E_{-1} = \mathbb{C}/(\mathbb{Z} + \mathrm{i}\mathbb{Z})$ *with complex multiplication by the Gauss numbers* $\mathbb{Q}(\mathrm{i})$*, its two-torsion points*

$$Q_0 = 0(\mathrm{mod}\, \mathbb{Z} + \mathrm{i}\mathbb{Z}), \quad Q_1 = \frac{1}{2}(\mathrm{mod}\, \mathbb{Z} + \mathrm{i}\mathbb{Z}), \quad Q_2 = \mathrm{i}Q_1, \quad Q_3 = Q_1 + Q_2$$

*the abelian surface* $A_{-1} = E_{-1} \times E_{-1}$*, the points*

$$Q_{ij} = (Q_i, Q_j) \in A_{2-\mathrm{tor}} \subset A_{-1}$$

*and the divisor* $T_{-1}^{(6,8)} = \sum_{i=1}^{8} T_i$ *with smooth elliptic irreducible components*

$$T_k = E_{\mathrm{i}^k,1} \quad \text{for} \quad 1 \le k \le 4$$

$$T_{m+4} = Q_m \times E_{-1}, \quad T_{m+6} = E_{-1} \times Q_m \quad \textit{for} \quad 1 \le m \le 2.$$

*Then* $\left(A_{-1}, T_{-1}^{(6,8)}\right)$ *is an abelian model of an arithmetic ball quotient* $\mathbb{B}/\Gamma_{-1}^{(6,8)}$*, defined over* $\mathbb{Q}(\mathrm{i})$.

**Corollary 1** (Holzapfel [9]). i) *In the notations from* Theorem 3*, the multiplications* $I = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & 1 \end{pmatrix}$, $J = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix}$ *by* $\mathrm{i} \in \mathbb{Z}[\mathrm{i}] = \mathrm{End}(E_{-1})$ *on the first,* *respectively, the second elliptic factor* $E_{-1}$ *of* $A_{-1}$ *are automorphisms of* $\left(A_{-1}, T_{-1}^{(6,8)}\right)$.

ii) *If* $\Gamma_{K3,-1}^{(6,8)}$ *is the ball lattice, containing* $\Gamma_{-1}^{(6,8)}$ *as a normal subgroup with quotient* $\Gamma_{K3,-1}^{(6,8)}/\Gamma_{-1}^{(6,8)} = \langle -I_2 = I^2J^2 \rangle \subset \mathrm{Aut}\left(A_{-1}, T_{-1}^{(6,8)}\right)$, *then the ball quotient* $\mathbb{B}/\Gamma_{K3,-1}^{(6,8)}$ *is birational to the Kummer surface* $X_{K3}$ *of* $A_{-1}$.

iii) *If* $\Gamma_{\mathrm{Rat},-1}^{(6,8)}$ *is the ball lattice, containing* $\Gamma_{-1}^{(6,8)}$ *as a normal subgroup with quotient* $\Gamma_{\mathrm{Rat},-1}^{(6,8)}/\Gamma_{-1}^{(6,8)} = \langle I, J \rangle \subseteq \mathrm{Aut}\left(A_{-1}, T_{-1}^{(6,8)}\right)$, *then the ball quotient* $\mathbb{B}/\Gamma_{\mathrm{Rat},-1}^{(6,8)}$ *is a rational surface.*

The entire automorphism group $G_{-1}^{(6,8)} = \mathrm{Aut}\left(A_{-1}, T_{-1}^{(6,8)}\right)$ is described in the next lemma.

**Lemma 4.** *In the notations from* Theorem 3, *the group* $G_{-1}^{(6,8)} = \mathrm{Aut}\left(A_{-1}, T_{-1}^{(6,8)}\right)$ *is generated by* $I = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & 1 \end{pmatrix}$, $J = \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{i} \end{pmatrix}$, *the transposition* $\theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *of the elliptic factors* $E_{-1}$ *of* $A_{-1}$ *and the translation* $\tau_{33}$ *by* $Q_{33}$. *The aforementioned generators are subject to the relations*

$$I^4 = \mathrm{Id}, \quad J^4 = \mathrm{Id}, \quad \theta^2 = \mathrm{Id}, \quad \tau_{33}^2 = \mathrm{Id}, \quad IJ = JI$$

$$\theta I = J\theta, \quad \theta J = I\theta, \quad I\tau_{33} = \tau_{33}I, \quad J\tau_{33} = \tau_{33}J, \quad \theta\tau_{33} = \tau_{33}\theta.$$

*and* $G_{-1}^{(6,8)}$ *is of order* 64.

