
Jost Bürgi and the discovery of the logarithms

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Preface

In the year of 1620 the printing office of the University of Prague published a 58-page table containing the values $a_n = (1.0001)^n$ for $0 \leq n \leq 23027$, rounded to 9 decimal digits. This table had been devised and computed about 20 years earlier by the Swiss-born astronomer and watchmaker Jost Bürgi in order to facilitate the multi-digit multiplications and divisions he needed for his astronomical computations. The “Progreß Tabulen”, as Bürgi called his tables, are considered to be one of the two independent appearances of

Jost Bürgi (1552–1632) aus Lichtensteig (Kanton St. Gallen, Schweiz) war ab 1579 Hofuhrmacher (mechanicus) von Landgraf Wilhelm IV von Hessen in Kassel. In dieser Stellung hat er wesentlich zur Wissenschaft und zum Instrumentenbau jener Zeit beigetragen: genaue Sekundenuhren, sich selbst bewegende Himmelsgloben, Sextanten, etc. Hier berichtet der Autor über die von Bürgi konzipierte und berechnete Logarithmentafel. Diese listet die Potenzen B^n der Basis $B = 1.0001$ für n von 0 bis 23 027 mit 9 bedeutsamen Ziffern, ohne systematische Fehler auf. Mit linearer Interpolation (beschrieben im *gründlichen Unterricht*) erreicht man volle Genauigkeit. Leider haben wohl die Wirren des 30-jährigen Krieges eine ordentliche Publikation verhindert. Die zwei heute bekannten Originale (München, Graz; das Danziger Exemplar ging im zweiten Weltkrieg verloren) sind wahrscheinlich Probeabzüge (Prag, 1620). Die hier präsentierten Fehlerlisten und -statistiken belegen die hohe Qualität von Bürgis Tafel: 91.5 % korrekte Rundungen, 7.3 % Rundungen auf die falsche Seite; die restlichen fehlerhaften Ziffern sind häufig aufgrund von Übertragungsfehlern des Setzers entstanden oder sind unleserlichem Druck zuzuschreiben.

logarithmic tables in the history of mathematics – the other one, due to John Napier (1550–1617), appeared in 1614.

There are only a few copies of the original printing extant: one of them is now in the Astronomisch-Physikalisches Kabinett in Munich. Based on a copy of this original, the terminal digits of all table entries were extracted and compared with the exact values of a_n , a matter of a split second on a modern computer.

In this presentation we give a brief account of the mathematical environment at the end of the 16th century as well as a detailed description of Bürgi's Progreß Tabulen and their application to numerical computations. We will also give a sketch of Bürgi's remarkable life and of his numerous achievements besides his most efficient construction of logarithmic tables.

Our main purpose, however, is to analyze the numerical errors in Bürgi's table. First of all, there are no systematic errors, e.g., the crux of the table, $1.0001^{23027.0022} = 10$, is correct with all 9 digits given. 91.5% of the table entries are entirely correct, and 7.3% of the values show round-off errors between 0.5 and 1 unit of the least significant digit. The remaining 1.17% of the table errors are mainly errors of transcription and illegible digits.

Statistics of the round-off errors leads to interesting conclusions concerning Bürgi's algorithms of generating his table and on his handling of the round-off errors, as well as on the computational effort involved.

1 Introduction

The origins of this report date back to 1976, when the existence of an original copy of Jost Bürgi's table of logarithms [2] in the Astronomisch-Physikalisches Kabinett, Munich, was pointed out to the author by the engineer and historian Wolfhard Pohl, Zürich. This original had been found in 1847 by Rudolf Wolf [32], [33] in the Royal Library (Königliche Bibliothek) of Munich [3].

Subsequently, W. Pohl [23] was allowed to copy the entire table, in black and white, such that it became possible to investigate the numerical errors in Bürgi's hand calculations for the first time. This copy is now in possession of the author. In 1994 these results were presented to the mathematical community and commented on the occasion of the International Congress of Mathematicians (Zürich) in the form of a souvenir watch displaying Bürgi's title page as its dial [29], see Fig. 6.

In 1998 the author was invited to present a short history of the discovery of the logarithms [30] at "Slide Rule '98", the Fourth International Meeting of Slide Rule Collectors in Huttwil, Switzerland, organized by H. Joss [12] (October 14 to 16, 1998). Here we present an expanded version of the proceedings article, containing more information on the statistics of Bürgi's round-off errors and on his algorithm for generating the table.

It is amusing to ponder about possible lines of development of 16th century mathematics if calculators had been available at that time. The construction of tables of logarithms as instruments of numerical calculations would not have been necessary, and the concept of logarithms might have arisen only centuries later, e.g., in connection with the development of calculus by Newton, Leibniz and Euler. Almost certainly, the slide rule (see [12]), the

leading calculating device for three centuries, would have been missed altogether. However, as it happened, in the 16th century no efficient calculating machines were available, and there was a strong need to develop good algorithms for the more tedious arithmetic operations such as multiplications, divisions, and square roots.

Compared to our time, the scientific environment in the 16th century was simple. Astronomy was by far the most advanced discipline of science, looking back onto a continuous history of at least 2000 years. With the introduction of the Gregorian Calendar (Pope Gregory XIII) in 1582 the length of the year was defined as $365.97/400$ days = 365.2425 days (the modern value of the tropical year is 365.2422 days). Obviously, the astronomers of that time already needed to perform long arithmetic calculations, and the precision of some of their data (as the length of the year) asked for a precision of at least 6 digits.

Nowadays, arithmetic operations with multi-digit numbers are a standard topic in elementary schools all over the world. Paradoxically, due to the ubiquity of calculators, the widespread proficiency in these algorithms may eventually get lost.

2 The idea of the logarithms

The historical process of the discovery of the logarithms extended over at least half a century. As we understand it now, the sole purpose of the invention was to speed up multiplications and divisions by means of tables that could be computed once for all (with a huge effort, though). Ironically, this aspect of the logarithms is now all but irrelevant: for modern calculators and computers multiplications pose no bigger problems than additions. The logarithms as a tool for computing (their key role for almost 400 years) have disappeared completely within a decade.

On the other hand, about a century after its original discovery the logarithm function was found to be the indefinite integral of c/x (with an appropriate value of $c > 0$). In this role the logarithm will always keep its importance in all of mathematics.

The idea that directly leads to the mathematical object we now refer to as logarithm can be found already in the works of Archimedes, 287–212 BC, (see, e.g., [28]). However Archimedes missed the final breakthrough, and unfortunately his ideas were only picked up much later. After the French mathematician Nicolas Chuquet (1445–1488) had introduced a good nomenclature for large integers (e.g., 10^6 = million, 10^{12} = billion), the time was ready for a systematic treatment of large and small (real) numbers. In 1544 Michael Stifel [27] rediscovered what had remained forgotten for 1800 years. He considered what we now call the geometric sequence a_n , with initial element 1 and quotient 2,

$$a_n = 2^n, \quad n = \dots, -2, -1, 0, 1, 2, \dots, \tag{1}$$

or written in tabular form as

n	...	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	...
$a_n = 2^n$...	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	32	64	128	...

(2)

In modern language the second row of Stifel’s table is a list of the (positive and negative) powers of the **base** $B = 2$.

The continuation of the table to the left by fractions was a novel aspect and led directly to the discovery of the general law of multiplying powers of the same base B :

$$B^m \cdot B^n = B^{m+n} \quad \text{or} \quad a_m \cdot a_n = a_{m+n} \quad \text{with} \quad a_n = B^n. \quad (3)$$

Stifel became aware of the fact that this law could be exploited for finding the product of the elements a_m, a_n of the table by looking up the element a_{m+n} in the same table. Hence the product of the table elements in positions m, n is the table element in position $m + n$; a multiplication is reduced to 3 table-look-ups and an addition.

Example: $B = 2$

$$\begin{array}{ccccccc}
 8 \cdot 16 & = & B^3 \cdot B^4 & = & B^{3+4} & = & B^7 = 128 \\
 \underbrace{\hspace{1.5cm}} & & \uparrow \quad \uparrow & & \uparrow & & \underbrace{\hspace{1.5cm}} \\
 \text{table bottom to top} & & \text{addition} & & \text{table top to bottom} & &
 \end{array} \quad (4)$$

The nomenclature still in use today may be understood from M. Stifel’s table (2), but was introduced only much later by Napier [21]: the numbers a_n in the bottom row are the principal entries of the table, still referred to as *numeri*; the integers n in the top row are merely used in order to denote the position of a_n within the table: n is called the *logarithm* (greek for the *word*, i.e., the *essence*, of the *number*) of a_n (with respect to the *base* $B = 2$). Due to the “exposed” position of n in the upper row of the table, M. Stifel referred to n as the *exponent*. We still write $a_n = B^n$, with the exponent n in “exposed” position.

3 Jost Bürgi

These few highlights characterize the scientific environment into which Jost Bürgi was born. He was the son of a renowned family in the town of Lichtensteig in the Toggenburg valley (Canton of St. Gallen, Switzerland), born on February 28, 1552. Almost nothing is known about his youth. In his home town Jost Bürgi most likely only received the modest education that was possible in a rural environment. It is conceivable that he entered apprenticeship with his father Lienz Bürgi who was a locksmith. Summaries of the known fragments of Bürgi’s early life may be found in [15], [14], [16], [19], [28]. Details on Bürgi’s biography were also given by Ph. Schöbi [25]. The most comprehensive study on Bürgi’s life and achievements published until now has been made by Fritz Staudacher [26], also showing new aspects of Bürgi’s connections and innovations.

The next known date of Bürgi’s life is that in 1579 he was appointed at the court of Duke (Landgraf) Wilhelm IV of Hessen (in the city of Kassel, Germany) as the court watchmaker and “mechanicus”. It is not known when Bürgi (Fig. 1) left his home town and how and where he acquired the extraordinary skills that made him eligible for the prestigious appointment in the duke’s observatory. Rudolf Wolf [32] speculates that Jost Bürgi might have participated in the construction of the astronomical clock in the cathedral of Straßburg, carried out 1570–1574 by the clockmakers Isaak and Josias Habrecht



Fig. 1 Jost Bürgi at the age of 67. Copper plate print from a drawing by Aegidius Sadeler (1619), taken from Benjamin Brammer: Bericht zu M. Jobsten Bürgi seligen geometrischen Triangular Instruments, Kassel 1648 ([1], p. 21).

from Dießenhofen (Canton of Thurgau, Switzerland). According to another hypothesis, recently brought up by F. Staudacher [26], the young Bürgi might have acquired his skills in Schaffhausen (Switzerland), where the Habrecht family had built clocks at least until 1572, and he also might have worked with Christian Heiden (1526–1576), a constructor of celestial globes in Nürnberg [14].

