



Uniform regularity results for critical and subcritical surface energies

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Received: 15 April 2018 / Accepted: 26 October 2018 / Published online: 30 November 2018
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Abstract

We establish regularity results for critical points to energies of immersed surfaces depending on the first and the second fundamental form exclusively. These results hold for a large class of intrinsic elliptic Lagrangians which are sub-critical or critical. They are derived using uniform ϵ -regularity estimates which do not degenerate as the Lagrangians approach the critical regime given by the Willmore integrand.

Mathematics Subject Classification 49Q10 · 53A30 · 53A05 · 58E15 · 35J35 · 35J48

1 Introduction and main results

Let $\vec{\Phi} : \Sigma \rightarrow \mathbb{R}^3$ denote the immersion of an oriented two-dimensional closed surface Σ into \mathbb{R}^3 . We denote by A and H the corresponding second fundamental form and mean curvature. In this paper, we study (weak) critical points of functionals of the types

$$\mathcal{W}_F := \int_{\Sigma} F(H^2) \, d\text{vol}_g \quad \text{and} \quad \mathcal{E}_F := \int_{\Sigma} F(|A|^2) \, d\text{vol}_g, \quad (1.1)$$

where $g := \vec{\Phi}^* g_{\mathbb{R}^3}$ is the pull-back of the Euclidean metric onto Σ . Here, F is a smooth function whose properties will be made more precise in due time. We shall denote by \vec{n} the unit Gauss map given by $\vec{\Phi}$ and the orientation on Σ . With this notation, one has

$$\forall X, Y \in T_x \Sigma \quad A_x(X, Y) = -(d\vec{n}_x X, \vec{Y})_{g_{\mathbb{R}^3}}$$

where $\vec{Y} := \vec{\Phi}_* Y$ and hence

$$|A|^2 = |d\vec{n}|_g^2.$$

Communicated by A. Malchiodi.

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When $F(H^2) = H^2$, the functional \mathcal{W}_F is the Willmore energy, which has generated much interest for more than two centuries. The critical points of \mathcal{W}_H are known as Willmore surfaces. The Willmore energy is invariant under the action of the non-compact Möbius group of conformal transformations \mathbb{R}^3 . As such, it does not satisfy the Palais–Smale condition and its critical points are subject to concentration compactness phenomena. For these reasons, there is some interest of approximating the Willmore energy by more coercive energies of the form \mathcal{W}_F or \mathcal{E}_F where $t \ll F(t)$ for $t \gg 1$ (for example $F(t) = (1 + t)^q$ for $q > 1$).

In [18], the second author introduced the notion of weak $W^{2,p}$ immersion for any $p \geq 2$. Let g_0 be any smooth reference metric on Σ , and set

$$W_{imm}^{2,p}(\Sigma, \mathbb{R}^3) := \left\{ \begin{array}{l} \vec{\Phi} \in W^{1,\infty} \cap W^{2,p}(\Sigma, \mathbb{R}^3) \quad ; \text{ denoting } g_{\vec{\Phi}}(X, Y) := (\vec{\Phi}_* X, \vec{\Phi}_* Y)_{g_{\mathbb{R}^3}} \\ \exists C_{\vec{\Phi}} > 1 \quad \text{s. t. } C_{\vec{\Phi}}^{-1} g_0 \leq g_{\vec{\Phi}} \leq C_{\vec{\Phi}} g_0 \end{array} \right\}$$

The first part of the paper has to do with regularity properties of functionals of the form \mathcal{W}_F . The first result of this paper gives the regularity of the critical points of \mathcal{W}_F under some assumptions on F .

Theorem 1.1 *Assume that $F : [0, \infty) \rightarrow [0, \infty)$ is a C^1 function satisfying¹:*

- (a) $C^{-1} t^p \leq F(t^2) \leq C(1 + t^p)$,
- (b) $tF'(t^2)$ is a smooth and invertible function of t ,
- (c) $C^{-1} t^{p-1} \leq tF'(t^2) \leq C(1 + t^{p-1})$,

where $C > 1$ and $p \geq 2$. Then any critical point $\vec{\Phi} \in W^{2,p}(\Sigma, \mathbb{R}^3)$ of

$$\mathcal{W}_F(\vec{\Phi}) = \int_{\Sigma} F(H^2) \, dvol_g$$

is smooth. If one drops assumption (b) one has at least that $H \in L^\infty$. □

A proof of this result has been established in the critical case $F(s) = s$ in [17] while a Proof of Theorem 1.1 can be found in [11] in the subcritical case for $F(s) = (1 + s)^q$ with $q > 1$.

The investigation of sub-critical curvature functionals involving L^p norms of the mean curvature (resp. of the second fundamental forms) with p larger than the dimension has already been addressed in the past. In [12] a compactness result for immersions of surfaces in the Euclidean space \mathbb{R}^3 with $L^p - > 2$ —second fundamental form is proved. In [4] the aforementioned result of Langer is generalized to any dimension and codimension. In [9] functionals of the type $\int |A|^p$ or $\int |H|^p$ are studied in the framework of varifolds. In [14] the minimizations of the functionals $\int |A|^p$ and $\int |H|^p$ are performed in the sub-critical regime $p > m$ among m -dimensional integral varifolds without boundary in Riemannian manifolds of arbitrary codimension. Finally, let us mention that, a Proof of Theorem 1.1 has been established in the critical case $F(s) = s$ in [17] while a Proof of Theorem 1.1 can be found in [11] in the subcritical case for $F(s) = (1 + s)^q$ with $q > 1$.

In order to establish Theorem 1.1 we are adopting the parametric approach of [17] which is based on the local existence of isothermal coordinates (a fact which holds “uniformly” for $p \geq 2$). In contrast, the proof in [11] is based on the fact that, for $p > 2$, any weak $W^{2,p}$ immersion is obviously locally a graph. This fails for $p = 2$, and the proof in [11] “blows-up” as $p \rightarrow 2$.

¹ $F'(s)$ is the derivative of $F(s)$ with respect to s .

Our next goal is to merge the case $p = 2$ with the case $p > 2$ in a single proof by establishing estimates which remain controlled as $p \rightarrow 2$. This is realized in the following result which is an ε -regularity independent of $p \in [2, q]$ for any $q > 2$.

Theorem 1.2 *Let $q \in (2, \infty)$. For any $m > 0$, there exists a constant $\varepsilon_0(m) > 0$ with the following property. If a conformal immersion $\vec{\Phi} \in W^{2,p}(D_2(0), \mathbb{R}^3)$ with $p \in [2, q]$ with conformal factor λ is a critical point of*

$$\mathcal{W}_p := \int (1 + |H|^2)^{p/2} dvol_g \quad \text{with} \quad w_p := \int_{D_2(0)} (1 + |H|^2)^{p/2} dvol_g < \infty$$

satisfying

$$m := \|\nabla\lambda\|_{L^{2,\infty}(D_2(0))} < \infty \quad \text{and} \quad \|\nabla\vec{n}\|_{L^2(D_2(0))} < \varepsilon_0(m),$$

then

$$\|H^p e^{2\lambda}\|_{L^\infty(D_1(0))} \leq C(m, w_p, q) \int_{D_2(0)} (1 + |H|^2)^{p/2} dvol_g,$$

for some positive constant $C(m, w_p, q)$ which is uniformly bounded as m, w_p and q are uniformly bounded. □

Remark 1.1 A uniform ε -regularity for relaxations of the Willmore energy has been first established by the second author in [19] for perturbations of the form

$$W_\sigma(\vec{\Phi}) := \int_\Sigma |\vec{H}_{\vec{\Phi}}|^2 dvol_{g_{\vec{\Phi}}} + \sigma^2 \int_\Sigma (1 + |\vec{H}_{\vec{\Phi}}|^2)^2 dvol_{g_{\vec{\Phi}}}$$

□

Combining Theorem 1.2 with the main weak compactness result of [18] gives the following corollary which can be seen as a ‘‘Sacks–Uhlenbeck type’’ concentration compactness counterpart theorem for weak immersions which is one of the main contribution of the present work

Corollary 1.1 *Let $\alpha_k > 0$ and $\alpha_k \rightarrow 0$ and let $\vec{\Phi}_k$ be a sequence of weak critical points of*

$$\mathcal{W}^k(\vec{\Phi}) := \int_\Sigma (1 + H^2)^{1+\alpha_k} dvol_g$$

Assuming

$$\limsup_{k \rightarrow +\infty} \mathcal{W}^k(\vec{\Phi}_k) < +\infty,$$

and assuming that the sequence of Riemann surfaces associated to $(\Sigma, g_{\vec{\Phi}_k})$ is pre-compact in the moduli space of Σ , then, there exists a subsequence k' , finitely many points $a_1 \dots a_Q \in \Sigma$ a sequence $\vec{p}_{k'} \in \mathbb{R}^3$, a sequence μ_k converging either to $+\infty$ or identically equal to 0 and a map $\vec{\Psi}_\infty \in C^\infty(\Sigma, \mathbb{R}^3)$ such that

$$e^{\mu_k} (\vec{\Phi}_{k'} - \vec{p}_{k'}) \longrightarrow \vec{\Psi}_\infty \quad \text{in } C^l_{loc}(\Sigma \setminus \{a_1 \dots a_Q\}, \mathbb{R}^3) \quad \forall l \in \mathbb{N}$$

moreover, either $\vec{\Psi}_\infty$ is a constant map or it is a smooth immersion of $\Sigma \setminus \{a_1 \dots a_Q\}$ critical point respectively of

$$\mathcal{W}(\vec{\Psi}) := \int_\Sigma (1 + H^2) dvol_{g_{\vec{\Phi}}} \quad \text{in case } \mu_k \equiv 0.$$

or

$$W(\vec{\Phi}) := \int_{\Sigma} H^2 \, d\text{vol}_{g_{\vec{\Phi}}} \quad \text{in case } \mu_k \rightarrow +\infty.$$

Moreover, if $\vec{\Psi}_{\infty}(\Sigma \setminus \{a_1 \dots a_Q\})$ is compact, then $\vec{\Psi}_{\infty}$ is a critical point respectively of \mathcal{W} or W . □

Remark 1.2 If one skips the assumption that $(\Sigma, g_{\vec{\Phi}_k})$ is pre-compact in the moduli space of Σ a similar result holds true on each “thick part” of the limiting nodal surface. This is obtained combining Theorem 1.2 with theorem 0.3 of [13]. □

Functionals of the type \mathcal{E}_p are more complicated. To make it short the variations of \mathcal{E}_p generate p -harmonic operator while the variations of \mathcal{W}_p are generating standard Laplacians. When $F(|A|^2) = (1+|A|^2)^{p/2}$, it is shown in [11] that critical points of $\int F \, d\text{vol}_g$ are smooth. In this paper, beyond regularity matters, we focus on estimates that remain stable in the limit $p \searrow 2$. In particular, we prove the following ε -regularity result.

Theorem 1.3 *There exist constants $\delta > 0$ and for any $m > 0$ there exists $\varepsilon_0(m) > 0$ with the following property. If a conformal immersion $\vec{\Phi} \in W^{2,p}(D_1(0), \mathbb{R}^3)$ for $p \in [2, 2 + \delta]$ with conformal factor λ is a critical point of*

$$\mathcal{E}_p(\vec{\Phi}) := \int_{D_1(0)} (1 + |d\vec{n}|^2)^{p/2} \, d\text{vol}_g \quad \text{with} \quad e_p := \int_{D_1(0)} |d\vec{n}|_g^p \, d\text{vol}_g < +\infty$$

satisfying

$$m := \|\nabla \lambda\|_{L^{2,\infty}(D_1(0))} < \infty \quad \text{and} \quad \|\nabla \vec{n}\|_{L^2(D_1(0))} < \varepsilon_0(m),$$

then there exists $q > 2 + \delta$ such that

$$\left\| |\nabla \vec{n}| e^{\lambda(2-p)/p} \right\|_{L^q(D_{1/2}(0))} \leq C(m, e_p) e_p^{1/p}. \tag{1.2}$$

for some positive constant $C(m, e_p)$. □

Combining Theorem 1.3 this time with the main weak compactness result of [18] gives the following concentration compactness theorem for weak immersions which is one of the main contribution of the present work.

Corollary 1.2 *Let $\alpha_k > 0$ and $\alpha_k \rightarrow 0$ and let $\vec{\Phi}_k$ be a sequence of weak critical points of*

$$\mathcal{E}^k(\vec{\Phi}) := \int_{\Sigma} (1 + |d\vec{n}|^2)^{1+\alpha_k} \, d\text{vol}_{g_{\vec{\Phi}}}$$

Assuming

$$\limsup_{k \rightarrow +\infty} \mathcal{E}^k(\vec{\Phi}_k) < +\infty,$$

and assuming that the sequence of Riemann surfaces associated to $(\Sigma, g_{\vec{\Phi}_k})$ is pre-compact in the moduli space of Σ , then, there exists a subsequence k' , finitely many points $a_1 \dots a_Q \in \Sigma$, a sequence $\vec{p}_{k'} \in \mathbb{R}^3$, a sequence μ_k converging either to $+\infty$ or identically equal to 0 and a map $\vec{\Psi}_{\infty} \in C^{\infty}(\Sigma, \mathbb{R}^3)$ such that

$$e^{\mu_k} (\vec{\Phi}_{k'} - \vec{p}_{k'}) \longrightarrow \vec{\Psi}_{\infty} \quad \text{in } C^1_{loc}(\Sigma \setminus \{a_1 \dots a_Q\}, \mathbb{R}^3).$$

Moreover, either $\vec{\Psi}_{\infty}$ is a constant map or it is a smooth immersion of $\Sigma \setminus \{a_1 \dots a_Q\}$ critical point respectively of

$$\mathcal{W}(\vec{\Psi}) := \int_{\Sigma} (1 + H^2) \, d\text{vol}_{g_{\vec{\Phi}}} \quad \text{in case } \mu_k \equiv 0.$$

or

$$W(\vec{\Phi}) := \int_{\Sigma} H^2 \, d\text{vol}_{g_{\vec{\Phi}}} \quad \text{in case } \mu_k \rightarrow +\infty.$$

Moreover, if $\vec{\Psi}_{\infty}(\Sigma \setminus \{a_1 \dots a_Q\})$ is compact, then $\vec{\Psi}_{\infty}$ is a critical point respectively of \mathcal{W} or W . □

Remark 1.3 Here again, if one skips the assumption that $(\Sigma, g_{\vec{\Phi}_k})$ is pre-compact in the moduli space of Σ a similar result holds true on each “thick part” of the limiting nodal surface. This is obtained combining Theorem 1.3 with theorem 0.3 of [13]. □

As a matter of clarity the whole work is dealing with immersions of surfaces into the 3 dimensional euclidian space. Using the computations performed in [15] the result above should be extendable with moderate efforts (but using some more heavy notations) to general codimension and general riemannian targets.

2 Preliminaries

2.1 First variations of \mathcal{W}_F and \mathcal{E}_F

In local coordinates $\{x_1, x_2\}$ on the unit disk, let the pull-back metric by $\vec{\Phi}$ be $g_{ij} := \partial_{x_i} \vec{\Phi} \cdot \partial_{x_j} \vec{\Phi}$. As usual, we let $|g|$ denote the determinant of the matrix (g_{ij}) . We let \vec{n} denote the outward-unit normal vector. We will suppose that $\vec{\Phi}$ is conformal, namely, in local coordinates $\{x_1, x_2\}$, we have $\partial_{x_i} \vec{\Phi} \cdot \partial_{x_j} \vec{\Phi} = e^{2\lambda} \delta_{ij}$. In order to derive the Euler–Lagrange equation associated with the energies \mathcal{W}_F and \mathcal{E}_F , we consider a variation of the type

$$\vec{\Phi}_t := \vec{\Phi} + t\vec{w} + o(t),$$

where \vec{w} is a normal vector. The variation of the outward unit normal vector \vec{n} is easily found to be

$$\vec{n}_t = \vec{n} + te^{-\lambda}(a_1 \partial_{x_1} \vec{\Phi} + a_2 \partial_{x_2} \vec{\Phi}) + o(t),$$

for some a_1 and a_2 . One easily verifies that

$$\left. \frac{d}{dt} \right|_{t=0} \vec{n}_t = -\langle \vec{n} \cdot d\vec{w}, d\vec{\Phi} \rangle_g. \tag{2.1}$$

The variation of the components of the metric $(g_t)_{ij} := \partial_{x_i} \vec{\Phi}_t \cdot \partial_{x_j} \vec{\Phi}_t$ is also easily found to be

$$\left. \frac{d}{dt} \right|_{t=0} (g_t)_{ij} = \partial_{x_i} \vec{\Phi} \cdot \partial_{x_j} \vec{w} + \partial_{x_j} \vec{\Phi} \cdot \partial_{x_i} \vec{w}. \tag{2.2}$$

From this and the fact that $(g_t)_{ki}(g_t)^{ij} = \delta_k^j$, it follows that the inverse metric coefficients vary according to

$$\left. \frac{d}{dt} \right|_{t=0} g_t^{ij} = -e^{-4\lambda}(\partial_{x_i} \vec{\Phi} \cdot \partial_{x_j} \vec{w} + \partial_{x_j} \vec{\Phi} \cdot \partial_{x_i} \vec{w}). \tag{2.3}$$

Note that the variation of the volume form is

$$\frac{d}{dt} \Big|_{t=0} d\text{vol}_{g_t} = \langle d\vec{\Phi}; d\vec{w} \rangle_g d\text{vol}_g. \tag{2.4}$$

The variations of the mean curvature and of the square of the second fundamental form satisfy respectively

$$\frac{d}{dt} \Big|_{t=0} H_t^2 = H(\langle d\vec{n}; d\vec{w} \rangle_g - d^{*g}(\vec{n} \cdot d\vec{w})) \tag{2.5}$$

and

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} |d\vec{n}_t|_g^2 &\equiv \frac{d}{dt} \Big|_{t=0} (g_t^{ij} \partial_{x_i} \vec{n}_t \cdot \partial_{x_j} \vec{n}_t) \\ &= -2\langle d\vec{\Phi} \dot{\otimes} d\vec{w}, d\vec{n} \dot{\otimes} d\vec{n} \rangle_g - 2\langle d\vec{n} \cdot d\vec{w}, d\vec{\Phi} \rangle_g; d\vec{n} \Big|_g, \end{aligned} \tag{2.6}$$

where we have used (2.3) and (2.1). Combining (2.4) and (2.5) gives

$$\frac{d}{dt} \Big|_{t=0} \mathcal{W}_F = \int_{\Sigma} \left[F(H^2) \langle d\vec{\Phi}; d\vec{w} \rangle_g + HF'(H^2) (\langle d\vec{n}; d\vec{w} \rangle_g - d^{*g}(\vec{n} \cdot d\vec{w})) \right] d\text{vol}_g.$$

where $F'(H^2)$ is understood as $dF(H^2)/dH^2$.

