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Uniqueness of tangent cones for calibrated 2-cycles and other regularity problems

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Abstract

This work investigates regularity properties of three different mathematical objects by establishing blow-up rates for them.

In the first chapter we study calibrated 2-cycles on a Riemannian manifold and show existence and uniqueness of tangent cones for them. Tangent cones are obtained by considering radial dilations around a point, called "blow-up", and existence of a limit for these dilations is derived from a monotonicity formula. The main part of this chapter is then devoted to proving that these tangent cones are independent of the subsequence chosen and a blow-up rate for the radial dilations will yield this result. The key step in obtaining such a blow-up rate is to observe that such currents are locally essentially J-holomorphic and thus can be pushed forward by the canonical projection $\pi : \mathbf{C}^m \setminus \{0\} \longrightarrow \mathbf{CP}^{m-1}$ to yield a calibrated 2-current in $\mathbf{CP}^{m-1}$. A general result for such currents on $\mathbf{CP}^{m-1}$ then allows us integrate by parts which reduces the situation to the boundary of the intersection of the current with a suitable small ball. Since this is a 1-dimensional cycle we can prove a Poincaré type inequality on it which enables us to estimate the mass of the dilated current. From a suitable choice of radii a blow-up rate is then deduced using an iteration argument.

In the second chapter we apply a similar technique to J-holomorphic maps between almost Kähler manifolds and obtain regularity results for them. First a monotonicity formula is used to prove an ϵ-regularity result giving an estimate on the size of the singular set. Then a slicing argument using the map π above together with the co-area formula yields a blow-up rate for radial dilations around each point. From this uniqueness of tangent maps is deduced.

The third chapter is devoted to proving regularity results for polyharmonic maps in higher dimensions. We derive their Euler-Lagrange equation and use a sharp Gagliardo-Nirenberg interpolation inequality proved in the appendix to show regularity of such maps under certain integrability conditions avoiding the use of moving frames. In view of the monotonicity formulae for stationary harmonic and biharmonic maps we infer partial regularity in these cases. The same technique is also applied to more general k-th order elliptic systems with critical growth.

Zusammenfassung

Diese Arbeit untersucht Regularitätseigenschaften dreier verschiedener mathematischer Objekte mittels Konvergenzraten für geeignete Skalierungen.

Das erste Kapitel behandelt kalibrierte 2-Zykeln auf Riemannschen Mannigfaltigkeiten, wofür Existenz und Eindeutigkeit des Tangentialkegels gezeigt werden. Wird der Zykel radial um einen fixierten Punkt gestreckt, folgt die Existenz eines Grenzwerts für eine Teilfolge von Radien aus einer Monotonieformel und jeder solche Grenzwert wird als Tangentialkegel bezeichnet. Allerdings ist a priori nicht klar, ob dieser Grenzwert von der Wahl der Teilfolge abhängt, weshalb der Hauptteil des Kapitels der Beantwortung dieser Frage gewidmet ist. Der wichtigste Schritt besteht darin, zu zeigen, dass solch ein Zykel beinahe J-holomorph ist, und daher mittels der kanonischen Projektion $\pi : \mathbf{C}^m \setminus \{0\} \longrightarrow \mathbf{CP}^{m-1}$ auf einen kalibrierten 2-Zykel auf $\mathbf{CP}^{m-1}$ abgebildet werden kann. Danach wird ein allgemeines Resultat für 2-Zykeln auf $\mathbf{CP}^{m-1}$ verwendet, um eine partielle Integration zu rechtfertigen, welche dann wiederum erlaubt, das Problem auf den Rand des Schnitts des ursprünglichen Zykels mit einem kleinen Ball zu reduzieren. Da dieser Schnitt selbst einen 1-dimensionalen Zykel darstellt, können wir darauf eine Poincaré Ungleichung zeigen, welche wiederum eine Abschätzung für die Masse des skalierten Zykels liefert. Eine geeignete Wahl von Radien erlaubt es dann eine Konvergenzrate gegen die Dichte des Stromes am fixierten Punkt zu finden, woraus wiederum die Eindeutigkeit des Grenzwerts folgt.

Im zweiten Kapitel wenden wir ähnliche Techniken auf Regularitätsfragen für J-holomorphe Abbildungen zwischen beinahe Kählermannigfaltigkeiten an. Zunächst wird dafür eine Monotonieformel bewiesen, welche die Herleitung von partieller Regularität erlaubt, woraus eine Abschätzung für die Dimension der Singularitätenmenge folgt. Desweiteren wird, mittels Schnitten durch die bereits oben erwähnte Abbildung π, eine Konvergenzrate der Energie der skalierten Abbildung um jeden Punkt bewiesen. Jene führt dann zur Existenz einer eindeutigen Tangentialabbildung an jedem Punkt.

Im dritten und letzten Kapitel werden Regularitätsresultate für polyharmonische Abbildungen in höheren Dimensionen bewiesen. Ausgehend von deren Euler-Lagrange Gleichung wird eine Interpolationsungleichung aus Appendix A verwendet, wodurch der Beweis im Wesentlichen mit elementaren Methoden geführt werden kann. Angesichts der Monotonieformeln für stationär harmonische und biharmonische Abbildungen, folgt in diesen Fällen sogar partielle Regularität. Ähnliche Techniken werden danach auch auf allgemeine elliptische Systeme mit kritischem Wachstum angewandt.

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Chapter 1

Calibrated 2-currents

Abstract

In this chapter we prove uniqueness of tangent cones for calibrated 2-currents. First calibrated 2-currents are introduced and an overview of previous results is given. Next we prove a monotonicity formula and derive first properties from it. The main theorem is obtained from a projection argument allowing us to deduce a rate of convergence from which uniqueness of tangent cones immediately follows. Finally, we give an outlook for future research directions and possible extensions of our techniques.

1.1 Introduction

We begin this chapter by introducing the notion of a calibration on a Riemannian manifold. Although all our arguments later on will be local and hence can be carried out in Euclidean space, the main motivation for studying calibrations comes from their relevance in applications in differential geometry, whence the more general setting on manifolds is used.

Let M be a smooth compact oriented m-dimensional manifold without boundary and let $\Omega^k(M)$ denote the smooth k-forms on M. Let g be a smooth Riemannian metric on M and denote the resulting volume form on M by vol_M. Given a smooth k-form ω on M we call ω a calibration if it satisfies

$$\omega|_\tau \leq \mathrm{vol}_M|_\tau$$

for all k-dimensional subspaces $\tau \subset T_pM$ and all $p \in M$. A smooth k-dimensional submanifold N is then called calibrated by ω if its volume form agrees with ω at all points, i.e.

$$\omega|_N = \mathrm{vol}_N \ .$$

Under the extra assumption on ω to be closed, calibrated submanifolds have the important property of being homologically volume minimising. To see this, simply let N' be a manifold in the same homology class as N and compute

$$\mathrm{vol}\,(N) = \int_N \omega|_N = \int_{N'} \omega|_N + \int_{N-N'} \omega|_N = \int_{N'} \omega|_N + \int_S d\omega \leq \int_{N'} \mathrm{vol}_{N'} = \mathrm{vol}\,(N') \ .$$

Historically, calibrations were introduced by R. Harvey and B. Lawson in [38] where they also gave the following important examples from differential geometry.

Let (M^{2m}, J, ω) be a Kähler manifold where ω is the closed Kähler form compatible with the complex structure J (this means that $\omega(\cdot, J\cdot)$ is a Riemannian metric on M). Then the $2k$-form given by $\frac{1}{k!}\omega^k$ is a calibration and any complex submanifold of M is calibrated by it. In fact, it suffices to have an almost Kähler manifold structure on M, i.e. one where the almost complex structure J need not be integrable. As an interesting extension of this scenario, in case of the complex projective space $\mathbf{CP}^n$ with its standard Kähler form $\omega_{\mathbf{CP}^n}$, the smooth parts of (complex) k-dimensional algebraic subvarieties can be seen as $2k$-dimensional submanifolds calibrated by $\frac{1}{k!}\omega^k_{\mathbf{CP}^n}$. The fact that algebraic subvarieties are homologically area-minimising had already been observed by H. Federer [24], well before the notion of a calibration was introduced. They serve as a first example to show that calibrations are of interest even for less regular objects than smooth submanifolds.

On a general closed almost complex manifold (M^{2m}, J) one can define a $k-k$-submanifold by requiring N^{2k} to satisfy $J_x(T_xN) = T_xN$ for any $x \in N$, i.e. to have almost complex tangent spaces at all points. Notice that the case of $1-1$-submanifolds is particularly interesting, since they arise as perturbations of J_0-holomorphic graphs and are generic from the existence point of view — see the introduction of [71] by T. Rivière and G. Tian. If we can find a closed form which is compatible with J, then we are back in the almost Kähler case above and $1-1$-submanifolds are homologically mass-minimising. It turns out that locally one can indeed find a 2-form ω which is compatible with J, but in general this ω will not be closed. In real dimensions at most 4, locally the form ω can in fact be constructed to be closed — see the appendix of [70] by T. Rivière and G. Tian. For higher dimensions R. Bryant [17] constructed an almost complex structure on S^6 which does not admit any compatible ω even locally. Hence in such a case $1-1$-submanifolds are still calibrated by a smooth 2-form ω but no longer area-minimising. However, naturally $1-1$-submanifolds are of geometric interest even in the absence of a symplectic form which is one of the reasons why later on we will not assume the calibration to be closed.

Another important example of a calibration is connected to Special Lagrangian submanifolds in Almost Calabi-Yau k-folds. An Almost Calabi-Yau k-fold is a $2k$-dimensional Kähler manifold (M^{2k}, J, ω) for which there exists a non-vanishing $(k,0)$-form Ω (similar to the holomorphic volume form on Calabi-Yau k-folds) satisfying

$$\frac{\omega^k}{k!} = (-1)^{\frac{k(k-1)}{2}} \left(\frac{i}{2}\right)^k \Omega \wedge \bar{\Omega} \, .$$

Then there exists a metric $\tilde{g}$ on M such that a real k-dimensional submanifold N is called Special Lagrangian if it is calibrated by $\operatorname{Re}\Omega$ — the real part of Ω. As a special case one can consider Special Lagrangian cones C in $\mathbf{C}^k$ which are calibrated by $\operatorname{Re}(dz_1 \wedge \ldots \wedge dz_k)$ and of the form $C = 0 \mathbin{⨳} T$ (see the book [25] by H. Federer for the notation), where T is a Special Legendrian $k-1$-submanifold in the sphere S^{2k-1}. These $k-1$-submanifolds are in turn calibrated by $\operatorname{Re}\left(\sum dz_1 \wedge \cdots \wedge z_i dz_{i+1} \wedge dz_{i-1} \wedge \cdots \wedge dz_k\right)$ and it is important to note that again this calibration is not a closed form.

Given a calibration on a manifold it is not at all clear whether corresponding calibrated submanifolds exist. In the situation of Special Lagrangians mentioned above a lot of effort is put into constructing actual examples of such submanifolds – see the lecture notes by D. Joyce [46] for an overview. One natural approach to resolve this issue, is to relax the notion of submanifolds and to prove existence results for less regular objects which have suitable compactness properties. This extension of smooth submanifolds leads to the notion of k-dimensional current described below.
For this we let M again be a smooth compact oriented m-dimensional manifold without boundary and let $\Omega_0^k(M)$ denote the smooth compactly supported k-forms on M. For a smooth Riemannian metric g on M we define the comass of a k-form ω to be

$$\|\omega\|_* := \sup_{x\in M} \sup_{e_1,\dots,e_k\in S_xM} |\langle \omega, e_1\wedge\ldots\wedge e_k\rangle| \ ,$$

where S_xM means the unit sphere in T_xM with respect to g_x. Given a smooth oriented k-dimensional submanifold of M we can integrate k-forms over it and hence view this submanifold as a linear functional on $\Omega_0^k(M)$. This feature will be used to extend the notion of 'k-dimensional submanifold' to objects which are no longer necessarily smooth. A k-dimensional current C in M is therefore given as a distribution on $\Omega_0^k(M)$, i.e. as a linear functional on $\Omega_0^k(M)$. The boundary of such a k-current is the $(k-1)$-current defined by $\partial C(\omega) := C(d\omega)$, where $\omega \in \Omega_0^{k-1}(M)$ and we say that a k-current C is a k-cycle if $\partial C = 0$. Using Stokes' theorem one can see that this notion of boundary indeed coincides with the usual one for smooth k-dimensional submanifolds.
The comass of k-forms allows us to extend the notion of volume of a submanifold by defining the mass $M(C)$ of a k-current C as

$$M(C) := \sup_{\omega\in\Omega_0^k(M), \|\omega\|_*\le 1} |\langle C, \omega\rangle| \ .$$

A k-cycle C is called a normal cycle if C satisfies $M(C) < +\infty$. More generally, a normal k-current is one where both $M(C)$ and $M(\partial C)$ are finite.
We call a current an integer multiplicity rectifiable k-current, if C is a normal k-current and there are k-Hausdorff measurable subsets $\mathcal{N}_j$ of oriented k-dimensional C^1-submanifolds N_j with $\mathcal{N}_i \cap \mathcal{N}_j = \emptyset$, $i \neq j$, and a multiplicity $\Theta : \bigcup_{j=1}^\infty \mathcal{N}_j \to \mathbf{Z}$ such that

$$\langle C, \psi\rangle = \sum_{j=1}^\infty \int \psi\,\Theta \; d\mathcal{H}^k \llcorner \mathcal{N}_j \ .$$

We abbreviate integer multiplicity rectifiable k-cycles by calling them integral k-cycles and, more generally, integral currents are currents for which both C and ∂C are integer multiplicity rectifiable currents. As integral currents consist of countably many pieces of C^1-submanifolds they have regularity properties in between smooth submanifolds — having the mildest — and general normal currents — having the wildest behaviour. In particular, the mass of an integral current C is given by $M(C) = \sum_{j=1}^\infty \int |\Theta| \; d\mathcal{H}^k \llcorner \mathcal{N}_j$.
The standard references for the terminology introduced above are the books by H. Federer [25] and by L. Simon [76].

In the definition of the mass of a k-current we took the supremum over all k-forms with comass at most 1 which, in general, is difficult to compute. However, we saw that for submanifolds a calibration played the rôle of the volume form which we now use to define calibrated currents.

Definition 1.1 *Let ω be a smooth k-form on (M, g) with comass equal to 1. Then we say that a k-current is calibrated by ω if for all open subsets $U \subset M$ we have*

$$\langle C \llcorner U, \omega \rangle = M(C \llcorner U) ,$$

where $C \llcorner U$ means the restriction of C to U.

Having obtained a calibrated cycle the next natural goal is to study its regularity. One intuitive reason why one can hope for extra regularity from the fact that the cycle is calibrated is that a calibration limits the possible tangent spaces such a current can have (albeit only at almost every point). The first step in looking for an improvement in regularity of these cycles is to analyse their blow-up around points, i.e. to prove whether such a blow-up exists and if it exists whether it is unique. The blow-up analysis of C around a point $x_0 \in M$ is done as follows: consider a dilation of C around x_0 which in normal coordinates near x_0 is given by the push-forward of the current C under the map $\frac{x-x_0}{r}$ — here we mean that ${\frac{x-x_0}{r}}_* C(\psi) := C({\frac{x-x_0}{r}}^* \psi)$. To analyse the behaviour of these dilations as $r \to 0$ we need to bound the mass of ${\frac{x-x_0}{r}}_* C \llcorner B_1(x_0)$ independently of r which is done using a monotonicity formula, stating that $r^{-k} M(C \llcorner B_r(x_0))$ is increasing in r, since $M({\frac{x-x_0}{r}}_* C \llcorner B_1(0)) \cong r^{-k} M(C \llcorner B_r(x_0))$. Using this and the fact that C is a cycle we apply the compactness theorem of H. Federer and W. Fleming (see [25] 4.2.17) to deduce that there exists a sequence $r_n \to 0$ and a normal k-current C_{∞,x_0} such that weakly

$${\frac{x-x_0}{r_n}}_* C \llcorner B_1 \rightharpoonup C_{\infty,x_0} .$$

It turns out that C_{∞,x_0} is a cone — called a tangent cone to C at x_0 — which is calibrated by ω_{x_0} (see section 1.3 of this chapter). The main questions related to such a construction are whether the blow-up limit is unique (i.e. independent of the subsequence of radii given by the compactness theorem) and whether the dilated currents converge to the limiting object at a certain rate.

These two questions are in fact related. From the monotonicity formula we know that the limiting density $\Theta(\|C\|, x_0) := \lim_{r \to 0} \alpha(k)^{-1} r^{-k} M(C \llcorner B_r(x_0))$ exists at all points $x_0 \in M$ (here $\alpha(k)$ means the volume of the k-dimensional unit ball) and that this limiting density is equal to the mass of any possible tangent cone. B. White [94] showed that a rate of convergence for $\alpha(k)^{-1} r^{-k} M(C \llcorner B_r(x_0))$ to the limiting density $\Theta(\|C\|, x_0)$ implies the uniqueness of the tangent cone to C at x_0 — see Theorem 3 in [94] —, i.e. if there exist constants $C_1 > 0$ and $\gamma > 0$ such that for all $r > 0$

$$\frac{M(C \llcorner B_r(x_0))}{\alpha(k) r^k} - \Theta(\|C\|, x_0) \le C_1 r^{\gamma} ,$$

then the blow-up limit is indeed independent of the subsequence chosen.