From that time on Jost Bürgi's life is relatively well documented, mainly by the numerous precision instruments for geometry and astronomy, and by his astronomical clocks that soon earned world fame. Most famous up to the present day are Bürgi's "Celestial Globes", celestial spheres as they would be seen by an outside observer, with a clockwork inside (Fig. 2, [15]). The fame of those masterpieces was so great that the emperor Rudolf II (1552–1612) invited Bürgi to his court in Prague in 1592.

Among the precision instruments we mention the "Proportional Compasses" (Fig. 3), a device invented and built by Bürgi that can be used for proportionally changing the scale



Fig. 2 The little "Himmelsglobus", completed by Bürgi in 1594, to be seen in the Swiss National Museum in Zürich, see [15], [19]. It is an extraordinary piece of early astronomy and precision craftsmanship, 142 mm in diameter and 255 mm of total height, accurately displaying the motion of the celestial sphere, the sun, and giving the time.

Fig. 3 Bürgi's "Proportional Circkel", a precision compass device invented and built by Jost Bürgi, first mentioned by Levin Hulsius in 1603 (published in 1607). It serves to reduce the scale of an object by a constant factor (from [19], p. 130).



of a drawing. Bürgi was not only a skilled watchmaker, astronomer and mathematician, he also knew to organize his private life: when he returned to Kassel in 1593 he had become the owner of the house he had been living in before. It is documented that Bürgi also became a successful real estate agent and banker. One of the few authentic documents in Bürgi's handwriting is a letter (dated Prague, 27.7.1616, signed Jost Bürgi, Uhrmacher) that deals, in addition to 7 other cases, with a loan in the amount of 500 guilders (Gulden) (Fig. 4).

Important events in Jost Bürgi's life as well as a few other relevant events are summarized in the following chronological chart in Fig. 5.

Not only by his scientific achievements, but also according to reports by contemporaries, mainly Kepler [13] and Bramer [1], Bürgi must have been a mathematician of remarkable stature. This is even more astounding since Bürgi did not have much of a formal education and must have discovered or rediscovered many relations on his own. Unfortunately, only a small number of Bürgi's theoretical results are documented; most of the evidence is of an indirect nature. As Staudacher [26] discovered in 2012, Bürgi and Kepler were restricted in publishing their knowledge about Bürgi's mathematical innovations by a mutual confidence agreement. A similar agreement existed between Brahe and Kepler [13].

As an example, Kepler [13] quotes a theorem of Cardano and mentions that Jost Bürgi has announced to be in possession of a proof.

"The secant of 89° plus the tangent of 89° equals the sum of the sines of all degrees of the semicircle".

Slightly generalized, and reformulated in modern notation with $N = 180$, $\delta = \pi/N$ this means

$$S(N) = \sum_{k=0}^N \sin(k\delta) = \sec\left(\frac{\pi}{2} - \delta\right) + \tan\left(\frac{\pi}{2} - \delta\right).$$

This is by no means an obvious relation, and Kepler's statement sheds some light onto Bürgi's mathematical skills, although, unfortunately, Bürgi's alleged proof has never been

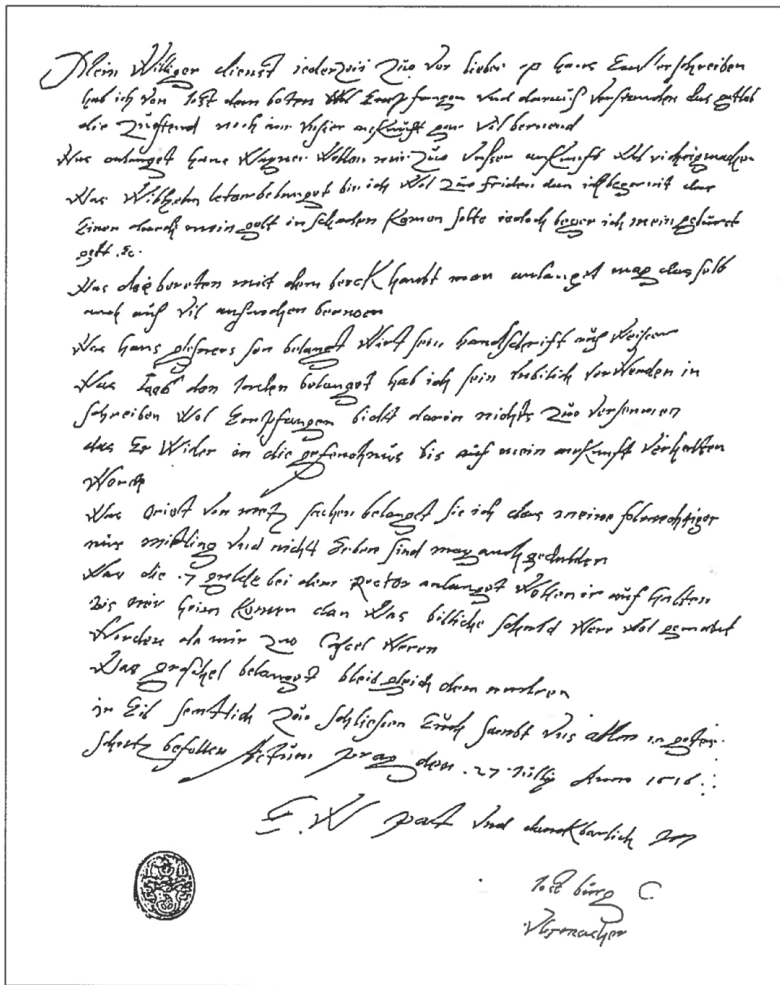


Fig. 4 Handwritten letter by Jost Bürgi, addressed to his attorney “Hans Dickhaut, meinen grossgunstigen freund und paten – Cassel”, dated “Actum Prag den 27. Julij Anno 1616” (line 22). Lines 6 and 7 refer to a loan of 500 guilders (“gulde”, see the 4th word on line 17): “iedoch beger ich mein gebüret gelt 5c”. Original in the Hessisches Staatsarchiv, Marburg an der Lahn, transcript see A.C. von Drach [6], p. 40.

found. Using geometric series and the technique of complex numbers, involving $i := \sqrt{-1}$, fully developed only much later by Euler (1707–1783) and others, the proof would look as follows (taking advantage of $\sin(N\delta) = 0$, $e^{iN\delta} = -1$):

$$\begin{aligned}
 S(N) &= \frac{1}{2i} \sum_{k=0}^{N-1} (e^{ik\delta} - e^{-ik\delta}) = \frac{1}{2i} \left(\frac{2}{1 - e^{i\delta}} - \frac{2}{1 - e^{-i\delta}} \right) \\
 &= \frac{1 + e^{i\delta}}{i(1 - e^{i\delta})} = \frac{e^{\frac{i\delta}{2}} + e^{-\frac{i\delta}{2}}}{\frac{1}{i}(e^{\frac{i\delta}{2}} - e^{-\frac{i\delta}{2}})} = \cotan \frac{\delta}{2}.
 \end{aligned}$$

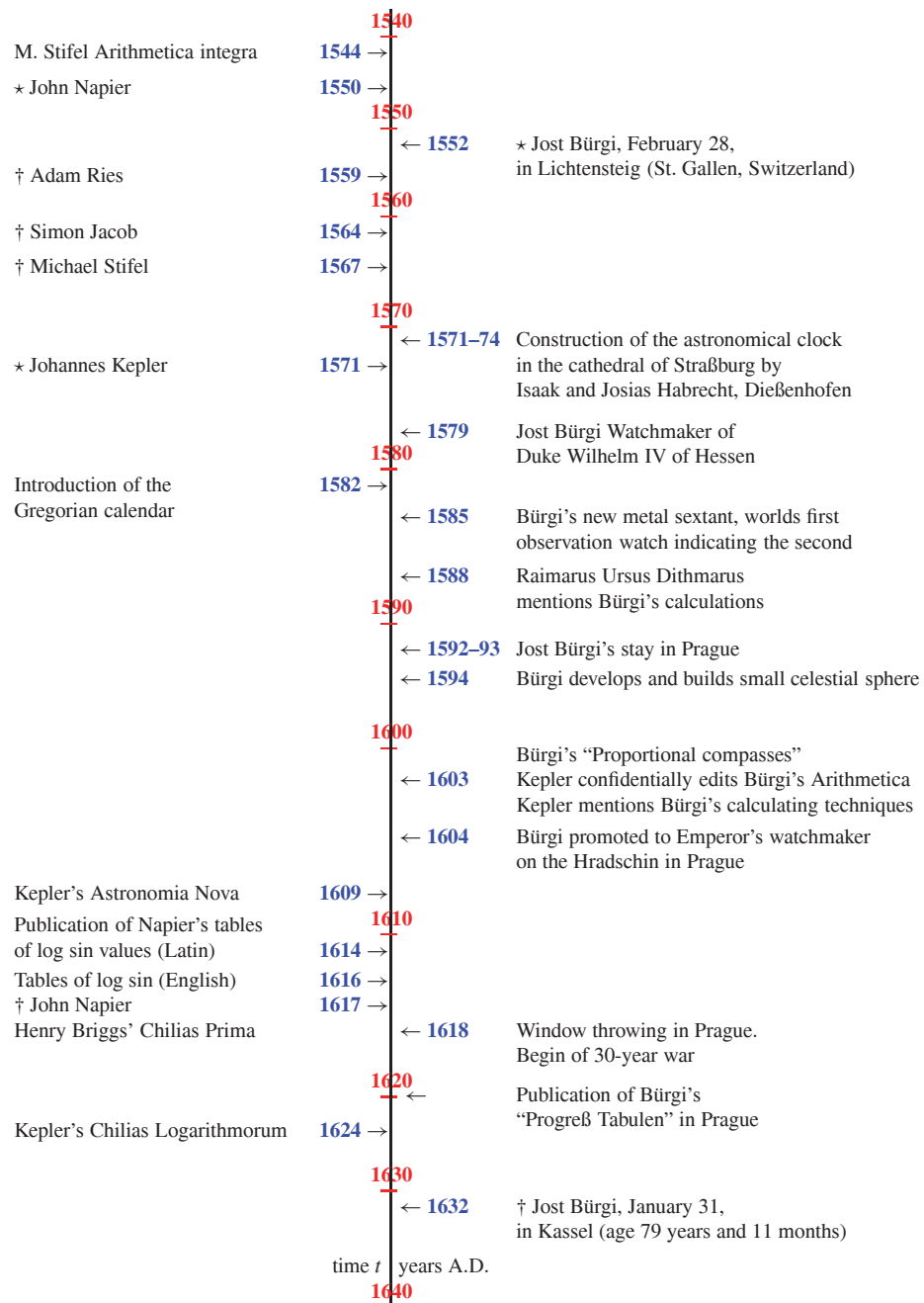


Fig. 5 Chronological chart 1544–1632.