It is a simple matter to integrate the latter by parts. Equating the resulting integral to zero and recalling that the obtained identity holds for all \vec{w} , one concludes that a critical point of \mathcal{W}_F must satisfy the Euler–Lagrange equation

$$d^{*g} \left[Fd\vec{\Phi} + HF'd\vec{n} - \vec{n} d(HF') \right] = 0. \tag{2.7}$$

Combining now (2.4) and (2.6) and using the fact that $|A|^2 = |d\vec{n}|_g^2$ gives

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} \mathcal{E}_F &= \int_{\Sigma} \left[F \langle d\vec{\Phi}; d\vec{w} \rangle_g - 2F'[\langle d\vec{\Phi} \dot{\otimes} d\vec{w}, d\vec{n} \dot{\otimes} d\vec{n} \rangle_g \right. \\ &\quad \left. + \langle d\vec{n} \cdot d\vec{w}, d\vec{\Phi} \rangle_g; d\vec{n} \Big|_g \right] d\text{vol}_g, \end{aligned} \tag{2.8}$$

where $F' := dF(|A|^2)/d|A|^2$.

It is a simple matter to integrate the latter by parts. Equating the resulting integral to zero and recalling that the obtained identity holds for all \vec{w} , one concludes that a critical point of \mathcal{E}_F must satisfy the Euler–Lagrange equation

$$d^{*g} \left[Fd\vec{\Phi} - 2F'(d\vec{n} \dot{\otimes} d\vec{n}) \lrcorner_g d\vec{\Phi} + 2(d^{*g} F'd\vec{n}) \cdot d\vec{\Phi} \right] \vec{n} = 0, \tag{2.9}$$

where $(d\vec{n} \dot{\otimes} d\vec{n}) \lrcorner_g d\vec{\Phi}$ is the contraction given in local conformal coordinates by

$$(d\vec{n} \dot{\otimes} d\vec{n}) \lrcorner_g d\vec{\Phi} = e^{-2\lambda} \sum_{i,j=1}^2 (\partial_{x_i} \vec{n} \cdot \partial_{x_j} \vec{n}) \partial_{x_j} \vec{\Phi} dx_i.$$

2.2 Conservation laws for critical points of \mathcal{W}_F

Proposition 2.1 *In a conformal chart, a critical point of \mathcal{W}_F , respectively \mathcal{E}_F , satisfies the Euler–Lagrange Eq. (2.7), which reads*

$$\text{div} \left[F(H^2) \nabla \vec{\Phi} + HF'(H^2) \nabla \vec{n} - \vec{n} \nabla (HF'(H^2)) \right] = 0. \tag{2.10}$$

Respectively for E_F the Eq. (2.9) which reads

$$\operatorname{div} \left[F \nabla \vec{\Phi} - 2 e^{-2\lambda} F' \sum_{j=1}^2 (\nabla \vec{n} \cdot \partial_{x_j} \vec{n}) \partial_{x_j} \vec{\Phi} + 2 e^{-2\lambda} \vec{n} (\operatorname{div}(F' \nabla \vec{n}) \cdot \nabla \vec{\Phi}) \right] = 0. \tag{2.11}$$

□

We shall now first concentrate on critical points of \mathcal{W}_F . The divergence form (2.10) may be locally integrated to yield the existence of a potential function \vec{L} satisfying

$$\nabla^\perp \vec{L} = H F'(H^2) \nabla \vec{n} - \vec{n} \nabla(H F'(H^2)) + F(H^2) \nabla \vec{\Phi}, \tag{2.12}$$

where, in terms of the local coordinates $\{x_1, x_2\}$, we have set $\nabla^\perp := (-\partial_{x_2}, \partial_{x_1})$.

Note that

$$\nabla \vec{n} \cdot \nabla \vec{\Phi} = -\vec{n} \cdot \Delta \vec{\Phi} = -2e^{2\lambda} H.$$

Hence (2.12) yields

$$\nabla \vec{\Phi} \cdot \nabla^\perp \vec{L} = 2 e^{2\lambda} (F - H^2 F').$$

For the sake of our future needs, let Y be a solution of

$$-\Delta Y = 2e^{2\lambda} (F - H^2 F'). \tag{2.13}$$

Equation (2.13) states that

$$\operatorname{div}(\vec{L} \cdot \nabla^\perp \vec{\Phi} - \nabla Y) = 0.$$

This identity is integrated to give the existence of a function S satisfying

$$\nabla S = \vec{L} \cdot \nabla \vec{\Phi} + \nabla^\perp Y. \tag{2.14}$$

Next, using that $\nabla \vec{\Phi} \times \nabla \vec{n} = \operatorname{div}(\nabla \vec{\Phi} \times \vec{n}) = \operatorname{div} \nabla^\perp \vec{\Phi} = 0$, we obtain from (2.12) that

$$\nabla \vec{\Phi} \times \nabla^\perp \vec{L} = -\nabla^\perp \vec{\Phi} \cdot \nabla(H F') = -\operatorname{div}(H F' \nabla^\perp \vec{\Phi}),$$

which is an exact divergence equation and may thus be integrated to give the existence of a potential function \vec{V} satisfying

$$\nabla \vec{V} = \vec{L} \times \nabla \vec{\Phi} + H F' \nabla \vec{\Phi}. \tag{2.15}$$

We can summarize what we have established so far in the following proposition.

Proposition 2.2 *Let $\vec{\Phi}$ be a weak critical point of*

$$\mathcal{W}_F(\vec{\Phi}) := \int_{\Sigma} F(H^2) \, d\operatorname{vol}_g.$$

In local conformal coordinate we introduce² \vec{L} such that

$$\nabla^\perp \vec{L} = H F'(H^2) \nabla \vec{n} - \vec{n} \nabla(H F'(H^2)) + F(H^2) \nabla \vec{\Phi},$$

Then the following two identities hold

$$\begin{cases} \nabla \vec{\Phi} \cdot \nabla^\perp \vec{L} = 2 e^{2\lambda} (F(H^2) - H^2 F'(H^2)) \\ \nabla \vec{\Phi} \times \nabla^\perp \vec{L} = -\operatorname{div}(H F'(H^2) \nabla^\perp \vec{\Phi}), \end{cases} \tag{2.16}$$

where $e^\lambda := |\partial_{x_1} \vec{\Phi}| = |\partial_{x_2} \vec{\Phi}|$.

□

² The local existence of \vec{L} is given by Proposition 2.1.

We now derive central identities linking together the potentials S and \vec{V} . Note first that

$$\vec{n} \cdot \nabla \vec{V} = \vec{n} \cdot (\vec{L} \times \nabla \vec{\Phi}) = \vec{L} \cdot (\nabla \vec{\Phi} \times \vec{n}) = \vec{L} \cdot \nabla^\perp \vec{\Phi} = \nabla^\perp S + \nabla Y. \tag{2.17}$$

The tangential and normal parts of $\nabla \vec{V}$ are

$$\pi_T \nabla \vec{V} = -(\vec{L} \cdot \vec{n}) \nabla^\perp \vec{\Phi} + HF' \nabla \vec{\Phi} \quad \text{and} \quad \pi_{\vec{n}} \nabla \vec{V} = (\nabla^\perp S + \nabla Y) \vec{n}.$$

Hence

$$\begin{aligned} \vec{n} \times \nabla \vec{V} &\equiv \vec{n} \times \pi_T \nabla \vec{V} = -(\vec{L} \cdot \vec{n}) \nabla \vec{\Phi} - HF' \nabla^\perp \vec{\Phi} \\ &= -\pi_T \nabla^\perp \vec{V} = -\nabla^\perp \vec{V} - (\nabla S - \nabla^\perp Y) \vec{n}. \end{aligned} \tag{2.18}$$

We formally decompose à la Hodge the quantity

$$\vec{n} \nabla^\perp Y = \nabla \vec{v} + \nabla^\perp \vec{u}, \tag{2.19}$$

and set $\vec{R} := \vec{V} - \vec{u}$ into (2.17) and into (2.18) to discover the following proposition

Proposition 2.3 *With the previous notations, the following equation hold*

$$\vec{n} \times \nabla \vec{R} = -\nabla^\perp \vec{R} - \vec{n} \nabla S - \vec{n} \times \nabla^\perp \vec{v} + \nabla \vec{v}. \tag{2.20}$$

□

2.3 Control of the conformal factor

Using F. Hélein’s method of moving Coulomb frames [16], a weak immersion $\vec{\Phi} \in W_{imm}^{2,2}(D_2(0), \mathbb{R}^3)$ of the unit disk $D_2(0)$ into \mathbb{R}^3 can be re-parametrized by a diffeomorphism of $D_2(0)$ to become conformal. Our functional being independent of parametrization, we will without loss of generality suppose that $\vec{\Phi}$ is conformal with parameter λ , namely:

$$\partial_{x_i} \vec{\Phi} \cdot \partial_{x_j} \vec{\Phi} = e^{2\lambda} \delta_{ij}.$$

We will henceforth use the notation ∇ , div , and Δ to denote the usual gradient, divergence, and Laplacian operators in flat local coordinates $\{x_1, x_2\}$.

Assume

$$\int_{D_2(0)} |\nabla \vec{n}|^2 dx^2 \leq 8\pi/3 \quad \text{and} \quad \|\nabla \lambda\|_{L^2, \infty(D_2(0))} = m < +\infty.$$

We can call upon Lemma 5.1.4 in [8] to deduce the existence of an orthonormal frame $\{\vec{e}_1, \vec{e}_2\} \in W^{1,2}(D_1(0))$ satisfying $\vec{n} = \vec{e}_1 \times \vec{e}_2$ and

$$\|\nabla \vec{e}_1\|_{L^2(D_2(0))} + \|\nabla \vec{e}_2\|_{L^2(D_2(0))} \leq C \|\nabla \vec{n}\|_{L^2(D_2(0))}, \tag{2.21}$$

As is easily verified, the conformal parameter satisfies

$$\Delta \lambda = \nabla \vec{e}_1 \cdot \nabla^\perp \vec{e}_2 \quad \text{in } D_2(0). \tag{2.22}$$

Let μ satisfy

$$\begin{cases} \Delta \mu = \nabla \vec{e}_1 \cdot \nabla^\perp \vec{e}_2, & \text{in } D_2(0) \\ \mu = 0, & \text{on } \partial D_2(0). \end{cases} \tag{2.23}$$

Standard Wente estimates (cf. Theorem 3.4.1 in [8]) give

$$\|\mu\|_{L^\infty(D_2(0))} + \|\nabla\mu\|_{L^2(D_2(0))} \leq \|\nabla\vec{e}_1\|_{L^2(D_2(0))} \|\nabla\vec{e}_2\|_{L^2(D_2(0))} \leq C \|\nabla\vec{n}\|_{L^2(D_2(0))}^2. \tag{2.24}$$

The harmonic function $v := \lambda - \mu$ satisfies the usual estimate

$$\int_{D_{3/2}(0)} |v - \bar{v}| dx^2 \leq C \|\nabla v\|_{L^1(D_2(0))} \leq C \|\nabla v\|_{L^{2,\infty}(D_2(0))},$$

where \bar{v} denotes the average of v on $D_1(0)$. Hence

$$\|v - \bar{v}\|_{L^\infty(D_{3/2}(0))} \leq C \|\nabla v\|_{L^{2,\infty}(D_2(0))},$$

Combining the latter to (2.24) yields now

$$\|\lambda - \bar{\lambda}\|_{L^\infty(D_{3/2}(x))} \leq C \|\nabla\lambda\|_{L^{2,\infty}(D_2(0))} + C \|\nabla\vec{n}\|_{L^2(D_2(0))}^2 \leq C(m),$$

where $\bar{\lambda}$ denotes the average of λ on $D_{3/2}(0)$, and m is the $L^{2,\infty}$ norm of $\nabla\lambda$ on $D_2(0)$, assumed to be finite. We can summarize this subsection by stating the following lemma.

Lemma 2.1 *Let $\vec{\Phi} \in W_{imm}^{2,2}(D_2(0), \mathbb{R}^3)$ be a conformal weak immersion such that*

$$\int_{D_2(0)} |\nabla\vec{n}|^2 dx^2 \leq 8\pi/3 \quad \text{and} \quad \|\nabla\lambda\|_{L^{2,\infty}(D_2(0))} = m < +\infty.$$

where $e^\lambda := |\partial_{x_1}\vec{\Phi}| = |\partial_{x_2}\vec{\Phi}|$. Then the following estimate holds

$$\begin{aligned} \|\nabla\lambda\|_{L^2(D_{3/2}(0))} + \left| \lambda - \frac{1}{|D_{3/2}(0)|} \int_{D_{3/2}(0)} \lambda(x) dx^2 \right| &\leq C \|\nabla\lambda\|_{L^{2,\infty}(D_2(0))} \\ &+ C \|\nabla\vec{n}\|_{L^2(D_2(0))}^2 \leq C(m). \end{aligned} \tag{2.25}$$

□

We shall now prove the following extension to general exponents $p \in [2, +\infty)$.

Lemma 2.2 *Let $2 \leq p < q$ $\vec{\Phi} \in W_{imm}^{2,p}(D_2(0), \mathbb{R}^3)$ be a conformal weak immersion such that*

$$\int_{D_2(0)} |\nabla\vec{n}|^2 dx^2 \leq 8\pi/3 \quad \text{and} \quad \|\nabla\lambda\|_{L^{2,\infty}(D_2(0))} = m < +\infty.$$

where $e^\lambda := |\partial_{x_1}\vec{\Phi}| = |\partial_{x_2}\vec{\Phi}|$. Then the following estimate holds

$$\|\nabla\lambda\|_{L^p(D_{3/2}(0))} \leq C(q) \|\nabla\lambda\|_{L^{2,\infty}(D_2(0))} + C(q) \|\nabla\vec{n}\|_{L^p(D_2(0))}. \tag{2.26}$$

where $C(q) > 0$ is independent of $p \in [2, q]$. □

Proof of Lemma 2.2 The first step consists in constructing an orthonormal frame (\vec{e}_1, \vec{e}_2) in $W^{1,p}(D_2(0))$ satisfying $\vec{n} = \vec{e}_1 \times \vec{e}_2$ and

$$\begin{aligned} \|\nabla\vec{e}_1\|_{L^2(D_2(0))} + \|\nabla\vec{e}_2\|_{L^2(D_2(0))} &\leq C \|\nabla\vec{n}\|_{L^2(D_2(0))} \\ \|\nabla\vec{e}_1\|_{L^p(D_2(0))} + \|\nabla\vec{e}_2\|_{L^p(D_2(0))} &\leq C(q) \|\nabla\vec{n}\|_{L^p(D_2(0))}. \end{aligned} \tag{2.27}$$

We follow step by step the proof of lemma 5.1.4 in [8] where p is replacing 2. The main ingredient is the use of Lemma A.2 instead of theorem 3.1.7 of [8]. Then the whole argument

goes through without any further modification. Once \vec{e}_i satisfying (2.27) is constructed we recall that λ satisfies on $D_2^2(0)$ the equation (2.22). We can then decompose λ into the sum of its harmonic extension λ^0 on D_2^2 and the function μ satisfying (2.22). Applying one more time Lemma A.2 but to μ this time one obtains

$$\begin{aligned} \|\nabla\mu\|_{L^p(D_2(0))} &\leq C(q) \|\nabla\vec{e}_1\|_{L^2(D_2(0))} \|\nabla\vec{e}_2\|_{L^p(D_2(0))} \\ &\leq C(q) \|\nabla\vec{n}\|_{L^2(D_2(0))} \|\nabla\vec{n}\|_{L^p(D_2(0))}. \end{aligned} \tag{2.28}$$

Standard estimates on harmonic functions give also

$$\|\nabla\lambda^0\|_{L^p(D_{3/2}(0))} \leq C(q) \|\nabla\lambda\|_{L^{2,\infty}(D_2(0))} \tag{2.29}$$

Combining (2.28) and (2.29) give (2.26) and Lemma 2.2 is proved. \square

2.4 Controlling the L^p norm of $\nabla\vec{n}$

We shall now prove the following lemma.