The main result of this chapter will make use of this idea. We will deduce the following theorem:

Theorem 1.1 *Let (M, g) be a closed m-dimensional Riemannian manifold and let ω be a calibrating 2-form which is C^2 but not necessarily closed. Let $x_0 \in M$ and let C be an integer rectifiable 2-cycle calibrated by ω. Then there exists a unique tangent cone to C at x_0.*

To apply B. White's result we have to establish a rate of convergence first, to which much of the remainder of this chapter is devoted. It is instructive to compare our next result below with Morrey decay rates obtained for harmonic and J-holomorphic maps and refer to chapter 2 for details.

Theorem 1.2 *Let (M, g) be a closed m-dimensional Riemannian manifold and let ω be a calibrating 2-form which is C^2 but not necessarily closed. Then for any $x_0 \in M$ there exist $r_0 > 0$, $C_1 > 0$ and $\gamma \in (0, 1]$ such that if C is an integer rectifiable 2-cycle calibrated by ω and if $r \in (0, r_0)$ we have*

$$\frac{M(C \llcorner B_r(x_0))}{\pi r^2} - \Theta(\|C\|, x_0) \leq C_1 r^\gamma \ .$$

In the following paragraph we briefly sketch the main ideas of the proof of Theorem 1.2. First we prove an almost monotonicity formula in section 1.2. Furthermore we show that calibrated 2-cycles are essentially calibrated by a constant 2-form and hence almost holomorphic cycles in some $\mathbf{C}^m$. The key observation is that due to the monotonicity formula the restriction of such cycles to small balls around x_0 can be pushed forward onto $\mathbf{CP}^{m-1}$ under the canonical projection $\pi : \mathbf{C}^m \setminus \{0\} \longrightarrow \mathbf{CP}^{m-1}$. The resulting 2-current on $\mathbf{CP}^{m-1}$ can then be shown to have small boundary and we prove a general result for such currents on $\mathbf{CP}^{m-1}$ to show that their support cannot fill all of $\mathbf{CP}^{m-1}$. We use this to perform an integration by parts which allows us to reduce to the boundary of the intersection of C with the small ball around x_0. This intersection yields a 1-dimensional integral cycle on which a Poincaré inequality can be proved (see also section 1.9 of this chapter). Hence we are able to estimate the mass of the dilated current viewed as a function of the dilating radius r in terms of its derivative. From a suitable choice of radii this is in turn estimated by the mass of the dilated current albeit with a gap proportional to r around x_0. Filling this hole and applying a standard iteration argument we obtain the theorem.

Although the problem of uniqueness of tangent cones is such a fundamental question about singularities of area-minimising or calibrated currents only few results have been obtained so far. The main result related to our work is the one by B. White mentioned above [94]. Following an idea of E. Reifenberg in [65], he deduced uniqueness of tangent cones for area-minimising 2-cycles using an epiperimetric inequality coming from a comparison argument — a completely different argument to the one we use in our proof. His result was extended to arbitrary Riemannian manifolds by S. Chang in the appendix of [19]. The problem of uniqueness of tangent cones was also studied by W. Allard and F. Almgren in [4] for 1-dimensional varifolds and in [5] for general area-minimising currents in case one of the tangent cones has a special structure. For 2-dimensional area-minimising flat chains modulo 3 in $\mathbf{R}^3$ and soap-film like minimal surfaces uniqueness results were obtained by J. Taylor in [88] and [89] respectively. Furthermore, L. Simon [77] obtained, amongst many

other results, uniqueness of tangent cones in case the tangent cone is of multiplicity 1 with an isolated singularity at 0 using ODE techniques (this work also established uniqueness of tangent maps for harmonic maps between Riemannian manifolds which is related to our results in chapter 2 below). Also he established uniqueness of some cylindrical tangent cones in [78].
In case M is an almost complex manifold with integrable almost complex structure and compatible calibrating 2-form, our uniqueness result follows from the much stronger regularity result by Y.-T. Siu in [80] using methods from several complex variables theory. Further uniqueness and non-uniqueness results in the realm of complex analytic currents and their relation to our work will be discussed in detail in section 1.10 of this chapter.
Applications of uniqueness of tangent cones results to the regularity of area-minimising 2-currents were given by S. Chang [19] who combined B. White's result [94] with techniques from F. Almgren's long regularity paper [6]. He proved that area-minimising 2-cycles are smooth aside from isolated singular points. Furthermore, T. Rivière and G. Tian in [70] and [71] showed that calibrated $1-1$-cycles are smooth aside from finitely many singular points (see also the paper [86] by C. Taubes). Although their work is in part also based on B. White's result they manage to avoid the use of F. Almgren's monumental work and therefore give a much more direct proof.

The chapter is organised as follows. In section 1.2 we prove an almost monotonicity for calibrated 2-cycles. Existence and structure of tangent cones is derived in section 1.3. Section 1.4 shows that the current can be projected, whereas section 1.5 proves properties of currents in $\mathbf{CP}^{m-1}$. In section 1.6 Theorem 1.2 is proved in a special case which is then generalised in section 1.7. Section 1.8 deduces Theorem 1.1 from this. Finally, in sections 1.9 and 1.10 we explain possible extensions and future research problems related to our proof of Theorem 1.2.

Acknowledgement: The work in this chapter is the outcome of joint work with my advisor Tristan Rivière in [64].

1.2 Monotonicity formula

In this section we will compute a monotonicity formula for general calibrated k-cycles. Although later on we will only need the situation where the cycle is integer rectifiable and 2-dimensional, we will state and prove the monotonicity formula in its full generality. In the special case where ω is closed and the cycle C is integer rectifiable the results for area-minimising currents apply. Thus from [25] Theorem 5.4.3 (2) we know that C satisfies a monotonicity formula. In the case where C is a normal cycle calibrated by a constant calibration, R. Harvey and B. Lawson proved a monotonicity formula depending only on this first order information (see [38] Theorem 5.7). Based on their proof we will now deduce an almost monotonicity formula in the general case.

As mentioned in the introduction from now on all arguments will be local and can therefore be carried out in Euclidean space. For the remainder we therefore assume that the calibration ω is a form on $\mathbf{R}^m$ which is at least C^2 but again not necessarily closed.

Thus we know that for any point $x_0 \in \mathbf{R}^m$ there exists a neighbourhood depending only on the C^2-norm of ω such that in this neighbourhood we can write $\omega(x) = \omega_0(x) + \omega_1(x)$ with $\omega_0(x) = \omega(x_0)$ and $\|\omega_1(x)\|_{C^2} = O(|x - x_0|)$, i.e. we view the form ω as C^2 perturbation of a constant form.

Proposition 1.3 *Let C be a k-dimensional normal cycle in $\mathbf{R}^m$. Assume that C is calibrated by a comass 1 k-form ω. Then there exist $C_1 > 0$, $r_0 > 0$ depending only on the C^2-norm of ω such that given $x_0 \in \mathbf{R}^m$ for any $0 < s < r \leq r_0$ we have*

$$\begin{aligned} \frac{e^{C_1 r} + C_1 r}{r^k} M(C \llcorner B_r(x_0)) \quad &- \quad \frac{e^{C_1 s} + C_1 s}{s^k} M(C \llcorner B_s(x_0)) \\ &\geq \int_{B_r \setminus B_s(x_0)} \frac{1}{|x - x_0|^k} \sum_{i=1}^{N(x)} \lambda_i(x) \left| \xi_i(x) \wedge \frac{\partial}{\partial r} \right|^2 d\|C\| \ , \end{aligned} \tag{1.1}$$

where C is represented by $\int \langle \tau, \cdot \rangle d\|C\|$ and $\sum_{i=1}^{N(x)} \lambda_i(x) \xi_i(x)$ is the decomposition of $\tau(x)$ into a convex sum of calibrated simple k-vectors.

Proof. Without loss of generality we can assume x_0 to be 0. Using the setting established above we write $\omega(x) = \omega_0(x) + \omega_1(x)$ where $\omega_0(x) = \omega(x_0)$ and $\|\omega_1(x)\|_{C^2} = O(|x|)$. Note that since ω_0 is a constant k-form we know that $\omega_0 = \frac{1}{k} d\big(\frac{\partial}{\partial r} \lrcorner \omega_0\big) = \frac{1}{k} L_{\frac{\partial}{\partial r}} \omega_0$, where $L_{\frac{\partial}{\partial r}}$ denotes the Lie derivative in the direction of $\frac{\partial}{\partial r}$ with $r = |x|$. Next we take a smooth cut-off function $\phi : \mathbf{R} \to \mathbf{R}$ such that $\phi(t) = 1$ for $t \leq \frac{1}{2}$, $\phi(t) = 0$ for $t \geq 1$ and $\phi'(t) \leq 0$. Setting $\gamma(x) := \phi\big(\frac{r}{\rho}\big)$ we obtain a smooth version of the characteristic function of $B_\rho(0)$. To replace $M(C \llcorner B_\rho(0))$ define $I(\rho) := \int_{\mathbf{R}^m} \gamma(x) \langle \tau, \omega \rangle d\|C\|$ and, furthermore, denote the tangential part of ω by

$$\omega^t := \frac{\partial}{\partial r} \lrcorner \left(\frac{\partial}{\partial r} \wedge \omega \right) \ .$$

Using this notation and the fact that C is a cycle we compute

$$\begin{aligned} kI(\rho) &= k \int_{\mathbf{R}^m} \gamma(x) \langle \tau, \omega_0 \rangle d\|C\| + k \int_{\mathbf{R}^m} \gamma(x) \langle \tau, \omega_1 \rangle d\|C\| \\ &= \int_{\mathbf{R}^m} \left\langle \tau, \gamma(x) d \left(\frac{\partial}{\partial r} \lrcorner \omega_0 \right) \right\rangle d\|C\| + k \int_{\mathbf{R}^m} \gamma(x) \langle \tau, \omega_1 \rangle d\|C\| \\ &= \int_{\mathbf{R}^m} \left\langle \tau, d \left(\gamma(x) \left(\frac{\partial}{\partial r} \lrcorner \omega_0 \right) \right) \right\rangle d\|C\| \\ &\quad - \int_{\mathbf{R}^m} \left\langle \tau, \frac{d}{dr}(\gamma(x)) dr \wedge d \left(\frac{\partial}{\partial r} \lrcorner \omega_0 \right) \right\rangle d\|C\| + k \int_{\mathbf{R}^m} \gamma(x) \langle \tau, \omega_1 \rangle d\|C\| \\ &= \rho \int_{\mathbf{R}^m} \left\langle \tau, \frac{d}{d\rho} \phi \left(\frac{r}{\rho} \right) \left[\frac{dr}{r} \wedge \frac{\partial}{\partial r} \lrcorner \omega_0 \right] \right\rangle d\|C\| + k \int_{\mathbf{R}^m} \phi \left(\frac{r}{\rho} \right) \langle \tau, \omega_1 \rangle d\|C\| \\ &= \rho \int_{\mathbf{R}^m} \left\langle \tau, \frac{d}{d\rho} \phi \left(\frac{r}{\rho} \right) (\omega_0 - \omega_0^t) \right\rangle d\|C\| + k \int_{\mathbf{R}^m} \phi \left(\frac{r}{\rho} \right) \langle \tau, \omega_1 \rangle d\|C\| \ . \end{aligned}$$

Finally, we obtain

$$\begin{aligned} kI(\rho) &= \rho\int_{\mathbf{R}^m}\left\langle \tau, \frac{d}{d\rho}\phi\left(\frac{r}{\rho}\right)\omega\right\rangle d\|C\| - \rho\int_{\mathbf{R}^m}\left\langle \tau, \frac{d}{d\rho}\phi\left(\frac{r}{\rho}\right)\omega^t\right\rangle d\|C\| \\ &\quad -\rho\int_{\mathbf{R}^m}\left\langle \tau, \frac{d}{d\rho}\phi\left(\frac{r}{\rho}\right)\omega_1\right\rangle d\|C\| + \rho\int_{\mathbf{R}^m}\left\langle \tau, \frac{d}{d\rho}\phi\left(\frac{r}{\rho}\right)\omega_1^t\right\rangle d\|C\| \\ &\quad +k\int_{\mathbf{R}^m}\phi\left(\frac{r}{\rho}\right)\langle\tau,\omega_1\rangle d\|C\| \ . \end{aligned}$$

Setting $J(\rho) := \int_{\mathbf{R}^m}\langle\tau, \phi(\frac{r}{\rho})\omega^t\rangle d\|C\|$ the above computation can be summarised as

$$\begin{aligned} -\frac{kI(\rho)}{\rho^{k+1}} + \frac{I'(\rho)}{\rho^k} - \frac{J'(\rho)}{\rho^k} &= \frac{1}{\rho^k}\int_{\mathbf{R}^m}\left\langle \tau, \frac{d}{d\rho}\phi\Big(\frac{r}{\rho}\Big)(\omega_1-\omega_1^t)\right\rangle d\|C\| \\ &\quad - \frac{k}{\rho^{k+1}}\int_{\mathbf{R}^m}\phi\Big(\frac{r}{\tau}\Big)\langle\tau,\omega_1\rangle d\|C\| \ . \end{aligned}$$

Choosing $\rho > 0$ is sufficiently small depending only on the C^2-norm of ω and using the definition of ϕ we obtain the estimate

$$\begin{aligned} \left|-\frac{kI(\rho)}{\rho^{k+1}} + \frac{I'(\rho)}{\rho^k} - \frac{J'(\rho)}{\rho^k}\right| &\leq \\ &\leq \frac{C_1}{\rho^k}\int_{\mathbf{R}^m}\left|\frac{d}{d\rho}\phi\left(\frac{r}{\rho}\right)\right|\cdot|x|d\|C\| + \frac{C_2}{\rho^{k+1}}\int_{\mathbf{R}^m}\phi\left(\frac{r}{\tau}\right)|x|d\|C\| \\ &\leq \frac{C_1}{\rho^{k-1}}\int_{\mathbf{R}^m}\frac{d}{d\rho}\phi\left(\frac{r}{\rho}\right)d\|C\| + \frac{C_2}{\rho^k}\int_{\mathbf{R}^m}\phi\left(\frac{r}{\tau}\right)d\|C\| \\ &= \frac{C_1}{\rho^{k-1}}I'(\rho) + \frac{C_2}{\rho^k}I(\rho) \\ &= \frac{C_3}{\rho^{k-1}}I(\rho) + C_3\frac{d}{d\rho}\left(\frac{I(\rho)}{\rho^{k-1}}\right) \ . \end{aligned} \tag{1.2}$$

From this estimate we deduce

$$\frac{d}{d\rho}\left(\frac{I(\rho)}{\rho^k}\right) + C_3\frac{I(\rho)}{\rho^k} \geq \frac{1}{\rho^k}\frac{d}{d\rho}J(\rho) - C_3\frac{d}{d\rho}\left(\frac{I(\rho)}{\rho^{k-1}}\right) \ ,$$

which for $\rho > 0$ possibly chosen even smaller becomes

$$\frac{d}{d\rho}\left(\frac{e^{C_3\rho}I(\rho)}{\rho^k}\right) \geq \frac{1}{\rho^k}\frac{d}{d\rho}J(\rho) - C_3\frac{d}{d\rho}\left(\frac{I(\rho)}{\rho^{k-1}}\right) \ .$$

This implies that for small enough $\rho > 0$ we have

$$\frac{d}{d\rho}\left(\frac{e^{C_3\rho}+C_3\rho}{\rho^k}I(\rho)\right) \geq \frac{1}{\rho^k}\frac{d}{d\rho}J(\rho) \ .$$

Letting ϕ increase to the characteristic function of $(-\infty, 1)$, we obtain that the above inequality continues to hold in the sense of distributions, i.e.

$$\frac{d}{d\rho}\left(\frac{e^{C_3\rho}+C_3\rho}{\rho^k}M(C \llcorner B_\rho(0))\right) \geq \frac{d}{d\rho}\int_{B_\rho(0)}\frac{1}{|x|^k}\langle\tau,\omega^t\rangle d\|C\| \ .$$

Since $\tau(x)$ is calibrated by $\omega(x)$ for $\|C\|$-a.e. $x \in B_\rho(0)$, with the help of Lemma 5.11 in [38] (see also Lemma C.2 in appendix C) we can express the integrand on the right-hand side above as a positive quantity:

$$\langle \tau(x), \omega^t(x) \rangle = \sum_{i=1}^{N(x)} \lambda_i(x) \left| \xi_i(x) \wedge \frac{\partial}{\partial r} \right|^2 ,$$

where $\tau(x) = \sum_{i=1}^{N(x)} \lambda_i(x)\xi_i(x)$ and the $\xi_j(x)$ are simple k-vectors calibrated by $\omega(x)$. Integration from $0 < s < r \le r_0$ finishes the proof of the proposition. □

If we use the other inequality given by (1.2) virtually the same proof also shows

$$\begin{aligned} \frac{e^{C_1 r} - C_1 r}{r^k} M(C \llcorner B_r(x_0)) \quad &- \quad \frac{e^{C_1 s} - C_1 s}{s^k} M(C \llcorner B_s(x_0)) \\ &\le \quad C_1 \int_{B_r \setminus B_s(x_0)} \frac{1}{|x - x_0|^k} \sum_{i=1}^{N(x)} \lambda_i(x) \left| \xi_i(x) \wedge \frac{\partial}{\partial r} \right|^2 d\|C\| , \qquad (1.3) \end{aligned}$$

which will be used in the proof of Theorem 1.2. Since we will only look at the situation where the radii become small the perturbation terms will not play a significant rôle.