**Proof:** Any $g \in G_{-1}^{(6,8)}$ leaves invariant

$$\left(T_{-1}^{(6,8)}\right)^{\mathrm{sing}} = \sum_{1 \leq i < j \leq 8} T_i \cap T_j = \sum_{m=1}^{2}\sum_{n=1}^{2} Q_{mn} + Q_{00} + Q_{33}.$$

Thus, $g(T_i) = T_j$ implies $s_i = \mathrm{card}(T_i \cap T^{\mathrm{sing}}) = \mathrm{card}(T_j \cap T^{\mathrm{sing}}) = s_j$, according to the bijectiveness of $g$. In the case under consideration, $s_1 = s_2 = s_3 = s_4 = 4$ and $s_5 = s_6 = s_7 = s_8 = 2$, so that $G_{-1}^{(6,8)}$ permutes separately $T_1, \ldots, T_4$ and $T_5, \ldots, T_8$. In particular, the intersection $\cap_{i=1}^{4} T_i = \{Q_{00}, Q_{33}\}$ is $G_{-1}^{(6,8)}$-invariant and any $g = \tau_{(U,V)} g_o \in G_{-1}^{(6,8)}$ transforms the origin $\check{o}_{A_{-1}} = Q_{00}$ into $g(\check{o}_{A_{-1}}) = (U_1, U_2) \in \{Q_{00}, Q_{33}\}$. Straightforwardly, $\tau_{33}(T_i) = T_i$ for $1 \leq i \leq 4$ and $\tau_{33}(T_{m+2n}) = T_{3-m+2n}$ for $1 \leq m \leq 2, \; 2 \leq n \leq 3$ imply that $\tau_{33} \in G_{-1}^{(6,8)}$. Therefore $G_{-1}^{(6,8)}$ is generated by $G_{-1}^{(6,8)} \cap \mathrm{GL}_2(\mathrm{End}(E_{-1})) = G_{-1}^{(6,8)} \cap \mathrm{GL}_2(\mathbb{Z}[\mathrm{i}])$ and $\tau_{33}$. Note that $\theta \in \mathrm{Aut}(A_{-1})$ acts on $T_{-1}^{(6,8)}$ and induces the permutation $(T_1, T_3)(T_5, T_7)(T_6, T_8)$ of its irreducible components. Therefore

$\theta \in G_{-1}^{(6,8)}$ and $\langle I, J, \theta \rangle$ is a subgroup of $G_{-1}^{(6,8)} \cap \mathrm{GL}_2(\mathbb{Z}[\mathrm{i}])$. On the other hand, any $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G_{-1}^{(6,8)} \cap \mathrm{GL}_2(\mathbb{Z}[\mathrm{i}])$ acts on $T_5, \ldots, T_8$ and, therefore, on the set $\{\widetilde{T_5} = \widetilde{T_6} = 0 \times \mathbb{C}, \widetilde{T_7} = \widetilde{T_8} = \mathbb{C} \times 0\}$ of the corresponding universal covers. If $g(0 \times \mathbb{C}) = 0 \times \mathbb{C}$, $g(\mathbb{C} \times 0) = \mathbb{C} \times 0$ then $\beta = \gamma = 0$, so that $\alpha, \delta \in \mathrm{End}(E_{-1}) = \mathbb{Z}[\mathrm{i}]$ and $\det(g) = \alpha\delta \in \mathrm{End}^*(E_{-1}) = \langle \mathrm{i} \rangle = \mathbb{C}_4$ imply $g = I^k J^l$ for some $0 \leq k, l \leq 3$. Similarly, for $g(0 \times \mathbb{C}) = \mathbb{C} \times 0$, $g(\mathbb{C} \times 0) = 0 \times \mathbb{C}$ one has $\alpha = \delta = 0$, whereas $\beta, \gamma \in \mathbb{Z}[\mathrm{i}]$, $\beta\gamma \in \mathbb{Z}[\mathrm{i}]^* = \langle \mathrm{i} \rangle$ and $g = I^k J^l \theta$ for some $0 \leq k, l \leq 3$. Consequently, $G_{-1}^{(6,8)} \cap \mathrm{GL}_2(\mathbb{Z}[\mathrm{i}]) = \langle I, J, \theta \rangle$ and $G_{-1}^{(6,8)} = \langle I, J, \theta, \tau_{33} \rangle$. The announced relations among $\tau_{33}$, $I$, $J$, $\theta$ imply that