Lemma 2.3 *Let $2 < q < +\infty$ and consider $\vec{\Phi} \in W_{imm}^{2,p}(D_2(0), \mathbb{R}^3)$ for some $2 \leq p \leq q$. If*

$$\int_{D_2(0)} |\nabla\vec{n}|^2 dx^2 \leq 8\pi/3 \quad \text{and} \quad \|\nabla\lambda\|_{L^{2,\infty}(D_2(0))} = m < +\infty,$$

where $e^\lambda := |\partial_{x_1}\vec{\Phi}| = |\partial_{x_2}\vec{\Phi}|$. Then, there exists $C(q, m) > 0$ such that

$$\begin{aligned} \int_{D_1(0)} |\nabla\vec{n}|^p dx^2 &\leq C(m, q) \left[\int_{D_{3/2}(0)} e^{2\lambda} dx^2 \right]^{(p-2)/2} \\ &\times \int_{D_{3/2}(0)} |\vec{H}|^p e^{2\lambda} dx^2 + C(m, q). \end{aligned} \tag{2.30}$$

\square

Proof of Lemma 2.3 Denote $\bar{\lambda}$ the average of λ on $D_{3/2}(0)$. Using (2.25) together with the fact that

$$\Delta\vec{\Phi} = 2 e^{2\lambda} \vec{H}$$

we obtain

$$\int_{D_{3/2}(0)} |\Delta\vec{\Phi}|^p dx^2 \leq C(m, q) e^{2\bar{\lambda}(p-1)} \int_{D_{3/2}(0)} |\vec{H}|^p e^{2\lambda} dx^2. \tag{2.31}$$

Using classical elliptic estimates we have

$$\int_{D_1(0)} |\nabla^2\vec{\Phi}|^p dx^2 \leq C(q) \int_{D_{3/2}(0)} |\Delta\vec{\Phi}|^p dx^2 + C(q) \left[\int_{D_{3/2}(0)} |\nabla\vec{\Phi}|^2 dx^2 \right]^{p/2}. \tag{2.32}$$

The following pointwise estimate holds

$$|\nabla\vec{n}|(x) = \left| \nabla \left(\frac{\partial_{x_1}\vec{\Phi} \times \partial_{x_2}\vec{\Phi}}{|\partial_{x_1}\vec{\Phi} \times \partial_{x_2}\vec{\Phi}|} \right) \right|(x) \leq 4 |\nabla^2\vec{\Phi}|(x) e^{-\lambda(x)}. \tag{2.33}$$

This gives obviously using one more time (2.25)

$$\int_{D_1(0)} |\nabla \vec{n}|^p dx^2 \leq C(m, q) e^{-p\bar{\lambda}} \int_{D_1(0)} |\nabla^2 \vec{\Phi}|^p dx^2. \tag{2.34}$$

Combining (2.32) and (2.34) we obtain

$$\begin{aligned} \int_{D_1(0)} |\nabla \vec{n}|^p dx^2 &\leq C(m, q) e^{-p\bar{\lambda}} \int_{D_{3/2}(0)} |\Delta \vec{\Phi}|^p dx^2 \\ &\quad + C(m, q) e^{-p\bar{\lambda}} \left[\int_{D_{3/2}(0)} |\nabla \vec{\Phi}|^2 dx^2 \right]^{p/2} \\ &\leq C(m, q) e^{-p\bar{\lambda}} \int_{D_{3/2}(0)} |\Delta \vec{\Phi}|^p dx^2 + C(m, q) \end{aligned} \tag{2.35}$$

Finally combining (2.31) and (2.35) we obtain

$$\int_{D_1(0)} |\nabla \vec{n}|^p dx^2 \leq C(m, q) e^{\bar{\lambda}(p-2)} \int_{D_{3/2}(0)} |\vec{H}|^p e^{2\lambda} dx^2 + C(m, q). \tag{2.36}$$

This concludes the proof of Lemma 2.3. □

3 Proof of Theorem 1.1

From now on, we take $F(t)$ as in the statement of Theorem 1.1. In this subsection we establish regularity from a qualitative perspective exclusively. We shall post-pone quantitative estimates to the next subsection. We treat the case $p > 2$ in this subsection. For the most delicate case $p = 2$ (already treated in [17]) quantitative estimates as the one derived in the next subsection are needed.

By hypothesis, $\nabla^2 \vec{\Phi}$ lies in L^p , and thus

$$\nabla \vec{n} \in L^p. \tag{3.1}$$

Hence H lie in L^p . From hypothesis (a) on F , we find that $F(H^2)e^{2\lambda}$, and thus $F(H^2)$, are integrable. Per hypothesis (c), it follows that

$$|HF'| \in L^{p'}(D), \quad \text{where} \quad \frac{1}{p'} := 1 - \frac{1}{p}, \tag{3.2}$$

so that

$$H^2 F'(H^2) \in L^p \cdot L^{p'} \in L^1. \tag{3.3}$$

Accordingly, writing (2.12) in the form

$$\nabla^\perp \vec{L} = -\nabla(HF'\vec{n}) + 2HF'\nabla\vec{n} + F\nabla\vec{\Phi}$$

shows that

$$\vec{L} \in L^{p'} + W^{1,1} \subset L^{p'}. \tag{3.4}$$

We also get from (2.13) that ΔY is integrable, so that³

$$\nabla Y \in L^{2,\infty}. \tag{3.5}$$

³ $L^{2,\infty}$ is the weak Marcinkiewicz space. Refer to the Appendix for details. That ∇Y lies in $L^{2,\infty}$ follows from the fact that ΔY lies in L^1 . See Theorem 3.3.6 in [8].

Bringing (3.4) and (3.5) into (2.14) and (4.18) shows that

$$\nabla S, \nabla \vec{V} \in L^{p'} + L^{2,\infty} \subset L^{p'}, \tag{3.6}$$

since $p' < 2$.

From the Hodge decomposition (2.19), we see that

$$\Delta \vec{v} = \nabla^\perp Y \cdot \nabla \vec{n}. \tag{3.7}$$

According to the conditions (3.5) and (3.1), since $p > 2$, by classical rules on products in Lorentz spaces, we have that $\nabla^\perp Y \cdot \nabla \vec{n} \in L^{2p/(p+2),p}$ and by Lorentz–Sobolev embeddings we obtain

$$\nabla \vec{v} \in L^p \quad \text{and} \quad \nabla^2 \vec{v} \in L^r \quad \text{for all } 1 \leq r < \frac{2p}{p+2}, \tag{3.8}$$

which, along with (3.5) and (2.19), gives us

$$\nabla \vec{u} \in L^{2,\infty}. \tag{3.9}$$

As $\vec{R} := \vec{V} - \vec{u}$, we obtain from the latter and (3.6) that

$$\nabla S, \nabla \vec{R} \in L^{p'}. \tag{3.10}$$

Next, we differentiate the first order system (2.20) to obtain the second order system in divergence form

$$\begin{cases} \Delta \vec{R} = \nabla \vec{n} \times \nabla^\perp \vec{R} + \nabla \vec{n} \cdot \nabla^\perp S - \operatorname{div}(\vec{n} \times \nabla \vec{v}) \\ \Delta S = -\nabla \vec{n} \cdot \nabla^\perp \vec{R} + \operatorname{div}(\vec{n} \cdot \nabla \vec{v}). \end{cases} \tag{3.11}$$

We will now call upon another result of integration by compensation (see [5]).

Lemma 3.1 *Let $D := D_r(x)$ be an arbitrary disk in \mathbb{R}^2 . Let $q \in [2, \infty)$, $2 \leq p \leq q$ and let $p' = \frac{p}{p-1}$. Consider two functions $a \in W^{1,p}(D)$ and $b \in W^{1,p'}(D)$. The solution of the problem*

$$\begin{cases} \Delta \varphi = \nabla^\perp a \cdot \nabla b, & \text{in } D \\ \varphi = 0, & \text{on } \partial D, \end{cases}$$

satisfies

$$\|\nabla^2 \varphi\|_{L^1(D)} \leq C_q \|\nabla a\|_{L^p(D)} \|\nabla b\|_{L^{p'}(D)},$$

for some constant $C_q > 0$ depending only on q . □

With this result at our disposal, we can now use the hypotheses (3.1), (3.10) and simple regularity estimates to derive from the system (3.11) that

$$\nabla^2 S, \nabla^2 \vec{R} \in L^1. \tag{3.12}$$

In order to weave this information back to the level of curvature, we need a new identity. On one hand, from (2.19), we find

$$\begin{aligned} \nabla \vec{V} \times \nabla^\perp \vec{\Phi} &= \nabla \vec{R} \times \nabla^\perp \vec{\Phi} + \nabla \vec{u} \times \nabla^\perp \vec{\Phi} = \nabla \vec{R} \times \nabla^\perp \vec{\Phi} + (\nabla \vec{v} - \vec{n} \nabla^\perp Y) \times \nabla \vec{\Phi} \\ &= \nabla \vec{R} \times \nabla^\perp \vec{\Phi} + \nabla \vec{v} \times \nabla \vec{\Phi} + \nabla Y \cdot \nabla \vec{\Phi}. \end{aligned} \tag{3.13}$$

On the other hand, using (2.15) and (2.14), there holds

$$\begin{aligned} \nabla \vec{V} \times \nabla^\perp \vec{\Phi} &= (\vec{L} \times \nabla \vec{\Phi}) \times \nabla^\perp \vec{\Phi} + HF'(H^2) \nabla \vec{\Phi} \times \nabla^\perp \vec{\Phi} \\ &= (\vec{L} \cdot \nabla^\perp \vec{\Phi}) \cdot \nabla \vec{\Phi} - 2e^{2\lambda} HF'(H^2) \vec{n} \\ &= \nabla^\perp S \cdot \nabla \vec{\Phi} + \nabla Y \cdot \nabla \vec{\Phi} - 2e^{2\lambda} HF'(H^2) \vec{n}. \end{aligned}$$

Combining the latter to (3.13) yields the identity

$$- 2e^{2\lambda} HF'(H^2) \vec{n} = \nabla S \cdot \nabla^\perp \vec{\Phi} + \nabla \vec{R} \times \nabla^\perp \vec{\Phi} + \nabla \vec{v} \times \nabla \vec{\Phi}. \tag{3.14}$$

Recall that the regularity of the Gauss curvature is tied to that of the Gauss map, namely, $e^{2\lambda} K$ is as regular as $\nabla \vec{n}$ is. In particular, the Liouville equation

$$- \Delta \lambda = e^{2\lambda} K \tag{3.15}$$

and the fact that $\vec{n} \in W^{1,p}$ show that $\lambda \in W^{2,p} \hookrightarrow L^\infty$, and thus that

$$e^{\pm \lambda} \in W^{2,p} \quad \text{and} \quad \nabla \vec{\Phi} \in W^{2,p}. \tag{3.16}$$

Combining (3.8), (3.12), and (3.16) into (3.15) shows that

$$HF'(H^2) \in W^{1,1} \subset L^2.$$

Since $|HF'(H^2)| \gtrsim |H|^{p-1}$ by hypothesis (c), it follows that

$$H \in L^{2(p-1)}, \tag{3.17}$$

which, because $p > 2$, is an improvement to the original assumption that $H \in L^p$. Recall that the Gauss map satisfies a perturbed harmonic map equation,⁴ namely:

$$\Delta \vec{n} = - |\nabla \vec{n}|^2 \vec{n} - 2 \operatorname{div}(H \nabla \vec{\Phi}) + 4 e^{2\lambda} H^2 \vec{n}. \tag{3.18}$$

Owing to $\nabla \vec{n}$ being in L^p (for $p > 2$) and to (3.17), it follows easily that $\nabla \vec{n} \in L^{2(p-1)}$. We now introduce this information along with (3.12) and (3.8) into the system (3.11) to discover that ΔS and ΔR lie in L^r for all $1 \leq r < \frac{2p}{p+2}$. As $p > 2$, we obtain the improvement

$$\nabla^2 S, \nabla^2 \vec{R} \in L^r, \quad \forall 1 \leq r < \frac{2p}{p+2}. \tag{3.19}$$

Just as we did above, we conclude that

$$HF'(H^2) \in W^{1,r}, \quad \forall r < \frac{2p}{p+2}, \tag{3.20}$$

and thus

$$H \in L^q \quad \forall q < p(p-1).$$

Put into (3.18), the latter easily shows that

$$\nabla \vec{n} \in L^q \quad \forall q < p(p-1). \tag{3.21}$$

⁴ cf. Appendix B in [3].

We also have that $F(H^2)$ and $H^2F'(H^2)$ lie in the space L^s for $s < p - 1$. Per (2.13), it follows that ∇Y lies in the space $W^{1,s}$, for $s < p - 1$. Coupled to (3.21) and brought into (3.7), this information yields now

$$\nabla^2 \vec{v} \in L^a \quad \forall \begin{cases} a < \frac{2p(p-1)}{-p^2 + 3p + 2} & \text{if } p \leq 3 \\ a < p(p-1) & \text{if } p > 3. \end{cases}$$

This can now be bootstrapped along with (3.21) and (3.19) back into the system (3.11) to obtain that

$$\nabla^2 S, \nabla^2 \vec{R} \in L^s, \quad \forall 1 \leq s < p - 1,$$

which is an improvement to (3.19).

The above routine can of course be run again, improving with each step the integrability of the quantities involved. Without much effort, one eventually reaches that

$$HF'(H^2) \in W^{1,s}, \quad \forall s < \infty. \tag{3.22}$$

To transfer this information back to H , it suffices to use hypothesis (b), which states that $HF'(H^2)$ is a smooth and invertible function of H . In particular,

$$H \in W^{1,s}, \quad \forall s < \infty.$$

The equation $\Delta \vec{\Phi} = 2e^{2\lambda} H \vec{n}$ now gives that $\vec{\Phi} \in W^{3,s}$ for all finite s . Classical methods easily imply that the immersion $\vec{\Phi}$ is in fact smooth, by repeatedly differentiating all equations and applying standard elliptic estimates. This concludes the Proof of Theorem 1.1.

Remark 3.1 In the special case when $F(H) = |H|^p$, we have $HF'(H^2) = \frac{p}{2}|H|^{p-1}$, which is not invertible near $H = 0$. A glance at the proof reveals however that (3.22) still holds. In particular, H is a bounded function.

4 Proof of Theorem 1.2

The proof is accomplished in several steps.

4.1 Improved integrability: from $L^{p',\infty}$ to $L^{p'}$

We will work within the context of *weak immersions with bounded second fundamental form*. That is, we require $\vec{\Phi} : \Sigma \rightarrow \mathbb{R}^3$ to be Lipschitz and have a non-degenerate pull-back metric $g := \vec{\Phi}^* g_{\mathbb{R}^3}$. Moreover, we demand that the second fundamental form of the immersion lie in the space L^2 :

$$\int_{\Sigma} |d\vec{n}|_g^2 d\text{vol}_g < \infty.$$

These immersions are well-understood [16]. In particular, it is known since the work of F. Hélein, that locally about every point, it is possible to *conformally* reparametrize the surface. Since our results will be local in nature, we will thus assume that $\vec{\Phi}$ is conformal, namely, $g = e^{2\lambda} \delta$, where δ denotes the Cartesian metric on \mathbb{R}^2 , and λ is the conformal parameter.

Localizing, we assume that there holds⁵

$$\int_{D_2(0)} |\nabla \vec{n}|^2 \leq \varepsilon_0 \tag{4.1}$$

for some suitably chosen constant $8\pi/3 > \varepsilon_0 > 0$ (to be made precise in the sequel). We will also suppose that

$$m := \|\nabla \lambda\|_{L^{2,\infty}(D_2(0))} < \infty.$$

From Lemma 2.1 we have

$$\left| \lambda - \frac{1}{|D_{3/2}(0)|} \int_{D_{3/2}(0)} \lambda(x) dx \right| \leq C(m). \tag{4.2}$$

We focus on values of p within the interval $[2, q]$ for some fixed $q > 2$. We set $p' := p/(p - 1)$. Using Lemma 2.3 we have

$$\begin{aligned} \|\nabla \vec{n}\|_{L^p(D_1(0))} &\leq C(m, q) e^{\bar{\lambda}(1-2/p)} \left[\int_{D_{3/2}(0)} (1 + H^2)^{p/2} e^{2\lambda} dx \right]^{1/p} + C(m, q) \\ &\leq C(m, q) [e^{\bar{\lambda}(1-2/p)} w_p^{1/p} + 1] \leq C(m, q) \sqrt{w_p}. \end{aligned} \tag{4.3}$$

where we used $e^{2\bar{\lambda}} \leq w_p$. In this subsection, we study critical points $\vec{\Phi}$ of the energy

$$\mathcal{W}_p(\Sigma) := \int_{\Sigma} (1 + |H|^2)^{p/2} e^{2\lambda} dx \quad \text{with} \quad w_p := \int_{D_{3/2}(0)} (1 + |H|^2)^{p/2} e^{2\lambda} dx < \infty.$$

As we have seen in the first section, critical points of the energy \mathcal{W}_p satisfy

$$\operatorname{div}(-\nabla(HF'\vec{n}) + 2HF'\nabla\vec{n} + F\nabla\vec{\Phi}) = 0 \quad \text{on } D_2(0), \tag{4.4}$$

with $F(H^2) = (1 + H^2)^{p/2}$. Note that

$$F' = \frac{p}{2}(1 + H^2)^{p/2-1} \quad \text{satisfies} \quad \|F'\|_{L^{p/(p-2)}(D_1(0))} \leq C(q) e^{-(2-4/p)\bar{\lambda}} w_p^{1-2/p}.$$

and we have moreover

$$\|HF'\|_{L^{p'}(D_1(0))} \leq e^{-2\bar{\lambda}(1-1/p)} w_p^{1-1/p} \tag{4.5}$$

Owing to (4.3), we also have

$$\|HF'\nabla\vec{n}\|_{L^1(D_1(0))} \leq \|HF'\|_{L^{p'}(D_1(0))} \|\nabla\vec{n}\|_{L^p(D_1(0))}. \tag{4.6}$$

We can call upon Lemma A.12 to integrate (4.4) on the disk $D_1(0)$ and find there exists an element \vec{L} (unique up to the addition of a constant) satisfying

$$\nabla^\perp \vec{L} = -\nabla(HF'\vec{n}) + 2HF'\nabla\vec{n} + F\nabla\vec{\Phi} \quad \text{on } D_1(0), \tag{4.7}$$

⁵ For simplicity, we have switched to the “flat” coordinates notation: $|d\vec{n}|_g^2 d\operatorname{vol}_g = |\nabla\vec{n}|^2 dx_1 dx_2$, where $\nabla := (\partial_{x_1}, \partial_{x_2})$.

and such that

$$\begin{aligned} \|\vec{L}\|_{L^{p',\infty}(D_1(0))} &\leq C_q \|HF'\|_{L^{p'}(D_1(0))} + \|HF'\nabla\vec{n}\|_{L^1(D_1(0))} + \|F e^\lambda\|_{L^1(D_1(0))} \\ &\leq C(m, q) e^{-2\bar{\lambda}(1-1/p)} w_p^{1-1/p} [1 + \|\nabla\vec{n}\|_{L^p(D_1(0))}] + e^{-\bar{\lambda}} w_p \end{aligned} \tag{4.8}$$

where we are choosing \vec{L} to have average 0 on $D_1(0)$.

