Geometry, Integrability and Quantization
September 1–10, 1999, Varna, Bulgaria
Ivaïlo M. Mladenov and Gregory L. Naber, Editors
**Coral Press**, Sofia 2000, pp 7-16

# ON THE STRUCTURE OF AUTOMORPHISMS OF MANIFOLDS*

KÖJUN ABE† and KAZUHIKO FUKUI‡

†*Department of Mathematical Sciences, Shinshu University
Matsumoto 390-8621, Japan*

‡*Department of Mathematics, Kyoto Sangyo University
Kyoto 603-8555, Japan*

**Abstract**. Thurston [16] proved that the group $\mathrm{Diff}^\infty(M)$ of a smooth manifold $M$ is perfect, which implies the first homology group is trivial. If $M$ has a geometric structure, then the first homology of the group of automorphisms of $M$ preserving the geometric structure is not necessarily trivial. There are many results concerning this field. In this paper, we shall summarize the results of the first homology groups of automorphisms of manifolds with geometric structure.

## Introduction

In this paper we shall report on the first homology group of the group of automorphisms of a manifold with geometric structure. Here the first homology group of a group is the quotient group of the group by its commutator subgroup. Let $M$ be a connected closed smooth manifold. Let $\mathrm{Diff}^\infty(M)$ denote the group of $C^\infty$-diffeomorphisms of $M$ which are isotopic to the identity. Thurston [16] proved that $\mathrm{Diff}^\infty(M)$ is perfect which implies the first homology group is trivial. The result is related to the topology of the classifying space of foliations. There are many analogous results on the group of automorphisms of a manifold $M$ which preserve a geometric structure on $M$ such as volume structure, symplectic structure, submanifold structure, foliated structure, $G$-manifold structure. In those cases the first homology groups are not necessarily trivial. Then the calculation of the first homology is the next problem. The first homology

*Dedicated to Professor Fuichi Uchida on his 60-th birthday.

groups are interesting for us because they are expected relating mostly to the fundamental geometric structures on the manifolds.

The first homology group of the group of automorphisms has been studied when $M$ is a topological manifold or Lipschitz manifold with geometric structure. In each category we need alternative methods to analyze the first homology, which reflects the topological properties of the geometric structure of $M$.

The purpose of this paper is to try to summarize the results of the first homology groups of the automorphisms stated above. We shall mainly state the results relating to our work. Many interesting results concerning those fields are found in Banyaga [6].

In Sect. 1 we review the results by Thurston [17] and Banyaga [5], [6] studying volume form or symplectic form preserving diffeomorphisms of manifolds. Section 4 is devoted to studying the commutators of homeomorphisms or Lipschitz homeomorphisms of manifolds with geometric structures. First we treat the case of the group of homeomorphisms preserving a submanifold. Secondly we discuss the case of the group of foliation preserving homeomorphisms.

In Sect. 3 we consider the equivariant diffeomorphisms of smooth $G$-manifolds. In this case the first homology is depend on the orbit structure of the $G$-manifold and the smooth structure of the orbit space. We also study the commutators of the equivariant Lipschitz homeomorphisms of Lipschitz principal bundles.

## 1. Commutators of Volume (or Symplectic) Preserving Diffeomorphisms

In this section we review the results on commutators of diffeomorphisms preserving a volume form or a symplectic form of manifolds which have been investigated by Thurston [17] and Banyaga [5], [6].

Let $M$ be a compact $m$-dimensional $C^\infty$-manifold without boundary and let $\omega$ be a closed $p$-form on $M$. Let $\mathrm{Diff}^\infty(M)$ denote the group of all $C^\infty$-diffeomorphisms of $M$ with the compact open $C^\infty$-topology. We denote by $\mathrm{Diff}^\infty_\omega(M)$ the subgroup of $\mathrm{Diff}^\infty(M)$ consisting of $\omega$ preserving diffeomorphisms. For an isotopy $\varphi_t \in \mathrm{Diff}^\infty_\omega(M)$ with $\varphi_0 = 1_M$, put

$$\dot{\varphi}_t = \frac{\mathrm{d}\varphi_t}{\mathrm{d}t}, \quad I_{\varphi_t}(\alpha) = \int_0^1 \varphi_t^* \iota(\dot{\varphi}_t)\alpha \, \mathrm{d}t,$$

where $\iota$ denotes the inner product. Then $I_{\varphi_t}(\omega)$ is a closed $(p-1)$-form.