1.3 Existence and structure of tangent cones

We now look at some implications of the monotonicity formula for the blow-up behaviour of calibrated currents. First of all note that the monotonicity formula implies that $r^{-k} M(C \llcorner B_r(x_0))$ is almost increasing in r and remains bounded for small radii. Hence we know that the density $\Theta(\|C\|, x_0)$ of C at x_0 defined by the limit $\Theta(\|C\|, x_0) := \lim_{r \to 0} \alpha(k)^{-1} r^{-k} M(C \llcorner B_r(x_0))$ exists for all $x_0 \in \mathbf{R}^m$. Furthermore, we dilate the current C around x_0 by setting

$$C_{r,x_0} := (\lambda_*^{r,x_0} C) \llcorner B_1(x_0) ,$$

where $\lambda_*^{r,x_0} C$ means the push-forward of C by $\lambda^{r,x_0}(x) = \frac{x - x_0}{r}$. The important fact that $M(C_{r,x_0}) = r^{-k} M(C \llcorner B_r(x_0))$ then allows us to conclude that $M(C_{r,x_0})$ is uniformly bounded as r tends to 0. From the cycle condition on C we get that $\partial C_{r,x_0} \llcorner B_1(x_0) = 0$ and hence that $\mathbf{N}(C_{r,x_0}) := M(C_{r,x_0}) + M(\partial C_{r,x_0} \llcorner B_1(x_0))$ is uniformly bounded in r. Therefore the compactness theorem of H. Federer and W. Fleming (see [26] or 4.2.17 (1) in [25]) implies that for any sequence of radii $\{r_n\}$ tending to 0, there exists a subsequence $\{r_{n'}\}$ such that as $n' \to \infty$

$$C_{r_{n'},x_0} \xrightarrow{\mathbf{F}} C_{\infty,x_0} ,$$

for some normal k-current C_{∞,x_0}. For reasons that will become apparent later, we call such a limiting current C_{∞,x_0} a tangent cone to C at x_0, whose existence at any point x_0 is a direct consequence of the monotonicity formula. Note that a-priori the limiting object might very well depend on the sequence of radii chosen and the main part of this chapter is devoted to showing that this is not the case, i.e. that the blow-up limit is a unique tangent cone.

Since we have already obtained existence, the next step is to investigate the structure of tangent cones. Towards this we now show that tangent cones to a calibrated k-current are again calibrated k-currents themselves. First note that the lower semi-continuity of mass under weak convergence implies that

$$\lim_{r\to 0} \frac{M(C \llcorner B_r(x_0))}{r^k} = \lim_{r\to 0} M(C_{r,x_0}) \geq M(C_{\infty,x_0}) \ . \tag{1.4}$$

For currents calibrated by a k-form ω the above inequality can in fact be improved to an equality. Since by definition of a calibration $M(C \llcorner B_r(x_0)) = C \llcorner B_r(x_0)(\omega)$ we have

$$M(C_{r,x_0}) = \frac{1}{r^k} C \llcorner B_r(x_0)(\omega) = C_{r,x_0}(r^k (\lambda^{r,x_0})^* \omega) \ ,$$

i.e. that C_{r,x_0} is calibrated by $r^k(\lambda^{r,x_0})^*\omega$. Since $M(C_{r,x_0}) \leq C_1 < \infty$ and ω is in C^2 we conclude that as $r \to 0$

$$|C_{r,x_0}(r^k(\lambda^{r,x_0})^*\omega - \omega_0)| \leq C_1 \|r^k(\lambda^{r,x_0})^*\omega - \omega_0\|_* \to 0 \ .$$

Therefore we obtain

$$\lim_{r\to 0} M(C_{r,x_0}) = \lim_{n'\to\infty} C_{r_{n'},x_0}(\omega_0) = C_{\infty,x_0}(\omega_0) \ . \tag{1.5}$$

Combining this fact with (1.4) we deduce that

$$M(C_{\infty,x_0}) \leq \lim_{n'\to\infty} M(C_{r_{n'},x_0}) = C_{\infty,x_0}(\omega_0) \ .$$

Since ω_0 has comass equal to 1, we also conclude that $C_{\infty,x_0}(\omega_0) \leq M(C_{\infty,x_0})$ and hence $C_{\infty,x_0}(\omega_0) = M(C_{\infty,x_0})$, i.e. C_{∞,x_0} is calibrated by ω_0. Note that independent of the properties of ω the constant form ω_0 is always closed, so even though calibrated k-cycles may not be (homologically) mass-minimising their tangent cones always are.

We continue our discussion of tangent cones by looking at the density of a tangent cone at the origin. Below we will use the notation $\|C\|$ for the Radon measure associated with a normal current and the fact that $M(C \llcorner B_r(x_0)) = \|C\|(B_r(x_0))$. From the discussion above we get that

$$\begin{aligned}
\frac{1}{r^k}\|C_{\infty,x_0}\|(B_r(0)) &= \lim_{n'\to\infty} \frac{1}{r^k}\|C_{r_{n'},x_0}\|(B_r(x_0)) \\
&= \lim_{n'\to\infty} \frac{1}{(rr_{n'})^k}\|C\|(B_{rr_{n'}}(x_0)) \\
&= \alpha(k)\Theta(\|C\|, x_0) \ ,
\end{aligned}$$

where $\alpha(k)$ again denotes the volume of the unit ball in $\mathbf{R}^k$. Thus we conclude that for all radii $r > 0$

$$\frac{1}{\alpha(k) r^k}\|C_{\infty,x_0}\|(B_r(0)) = \Theta(\|C_{\infty,x_0}\|, 0) = \Theta(\|C\|, x_0) \ ,$$

i.e. that the density of C at x_0 equals the density of C_{∞,x_0} at the origin. We will now use this identity together with the monotonicity formula to justify the notion of a tangent cone. First note that in case a current is calibrated by a constant form the perturbation terms in the monotonicity formula disappear. In particular, the right-hand side of (1.2) vanishes so that instead of an inequality we end up with an equality. Continuing the proof of proposition 1.3 with this identity one can see that in this easier case (1.1) can be replaced by the simpler monotonicity formula for $0 < s < r$

$$\frac{1}{r^k}\|C_{\infty,x_0}\|(B_r(0)) - \frac{1}{s^k}\|C_{\infty,x_0}\|(B_s(0)) = \int_{B_r(0)\setminus B_s(0)} \frac{1}{|x|^k}\sum_{j=1}^{N(x)} \lambda_j(x)\left|\xi_j(x)\wedge\frac{\partial}{\partial r}\right|^2 d\|C_{\infty,x_0}\| \,,$$

where as above $\tau^\infty(x) = \sum_{j=1}^{N(x)} \lambda_j(x)\xi_j(x)$ — alternatively one can use Theorem 5.7 in the paper by R. Harvey and B. Lawson [38]. The identity for the density above then ensures the right-hand side in the monotonicity formula to vanish, i.e. that $\sum_{j=1}^{N(x)} \lambda_j(x)\left|\xi_j(x)\wedge \frac{\partial}{\partial r}\right|^2 = 0$ at $\|C_{\infty,x_0}\|$-a.e. x. Therefore we get that $\tau^\infty(x)\wedge\frac{\partial}{\partial r} = 0$, $\|C_{\infty,x_0}\|$-a.e., which by the homotopy formula 4.1.9 in [25] (applied to the affine homotopy from λ^{1,x_0} to λ^{r,x_0}) implies that C_{∞,x_0} is indeed a cone, i.e. that $\lambda^{r,0}_* C_{\infty,x_0} = C_{\infty,x_0}$ for all $r > 0$.

For the remainder of this section we will go back to the special case where the calibration ω_0 is a constant 2-dimensional form and investigate the support and structure of a tangent cone calibrated by it. To do this, we first recall the structure theorem for constant 2-forms of unit comass on $\mathbf{R}^m$ (see Theorem 7.16 in [38] or Theorem C.3 in appendix C). For such a 2-form ω_0 we know that there exist coordinates and an almost complex structure J on a subspace $\mathbf{R}^{2n}$ for some n, such that J is compatible with the Euclidean metric and such that ω_0 is the standard symplectic form for J on $\mathbf{R}^{2n}$. This implies that, without loss of generality, for any $x_0 \in \operatorname{spt}\|C\|$ we can assume that the coordinates are chosen so that $x_0 = 0$, $J = J_0$ and ω_0 is the standard symplectic form on $\mathbf{R}^{2n}\subset\mathbf{R}^m$ for some n. Therefore, calibrated 2-vectors have to vanish in the $\mathbf{R}^{m-2n}$-direction. Using this together with the fact that $\tau^\infty(x)\wedge\frac{\partial}{\partial r}(x) = 0$ for $\|C_{\infty,x_0}\|$-a.e. $x\in\mathbf{R}^m$, we deduce that the set of $x \in \operatorname{spt}\|C_{\infty,x_0}\|\cap\mathbf{R}^{m-2n}$ has $\|C_{\infty,x_0}\|$-measure 0 (because of the above for these x, $\tau^\infty(x)\wedge\frac{\partial}{\partial r}(x)\neq 0$). Thus the support of $\|C_{\infty,x_0}\|$ can be assumed to be contained in $\mathbf{R}^{2n}\subset\mathbf{R}^m$.
Furthermore, since $\tau^\infty(x)$ is calibrated by ω_0, we know that the approximate tangent planes are J_0-holomorphic and thus from $\tau^\infty(x)\wedge\frac{\partial}{\partial r}(x) = 0$ we deduce that $\tau^\infty(x)\wedge J_0\frac{\partial}{\partial r}(x) = 0$. Hence $\tau^\infty(x) = \frac{\partial}{\partial r}\wedge J_0\frac{\partial}{\partial r}(x)$ for $\|C_{\infty,x_0}\|$-a.e. $x\in\mathbf{R}^m$. The above arguments immediately give the following proposition:

Proposition 1.4 *Let C_{∞,x_0} be a tangent cone to C at x_0 which is calibrated by the 2-form ω_0. Then there are coordinates centred at x_0 such that for any 2-form ψ, $C_{\infty,x_0}(\psi)$ is of the form:*

$$C_{\infty,x_0}(\psi) = \left\langle \phi\,,\psi\left(\frac{\partial}{\partial r}\wedge J_0\frac{\partial}{\partial r}\right)\right\rangle\,,$$

where ϕ is a distribution in $\mathcal{D}'(\mathbf{R}^m)$ with support in $\mathbf{R}^{2n}\subset\mathbf{R}^m$.

From the fact that C_{∞,x_0} is a J_0-holomorphic cone we can deduce more information on the structure of the distribution ϕ above:

Proposition 1.5 *Let ϕ be the distribution given by Proposition 1.4. Then ϕ is of the form*

$$\langle \phi \, , f \rangle = \int_0^1 \frac{1}{t} \Big\langle \Gamma \, , \int_{H^{-1}(p)} f(t,\theta) \; d\mathcal{H}^1(\theta) \Big\rangle d\mathcal{H}^1(t) \ ,$$

where $H : S^{2n-1} \to \mathbf{CP}^{n-1}$ is the Hopf fibration and Γ is a distribution on $\mathbf{CP}^{n-1}$ determining ϕ.

Proof. From the fact that C_{∞,x_0} is a cone we know that $\lambda_*^{0,r} C_{\infty,x_0} = C_{\infty,x_0}$ for any positive r. Thus ϕ also satisfies $\langle \phi \, , f(t,\theta)\rangle = \langle \phi \, , f(rt,\theta)\rangle$ and hence as a distribution ϕ is independent of r, i.e. $\frac{\partial \phi}{\partial r} = 0$. From this one immediately deduces that

$$\langle \phi \, , f(t,\theta)\rangle = \int_0^1 \frac{1}{t} \langle \Sigma \, , f(t,\theta)\rangle d\mathcal{H}^1(t) \ ,$$

where Σ is a distribution on $S^{2n-1} \subset \mathbf{R}^{2n}$ and for fixed t we view $f(t,\theta)$ as a function on S^{2n-1}. Using the fact that C_{∞,x_0} is also J_0-holomorphic we get that $\langle \Sigma \, , f(t,\theta)\rangle = \langle \Sigma \, , f(t, J_0\theta)\rangle$ and hence that for any $s \in [0, 2\pi]$, $\langle \Sigma \, , f(t,\theta)\rangle = \langle \Sigma \, , f(t, e^{is} \cdot \theta)\rangle$, where by $e^{is} \cdot \theta$ we mean the multiplication of each component by e^{is}. Thus Σ is invariant along the fibres of the Hopf fibration given as $H^{-1}(p)$ for $p \in \mathbf{CP}^{n-1}$. Then Σ defines a distribution Γ on $\mathbf{CP}^{n-1}$ given by

$$\Big\langle \Gamma \, , \int_{H^{-1}(p)} f(t,\tilde\theta) d\mathcal{H}^1(\tilde\theta) \Big\rangle := \Big\langle \Sigma \, , \int_{H^{-1}(H(\theta))} f(t,\tilde\theta) d\mathcal{H}^1(\tilde\theta) \Big\rangle$$

and the proposition holds. □

So far the arguments in this section were valid even for normal cycles. If one assumes that the cycles is integer rectifiable even more can be said about its tangent cones. In the case of tangent cones to integral area-minimising 2-cycles, F. Morgan [58] proved that then C_{∞,x_0} is a finite union of 2-dimensional disks through x_0. For J-holomorphic integral $1-1$-cycles a more direct proof of the analogous result was given by T. Rivière and G. Tian in section 2 of [71], where in this case the disks are J_{x_0}-holomorphic. In Proposition 1.5 this corresponds to the situation where the distribution Γ on $\mathbf{CP}^{n-1}$ is given by a sum of Dirac masses with (positive) integer weights concentrated at points in $\mathbf{CP}^{n-1}$.

1.4 Projecting the current onto $\mathbf{CP}^{m-1}$

In this section we will discuss the geometric observation which we will use in the proof of Theorem 1.2. First we illustrate this in the case where the manifold M is $2m$-dimensional and for each point $x_0 \in M$ there are coordinates such that the calibration is the standard symplectic 2-form ω_0 on $\mathbf{R}^{2m}$. Thus there exists an almost complex structure J on M compatible with ω. The extension to the general case will be given at the end of section 1.6.

The main tool is the canonical projection of $\mathbf{C}^m \setminus \{0\}$ onto $\mathbf{CP}^{m-1}$ denoted by π. Thus, using homogeneous coordinates on $\mathbf{CP}^{m-1}$, $\pi : \mathbf{C}^m \setminus \{0\} \longrightarrow \mathbf{CP}^{m-1}$ sends $(z_1, \ldots, z_m)$ to $[z_1, \ldots, z_m]$. Note that the push-forward under this map will send any tangent cone

to C at 0 to the union of points at which the Dirac masses representing it are concentrated. As the outcome of several lemmas we relate the mass of the push-forward under π of $C \llcorner B_r(0)$ to the monotonicity formula allowing us to conclude that the projection of $C \llcorner B_r(0)$ onto $\mathbf{CP}^{m-1}$ is well-defined and gives a current of small mass. Furthermore, we also show that the push-forward of $\partial\,[C \llcorner B_r(0)]$ is controlled by the monotonicity formula yielding that $\pi_* C \llcorner B_r(0)$ is an integral 2-current on $\mathbf{CP}^{m-1}$ with small flat norm.
Since we will be working with an arbitrary complex structure J on M determined by ω which is also compatible with it, we need to construct a similar projection from $(\mathbf{R}^{2m}, J)$ onto $\mathbf{CP}^{m-1}$ where this time the pre-images of points in $\mathbf{CP}^{m-1}$ are J-holomorphic surfaces rather than holomorphic planes. The construction of this map will follow the construction given by T. Rivière and G. Tian in appendix A of [70] and can be found in our appendix D.

The first Lemma towards the above mentioned projection result is a general estimate for norms of wedge-products with J-holomorphic vectors and is a crucial observation used throughout this chapter.

Lemma 1.6 *Let τ be a simple 2-vector which calibrated by ω_0 on $\mathbf{R}^{2m}$. There exists a constant $C_{2m} > 0$ depending only on the dimension $2m$ such that for any vector $\zeta \in \mathbf{R}^{2m}$ we have*

$$\frac{1}{C_{2m}}|\tau \wedge \zeta|^2 \leq |\tau \wedge \zeta \wedge J\zeta| \leq C_{2m}|\tau \wedge \zeta|^2 \ .$$

Proof. By Wirtinger's inequality (see 1.8.2 in [25] by H. Federer) we know that $\tau = \xi_1 \wedge J\xi_1$ for some $\xi_1 \in \mathbf{R}^{2m}$ of unit length. Since ξ_1 and $J\xi_1$ are orthonormal we can extend them to an ordered orthonormal basis $\{\xi_1, J\xi_1, \xi_2, J\xi_2, \ldots, \xi_m, J\xi_m\}$ of $\mathbf{R}^{2m}$. Writing an arbitrary vector $\zeta = \sum_{l=1}^{2m} a_l \xi_l$ in this basis we get that

$$|\tau \wedge \zeta|^2 = \left|\sum_{l=1}^{2m} a_l \tau \wedge \xi_l\right|^2 = \left|\sum_{l=3}^{2m} a_l \xi_1 \wedge J\xi_1 \wedge \xi_l\right|^2 = \sum_{l=3}^{2m} |a_l|^2 \ .$$

Next we compute $|\tau \wedge \zeta \wedge J\zeta|$. Writing $\zeta = \sum_{k=1}^{m}(a_{2k-1}\xi_{2k-1} + a_{2k}\xi_{2k})$ and $J\zeta = \sum_{l=1}^{m}(a_{2l-1}\xi_{2l} - a_{2l}\xi_{2l-1})$ we obtain

$$\begin{aligned}
|\tau \wedge \zeta \wedge J\zeta| \ &= \ \Bigg|\xi_1 \wedge J\xi_1 \wedge \Bigg[\sum_{k,l=1}^{m}(a_{2k-1}a_{2l-1}\xi_{2k-1} \wedge \xi_{2l} - \\
&\quad -(a_{2k-1}a_{2l})\xi_{2k-1} \wedge \xi_{2l-1} + (a_{2k}a_{2l-1})\xi_{2k} \wedge \xi_{2l} - \qquad (1.6)\\
&\quad -(a_{2k}a_{2l})\xi_{2k} \wedge \xi_{2l-1}\Bigg]\Bigg| \\
&\leq \ C_{2m}\left|\sum_{3 \leq l < k}^{2m} a_l^2 \xi_1 \wedge J\xi_1 \wedge \xi_l \wedge \xi_k\right| \\
&\leq \ C_{2m}\sum_{3=l}^{2m}|a_l|^2 = C_{2m}|\tau \wedge \zeta|^2 \ .
\end{aligned}$$

For the remaining estimate first note that $|\tau \wedge \zeta|^4$ can be bounded by $C_{2m} \sum_{l=3}^{2m} |a_l|^4$ and thus it remains to bound this by $|\tau \wedge \zeta \wedge J\zeta|^2$. From the identity in equation 1.6 one can see that $\tau \wedge \zeta \wedge J\zeta$ contains all terms of the form $(a_{2l-1})^2 \xi_{2l-1} \wedge \xi_{2l} - (a_{2l})^2 \xi_{2l} \wedge \xi_{2l-1}$ when $3 \leq l \leq m$ and no other terms contain these combinations of $\xi_k \wedge \xi_l$. Therefore $C_{2m}|\tau \wedge \zeta \wedge J\zeta|^2$ is larger or equal than $\sum_{l=3}^{2m} |a_l|^4$, which completes the argument. □

In the next Lemma we combine the above result with the monotonicity formula from Proposition 1.3 to estimate the mass of $\pi_*[C \llcorner B_r \setminus B_s(x_0)]$ for small enough radii.