$$G_{-1}^{(6,8)} = \{\tau_{33}^n I^k J^l \theta^m \ ; \ 0 \leq k, l \leq 3, \quad 0 \leq m, n \leq 1\}$$

is of order 64.

□

**Theorem 4.** *In the notations from* Lemma 3, Theorem 3 *and* Lemma 4, *let us consider the subgroups* $H_{HE} = \langle \tau_{33} J^2 \rangle$, $H_{\mathrm{Enr}} = \langle -I_2, \tau_{33} I^2 \rangle$, $H_{\mathrm{Rul}} = \langle J^2 \rangle$ *of* $G_{-1}^{(6,8)} = \mathrm{Aut}\left(A_{-1}, T_{-1}^{(6,8)}\right)$, *their liftings* $\Gamma_{HE,-1}^{(6,8)}$, $\Gamma_{\mathrm{Enr},-1}^{(6,8)}$, $\Gamma_{\mathrm{Rul},-1}^{(6,8)}$ *to ball lattices and the blow-up* $A_{\widehat{2-\mathrm{tor}}}$ *of* $A_{-1}$ *at the two-torsion points* $A_{2-\mathrm{tor}}$. *Then*

i) $\mathbb{B}/\Gamma_{HE,-1}^{(6,8)}$ *is a smooth ball quotient, birational to the smooth hyperelliptic surface* $A_{-1}/H_{HE}$

ii) $\mathbb{B}/\Gamma_{\mathrm{Enr},-1}^{(6,8)}$ *is a ball quotient with one double point* $\mathrm{Orb}_{H_{\mathrm{Enr}}}(Q_{03})$, *which is birational to the smooth Enriques surface* $A_{\widehat{2-\mathrm{tor}}}/H_{\mathrm{Enr}}$

iii) $\mathbb{B}/\Gamma_{\mathrm{Rul},-1}^{(6,8)}$ *is a smooth ball quotient, birational to the smooth trivial ruled surface* $A_{-1}/H_{\mathrm{Rul}} = E_{-1} \times \mathbb{P}^1$ *with an elliptic base* $E_{-1}$.

**Proof:** i) Recall that the $\mathbb{Z}$-module $\pi_1(E_{-1}) = \mathbb{Z} + \mathrm{i}\mathbb{Z} = \mathbb{Z} + (1+\mathrm{i})\mathbb{Z}$ is generated by $1, 1+\mathrm{i}$ and $Q_3 = \frac{1+\mathrm{i}}{2} (\mathrm{mod}\, \pi_1(E_{-1}))$. The translation $\tau_{Q_3} : E_{-1} \to E_{-1}$ is of order 2, as well as the morphism

$$\tau_{Q_3}(-1) : E_{-1} \longrightarrow E_{-1}$$

$$\tau_{Q_3}(-1)(P) = -P + Q_3$$

with four fixed points

$$\frac{1}{2} Q_3 + (E_{-1})_{2-\mathrm{tor}} = \frac{1}{2} Q_3 + \{Q_i \ ; \ 0 \leq i \leq 3\}.$$