**Theorem 1.1.** (cf. [6]) *Let $\omega$ be a closed $p$-form an a compact $m$-dimensional $C^\infty$-manifold $M$ and $\varphi_t \in \mathrm{Diff}^\infty_\omega(M)$ be an isotopy with $\varphi_0 = 1_M$. Then the* *cohomology class* $[I_{\varphi_t}(\omega)] \in H^{p-1}(M,\mathbb{R})$ *depends only on the homotopy class relatively to fixed ends of the isotopy* $\varphi_t \in \mathrm{Diff}^\infty_\omega(M)$.

*Let* $\widetilde{G_\omega(M)}$ *be the group consisting the homotopy classes* $[\varphi_t]$ *of isotopies* $\varphi_t \in \mathrm{Diff}^\infty_\omega(M)$ *relatively to fixed ends. The mapping* $[\varphi_t] \mapsto [I_{\varphi_t}(\omega)]$ *is a group homomorphism:*

$$\tilde{S}_\omega : \widetilde{G_\omega(M)} \to H^{p-1}(M,\mathbb{R}) .$$

If $\omega$ is a volume form or a symplectic form, then the group $\widetilde{G_\omega(M)}$ is the universal covering of the identity component $G_\omega(M)$ in $\mathrm{Diff}^\infty_\omega(M)$. Put $\Gamma_\omega = \tilde{S}_\omega(\pi_1(G_\omega(M)))$. Then we have a homomorphism $S_\omega : G_\omega(M) \to H^{p-1}(M,\mathbb{R})/\Gamma_\omega$ induced from $\tilde{S}_\omega$. Thurston proved the following.

**Theorem 1.2.** ([17]) *Let* $M$ *be a compact* $m$*-dimensional* $C^\infty$*-manifold without boundary with a volume form* $\omega$ $(m \geq 3)$. *Then* $\ker S_\omega$ *is a simple group equal to the commutator subgroup of* $G_\omega(M)$. *In particular*

$$H_1(G_\omega(M)) \cong H^{m-1}(M,\mathbb{R})/\Gamma_\omega .$$

Banyaga proved the following theorem applying the Thurston's idea to the symplectic case.

**Theorem 1.3.** ([6]) *Let* $M$ *be a compact* $m$*-dimensional* $C^\infty$*-manifold without boundary with a symplectic form* $\omega$. *Then* $\ker S_\omega$ *is a simple group equal to the commutator subgroup of* $G_\omega(M)$. *In particular*

$$H_1(G_\omega(M)) \cong H^1(M,\mathbb{R})/\Gamma_\omega .$$

## 2. Commutators of Homeomorphisms Preserving Geometric Structurs

In this section we discuss the commutators of homeomorphisms or Lipschitz homeomorphisms of manifolds with geometric structures.

Let $M$ be a compact $m$-dimensional CAT-manifold (CAT $\equiv TOP, LIP, C^\infty$). Let $G^{\mathrm{CAT}}(M)$ denote the identity component of the space of all CAT-homeomorphisms of $M$ with the suitable topology, that is , the topology is the compact open topology, the compact open Lipschitz topology, and the compact open $C^\infty$ topology respectively in the case of CAT $\equiv TOP$, $LIP$, $C^\infty$.

In this section we consider the subgroups of $G^{\mathrm{CAT}}(M)$ in the following.

i) Case of homeomorphisms preserving a submanifold

Let $N$ be a compact $n$-dimensional CAT-submanifold of $M$. We denote by $G^{\mathrm{CAT}}_N(M)$ the identity component of the subgroup of $G^{\mathrm{CAT}}(M)$ consisting of all CAT-homeomorphisms which map $N$ to itself.

A locally flat proper manifold pair $(M, N)$ is by definition a pair of topological manifolds such that $N$ is a locally flat submanifold of $M$, properly imbedded as a closed subset. Then we have the fragmentation lemma from the relative version of Corollary 1.3 of Edwards and Kirby [8]. Thus we have the following.