Lemma 1.7 *Let C be as in the main theorem. Then there exist constants $C_1 > 0$ and $r_0 > 0$ such that for any $0 < s < r \leq r_0$ and $x_0 \in M$ we have*

$$M(\pi_*[C \llcorner B_r \setminus B_s(x_0)]) \leq C_1 \left[\frac{M(C \llcorner B_r(x_0))}{r^2} - \frac{M(C \llcorner B_s(x_0))}{s^2} \right] .$$

Proof. Without loss of generality we can assume that the coordinates are centred at x_0 and chosen so that ω_0 is the standard symplectic form on $\mathbf{R}^{2m}$. Let $r_0 > 0$ be chosen so that the monotonicity formula is valid for balls of radius less than $r_0 > 0$. Since the estimates in Lemma 1.6 are invariant under isometries we can apply the Lemma to $\zeta = \frac{\partial}{\partial r}$ at each point in $B_r \setminus B_s(0)$ and deduce that

$$\begin{aligned} \frac{1}{C_{2m}} \int_{B_r \setminus B_s(0)} & \frac{1}{|x|^2} \left| \tau(x) \wedge \frac{\partial}{\partial r} \right|^2 d\|C\| \\ & \leq \int_{B_r \setminus B_s(0)} \frac{1}{|x|^2} \left| \tau(x) \wedge \frac{\partial}{\partial r} \wedge J \frac{\partial}{\partial r} \right| d\|C\| \\ & \leq C_{2m} \int_{B_r \setminus B_s(0)} \frac{1}{|x|^2} \left| \tau(x) \wedge \frac{\partial}{\partial r} \right|^2 d\|C\| . \end{aligned}$$

Combining this with the monotonicity formula we obtain that

$$\begin{aligned} \int_{B_r \setminus B_s(0)} & \frac{1}{|x|^2} \left| \tau(x) \wedge \frac{\partial}{\partial r} \wedge J \frac{\partial}{\partial r} \right| d\|C\| \\ & \leq C_{2m} \left[\frac{M(C \llcorner B_r(0))}{r^2} - \frac{M(C \llcorner B_s(0))}{s^2} \right] . \end{aligned}$$

Since π is only a small perturbation of $H \circ \frac{x}{|x|}$ we deduce that

$$\left\langle \tau(x), \bigwedge\nolimits_2 \pi(x) \right\rangle \cong \frac{1}{|x|^2} \left| \tau(x) \wedge \frac{\partial}{\partial r} \wedge J_0 \frac{\partial}{\partial r} \right|$$

and the proof of the lemma is completed. □

We remark that as a direct consequence of this lemma and the monotonicity formula, since $r^{-2} M(C \llcorner B_r(x_0))$ is almost increasing in r, we get that

$$\lim_{r \to 0} \lim_{s \to 0} M(\pi_*[C \llcorner B_r \setminus B_s(x_0)]) = 0 . \tag{1.7}$$

This already shows that the push-forward of $C \llcorner B_r(0)$ under π is well-defined and has small mass. The next step is to prove that for certain radii we will also be able to control the mass of the push-forward of $\partial[C \llcorner B_r(0)]$.

To do this, we note that it will suffice to prove Theorem 1.2 for radii of the form $r = \frac{1}{2^j}$ where $j \in \mathbf{N}$. From the previous Lemma combined with (1.1) and (1.3) we know that a rate of convergence for $r^{-2}M(C \llcorner B_r(x_0))$ is equivalent to a rate of convergence for $M(\pi_*[C \llcorner B_r(x_0)])$ since

$$\begin{aligned} M(\pi_* C \llcorner B_r(x_0)) &\lesssim \frac{M(C \llcorner B_r(x_0))}{r^2} - \pi\Theta(\|C\|, x_0) \\ &\simeq \int_{B_r(x_0)} \frac{1}{|x - x_0|^2} \left| \tau \wedge \frac{\partial}{\partial r} \right|^2 d\|C\| \\ &\lesssim M(\pi_* C \llcorner B_r(x_0)) \ . \end{aligned}$$

To prove a rate of convergence for $M(\pi_* C \llcorner B_r(x_0))$ to 0 we want to find a constant $\theta \in (0,1)$ independent of j such that for all j

$$M\left(\pi_* \left[C \llcorner B_{\frac{1}{2^{j+1}}}(x_0)\right]\right) \le \theta M\left(\pi_* \left[C \llcorner B_{\frac{1}{2^j}}(x_0)\right]\right) \ , \tag{1.8}$$

since then the result follows by a standard iteration argument.
We begin by partitioning the set of indices into two subset as follows:

$$A := \left\{ j \in \mathbf{N} \ : \ M\left(\pi_* \left[C \llcorner B_{\frac{1}{2^{j+1}}}(x_0)\right]\right) \le \frac{1}{2} M\left(\pi_* \left[C \llcorner B_{\frac{1}{2^j}}(x_0)\right]\right) \right\} \ ,$$

$$B := \left\{ j \in \mathbf{N} \ : \ M\left(\pi_* \left[C \llcorner B_{\frac{1}{2^{j+1}}}(x_0)\right]\right) > \frac{1}{2} M\left(\pi_* \left[C \llcorner B_{\frac{1}{2^j}}(x_0)\right]\right) \right\} \ .$$

Note that for $j \in A$ estimate 1.8 holds true with $\theta = \frac{1}{2}$. Thus it remains to show 1.8 when $j \in B$. To do this we will work on 'good slices' of C and we will define these and prove their existence in the next Lemma.

Lemma 1.8 *Given $0 < r < r_0$, there exists $\rho_0 \in \left[\frac{r}{2}, r\right]$ such that the following hold:*

1. $M(\langle C, |\cdot|, \rho_0 \rangle) \le C_1 \rho_0$,
2. $\int_{\partial B_{\rho_0}(x_0)} |\nabla \pi|_C|^2 d\langle C, |\cdot|, \rho_0 \rangle \le \frac{1}{\rho_0} \int_{B_r \setminus B_{\frac{r}{2}}(x_0)} |\nabla \pi|_C|^2 d\|C\|$,
3. $M(\pi_* \partial [C \llcorner B_{\rho_0}(x_0)]) \le C_1 \fint_{\frac{r}{2}}^{r} \left[\int_{\partial B_\rho} \frac{1}{|x|} \left| \tau \wedge \frac{\partial}{\partial r} \right| d\langle C, |\cdot|, \rho \rangle \right] d\rho$,

where $\langle C, |\cdot|, \rho \rangle$ denotes the slice current of C at the radius ρ (see [25] chapter 4.3 for the definition). For such ρ_0 the slice $\langle C, |\cdot|, \rho_0 \rangle$ will be called a 'good slice'.

Proof. For the first estimate, from 4.2.1 in [25] by H. Federer and the almost monotonicity formula we know that there exists a constant $C_1 > 0$ such that

$$\begin{aligned} \int_{\frac{r}{2}}^{r} \frac{1}{\rho} M(\langle C, |\cdot|, \rho \rangle) d\rho &\lesssim \frac{2}{r} M(C \llcorner B_r \setminus B_{\frac{r}{2}}) \lesssim \frac{2}{r} M(C \llcorner B_r) \\ &\le C_1 \frac{r}{2} \ , \end{aligned}$$

i.e. that

$$\fint_{\frac{r}{2}}^{r} \frac{1}{\rho} M(\langle C, |\cdot|, \rho\rangle) d\rho \leq C_1 \ .$$

For the second one, set $\phi(r) := \int_{B_r \setminus B_{\frac{r}{2}}(x_0)} |\nabla \pi|_C|^2 d\|C\|$ to obtain

$$\int_{\frac{r}{2}}^{r} \frac{r}{\phi(r)} \int_{\partial B_\rho(x_0)} |\nabla \pi|_C|^2 d\langle C, |\cdot|, \rho\rangle d\rho \leq \frac{r}{\phi(r)} \phi(r) = r \ .$$

For the last estimate we denote the orienting vector for $\pi_* \partial[C \llcorner B_\rho]$ by t and apply lemma 1.6 with $\tau = \frac{\partial}{\partial r} \wedge J \frac{\partial}{\partial r}$ and $\zeta = t$ at each point. Therefore,

$$\begin{aligned} \left| t \wedge \frac{\partial}{\partial r} \wedge J \frac{\partial}{\partial r} \right|^2 &\cong \left| t \wedge \frac{\partial}{\partial r} \wedge J \frac{\partial}{\partial r} \right| \cdot \left| Jt \wedge \frac{\partial}{\partial r} \wedge J \frac{\partial}{\partial r} \right| \\ &\cong \left| t \wedge Jt \wedge \frac{\partial}{\partial r} \wedge J \frac{\partial}{\partial r} \right| \\ &\cong \left| \tau \wedge \frac{\partial}{\partial r} \wedge J \frac{\partial}{\partial r} \right| \\ &\cong \left| \tau \wedge \frac{\partial}{\partial r} \right|^2 \ . \end{aligned}$$

As in the proof of lemma 1.7 we use this to compute

$$\begin{aligned} M(\pi_* \partial[C \llcorner B_\rho(x_0)]) &\leq C_1 \int_{\partial B_\rho} \frac{1}{|x|} \left| t \wedge \frac{\partial}{\partial r} \wedge J \frac{\partial}{\partial r} \right| d\langle C, |\cdot|, \rho\rangle \\ &\leq C_1 \int_{\partial B_\rho} \frac{1}{|x|} \left| \tau \wedge \frac{\partial}{\partial r} \right| d\langle C, |\cdot|, \rho\rangle \ . \end{aligned}$$

Now as for the previous case, set $\psi(r) := \fint_{\frac{r}{2}}^{r} \left[\int_{\partial B_\rho} \frac{1}{|x|} \left| \tau \wedge \frac{\partial}{\partial r} \right| d\langle C, |\cdot|, \rho\rangle \right] d\rho$ to get that

$$\fint_{\frac{r}{2}}^{r} \frac{1}{\psi(r)} M(\pi_* \partial[C \llcorner B_\rho(x_0)]) d\rho \leq C_1 \ .$$

To get a radius $\rho_0 \in \left[\frac{r}{2}, r\right]$ which works in all three situations, simply add the three average integrals together to obtain that there is ρ_0 with

$$\frac{1}{\rho_0} M(\langle C, |\cdot|, \rho_0\rangle) + \frac{r}{\phi(r)} \int_{\partial B_{\rho_0}(x_0)} |\nabla \pi|_C|^2 d\langle C, |\cdot|, \rho_0\rangle + \frac{1}{\psi(r)} M(\pi_* \partial[C \llcorner B_{\rho_0}(x_0)]) \leq C_2 \ .$$

This establishes the assertions of the Lemma. □

In the next Lemma we will use estimate *3.* in the previous Lemma 1.8 to bound $M(\pi_* \partial[C \llcorner B_{\rho_j}])$ in case $j \in B$.

Lemma 1.9 *There exists $C_1 > 0$ such that if $j \in B$ and $\rho_j \in \left[\frac{1}{2^{j+1}}, \frac{1}{2^j}\right]$ denotes a good slice from Lemma 1.8. Then*

$$M(\pi_* \partial[C \llcorner B_{\rho_j}]) \leq C_1 [M(\pi_* C \llcorner B_{\rho_j}]^{\frac{1}{2}} \ .$$

Proof. In estimate *3.* in Lemma 1.8 we first apply Cauchy-Schwarz to the inner integral to obtain

$$\int_{\partial B_\rho} \frac{1}{|x|} \left| \tau \wedge \frac{\partial}{\partial r} \right| d\langle C, |\cdot|, \rho\rangle \le \left(\int_{\partial B_\rho} \frac{1}{|x|^2} \left| \tau \wedge \frac{\partial}{\partial r} \right|^2 d\langle C, |\cdot|, \rho\rangle \right)^{\frac{1}{2}} \left(M(\langle C, |\cdot|, \rho\rangle\right)^{\frac{1}{2}} .$$

Next we apply Cauchy-Schwarz to the average integral

$$\begin{aligned}
&\fint_{\frac{r}{2}}^{r} \left[\int_{\partial B_\rho} \frac{1}{|x|} \left| \tau \wedge \frac{\partial}{\partial r} \right| d\langle C, |\cdot|, \rho\rangle \right] d\rho \\
&\qquad \le \frac{2}{r} \left(\int_{\frac{r}{2}}^{r} \left[M(\langle C, |\cdot|, \rho\rangle)^{\frac{1}{2}} \left(\int_{\partial B_\rho} \frac{1}{|x|^2} \left| \tau \wedge \frac{\partial}{\partial r} \right|^2 d\langle C, |\cdot|, \rho\rangle \right)^{\frac{1}{2}} \right] d\rho \right) \\
&\qquad \le \sqrt{\frac{2}{r}} \left(\fint_{\frac{r}{2}}^{r} M(\langle C, |\cdot|, \rho\rangle) d\rho \right)^{\frac{1}{2}} \left(\int_{\frac{r}{2}}^{r} \int_{\partial B_\rho} \frac{1}{|x|^2} \left| \tau \wedge \frac{\partial}{\partial r} \right|^2 d\langle C, |\cdot|, \rho\rangle d\rho \right)^{\frac{1}{2}} .
\end{aligned}$$

For the first integral we use the monotonicity formula like in the proof of the first estimate of Lemma 1.8 to obtain

$$\left(\fint_{\frac{r}{2}}^{r} M(\langle C, |\cdot|, \rho\rangle) d\rho \right)^{\frac{1}{2}} \le C_1 \sqrt{\frac{r}{2}} .$$

Since the second integral is bounded by $[M(\pi_*[C \llcorner B_r \setminus B_{\frac{r}{2}}(x_0)]^{\frac{1}{2}}$ we can combine these to get

$$M(\pi_* \partial [C \llcorner B_{\rho_j}]) \le C_1 [M(\pi_*[C \llcorner B_r \setminus B_{\frac{r}{2}}])]^{\frac{1}{2}} .$$

By assumption $r = \frac{1}{2^j}$ with $j \in B$ so that the defining property of B gives $M(\pi_*[C \llcorner B_{\frac{1}{2^j}}]) \le 2M(\pi_* \partial [C \llcorner B_{\frac{1}{2^{j+1}}}])$. Applying first this to the above estimate and then the monotonicity formula to the resulting inequality (using that $\frac{1}{2^{j+1}} \le \rho_j$) we get

$$\begin{aligned}
M\left(\pi_* \partial \left[C \llcorner B_{\rho_j}\right]\right) &\le C_1 \left[M\left(\pi_* \left[C \llcorner B_r \setminus B_{\frac{r}{2}}\right]\right)\right]^{\frac{1}{2}} \\
&\le C_1 \left[M\left(\pi_* \left[C \llcorner B_{\frac{r}{2}}\right]\right)\right]^{\frac{1}{2}} \\
&\le C_1 \left[M\left(\pi_* \left[C \llcorner B_{\rho_j}\right]\right)\right]^{\frac{1}{2}} ,
\end{aligned}$$

which is exactly what we wanted to show and the Lemma holds true. □

1.5 Calibrated 2-currents on $\mathbf{CP}^{m-1}$

The aim of this section is to show that the projected currents $\pi_* \left[C \llcorner B_{\rho_j}\right]$ for $j \in B$ will not cover all of $\mathbf{CP}^{m-1}$ which will be needed later on to integrate by parts. More precisely, given a small constant $\eta > 0$ we will prove that there exists a radius $\epsilon > 0$ and a region $A_\epsilon \subset \mathbf{CP}^{m-1}$ such that the current $\pi_* \left[C \llcorner B_{\rho_j}\right] \llcorner A_\epsilon$ will have mass less than $\eta M(\pi_* \left[C \llcorner B_{\rho_j}\right])$. This will be deduced from a general Lemma for j-holomorphic integral 2-currents T in $\mathbf{CP}^{m-1}$ with small flat norm which we will then apply to $T = \pi_*[C \llcorner B_{\rho_j}]$.