According to [5], the quotient $A_{-1}/H_{HE}$ by the cyclic group

$$H_{HE} = \langle \tau_{Q_3} \times \tau_{Q_3}(-1) \rangle$$

of order 2 is a smooth hyperelliptic surface. Lemma 3 ii) implies that $\mathbb{B}/\Gamma^{(6,8)}_{HE,-1}$ is a smooth ball quotient, birational to $A_{-1}/H_{HE}$.
ii) The quotient $X_{K3} = A_{\widehat{2-\mathrm{tor}}}/\langle -I_2\rangle$ is a smooth K3 surface, called the Kummer surface of $A_{-1}$. We claim that the involution $\tau_{33}I^2$ acts on $A_{\widehat{2-\mathrm{tor}}}$ and determines an unramified double cover

$$\zeta : X_{K3} = A_{\widehat{2-\mathrm{tor}}}/\langle -I_2\rangle \to A_{\widehat{2-\mathrm{tor}}}/\langle -I_2, \tau_{33}I^2\rangle = A_{\widehat{2-\mathrm{tor}}}/H_{\mathrm{Enr}}.$$

More precisely, $\tau_{33}I^2 = \tau_{Q_3}(-1) \times \tau_{Q_3}$ leaves invariant the two-torsion points $A_{2-\mathrm{tor}} = \{Q_{ij}\ ;\ 0 \leq i,j \leq 3\}$ and any choice of an automorphism of $\mathbb{P}^1$ extends $\tau_{33}I^2$ to an automorphism of $A_{\widehat{2-\mathrm{tor}}}$. Note that $\tau_{33}I^2(-I_2) = (-I_2)\tau_{33}I^2$, so that $\tau_{33}I^2$ normalizes $\langle -I_2\rangle$ and there is a well defined quotient group $H_{\mathrm{Enr}}/\langle -I_2\rangle = \langle \tau_{33}I^2\rangle$ of order 2. That allows to define $\zeta : X_{K3} \to A_{\widehat{2-\mathrm{tor}}}/H_{\mathrm{Enr}}$ as $H_{\mathrm{Enr}}/\langle -I_2\rangle$-Galois cover. We claim that $\tau_{33}I^2$ is a fixed point free involution on $X_{K3}$, in order to conclude that $A_{\widehat{2-\mathrm{tor}}}/H_{\mathrm{Enr}}$ is a smooth Enriques surface. More precisely, the fixed points of $\tau_{33}I^2$ on the set $X_{K3}$ of the $\langle -I_2\rangle$-orbits on $A_{\widehat{2-\mathrm{tor}}}$ lift to $\varepsilon$-fixed points of $\tau_{33}I^2$ on $A_{\widehat{2-\mathrm{tor}}}$ for $\varepsilon = \pm 1$. The $\varepsilon$-fixed points $(P,Q) \in A_{-1}$ are subject to

$$\begin{aligned} -P + Q_3 &= \varepsilon P \\ Q + Q_3 &= \varepsilon Q. \end{aligned}$$

For $\varepsilon = 1$ the equality $Q + Q_3 = Q$ has no solution $Q \in E_{-1}$, while for $\varepsilon = -1$ the equation $-P + Q_3 = -P$ on $P \in E_{-1}$ is inconsistent. Therefore $\tau_{33}I^2$ has no $\varepsilon$-fixed points on $A_{-1}$. By the very definition of the $\tau_{33}I^2$-action on $A_{\widehat{2-\mathrm{tor}}}$, there are no $\varepsilon$-fixed points for $\tau_{33}I^2$ on $A_{\widehat{2-\mathrm{tor}}}$ and $\tau_{33}I^2 : X_{K3} \to X_{K3}$ is a fixed point free involution. As a result, $A_{\widehat{2-\mathrm{tor}}}/H_{\mathrm{Enr}}$ is a smooth Enriques surface.