Using that $\nabla\vec{n} \cdot \nabla\vec{\Phi} = -2e^{2\lambda}H$, we find from (4.7)

$$\nabla\vec{\Phi} \cdot \nabla^\perp\vec{L} = 2HF'\nabla\vec{n} \cdot \nabla\vec{\Phi} + F|\nabla\vec{\Phi}|^2 = 4e^{2\lambda}(F - H^2F'). \tag{4.9}$$

Let Y be the solution of

$$\begin{cases} \Delta Y = 4e^{2\lambda}(F - H^2F') & \text{in } D_1(0) \\ Y = 0 & \text{on } \partial D_1(0). \end{cases} \tag{4.10}$$

Clearly,

$$\|\Delta Y\|_{L^1(D_1(0))} \leq C \|F e^{2\lambda}\|_{L^1(D_1(x))} + C \|H^2F'(H) e^{2\lambda}\|_{L^1(D_1(0))} \leq C(m) w_p,$$

so that, owing to a classical linear elliptic result (see e.g. Theorem 3.3.6 in [8]):

$$\|\nabla Y\|_{L^{2,\infty}(D_1(0))} \leq C(m) w_p. \tag{4.11}$$

Let next \vec{v} be the unique solution of the following problem:

$$\begin{cases} \Delta\vec{v} = \nabla^\perp Y \cdot \nabla\vec{n} & \text{in } D_1(0) \\ \vec{v} = 0, & \text{on } \partial D_1(0). \end{cases} \tag{4.12}$$

To obtain estimates on \vec{v} , we will use Lemma A.2 in the range $p \in [2, q]$ to the system (4.12) using (4.3) and (4.11). This yields the estimate

$$\|\nabla\vec{v}\|_{L^p(D_1(0))} \leq C_q \|\nabla Y\|_{L^{2,\infty}(D_1(0))} \|\nabla\vec{n}\|_{L^p(D_1(0))} \leq C(m) w_p \|\nabla\vec{n}\|_{L^p(D_1(0))}. \tag{4.13}$$

and

$$\|\nabla\vec{v}\|_{L^2(D_1(0))} \leq C_q \|\nabla Y\|_{L^{2,\infty}(D_1(0))} \|\nabla\vec{n}\|_{L^2(D_1(0))} \leq C(q) w_p \varepsilon_0. \tag{4.14}$$

Note that $\text{div}(\vec{n}\nabla^\perp Y - \nabla\vec{v}) \equiv 0$ holds on $D_1(0)$. This equation can be integrated to yield the existence of \vec{u} satisfying

$$\nabla^\perp\vec{u} = \vec{n}\nabla^\perp Y - \nabla\vec{v}. \tag{4.15}$$

Owing to the fact that $L^2 \subset L^{2,\infty}$. Combining (4.11) and (4.13) then gives

$$\|\nabla\vec{u}\|_{L^{2,\infty}(D_1(0))} \leq C(m, q) w_p. \tag{4.16}$$

Let us return to (4.9) and (4.10). Together they give that $\text{div}(\vec{L} \cdot \nabla^\perp\vec{\Phi} - \nabla Y) \equiv 0$. Integrating this equation yields the existence of a potential S satisfying

$$\nabla^\perp S = \vec{L} \cdot \nabla^\perp\vec{\Phi} - \nabla Y = \vec{L} \cdot \nabla^\perp\vec{\Phi} + \vec{n} \cdot (\nabla\vec{u} - \nabla^\perp\vec{v}), \tag{4.17}$$

where we have used (4.15). Next, from the definition of $\nabla\vec{L}$ given in (4.7) and using the fact that $\vec{n} \times \nabla\vec{\Phi} = -\nabla^\perp\vec{\Phi}$ (and hence that $\nabla\vec{n} \times \nabla\vec{\Phi} = 0$), we find

$$\text{div}(-\vec{L} \times \nabla^\perp\vec{\Phi}) = \nabla^\perp\vec{L} \times \nabla\vec{\Phi} = \text{div}(HF'\nabla^\perp\vec{\Phi}).$$

This exact divergence form is also locally integrated to yield a potential function \vec{V} satisfying

$$\nabla \vec{V} = \vec{L} \times \nabla \vec{\Phi} + HF' \nabla \vec{\Phi}. \tag{4.18}$$

As in the previous section, we set $\vec{R} := \vec{V} - \vec{u}$ to obtain the system

$$\begin{cases} \Delta \vec{R} = -\nabla^\perp \vec{n} \times \nabla \vec{R} - \nabla^\perp \vec{n} \cdot \nabla S - \operatorname{div}(\vec{n} \times \nabla \vec{v}) \\ \Delta S = \nabla^\perp \vec{n} \cdot \nabla \vec{R} + \operatorname{div}(\vec{n} \cdot \nabla \vec{v}). \end{cases} \tag{4.19}$$

We shall denote

$$S_p := e^{(1-2/p)\bar{\lambda}} S, \quad \vec{R}_p := e^{(1-2/p)\bar{\lambda}} \vec{R} \quad \text{and} \quad \vec{v}_p := e^{(1-2/p)\bar{\lambda}} \vec{v}. \tag{4.20}$$

and by homogeneity we have

$$\begin{cases} \Delta \vec{R}_p = -\nabla^\perp \vec{n} \times \nabla \vec{R}_p - \nabla^\perp \vec{n} \cdot \nabla S_p - \operatorname{div}(\vec{n} \times \nabla \vec{v}_p) \\ \Delta S_p = \nabla^\perp \vec{n} \cdot \nabla \vec{R}_p + \operatorname{div}(\vec{n} \cdot \nabla \vec{v}_p). \end{cases} \tag{4.21}$$

Using the definition of S and S_p , combining (4.8) and (4.11) we obtain

$$\|\nabla S_p\|_{L^{p',\infty}(D_1(0))} \leq C(m, q) w_p^{1-1/p} [1 + \|\nabla \vec{n}\|_{L^p(D_1(0))}] + C(m) e^{(1-2/p)\bar{\lambda}} w_p \tag{4.22}$$

Recall that $e^{2\bar{\lambda}} \leq w_p$, hence, combining (4.3) with (4.22) we obtain

$$\begin{aligned} \|\nabla S_p\|_{L^{p',\infty}(D_1(0))} &\leq C(m, q) w_p^{1-1/p} [e^{(1-2/p)\bar{\lambda}} w_p^{1/p} + 1] + C(m) e^{(1-2/p)\bar{\lambda}} w_p \\ &\leq C(m, q) w_p^{1-1/p} [\sqrt{w_p} + 1] \end{aligned} \tag{4.23}$$

Combining (4.13) and (4.3) we obtain

$$\|\nabla \vec{v}_p\|_{L^p(D_1(0))} \leq C(m) e^{(1-2/p)\bar{\lambda}} w_p \|\nabla \vec{n}\|_{L^p(D_1(0))} \leq C(m) w_p^{1-1/p} [w_p + \sqrt{w_p}] \tag{4.24}$$

and using (4.14) we obtain

$$\|\nabla \vec{v}_p\|_{L^2(D_1(0))} \leq C(m) w_p^{1-1/p} \sqrt{w_p}. \tag{4.25}$$

We also obtain

$$\|\nabla \vec{u}_p\|_{L^{2,\infty}(D_1(0))} \leq C(m, q) w_p^{1-1/p} \sqrt{w_p}. \tag{4.26}$$

Similarly to S_p we have

$$\|\nabla \vec{V}_p\|_{L^{p',\infty}(D_1(0))} \leq C(m, q) w_p^{1-1/p} [\sqrt{w_p} + 1] \tag{4.27}$$

and hence we deduce from (4.26) and (4.27)

$$\|\nabla \vec{R}_p\|_{L^{p',\infty}(D_1(0))} \leq C(m, q) w_p^{1-1/p} [\sqrt{w_p} + 1] \tag{4.28}$$

Applying Lemma A.4 for $q = \infty$ to the system (4.21) with (4.3), (4.26) and (4.28) we first find

$$\|\nabla S_p\|_{L^{p',2}(D_{4/5}(0))} + \|\nabla \vec{R}_p\|_{L^{p',2}(D_{4/5}(0))} \leq C(m, q) w_p^{1-1/p} [w_p + 1]. \tag{4.29}$$

and then, applying the same lemma but for $q = 2$ we obtain

$$\|\nabla S_p\|_{L^{p',1}(D_{3/4}(0))} + \|\nabla \vec{R}_p\|_{L^{p',1}(D_{3/4}(0))} \leq C(m, q) w_p^{1-1/p} [w_p + 1]. \tag{4.30}$$

4.2 Power decays

The case $p - 2$ away from 0 We assume first that there exists $1 > \delta > 0$ such that $0 < \delta \leq p - 2 \leq \delta^{-1} < +\infty$. In this case each step of the bootstrap argument in the Proof of Theorem 1.1 can be made quantitative with constants depending only on δ .

The case $p - 2$ sufficiently small This is the most delicate case. Let $x \in D_{1/2}(0)$ and $r < r_0$ where r_0 will be fixed later independent of x , $\vec{\Phi}$ and $p \in [2, q]$. We decompose S_p and \vec{R}_p in $D_r(x)$ as follows

$$S_p = S_p^0 + S_p^1 \quad \text{where} \quad \begin{cases} \Delta S_p^0 = 0 & \text{in } D_r(x) \\ S_p^0 = S_p & \text{on } \partial D_r(x) \end{cases} \quad \text{and} \quad \vec{R}_p = \vec{R}_p^0 + \vec{R}_p^1$$

$$\text{where} \quad \begin{cases} \Delta \vec{R}_p^0 = 0 & \text{in } D_r(x) \\ \vec{R}_p^0 = \vec{R}_p & \text{on } \partial D_r(x) \end{cases}$$

Lemma A.1 implies that for any $t \in (0, 1/2)$

$$\int_{D_{tr}(x)} |\nabla S_p^0|^{p'}(y) + |\nabla \vec{R}_p^0|^{p'}(y) dy^2 \leq C t^2 \int_{D_r(x)} |\nabla S_p^0|^{p'}(y) + |\nabla \vec{R}_p^0|^{p'}(y) dy^2 \tag{4.31}$$

Using Lemma A.2 as well as standard elliptic estimates we have

$$\int_{D_r(x)} |\nabla S_p^1|^{p'}(y) + |\nabla \vec{R}_p^1|^{p'}(y) dy^2 \leq C_q \varepsilon_0^{p'} \int_{D_r(x)} |\nabla S_p|^{p'}(y) + |\nabla \vec{R}_p|^{p'}(y) dy^2$$

$$+ C_q \int_{D_r(x)} |\nabla \vec{v}_p|^{p'}(y) dy^2 \tag{4.32}$$

Combining (4.31) and (4.32) we obtain

$$\int_{D_{tr}(x)} |\nabla S_p|^{p'}(y) + |\nabla \vec{R}_p|^{p'}(y) dy^2$$

$$\leq C_q (\varepsilon_0^{p'} + t^2) \int_{D_r(x)} |\nabla S_p|^{p'}(y) + |\nabla \vec{R}_p|^{p'}(y) dy^2$$

$$+ C_q \int_{D_r(x)} |\nabla \vec{v}_p|^{p'}(y) dy^2 \tag{4.33}$$

We return to (4.10). Recall that

$$F - H^2 F' = F^{(p-2)/p} + \frac{1}{p}(p-2)H^2 F'$$

hence

$$\Delta Y = 4e^{2\lambda} F^{(p-2)/p} + \frac{4}{p}(p-2)e^{2\lambda} H^2 F' \quad \text{on } D_1(0), \quad \text{with } Y|_{\partial D_1(0)} = 0. \tag{4.34}$$

We denote $Y_p := e^{(1-2/p)\bar{\lambda}} Y$ and decompose $Y_p = Z_p + W_p$ where

$$\begin{cases} \Delta Z_p = 4e^{(1-2/p)\bar{\lambda}} e^{2\lambda} F^{(p-2)/p} & \text{in } D_1(0) \\ Z_p = 0 & \text{on } \partial D_1(0) \end{cases} \tag{4.35}$$

We have, for $p/(p - 2) > 4/3$

$$\|\nabla Z_p\|_{L^4(D_1(0))} \leq C e^{(1+2/p)\bar{\lambda}} \left[\int_{D_1(0)} F(H^2) e^{2\lambda} dx^2 \right]^{(p-2)/p} \leq C w_p^{1-1/p} \sqrt{w_p}. \tag{4.36}$$

This gives in particular that

$$\sup_{x \in D_{1/2}(0), r < r_0} r^{-1} \|\nabla Z_p\|_{L^2(D_r(x))} \leq \|\nabla Z_p\|_{L^4(D_1(0))} \leq C w_p^{1-1/p} \sqrt{w_p}. \tag{4.37}$$

Decomposing as above the function $W_p = W_p^0 + W_p^1$ on the ball $D_r(x)$ into an harmonic and a trace zero function we obtain, using Lemma A.1, for any $t \in (0, 1/2)$

$$\begin{aligned} \|\nabla W_p\|_{L^{2,\infty}(D_{tr}(x))} &\leq C t \|\nabla W_p\|_{L^{2,\infty}(D_r(x))} \\ &\quad + C(m, q) (p - 2) e^{(1-2/p)\bar{\lambda}} \int_{D_r(x)} H^2 F'(H^2) e^{2\lambda} dy^2 \end{aligned} \tag{4.38}$$

We write $H^2 F'(H^2) = H H F'(H^2)$ and from (3.14) we deduce

$$\begin{aligned} &\int_{D_r(x)} H^2 F'(H^2) e^{2\lambda} dy^2 \\ &\leq w_p^{1/p} \left[\|\nabla S_p\|_{L^{p'}(D_r(x))} + \|\nabla \vec{R}_p\|_{L^{p'}(D_r(x))} + \|\nabla \vec{v}_p\|_{L^{p'}(D_r(x))} \right] \end{aligned} \tag{4.39}$$

Hence combining (4.37), (4.38) and (4.39) we obtain

$$\begin{aligned} \|\nabla Y_p\|_{L^{2,\infty}(D_{tr}(x))} &\leq C t \|\nabla Y_p\|_{L^{2,\infty}(D_r(x))} + C w_p^{1-1/p} \sqrt{w_p} r \\ &\quad + C(m, q) (p - 2) \sqrt{w_p} \left[\|\nabla S_p\|_{L^{p'}(D_r(x))} + \|\nabla \vec{R}_p\|_{L^{p'}(D_r(x))} \right. \\ &\quad \left. + \|\nabla \vec{v}_p\|_{L^{p'}(D_r(x))} \right] \end{aligned} \tag{4.40}$$

Decomposing as above the map $\vec{v}_p = \vec{v}_p^0 + \vec{v}_p^1$ on the ball $D_r(x)$ into an harmonic and a trace zero function we obtain, using Lemma A.1, for any $t \in (0, 1/2)$ we obtain, using also Lemma A.2,

$$\|\nabla \vec{v}_p\|_{L^{p'}(D_{tr}(x))}^{p'} \leq C t^2 \|\nabla \vec{v}_p\|_{L^{p'}(D_r(x))}^{p'} + C \varepsilon_0^{p'} \|\nabla Y_p\|_{L^{2,\infty}(D_r(x))}^{p'} \tag{4.41}$$

Combining now (4.33), (4.40) as well as (4.41) for r and for r/t we obtain

$$\begin{aligned} &\int_{D_{tr}(x)} |\nabla S_p|^{p'}(y) + |\nabla \vec{R}_p|^{p'}(y) + |\nabla \vec{v}_p|^{p'}(y) dy^2 + \|\nabla Y_p\|_{L^{2,\infty}(D_{tr}(x))}^{p'} \\ &\leq C(m, q) \left(\varepsilon_0^{p'} + t^2 + (p - 2)^{p'} w_p^{p'/2} \right) \int_{D_r(x)} |\nabla S_p|^{p'}(y) \\ &\quad + |\nabla \vec{R}_p|^{p'}(y) + |\nabla \vec{v}_p|^{p'}(y) dy^2 \\ &\quad + C_q t^{p'} \int_{D_{r/t}(x)} |\nabla \vec{v}_p|^{p'} dy^2 + C (\varepsilon_0 + t^{p'}) \|\nabla Y_p\|_{L^{2,\infty}(D_{r/t}(x))}^{p'} + C w_p w_p^{p'/2} r^{p'} \end{aligned} \tag{4.42}$$

For ε_0, t and $p - 2$ chosen to be small enough we then have

$$\begin{aligned} & \int_{D_{tr}(x)} |\nabla S_p|^{p'}(y) + |\nabla \vec{R}_p|^{p'}(y) + |\nabla \vec{v}_p|^{p'}(y) dy^2 + \|\nabla Y_p\|_{L^{2,\infty}(D_{tr}(x))}^{p'} \\ & \leq 2^{-1} \int_{D_{r/t}(x)} |\nabla S_p|^{p'}(y) + |\nabla \vec{R}_p|^{p'}(y) + |\nabla \vec{v}_p|^{p'}(y) dy^2 \\ & \quad + \|\nabla Y_p\|_{L^{2,\infty}(D_{r/t}(x))}^{p'} + C w_p w_p^{p'/2} r^{p'} \end{aligned} \tag{4.43}$$

We choose $r_0/t < 1/4$ and by a standard iteration argument, using (4.25) and (4.30), we deduce the following power decay

$$\begin{aligned} & \int_{D_r(x)} |\nabla S_p|^{p'}(y) + |\nabla \vec{R}_p|^{p'}(y) + |\nabla \vec{v}_p|^{p'}(y) dy^2 + \|\nabla Y_p\|_{L^{2,\infty}(D_r(x))}^{p'} \\ & \leq C(m, q) r^\alpha w_p [w_p + 1]^{p'/2} \end{aligned} \tag{4.44}$$

where $\alpha > 0$ only depends on q and not on $p \in [2, q]$.