Lemma 1.10 *Given a constant $C_1 > 0$ and $\eta > 0$ small, there exist $\delta > 0$ and $\epsilon > 0$ with the following properties:*
For any j-holomorphic integral 2-current $T \in I_2(\mathbf{CP}^{m-1})$ satisfying:

1. *$M(T) + M(\partial T) < \delta$,*
2. *$M(\partial T) \leq C_1[M(T)]^{\frac{1}{2}}$,*

there exists a tubular neighbourhood $A_\epsilon \subset \mathbf{CP}^{m-1}$ of width ϵ such that

$$\frac{M(T \llcorner A_\epsilon)}{\epsilon^2} \leq \eta M(T) \ , \tag{1.9}$$

where the core of the tube A_ϵ consists of a $\mathbf{CP}^{m-2}$ inside $\mathbf{CP}^{m-1}$, i.e. a complex $(m-2)$-dimensional hyperplane in $\mathbf{CP}^{m-1}$.

Proof. We will argue by contradiction. If the Lemma is false, then there are $\eta > 0$ and $C_1 > 0$ such that for any $\delta > 0$ and any $\epsilon > 0$ there exists T with the above properties, but for which for all tubes $A_\epsilon \subset \mathbf{CP}^{m-1}$,

$$\frac{M(T \llcorner A_\epsilon)}{\epsilon^2} > \eta M(T) \ . \tag{1.10}$$

We will use the above assertion on T to derive a lower bound for δ which will be given purely in terms of η, C_1 and ϵ, thus in particular independent of T. However, since we are free to choose δ as small as we wish this will lead to a contradiction.
To get this estimate we begin by letting $\mathcal{N}$ denote the total space of the smooth fibre bundle of anchored $\mathbf{CP}^{m-2}$ in $\mathbf{CP}^{m-1}$. By this we mean that $\mathcal{N}$ is made up of pairs of the form (x, E), where $x \in \mathbf{CP}^{m-1}$ and $E \in G(m-1, m)$, the Grassmanian of complex $(m-1)$-planes in $\mathbf{C}^m$. Any such complex hyperplane can be projected down onto $\mathbf{CP}^{m-1}$ and its projection is diffeomorphic to a $\mathbf{CP}^{m-2}$. Thus a point $p \in \mathcal{N}$ consists of a point $x \in \mathbf{CP}^{m-1}$ and a plane E containing x. By construction $\mathcal{N}$ is compact and we let μ denote the volume measure induced by the standard metric on $\mathbf{C}^m$ so that $\mu(\mathcal{N}) < \infty$. Furthermore, to each point $p \in \mathcal{N}$ and for any $\epsilon > 0$ there exists an open tube $A_\epsilon(p)$, which is the open tube with core given by p and width ϵ.
Next we partition $\mathcal{N}$ into two sets P_1 and P_2 where we have

$$P_1 := \{p \in \mathcal{N} \mid M(\langle T, |\cdot|_p, s_p\rangle) = 0 \text{ for some } s_p \in [\epsilon, 2\epsilon]\} \ , \tag{1.11}$$

where $|\cdot|_p$ denotes the distance function from the core of the tube given by p. The other set is then $P_2 := \mathcal{N} \setminus P_1$.
We will now analyse P_1 and P_2 independently. If $p \in P_1$, we know that there exists $s_p \in [\epsilon, 2\epsilon]$ so that

$$M(\partial[T \llcorner A_{s_p}(p)] - (\partial T) \llcorner A_{s_p}(p)) = M(\langle T, |\cdot|_p, s_p\rangle) = 0 \ ,$$

i.e. $\partial[T \llcorner A_{s_p}(p)] = (\partial T) \llcorner A_{s_p}(p)$ and hence by restricting T to $A_{s_p}(p)$ we do not introduce any extra boundary. Since T is calibrated by $\omega_{\mathbf{CP}^{m-1}}$ we can apply the isoperimetric inequality 4.2.10 in [25] by H. Federer to $T \llcorner A_{s_p}(p)$ to get

$$\begin{aligned} M(T \llcorner A_{s_p}(p))^{\frac{1}{2}} &\leq C_1 M(\partial[T \llcorner A_{s_p}(p)]) \\ &= C_1 M((\partial T) \llcorner A_{s_p}(p)) \ . \end{aligned}$$

Combining this with the assumption that the lemma is false (1.10), we deduce

$$\begin{aligned} \epsilon^2 \eta M(T) &< M(T \llcorner A_\epsilon(p)) \\ &\leq M(T \llcorner A_{s_p}(p)) \\ &\leq C_1 M((\partial T) \llcorner A_{s_p}(p))^2 \\ &\leq C_1 M((\partial T) \llcorner A_{2\epsilon}(p))^2 \ , \end{aligned}$$

i.e.

$$\epsilon \sqrt{\eta} \sqrt{M(T)} \leq C_1 M((\partial T) \llcorner A_{2\epsilon}(p)) \ .$$

We now derive an estimate for $\mu(P_1)$ by integrating the above over all of P_1

$$\mu(P_1) \epsilon \sqrt{\eta} \sqrt{M(T)} \leq C_1 \int_{P_1} M((\partial T) \llcorner A_{2\epsilon}(p)) \, d\mu(p) \ .$$

Analysing the right-hand side of this expression, we have

$$\int_{P_1} M((\partial T) \llcorner A_{2\epsilon}(p)) \, d\mu(p) \leq \int_{(x,E) \in N} \int_{y \in \partial T} \chi_{A_\epsilon(x,E)}(y) \, d\|\partial T\|(y) d\mu(x,E) \ ,$$

and since there exists $E(x,y)$ such that $\chi_{A_\epsilon(x,E)}(y) = \chi_{A_\epsilon(y,E(x,y))}(x)$, we get that the above is less than or equal to

$$\int_{E \in G(m-1,m)} \int_{x \in \mathbf{CP}^{m-1}} \int_{y \in \partial T} \chi_{A_\epsilon(y,E)}(x) \, d\|\partial T\|(y) d\mu_1(x) d\mu_2(E) \ ,$$

which we estimate by interchanging the order of integration and using the fact that $\mu(A_\epsilon(p)) \leq C_1 \epsilon^2$ independent of p

$$\int_{y \in \partial T} \int_{E \in G(m-1,m)} \int_{x \in \mathbf{CP}^{m-1}} \chi_{A_\epsilon(y,E)}(x) d\mu_1(x) d\mu_2(E) d\|\partial T\|(y) \leq C_1 \epsilon^2 M(\partial T) \ .$$

Hence the estimate on $\mu(P_1)$ is

$$\mu(P_1) \epsilon \sqrt{\eta} \sqrt{M(T)} \leq C_1 \epsilon^2 M(\partial T) \leq C_1 \epsilon^2 \sqrt{M(T)} \ ,$$

since by assumption we can estimate $M(\partial T)$ by $\sqrt{M(T)}$. This implies that

$$\mu(P_1) \leq \epsilon \frac{C_1}{\sqrt{\eta}} \ ,$$

so that from the fact that $4\pi \leq \mu(N) < \infty$ we deduce the estimate for $\mu(P_2)$,

$$\mu(P_2) = \mu(N) - \mu(P_1) \geq \frac{1}{2} \ , \qquad (1.12)$$

when ϵ is chosen to be sufficiently small. It is important to observe that any dependence on T and δ has disappeared. Furthermore, this estimate readily implies that $P_2 \subset \mathcal{N}$ cannot be empty.

It remains to investigate $M(T \llcorner A_{2\epsilon}(p)) + M((\partial T) \llcorner A_{2\epsilon}(p))$ for $p \in P_2$ and we will prove that at least one of these two terms has to have a lower bound away from 0. Precisely, we will prove that there exists $C_1 > 0$ so that the following estimate for this quantity holds

$$M(T \llcorner A_{2\epsilon}(p)) + M((\partial T) \llcorner A_{2\epsilon}(p)) \geq C_1 \frac{\epsilon^2}{4} \ . \tag{1.13}$$

To see this, first observe that by definition of P_2 we know that $M(\langle T, |\cdot|_p, s\rangle) > 0$ for any $s \in [\epsilon, 2\epsilon]$. Thus for any interval $(\rho_1, \rho_2) \subset [\epsilon, 2\epsilon]$ we have

$$M(T \llcorner A_{\rho_1,\rho_2}(p)) \geq \int_{\rho_1}^{\rho_2} M(\langle T, |\cdot|_p, s\rangle) \, d\mathcal{L}^1 > 0 \ ,$$

where we use the notation $A_{\rho_1,\rho_2}(p) := A_{\rho_2} \setminus A_{\rho_1}(p)$ for the annular tube with core p. We will now slice the annular tube $A_{\epsilon,2\epsilon}(p)$ by the distance function to the core, $|\cdot|_p$, to get regions with and without boundary and derive estimate (1.13) from an estimate on the measure of them. Precisely, we partition the interval $[\epsilon, 2\epsilon]$ into the set

$$Q_1 := \{s \in [\epsilon, 2\epsilon] \mid M(\langle \partial T, |\cdot|_p, s\rangle) \geq 1\} \ ,$$

and $Q_2 := [\epsilon, 2\epsilon] \setminus Q_1$ so that

$$Q_2 := \{s \in [\epsilon, 2\epsilon] \mid M(\langle \partial T, |\cdot|_p, s\rangle) = 0\} \ ,$$

since $\langle \partial T, |\cdot|_p, s\rangle$ is a 0-dimensional integer rectifiable current. Next we take Q_1^ϵ to be an open set containing Q_1 so that $\mathcal{L}^1(Q_1^\epsilon \setminus Q_1) \leq \frac{\epsilon}{1000}$, say. If $\mathcal{L}^1(Q_1^\epsilon) \geq \frac{\epsilon}{2}$ we are done, since then

$$\begin{aligned} M((\partial T) \llcorner A_{2\epsilon}(p)) \ &\geq \ \int_{s \in Q_1} M(\langle \partial T, |\cdot|_p, s\rangle) \, d\mathcal{L}^1(s) \\ &\geq \ \mu(Q_1) \geq \frac{\epsilon}{2} - \frac{\epsilon}{1000} \geq \frac{\epsilon}{4} \geq \frac{\epsilon^2}{4} \ , \end{aligned}$$

since ϵ is assumed to be small.
Thus we can assume $\mathcal{L}^1(Q_1^\epsilon) < \frac{\epsilon}{2}$, so that since Q_1^ϵ is open, $\overline{Q_1^\epsilon} \neq [\epsilon, 2\epsilon]$ and hence Q_2^ϵ is defined by $Q_2^\epsilon := [\epsilon, 2\epsilon] \setminus \overline{Q_1^\epsilon}$ so that it has $\mathcal{L}^1(Q_2^\epsilon) \geq \frac{\epsilon}{2}$ and $M(\langle T, |\cdot|_p, s\rangle) = 0$ for any $s \in Q_2^\epsilon$.
Writing Q_2^ϵ as union of maximal disjoint intervals $I_j = (a_j, b_j)$ we note that for each j, $T \llcorner A_{a_j,b_j}(p)$ is a cycle in $A_{a_j,b_j}(p)$ and hence satisfies a monotonicity formula. Since $T \in I_2(\mathbf{CP}^{m-1})$ this means that for each $x \in A_{a_j,b_j}(p) \cap \operatorname{spt} \|T\|$ we have $\Theta(\|T\|, x) \geq 1$. Hence

$$M(T \llcorner A_{a_j,b_j}(p)) \geq C_1 \operatorname{vol}(A_{a_j,b_j}(p)) \ .$$

Therefore we conclude that

$$\begin{aligned} M(T \llcorner A_\epsilon(p)) \ &\geq \ \sum_i M(T \llcorner A_{a_j,b_j}(p)) \\ &\geq \ C_1 \sum_i \operatorname{vol}(A_{a_j,b_j}(p)) \geq C_1 \mathcal{L}^1(Q_2^\epsilon)^2 \geq C_1 \frac{\epsilon^2}{4} \ , \end{aligned}$$

which concludes the proof of estimate (1.13).

Combining this estimate with (1.12) bounding $\mu(P_2)$ from below we finally deduce the desired estimate on δ

$$\begin{aligned}\delta &> M(T) + M(\partial T) \\ &\geq C_1 \int_{p\in P_2} M(T \llcorner A_{2\epsilon}(p)) + M((\partial T) \llcorner A_{2\epsilon})\, d\mu(p) \\ &\geq C_1 \frac{\epsilon^2}{4} \mu(P_2) \geq C_1 \frac{\epsilon^2}{8} \ .\end{aligned}$$

Since this estimate is independent of T, it has to hold for all $\delta \in (0, \frac{1}{2})$. However, for ϵ sufficiently small but fixed, δ cannot be chosen arbitrarily small, which gives the desired contradiction. □

In the above lemma the tube $A_\epsilon \subset \mathbf{CP}^{m-1}$ was chosen in such a way that $\mathbf{CP}^{m-1} \setminus A_\epsilon$ satisfies $H^2(\mathbf{CP}^{m-1} \setminus A_\epsilon) = 0$. This allows us to construct a 1-form α_A on $\mathbf{CP}^{m-1} \setminus A_\epsilon$ such that $\omega_{\mathbf{CP}^{m-1}} = d\alpha_A$ there. More precisely, we have the following Lemma:

Lemma 1.11 *Given $\epsilon > 0$ and a tube $A_\epsilon \subset \mathbf{CP}^{m-1}$ as above, there is a smooth 1-form α such that:*

1. $\omega_{\mathbf{CP}^{m-1}} = d\alpha_A \quad$ *on* $\mathbf{CP}^{m-1} \setminus A_\epsilon$,
2. $\alpha_A = 0 \quad$ *on* $A_{\frac{\epsilon}{2}}$.

Furthermore, there is a constant $C_1 > 0$ independent of A_ϵ such that $\|\omega_{\mathbf{CP}^{m-1}} - d\alpha_A\|_* \leq C_1 \frac{1}{\epsilon^2}$ and $\|\alpha_A\| \leq C_1$.

Proof. Given the tube A_ϵ, let ϕ_ϵ be a cut-off function which is 1 on $\mathbf{CP}^{m-1} \setminus A_{\frac{3\epsilon}{4}}$, 0 on $A_{\frac{\epsilon}{2}}$ and decreasing in $A_{\frac{\epsilon}{2},\frac{3\epsilon}{4}}$, with $\|\nabla_h \phi_\epsilon\| = 0$ and $\|\nabla_v \phi_\epsilon\| \leq \frac{C_1}{\epsilon^2}$, where $\nabla_h \phi_\epsilon$ means the derivative in the direction of the core and $\nabla_v \phi_\epsilon$ the derivatives perpendicular to the core. Then we have that $\phi_\epsilon \omega_{\mathbf{CP}^{m-1}}$ is a closed form on $\mathbf{CP}^{m-1} \setminus A_\epsilon$ which is topologically trivial, whence there exists a form α_A with $d\alpha_A = \phi_\epsilon \omega_{\mathbf{CP}^{m-1}}$. We extend α_A to A_ϵ by extending the coefficient functions α_A^i radially to the core by $\phi_\epsilon \alpha_A^i$. By an abuse of notation we still denote the extended form by α_A and we immediately see that $\alpha_A = 0$ on $A_{\frac{\epsilon}{2}}$ and $\|\omega - d\alpha_A\|_* \leq C_1 \|\nabla_v \phi_\epsilon\| \leq \frac{C_1}{\epsilon^2}$, which establishes the Lemma. □

For later applications of this lemma it is important to note that the estimates given are independent of the specific tube $A_\epsilon \subset \mathbf{CP}^{m-1}$ chosen and we will therefore be free to choose A_ϵ according to Lemma 1.10.

1.6 Rate of convergence

As mentioned in section 1.4 we will deduce the rate for $M(\pi_*[C \llcorner B_r(x_0)])$ from a rate for

$$\int_{B_r(x_0)} \pi^* \omega_{\mathbf{CP}^{m-1}}|_C \, d\|C\| \ ,$$

where from now on $\omega_{\mathbf{CP}^{m-1}}$ denotes the standard symplectic 2-form which is compatible with the Fubini-Study metric on $(\mathbf{CP}^{m-1}, j)$.

First note that, from the monotonicity formula and Lemma 1.7, for all $0 < r \leq r_0$

$$\begin{aligned}\int_{B_r(x_0)} \pi^*\omega_{\mathbf{CP}^{m-1}}|_C \, d\|C\| &= \pi_*[C \llcorner B_r(x_0)](\omega_{\mathbf{CP}^{m-1}}) \\ &\leq M(\pi_*[C \llcorner B_r(x_0)]) \, \|\omega_{\mathbf{CP}^{m-1}}\|_* \\ &\leq C_1 \left[\frac{M(C \llcorner B_{r_0}(x_0))}{{r_0}^2} - \pi\Theta(\|C\|, x_0)\right] < +\infty \ .\end{aligned}$$

Furthermore, since the map π is J-j_0-holomorphic and C is J holomorphic, we know that pointwise $\|C\|$-a.e. we have $|\nabla\pi|_C|^2(x) = \pi^*\omega_{\mathbf{CP}^{m-1}}|_C(x)$, i.e. that for all $0 < r \leq r_0$ we have

$$\int_{B_r(x_0)} |\nabla\pi|_C|^2 d\|C\| = \int_{B_r(x_0)} \pi^*\omega_{\mathbf{CP}^{m-1}}|_C \, d\|C\| \leq C_1 < +\infty \ .$$

For the proof of the estimate assume that, using the notation from section 1.4, $r = \frac{1}{2^j}$ for $j \in B$ and let $\rho_j \in [\frac{r}{2}, r]$ be the corresponding radius given in Lemma 1.8. The same lemma also implies that for any $k \in \mathbf{N}$ there exists as sequence of radii $s_k \in [2^{-k}, 2^{-k+1}]$ such that $M(\pi_*\partial[C \llcorner B_{s_k}]) \to 0$ as $k \to \infty$. Together with Lemma 1.7 for any r we can therefore find s_0 sufficiently small such that

$$M(\pi_*[C \llcorner B_{s_0}(x_0)]) + M(\pi_*\partial[C \llcorner B_{s_0}(x_0)]) \leq M(\pi_*[C \llcorner B_r \setminus B_{\frac{r}{2}}(x_0)]) \ . \tag{1.14}$$

The main estimate will follow provided we can prove that there exists a constant $C_1 > 0$ such that

$$\int_{B_{\frac{r}{2}} \setminus B_{s_0}(x_0)} |\nabla\pi|_C|^2 d\|C\| \leq C_1 \int_{B_r \setminus B_{\frac{r}{2}}(x_0)} |\nabla\pi|_C|^2 d\|C\| \ , \tag{1.15}$$

for then the estimate on $M(\pi_*[C \llcorner B_{s_0}(x_0)])$ in (1.14) implies

$$\int_{B_{\frac{r}{2}}(x_0)} |\nabla\pi|_C|^2 d\|C\| \leq (C_1 + 1) \int_{B_r \setminus B_{\frac{r}{2}}(x_0)} |\nabla\pi|_C|^2 d\|C\| \ .$$

Hence we obtain

$$\int_{B_{\frac{r}{2}}(x_0)} |\nabla\pi|_C|^2 d\|C\| \leq \frac{C_1 + 1}{C_1 + 2} \int_{B_r(x_0)} |\nabla\pi|_C|^2 d\|C\|$$

from which we deduce (1.8) with $\theta = \frac{C_1+1}{C_1+2}$. From this the desired rate of convergence will follow completing the proof of Theorem 1.2.