Recall that the exceptional divisor $\xi^{-1}_{2-\mathrm{tor}}(A_{2-\mathrm{tor}})$ of the blow-up

$$\xi_{2-\mathrm{tor}} : A_{\widehat{2-\mathrm{tor}}} \to A_{-1}$$

of $A_{-1}$ at $A_{2-\mathrm{tor}}$ is $H_{\mathrm{Enr}}$-invariant, so that $\xi_{2-\mathrm{tor}}$ descends to the contraction $\overline{\xi_{2-\mathrm{tor}}} : A_{\widehat{2-\mathrm{tor}}}/H_{\mathrm{Enr}} \to A_{-1}/H_{\mathrm{Enr}}$ of $\xi^{-1}_{2-\mathrm{tor}}(A_{2-\mathrm{tor}})/H_{\mathrm{Enr}}$ to $A_{2-\mathrm{tor}}/H_{\mathrm{Enr}}$. In particular, the smooth Enriques surface $A_{\widehat{2-\mathrm{tor}}}/H_{\mathrm{Enr}}$ is birational to $A_{-1}/H_{\mathrm{Enr}}$. The singular locus $(A_{-1}/H_{\mathrm{Enr}})^{\mathrm{sing}} \subseteq (A_{2-\mathrm{tor}}/H_{\mathrm{Enr}})$, according to the smoothness of $A_{\widehat{2-\mathrm{tor}}}/H_{\mathrm{Enr}}$. On the other hand, $\tau_{33}I^2$ has no fixed points on $A_{2-\mathrm{tor}}$, so that $A_{2-\mathrm{tor}}/H_{\mathrm{Enr}}$ consists of eight double points

$$\mathrm{Orb}_{H_{\mathrm{Enr}}}(Q_{ij}) = \mathrm{Orb}_{H_{\mathrm{Enr}}}(Q_{3-i,3-j}), \quad 0 \leq i,j \leq 3$$

and $(A_{-1}/H_{\mathrm{Enr}})^{\mathrm{sing}} = A_{2-\mathrm{tor}}/H_{\mathrm{Enr}}$. Note that

$$\left(T^{(6,8)}_{-1}\right)^{\mathrm{sing}} = \{\mathrm{Orb}_{H_{\mathrm{Enr}}}(Q_{00}), \mathrm{Orb}_{H_{\mathrm{Enr}}}(Q_{11}), \mathrm{Orb}_{H_{\mathrm{Enr}}}(Q_{12})\}$$

is contained in $(A_{-1}/H_{\mathrm{Enr}})^{\mathrm{sing}}$ and the birational morphism

$$\xi_{H_{\mathrm{Enr}}} : \left( \overline{\mathbb{B}/\Gamma^{(6,8)}_{\mathrm{Enr},-1}} \right) \to A_{-1}/H_{\mathrm{Enr}}$$

resolves $\left(T^{(6,8)}_{-1}\right)^{\mathrm{sing}}$ by smooth rational curves of self-intersection $(-2)$. Therefore $\left( \overline{\mathbb{B}/\Gamma^{(6,8)}_{\mathrm{Enr},-1}} \right)^{\mathrm{sing}}$ consists of the following five double points:

$$\mathrm{Orb}_{H_{\mathrm{Enr}}}(Q_{01}),\ \mathrm{Orb}_{H_{\mathrm{Enr}}}(Q_{10}),\ \mathrm{Orb}_{H_{\mathrm{Enr}}}(Q_{02}),\ \mathrm{Orb}_{H_{\mathrm{Enr}}}(Q_{20}),\ \mathrm{Orb}_{H_{\mathrm{Enr}}}(Q_{03}).$$

Since

$$\mathrm{Orb}_{H_{\mathrm{Enr}}}(Q_{0,m}) \in \left[ T_{m+6} \setminus \left(T^{(6,8)}_{-1}\right)^{\mathrm{sing}} \right] / H_{\mathrm{Enr}} = \left( T'_{m+6} \setminus L \right) / H_{\mathrm{Enr}}$$

$$\mathrm{Orb}_{H_{\mathrm{Enr}}}(Q_{m,0}) \in \left[ T_{m+4} \setminus \left(T^{(6,8)}_{-1}\right)^{\mathrm{sing}} \right] / H_{\mathrm{Enr}} = \left( T'_{m+4} \setminus L \right) / H_{\mathrm{Enr}}$$