Note that Cardano, Kepler, and probably Bürgi missed the simpler form of $S(N) = \tan(89.5^\circ)$ found above; in fact:

$$\tan\left(\frac{\pi}{2} - \frac{\delta}{2}\right) = \cotan \frac{\delta}{2} = \frac{1 + \cos \delta}{\sin \delta} = \sec\left(\frac{\pi}{2} - \delta\right) + \tan\left(\frac{\pi}{2} - \delta\right).$$

Note also that the modern proof heavily uses the exponential function which Bürgi was about to discover (in real numbers only, though).

4 The Progreß Tabulen

There is no question that Bürgi's diversified work in astronomy, triangulation, geometry, clockmaking etc. required extended calculations with multi-digit numbers. It is not known to what extent Bürgi was familiar with M. Stifel's geometric sequence ("progression") $a_n = B^n$ with $B = 2$. However, in his instructions ("Gründlicher Unterricht") [10], p. 27, [18], Bürgi refers to Simon Jacob (1510–1564), who himself had summarized Michael Stifel's ideas. In any case, Bürgi must have had the clear insight how the power law (3) could be exploited for speeding up multiplications. At the same time he came up with a surprisingly simple and effective means for overcoming the obvious drawback of Stifel's table: the scarcity of the table entries a_n . Bürgi simply chose the base B as an appropriate number close to 1, namely

$$B = 1 + \frac{1}{10000} = 1.0001, \quad (5)$$

and tabulated the exponential function with base B ,

$$a_n := B^n, \quad n = 0, 1, \dots, 23027, \quad (6)$$

in 9-digit precision [2], [3], [4]. The use of the table for multiplying is exactly the same as in the example of Equ. (4).

In this section we briefly describe the table and comment on the old dispute about the priority between Jost Bürgi and John Napier for the invention of the logarithms. Mathematical considerations on Bürgi's table will be collected in Section 5.

In Bürgi's table the sequences ("progressions") are arranged in columns with 50 entries per column and 8 columns per page. The index n (i.e., the logarithm) is printed in red (die Rote Zahl) as $10n = 0, 10, 20, \dots, 500, 510, \dots$, see Fig. 7. The corresponding table entry $a_n = 1.0001^n$ is printed in black (die Schwartze Zahl) by omitting the decimal point, i.e., as the integer $10^8 \cdot a_n$. The entire table consisting of 23028 entries thus extends over 58 pages, covering the entire range from $a_n = 1.0000\ 0000$ to $a_{23027} = 9.9999\ 9779$ (see Fig. 7, Fig. 8). In [17] and [20] the question of the base of the exponential function tabulated by Bürgi is discussed. Bürgi's base B can be identified unambiguously as $B = 1.0001$, the constant quotient of two consecutive table entries. Therefore it is natural to consider the entry a_n as the n th power of B , which implies $a_0 = 1$. The natural choice for the "red number" corresponding to the table entry $a_n = B^n$ is the exponent n . It can only be speculated why Bürgi used the 10-fold exponents, $10n$, instead, e.g., $0, 10, 20, \dots$

for the beginning of the table (Fig. 7). A possible explanation for Bürgi's increase of the precision of n by one decimal digit is that he wanted to suggest (linear) interpolation in the accuracy of one digit (see Section 5). The difference of two consecutive table entries, $a_{n+1} - a_n = a_n/10000$, needed for this operation is readily available. A skilled user like Bürgi can do this division and the subsequent proportionality computation in his head.

In the following, we take the liberty to refer to the table arguments and entries by means of numbers n and a_n lying in the intervals

$$0.0 \leq n \leq 23027.0 \quad \text{and} \quad 1.00000000 \leq a_n < 10.00000000 ,$$

adopting the modern usage of the decimal point.

Bürgi concluded that the table should ideally terminate at the N th entry where

$$a_N = 10.0000\ 0000 .$$

By two refinements of the table on the last page (Fig. 8) he correctly finds the fractional value

$$N = 23027.0022 ,$$

referred to as the “whole Red Number”. A more precise value is

$$N := \frac{\log(10)}{\log(1.0001)} = \log_{1.0001}(10) = 23027.00220\ 32997 . \quad (7)$$

The title page (Fig. 6) summarizes the table by listing each 500th entry, as well as the whole Red Number N . In accordance with tabulating the index as $10n$, the “whole red number” N is given as 23027^o0022 with the superimposed ^o marking the digit with unit value, which corresponds to $10N$ with N from Equ. (7) (see also the final sentence of the legend of Fig. 8).

The arrangement of the entries in a circular dial clearly shows Bürgi's genius since it documents his insight that the next decade, e.g., [10, 100) is a mere repetition in 10-fold size of the current one, e.g., [1, 10). A modern mathematician unwillingly is reminded of Euler's famous relationship $\exp(ix) = \cos x + i \sin x$ between exponential and circular functions, discovered over 100 years later.

Example of the use of Bürgi's table (cf. Fig. 6, 8)

In order to illustrate the use of Bürgi's table for multidigit arithmetic we give a (constructed) example merely using the title page (Fig. 6) and later page 1 (reproduced as Fig. 7): **Compute the 5th power of 6.35923131.** The arrows mark the direction from red to black in Bürgi's table.

black (numerus)	← red (logarithm)
$c = 6.35923131$ given data	← $n = 18500.0$
$c^5 = ?$ to be computed	← $5n = 92500.0$

10.00000000	the whole red number	←	$N = 23027.0022$
10000.00000	10^4	←	$4 N = 92108.0088$
$\frac{c^5}{10^4}$	scaled result	←	$5 n - 4 N = 391.9912$

The scaling factor 10^4 had to be chosen such that the scaled result is in the range of the table. A crude answer may now be read off page 1 directly: Approximate the logarithm $5 n - 4 N$ of the scaled result as $5 n - 4 N \approx 392.0 = 350.0 + 42.0$. Then page 1 yields $c^5/10^4 \approx a_{392} = 1.03997642$. Due to the illegible digits we have to be satisfied with $c^5 \approx 10399$.

Owing to the transparent structure of Bürgi’s table, the illegible digits can easily be reconstructed from $a_{391} = 1.03987243$:

$$a_{392} = a_{391} + \frac{a_{391}}{10000} = 1.03997642 .$$

For linear interpolation (see also Fig. 15) this value is not even needed:

red	black
391.0	a_{391}
391.9912	$\frac{c^5}{10^4}$
392.0	$a_{391} + \frac{a_{391}}{10000}$

Simple proportionality yields

$$\frac{c^5}{10^4} = a_{391} + 0.9912 \cdot \frac{a_{391}}{10000} = 1.03987243 + 0.00010400 - 0.00000092 = 1.03997551 .$$

The result $c^5 = \mathbf{10399.7551}$ is correct with all digits given.

Presently, the existence of two originals of Bürgi’s table has been confirmed. Besides the Munich copy [3], which is the basis of this work, an original had been found 1985 in the library of Paul Guldin (1577–1643); it is now in the library of the University of Graz, Austria [4] (see Gerlinde Faustmann [7], [8], [9]). Unfortunately, the Danzig original (found 1855 by H. R. Gieswald [10]), which is the source of the well-known reproductions in [13], [28] (Fig. 7, Fig. 8) is lost since Word War II. Rumours that it is in Prague now, and that a fourth original exists in the Vatican have not been confirmed so far. The quality of the printing in all known copies is less than perfect. The abundant illegible digits are not due to imperfections of the facsimile reproductions. The Munich Copy seems to be of better quality than the Danzig copy.

Bürgi was very reluctant in publishing his table. It appeared in print as late as 1620 (Fig. 6), only after Johannes Kepler (1571–1630) had been urging him for a long time to publish it.



Fig. 6 Title page of Bürgi's Progreß Tabulen, Prague, 1620, as reproduced in [10], [28] from the Danzig (Gdansk, Poland) Copy. It summarizes the table by listing every 500th table entry and states the logarithm N of 10, the "gantze Rote Zahl", $N := \log 10 / \log 1.0001 = 23027.0022$. A typesetting error in the entry for 500.0 (not contained in the table) has been corrected by hand from ...26407 to ...26847.

The reason for the scarcity of the original copies may be that only a few preliminary copies were printed. Wars or financial problems might have interrupted the publishing process.