**Theorem 2.1.** ([10]) *$G_N^{TOP}(M)$ is perfect.*

Let $(M, N)$ be a Lipschitz manifold pair. Then we have the following fragmentation lemma from the relative version of Corollary 4.3 of [3].

**Lemma 2.2.** ([10] fragmentation lemma) *For any $f \in G_N^{LIP}(M))$, there are $f_i \in G_N^{LIP}(M)$ $(i = 1, 2, \dots, k)$ such that $(1)$ the support of each $f_i$ is contained either in a small ball $U$ with $U \cap N = \emptyset$ or in a small ball $U$ with $U \cap N \neq \emptyset$ and $(2)$ $f = f_k \circ f_{k-1} \circ \cdots \circ f_1$.*

Let $G^{LIP}(\mathbb{R}^m)$ be the identity component of the group of all Lipschitz homeomorphisms of $\mathbb{R}^m$ with compact support and $G_{\mathbb{R}^n}^{LIP}(\mathbb{R}^m)$ the identity component of the subgroup of $G^{LIP}(\mathbb{R}^m)$ consisting of Lipschitz homeomorphisms which map $\mathbb{R}^n$ to itself. Then we have the following theorem following Mather [13].

**Theorem 2.3.** ([3] Corollary 2.4) *$G^{LIP}(\mathbb{R}^m)$ and $G_{\mathbb{R}^n}^{LIP}(\mathbb{R}^m)$ are perfect groups for $n > 0$.*

**Theorem 2.4.** ([3] Theorem 3.1) *$G_N^{LIP}(M)$ is perfect for $\dim N > 0$.*

**Proof:** This follows from Lemma 2.2 and Theorem 2.3.

For the case of $\dim M = 1$, Tsuboi proved the following.

**Theorem 2.5.** ([18]) *$G_{\{0,1\}}^{LIP}([0, 1])$ is uniformly perfect.*

On the other hand, it seems not to be known the corresponding result in the $C^\infty$-case except for the following.

**Theorem 2.6.** ([9] Theorem 2.6) $H_1(G_{\{pt\}}^\infty(M)) \cong \mathbb{R}$.

ii) Case of foliation preserving homeomorphisms

Let $M$ be a compact topological manifold and $\mathcal{F}$ a codimension $p$ topological foliation of $M$. A homeomorphism $f\colon M \to M$ is called *a foliation preserving homeomorphism* (resp. *a leaf preserving homeomorphism*) if for each point $x$ of $M$, the leaf through $x$ is mapped into the leaf through $f(x)$ (resp. $x$), that is, $f(L_x) = L_{f(x)}$ (resp. $f(L_x) = L_x$), where $L_x$ is the leaf of $\mathcal{F}$ which contains $x$. Let $G^{TOP}(M, \mathcal{F})$ (resp. $G_L^{TOP}(M, \mathcal{F})$) denote the identity component of the subgroup of $G^{TOP}(M)$ consisting of foliation (resp. leaf) preserving homeomorphisms of $(M, \mathcal{F})$.

Then we can prove the following fragmentation lemma applying the deep argument of Edwards and Kirby [8] to the foliated case.

**Lemma 2.7.** ([11] Theorem 3.1) *Let $(M,\mathcal{F})$ be a foliated manifold. Any $f \in G_L^{TOP}(M,\mathcal{F})$ can be expressed as $f = f_k \circ f_{k-1} \circ \cdots \circ f_1$, where each $f_i$ is a leaf preserving homeomorphism with support in a small ball.*

Let $\mathcal{F}_0$ be the $p$-dimensional foliation of $\mathbb{R}^m$ whose leaves are defined by $x_{p+1} = \text{const}, \ldots, x_m = \text{const}$ $(1 \leq p \leq m)$. Let $G_L^{TOP}(\mathbb{R}^m, \mathcal{F}_0)$ denote the identity component of the group of leaf preserving homeomorphisms of $(\mathbb{R}^m, \mathcal{F}_0)$ with compact support. Then we have the following.