4.3 Improved integrability and uniform ε -regularity estimate

The obtained power decay, by the mean of (3.14), namely

$$-2e^{2\lambda} H F' \vec{n} = \nabla S \cdot \nabla^\perp \vec{\Phi} + \nabla \vec{R} \times \nabla^\perp \vec{\Phi} + \nabla \vec{v} \times \nabla \vec{\Phi}.$$

gives

$$e^{2\bar{\lambda}/p'} |H|^{p/p'} \leq C(m, q) \left[|\nabla S_p| + |\nabla \vec{R}_p| + |\nabla \vec{v}_p| \right] \tag{4.45}$$

Combining (4.44) and (4.45) gives then

$$\sup_{x \in D_{1/2}(0); r < r_0} r^{-\alpha} \int_{D_r(x)} H^p e^{2\lambda} dy^2 \leq C(m, q) w_p [w_p + 1]^{p'/2}. \tag{4.46}$$

Combining (4.46) with Lemma 2.3 we obtain

$$\begin{aligned} & \sup_{x \in D_{1/2}(0); r < r_0} r^{-\alpha/p'} \int_{D_r(x)} |H F'(H) \nabla \vec{n}| dy^2 \\ & \leq C(m, q) e^{-2\lambda/p'} w_p^{1/p'} (\sqrt{w_p} + 1) \sqrt{w_p}. \end{aligned} \tag{4.47}$$

moreover we have

$$\sup_{x \in D_{1/2}(0); r < r_0} r^{-\alpha} \int_{D_r(x)} F(H) |\nabla \vec{\Phi}| dy^2 \leq C(m, q) e^{-\bar{\lambda}} w_p [w_p + 1]^{p'/2}. \tag{4.48}$$

Inserting these power decays in the r-h-s of the Willmore PDE

$$\Delta(H F' \vec{n}) = \operatorname{div}(2H F' \nabla \vec{n} + F \nabla \vec{\Phi}) \quad \text{on } D_{3/4}(0), \tag{4.49}$$

gives, by the mean of Adams estimates [1], a higher integrability to $H F' \vec{n}$ of the form

$$\|H F'\|_{L^r(D_{1/4}(0))} \leq C(m, q) \left[e^{-2\lambda/p'} w_p^{1/p'} (\sqrt{w_p} + 1) \sqrt{w_p} + e^{-\bar{\lambda}} w_p [w_p + 1]^{p'/2} \right]$$

for some $r > p'$. We can now bootstrap this estimate in (4.49) to reach the desired ϵ -regularity statement

$$e^{2\lambda(0)} H^p(0) \leq C(m, q) w_p \left[(1 + \sqrt{w_p})^{p'} + w_p^{p'/2} [1 + w_p]^{(p')^2/2} \right]. \tag{4.50}$$

4.4 Proof of Corollary 1.1

Let $\vec{\Phi}_k$ be a sequence of conformal immersions from a constant curvature surface of volume 1 (Σ, h_k) such that $\vec{\Phi}_k$ is a critical point of \mathcal{W}^k and such that

$$\limsup_{k \rightarrow +\infty} \mathcal{W}^k(\vec{\Phi}_k) = \limsup_{k \rightarrow +\infty} \int_{\Sigma} (1 + H^2)^{1+\alpha_k} \, \text{dvol}_{g_{\vec{\Phi}_k}} < +\infty \quad \text{where } \alpha_k \rightarrow 0.$$

We moreover assume that (Σ, h_k) is compact in the Moduli Space.

A now classical argument (see for instance [16]) gives the existence of a subsequence that we keep denote $\vec{\Phi}_k$, the existence of $\mu_k \in \mathbb{R}_+$ and the existence of finitely many points $\{a_1 \dots a_Q\}$ in Σ such that, for any compact $K \subset \Sigma \setminus \{a_1 \dots a_Q\}$

$$\limsup_{k \rightarrow +\infty} \|\lambda_k + \mu_k\|_{L^\infty(K)} < +\infty \quad \text{and } \mu_k \equiv 0 \quad \text{or } \mu_k \rightarrow +\infty. \tag{4.51}$$

where

$$g_{\vec{\Phi}_k} = e^{\lambda_k} h_k.$$

Moreover, there exists $\vec{p}_k \in \mathbb{R}^3$ such that

$$e^{\mu_k} (\vec{\Phi}_k - \vec{p}_k) \rightharpoonup \vec{\Psi}_\infty \quad \text{weakly in } W_{loc}^{2,2} \cap (W^{1,\infty})_{loc}^* (\Sigma \setminus \{a_1 \dots a_Q\}).$$

To simplify the presentation we give the proof in the case $\mu_k \equiv 0$ and $\vec{p}_k \equiv \vec{0}$. Let ϕ_k be a conformal chart from $U \subset K$ into D^2 such that the expression of h_k strongly converge in any C^l norm in this chart. We keep denoting $\vec{\Phi}_k$ the composition of $\vec{\Phi}_k$ with ϕ_k^{-1} and $e^{\lambda_k} := |\partial_{x_1} \vec{\Phi}_k| = |\partial_{x_2} \vec{\Phi}_k|$. Using (4.50) and (4.51) we have

$$\limsup_{k \rightarrow +\infty} \|\lambda_k\|_{L^\infty(D^2)} + \|H_k\|_{L^\infty(D^2)} < +\infty \implies \limsup_{k \rightarrow +\infty} \|\Delta \vec{\Phi}_k\|_{L^\infty(D^2)} < +\infty. \tag{4.52}$$

Hence, since $\|\nabla \vec{\Phi}_k\|_{L^2(D_2)}$ is uniformly bounded, standard elliptic estimates gives a uniform upper-bound on $\|\nabla^2 \vec{\Phi}_k\|_{L^p(D_{1/2}^2)}$ for any $p < +\infty$ and consequently

$$\limsup_{k \rightarrow +\infty} \left\| \nabla \vec{n}_{\vec{\Phi}_k} \right\|_{L^p(D_{1/2}^2)} < +\infty \quad \text{for any } p < +\infty. \tag{4.53}$$

Using now (4.49) and classical elliptic estimates we deduce that

$$\limsup_{k \rightarrow +\infty} \left\| \nabla (H_k F'_k(H_k^2) \vec{n}_k) \right\|_{L^p(D_{1/4}^2)} < +\infty, \quad \forall p < +\infty. \tag{4.54}$$

Recall that $F'(t) = (1 + \alpha_k) (1 + t)^{\alpha_k}$. Hence we have

$$\begin{aligned} \nabla (H_k F'_k(H_k^2) \vec{n}_k) &= (1 + \alpha_k) \nabla (H_k (1 + H_k^2)^{\alpha_k} \vec{n}_k) \\ &= (1 + \alpha_k) [(1 + H_k^2)^{\alpha_k} + 2\alpha_k H_k^2 (1 + H_k^2)^{\alpha_k - 1}] \nabla H_k \vec{n}_k + H_k (1 + H_k^2)^{\alpha_k} \nabla \vec{n}_k \end{aligned}$$

Hence we deduce from (4.53) and (4.54)

$$\limsup_{k \rightarrow +\infty} \|\nabla H_k\|_{L^p(D_{1/4}^2)} < +\infty \quad \text{for any } p < +\infty.$$

By taking successive derivatives of the PDE $\Delta \vec{\Phi} = 2 H \partial_{x_1} \vec{\Phi} \times \partial_{x_2} \vec{\Phi}$ as well as successive derivatives of (4.49), iterating the arguments above we finally obtain

$$\forall l \in \mathbb{N} \quad \limsup_{k \rightarrow +\infty} \|\nabla^l \vec{\Phi}_k\|_{C^l_{loc}(\Sigma \setminus \{a_1 \dots a_Q\})} < +\infty$$

which implies Corollary 1.1. □

5 Proof of Theorem 1.3

5.1 Rewriting the Euler Lagrange Equation of \mathcal{E}_p and preliminary estimates

We place ourselves in the same context as in Sect. 4. We assume $2 \leq p \leq 3$. We consider a local conformal parametrization $\vec{\Phi} : D_2(0) \rightarrow \mathbb{R}^3$ with conformal parameter λ . We suppose that

$$\int_{D_2(0)} |\nabla \vec{n}|^2 \leq \varepsilon_0^2, \tag{5.1}$$

for some suitably chosen constant $\varepsilon_0 > 0$ (to be made precise in the sequel). We will also suppose that

$$m := \|\nabla \lambda\|_{L^{2,\infty}(D_2(0))} < \infty.$$

Following Lemma 2.1 we have

$$\begin{aligned} & \|\nabla \lambda\|_{L^2(D_{3/2}(0))} + \left| \lambda - \frac{1}{|D_{3/2}(0)|} \int_{D_{3/2}(0)} \lambda(x) dx \right| \\ & \leq C \|\nabla \lambda\|_{L^{2,\infty}(D_2(0))} + C \|\nabla \vec{n}\|_{L^2(D_2(0))}^2. \end{aligned} \tag{5.2}$$

Hélein’s construction of the moving frame (see [8] lemma 5.1.4) gives not only a Coulomb frame (\vec{e}_1, \vec{e}_2) satisfying (2.21) but it also satisfies

$$\|\nabla \vec{e}_1\|_{L^p(D_2(0))} + \|\nabla \vec{e}_2\|_{L^p(D_2(0))} \leq C \|\nabla \vec{n}\|_{L^p(D_2(0))} \tag{5.3}$$

where C is independent of $p \in [2, 3]$. Indeed, it is proved that there exists $v \in W^{1,2}_0(D_2)$ such that

$$\nabla v = (\vec{e}_1, \nabla^\perp \vec{e}_2)$$

Hence v is the solution of

$$\begin{cases} \Delta v = (\nabla \vec{e}_1; \nabla^\perp \vec{e}_2) & \text{in } D_2(0) \\ v = 0 & \text{on } \partial D_2(0) \end{cases}$$

Using Lemma A.2 we have the a-priori estimate (using (2.21))

$$\|\nabla v\|_{L^p(D_2(0))} \leq C \|\nabla \vec{e}_1\|_{L^2(D_2(0))} \|\nabla \vec{e}_2\|_{L^p(D_2(0))} \leq C \varepsilon_0 \|\nabla \vec{e}_2\|_{L^p(D_2(0))} \tag{5.4}$$

where C is independent of $p \in [2, 3]$. We have also the pointwise identity

$$|\nabla \vec{e}_1|^2 + |\nabla \vec{e}_2|^2 = 2|\nabla v|^2 + |\nabla \vec{n}|^2. \tag{5.5}$$

Hence we deduce the a-priori inequality

$$\|\nabla \vec{e}_1\|_{L^p(D_2(0))} + \|\nabla \vec{e}_2\|_{L^p(D_2(0))} \leq C \varepsilon_0 \|\nabla \vec{e}_2\|_{L^p(D_2(0))} + \|\nabla \vec{n}\|_{L^p(D_2(0))} \tag{5.6}$$

and (5.3) follows by a density argument as in [8]. With (5.3) at hand, using one more time the full strength of Lemma A.2, we can adapt step by step the Proof of Lemma 2.1 to deduce

$$\|\nabla\lambda\|_{L^p(D_{3/2}(0))} \leq C \|\nabla\lambda\|_{L^{2,\infty}(D_2(0))} + C \|\nabla\bar{n}\|_{L^p(D_2(0))}, \tag{5.7}$$

where C is independent of $p \in [2, 3]$.

As in the Proof of Lemma 2.3 we have

$$\begin{aligned} \int_{D_{6/5}(0)} |\nabla^2\bar{\Phi}|^p dx^2 &\leq C(m) e^{2\bar{\lambda}(p-1)} \int_{D_2(0)} |\bar{H}|^p e^{2\lambda} dx^2 \\ &+ C \left[\int_{D_{3/2}(0)} |\nabla\bar{\Phi}|^2 dx^2 \right]^{p/2}. \end{aligned} \tag{5.8}$$

We set $p' := p/(p - 1)$, and in this section, we study critical points $\bar{\Phi}$ of the energy

$$\mathcal{E}_p(\Sigma) := \int_{\Sigma} (1 + |A|^2)^{p/2} d\text{vol}_g \quad \text{with} \quad e_p := \int_{D_2(0)} (1 + |A|^2)^{p/2} e^{2\lambda} dx < \infty.$$

For notational convenience, we will set

$$F = (1 + |A|^2)^{p/2} \quad \text{and} \quad F' = \frac{p}{2} (1 + |A|^2)^{p/2-1}.$$

where $|A|^2 = e^{-2\lambda} |\nabla\bar{n}|^2$. With these notations we have

$$\|\nabla^2\bar{\Phi}\|_{L^p(D_1(0))} \leq C(m) e^{2\bar{\lambda}/p'} e_p^{1/p} + C e^{\bar{\lambda}} \tag{5.9}$$

For future reference, note that

$$\|F'(|A|^2) |\nabla\bar{n}|^2\|_{L^1(D_1(0))} \leq e_p \tag{5.10}$$

and

$$\|F'(|A|^2) |\nabla\bar{n}\|_{L^{p'}(D_1(0))} \leq e^{\bar{\lambda}(1-2/p')} e_p^{1/p'} \tag{5.11}$$

As we have seen in Sect. 2.1, critical points of \mathcal{E}_p satisfy (2.9):

$$d^{*g} \left[Fd\bar{\Phi} - 2F'(d\bar{n} \otimes d\bar{n}) \lrcorner_g d\bar{\Phi} + 2(d^{*g}(F'd\bar{n}) \cdot d\bar{\Phi})\bar{n} \right] = 0.$$

In conformal coordinates, this expression reads

$$\text{div} \left[F\nabla\bar{\Phi} - 2e^{-2\lambda} F' \sum_{j=1}^2 (\nabla\bar{n} \cdot \partial_{x_j}\bar{n}) \partial_{x_j}\bar{\Phi} + 2e^{-2\lambda}\bar{n} (\text{div}(F'\nabla\bar{n}) \cdot \nabla\bar{\Phi}) \right] = 0. \tag{5.12}$$

We have

$$\left| F\nabla\bar{\Phi} - 2e^{-2\lambda} F' \sum_{j=1}^2 (\nabla\bar{n} \cdot \partial_{x_j}\bar{n}) \partial_{x_j}\bar{\Phi} \right| \leq e^\lambda F + 2e^{-\lambda} |\nabla\bar{n}|^2 F',$$

so that owing to (5.10),

$$\left\| F\nabla\bar{\Phi} - 2e^{-2\lambda} F' \sum_{j=1}^2 (\nabla\bar{n} \cdot \partial_{x_j}\bar{n}) \partial_{x_j}\bar{\Phi} \right\|_{L^1(D_1(0))} \leq C(m) e^{-\bar{\lambda}} e_p. \tag{5.13}$$

where we have used that $p' \leq 2$. We also have

$$\|\text{div}(F'\nabla\bar{n})\|_{W^{-1,p'}(D_1(0))} \leq \|F'\nabla\bar{n}\|_{L^{p'}(D_1(0))} \leq e^{\bar{\lambda}(1-2/p')} e_p^{1/p'}, \tag{5.14}$$

where we have used (5.11). From (5.7) we have

$$\|e^\lambda\|_{L^\infty \cap W^{1,p}(D_1(0))} \leq C(m) \left[e^{\bar{\lambda}} + e^{2\bar{\lambda}/p'} e_p^{1/p'} \right], \tag{5.15}$$

from (5.9)

$$\|\nabla \vec{\Phi}\|_{L^\infty \cap W^{1,p}(D_1(0))} \leq C(m) \left[e^{\bar{\lambda}} + e^{2\bar{\lambda}/p'} e_p^{1/p'} \right], \tag{5.16}$$

and

$$\|\vec{n}\|_{L^\infty \cap W^{1,p}(D_1(0))} \leq 1 + e^{-2\bar{\lambda}/p} e_p^{1/p} \tag{5.17}$$

where $C(m)$ is independent of p . We have also

$$\|e^{-2\lambda} \vec{n} \otimes \nabla \vec{\Phi}\|_{L^\infty \cap W^{1,p}(D_1(0))} \leq C(m) \left[e^{-\bar{\lambda}} + e^{-2\bar{\lambda}/p} e_p^{1/p'} \right]. \tag{5.18}$$

Recall that there exists a constant C independent of $p \in [2, 3]$ such that for all $f \in C_0^\infty$

$$\|f\|_{L^\infty(D_1(0))} \leq C \|\nabla f\|_{L^{p,1}(D_1(0))}.$$

Hence, combining (5.11) and (5.18)

$$\begin{aligned} & \sup_{\|f\|_{W_0^{1,(p,1)}(D_1(0))} \leq 1} \int_{D_1(0)} f(x) e^{-2\lambda} \vec{n} (\operatorname{div}(F' \nabla \vec{n}) \cdot \nabla \vec{\Phi}) dx^2 \\ & \leq C(m) \left[e^{-2\bar{\lambda}/p'} e_p^{1/p'} + e^{-\bar{\lambda}} e_p \right] \end{aligned} \tag{5.19}$$

Hence

$$\|e^{-2\lambda} \vec{n} (\operatorname{div}(F' \nabla \vec{n}) \cdot \nabla \vec{\Phi})\|_{W^{-1,(p',\infty)}(D_1(0))} \leq C(m) \left[e^{-2\bar{\lambda}/p'} e_p^{1/p'} + e^{-\bar{\lambda}} e_p \right] \tag{5.20}$$

Let \vec{L} be the distribution (unique up to a constant) such that

$$\nabla^\perp \vec{L} = F \nabla \vec{\Phi} - 2e^{-2\lambda} F' \sum_{j=1}^2 (\nabla \vec{n} \cdot \partial_{x_j} \vec{n}) \partial_{x_j} \vec{\Phi} + 2e^{-2\lambda} \vec{n} (\operatorname{div}(F' \nabla \vec{n}) \cdot \nabla \vec{\Phi}) \tag{5.21}$$

Combining (5.13) and (5.20) we obtain

$$\|\nabla^\perp \vec{L}\|_{L^1(D_1(0)) + W^{-1,(p',\infty)}(D_1(0))} \leq C(m) \left[e^{-2\bar{\lambda}/p'} e_p^{1/p'} + e^{-\bar{\lambda}} e_p \right], \tag{5.22}$$

from which we deduce

$$\|\vec{L}\|_{L^{p',\infty}(D_1(0))} \leq C(m) \left[e^{-2\bar{\lambda}/p'} e_p^{1/p'} + e^{-\bar{\lambda}} e_p \right] \tag{5.23}$$

where we normalized \vec{L} to have average 0 on $D_1(0)$. We compute

$$\begin{aligned} \nabla \vec{\Phi} \cdot \nabla^\perp \vec{L} &= 2e^{2\lambda} F - 2F' |\nabla \vec{n}|^2 \\ &= e^{2\lambda} (1 + e^{-2\lambda} |\nabla \vec{n}|^2)^{p/2-1} (2 + (2-p)e^{-2\lambda} |\nabla \vec{n}|^2). \end{aligned} \tag{5.24}$$

where we have used that $|\nabla \vec{n}|^2 = e^{2\lambda} A^2$. Let us next introduce

$$\begin{cases} -\Delta Y = 2e^{2\lambda} F - 2F' |\nabla \vec{n}|^2, & \text{in } D_1(0) \\ Y = 0, & \text{on } \partial D_1(0). \end{cases} \tag{5.25}$$