Kepler reported in 1603 (1609 at latest according to different sources) that Bürgi was in possession of an efficient method to carry out multiplications and divisions. Even earlier, in 1588, the astronomer Raimarus Ursus Dithmarus (quoted by Rudolf Wolf [33]) reported that Bürgi was using a method to greatly simplify his calculations. It cannot be excluded that these statements mean that Bürgi's tables were operational as early as 1588. It is con-

	0	500	1000	1500	2000	2500	3000	3500
0	100000000	100501227	101004966	101511230	102020032	102531384	103045299	103561779
10	100000000	10011277	10015067	10021381	10030234	10041637	10055603	10072141
20	100000000	10022318	10027607	10035397	10045700	10058637	10074203	10092541
30	100000000	10033880	10040571	10049862	10061864	10075701	10092367	10111864
40	100000000	10045433	10053574	10064576	10078817	10095401	10114437	10136024
50	100000000	10057447	10067047	10079549	10094352	10111601	10131407	10156884
60	100000000	10069843	10081043	10095045	10111248	10129852	10151067	10176904
70	100000000	10082699	10095499	10111299	10129803	10150317	10172941	10198784
80	100000000	10096018	10109418	10127618	10148022	10170838	10196264	10224311
90	100000000	10110714	10125114	10144914	10167418	10191837	10219077	10249134
100	100100045	100601773	101106017	10161287	10212098	10263961	10316991	10371191
110	10010015	10061834	10112634	10164440	10216310	10269251	10323361	10378631
120	10010066	10063553	10115453	10167369	10220239	10274399	10329791	10386261
130	10010078	10065272	10118272	10170287	10223258	10278638	10335421	10393101
140	10010091	10067001	10121101	10173201	10226217	10282597	10341261	10400151
150	10010105	10068740	10123940	10176120	10229176	10288636	10347311	10407311
160	10010110	10070489	10126789	10179049	10232135	10294775	10353461	10414571
170	10010116	10072248	10129648	10182018	10235094	10300914	10359711	10421931
180	10010123	10074007	10132507	10185007	10238053	10307963	10366061	10429391
190	10010131	10075766	10135366	10188016	10241012	10315012	10372511	10436951
200	10010140	10077525	10138225	10191045	10244071	10322061	10379061	10444611
210	10010140	10079284	10141084	10194074	10247130	10329110	10386110	10452271
220	10010141	10081043	10143943	10197103	10250189	10336159	10393159	10459931
230	10010143	10082802	10146802	10200132	10253248	10343208	10400208	10467591
240	10010146	10084561	10149661	10203171	10256307	10350257	10407257	10475251
250	10010149	10086320	10152520	10206210	10259366	10357306	10414306	10482911
260	10010152	10088079	10155379	10209259	10262425	10364355	10421355	10490571
270	10010155	10089838	10158238	10212308	10265484	10371404	10428404	10498231
280	10010158	10091597	10161097	10215357	10268543	10378453	10435453	10505891
290	10010161	10093356	10163956	10218406	10271602	10385502	10442502	10513551
300	10010164	10095115	10166815	10221455	10274651	10392551	10449551	10521211
310	10010167	10096874	10169674	10224504	10277700	10399600	10456600	10528871
320	10010170	10098633	10172533	10227553	10280749	10406649	10463649	10536531
330	10010173	10099392	10175392	10230602	10283798	10413698	10470698	10544191
340	10010176	10101151	10178251	10233651	10286847	10420747	10477747	10551851
350	10010179	10102910	10181110	10236700	10289896	10427796	10484796	10559511
360	10010182	10104669	10183969	10239749	10292945	10434845	10491845	10567171
370	10010185	10106428	10186828	10242798	10295994	10441894	10498894	10574831
380	10010188	10108187	10189687	10245847	10299043	10448943	10505943	10582491
390	10010191	10109946	10192546	10248896	10302092	10455992	10512992	10590151
400	10010194	10111705	10195405	10251945	10305141	10463041	10520041	10597811
410	10010197	10113464	10198264	10254994	10308190	10470090	10527090	10605471
420	10010200	10115223	10201123	10258043	10311239	10477139	10534139	10613131
430	10010203	10116982	10203982	10261092	10314288	10484188	10541188	10620791
440	10010206	10118741	10206841	10264141	10317337	10491237	10548237	10628451
450	10010209	10120499	10209700	10267190	10320386	10498286	10555286	10636111
460	10010212	10122258	10212559	10270239	10323435	10505335	10562335	10643771
470	10010215	10124017	10215418	10273288	10326484	10512384	10569384	10651431
480	10010218	10125776	10218277	10276337	10329533	10519433	10576433	10659091
490	10010221	10127535	10221136	10279386	10332582	10526482	10583482	10666751
500	10010224	10129294	10223995	10282435	10335631	10533531	10590531	10674411

Fig. 7 First page of Bürgi’s Progreß Tabulen as reproduced in [10], [28] from the Danzig Copy. Most illegible digits are no better in the original. The Munich Copy (see Section 5) is in much better condition. E.g., the illegible entry for 102.0 (3rd entry in the 3rd column) clearly reads as 1.010 25168 (= 1.0001¹⁰²) in the Munich Copy. The last entry of the page, 1.040 80816, is erroneous by a transmission error. The (supposedly identical) leading entry of the following page, $a_{400} = 1.040 80869$, is entirely correct.

ceivable, however, that Kepler and Dithmarus refer to the use of trigonometric identities such as

$$\cos(x) \cdot \cos(y) = \frac{1}{2} (\cos(x + y) + \cos(x - y))$$

for reducing multiplications to additions and table-look-ups. According to many documents, this technique, referred to as *prosthapharesis* (“auxiliary separation”), was quite common among human calculators in the 16th century, [10], [16]. In any case, it seems

	228000	228500	229000	229500	230000	230000
0	977556601	982456378	987380714	992319732	997258750	997303557
10	...654356	...554623	...475452	...425965	...100000	...403287
20	...752122	...652879	...578200	...528208	...100000	...503027
30	...849897	...751144	...676958	...627461	...100000	...602778
40	...947682	...849419	...775726	...726724	...470000	...702538
50	978045477	...547704	...874503	...825996	...500000	...802308
60	...143281	983045999	...973291	...925279	...100000	...902088
70	...241096	...144304	985072088	993024572	...700000	998001879
80	...338920	...241618	...170895	...123874	...300000	...101679
90	...436754	...340942	...269712	...223187	...100000	...201489
100	...534597	...439276	...368539	...322509	230100000	...301309
110	...632451	...537620	...467376	...421841	...100000	...401139
120	...730314	...635974	...566223	...521183	...100000	...500979
130	...828187	...734338	...665080	...620535	...300000	...600829
140	...926050	...832711	...763946	...719898	...400000	...700690
150	979023962	...931094	...862822	...819269	...500000	...800560
160	...121865	984029488	...961709	...918651	...600000	...900440
170	...219777	...121865	986060605	994018043	...700000	999000330
180	...317699	...220303	...159511	...117445	...300000	...100230
190	...415631	...314726	...258427	...216857	...900000	...200140
200	...513571	...413158	...357353	...316278	230200000	...300060
210	...611524	...511601	...456288	...415710	...100000	...399990
220	...709485	...620053	...555234	...515152	...200000	...499930
230	...807456	...718515	...654190	...614603	...300000	...599880
240	...905437	...816987	...753155	...714065	...400000	...699840
250	980003427	...915468	...852130	...813536	...500000	...799810
260	...101427	985013360	...951115	...913017	...600000	...899790
270	...99438	...112461	990050111	995012509	230200000	999999779
280	...297457	...210973	...149116	...112010	...100000	...100000
290	...395487	...309464	...248130	...211521	...200000	...200000
300	...493527	...408025	...347155	...311042	...300000	...300000
310	...591576	...506565	...446190	...410573	230270022	999999999
320	...689635	...605116	...545235	...510115	...400000	...400000
330	...787704	...703677	...644289	...609666	...500000	...500000
340	...885783	...802247	...743353	...709227	...600000	...600000
350	...983872	...900827	...842428	...808797	...700000	...700000
360	981081970	...99417	...941512	...908378	...800000	...800000
370	...180078	986094017	991040606	996007969	...900000	...900000
380	...278156	...186827	...139711	...107570	...100000	...100000
390	...376324	...295247	...238825	...207181	...200000	...200000
400	...474462	...393876	...337948	...306801	...300000	...300000
410	...572609	...492516	...437082	...406432	...400000	...400000
420	...670766	...591165	...536226	...506073	...500000	...500000
430	...768934	...689824	...635350	...605723	...600000	...600000
440	...867110	...788493	...734543	...705384	...700000	...700000
450	...965297	...887172	...833717	...805054	...800000	...800000
460	982063494	...985861	...932900	...904735	...900000	...900000
470	...161700	987084559	992031093	997004425	...1000000	...1000000
480	...259916	...183268	...131296	...104126	...1000000	...1000000
490	...358142	...281986	...230510	...203836	...1000000	...1000000
500	...456378	...380714	...329732	...303557	...1000000	...1000000

Die Enden sich die
 zwei Summen
 in 9. Ziffern/
 ist die Note
 230270022-
 230270023+
 Die Schwärze
 aber ist ganz mit 9.
 nellen als 1000000000
 und so dieselben ganz
 gen Zahlen/nicht genug
 geben mögen / so mag
 man dieselben 2.3.4.
 5.6.7.8.9. zusammen
 addieren.

Fig. 8 Last page (p. 58) of Bürgi's Progreß Tabulen reproduced in black and white from the Munich Copy [3] (Astronomisch-Physikalisches Kabinett). The last column is supplemented by an additional column of "red numbers", indicating a gradual decrease of the table interval in order to interpolate to the final value of 10.0000 0000 (or 9.9999 9999). The logarithm N of 10 is stated in the final comment as lying in the interval $23027.0022 < N < 23027.0023$, cf. Equ. (7).

possible for a single human calculator to generate the Bürgi table within a few months, as will be explained in Section 5.



Fig. 9 Copper plate on the title page of John Napier's table of log-sin values, Edinburgh 1614, from [19], p. 28. An English version, [22], of the Latin original appeared in 1616.

While Bürgi could well have been a daily user of his own tables, similar ideas developed on the other side of the channel and approached their completion. In 1614, John Napier published tables of log-sin values, combining his concept of logarithms with trigonometric functions, [21] (Fig. 9). Clearly, the generation of these tables was more involved than Bürgi's tabulation of an exponential function; it took Napier more than 20 years of tedious calculations. In his tables Napier managed to grasp many advanced aspects of the modern natural logarithms. In this respect he was ahead of his contemporaries, but with the laborious computations he paid a high price for it. Neither Napier's nor Bürgi's table was free of errors.

Both authors saw simplification of multiplications and divisions as their main goal. Bürgi's table was designed for general arithmetics, whereas Napier's was specialized for trigonometric calculations needed in astronomy. The choice of the basis (Bürgi: 1.0001, Napier: the reciprocal of the Euler number, $e^{-1} = 0.367879\dots$) is irrelevant for this goal. Since applications involve reading the tables in both directions, Napier's approach (equal steps in the angles, tabulation of a logarithm function) is no better and no worse than Bürgi's (equal steps in the logarithms, tabulation of an exponential function).