**Theorem 2.8.** ([11] Corollary 2.3) $G_L^{TOP}(\mathbb{R}^m, \mathcal{F}_0)$ *is perfect.*

**Theorem 2.9.** ([11] Theorem 3.2) *$(M,\mathcal{F})$ be a foliated manifold. Then $G_L^{TOP}(M,\mathcal{F})$ is perfect.*

**Proof:** This follows from Lemma 2.7 and Theorem 2.8.

We consider the case of codimension one foliations. Let $\mathcal{F}$ be a codimension one foliation of a compact topological manifold $M$. By Theorem 6.26 of Siebenmann [15], there exists a one dimensional foliation $\mathcal{T}$ of $M$ transverse to $\mathcal{F}$. We define the subset $S_0$ of $M$ by

$$S_0 = \{x \in M;\ \text{there exists an element } f \text{ of } G^{TOP}(M,\mathcal{F}) \text{ such that } f(L_x) \neq L_x\}.$$

Then we see that $S_0$ is an open $\mathcal{F}$-saturated set and all leaves in $S_0$ have trivial holonomy.

**Theorem 2.10.** ([11] Theorem 4.3) *Let $S$ be a connected component of $S_0$. Then clearly $S$ is invariant under the action of $G^{TOP}(M,\mathcal{F})$ and $S$ is one of the following three types.*

Type $P$: *$S$ is homeomorphic to $L \times (0,1)$ and the foliations $\mathcal{F}|_S$ and $\mathcal{T}|_S$ correspond to the product structure of $L \times (0,1)$.*

Type $R$: *There exists a closed transverse curve $C$ in $S$ such that $C$ meets each leaf of $\mathcal{F}|_S$ at exactly one point and the natural map*

$$p : S \to C, \quad p(x) = L_x \cap C$$

*is a fibration and $\mathcal{T}|_S$ is a connection of the fibration $p$.*

Type $D$: *All leaves of $\mathcal{F}$ in $S$ are dense in $S$ and there exists a one parameter subgroup $\{\varphi_t\}$ of $G^{TOP}(M,\mathcal{F}|_S)$ whose orbits are leaves of $\mathcal{T}|_S$.*

Then we have the following.

**Theorem 2.11.** ([11] Theorem 4.6) *Let $\mathcal{F}$ be a codimension one foliation of a compact manifold $M$. Suppose that $\mathcal{F}$ has no components of type $D$ and has only a finite number of components of type $R$. Then $G^{TOP}(M, \mathcal{F})$ is perfect.*

**Remark 2.12.** From Theorem 2.11, we see that $G^{TOP}(S^3, \mathcal{F}_R)$ is perfect for the Reeb foliation $\mathcal{F}_R$ of $S^3$. In contrast with topological case, differentiable case is as follows. Let $G^\infty(S^3, \mathcal{F}_R)$ be the group of foliation preserving $C^\infty$-diffeomorphisms of $(S^3, \mathcal{F}_R)$ isotopic to the identity by a foliation preserving isotopy. Then Lemma 1 of [12] implies that $G^\infty(S^3, \mathcal{F}_R)$ is not perfect.
For a type $D$-component $S$, we define a submodule $\mathrm{Per}(S)$ of $\mathbb{R}$ by

$$\mathrm{Per}(S) = \{t \in \mathbb{R};\ \varphi_t(L) = L \text{ for one and all leaves } L \text{ in } S\}.$$

$\mathrm{Per}(S)$ depends on the parametrization of $\{\varphi_t\}$ but the quotient group $\mathbb{R}/\mathrm{Per}(S)$ is determined by $\mathcal{F}|_S$ and, as a set, this is the space of leaves of $\mathcal{F}|_S$.

**Theorem 2.13.** ([11] Theorem 4.8) *Let $S$ be a type $D$-component. Then there exists a homomorphism $\pi$ of $G^{TOP}(M, \mathcal{F})$ onto $\mathbb{R}/\mathrm{Per}(S)$ and we have*

$$\ker \pi = \{f \in G^{TOP}(M, \mathcal{F});\ f(L) = L \textit{ for any leaf } L \textit{ in } S\}.$$

Let $\pi\colon F(\mathcal{F}) \to \prod \mathbb{R}/\mathrm{Per}(S_i)$ denote the homomorphism defined by $\pi(f) = \prod \pi_i(f)$ for $f \in G^{TOP}(M, \mathcal{F})$, where $\pi_i$ is a homomorphism in the above Theorem for a type $D$-component $S_i$ and the product is taken for all type $D$-components $S_i$ of $\mathcal{F}$. Then $\pi$ induces a homomorphism $\pi_*$ of $H_1(G^{TOP}(M, \mathcal{F}))$ to $H_1(\prod \mathbb{R}/\mathrm{Per}(S_i)) \cong \prod \mathbb{R}/\mathrm{Per}(S_i)$. Then we have the following which is is an easy consequence of Theorem 2.13 and a non-zero element of $\ker \pi_*$ is represented by a leaf preserving homeomorphism which is not isotopic to the identity via leaf preserving homeomorphisms.