To prove the remaining estimate (1.15) first note that

$$
\begin{aligned}
\int_{B_{\frac{r}{2}}\setminus B_{s_0}(x_0)} |\nabla\pi|_C|^2 d\|C\| &\leq \int_{B_{\rho_j}\setminus B_{s_0}(x_0)} |\nabla\pi|_C|^2 d\|C\| \\
&= \int_{B_{\rho_j}\setminus B_{s_0}(x_0)} \pi^*\omega_{\mathbf{CP}^{m-1}} d\|C\| \\
&= \int_{B_{\rho_j}\setminus B_{s_0}(x_0)} \pi^*(d\alpha_A) d\|C\| \\
&\quad + \int_{B_{\rho_j}\setminus B_{s_0}(x_0)} \pi^*(\omega_{\mathbf{CP}^{m-1}} - d\alpha_A) d\|C\| \ , \qquad (1.16)
\end{aligned}
$$

where $A\epsilon \subset \mathbf{CP}^{m-1}$ will be the tube given by Lemma 1.10 for some $\eta > 0$ to be determined later. Using this Lemma and Lemma 1.11 the second term on the right-hand side can be estimated by $\eta \int_{B_r(x_0)} |\nabla\pi|_C|^2 d\|C\|$.
For the first term note that since π is smooth on $B_r \setminus B_{s_0}(x_0)$

$$
\begin{aligned}
\int_{B_{\rho_j}\setminus B_{s_0}(x_0)} \pi^*(d\alpha_A) d\|C\| &= \int_{B_{\rho_j}\setminus B_{s_0}(x_0)} d(\pi^*\alpha_A) d\|C\| \\
&= \int_{\partial B_{\rho_j}(x_0)} \pi^*\alpha_A d\langle C, |\cdot|, \rho_j\rangle \\
&\quad - \int_{\partial B_{s_0}(x_0)} \pi^*\alpha_A d\langle C, |\cdot|, s_0\rangle .
\end{aligned}
$$

By our choice of s_0 the second term is bounded by

$$
\begin{aligned}
\int_{\partial B_{s_0}(x_0)} \pi^*\alpha_A d\langle C, |\cdot|, s_0\rangle &\leq \|\alpha_A\|_* \, M(\pi_*\langle C, |\cdot|, s_0\rangle) \\
&\leq C_1 M(\pi_* C \llcorner B_r \setminus B_{\frac{r}{2}}(x_0)) \ . \qquad (1.17)
\end{aligned}
$$

Hence it remains to estimate $\int_{\partial B_{\rho_j}(x_0)} \pi^*\alpha_A d\langle C, |\cdot|, \rho_j\rangle$ which we will do by a Poincaré-type inequality on $\partial B_{\rho_j}(x_0)$.
Note that, since $\partial B_{\rho_j}(x_0)$ is an integral 1-cycle, we can apply the decomposition theorem for integral 1-cycles, Theorem 4.2.25 in [25]. We therefore write $\langle C, |\cdot|, \rho_j\rangle$ as $\langle C, |\cdot|, \rho_j\rangle = \sum_{i=1}^\infty T_i$ where each T_i is an indecomposable 1-cycle and $M(\langle C, |\cdot|, \rho_j\rangle) = \sum_{i=1}^\infty M(T_i)$. Furthermore, for each $i \in \mathbf{N}$ there exists $f_i : \mathbf{R} \to \mathbf{R}^{2m}$ with $\mathrm{Lip}(f_i) \leq 1$ such that $T_i = f_{i*}[[0, M(T_i)]]$. Having done this the desired estimate for $\int_{\partial B_{\rho_j}(x_0)} \pi^*\alpha_A d\langle C, |\cdot|, \rho_j\rangle$ will follow from estimates for

$$\int_{T_i} \pi^*\alpha_A d\|T_i\| \ .$$

Since α is a 1-form on $\mathbf{CP}^{m-1}$ and we can work on $\mathbf{CP}^{m-1} \setminus A_\epsilon$, we can write α in coordinates on $\mathbf{CP}^{m-1}$ so that $\langle \pi^*\alpha, \tau_i\rangle$, where τ_i denotes the oriented tangent vector to T_i, is of the form

$$\langle \pi^*\alpha(x), \tau_i(x)\rangle = \sum_{k=1}^{2m-2} \alpha_k(\pi(x)) \langle dy_k, d\pi(x)\tau_i\rangle \ .$$

Since each of the T_i is closed, integrating the above expression is unchanged when $\alpha_k(\pi(x))$ is replaced by $\alpha_k(\pi(x)) - C_k$, where the C_k are arbitrary constants. Applying Cauchy-Schwarz to the resulting integral, we obtain

$$\begin{aligned}\int_{T_i} \langle \pi^*\alpha, \tau_i \rangle dT_i &\leq \sum_{k=1}^{2m-2} \int_{T_i} \left| \alpha_k(\pi(x)) - \overline{\alpha_k(\pi)} \right| \, |\nabla\pi|_{T_i}| \, d\|T_i\|(x) \\ &\leq \sum_{k=1}^{2m-2} \left(\int_{T_i} \left| \alpha_k(\pi(x)) - \overline{\alpha_k(\pi)} \right|^2 d\|T_i\|(x) \right)^{\frac{1}{2}} \cdot \\ &\quad \cdot \left(\int_{T_i} |\nabla\pi|_{T_i}|^2 \, d\|T_i\|(x) \right)^{\frac{1}{2}} ,\end{aligned}$$

where we have chosen the constants C_k equal to $\overline{\alpha_k(\pi)}$ which denotes the average of $\alpha_k(\pi)$ on T_i. We estimate the first term on the right-hand side of this expression using a Poincaré-type inequality on T_i, where it is important to note that because of $\mathrm{Lip}(f_i) \leq 1$ we can take the constants in the estimate equal to 1 independent of i. This implies

$$\left(\int_{T_i} \left| \alpha_k(\pi) - \overline{\alpha_k(\pi)} \right|^2 dT_i \right)^{\frac{1}{2}} \leq M(T_i) \left(\int_{T_i} |\nabla\alpha_k(\pi)|_{T_i}|^2 \, d\|T_i\| \right)^{\frac{1}{2}} .$$

From the fact that the α_k are smooth we deduce that $\int_{T_i} |\nabla\alpha_k(\pi)|_{T_i}|^2 d\|T_i\|$ is bounded by $C_1 \int_{T_i} |\nabla\pi|_{T_i}|^2 d\|T_i\|$, where C_1 is again a constant independent of T_i. Thus, combining the above steps with the fact that $M(T_i) \leq M(\langle C, |\cdot|, \rho_j \rangle)$, we can sum over all i to obtain the important inequality

$$\begin{aligned}\int_{\partial B_{\rho_j}(x_0)} \pi^*\alpha d\langle C, |\cdot|, \rho_j \rangle &\leq C_1 \sum_{i=0}^{\infty} M(T_i) \left(\int_{T_i} |\nabla\pi|_{T_i}|^2 \, d\|T_i\| \right)^{\frac{1}{2}} \cdot \\ &\quad \cdot \left(\int_{T_i} |\nabla\pi|_{T_i}|^2 \, dT_i \right)^{\frac{1}{2}} \\ &\leq C_1 M(\langle C, |\cdot|, \rho_j \rangle) \int_{\partial B_{\rho_j}(x_0)} |\nabla\pi|_C|^2 \, d\langle C, |\cdot|, \rho_j \rangle .\end{aligned}$$

Together with Lemma 1.8 we get that

$$\begin{aligned}\int_{\partial B_{\rho_j}(x_0)} (\pi^*\alpha_A)|_C d\langle C, |\cdot|, \rho_j \rangle &\leq C_1 M(\langle C, |\cdot|, \rho_j \rangle) \int_{\partial B_{\rho_j}(x_0)} |\nabla\pi|_C|^2 d\langle C, |\cdot|, \rho_j \rangle \\ &\leq C_1 \rho_j \, \frac{1}{\rho_j} \int_{B_r(x_0) \setminus B_{\frac{r}{2}}(x_0)} |\nabla\pi|_C|^2 d\|C\| . \qquad (1.18)\end{aligned}$$

Combining estimates (1.16), (1.17) and (1.18) we obtain

$$\int_{B_{\frac{r}{2}}(x_0)} |\nabla\pi|_C|^2 d\|C\| \leq C_1 \int_{B_r \setminus B_{\frac{r}{2}}(x_0)} |\nabla\pi|_C|^2 d\|C\| + \eta \int_{B_r(x_0)} |\nabla\pi|_C|^2 d\|C\| .$$

Since we assumed η to be smaller than 1, we can subtract $\eta \int_{B_{\frac{r}{2}}(x_0)} |\nabla\pi|_C|^2 d\|C\|$ from both sides resulting in the estimate

$$\int_{B_{\frac{r}{2}}(x_0)} |\nabla\pi|_C|^2 d\|C\| \leq \frac{C_1}{(1-\eta)} \int_{B_r \setminus B_{\frac{r}{2}}(x_0)} |\nabla\pi|_C|^2 d\|C\| \ .$$

Filling the hole by adding $\frac{C_1}{(1-\eta)}$ times the left-hand side of the above, we get that there is a constant $C_1 > 0$ (now also depending on $\eta > 0$) such that

$$\int_{B_{\frac{r}{2}}(x_0)} |\nabla\pi|_C|^2 d\|C\| \leq \frac{C_1}{C_1+1} \int_{B_r(x_0)} |\nabla\pi|_C|^2 d\|C\| \ .$$

From this a standard iteration argument (see the book by M. Giaquinta [28] for details) proves Theorem 1.2 in case the calibration ω is compatible with an almost complex structure J on M^{2m}.

1.7 Perturbation argument

The next step is to extend the result to the general case which, since our proof is very robust, will follow from a perturbation argument. Hence from now on we are in the situation where the manifold is of arbitrary dimension m and the calibrating 2-form may not be closed. Fixing $x_0 \in \operatorname{spt}\|C\| \subset M$ we want to find a similar projection to π before. We can assume that we have chosen coordinates so that $\omega_0 = \omega(x_0)$ is the standard symplectic form on $\mathbf{R}^{2n} \subset \mathbf{R}^m$ (see section 1.3). Furthermore, we can assume to be working in a small ball $B_{r_0}(0)$ so that on this ball we have $\|\omega(x) - \omega_0\|_{C^2} = O(|x|)$ and the almost monotonicity formula holds true. Since ω_0 is the standard symplectic 2-form on $\mathbf{R}^{2n}$, it is compatible with the standard complex structure J_0. Therefore, as before we have the natural projection $\pi : \mathbf{C}^n \setminus \{0\} \longrightarrow \mathbf{CP}^{n-1}$, where we make the usual identification $\mathbf{C}^n = (\mathbf{R}^{2n}, J_0)$. We will now extend this to a new projection $\tilde{\pi}$ on all of $\mathbf{R}^m = \mathbf{R}^{2n} \times \mathbf{R}^{m-2n}$ by defining

$$\begin{aligned} \tilde{\pi} : \quad & \mathbf{R}^{2n} \times \mathbf{R}^{m-2n} \setminus \{0\} \times \mathbf{R}^{m-2n} \longrightarrow \mathbf{CP}^{n-1} \\ & (x,y) \longmapsto \pi(x) \ . \end{aligned}$$

We now aim at showing that the proof of Theorem 1.2 can still be carried out using this new projection instead of π. The first step in this direction is the Lemma below where we analyse the structure of the simple vectors defining C which are calibrated by ω near 0.

Lemma 1.12 *Let $x \in \operatorname{spt}\|C\|$ such that $\tau(x)$ exists and is calibrated by $\omega(x) = \omega_0 + \omega_1(x)$, with ω_0 as above. Then we can write $\tau(x)$ as $\tau_0(x) + \tau_1(x)$ so that $\|\tau_0(x)\| = 1$, $\omega_0(\tau_0(x)) = 1$ and $\|\tau_1(x)\| = O(|x|^{\frac{1}{2}})$.*

Proof. From the fact that $\omega(x)$ calibrates $\tau(x)$ and $\omega(x) = \omega_0 + \omega_1(x)$ we deduce that $\omega_0(\tau(x)) = 1 + O(|x|)$. Hence τ is not perpendicular to $\mathbf{R}^{2n} \subset \mathbf{R}^m$. To construct a simple 2-vector τ_0 close to τ we first orthogonally project $\tau(x)$ onto $\mathbf{R}^{2n} \subset \mathbf{R}^m$ to obtain a simple 2-vector $\tilde{\tau}(x)$. Then we get that $\omega_0(\tilde{\tau}(x)) = 1 + O(|x|)$ and that therefore $\|\tau(x) - \tilde{\tau}(x)\| = O\left(|x|^{\frac{1}{2}}\right)$. Setting $\bar{\tau}(x) := \frac{\tilde{\tau}}{\|\tilde{\tau}\|}$ from Lemma 6.13 in the paper by R. Harvey and B. Lawson (see Lemma C.4 in appendix C) we know that there exists a unitary basis $e_1, J_0e_1, \ldots, e_n, J_0e_n$ of $\mathbf{R}^{2n}$ such that $\bar{\tau}(x) = e_1 \wedge (J_0e_1 \cos\theta + e_2 \sin\theta)$ for some angle $\theta \in [0, 2\pi]$. Then the 2-vector $\tau_0(x) := e_1 \wedge J_0e_1$ is of mass 1 and calibrated by ω_0. Thus it remains to show that $\|\tau(x) - \tau_0\| = O\left(|x|^{\frac{1}{2}}\right)$. To see this note that it suffices to estimate $\|\tilde{\tau}(x) - \tau_0\|^2 = \|\, \|\tilde{\tau}\| e_2 - J_0e_1\|^2$. From the construction of $\tilde{\tau}$ we already know that $\omega_0(\tilde{\tau}(x)) = 1 + O(|x|)$, therefore deduce $\|\, \|\tilde{\tau}\| e_2 - J_0e_1\|^2 = O(|x|)$ and the proof of the Lemma is completed. $\square$

Next we will show that if dilated sufficiently far, the current will avoid a conical region near $\{0\} \times \mathbf{R}^{m-2n}$, i.e. where the map $\tilde{\pi}$ is not defined. Precisely, we have the following Lemma:

Lemma 1.13 *Let $\mathbf{R}^{2n} \subset \mathbf{R}^m$, C and ω be as above. Then there exists a radius $r_0 > 0$ such that for all radii $0 < r \leq r_0$ and any smooth compactly supported 2-form $\psi \in \bigwedge^2 \left(B_1(0) \setminus \left\{x \in B_1(0) \ :\ \mathrm{dist}\,(x, \mathbf{R}^{2n}) \leq \frac{1}{2}|x|\right\}\right)$*

$$C_{r,0}(\psi) = 0 \ .$$

Proof. We argue by contradiction. If the Lemma is false, there exist sequences of radii $r_n \longrightarrow 0$ and smooth 2-forms $\psi_n \in \bigwedge^2(B_1(0))$ such that $C_{r_n,0}$ converges to some tangent cone $C_{\infty,0}$ and

$$\mathrm{spt}\,\psi_n \subset E^c \ ,$$

with $E := \left\{x \in B_1(0) \ :\ \mathrm{dist}\,(x, \mathbf{R}^{2n}) \leq \frac{1}{2}|x|\right\}$, yet for all $n \in \mathbf{N}$

$$C_{r_n,0}(\psi_n) \neq 0 \ .$$

Then there exist $x_n \in E^c$ with $\lim_{r\to 0} M(C_{r,x_n}) \neq 0$ and from the monotonicity formula we get that

$$M\left(C_{r_n|x_n|,0} \llcorner B_{\frac{1}{2}}\left(\frac{x_n}{|x_n|}\right)\right) \geq \frac{1}{16}\pi \ .$$

Taking a subsequence so that $\frac{x_n}{|x_n|} \longrightarrow x_\infty$ (with $x_\infty \in E^c$) we have

$$M\big(C_{r_n|x_n|,0} \llcorner B_{\frac{3}{8}}(x_\infty)\big) \geq \frac{1}{16}\pi \ .$$

Since we also have

$$\begin{aligned}
&\left|C_{r_n|x_n|,0} \llcorner B_{\frac{3}{8}}(x_\infty)\big((r_n|x_n|)^2 \lambda^{r_n|x_n|,0^*}\omega - \omega_0\big)\right| \\
&\qquad \leq M(C_{r_n|x_n|,0} \llcorner B_{\frac{3}{8}}(x_\infty))\left\|(r_n|x_n|)^2 \lambda^{r_n|x_n|,0^*}\omega - \omega_0\right\|_\infty \longrightarrow 0 \ ,
\end{aligned}$$

we conclude that we would get

$$C_{\infty,0} \llcorner B_{\frac{3}{8}}(x_\infty)(\omega_0) \geq \frac{1}{16}\pi \ .$$

This contradicts the fact from Proposition 1.5 and the remark following it, namely that $C_{\infty,0}$ is the union of ω_0-calibrated disks which are supported in $\mathbf{R}^{2n}$ and thus in E. Therefore the Lemma holds true. $\square$

The proof is an adaptation of the proof of (III.3) in Lemma III.1 in the paper by T. Riviére and G. Tian [71] where they arrived at a much stronger conclusion using the additional fact that the tangent cones are unique which at this point is clearly unavailable for us.