Some authors, e.g., Lutstorf [16], [17], Wolf [32], [33] suspect that Bürgi's tables might have been operational earlier than Napier's. In this article, we see the inventor/discoverer of logarithms as the one who first made a practical table based on the exponents property, for the purpose of simplifying multiplications. Denis Roegel [24], on the other hand, considers the invention of the abstract notion of logarithm, independently of the construction of tables, and therefore does not come to the same conclusion. Almost certainly, neither of the human calculators knew about the work of the other. It seems to be fair, therefore, to consider Jost Bürgi and John Napier as the two simultaneous and independent discoverers of the logarithms, see also Clark and Montelle [5], Weiss [31], and Gronau [11].

5 Mathematical aspects of Bürgi's Progreß Tabulen

With Bürgi's choice $B = 1.0001$ according to (5) as the base, the obvious algorithm for generating the table is

$$a_0 = 1, \quad a_{n+1} = B \cdot a_n = a_n + \frac{a_n}{10000}, \quad n = 0, 1, \dots, 23027, \quad (8)$$

as follows from (6). The single step is as simple as it could possibly be: in order to calculate the next table entry, augment the current one by its 10000th part (right-shift by 4 digits). Just do this 23000 times, and you're done.

5.1 Checks

Leading a formidable task like this to a reliable result necessarily requires careful checks for computational errors. The multiplication rule for powers, Equ. (3), provides a simple and effective tool for detecting computational errors.

Bürgi must have taken care of this perfectly; otherwise the "whole red number" N of Equ. (7) could not have been correctly determined to lie in the interval $23027.0022 < N < 23027.0023$ (Fig. 8). Since Bürgi's method of checking is not known, we may ponder about possible checks.

Assume that the final entry of the first page, $a_{400} = 1.04080869$, has been confirmed, e.g., by checking $a_{200} = a_{100}^2$, $a_{400} = a_{200}^2$, etc., or, as we would do it now, by the binomial formula

$$(1 + \varepsilon)^{400} = \sum_{k=0}^K \binom{400}{k} \varepsilon^k + R_K, \quad \varepsilon = 10^{-4}.$$

With $K = 5$, the result $a_{400} = 1.04080869271$ (all 12 digits correct) may be obtained with negligible computational effort, using paper and pencil only.

Then, every subsequent page can be checked by one long-hand multiplication as follows. Let a_i be the initial entry of any page. Then the final entry, $a_f = a_{i+400}$, must satisfy $a_f = a_i \cdot a_{400}$. If the check fails, one needs to locate the error by testing $a_{i+d} = a_i \cdot a_d$ for some values $d < 400$, e.g., for $d = 200, 100$, etc., and redo (part of) the page.

Another possibility is to take advantage of the entry

$$a_{431} = 1.044040044101,$$

a rather unlikely curiosity. Checking the relation $a_{n+431} = 1.044040044101 \cdot a_n$ may be done by one long-hand multiplication by a_{431} which reduces to one quadrupling and a sum of 6 quickly decreasing terms.

5.2 Guard Digits

Besides erroneous calculation, the slow accumulation of round-off errors may be a problem. Here the only remedy is introducing additional digits of precision, called *guard digits*.

Example:

	2 guard digits	no guard digits
$a_{12345} =$	$\begin{array}{r} 3.4364\ 4765\ 40 \\ \quad 3\ 4364\ 48 \\ \hline \end{array}$	$\begin{array}{r} 3.4364\ 4765 \\ \quad 3\ 4364 \\ \hline \end{array}$
$a_{12346} =$	$\begin{array}{r} 3.4367\ 9129\ 88 \\ \hline \end{array}$	$\begin{array}{r} 3.4367\ 9129 \\ \hline \end{array}$

It is seen that guard digits must be carried along in order to avoid accumulation of round-off errors. Without this precaution (e.g., by strictly rounding the 8th digits after the decimal point) the erroneously rounded value 3.4367 9129 would have been found for a_{12346} from the correct value for a_{12345} . Fig. 10 shows the accumulated round-off error if the table is computed according to (8) using g guard digits.

We first define our notion of guard digits and describe the modification of the algorithm (8) for simulating rounded calculation with g guard digits. Then, conclusions on the number of guard digits necessary for guaranteeing an accurate table will be drawn.

As before, we denote the exact powers of B by $a_n := B^n$; furthermore, \tilde{a}_n is the correctly rounded table entry satisfying $|\tilde{a}_n - a_n| \leq \frac{1}{2} \delta$, $\delta = 10^{-8}$. Recall that Bürgi tabulated the integer values $f \tilde{a}_n$ with $f = 1/\delta = 10^8$.

Attaching g guard digits to the table entry \tilde{a}_n increases its resolution to $1/(f G)$ where $G := 10^g$, resulting in the new approximation

$$A_n = \tilde{a}_n + \frac{1}{f G} c, \quad |c| \leq \frac{G}{2},$$

where c stands for the (positive or negative) integer formed by the guard digits. Now the algorithm (8) becomes

$$f G A_{n+1} = \text{round}(1.0001 f G A_n), \tag{9}$$

where $\text{round}(x)$ stands for the integer closest to x . The rounded table values are then obtained as $\tilde{a}_n = \text{round}(A_n f) / f$.

In the above form the algorithm is not restricted to natural numbers g ; any value of $G > 1$ is allowed. In Fig. 10 we have used $G = 1, 3, 10, 30, 100, 300$ (approximately) corresponding to $g = 0, 0.5, 1, 1.5, 2, 2.5$ guard digits. E.g., 1.5 guard digits means that the

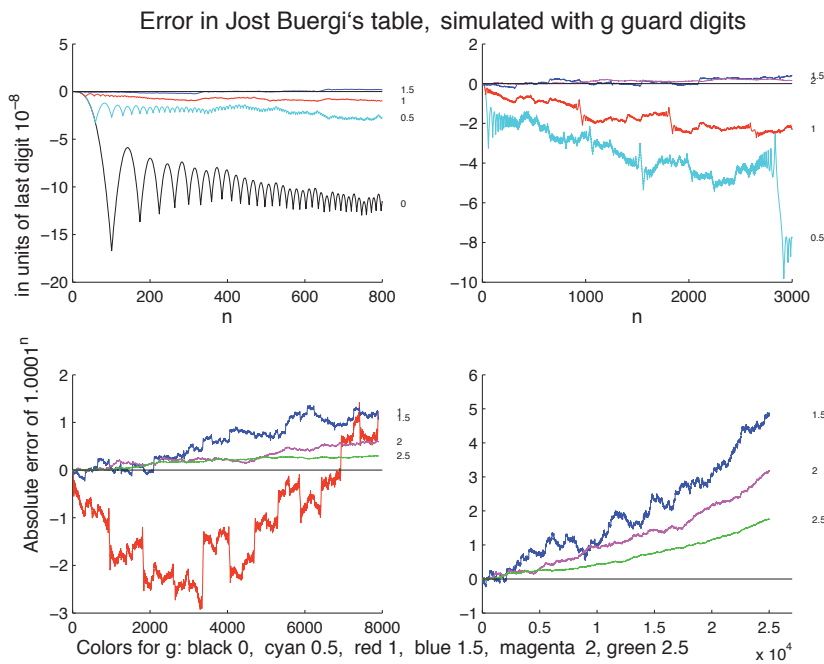


Fig. 10 Plots of $A_n - a_n$ in units of the least significant digit, 10^{-8} , in four intervals of n , as an illustration of the effect of g guard digits. Colours for g : black 0, cyan 0.5, red 1, blue 1.5, magenta 2, green 2.5.

values of A_n are always rounded to a precision of $\frac{1}{30}$, not practical for hand calculations, though.

It is seen that guard digits are an absolute necessity: $g = 0$ results in the loss of 1 digit already after 100 steps, just as the theory of random walks predicts. If checks are made after every page, 1 guard digit suffices in the lower part of the table. However, since in the upper part of the table the accuracy requirements are up to 10-fold, two guard digits are needed there in order to guarantee 9-digit accuracy. Two guard digits suffice to keep the accumulated round-off error below 3 units of the last digit. To guarantee an accurate table free of errors occasional checks according to Section 5.1 must be carried out.

5.3 Table errors

In 1976 the Astronomisch-Physikalisches Kabinett in Munich allowed W. Pohl [23] to copy their Progreß Tabula entirely. Based on this excellent material the Bürgi table was analyzed subsequently, see the Internal Note [29].

First, it was established that the table contains no systematic errors by checking the “whole Red Number” N , see Equ. (7). Also, the numbers given on the title page (Fig. 6), were found to be correct in every digit, except for the entries at 12000.0, 16000.0, 19000.0 which have a round-off error of slightly more than a half-unit of the least significant digit.

Assuming that the table contains no large systematic errors, the actually given (and possibly erroneous) table values \tilde{a}_n may be reconstructed from the index n and the terminal digit (TD). In this way it was possible to check the entire table by computer after having keyed in the 23028 terminal digits, which is the workload of about a week, including attempts to decypher barely legible digits.

The following counts of table errors $\Delta_n := 10^8(\tilde{a}_n - B^n)$ were obtained:

Type		counts	%
0	correctly rounded ($ \Delta_n < 0.5$)	21081	91.54
1	TD rounded to wrong side ($0.5 \leq \Delta_n < 1$)	1679	7.29
2	TD erroneous ($ \Delta_n \geq 1$)	87	.38
3	TD illegible	181	.79
		23028	100

The distribution of the 87 large errors (Type 2) is shown in the upper histogram of Fig. 11. Clearly, large errors are more abundant in the upper part of the table, where its relative accuracy is higher (up to 1 more digit of relative accuracy). The distribution of the 181 illegible terminal digits (Type 3) in the lower histogram of Fig. 11 is more or less uniform.