**Theorem 2.14.** ([11] Theorem 4.9) *The homomorphism*

$$\pi_* \textit{ of } H_1(G^{TOP}(M, \mathcal{F})) \textit{ to } \prod \mathbb{R}/\mathrm{Per}(S_i)$$

*is surjective.*

For a very simple case, we have the following.

**Theorem 2.15.** ([11] Theorem 4.10) *Let $\mathcal{F}$ be a foliation of a torus $T^n$ defined by a 1-form $\omega = \sum a_i dx_i$. If one of $a_i/a_j$ is irrational, then*

$$H_1(G^{TOP}(M, \mathcal{F})) \textit{ is isomorphic to } \mathbb{R}/a_1\mathbb{Z} + \cdots + a_n\mathbb{Z}.$$

## 3. Automorphism Groups of $G$-manifolds

In this section we shall consider the case when a compact Lie group $G$ acts on a manifold $M$.

First we treat the case when a compact Lie group $G$ acts smoothly and freely on a closed connected smooth manifold $M$. Let $\mathrm{Diff}^r_G(M)$ denote the equivariant $C^r$-diffeomorphism group of a $G$-manifold $M$ whose elements are $G$-isotopic to the identity. Banyaga proved the following.

**Theorem 3.1.** ([5]) *Let $M$ be a smooth closed and connected manifold of dimension $m$ on which the torus $T^q$ acts smoothly and freely. For $m \geq q+1$, $1 \leq r \leq \infty, r \neq m-q+1$, $\mathrm{Diff}^r_{T^q}(M)$ is a perfect group.*

We have proved the result when the group is any compact Lie group.

**Theorem 3.2.** ([2]) *Let $M$ be a closed manifold on which a compact Lie group $G$ acts smoothly and freely. If $1 \leq r \leq \infty, r \neq m-q+1$, and $m-q \geq 1$, then $\mathrm{Diff}^r_G(M)$ is perfect, where $m = \dim M$ and $q = \dim G$.*

From Theorem 3.2 we have the following.

**Corollary 3.3.** ([2]) *Let $M$ be a closed manifold on which a compact Lie group $G$ acts smoothly with one orbit type. If $1 \leq r \leq \infty, r \neq \dim M/G + 1$, and $\dim M/G \geq 1$, then $\mathrm{Diff}^r_G(M)$ is perfect.*

We remark that in those cases each group $\mathrm{Diff}^r_G(M)$ is perfect but is not simple. The following lemmas play key roles in the proof of Theorem 3.2.

**Lemma 3.4.** (fragmentation lemma, cf. [2] Lemma 1) *Let $\{V_i; 1 \leq i \leq n\}$ be a $G$-invariant finite open covering of $M$ and $\mathcal{N}$ be an open neighborhood of the identity in $\mathrm{Diff}^r_G(M)$. Then there exists an open neighborhood $\mathcal{N}_0 \subset \mathcal{N}$ of the identity with the following properties: For any $f \in \mathcal{N}_0$, there exist $f_i \in \mathcal{N}$, $1 \leq i \leq n$, such that*

1) *$f_i$ is $G$-isotopic to the identity through equivariant $C^r$ diffeomorphisms whose support are contained in $V_i$, and*

2) $f = f_n \circ f_{n-1} \circ \cdots \circ f_1$.