According to (5.24), we thus have an exact divergence form which may be locally integrated to yield a new potential S satisfying

$$\nabla S = \vec{L} \cdot \nabla \vec{\Phi} + \nabla^\perp Y. \tag{5.26}$$

We next compute

$$\begin{aligned}
 \nabla \vec{\Phi} \times \nabla^\perp \vec{L} &= -2e^{-2\lambda} (\nabla \vec{\Phi} \cdot \operatorname{div}(F' \nabla \vec{n})) \cdot (\vec{n} \times \nabla \vec{\Phi}) \\
 &= -2e^{-2\lambda} \sum_{j=1}^2 (\partial_{x_j} \vec{\Phi} \cdot \operatorname{div}(F' \nabla \vec{n})) (\vec{n} \times \partial_{x_j} \vec{\Phi}) \\
 &= -2\vec{n} \times \operatorname{div}(F' \nabla \vec{n}) = -\operatorname{div} [2F' \vec{n} \times \nabla \vec{n}] \\
 &= -\operatorname{div} [2F' \nabla^\perp \vec{n} + 4HF' \nabla^\perp \vec{\Phi}], \tag{5.27}
 \end{aligned}$$

where we have used that $\nabla \vec{\Phi} \times \nabla \vec{n} = 0$ and that $\nabla \vec{\Phi} \times \vec{n} = \nabla^\perp \vec{\Phi}$. We have also used the elementary identity

$$\nabla^\perp \vec{n} = \vec{n} \times \nabla \vec{n} - 2H \nabla^\perp \vec{\Phi}. \tag{5.28}$$

Equivalently, (5.27) reads

$$\operatorname{div} [\vec{L} \times \nabla^\perp \vec{\Phi} + 2F' \nabla^\perp \vec{n} + 4HF' \nabla^\perp \vec{\Phi}] = 0. \tag{5.29}$$

Integrating this identity yields a potential \vec{V} satisfying

$$\nabla^\perp \vec{V} = \vec{L} \times \nabla^\perp \vec{\Phi} + 2F' \nabla^\perp \vec{n} + 4HF' \nabla^\perp \vec{\Phi}. \tag{5.30}$$

Observe that

$$\vec{n} \cdot \nabla \vec{V} = \vec{n} \cdot (\vec{L} \times \nabla \vec{\Phi}) = \vec{L} \cdot (\nabla \vec{\Phi} \times \vec{n}) = \vec{L} \cdot \nabla^\perp \vec{\Phi} = \nabla^\perp S + \nabla Y \tag{5.31}$$

We decompose

$$2F'(|A|^2) \nabla \vec{n} = -\nabla \vec{\sigma} + \nabla^\perp \vec{\tau} \tag{5.32}$$

where $\vec{\tau}$ is zero on $\partial D_1(0)$ and

$$\vec{n} \nabla Y = \nabla \vec{u} - \nabla^\perp \vec{w} \tag{5.33}$$

where \vec{w} is equal to zero on $\partial D_1(0)$. We have

$$\begin{aligned}
 \nabla(\vec{V} + \vec{\sigma}) &= \vec{L} \times \nabla \vec{\Phi} + 4HF'(|A|^2) \nabla \vec{\Phi} + \nabla^\perp \vec{\tau} \\
 &= -(\vec{L} \cdot \vec{n}) \nabla^\perp \vec{\Phi} + \vec{L} \cdot \nabla^\perp \vec{\Phi} \vec{n} + 4HF'(|A|^2) \nabla \vec{\Phi} + \nabla^\perp \vec{\tau} \\
 &= -(\vec{L} \cdot \vec{n}) \nabla^\perp \vec{\Phi} - \vec{n} \nabla^\perp S - \vec{n} \nabla Y + 4HF'(|A|^2) \nabla \vec{\Phi} + \nabla^\perp \vec{\tau} \\
 &= -(\vec{L} \cdot \vec{n}) \nabla^\perp \vec{\Phi} + 4HF'(|A|^2) \nabla \vec{\Phi} - \vec{n} \nabla^\perp S - \nabla \vec{u} \\
 &\quad + \nabla^\perp (\vec{\tau} + \vec{w}) \tag{5.34}
 \end{aligned}$$

Denote

$$\vec{R} := \vec{V} + \vec{\sigma} + \vec{u} \quad \text{and} \quad \vec{v} := \vec{\tau} + \vec{w}$$

We have the following

Proposition 5.1 *With the previous notations the following equation hold*

$$\vec{n} \times \nabla \vec{R} = -\nabla^\perp \vec{R} + \vec{n} \nabla S - \nabla \vec{v} + \vec{n} \times \nabla^\perp \vec{v}. \tag{5.35}$$

□

Proof of Proposition 5.1 Observe that

$$\begin{cases} \vec{n} \times \nabla^\perp \vec{\Phi} = \nabla \vec{\Phi} \\ \vec{n} \times \nabla \vec{\Phi} = -\nabla^\perp \vec{\Phi} \end{cases} \tag{5.36}$$

Using these identities we deduce in one hand

$$\vec{n} \times \nabla \vec{R} = -(\vec{L} \cdot \vec{n}) \nabla \vec{\Phi} - 4 H F'(|A|^2) \nabla^\perp \vec{\Phi} + \vec{n} \times \nabla^\perp \vec{v}, \tag{5.37}$$

and in the other hand

$$\nabla^\perp \vec{R} = (\vec{L} \cdot \vec{n}) \nabla \vec{\Phi} + 4 H F'(|A|^2) \nabla^\perp \vec{\Phi} + \vec{n} \nabla S - \nabla \vec{v}. \tag{5.38}$$

Combining the two previous lines gives the proposition. □

5.2 Improved integrability: from $L^{p',\infty}$ to $L^{p'}$

Let $r_0 < 1/4$ to be fixed later. We proceed to the following renormalization

$$Y_p := e^{(1-2/p)\bar{\lambda}} Y, \quad S_p := e^{(1-2/p)\bar{\lambda}} S, \quad \vec{R}_p := e^{(1-2/p)\bar{\lambda}} \vec{R} \quad \text{and} \\ \vec{v}_p := e^{(1-2/p)\bar{\lambda}} \vec{v}.$$

Using (5.25) we have

$$\|\nabla Y_p\|_{L^{2,\infty}(D_1(0))} \leq C e_p^{1-1/p} \sqrt{e_p}. \tag{5.39}$$

Using (5.23), (5.26) and (5.39) we have

$$\|\nabla S_p\|_{L^{p',\infty}(D_1(0))} \leq C(m) e_p^{1/p'} + C e_p^{1-1/p} \sqrt{e_p}. \tag{5.40}$$

Using (5.11), (5.23), (5.30) and the fact that $|H|^2 \leq e^{-2\lambda} |\nabla \vec{n}|^2$ we have

$$\|\nabla \vec{v}_p\|_{L^{p',\infty}(D_1(0))} \leq C(m) e_p^{1/p'} + C e_p^{1-1/p} \sqrt{e_p}. \tag{5.41}$$

From (5.39) we deduce using standard elliptic estimates

$$\|\nabla \vec{u}_p\|_{L^{2,\infty}(D_1(0))} + \|\nabla \vec{w}_p\|_{L^{2,\infty}(D_1(0))} \leq C \|\nabla Y_p\|_{L^{2,\infty}(D_1(0))} \leq C e_p^{1-1/p} \sqrt{e_p}. \tag{5.42}$$

where we used the fact that \vec{w}_p satisfies the following system

$$\begin{cases} -\Delta \vec{w}_p = \nabla^\perp Y_p \nabla \vec{n} & \text{in } D_1(0) \\ \vec{w}_p = 0 & \text{on } D_1(0) \end{cases} \tag{5.43}$$

Using Lemma A.2 we deduce

$$\|\nabla \vec{w}_p\|_{L^2(D_1(0))} \leq C \|\nabla Y_p\|_{L^{2,\infty}(D_1(0))} \|\nabla \vec{n}\|_{L^2(D_1(0))} \leq C \varepsilon_0 e_p^{1-1/p} \sqrt{e_p} \tag{5.44}$$

and from (5.11) and (5.32) we have using again standard elliptic estimates

$$\begin{aligned} \|\nabla \vec{\sigma}_p\|_{L^{p'}(D_1(0))} + \|\nabla \vec{\tau}_p\|_{L^{p'}(D_1(0))} &\leq C e^{\bar{\lambda}(1-2/p)} \|F'(|A|^2) |\nabla \vec{n}|\|_{L^{p'}(D_1(0))} \\ &\leq C(m) e_p^{1/p'} \end{aligned} \tag{5.45}$$

where the constant C is independent on $p \in [2, 3]$. Hence we deduce from (5.41), (5.42) and (5.45)

$$\|\nabla \vec{R}_p\|_{L^{p',\infty}(D_1(0))} \leq C(m) e_p^{1/p'} + C e_p^{1-1/p} \sqrt{e_p}. \tag{5.46}$$

Combining (5.44) and (5.45) we deduce

$$\|\nabla \vec{v}_p\|_{L^{p'}(D_1(0))} \leq C(m) e_p^{1/p'} + C \varepsilon_0 e_p^{1-1/p} \sqrt{e_p}. \tag{5.47}$$

From Proposition 5.1 we have that (S_p, \vec{R}_p) satisfies the following system

$$\begin{cases} \Delta \vec{R}_p = \nabla S_p \cdot \nabla^\perp \vec{n} + \nabla \vec{R}_p \times \nabla^\perp \vec{n} + \operatorname{div}(\vec{n} \times \nabla \vec{v}_p) \\ \Delta S_p = \nabla \vec{R}_p \cdot \nabla^\perp \vec{n} + \operatorname{div}(\vec{n} \cdot \nabla \vec{v}_p) \end{cases} \tag{5.48}$$

Applying Lemma A.4 for $q = \infty$ to the system (5.48) with (5.40), (5.46) and (5.47)

$$\|\nabla S_p\|_{L^{p',2}(D_{4/5}(0))} + \|\nabla \vec{R}_p\|_{L^{p',2}(D_{4/5}(0))} \leq C(m) e_p^{1-1/p} [e_p + 1]. \tag{5.49}$$

and then, applying the same lemma but for $q = 2$ we obtain

$$\|\nabla S_p\|_{L^{p',1}(D_{3/4}(0))} + \|\nabla \vec{R}_p\|_{L^{p',1}(D_{3/4}(0))} \leq C(m) e_p^{1-1/p} [e_p + 1]. \tag{5.50}$$

5.3 Power decays

Consider now an arbitrary point $x \in D_{1/2}(0)$ and a radius $0 < r < r_0$ where $r_0 < 1/4$ will be fixed later. Exactly as in the proof of (4.33), by exploiting the system (5.48) one first establishes that for $t \in (0, 1)$

$$\begin{aligned} \int_{D_{tr}(x)} |\nabla S_p|^{p'}(y) + |\nabla \vec{R}_p|^{p'}(y) dy^2 &\leq C (\varepsilon_0^{p'} + t^2) \int_{D_r(x)} |\nabla S_p|^{p'}(y) \\ &\quad + |\nabla \vec{R}_p|^{p'}(y) dy^2 \\ &\quad + C \int_{D_r(x)} |\nabla \vec{v}_p|^{p'}(y) dy^2 \end{aligned} \tag{5.51}$$

where C is independent of $p \in [2, 3]$.

As in the previous section for the Lagrangian \mathcal{W}_p , we proceed to the following decomposition $Y_p := Z_p + W_p$ where

$$\begin{cases} -\Delta Z_p = 2 e^{(1-2/p)\bar{\lambda}} e^{2\lambda} (1 + e^{-2\lambda} |\nabla \vec{n}|^2)^{p/2-1} & \text{in } D_1(0) \\ Z_p = 0 & \text{on } \partial D_1(0) \end{cases}$$

We have for $p/(p-2) > 4/3$

$$\begin{aligned} \|\nabla Z_p\|_{L^4(D_1(0))} &\leq C(m) e^{(1+2/p)\bar{\lambda}} \left[\int_{D_1(0)} (1 + e^{-2\lambda} |\nabla \vec{n}|^2)^{p/2} e^{2\lambda} dx^2 \right]^{(p-2)/p} \\ &\leq C(m) e_p^{1-1/p} \sqrt{e_p}. \end{aligned} \tag{5.52}$$

This implies in particular

$$\sup_{x \in D_{1/2}(0); r < r_0} r^{-1} \|\nabla Z_p\|_{L^2(D_r(x))} \leq C(m) e_p^{1-1/p} \sqrt{e_p}. \tag{5.53}$$

By decomposing $W_p = W_p^0 + W_p^1$ as in the previous section into the sum of an harmonic and a trace free part on $\partial D_r(x)$ we obtain a similar estimate as (4.38)

$$\begin{aligned} \|\nabla W_p\|_{L^{2,\infty}(D_{tr}(x))} &\leq C t \|\nabla W_p\|_{L^{2,\infty}(D_r(x))} \\ &\quad + C(m) (p-2) e^{(1-2/p)\bar{\lambda}} \int_{D_r(x)} (1 + e^{-2\lambda} |\nabla \vec{n}|^2)^{p/2} e^{2\lambda} dy^2. \end{aligned} \tag{5.54}$$

Combining (5.53) and (5.54) gives then

$$\begin{aligned} \|\nabla Y_p\|_{L^{2,\infty}(D_{t,r}(x))} &\leq C t \|\nabla Y_p\|_{L^{2,\infty}(D_r(x))} \\ &+ C(m) (p - 2) e^{(1-2/p)\bar{\lambda}} \int_{D_r(x)} (1 + e^{-2\lambda} |\nabla \bar{n}|^2)^{p/2} e^{2\lambda} dy^2 + C r. \end{aligned} \tag{5.55}$$

Taking the dot of (5.38) with $\nabla^\perp \bar{\Phi}$ we obtain

$$8 e^{2\lambda} H F'(|A|^2) = \nabla \bar{\Phi} \cdot \nabla \bar{R} - \nabla \bar{\Phi} \cdot \nabla^\perp \bar{v}, \tag{5.56}$$

from which we deduce

$$\int_{D_r(x)} |H|^p e^{2\lambda} dy^2 \leq C(m) \int_{D_r(x)} |\nabla \bar{R}_p|^{p'} + |\nabla \bar{v}_p|^{p'} dy^2. \tag{5.57}$$

Recall the structural equation

$$-\nabla \bar{n} = \bar{n} \times \nabla^\perp \bar{n} + 2 H \nabla \bar{\Phi}. \tag{5.58}$$

Taking the divergence gives

$$-\Delta \bar{n} = \nabla \bar{n} \times \nabla^\perp \bar{n} + 2 \operatorname{div}(H \nabla \bar{\Phi}) \tag{5.59}$$

Decomposing again \bar{n} into the sum of an harmonic part and a trace free part in $D_r(x)$ and making use of the Lemma A.2 as well as Lemma A.1 we obtain for any $t \in (0, 1)$

$$\begin{aligned} \int_{D_{t,r}(x)} |\nabla \bar{n}|^p e^{-\lambda p} e^{2\lambda} &\leq C(m) (t^2 + \varepsilon^2) \int_{D_r(x)} |\nabla \bar{n}|^p e^{-\lambda p} e^{2\lambda} \\ &+ C(m) \int_{D_r(x)} |H|^p e^{2\lambda} dy^2 \end{aligned} \tag{5.60}$$

Combining (5.57) and (5.60) gives then

$$\begin{aligned} \int_{D_{t,r}(x)} |\nabla \bar{n}|^p e^{-\lambda p} e^{2\lambda} dy^2 &\leq C(m) (t^2 + \varepsilon^2) \int_{D_r(x)} |\nabla \bar{n}|^p e^{-\lambda p} e^{2\lambda} \\ &+ C(m) \int_{D_r(x)} |\nabla \bar{R}_p|^{p'} + |\nabla \bar{v}_p|^{p'} dy^2. \end{aligned} \tag{5.61}$$

Observe that since $\lambda - \bar{\lambda}$ is bounded by a constant $C(m)$ one has

$$\int_{D_{t,r}(x)} e^{2\lambda} dy^2 \leq C(m) t^2 \int_{D_r(x)} e^{2\lambda} dy^2 \tag{5.62}$$

Hence, combining (5.61) and (5.62) we have

$$\begin{aligned} \int_{D_{t,r}(x)} (1 + |\nabla \bar{n}|^2 e^{-2\lambda})^{p/2} e^{2\lambda} dy^2 &\leq C(m) (t^2 + \varepsilon^2) \int_{D_r(x)} (1 + |\nabla \bar{n}|^2 e^{-2\lambda})^{p/2} e^{2\lambda} \\ &+ C(m) \int_{D_r(x)} |\nabla \bar{R}_p|^{p'} + |\nabla \bar{v}_p|^{p'} dy^2. \end{aligned} \tag{5.63}$$

Now it remains to establish some decay of the $L^{p'}$ norm of $\nabla \vec{v}_p$. Recall $\nabla \vec{v}_p := \nabla \vec{w}_p + \nabla \vec{\tau}_p$. For the $\nabla \vec{w}_p$ the argument follows the lines similar to what we did in the previous section for \vec{v}_p : Using equation (5.43) together with Lemma A.1, for any $t \in (0, 1)$ we obtain, using also Lemma A.2, exactly as we did for proving (4.41)

$$\|\nabla \vec{w}_p\|_{L^{p'}(D_{tr}(x))}^{p'} \leq C t^2 \|\nabla \vec{w}_p\|_{L^{p'}(D_r(x))}^{p'} + C \varepsilon_0^{p'} \|\nabla Y_p\|_{L^{2,\infty}(D_r(x))}^{p'} \tag{5.64}$$

where C is independent of $p \in [2, 3]$.