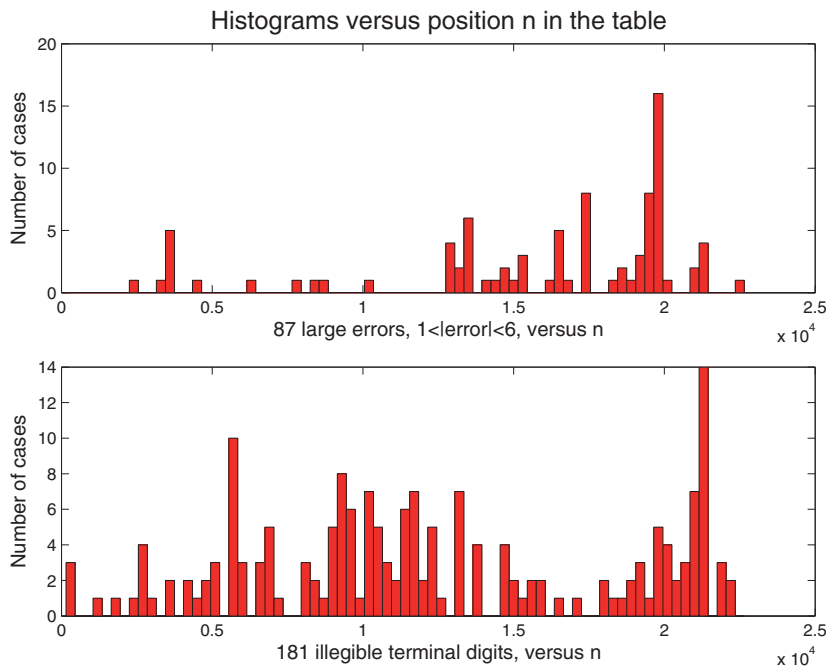


Fig. 11 Histograms of table errors (in units of the least significant digit) versus position n .
 Upper plot: 87 large errors, Type 2, 1 to 6 units.
 Lower plot: 181 illegible terminal digits, Type 3.

Among the 87 large errors (Type 2) about 25 seem to be errors of transcription (such as $n = 12870$: $\tilde{a}_n = 3.621671\mathbf{84}$ instead of the correct value $B_n = 3.621671\mathbf{48}$), see Appendix A. For the other 62 cases we have $1 \leq |\Delta_n| \leq 2.64$. The longest interval of systematically erroneous values is $19890 \leq n \leq 19904$ with $1.73 \leq \Delta_n \leq 2.64$ (Appendix B). For the 22759 errors of Types 0 and 1 the histogram given below is also plotted in Fig. 12.

Δ_n	-1	-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1	0
cases	10	8	14	45	195	1145	2036	2231	2158	2343	
	272					9913					
Δ_n	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
cases	2314	2343	2263	2153	2095	1126	196	48	19	17	
	11168					1406					

The bias of the small rounding errors towards the positive side seen in Fig. 12 may be a consequence of permanent upwards rounding of the single guard digit 5. This hypothesis may be corroborated by simulations of various rounding algorithms for generating Bürgi's table. The third histogram of Fig. 13, generated by permanent upwards rounding of the single guard digit 5, is an ideal form of the histogram of Bürgi's rounding errors (Fig. 12). In contrast, the symmetric rounding (half-integers to nearest even) produces the symmetric fourth histogram of Fig. 13.

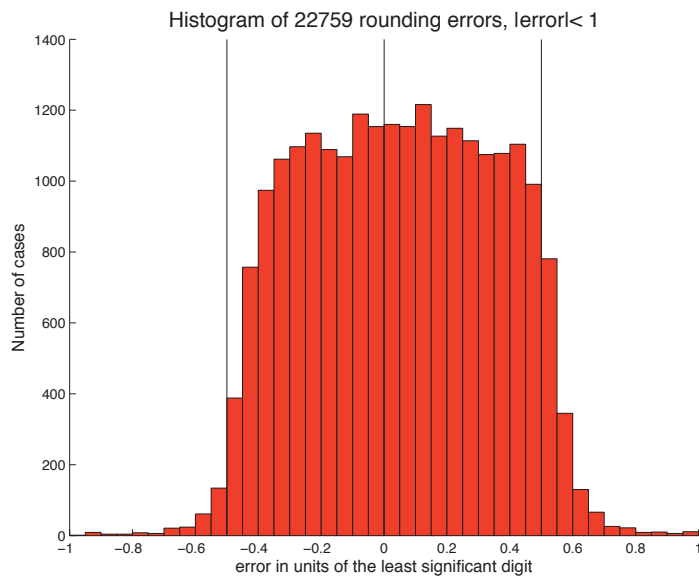


Fig. 12 Histogram of the 22759 rounding errors of Types 0 and 1.

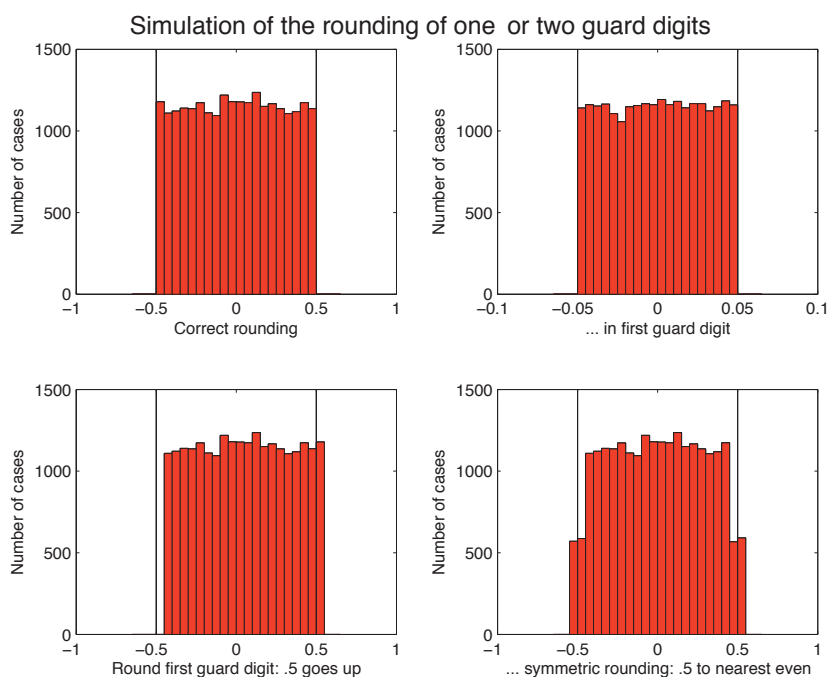


Fig. 13 Histograms of the errors in four simulated rounding models for the entire table (rounding to 9 significant digits).

- (1) Direct rounding of the exact values.
- (2) Rounding in the first guard digit, absolute errors < 0.05 .
- (3) Rounding of the first guard digit: 0.5 goes up.
- (4) Symmetric rounding of first guard digit: 0.5 to nearest even.

In Appendix B the only three sequences of consecutive seriously erroneous entries are listed. The comment “correct transition” expresses the hypothesis that the transition from \tilde{a}_{n-1} to $\tilde{a}_n = \text{round}(1.0001 \cdot \tilde{a}_{n-1})$ was done correctly; presumably, these errors were caused by erroneous transmissions. Appendix C lists a few sequences of consecutive roundings to the wrong side.

5.4 Linear interpolation

Linear interpolation was suggested by Bürgi himself, and the “user’s manual” that was to go with the table does in fact contain instructions for accurate or approximate linear interpolation. Unfortunately, those instructions, announced on the title page (Fig. 6) by the words *sambt gründlichem Unterricht*, “with thorough instruction”, have not been published with the table. Only much later, Gieswald rediscovered their manuscript in the archives of Danzig und published it in 1856 [10], [16].

Fig. 14 shows an excerpt from Bürgi’s instructions as reproduced in [18]. In the example Bürgi explains how to find the accurate red number (in modern language: the logarithm to

Kurzer Bericht der Progress Tabulen, Wie dieselbigen nützlich in allerley Rechnungen zu gebrauchen.

II. Mann soll zum Exempel die wahre rothe Zahl von 36 suchen, so setzet man noch Sieben 0 für, damit ich 9 Biffern bekomme, denn alle schwarze Zahlen haben in unser Tabula nicht weniger als 360000000 Darnach sucht man in der Tabul unter den schwarzen Zahl Die 2 nächst kleiner und nächst größer ist dann 360000000 bis finde ich am 33 blat in der columna 12500 und auf der linken seite, nun felt mir die schwarze als 360000000 zwischen

9 dieße hat schwarz 359964763 dieße ist zu klein
 10 die Differenz 35996 die Differenz
 dieße hat schwarz 36000759 dieß ist zu groß
 dieße kleinere Zahl von 359964763 Subtrahire
 von meiner gegebenen Zahl 360000000

000035237				
Wie sich helt die	Differenz	zu der	rothen	also helt sich die 3 zur 4
	35996		12500	35237 als 9759

Dieß Viert Vierte addier zu der kleinen rothen Zahl

Die kleine rothe Zahl ist 90

Die Zahl der columna 12500

Dieß ist der Schwarzen Zahl von 360000000 ihr rote 122099789

Es sol gleichwol so verstand worden 36 haben ihr rothe 125000 $\frac{78}{100}$

und werden alle Zeit bis unter die 9 ganze verstanden und die folgen der Bruch. 67

Fig. 14 In the example Bürgi explains backwards reading of the table, i.e., given a black number a_n , find the corresponding red number $n = \log_{1.0001} a_n = \log(a_n)/\log(1.0001)$. In particular $a_n = 3.6$ yields $n = 12809.9789\ 1087$. Bürgi gives 9 correct digits (third line from bottom).

base 1.0001) of a given number a_n ,

$$n = \log_{1.0001}(a_n).$$

He therefore shows how to cheaply find logarithms from his table of an exponential function.

For generally discussing linear interpolation in this context, let x be a real variable, and consider the exponential function

$$f(x) := B^x \text{ with } B = 1 + \varepsilon, \quad \varepsilon = 0.0001 \quad (10)$$

in the range $0 < x < \log 10/\log(1 + \varepsilon)$. Since the table interval is 1, define $n := \text{floor}(x)$, $t := x - n$. Then linear interpolation in the interval $n \leq x \leq n + 1$ yields the approximation

$$f^* = t f(n + 1) + (1 - t) f(n) \quad (11)$$

of $f(x)$. Due to the convex curvature of the graph of f the relative interpolation error is nonnegative and roughly bounded by $\varepsilon^2/8$,

$$0 \leq (f^* - f(x))/f(x) \leq \varepsilon^2/8. \quad (12)$$

This implies $0 \leq f^* - f(x) \leq 1.25 \cdot 10^{-8}$ if $1 \leq f(x) \leq 10$.