**Lemma 3.5.** ([2] Lemma 2) *For $\delta > 0$, let $u : \mathbb{R} \to \mathbb{R}$ be a $C^r$ function supported in $(-\delta, \delta)$ which is $C^1$-close to the zero map. Then there exist a $C^r$ function $v : \mathbb{R} \to \mathbb{R}$ and a $C^r$ diffeomorphism $\phi : \mathbb{R} \to \mathbb{R}$ such that*

1) *$supp(v) \subset (-2\sqrt{3}\delta, 2\sqrt{3}\delta), |v(x)| \leq 3\delta,$*
2) *$supp(\phi) \subset (-\delta, \delta)$ and $\phi$ is isotopic to the identity through $C^r$ diffeomorphisms supported in $(-\delta, \delta)$,*
3) *$u = v \circ \phi - v$.*

If a $G$-manifold $M$ has more than two orbit types, then $\mathrm{Diff}^r_G(M)$ is not perfect and therefore the first homology $H_1(\mathrm{Diff}^r_G(M))$ is not necessarily trivial. In this case it is difficult to calculate the first homology in general.

We consider the case when $M$ is a smooth connected closed $G$-manifold with codimension one orbit. Then the orbit space $M/G$ of $M$ is homeomorphic to the circle $S^1$ or the unit interval $[0,1]$. If $M/G$ is homeomorphic to $S^1$, then $M$ has one orbit type and it follows from Corollary 3.3 that $\mathrm{Diff}^\infty_G(M)$ is perfect. If $M/G$ is homeomorphic to $[0,1]$, then $M$ has two or three orbit types. In this case $\mathrm{Diff}^\infty_G(M)$ is not perfect.

From the differentiable slice theorem, $M$ is equivariantly diffeomorphic to the union of two disc bundles $G\times_{K_i} D(V_i)$ $(i=0,1)$ with the boudaries identified under an equivariant diffeomorphism. Here $K_i$ is an singular isotropy subgroup and $D(V_i)$ is a unit disc of a linear slice $V_i$ at a point of the singular orbit. Let $H$ be a principal isotropy subgroup of $M$. We can assume that $H$ is a subgroup of $K_1 \cap K_2$. Let $N(H)$ denote the normalizer of $H$ in $G$. Put $W(M) = ((N(H)\cap N(K_0))/H \times (N(H)\cap N(K_1))/H)_0$. Then our result is

**Theorem 3.6.** ([3] Theorem 4.2)

$$H_1(\mathrm{Diff}^\infty_G(M)) \cong \mathbb{R}^2 \times H_1(W(M))\,.$$

The proof of Theorem 3.6 is based on a differentiable structure of the orbit space $M/G$ of the $G$-manifold $M$ such that the functional structure of $M/G$ is induced from that of $M$ [1]. Then we have a natural homeomorphisms $P\colon \mathrm{Diff}^\infty_G(M)_0 \to \mathrm{Diff}^\infty[0,1]_0$ which is defined by the orbit map. With this functional structure and Baker-Campbell-Hausdorff formula, we can analyze the $G$-diffeomorphisms around singular orbits and determine the group of the orbit preserving $G$-diffeomorphisms of $M$. Let $C^\infty_\infty([0,1], N(H)/H)_0$ denote the subgoup of $C^\infty([0,1],(N(H)/H)_0)$ which are infinitely tangent to $N(K_0)\cap N(H)/H$ at 0 and infinitely tangent to $N(K_1)\cap N(H)/H$ at 1. The following theorem determines the group structure of the orbit preserving $G$-diffeomorphisms of $M$, and plays an important role of the proof of Theorem 3.6.

**Theorem 3.7.** ([3] Corollary 3.3)

$$\ker P \;\cong\; C^\infty_\infty([0,1], N(H)/H)_0\,.$$

Next we consider the group of equivariant Lipschitz homeomorphisms of a principal $G$-bundle over a Lipschitz manifold.

Let $G$ be a compact Lie group of dimension $q$ and $M$ the total space of a principal $G$-bundle over a closed $(m-q)$-dimensional Lipschitz manifold $B$ such that each transition function is Lipschitz. Let $\mathcal{H}_{LIP,G}(M)$ denote the identity component of the group of equivariant Lipschitz homeomorphisms of $M$.

**Theorem 3.8.** ([4] Theorem 5.1) *$\mathcal{H}_{LIP,G}(M)$ is perfect for $\dim B > 0$. For $\delta > 0$, put $B_\delta = \{x \in \mathbb{R}^m;\ ||x|| \le \delta\}$.*

Similarly as in $C^r$ case, in order to prove Theorem 3.8 we need the following key lemmas.