Since the dilated current $C_{r_0,0}$ is a 2-cycle calibrated by $r_0^2\,(\lambda^{r_0,0})^*\,\omega$ which equals ω_0 at the origin, without loss of generality we can assume that the radius r_0 given by Lemma 1.13 is in fact equal to 1 for otherwise we replace C by $C_{r_0,0}$. Consequently, the $\|C\|$-measure of E^c in the proof above equals 0. The next step in the proof is to combine Lemmas 1.6, 1.12 and 1.13 to obtain an analogous version of (1.7) for $\tilde{\pi}$ instead of π. From Lemmas 1.6 and 1.12 we obtain that point-wise $\|C\|$-a.e. for r_0 sufficiently small

$$\begin{aligned}
\left\langle \tau, \textstyle\bigwedge_2 \tilde{\pi} \right\rangle &\cong (1 \pm C_1|x|^{\frac{1}{2}}) \left\langle \bar{\tau}, \textstyle\bigwedge_2 \pi \right\rangle \\
&\cong \left\langle e_1 \wedge (J_0 e_1 \cos\theta + e_2 \sin\theta), \textstyle\bigwedge_2 \pi \right\rangle \\
&\cong \frac{1}{|x|^2}\left[|\cos\theta| \left| e_1 \wedge J_0 e_1 \wedge \frac{\partial}{\partial r} \wedge J_0 \frac{\partial}{\partial r} \right| \pm |\sin\theta| \left| e_1 \wedge \frac{\partial}{\partial r} \wedge J_0 \frac{\partial}{\partial r} \right| \right] \\
&\cong \frac{1}{|x|^2}\left[\left| \tau_0 \wedge \frac{\partial}{\partial r} \right|^2 \pm C_1 |x|^{\frac{1}{2}} \left| \tau_0 \wedge \frac{\partial}{\partial r} \right| \right] . \qquad (1.19)
\end{aligned}$$

To show (1.7) for $\tilde{\pi}$ we first estimate the last term of (1.19) by $C_1|x|^{\frac{1}{2}}$. For the resulting term we have that for $0 < s < r \le r_0$ (with r_0 the radius given by the almost monotonicity formula)

$$\int_{B_r \setminus B_s(0)} \frac{1}{|x|^{\frac{3}{2}}}\, d\|C\| \lesssim \frac{M(C \llcorner B_r(0))}{s^{\frac{3}{2}}} \lesssim r^{\frac{1}{2}} \left(\frac{r}{s}\right)^{\frac{3}{2}} \frac{M(C \llcorner B_{r_0}(0))}{r_0^2} .$$

Consequently, iterating this for $s \in [r/2, r]$

$$\begin{aligned}
\int_{B_r \setminus B_{\frac{r}{2^n}}(0)} \frac{1}{|x|^{\frac{3}{2}}}\, d\|C\| &\lesssim \sum_{i=0}^{n-1} \left(\frac{r}{2^i}\right)^{\frac{1}{2}} \\
&\le C_1 \frac{r^{\frac{1}{2}}}{1 - 2^{-\frac{1}{2}}} .
\end{aligned}$$

Since the right-hand side is independent of $n \in \mathbf{N}$ we can take the limit $n \longrightarrow \infty$ and obtain the estimate

$$\int_{B_r(0)} \frac{1}{|x|^{\frac{3}{2}}}\, d\|C\| \le C_1 r^{\frac{1}{2}} .$$

This allows us to deduce

$$\begin{aligned}
M(\tilde{\pi}_*[C \llcorner B_r \setminus B_s(0)]) &= \int_{B_r \setminus B_s(0)} \left\langle \tau, \textstyle\bigwedge_2 \tilde{\pi} \right\rangle d\|C\| \\
&\cong \int_{B_r \setminus B_s(0)} \frac{1}{|x|^2} \left| \tau_0 \wedge \frac{\partial}{\partial r} \right|^2 d\|C\| \pm C_1 r^{\frac{1}{2}} .
\end{aligned}$$

Thus, to prove (1.7) for $\tilde{\pi}$ it suffices to show this for $\int_{B_r(0)} \frac{1}{|x|^2} \left|\tau_0 \wedge \frac{\partial}{\partial r}\right|^2 d\|C\|$. However, essentially the same argument relates this term to the almost monotonicity formula leading to

$$M(\tilde{\pi}_*[C \llcorner B_r \setminus B_s(0)]) \pm C_1 r^{\frac{1}{2}} \cong \frac{M(C \llcorner B_r(0))}{r^2} - \frac{M(C \llcorner B_s(0))}{s^2}$$

therefore proving (1.7) for $\tilde{\pi}$.
In fact, we can prove even more. Going back to the second term on the right-hand side of (1.19) we note that Cauchy-Schwarz' inequality and an argument analogous to the one above shows

$$\begin{aligned}\int_{B_r(0)} \frac{1}{|x|^{\frac{3}{2}}} \left|\tau_0 \wedge \frac{\partial}{\partial r}\right| d\|C\| &\leq \left(\int_{B_r(0)} \frac{1}{|x|^2} \left|\tau_0 \wedge \frac{\partial}{\partial r}\right|^2 d\|C\|\right)^{\frac{1}{2}} \left(\int_{B_r(0)} \frac{1}{|x|} d\|C\|\right)^{\frac{1}{2}} \\ &\leq C_1 \left(\int_{B_r(0)} \frac{1}{|x|^2} \left|\tau_0 \wedge \frac{\partial}{\partial r}\right|^2 d\|C\|\right)^{\frac{1}{2}} r^{\frac{1}{2}} .\end{aligned}$$

Combining this with (1.19) we obtain

$$\begin{aligned}M(\tilde{\pi}_*[C \llcorner B_r(0)]) &\cong \left(\int_{B_r(0)} \frac{1}{|x|^2} \left|\tau_0 \wedge \frac{\partial}{\partial r}\right|^2 d\|C\|\right) \\ &\quad \pm C_1 \left(\int_{B_r(0)} \frac{1}{|x|^2} \left|\tau_0 \wedge \frac{\partial}{\partial r}\right|^2 d\|C\|\right)^{\frac{1}{2}} r^{\frac{1}{2}} \end{aligned} \tag{1.20}$$

and analogously for $\omega_{\mathbf{CP}^{n-1}}$

$$\begin{aligned}\langle \tilde{\pi}_*[C \llcorner B_r(0)], \omega_{\mathbf{CP}^{n-1}} \rangle &\cong \left(\int_{B_r(0)} \frac{1}{|x|^2} \left|\tau_0 \wedge \frac{\partial}{\partial r}\right|^2 d\|C\|\right) \\ &\quad \pm C_1 \left(\int_{B_r(0)} \frac{1}{|x|^2} \left|\tau_0 \wedge \frac{\partial}{\partial r}\right|^2 d\|C\|\right)^{\frac{1}{2}} r^{\frac{1}{2}} . \end{aligned} \tag{1.21}$$

We also have the following version of Lemma 1.8 with exactly the same proof:

Lemma 1.14 *Given $0 < r < r_0$, there exists $\rho_0 \in \left[\frac{r}{2}, r\right]$ such that the following hold:*

1. $M(\langle C, |\cdot|, \rho_0 \rangle) \leq C_1 \rho_0$,

2. $\int_{\partial B_{\rho_0}(x_0)} |\langle \tau, \bigwedge_2 \tilde{\pi} \rangle| d\langle C, |\cdot|, \rho_0 \rangle \leq \frac{1}{\rho_0} \int_{B_r \setminus B_{\frac{r}{2}}(x_0)} |\langle \tau, \bigwedge_2 \tilde{\pi} \rangle| d\|C\|$,

3. $M(\tilde{\pi}_* \partial [C \llcorner B_{\rho_0}(x_0)]) \leq C_1 \fint_{\frac{r}{2}}^{r} \left[\int_{\partial B_\rho} \frac{1}{|x|} \left|\tau_0 \wedge \frac{\partial}{\partial r}\right| d\langle C, |\cdot|, \rho \rangle\right] d\rho$.

Before we come to the version of Lemma 1.9 we again restrict to radii of the form $r = \frac{1}{2^j}$. Again we partition the set of indices into two subsets:

$$\begin{aligned}
A &:= \Bigg\{ j \in \mathbf{N} \;:\; M\left(\tilde{\pi}_* \left[C \llcorner B_{\frac{1}{2^{j+1}}}(x_0)\right]\right) \leq \frac{1}{2} M\left(\tilde{\pi}_* \left[C \llcorner B_{\frac{1}{2^j}}(x_0)\right]\right) \\
&\qquad \text{or} \left(\int_{B_{\frac{1}{2^{j+1}}}(0)} \frac{1}{|x|^2} \left| \tau_0 \wedge \frac{\partial}{\partial r} \right|^2 d\|C\| \right)^{\frac{1}{2}} \leq C_1 \cdot 1000 \cdot \left(\frac{1}{2^{j+1}} \right)^{\frac{1}{2}} \Bigg\}, \\
B &:= \Bigg\{ j \in \mathbf{N} \;:\; M\left(\tilde{\pi}_* \left[C \llcorner B_{\frac{1}{2^{j+1}}}(x_0)\right]\right) > \frac{1}{2} M\left(\tilde{\pi}_* \left[C \llcorner B_{\frac{1}{2^j}}(x_0)\right]\right) \\
&\qquad \text{and} \left(\int_{B_{\frac{1}{2^{j+1}}}(0)} \frac{1}{|x|^2} \left| \tau_0 \wedge \frac{\partial}{\partial r} \right|^2 d\|C\| \right)^{\frac{1}{2}} > C_1 \cdot 1000 \cdot \left(\frac{1}{2^{j+1}} \right)^{\frac{1}{2}} \Bigg\}.
\end{aligned}$$

As before, the aim of the remainder of this section will be to prove the existence of a constant $\theta \in (0,1)$ for an analogous version of estimate (1.8) for $j \in B$. To see why this leads to a rate of convergence first note that for $j \in A$ either estimate (1.8) holds true with $\theta = \frac{1}{2}$ or (1.20) implies that $M(\tilde{\pi}_*[C \llcorner B_{\frac{1}{2^{j+1}}}(0)]) \leq C_1 \frac{1}{2^{j+1}}$. Thus for any $j \in \mathbf{N}$ we have

$$\begin{aligned}
M(\tilde{\pi}_*[C \llcorner B_{\frac{1}{2^j}}(0)]) &\leq \begin{cases} C_1 \theta^k 2^{-j+k} & \text{for some } k \in [1, j-1] \\ \theta^j M(\tilde{\pi}_*[C \llcorner B_1(0)]) & \text{otherwise} \end{cases} \\
&\leq C_1 \left(\frac{1}{2^j} \right)^{\gamma},
\end{aligned}$$

where we choose γ so small that $\theta \leq 2^{-\gamma}$. Thus it remains to work with $j \in B$ which means that from (1.20) we can assume

$$M(\tilde{\pi}_*[C \llcorner B_{\frac{1}{2^{j+1}}}(0)]) \cong C_1 \left(1 \pm \frac{1}{1000} \right) \left(\int_{B_{\frac{1}{2^{j+1}}}(0)} \frac{1}{|x|^2} \left| \tau_0 \wedge \frac{\partial}{\partial r} \right|^2 d\|C\| \right).$$

With this we immediately have the result of Lemma 1.9 for $\tilde{\pi}$

Lemma 1.15 *There exists $C_1 > 0$ such that if $j \in B$ and $\rho_j \in \left[\frac{1}{2^{j+1}}, \frac{1}{2^j}\right]$ denotes a good slice from Lemma 1.14. Then*

$$M(\tilde{\pi}_* \partial [C \llcorner B_{\rho_j}]) \leq C_1 \left[M(\tilde{\pi}_* C \llcorner B_{\rho_j} \right]^{\frac{1}{2}} .$$

We now have all the necessary ingredients to formulate the adapted version of Lemma 1.5.

Lemma 1.16 *Given a constant $C_1 > 0$ and $\eta > 0$ small, there exist $\delta > 0$ and $\epsilon > 0$ with the following properties:*
Let $T \in \mathbf{I}_2(\mathbf{CP}^{n-1})$ be an integral 2-current satisfying

1. *T is almost calibrated by $\omega_{\mathbf{CP}^{n-1}}$, i.e. $M(T \llcorner U) \cong \left(1 \pm \frac{1}{1000}\right) \langle T \llcorner U, \omega_{\mathbf{CP}^{n-1}} \rangle$ for all $U \subset \mathbf{CP}^{n-1}$,*

2. $M(T) + M(\partial T) < \delta$,

3. $M(\partial T) \leq C_1[M(T)]^{\frac{1}{2}}$.

Then there exists a tubular neighbourhood $A_\epsilon \subset \mathbf{CP}^{n-1}$ *of width* ϵ *with core consisting of a* $\mathbf{CP}^{n-2} \subset \mathbf{CP}^{n-1}$ *such that*

$$\frac{M(T \llcorner A_\epsilon)}{\epsilon^2} \leq \eta M(T) \ .$$

The only effect on the proof comes in the step where the isoperimetric inequality is applied. However, suppose that T is as above and that $\partial(T \llcorner A_\epsilon) = (\partial T) \llcorner A_\epsilon$ for some tube $A_\epsilon \subset \mathbf{CP}^{n-1}$. Then there exists an integral 2-current $R \in \mathbf{I}_2(\mathbf{CP}^{n-1})$ such that $\partial R = \partial(T \llcorner A_\epsilon)$ with $M(R)^{\frac{1}{2}} \leq C_1 M(\partial(T \llcorner A_\epsilon))$. Since R is homologous to $T \llcorner A_\epsilon$, i.e. there exists an integral 3-current $S \in \mathbf{I}_3(\mathbf{CP}^{n-1})$ such that $\partial S = T \llcorner A_\epsilon - R$, we have

$$\begin{aligned} M(T \llcorner A_\epsilon) \leq C_1 \langle T \llcorner A_\epsilon, \omega_{\mathbf{CP}^{n-1}} \rangle &= C_1 \langle T \llcorner A_\epsilon - R, \omega_{\mathbf{CP}^{n-1}} \rangle + C_1 \langle R, \omega_{\mathbf{CP}^{n-1}} \rangle \\ &\leq C_1 \langle S, d\omega_{\mathbf{CP}^{n-1}} \rangle + C_1 M(R) = C_1 M(R) \end{aligned}$$

showing that $M(T \llcorner A_\epsilon)^{\frac{1}{2}} \leq C_1 M(\partial(T \llcorner A_\epsilon))$. Other than that the argument can be carried over word for word.
Furthermore, since all the relevant Lemmas are still valid for $\tilde{\pi}$, the proof of the rate of convergence in section 1.6 is not affected by the small perturbation. Hence the proof of Theorem 1.2 is completed.