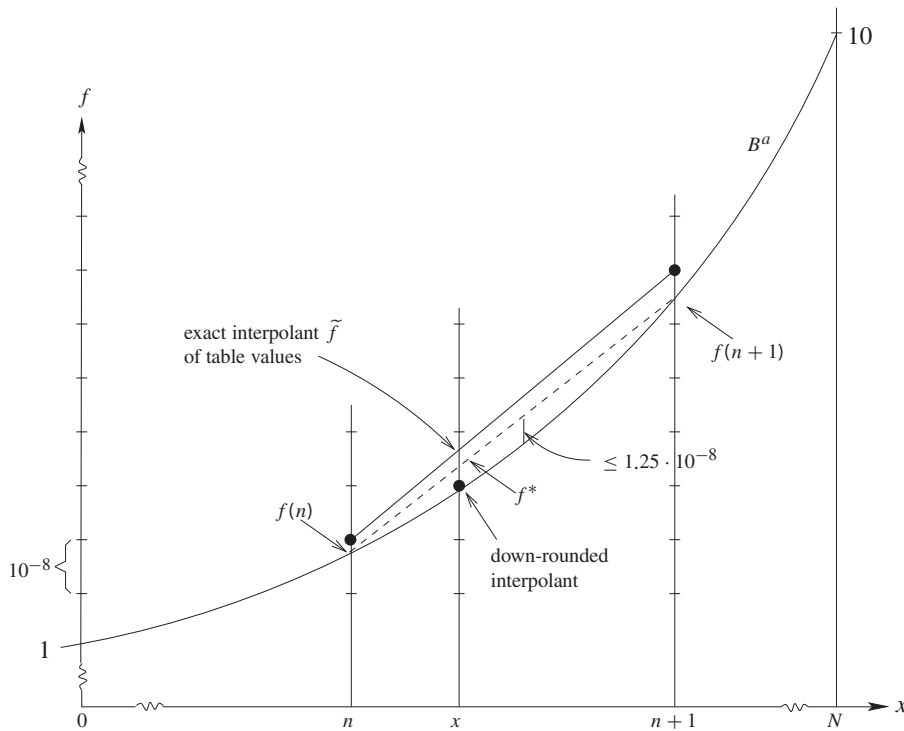


Fig. 15 Linear interpolation.

Solid curve: graph of $f(x) := B^x, 0 \leq x \leq N$ (Equ. (10)).

Dashed line: linear interpolation polynomial of exact function values for $n \leq x \leq n + 1$, with exact interpolant \tilde{f} .

Black dots: rounded table entries to be interpolated.

Solid line: linear interpolation polynomial of rounded table entries. The down-rounded approximation of \tilde{f} is also marked by a black dot.

The conclusion is that Bürgi’s choice of tabulating the values $f(n)$ with 9 significant digits is optimal in the sense that the error due to linear interpolation is never more than 1.25 units at the least significant digit. This error bound is reached near the upper end of the table, whereas near the lower end of the table the error is bounded by $0.125 \cdot 10^{-8}$.

The linear interpolation process of the exponential function (10) in the interval $n \leq x \leq n + 1$ is sketched in Fig. 15. The tickmarks on the vertical lines indicate the grid of the table entries. \tilde{f} denotes the exact interpolant of the rounded table entries. In the upper half of the table the positive bias of the interpolation error may almost be compensated by always rounding \tilde{f} downwards.

This is exemplified in Fig. 16, where we plot the interpolation error in the interval $23000 \leq x \leq 23025$ near the upper end of the table. The correctly rounded table values of $f(x)$ were used as input data. Three rounding strategies were used: (1) \tilde{f} exact (black), (2) \tilde{f} rounded to nearest integer (cyan), (3) \tilde{f} down-rounded (red). In the upper half of the table Strategy 3 is preferable (still with a slightly positive bias), in the lower half Strategy 2 is better.

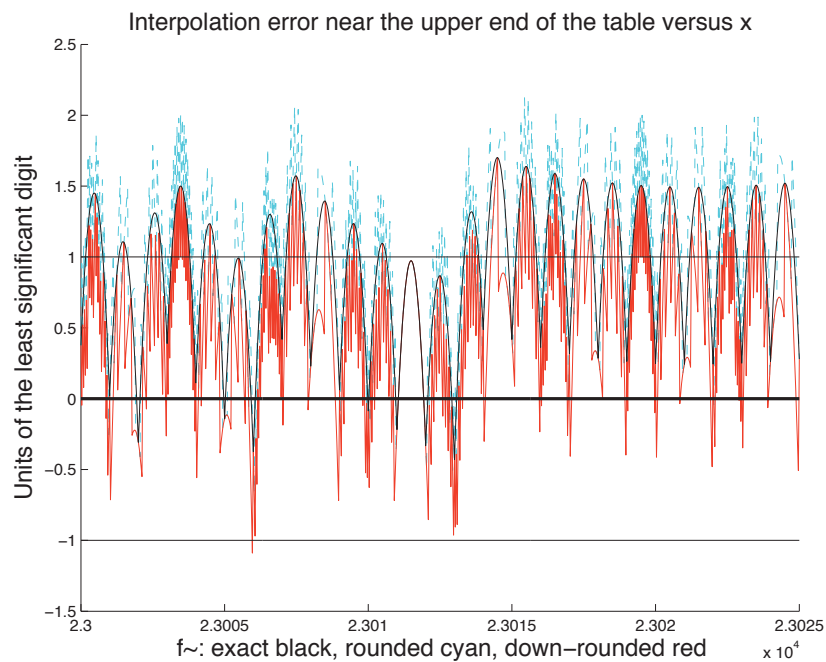


Fig. 16 Interpolation error near the upper end of the table versus x . Black line (Strategy 1): error of the interpolant \tilde{f} of the values $f(n) = \text{round}(B^n)$. Cyan/Red (Strategies 2/3): errors of correctly rounded/down-rounded interpolants of rounded table entries.

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Appendix A: Summary of isolated large errors

25 computational errors or transcription errors with $|\text{error}| > 10^{-8}$
 $a_n = 1.0001^n$, “Bürgi”: 4 terminal digits, “Bü-ex”: units of last digit

n	$\text{ex} = 10^8 \cdot a_n$	Bürgi	Bü-ex	error	comments
2363	126653925.6	3929	3.38	6 → 9	last digit upside down
4531	157314585.9	4583	-2.92	6 → 3	3 clear
6299	187736370.3	6375	4.70	0 → 5	5 clear but distorted
8795	240958860.1	8867	-6.90		previous entry 240934767
12869	362130935.0	0938	3.02	5 → 8	8 clear
12870	362167148.1	7184		48 → 84	two digits interchanged
12952	365148978.4	8975	-3.40	8 → 5	5 clear
13017	367530057.8	0053	-4.83	8 → 3	3 clear
13335	379404726.6	4724	-2.57	7 → 4	4 clear
13516	386334127.6	4125	-2.57	8 → 5	5 clear
14422	422968657.6	8460	2.40	658 → 460	460 clear
14950	445900325.2	5740			previous entry 445855740
15198	457096350.4	6354	3.60	50.4 → 54	4 clear, guard digit?
16350	512903870.9	3817		71 → 17	two digits interchanged
18504	636177538.0	7535	-2.98	8 → 5	5 clear
18713	649612887.1	2883	-4.14	8 → 3	3 clear
18925	663530995.1	0999	3.85	5 → 9	9 clear
19414	696782369.0	2362	-7.04	9 → 2	2 clear
19464	700274830.1	4839	8.86	30 → 39	39 clear
19574	708019986.3	9988	1.71	6 → 8	8 clear
19822	725797523.4	7537		23 → 37	37 clear
20997	816286346.3	6341	-5.30	6 → 1	1 clear
21255	837619481.6	9480	-1.56	2 → 0	0 clear
21389	848918552.4	8554	1.62	2 → 4	4 clear
22544	952849980.9	9976	-4.87	81 → 76	76 clear

Appendix B: Sequences of consecutive seriously erroneous entries

3 sequences with 29 serious errors, $|\text{error}| > 10^{-8}$
 $a_n = 1.0001^n$, “Bürigi”: 4 terminal digits, “Bü-ex”: units of last digit

n	ex = $10^8 \cdot a_n$	Bürigi	Bü-ex	error	comments
16387	514805035.1	5034	-1.14		1 unit low
16388	514856515.6	6516	0.35		correctly rounded
16389	514908001.3	7981	-20.30	8001→7981	20 units low
16390	514959492.1	9392	-100.10	9492→9392	100 units low
16391	515010988.0	8888		0888→8888	after transition
16392	515062489.1	0390			correct transition
16393	515113995.4	1897			correct transition
16394	515165506.8	3409			correct transition
16395	515217023.3	7023	-0.35		recovered
16396	515268545.0	8545	-0.05		recovered
16397	515320071.9	0702	0.10	0072→0702	digits 0,7 interchanged
19245	685106271.7	6272	0.30		correctly rounded
19246	685174782.3	4790	7.67	82→90	8 units high
19247	685243299.8	3308	8.19	00→08	correct transition
19248	685311824.1	1832	7.86	24→32	correct transition
19249	685380355.3	0365	9.68	0363→0365	after correct transition
19250	685448893.3	8893	-0.36		recovered, 3 distorted
19251	685517438.2	7438	-0.25		recovered, corr.rounded
19887	730530335.7	0335	-0.68		1 unit low
19888	730603388.7	3388	-0.72		1 unit low
19889	730676449.1	6449	-0.06		correctly rounded
19890	730749516.7	9519	2.30		2 units high
19891	730822591.7	2594	2.35		2 units high
19892	730895673.9	5676	2.09		2 units high
19893	730968763.5	8766	2.52		3 units high
19894	731041860.4	1863	2.64		3 units high
19895	731114964.5	4967	2.46		2 units high
19896	731188076.0	8078	1.96		2 units high
19897	731261194.8	1197	2.15		2 units high
19898	731334321.0	4323	2.03		2 units high
19899	731407454.4	7457	2.60		3 units high
19900	731480595.1	0597	1.86		2 units high
19901	731553743.2	3745	1.80		2 units high
19902	731626898.6	6901	2.42		2 units high
19903	731700061.3	0063	1.73		2 units high
19904	731773231.3	3233	1.73		2 units high
19905	731846408.6	6409	0.40		recovered, corr.rounded