In order to establish some interesting decay for the $L^{p'}$ norm of $\vec{\tau}_p$ we have to use some fundamental lemma in p -harmonic theory which is the only new ingredient in this section in comparison with the previous one. We decompose $\vec{\tau}_p := \vec{\tau}_p^0 + \vec{\tau}_p^1$ where $\vec{\tau}_p^0$ is harmonic in $D_r(x)$ and equal to $\vec{\tau}_p$ on $\partial D_r(x)$. Hence, using (5.32), $\vec{\tau}_p^1$ satisfies

$$\begin{cases} -\Delta \vec{\tau}_p^1 = p e^{\frac{2-p}{p'} \bar{\lambda}} \operatorname{div} \left(\left(e^{2\bar{\lambda}} + e^{-2(\lambda-\bar{\lambda})} |\nabla^\perp \vec{n}|^2 \right)^{p/2-1} \nabla^\perp \vec{n} \right) & \text{in } D_r(x) \\ \vec{\tau}_p^1 = 0 & \text{on } \partial D_r(x) \end{cases} \tag{5.65}$$

Using (A.27) we have for any $p - 2 < \alpha < 1/2$

$$\begin{aligned} &\|\nabla \vec{\tau}_p^1\|_{L^{p/(p-1)}(D_r^2(x))} \\ &\leq C e^{\frac{2-p}{p'} \bar{\lambda}} \frac{(p-2)}{\alpha} \left(\|\nabla \vec{n}\|_{L^p(D_r^2(x))}^{p-1} + (r^{2/p} e^{\bar{\lambda}})^{(p-2)+\alpha} \|\nabla \vec{n}\|_{L^p(D_r^2(x))}^{1-\alpha} \right), \end{aligned} \tag{5.66}$$

Observe that $1 - (1 - \alpha)/(p - 1) = (p - 2 + \alpha)/(p - 1)$ thus $a^{p-2+\alpha} b^{(1-\alpha)/(p-1)} \leq C_\alpha [a^{p-1} + b]$ where C_α is uniformly bounded for $\alpha < 1/2$ bounded from below away from 0. We choose $\alpha = 1/4$ and we have

$$\|\nabla \vec{\tau}_p^1\|_{L^{p/(p-1)}(D_r^2(x))} \leq C(m) e^{\frac{2-p}{p'} \bar{\lambda}} (p-2) \left(\|\nabla \vec{n}\|_{L^p(D_r^2(x))}^{p-1} + (r^2 e^{p\bar{\lambda}})^{(p-1)/p} \right). \tag{5.67}$$

Which gives for any $t < 1$

$$\begin{aligned} &\int_{D_{tr}(x)} |\nabla \vec{\tau}_p|^{p'}(y) dy^2 \leq C t^2 \int_{D_r(x)} |\nabla \vec{\tau}_p|^{p'}(y) dy^2 \\ &\quad + C [p-2]^{p'} e^{(2-p)\bar{\lambda}} \int_{D_r(x)} |\nabla \vec{n}|^p dy^2 + C (p-2)^{p'} r^2 e^{2\bar{\lambda}} \\ &\leq C t^2 \int_{D_r(x)} |\nabla \vec{\tau}_p|^{p'}(y) dy^2 + C [p-2]^{p'} \int_{D_r(x)} (1 + |\nabla \vec{n}|^2 e^{-2\lambda})^{p/2} e^{2\lambda} dy^2 \end{aligned} \tag{5.68}$$

Combining (5.51), (5.55), (5.63), (5.64) and (5.68) we obtain

$$\begin{aligned} &\int_{D_{tr}(x)} |\nabla S_p|^{p'} + |\nabla \vec{R}_p|^{p'} + |\nabla \vec{w}_p|^{p'} + |\nabla \vec{\tau}_p|^{p'} + (1 + |\nabla \vec{n}|^2 e^{-2\lambda})^{p/2} e^{2\lambda} dy^2 \\ &\quad + \|\nabla Y_p\|_{L^{2,\infty}(D_{tr})}^{p'} \\ &\leq \delta \left[\int_{D_{\frac{r}{t}}(x)} |\nabla S_p|^{p'} + |\nabla \vec{R}_p|^{p'} + |\nabla \vec{w}_p|^{p'} + |\nabla \vec{\tau}_p|^{p'} \right. \\ &\quad \left. + (1 + |\nabla \vec{n}|^2 e^{-2\lambda})^{p/2} e^{2\lambda} dy^2 + \|\nabla Y_p\|_{L^{2,\infty}(D_{\frac{r}{t}})}^{p'} \right] + C r^{p'} \end{aligned} \tag{5.69}$$

where $\delta = C(m) (\varepsilon_0^{p'} + t^2 + t^{p'} + (p - 2)^{p'})$. Taking ε_0, t and $p - 2$ small enough we can choose $\delta = 1/2$ and we deduce, as in the previous section, the following power decay

$$\begin{aligned} & \sup_{x \in D_{1/2}(0); r < 1/4} r^{-\alpha} \int_{D_r^2(x)} |\nabla S_p|^{p'} + |\nabla \vec{R}_p|^{p'} + |\nabla \vec{w}_p|^{p'} + |\nabla \vec{\tau}_p|^{p'} \\ & \quad + (1 + |\nabla \vec{n}|^2 e^{-2\lambda})^{p/2} e^{2\lambda} dy^2 \\ & \leq C(m) e_p [1 + e_p^{p'}] \end{aligned} \tag{5.70}$$

for some $\alpha > 0$ independent of $p \simeq 2$, where we also used (5.47) and (5.50).

5.4 Improved integrability and uniform ε -regularity estimate

First of all we establish improved integrability for the following quantities : $\vec{n} \cdot \nabla \vec{R}_p, \vec{n} \cdot \nabla \vec{v}_p, \vec{n} \times \nabla \vec{R}_p + \nabla \vec{v}_p, \nabla S_p$. We deduce from (5.21)

$$\vec{n} \cdot \nabla^\perp \vec{R}_p = \nabla S_p - \vec{n} \cdot \nabla \vec{v}_p \tag{5.71}$$

This gives in particular

$$\begin{cases} \operatorname{div} (\vec{n} \cdot \nabla \vec{R}_p) = \nabla \vec{n} \cdot \nabla^\perp \vec{v}_p \\ \operatorname{curl} (\vec{n} \cdot \nabla \vec{R}_p) = -\nabla \vec{n} \cdot \nabla^\perp \vec{R}_p \end{cases} \tag{5.72}$$

Observe that

$$\begin{aligned} & \int_{D_2^2} (1 + |\nabla \vec{n}|^2 e^{-2\lambda})^{p/2} e^{2\lambda} dy^2 \leq e_p \Rightarrow e^{2\bar{\lambda}} \leq C(m) e_p \text{ and} \\ & \int_{D_1^2} |\nabla \vec{n}|^p dx^2 \leq C(m) e_p^{p/2}. \end{aligned} \tag{5.73}$$

Combining (5.70) and (5.72) and (5.73) gives

$$\sup_{x \in D_{1/2}(0); r < 1/4} r^{-\alpha/p'} \int_{D_r^2(x)} \left| \operatorname{div} (\vec{n} \cdot \nabla \vec{R}_p) \right| + \left| \operatorname{curl} (\vec{n} \cdot \nabla \vec{R}_p) \right| \leq C(m) e_p^{1/p'} [1 + e_p] e_p^{1/2} \tag{5.74}$$

Adams elliptic estimates from [1] gives then that for any $q < \frac{2-\alpha/p'}{1-\alpha/p'}$

$$\|\vec{n} \cdot \nabla \vec{R}_p\|_{L^q(D_{1/4}^2)} \leq C(m) e_p^{1/p'} [1 + e_p] \left[1 + e_p^{1/2} \right]. \tag{5.75}$$

From (5.32) and (5.33) we have

$$\vec{n} \cdot \nabla^\perp \vec{v}_p = -\nabla Y_p + \vec{n} \cdot \nabla (\vec{\sigma}_p + \vec{u}_p) \tag{5.76}$$

This gives in particular

$$\begin{cases} \operatorname{div} (\vec{n} \cdot \nabla \vec{v}_p) = -\nabla \vec{n} \cdot \nabla^\perp (\sigma_p + \vec{u}_p) \\ \operatorname{curl} (\vec{n} \cdot \nabla \vec{v}_p) = -\nabla \vec{n} \cdot \nabla^\perp \vec{v}_p \end{cases} \tag{5.77}$$

and arguing exactly as for $\vec{n} \cdot \nabla \vec{R}_p$ we obtain that for any $q < \frac{2-\alpha/p'}{1-\alpha/p'}$

$$\|\vec{n} \cdot \nabla \vec{v}_p\|_{L^q(D_{1/4}^2)} \leq C(m) e_p^{1/p'} [1 + e_p] \left[1 + e_p^{1/2} \right]. \tag{5.78}$$

Combining (5.71), (5.75) and (5.78) we obtain that for any $q < \frac{2-\alpha/p'}{1-\alpha/p'}$

$$\|\nabla S_p\|_{L^q(D_{1/4}^2)} \leq C(m) e_p^{1/p'} [1 + e_p] [1 + e_p^{1/2}]. \tag{5.79}$$

Using now (5.35), we have

$$\begin{cases} \operatorname{div}(\vec{n} \times \nabla \vec{R}_p + \nabla \vec{v}_p) = -\nabla \vec{n} \times \nabla^\perp \vec{v}_p + \operatorname{div}(\nabla S_p \vec{n}) \\ \operatorname{curl}(\vec{n} \times \nabla \vec{R}_p + \nabla \vec{v}_p) = -\nabla \vec{n} \cdot \nabla^\perp \vec{R}_p \end{cases} \tag{5.80}$$

Combining (5.80) and (5.79) we obtain as before that for any $q < \frac{2-\alpha/p'}{1-\alpha/p'}$

$$\|\vec{n} \times \nabla \vec{R}_p + \nabla \vec{v}_p\|_{L^q(D_{1/4}^2)} \leq C(m) e_p^{1/p'} [1 + e_p] [1 + e_p^{1/2}]. \tag{5.81}$$

and combining (5.75) and (5.81) we obtain that for any $q < \frac{2-\alpha/p'}{1-\alpha/p'}$

$$\|\vec{n} \times \nabla \vec{v}_p - \nabla \vec{R}_p\|_{L^q(D_{1/4}^2)} \leq C(m) e_p^{1/p'} [1 + e_p] [1 + e_p^{1/2}]. \tag{5.82}$$

Combining (5.74) and (5.43) we obtain

$$\sup_{x \in D_{1/2}(0); r < 1/4} r^{-\alpha/p'} \int_{D_r^2(x)} |\Delta \vec{w}_p| dy^2 \leq C(m) e_p^{1/p'} [1 + e_p] e_p^{1/2} \tag{5.83}$$

and we deduce as before that for any $q < \frac{2-\alpha/p'}{1-\alpha/p'}$

$$\|\nabla \vec{w}_p\|_{L^q(D_{1/4}^2)} \leq C(m) e_p^{1/p'} [1 + e_p] [1 + e_p^{1/2}]. \tag{5.84}$$

From (5.59), using again the decomposition of $\vec{n} = \vec{n}^0 + \vec{n}^1$ into an harmonic part and a trace free part, we deduce by standard elliptic estimates that, for any $x \in D_{1/8}(0)$ and any $r < 1/16$,

$$\int_{D_{r/2}(x)} |\nabla \vec{n}|^p \leq C \left[\int_{D_r(x)} |\nabla \vec{n}|^{2p/3} dy^2 \right]^{3/2} + C \int_{D_r(x)} H^p e^{p\lambda} dy^2 \tag{5.85}$$

From which we deduce (still for a constant $C(m) > 0$ independent of r, x and $3 \geq p \geq 2$ but possibly depending on m)

$$\begin{aligned} \int_{D_{r/2}(x)} |\nabla \vec{n}|^p e^{(2-p)\lambda} dy^2 &\leq C(m) \left[\int_{D_r(x)} |\nabla \vec{n}|^{2p/3} e^{(4-2p)/3 \lambda} dy^2 \right]^{3/2} \\ &+ C(m) \int_{D_r(x)} H^p e^{2\lambda} dy^2 \end{aligned} \tag{5.86}$$

Using (5.57), we have also

$$\int_{D_r(x)} |H|^p e^{2\lambda} dy^2 \leq C(m) \int_{D_r(x)} \left[|\nabla \vec{R}_p - \vec{n} \times \nabla \vec{v}_p|^{p'} + |\nabla \vec{w}_p|^{p'} + |\nabla \vec{\tau}_p|^{p'} \right] dy^2. \tag{5.87}$$

Decomposing again $\vec{\tau} := \vec{\tau}^0 + \vec{\tau}^1$ into an harmonic part and a trace free part, using (5.67), we deduce from (5.67) by standard elliptic estimates that, for any $x \in D_{1/8}(0)$ and any $r < 1/16$,

$$\begin{aligned} \int_{D_r(x)} |\nabla \vec{\tau}_p|^{p'} dy^2 &\leq C(m) \left[\int_{D_{2r}(x)} |\nabla \vec{\tau}_p|^{3/2} dy^2 \right]^{2p'/3} \\ &+ C(m) [p - 2]^{p'} \int_{D_{2r}(x)} (1 + |\nabla \vec{n}|^2 e^{-2\lambda})^{p/2} e^{2\lambda} dy^2 \end{aligned} \tag{5.88}$$

Combining (5.86), (5.87) and (5.88) we obtain

$$\begin{aligned} &\int_{D_{r/2}(x)} \left[(1 + |\nabla \vec{n}|^2 e^{-2\lambda})^{p/2} e^{2\lambda} + |\nabla \vec{\tau}_p|^{p'} \right] dy^2 \\ &\leq C(m) \left[\int_{D_{2r}(x)} \left[(1 + |\nabla \vec{n}|^2 e^{-2\lambda})^p e^{2\lambda} + |\nabla \vec{\tau}_p|^{p'} \right]^{2/3} dy^2 \right]^{3/2} \\ &+ C(m) \int_{D_{2r}(x)} \left[|\nabla \vec{R}_p - \vec{n} \times \nabla \vec{v}_p|^{p'} + |\nabla \vec{w}_p|^{p'} \right] dy^2 \\ &+ C(m) [p - 2]^{p'} \int_{D_{2r}(x)} (1 + |\nabla \vec{n}|^2 e^{-2\lambda})^{p/2} e^{2\lambda} dy^2 \end{aligned} \tag{5.89}$$

Using Gehring type lemma given by proposition 1.1 of [7] as well as estimates (5.82) and (5.84), we obtain for $p - 2$ small enough the existence of $q > \max\{p, 2 + \beta\}$ where β is independent of p larger or equal than 2 and sufficiently close to 2 such that

$$\left\| |\nabla \vec{n}| e^{\lambda(2-p)/p} \right\|_{L^q(D_{1/8}(0))} \leq C(m) e_p^{1/p'} [1 + e_p] [1 + e_p^{1/2}]. \tag{5.90}$$

This implies Theorem 1.3. □

Acknowledgements This work has been initiated while the first author was visiting the Forschungsinstitut für Mathematik at the ETH in Zürich. He would like to thank the institute for the excellent working conditions and stimulating environment. The authors would like to thank the referee for useful suggestions and comments.

A Appendix

The following lemma is well known but we give a proof for the reader’s convenience.

Lemma A.1 *Let f be an harmonic function on the unit disc $D_1(0)$, then for any $p \in [1, 2]$ and any $t \in (0, 1)$ the following inequality holds*

$$\int_{D_t(0)} |f|^p dx^2 \leq C t^2 \int_{D_1(0)} |f|^p dx^2, \tag{A.1}$$

where $C > 0$ is universal. Similarly, the following inequality holds for any $t \in (0, 1)$

$$\|f\|_{L^{2,\infty}(D_t(0))} \leq C t \|f\|_{L^{2,\infty}(D_1(0))}. \tag{A.2}$$

□

Proof of Lemma A.1

Since f is harmonic we have

$$\Delta |f|^2 = 2 |\nabla f|^2 \geq 0$$

This implies that

$$\int_{\partial D_r(0)} \partial_r |f|^2 \geq 0$$

from which we deduce that

$$\frac{d}{dr} \left[\frac{1}{r^2} \int_{D_r(0)} |f|^2 dx^2 \right] = \frac{d}{dr} \left[\int_{D_1(0)} |f(rx)|^2 dx^2 \right] = r \int_0^1 ds \int_{\partial D_s(0)} \partial_r |f|^2(rx) \geq 0$$

Let $t < 1/2$ we have using Hölder for $p \in [1, 2]$

$$\begin{aligned} \int_{D_t(0)} |f|^p dx^2 &\leq \pi^{1-p/2} t^{2-p} \left[\int_{D_t(0)} |f|^2(x) dx^2 \right]^{p/2} \\ &\leq \pi^{1-p/2} t^{2-p} \left[4t^2 \int_{D_{1/2}(0)} |f|^2(x) dx^2 \right]^{p/2} \end{aligned}$$

Since f is harmonic, classical elliptic estimates give

$$\left[\int_{D_{1/2}(0)} |f|^2(x) dx^2 \right]^{p/2} \leq C \int_{D_1(0)} |f|^p(x) dx^2.$$

where C is a universal constant. Combining the two previous identities gives (A.1). The inequality (A.2) is derived in a similar way. □

We now prove the following integrability by compensation result due to Y. Ge in the case $p = 2$ [6] and to the authors for $p \neq 2$ (Lemma IV.2 in [2]).

Lemma A.2 *Let D be a disk. Consider the divergence-form problem*

$$\begin{cases} \Delta \varphi = \nabla^\perp a \cdot \nabla b, & \text{in } D \\ \varphi = 0, & \text{on } \partial D, \end{cases}$$

where $\nabla a \in L^{2,\infty}(D)$ and $\nabla b \in L^p(D)$, for some $p \in (1, \infty)$. There holds

$$\|\nabla \varphi\|_{L^p(D)} \leq C_p \|\nabla a\|_{L^{2,\infty}(D)} \|\nabla b\|_{L^p(D)}, \tag{A.3}$$

for some constant $C_p > 0$ depending only on p which satisfies the estimate

$$C_p \leq C(p_0, p_1) \quad \forall 1 < p_0 \leq p \leq p_1 < \infty, \tag{A.4}$$

for some constant $C(p_0, p_1)$ independent of p .

Proof The inequality (A.3) is proved in details in the aforementioned references. The estimate (A.4) can be reached using the Marcinkiewicz interpolation theorem. Let $1 < p_0 \leq p \leq p_1 < \infty$. As the operator $\nabla b \mapsto \|\nabla a\|_{L^{2,\infty}}^{-1} \nabla \varphi$ is strongly continuous from $L^{(p_0+1)/2}$ to itself and from L^{p_1+1} to itself with norms $M_{(p_0+1)/2}$ and M_{p_1+1} respectively, then it is also strongly continuous from L^p to L^p with norm

$$p^{1/p} \left(\frac{M_{(p_0+1)/2}^{(p_0+1)/2}}{p - (p_0 + 1)/2} + \frac{M_{p_1+1}^{p_1+1}}{p_1 + 1 - p} \right) \leq \frac{3}{2} \left(\frac{2}{p_0 - 1} M_{(p_0+1)/2}^{(p_0+1)/2} + M_{p_1+1}^{p_1+1} \right),$$

which depends only on p_0 and p_1 , as desired. □

We will need a Wente-type result.