**Lemma 3.9.** (fragmentation lemma, [4] Corollary 5.5) *For any $f \in \mathcal{H}_{LIP,G}(M)$, there are $f_i \in \mathcal{H}_{LIP,G}(M)$ $(i = 1, 2, \ldots, k)$ such that*
1) *$f = f_k \circ f_{k-1} \circ \cdots \circ f_1$ and*
2) *the image of the support of each $f_i$ by $\pi$ is contained in a small ball in $B$.*

**Lemma 3.10.** ([4] Lemma 5.7) *For any $u \in C_{LIP}(\mathbb{R}^m, \mathbb{R})$ with $supp(u) \subset B_\delta$ and $lip(u) = k < 1$, there exist $v \in C_{LIP}(\mathbb{R}^m, \mathbb{R})$ with $supp(v) \subset B_{5\delta}$ and $\phi \in \mathcal{H}_{LIP}(\mathbb{R}^m)$ such that $u = v \circ \phi - v$.*

**Remark 3.11.** In the topological case, we can prove fragmentation lemma but we can not prove the lemma corresponding to Lemma 3.10. If it is valid in the topological case, we can prove the analogous result to Theorem 3.8 such that $M$ is a total space of a principal $G$-bundle over a closed $(m-q)$-dimensional topological manifold $B$.

**Acknowledgement**

This research was partially suported by Grant-in-Aid for Scientific Research (No. 09640101), Ministry of Education, Science and Culture, Japan.

## References

[1] Abe K., *On the Homotopy Type of the Groups of Equivariant Diffeomorphisms*, Publ. RIMS. Kyoto Univ., **16** (1980) 601–626.
[2] Abe K. and Fukui K., *On Commutators of Equivarint Diffeomorphisms*, Proc. Japan Acad., **54** (1978) 52–54.
[3] Abe K. and Fukui K., *On the Structure of the Group of Equivariant Diffeomorphisms of G-Manifolds with Codimension One Orbit*, preprint.
[4] Abe K. and Fukui K., *On the Structure of the Group of Lipschitz Homeomorphisms and its Subgroups*, preprint.
[5] Banyaga A., *On the Structure of the Group of Equivarinat Diffeomorphisms*, Topology, **16** (1977) 279–283.
[6] Banyaga A., *Sur la structure du groupe des difféomorphismes qui réservent une form symplectique*, Comment. Math. Helv., **53** (1978) 174–227.
[7] Banyaga A., *The Structure of Classical Diffeomorphism Groups*, Kluwer Academic Publishers, Amsterdam 1997.

[8] Edwards R. D., Kirby R. C., *Deformations of Spaces of Imbeddings*, Ann. of Math., **93** (1971) 63–88.
[9] Fukui K., *Homologies of the Group of* $\mathrm{Diff}^\infty(R^n, 0)$ *and its Subgroups*, J. Math. Kyoto Univ., **20** (1980) 475–487.
[10] Fukui K., *Commutators of Foliation Preserving Homeomorphisms for Certain Compact Foliations*, Publ. RIMS. Kyoto Univ., **34** (1998) 65–73.
[11] Fukui K., Imanishi H., *On Commutators of Foliation Preserving Homeomorphisms*, J. Math. Soc. Japan, **51**(1) (1999) 227–236.
[12] Fukui K., Ushiki S., *On the Homotopy Type of* $\mathrm{FDiff}(S^3, \mathcal{F}_R)$, J. Math. Kyoto Univ., **15**(1) (1975) 201–210.
[13] Mather J. N., *The Vanishing of the Homology of Certain Groups of Homeomorphisms*, Topology, **10** (1971) 297–298.
[14] Mather J. N., *Commutators of Diffeomorphisms I and II*, Comment. Math. Helv., **49** (1992) 512–528; **50** (1975) 33–40.
[15] Siebenmann L. C., *Deformations of Homeomorphisms on Stratified Sets*, Comment. Math. Helv., **47** (1972) 123–163.
[16] Thurston W., *Foliations and Group of Diffeomorphisms*, Bull. Amer. Math. Soc., **80** (1974) 304–307.
[17] Thurston W., *On the Structure of Volume Preserving Diffeomorphisms* (unpublised).
[18] Tsuboi T., *On the Prefectness of Groups of Diffeomorphisms of the Interval Tangent to the Identity at the Endpoints* (preprint).