Appendix C: Sequences of consecutive erroneous roundings

5 sequences with 37 harmless errors, $|\text{error}| < 1.6 \cdot 10^{-8}$
 $a_n = 1.0001^n$, “Bürgi”: 4 terminal digits, “Bü-ex”: units of last digit

n	$\text{ex} = 10^8 \cdot a_n$	Bürgi	Bü-ex	error	comments
3501	141918462.1	8463	0.89		1 unit high
3502	141932654.0	2655	1.05		1 unit high
3503	141946847.2	6848	0.78		1 unit high
3504	141961041.9	1043	1.10		1 unit high
3505	141975238.0	5239	0.99		1 unit high
3506	141989435.5	9437	1.47		1 unit high
3507	142003634.5	3635	0.53		1 unit high
3508	142017834.8	7836	1.16		1 unit high
3509	142032036.6	2038	1.38		1 unit high
3510	142046239.8	6240	0.18		recovered
13415	382451984.8	1984	-0.80		1 unit low
13416	382490230.0	0229	-1.00		1 unit low
13417	382528479.0	8478	-1.02		1 unit low
13418	382566731.9	6731	-0.87		1 unit low
13419	382604988.5	4988	-0.55		1 unit low
13420	382643249.0	3248	-1.04		1 unit low
13421	382681513.4	1512	-1.37		1 unit low
13422	382719781.5	9781	-0.52		1 unit low
13423	382758053.5	8053	-0.50		correctly rounded
13424	382796329.3	6328	-1.30		1 unit low
14786	438647557.8	7559	1.25		1 unit high
14787	438691422.5	1423	0.49		correctly rounded
14788	438735291.7	7835293	1.35	87→78	digits 8,7 interchanged
14789	438779165.2	9166	0.82		1 unit high
17389	569058503.4	8505	1.59		1 unit high
17390	569115409.3	5410	0.74		1 unit high
17391	569172320.8	2322	1.20		1 unit high
17392	569229238.0	9239	0.97		1 unit high
17393	569286161.0	6162	1.04		1 unit high
17394	569343089.6	3091	1.43		1 unit high
17395	569400023.9	0025	1.12		1 unit high
17396	569456963.9	6965	1.12		1 unit high
17397	569513909.6	3911	1.42		1 unit high
17398	569570861.0	0862	1.03		1 unit high
17399	569627818.1	7819	0.94		1 unit high
19525	704559346.9	9348	1.13		1 unit high
19526	704629802.8	9804	1.19		1 unit high
19527	704700265.8	0267	1.21		1 unit high
19528	704770735.8	0737	1.19		1 unit high
19529	704841212.9	1214	1.11		1 unit high

References

- [1] Bramer, Benjamin, 1648. *Bericht zu M. Jobsten Bürgi seligen geometrischen Triangular Instruments*. Gedruckt zu Cassel bey Jacob Gentsch. In Verlegung Johan. Schützen.
- [2] Bürgi, Jost, 1620. *Arithmetische und Geometrische Progreß-Tabulen*. Paul Sessen, Universitätsbuchdruckerei, Prag, 59 pp.
- [3] Bürgi, Jost, 1620. *Arithmetische und Geometrische Progreß-Tabulen*. Signature 1603/R 1620-001 in <http://www.ub.uni-muenchen.de/>.
- [4] Bürgi, Jost, 1620. *Arithmetische und Geometrische Progreß-Tabulen*. Hauptbibliothek SOSA, Guldin-Bibliothek, Signature I 18601 in <http://www.yorick.uni-graz.at:8991/F?RN=160090835>.
- [5] Clark, Kathleen M. and Montelle, Clemency, 2012. *Priority, parallel discovery, and pre-eminence. Napier, Bürgi and the early history of the logarithm relation*. *Revue d'Histoire des Mathématiques* **18**, 2, 223–270.
- [6] Drach, Alhard Carl von, 1884. *Jost Bürgi – Kammeruhrmacher Kaiser Rudolfs II. Beiträge zu seiner Lebensgeschichte und Nachrichten über Arbeiten desselben*. Jahrbuch der kunsthistorischen Sammlungen des allerhöchsten Kaiserhauses, **15**, 15–44.
- [7] Faustmann, Gerlinde, 1995. *Die Geschichte der Logarithmen*. Didaktikhefte der Österreichischen Mathematischen Gesellschaft **23**, 36–57.
- [8] Faustmann, Gerlinde, 1997. *Jost Bürgis Progress Tabulen und die Entwicklung der Logarithmen*. Acta Historiae Rerum Naturalium, New Series Vol. I, Prag.
- [9] Faustmann, Gerlinde, 1999. *Jost Bürgi. Rechenbücher und mathematische Texte der frühen Neuzeit*. Schriften des Adam-Ries-Bundes, Annaberg-Buchholz-Band **11**.
- [10] Gieswald, Hermann Robert, 1856. *Justus Byrg als Mathematiker, und dessen Einleitung zu seinen Logarithmen*. Programm der St. Johannisschule in Danzig.
- [11] Gronau Detlef, 1996. *The logarithm – From calculation to functional equations*. Notices of the South African Math. Soc. **28**, 60–66.
- [12] Joss, Heinz, 1998. *Messrechnen: 350 Jahre Rechenschieber*. *Elem. Math.*, **53**, 73–78.
- [13] Kepler, Johannes, 1603. *Arithmetica Bürgii*. Einschließlich Faksimile-Seiten Keplers und Bürgis der Pulkower Handschrift. In: List, Martha und Bialas, Volker, 1973. *Die Coss von Jost Bürgi in der Redaktion von Johannes Kepler. Ein Beitrag zur frühen Algebra*. Bayerische Akademie der Wissenschaften, Band V. Neue Folge *Nova Kepleriana*, Heft 154, Verlag C.H. Beck, München.
- [14] Leopold, John H., 1986. *Astronomen, Sterne, Geräte – Landgraf Wilhelm IV und seine sich selbst bewegendes Globen*. Edition Joseph Freimersdorf, Luzern, 239 pp.
- [15] Leopold, John H. and Pechstein, K., 1977. *Der kleine Himmelsglobus (1594) von Jost Bürgi*. Edition Joseph Freimersdorf, Luzern, 122 pp.
- [16] Lutstorf, Heinz and Walter, Max, 1992. *Jost Bürgis Progreß Tabulen, nachgerechnet und kommentiert*. Schriftenreihe der ETH-Bibliothek **28**, Zürich, 40 + 60 pp.
- [17] Lutstorf, Heinz, 2002. *Die Logarithmentafeln Jost Bürgis*. ETH-Bibliothek, Zürich, 153 pp.
- [18] Lutstorf, Heinz, 2005. *Die Logarithmentafeln Jost Bürgis. Bemerkungen zur Stellenwert- und Basisfrage, mit Kommentar zu Bürgis "Gründlichem Unterrichts"*. Schriftenreihe A, **3** (Wissenschaftsgeschichte) der ETH-Bibliothek, Zürich.
- [19] Mackensen, Ludolf von, 1979. *Die erste Sternwarte Europas mit ihren Instrumenten und Uhren. 400 Jahre Jost Bürgi in Kassel*. Callwey-Verlag, München, 158 pp.
- [20] Mautz, O., 1918/19 *Zur Basisbestimmung der Napierschen und Bürgischen Logarithmen*. Beilage zu den Jahresberichten des Gymnasiums, der Realschule und der Töchterschule, Basel.
- [21] Napier, John, 1614. *Mirifici Logarithmorum Canonis descriptio*. Andrew Hart, Edinburgh.
- [22] Napier, John, 1964. *A description of the admirable table of logarithms*. London, 1616, Reprint by Da Capo Press (a division of Plenum Publ. Corp.), Amsterdam, New York.

-
- [23] Pohl, Wolfhard, 1960. *Von Zeit und Zeitmessung in alter und neuer Zeit*. Schweiz. Techn. Zeitschrift STZ **28**, 557–566; **29**, 577–587.
- [24] Roegel, Denis, 2010. *Bürgi's Progress Tabulen (1620): logarithmic tables without logarithms*. Locomat Project Loria: <http://www.locomat.loria.fr> (Nancy 7.2.2013).
- [25] Schöbi-Fink, Philipp, 2008. *Georg Joachim Rheticus aus Feldkirch und Jost Bürgi, ein Genie aus dem Toggenburg*. Vierteljahresschrift der Rheticus-Gesellschaft **30**, 4, 5–20.
- [26] Staudacher, Fritz, 2013. *Jost Bürgi, Johannes Kepler und der Kaiser*. Verlag NZZ Libro, Zürich, 293 pp.
- [27] Stifel, Michael, 1544. *Arithmetica Integra*. Johannes Petreius, Nürnberg.
- [28] Voellmy, Erwin, 1948. *Jost Bürgi und die Logarithmen*. Elemente der Mathematik, Beiheft Nr. 5, Kurze Mathematiker-Biographie. Birkhäuser Verlag, Basel, Reprint 1974, 24 pp.
- [29] Waldvogel, Jörg, 1976. *Jost Bürgi's logarithms*. ETH, Applied Mathematics, Internal Note, 1976. Comments to the ICM94 watch, International Congress of Mathematicians, Zürich, 1994.
- [30] Waldvogel, Jörg, 1998. *Jost Bürgi, a Swiss Discoverer of the Logarithms*. In: H. Joss (ed.), Slide Rule '98 Proceedings, Desk Top Publishing, Dällikon/Zürich, 15–22.
- [31] Weiss, Stephan, 2010/2013. *Anmerkungen zur Idee der historischen Logarithmen*. <http://www.mechrech.info>.
- [32] Wolf, Rudolf, 1858. *Jost Bürgi von Lichtensteig*. In: Biographien zur Kulturgeschichte der Schweiz. Orell Füssli, Zürich, 57–80.
- [33] Wolf, Rudolf, 1872. *Johannes Kepler und Jost Bürgi*. Rathausvortrag 4.1.1872, Verlag Friedrich Schult-hess, Zürich, 30 pp.

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