Lemma A.3 *Let $q \in (2, \infty)$ and $p \in [2, q]$. Consider the divergence-form problem*

$$\begin{cases} \Delta\varphi = \nabla^\perp a \cdot \nabla b, & \text{in } D \\ \varphi = 0, & \text{on } \partial D, \end{cases}$$

where $\nabla a \in L^p(D)$ and $\nabla b \in L^{p',\infty}(D)$, for some $s \in (1, \infty)$ with $p' := p/(p - 1)$. There holds

$$\|\nabla\varphi\|_{L^{2,p}(D)} \leq C_q \|\nabla a\|_{L^p(D)} \|\nabla b\|_{L^{p',\infty}(D)}, \tag{A.5}$$

for some constant $C_q > 0$ depending only on q .

Proof The estimate (A.5) holds for $p = 2$, as shown by Y. Ge in [6]. Let now $p > 2$ so that $p' < 2$. By a continuous embedding of L. Tartar (see Lemma 31.2 in [21]) and the Hölder inequality for Lorentz spaces, we find:

$$b\nabla^\perp a \in W^{1,(p',\infty)} \cdot L^p \subset L^{\frac{2p}{p-2},\infty} \cdot L^{p,p} \subset L^{2,p}.$$

Since $\Delta\varphi = \text{div}(b\nabla^\perp a)$, the estimate (A.5) follows accordingly for $p > 2$.

Now, if $p < 2$ so that $p' > 2$, the Sobolev–Lorentz embedding theorem along with the Hölder inequality for Lorentz spaces, we find

$$a\nabla^\perp b \in W^{1,s} \cdot L^{p',\infty} \subset L^{\frac{2p}{2-p},p} \cdot L^{p',\infty} \subset L^{2,p},$$

which again proves (A.5) for $p < 2$, since $\Delta\varphi = -\text{div}(a\nabla^\perp b)$.

Consider the operator

$$B : (\nabla a, \nabla b) \longmapsto \nabla\varphi.$$

As we have seen above, B maps $L^{q'} \times L^{q,\infty}$ continuously into $L^{2,q'}$ with norm $M_0(q)$; and it maps continuously $L^q \times L^{q',\infty}$ into $L^{2,q}$ with norm $M_1(q)$. The bilinear interpolation result of Lions and Peetre given in Lemma 28.3 of [21] implies now that B maps continuously the spaces⁶

$$(L^{q'}, L^q)_{\theta,p} \times (L^{q,\infty}, L^{q',\infty})_{\theta,\infty} \equiv L^{\left(\frac{1-\theta}{q'} + \frac{\theta}{q}\right)^{-1},p} \times L^{\left(\frac{1-\theta}{q} + \frac{\theta}{q'}\right)^{-1},\infty}$$

into

$$(L^{2,q'}, L^{2,q})_{\theta,p} \equiv L^{2,p},$$

for all $0 \leq \theta \leq 1$ and $p^{-1} = (1 - \theta)/q' + \theta/q$. Equivalently, B maps continuously

$$L^p \times L^{p',\infty} \text{ into } L^{2,p} \quad \text{for } p \in [q', q]$$

with a norm $C_q = C_q(M_0(q), M_1(q), q)$ depending only on q , owing to the nature of interpolation. □

Lemma A.4 *Let D be a disk. Consider the divergence-form problem*

$$\begin{cases} \Delta\varphi = \nabla^\perp a \cdot \nabla b, & \text{in } D \\ \varphi = 0, & \text{on } \partial D, \end{cases}$$

where $\nabla a \in L^2(D)$ and $\nabla b \in L^{p,q}(D)$, for some $p \in (1, \infty)$ and $2 \leq q \leq \infty$. There holds

$$\|\nabla\varphi\|_{L^{p,q^*}(D)} \leq C_p \|\nabla a\|_{L^2(D)} \|\nabla b\|_{L^{p,q}(D)}, \tag{A.6}$$

⁶ See [21] for the notation and basic results about interpolation spaces.

where $1/q^* = 1/2 + 1/q$ for some constant $C_p > 0$ depending only on p which satisfies the estimate

$$C_p \leq C(p_0, p_1) \quad \forall 1 < p_0 \leq p \leq p_1 < \infty, \tag{A.7}$$

for some constant $C(p_0, p_1)$ independent of p . □

Proof of Lemma A.4 It suffices to consider the case when $\infty > p_1 > 2$ and $1 < p_0 < 2$. According to the Hölder inequality for Lorentz spaces, it holds

$$L^2 \cdot L^{p_1, q} = L^{2,2} \cdot L^{p_1, q} \subset L^{\frac{2p_1}{2+p_1}, \frac{2q}{2+q}}. \tag{A.8}$$

Since $p_1 > 2$, we call upon the usual Calderon-Zygmund theorem and the Sobolev–Lorentz embedding theorem of L. Tartar (Lemma 31.2 in [21]) to obtain that $\nabla\varphi \in W^{1, (\frac{2p_1}{2+p_1}, \frac{2q}{2+q})} \subset L^{p_1, \frac{2q}{2+q}}$ with the estimate (A.6) with a constant independent of q .

Next, for $1 < p_0 \leq 2$, we use the divergence-form structure of the equation. Note that

$$\Delta\varphi = \operatorname{div}(b \nabla^\perp a), \tag{A.9}$$

and that

$$b \nabla^\perp a \in W^{1, (p_0, q)} \cdot L^2 \subset L^{\frac{2p_0}{2-p_0}, q} \cdot L^{2,2} \subset L^{p_0, \frac{2q}{2+q}}, \tag{A.10}$$

where we have again used the Sobolev–Lorentz embedding theorem of J. Peetre and the Hölder inequality for Lorentz spaces. The desired (A.6) follows immediately from (A.9) and (A.10).

Now for $p \in (p_0, p_1)$ the estimate (A.7) can be reached using the Marcinkiewicz interpolation theorem just as is done in the proof of Lemma A.2. □

Lemma A.5 Let $a(x)$ be a positive function on the disc D^2 which satisfies $c_0 > a(x) > c_0^{-1}$, for some positive constant c_0 and let $A > 0$. Let $p \in [2, 3]$ and define the operator

$$S(f) := \left[\frac{(A^2 + a^2(x)|f|^2)^{(p-2)/2}}{(A^2 + \|f\|_p^2)^{(p-2)/2}} \right] f.$$

If T is a continuous operator from $L^s(D^2)$ to itself for any $s \in [4/3, 6]$ then, for any $\alpha \in (p - 2, 1/2)$ one has

$$\|TS(f) - S(Tf)\|_{p/(p-1)} \leq 2C\alpha^{-1}(p-2)\|T\| \left[\|f\|_p^\alpha + (A^2 + \|f\|_p^2)^{\alpha/2} \right] \|f\|_p^{1-\alpha}. \tag{A.11}$$

where $\|T\| := \sup_{s \in [4/3, 6]} \|T\|_{L^s \rightarrow L^s}$. □

Proof of Lemma A.5 We follow an idea from complex interpolation introduced in [20] and also used in [10]. For $z = a + ib \in D_{1/2}^2$ we introduce

$$S_z(f) := \left[\frac{(A^2 + a^2(x)|f|^2)^{1/2}}{(A^2 + \|f\|_p^2)^{1/2}} \right]^z f.$$

Denote

$$p_a := \frac{p}{1+a} \in (2p/3, 2p) \subset (4/3, 6) \quad \text{and} \quad p'_a = \frac{pa}{pa-1} = \frac{p'}{1-p'/p}.$$

It is not difficult to see that S_z maps L^p into L^{pa} and we shall now compute the norm of S_z between these two spaces. We have

$$\int_{D^2} |S_z(f)|^{pa} dx^2 \leq \int_{D^2} \left| \frac{(A^2 + a^2(x)|f|^2)^{1/2}}{(A^2 + \|f\|_p^2)^{1/2}} \right|^{ap/(1+a)} |f|^{p/(1+a)} dx^2$$

We first assume $\|f\|_p > A$. In this first case we have

$$\begin{aligned} \int_{D^2} |S_z(f)|^{pa} dx^2 &\leq C \|f\|_p^{-ap/(1+a)} \\ &\quad \times \int_{D^2} \left| (A^2 + a^2(x)|f|^2)^{1/2} \right|^{ap/(1+a)} |f|^{p/(1+a)} dx^2 \\ &\leq C \|f\|_p^{-ap/(1+a)} \int_{|f|>A} |f|^p dx^2 + C \|f\|_p^{-ap/(1+a)} \\ &\quad \times \int_{|f|\leq A} A^{ap/(1+a)} |f|^{p/(1+a)} dx^2. \end{aligned} \tag{A.12}$$

for $a \geq 0$ first one has

$$\int_{|f|\leq A} A^{ap/(1+a)} |f|^{p/(1+a)} dx^2 \leq \|f\|_p^{ap/(1+a)} \|f\|_p^{p/(1+a)}, \tag{A.13}$$

and combining (A.12) and (A.13) one obtains

$$\int_{D^2} |S_z(f)|^{pa} dx^2 \leq C \|f\|_p^{pa}. \tag{A.14}$$

Nor for $a < 0$, still under the assumption that $\|f\|_p > A$ one has

$$\int_{|f|\leq A} A^{ap/(1+a)} |f|^{p/(1+a)} dx^2 \leq \int_{|f|\leq A} |f|^p dx^2. \tag{A.15}$$

Combining (A.12) and (A.15) gives again (A.14). We assume now $\|f\|_p \leq A$. In this case we have

$$\begin{aligned} \int_{D^2} |S_z(f)|^{pa} dx^2 &\leq C A^{-ap/(1+a)} \int_{D^2} \left| (A^2 + a^2(x)|f|^2)^{1/2} \right|^{ap/(1+a)} |f|^{p/(1+a)} dx^2 \\ &\leq C A^{-ap/(1+a)} \int_{|f|>A} |f|^p dx^2 + C A^{-ap/(1+a)} \\ &\quad \times \int_{|f|\leq A} A^{ap/(1+a)} |f|^{p/(1+a)} dx^2. \end{aligned} \tag{A.16}$$

First, for $a > 0$, under this assumption $\|f\|_p \leq A$ one gets

$$A^{-ap/(1+a)} \int_{|f|>A} |f|^p dx^2 \leq \|f\|_p^{-ap/(1+a)} \|f\|_p^p = \|f\|_p^{pa}. \tag{A.17}$$

and in this case again we have (A.14). Now, for $a < 0$ and $\|f\|_p \leq A$ we have from (A.16)

$$\int_{D^2} |S_z(f)|^{pa} dx^2 \leq C \left(\frac{A}{\|f\|_p} \right)^{-ap/(1+a)} \|f\|_p^{pa}. \tag{A.18}$$

This gives in all cases

$$\|S_z(f)\|_{p_a} \leq C \left[1 + \left(\frac{\|f\|_p^2}{A^2 + \|f\|_p^2} \right)^{\alpha/2} \right] \|f\|_p. \tag{A.19}$$

Denote

$$Q_z(g) := \left(\frac{\|g\|_{p'}}{|g|} \right)^{\frac{\bar{z}p'}{p}} g.$$

Observe that we have

$$\|Q_z(g)\|_{p'_a} = \|g\|_{p'} \tag{A.20}$$

Let $f \in L^p(D^2)$ and $g \in L^{p'}(D^2)$ and let T be a continuous operator from $L^s(D^2)$ into $L^s(D^2)$ for any $s \in [4/3, 6]$ such that

$$\|T\| := \sup_{s \in [4/3, 6]} \|T\|_{L^s \rightarrow L^s} < +\infty.$$

We introduce the function defined on the disc D^2 by

$$\varphi(z) := \int_{D^2} [TS_z(f) - S_z(Tf)] \overline{Q_z(g)} dx^2.$$

φ is obviously holomorphic and one has

$$|\varphi(z)| \leq \|T\| \|S_z(f)\|_{p_a} \|Q_z(g)\|_{p'_a} \leq C \|T\| \left[1 + \left(\frac{\|f\|_p^2}{A^2 + \|f\|_p^2} \right)^{\Re(z)/2} \right] \|f\|_p \|g\|_{p'}. \tag{A.21}$$

Hence

$$\sup_{|z| \leq \alpha} |\varphi(z)| \leq C \|T\| \left[1 + \left(\frac{A^2 + \|f\|_p^2}{\|f\|_p^2} \right)^{\alpha/2} \right] \|f\|_p \|g\|_{p'} \tag{A.22}$$

Using Schwartz lemma we obtain the pointwise estimate on D_α^2

$$|\varphi(z)| \leq C \alpha^{-1} |z| \|T\| \left[1 + \left(\frac{A^2 + \|f\|_p^2}{\|f\|_p^2} \right)^{\alpha/2} \right] \|f\|_p \|g\|_{p'} \tag{A.23}$$

Hence, using (A.20), after observing that for $g = (h \|h\|_{p'_a}^{-\bar{z}p'/p})^{1/(1-\bar{z}p'/p)}$ one has $Q_z(g) = h$, we deduce that

$$\begin{aligned} & \sup_{\|h\|_{p'_a} \leq 1} \int_{D^2} [TS_z(f) - S_z(Tf)] \bar{h} dx^2 \\ & \leq 2C \alpha^{-1} |z| \|T\| \left[1 + \left(\frac{A^2 + \|f\|_p^2}{\|f\|_p^2} \right)^{\alpha/2} \right] \|f\|_p \end{aligned} \tag{A.24}$$

We apply (A.24) to $z = p - 2$ and we obtain

$$\begin{aligned} & \|TS_{p-2}(f) - S_{p-2}(Tf)\|_{p/(p-1)} \\ & \leq 2C \alpha^{-1} |z| \|T\| \left[\|f\|_p^\alpha + (A^2 + \|f\|_p^2)^{\alpha/2} \right] \|f\|_p^{1-\alpha}. \end{aligned} \tag{A.25}$$

which implies the lemma. □

Lemma A.6 *Let $p \in [2, 3)$, $r < 1$, $B \in \mathbb{R}$, $a(x) \in L^\infty(D_r^2)$ and \vec{n} such that $\nabla \vec{n} \in L^p(D_r^2)$ and assume $c_0^{-1} \leq a(x) \leq c_0$. Denote $\zeta \in L^{p/(p-1)}$ to be the solution of*

$$\begin{cases} -\Delta \vec{\zeta} = \operatorname{div} \left(\left(B^2 + a^2(x) |\nabla^\perp \vec{n}|^2 \right)^{p/2-1} \nabla^\perp \vec{n} \right) & \text{in } D_r^2 \\ \vec{\zeta} = 0 & \text{on } \partial D_r^2 \end{cases} \tag{A.26}$$

Then the following inequality holds for any $p - 2 < \alpha < 1/2$

$$\|\nabla \vec{\zeta}\|_{L^{p/(p-1)}(D_r^2)} \leq C \frac{(p-2)}{\alpha} \left(\|\nabla \vec{n}\|_{L^p(D_r^2)}^{p-1} + (r^{2/p} B)^{(p-2)+\alpha} \|\nabla \vec{n}\|_{L^p(D_r^2)}^{1-\alpha} \right), \tag{A.27}$$

where C is independent of p, α, r, B and \vec{n} . □

Proof of Lemma A.6 Let $\vec{n}_r(x) := \vec{n}(rx)$ and $\vec{\zeta}_r(x) := \vec{\zeta}(rx)$. The following system is satisfied

$$\begin{cases} -\Delta \vec{\zeta}_r = r^{-p+2} \operatorname{div} \left(\left(B^2 r^2 + a^2(x) |\nabla^\perp \vec{n}_r|^2 \right)^{p/2-1} \nabla^\perp \vec{n}_r \right) & \text{in } D_1^2 \\ \vec{\zeta}_r = 0 & \text{on } \partial D_1^2 \end{cases} \tag{A.28}$$

Let T be the operator which to an $L^s(D^2)$ vector Field X assigns ∇u the solution to

$$\begin{aligned} -\operatorname{div}(\nabla u) &= \operatorname{div}(X) && \text{on } \partial D^2 \\ u &= 0 && \text{on } \partial D^2 \end{aligned}$$

Calderon Zygmund Theory gives that T is continuous from $L^s(D^2)$ into $L^s(D^2)$ and $\sup_{s \in [4/3, 6]} \|T\|_{L^s \rightarrow L^s} = \|T\| < +\infty$. Observe that

$$T \left(\left(B^2 r^2 + a^2(x) |\nabla^\perp \vec{n}_r|^2 \right)^{p/2-1} \nabla^\perp \vec{n}_r \right) = r^{p-2} \nabla \vec{\zeta}_r \quad \text{and} \quad T(\nabla^\perp \vec{n}_r) = 0 \tag{A.29}$$

Denote $A := Br$ and

$$S(f) := \left(\frac{A^2 + a^2(x) |f|^2}{A^2 + \|f\|_p^2} \right)^{p/2-1} f$$

Combining (A.11) with (A.29) one obtains

$$\begin{aligned} & \frac{r^{p-2} \|\nabla \vec{\zeta}_r\|_{L^{p/(p-1)}(D_1^2)}}{\left(A^2 + \|\nabla \vec{n}_r\|_p^2 \right)^{p/2-1}} \\ & \leq 2 C \alpha^{-1} (p-2) \|T\| \left[\|\nabla \vec{n}_r\|_p^\alpha + \left(A^2 + \|\nabla \vec{n}_r\|_p^2 \right)^{\alpha/2} \right] \|\nabla \vec{n}_r\|_p^{1-\alpha}. \end{aligned} \tag{A.30}$$

Observe that

$$\begin{aligned} \|\nabla \vec{n}_r\|_{L^p(D_1^2)} &= r^{1-2/p} \|\nabla \vec{n}\|_{L^p(D_r^2)} \quad \text{and} \\ \|\nabla \vec{\zeta}_r\|_{L^{p/(p-1)}(D_1^2)} &= r^{-1+2/p} \|\nabla \vec{\zeta}\|_{L^{p/(p-1)}(D_r^2)}. \end{aligned}$$

Then

$$\|\nabla \vec{\zeta}\|_{L^{p/(p-1)}(D_r^2)} \leq 2C \frac{(p-2)}{\alpha} \|T\| \left(\|\nabla \vec{n}\|_{L^p(D_r^2)}^{p-1} + (r^{2/p} B)^{(p-2)+\alpha} \|\nabla \vec{n}\|_{L^p(D_r^2)}^{1-\alpha} \right) \quad (\text{A.31})$$

which implies the lemma. \square

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