for all $1 \leq m \leq 2$ belong to the compactifying divisor $T'/H_{\mathrm{Enr}}$, the ball quotient $\mathbb{B}/\Gamma^{(6,8)}_{\mathrm{Enr},-1}$ has only one singular point

$$\left( \mathbb{B}/\Gamma^{(6,8)}_{\mathrm{Enr},-1} \right)^{\mathrm{sing}} = \{ \mathrm{Orb}_{H_{\mathrm{Enr}}}(Q_{0,3}) \}.$$

iii) The quotient $X = A_{-1}/H_{\mathrm{Rul}} = E_{-1} \times [E_{-1}/\langle(-1)\rangle]$ of $A_{-1}$ by the reflection $J^2 = 1 \times (-1)$ is a smooth surface, birational to the smooth ball quotient $\mathbb{B}/\Gamma^{(6,8)}_{\mathrm{Rul},-1}$. It is well known that $C = E_{-1}/\langle -1 \rangle$ is a smooth projective curve. More precisely, if

$$\mathfrak{p}(t) = \frac{1}{t^2} + \sum_{\lambda \in (\mathbb{Z}+\mathrm{i}\mathbb{Z})\setminus\{0\}} \left[ \frac{1}{(t-\lambda)^2} - \frac{1}{\lambda^2} \right]$$

is the **Weierstrass $\mathfrak{p}$-function**, associated with the lattice $\mathbb{Z} + \mathrm{i}\mathbb{Z} = \pi_1(E_{-1})$, then the map

$$\psi : E_{-1} \setminus \{\check{o}_{E_{-1}}\} \longrightarrow \mathbb{P}^2$$

$$\psi(t + (\mathbb{Z}+\mathrm{i}\mathbb{Z})) = \left[1 : \mathfrak{p}(t + (\mathbb{Z}+\mathrm{i}\mathbb{Z})) : \mathfrak{p}'(t + (\mathbb{Z}+\mathrm{i}\mathbb{Z}))\right] = [1 : \mathfrak{p}(t) : \mathfrak{p}'(t)]$$

extends by $\psi(\check{o}_{E_{-1}}) = [0:0:1] = p_\infty$ to a projective embedding of $E_{-1}$. The image

$$\psi(E_{-1}) = \left\{ [z:x:y] \in \mathbb{P}^2 \ ; \ zy^2 = (x - \mathfrak{p}(Q_1))(x - \mathfrak{p}(Q_2))(x - \mathfrak{p}(Q_3)) \right\}$$

is a **cubic hypersurface** in $\mathbb{P}^2$. As far as $\mathfrak{p}(t)$ is even and $\mathfrak{p}'(t)$ is an odd function of $t$, the multiplication $\mu_{-1}$ by $-1$ on $E_{-1}$ acts on $\psi(E_{-1})$ by the rule

$$\mu_{-1}([z:x:y]) = [z:x:-y].$$

The fixed points of this action are $p_\infty$ and $\mathfrak{p}(Q_i)$ for $1 \leq i \leq 3$. The fibres of the projection

$$\Pi : \psi(E_{-1}) \setminus \{p_\infty\} \longrightarrow \mathbb{P}^1 \setminus \{q_\infty = [0:1]\}$$
$$\Pi([z:x:y]) = [z:x]$$

are exactly the $\mu_{-1}$-orbits on $\psi(E_{-1}) \setminus \{p_\infty\}$, so that its image

$$\mathbb{P}^1 \setminus \{q_\infty\} = \Pi(\psi(E_{-1}) \setminus \{p_\infty\}) = (\psi(E_{-1}) \setminus \{p_\infty\})/\langle \mu_{-1} \rangle$$

is the corresponding Galois quotient by the cyclic group $\langle \mu_{-1} \rangle$ of order 2. Thus,

$$\psi(E_{-1})/\langle \mu_{-1} \rangle = (\psi(E_{-1}) \setminus \{p_\infty\})/\langle \mu_{-1} \rangle \cup \{p_\infty\} = (\mathbb{P}^1 \setminus \{q_\infty\}) \cup \{p_\infty\} = \mathbb{P}^1.$$

□

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