1.8 Uniqueness of tangent cones

Although the passage from Theorem 1.2 to Theorem 1.1 had already been shown by B. White in [94] we will include his proof for the sake of completeness. The proof is directly taken from part (b) of Theorem 3 in [94].
As before we will assume that the coordinates are chosen so that in these coordinates x_0 is identified with 0. Let $F : \mathbf{R}^m \longrightarrow S^{m-1}$ be the radial projection on $\mathbf{R}^m$ sending x to $\frac{x}{|x|}$. If we can show that for $0 < s < r$ the quantity $M(F_*[C \llcorner B_r \setminus B_s(0)])$ tends to 0 as first s and then r tend to 0, i.e.

$$\lim_{r \to 0} \lim_{s \to 0} M\left(F_*\left[C \llcorner B_r \setminus B_s(0)\right]\right) = 0 \ , \tag{1.22}$$

then uniqueness of tangent cones will follow immediately — compare this with estimate (1.7) for the projection under π. To see this, first consider two sequences $\{\rho_i\}$ and $\{\bar{\rho}_i\}$ so that $\lambda_*^{\rho_i,0}[C \llcorner B_{\rho_i}(0)] \llcorner B_1(0)$ and $\lambda_*^{\bar{\rho}_i,0}[C \llcorner B_{\bar{\rho}_i}(0)] \llcorner B_1(0)$ converge in the $\mathcal{F}$-norm (see 4.1.24 in [25] for the definition) to tangent cones $C_{\infty,0}$ and $\bar{C}_{\infty,0}$ respectively. As an immediate consequence we also have that $\partial \lambda_*^{\rho_i,0}[C \llcorner B_{\rho_i}(0)] \llcorner B_1(0)$ and $\partial \lambda_*^{\bar{\rho}_i,0}[C \llcorner B_{\bar{\rho}_i}(0)] \llcorner B_1(0)$ converge in the $\mathcal{F}$-norm to $\partial C_{\infty,0}$ and $\partial \bar{C}_{\infty,0}$ respectively. By the structure of tangent cones result in Lemma 1.5 we know that the two tangent cones are equal provided we can show that $\partial C_{\infty,0} = \partial \bar{C}_{\infty,0}$ in the $\mathcal{F}$-norm. We will do this by proving that $\partial \lambda_*^{\rho_i,0}[C \llcorner B_{\rho_i}(0)] \llcorner B_1(0) - \partial \lambda_*^{\bar{\rho}_i,0}[C \llcorner B_{\bar{\rho}_i}(0)] \llcorner B_1(0)$ converges to 0. Since for almost all $0 < s < r$ we have

$$\partial F_*\left[C \llcorner B_r \setminus B_s(0)\right] = \partial \lambda_*^{r,0}\left[C \llcorner B_r(0)\right] - \partial \lambda_*^{s,0}\left[C \llcorner B_s(0)\right] \ ,$$

the definition of the $\mathcal{F}$-norm implies that

$$\partial\lambda_*^{\rho_i,0}[C\llcorner B_{\rho_i}(0)]\llcorner B_1(0) - \partial\lambda_*^{\bar\rho_i,0}[C\llcorner B_{\bar\rho_i}(0)]\llcorner B_1(0) \leq M(F_*\left[C\llcorner B_{\rho_i}\setminus B_{\bar\rho_i}(0)\right])\ ,$$

which from (1.22) we know tends to 0.
Therefore it remains to prove (1.22). Using first the definitions, Schwarz' inequality, and the monotonicity formula for all $0 < s < r \leq r_0$ we compute

$$\begin{aligned} M\left(F_*\left[C\llcorner B_r\setminus B_s(0)\right]\right) &= \int_{B_r\setminus B_s(0)} \left\langle \tau(x), \bigwedge_2 F(x)\right\rangle d\|C\| \\ &\lesssim \int_{B_r\setminus B_s(0)} \frac{1}{|x|^2}\left|\tau(x)\wedge\frac{\partial}{\partial r}\right| d\|C\| \\ &\lesssim \left(\int_{B_r\setminus B_s(0)} \frac{1}{|x|^2}\left|\tau(x)\wedge\frac{\partial}{\partial r}\right|^2 d\|C\|\right)^{\frac12} \\ &\quad\cdot\left(\int_{B_r\setminus B_s(0)}\frac{1}{|x|^2} d\|C\|\right)^{\frac12} \\ &\lesssim \left(\frac{M(C\llcorner B_r(0))}{r^2} - \Theta(\|C\|,0)\right)^{\frac12}\cdot\left(\frac{r}{s}\right)\left(\frac{M(C\llcorner B_r(0))}{r^2}\right)^{\frac12} \end{aligned}$$

From Theorem 1.2 we can bound the first term on the right-hand side above by $C_1 r^{\frac{\gamma}{2}}$ whilst the last one is taken care of by the monotonicity formula. Therefore, if we only consider $r/2 \leq s < r$, we obtain the estimate

$$M\left(F_*\left[C\llcorner B_r\setminus B_s(0)\right]\right) \leq C_1 r^{\frac{\gamma}{2}}\ . \tag{1.23}$$

Iterating this when $s = \frac{r}{2^n}$ for $n\in\mathbf{N}$ we have

$$\begin{aligned} M\left(F_*\left[C\llcorner B_r\setminus B_{\frac{r}{2^n}}(0)\right]\right) &= \sum_{i=0}^{n-1} M\left(F_*\left[C\llcorner B_{\frac{r}{2^i}}\setminus B_{\frac{r}{2^{i+1}}}(0)\right]\right) \\ &\leq \sum_{i=0}^{n-1} C_1\left(\frac{r}{2^i}\right)^{\frac{\gamma}{2}} \\ &\leq C_1\frac{r^{\frac{\gamma}{2}}}{1-2^{-\frac{\gamma}{2}}}\ . \end{aligned}$$

Since the right-hand side of this estimate is independent of n, we can immediately deduce estimate (1.22) which we wanted to show. Thus we proved that Theorem 1.1 is a direct consequence of Theorem 1.2.

1.9 Outlook I: Poincaré inequalities

In the following two sections we look at possible extensions of the above techniques. As a first attempt one would like to extend the proof to higher dimensional currents. In other words, let C be an integral k-cycle in $\mathbf{R}^m$ which is calibrated by a smooth closed k-form ω — for simplicity we restrict ourselves to the case of a smooth calibration on $\mathbf{R}^m$. In particular, to illustrate the ideas consider the case where C is calibrated by $\omega = \frac{1}{k!}\omega_0^k$ on $\mathbf{R}^{2m}$, where ω_0 is again the standard symplectic form on $\mathbf{R}^{2m}$. The monotonicity formula for such a $2k$-cycle is given by

$$\frac{M(C \llcorner B_r(x_0))}{\alpha(2k)r^{2k}} - \frac{M(C \llcorner B_s(x_0))}{\alpha(2k)s^{2k}} \cong \int_{B_r \setminus B_s(x_0)} \frac{1}{|x|^{2k}} \left| \tau \wedge \frac{\partial}{\partial r} \right|^2 d\|C\| \ .$$

As in the special case when $k = 1$ we can relate this to the mass of $\pi_* C \llcorner B_r \setminus B_s(x_0)$ which is calibrated by $\frac{1}{k!}\omega^k_{\mathbf{CP}^{m-1}}$ again since π is $J_0 - j_0$-holomorphic. We have

$$M(\pi_* C \llcorner B_r \setminus B_s(x_0)) \cong \langle C \llcorner B_r \setminus B_s(x_0), \pi^* \omega^k_{\mathbf{CP}^{m-1}} \rangle \cong \frac{M(C \llcorner B_r(x_0))}{\alpha(2k)r^{2k}} - \frac{M(C \llcorner B_s(x_0))}{\alpha(2k)s^{2k}} \ ,$$

showing that as before $M(\pi_* C \llcorner B_r(x_0))$ is well-defined and converges to 0 as $r \longrightarrow 0$. Therefore, one of the main ingredients of our proof is still valid in this higher dimensional setting relating a rate of convergence for C to a rate of convergence for $M(\pi_* C \llcorner B_r(x_0))$. Our proof in the special case when $k = 1$ above can be summarised in two main steps:

Step 1 (Integration by parts): Lemmas 1.5-1.11 allowed us to integrate by parts. More precisely, we could show the existence of a $(2k-1)$-form α on $\mathbf{CP}^{m-1}$ with $|\nabla \alpha| \leq C_1$ (component-wise in a fixed finite atlas) such that

$$\int_{B_r(x_0)} \frac{1}{k!} \pi^* \omega^k_{\mathbf{CP}^{m-1}} \, d\|C\| = \int_{B_r(x_0)} d(\pi^* \alpha) \, d\|C\| \ .$$

Step 2 (Poincaré inequality): The other main estimate in section 1.6 is the Poincaré-type inequality we proved on $\langle C, |\cdot|, r \rangle$ which in this more general case would ask for existence of a constant C_1 such that, when $\bar{\alpha}$ denotes the average of α on each connected component of $\langle C, |\cdot|, r \rangle$,

$$\int_{\partial B_r(x_0)} |\alpha - \bar{\alpha}|^{2k} \, d\langle C, |\cdot|, r \rangle \leq C_1 M(\partial [C \llcorner B_r(x_0)])^{\frac{2k}{2k-1}} \cdot \left(\int_{\partial B_r(x_0)} |\nabla \alpha|^{2k} \, d\langle C, |\cdot|, r \rangle \right) \ .$$

Combining these estimates as in section 1.6 together with a variant of Lemma 1.8 we would be able to deduce a rate of convergence as before.

Generalising this even further we suggest the following three open problems leading to a blow-up rate for k-cycles which are calibrated by certain smooth k-forms ω of unit comass.

Open Problem 1.1 (Geometric structure of calibrations) *For a smooth calibrating k-form ω on $\mathbf{R}^m$ and $x_0 \in \mathbf{R}^m$, denote the space of oriented k-planes through x which are calibrated by $\omega(x)$ by $G_{x_0}(\omega)$. Furthermore, denote the union of the $G_{x_0}(\omega)$ by $G(\omega)$. For which calibrations ω do there exist projections π from $\mathbf{R}^m \setminus \{x_0\}$ onto a smooth closed manifold N of dimension $n < m$ such that for some closed calibrating k-form $\tilde{\omega}$ on N*

1. $|\nabla \pi(x)| \cong \frac{1}{|x - x_0|}$ *for all $x \in \mathbf{R}^m$* ,

2. $\langle \tau, \pi^*\tilde{\omega}\rangle \cong \frac{1}{|x-x_0|^k}\left|\tau \wedge \frac{\partial}{\partial r}\right|^2$ *for all* $\tau \in G(\omega)$ *?*

Open Problem 1.2 (Integration by parts) *Using the same notation as above, does there exist a constant* C_1 *such that for any calibrated* k*-cycle there exists a smooth* $(k-1)$*-form* α *on* N *with* $|\nabla\alpha| \leq C_1$ *and*

$$\int_{B_r(x_0)} \langle \tau, \pi^*\tilde{\omega}\rangle \, d\|C\| = \int_{B_r(x_0)} \langle \tau, d(\pi^*\alpha)\rangle \, d\|C\| \quad ?$$

Open Problem 1.3 (Poincaré-type inequality) *Does there exist a constant* C_1 *such that for any* $(k-1)$*-form* α *above we have, when* $\bar{\alpha}$ *denotes the average of* α *on each connected component of* $C \llcorner B_{2r} \setminus B_r(x_0)$*,*

$$\int_{B_{2r}\setminus B_r(x_0)} |\alpha - \bar{\alpha}|^k \, d\|C\| \leq C_1 M(C \llcorner B_{2r}(x_0)) \cdot \left(\int_{B_{2r}\setminus B_r(x_0)} |\nabla\alpha|^k \, d\|C\|\right) \, ?$$

Assuming these steps we now show that a slight modification of the approach in section 1.6 will give a rate of convergence. To see this, first consider a cut-off function $\phi : (0,\infty) \longrightarrow [0,1]$ such that $\phi(\rho) = 1$ for $\rho \leq r$, $\phi(\rho) = 0$ for $\rho \geq 2r$ and $\phi(\rho) \leq 0$ with $|\phi'(\rho)| \leq C_1 \frac{1}{\rho}$. With this and the properties of ϕ we compute

$$\begin{aligned} \int_{B_r(x_0)} |\nabla\pi|^k \, d\|C\| &\leq \int_{B_{2r}(x_0)} |\nabla\pi|^k \phi(|x-x_0|) \, d\|C\| \\ &= \int_{B_{2r}(x_0)} \pi^*\tilde{\omega} \; \phi(|x-x_0|) \, d\|C\| \; . \end{aligned}$$

Integrating by parts we deduce

$$\begin{aligned} \int_{B_{2r}(x_0)} \pi^*\tilde{\omega} \; \phi(|x-x_0|) \, d\|C\| &= \int_{B_{2r}(x_0)} \pi^* d\alpha \; \phi(|x-x_0|) \, d\|C\| \\ &= \int_{B_{2r}(x_0)} d\pi^*\alpha \; \phi(|x-x_0|) \, d\|C\| \\ &= -\int_{B_{2r}\setminus B_r(x_0)} \pi^*\alpha \wedge d\phi(|x-x_0|) \, d\|C\| \end{aligned}$$

which we can estimate using the properties of ϕ and Hölder's inequality

$$\begin{aligned} -\int_{B_{2r}\setminus B_r(x_0)} \pi^*\alpha \wedge d\phi(|x-x_0|) \, d\|C\| &\leq C_1 \frac{1}{r} \int_{B_{2r}\setminus B_r(x_0)} |\alpha - \bar{\alpha}| \, |\nabla\pi|^{k-1} \, d\|C\| \\ &\leq C_1 \frac{1}{r} \left(\int_{B_{2r}\setminus B_r(x_0)} |\alpha - \bar{\alpha}|^k \, d\|C\|\right)^{\frac{1}{k}} \\ &\quad \cdot \left(\int_{B_{2r}\setminus B_r(x_0)} |\nabla\pi|^k \, d\|C\|\right)^{\frac{k-1}{k}} \; . \end{aligned}$$

Then the Poincaré-type inequality on $C \llcorner B_{2r} \setminus B_r(x_0)$ yields

$$\left(\int_{B_{2r}\setminus B_r(x_0)} |\alpha - \bar{\alpha}|^k \, d\|C\|\right)^{\frac{1}{k}} \leq C_1 \frac{1}{r} M(C \llcorner B_{2r}(x_0))^{\frac{1}{k}} \left(\int_{B_{2r}\setminus B_r(x_0)} |\nabla\alpha|^k \, d\|C\|\right)^{\frac{1}{k}}$$

which together with the fact that $|\nabla\alpha| \le C_1$, $M(C) \le C_1 < \infty$ and the monotonicity formula applied to $M(C \llcorner B_{2r}(x_0))$ gives

$$\int_{B_r(x_0)} |\nabla\pi|^k \, d\|C\| \le C_1 \frac{1}{r} \cdot r \int_{B_{2r}\setminus B_r(x_0)} |\nabla\pi|^k \, d\|C\| \; = C_1 \int_{B_{2r}\setminus B_r(x_0)} |\nabla\pi|^k \, d\|C\| \; .$$

Filling the hole, a rate of convergence for $M(\pi_* C \llcorner B_r(x_0))$ would follow as before.

Of these three problems, finding a Poincaré-type inequality seems to be the most difficult one and is in fact an interesting question in its own right. One possible attempt in this direction is to embed the problem into the context of analysis on metric spaces. To do this, first consider the support $\operatorname{spt}\|C\|$ of the integral k-cycle C in $\mathbf{R}^m$ which is calibrated by a smooth k-form ω. This is a compact subset of $\mathbf{R}^m$ which we will consider as metric space with the distance function d inherited from the usual distance function on $\mathbf{R}^m$ and the triple $(\operatorname{spt}\|C\|, d, \|C\|)$ forms a metric measure space. Furthermore, from the fact that $M(C) \le C_1 < \infty$, the fact that the density of $\|C\|$ is at least 1 almost everywhere and the monotonicity formula, for all $x_0 \in \operatorname{spt}\|C\|$ and all $r > 0$ we have

$$\|C\|(B_{2r}(x_0)) \le C_1\alpha(k)2^k r^k \le C_1\alpha(k)2^k r^k \frac{\|C\|(B_r(x_0))}{\alpha(k)r^k} = C_1 2^k \|C\|(B_r(x_0)) \; .$$

Therefore $\|C\|$ satisfies the doubling property at all $x_0 \in \operatorname{spt}\|C\|$. Having shown this doubling property we can extend the Open problem 1.3 to a Poincaré inequality for the restriction of arbitrary smooth functions on an open subset of $\operatorname{spt}\|C\|$.

Open Problem 1.4 *For an integral current C as before let $U \subset \mathbf{R}^m$ be a bounded open subset containing* $\operatorname{spt}\|C\|$ *and let $f \in C^1_0(U)$. Does there exist a constant $C_1 > 0$ depending only on k, m and the mass of C such that for all $x_0 \in \operatorname{spt}\|C\|$, all $r > 0$ and any $1 \le p < \infty$ (and thus in particular for $p = k$), when $\bar{f}$ denotes the average of f on each connected component of $C \llcorner B_r(x_0)$,*

$$\int_{B_r(x_0)} |f - \bar{f}|^p \, d\|C\| \le C_1 r^p \int_{B_r(x_0)} |\nabla f|^p \, d\|C\| \quad ?$$

For $1 \le p < k$ such an inequality seems to be a direct consequence of the monotonicity formula together with the lower density bound and was proved by J. Michael and L. Simon in [56] (see also chapter 18 in the book [76] by L. Simon). However, in the cases of $p \ge k$ more information on $\operatorname{spt}\|C\|$ is needed and such an inequality is not known to us, although the question of Poincaré inequalities on doubling metric measure spaces has been studied intensely. As introduction to this subject we recommend the book by L. Ambrosio and P. Tilli [7] for a general overview and the book by J. Heinonen [39] for the geometric issues involved in finding such an inequality. These and in particular the relevance of existence of large families of curves together with good estimates are discussed further in the lecture notes by G. David and S. Semmes [21] and the paper by S. Semmes [75]. The book by P. Hajłasz and P. Koskela [34] (see also their paper [35]) specialises on the relationship between Poincaré and Sobolev inequalities on doubling metric measure spaces and also serves as a very good starting point for delving deeper into the subject.

1.10 Outlook II: Normal currents

Another possible extension would be to reduce the regularity assumptions on the current. As we showed in section 1.2, the monotonicity formula for calibrated cycles is still valid for merely normal currents, dropping the assumptions that the support spt $\|C\|$ is a k-rectifiable set and that the density is almost everywhere at least 1. Apart from the monotonicity formula leading to the existence of tangent cones, with Lemma 1.7 another main ingredient of our approach to a rate of convergence and uniqueness of tangent cones remains available. However, as we will explain below, there is no hope that Lemmas 1.5 and the Poincaré inequality can both be extended to this case, since in the setting of complex analysis of several complex variables non-uniqueness results were obtained.

For the setting in the theory of several complex variables, we consider $\mathbf{C}^m$ with the closed $(1,1)$-form $\omega = \frac{i}{2}\sum_{j=1}^m dz_k \wedge d\bar{z}_k \in \bigwedge^{1,1}(\mathbf{C}^m)$ which is the same as the standard symplectic 2-form on $\mathbf{R}^{2m}$ we used before. We then define the (p,p)-currents, denoted $\mathcal{D}^{p,p}(\mathbf{C}^m)$, to be the continuous linear functionals on the compactly supported $(m-p, m-p)$-forms $\bigwedge_c^{m-p,m-p}(\mathbf{C}^m)$ and define a (p,p)-current T to be real if $T = \bar{T}$, i.e. if $\overline{\langle T, \psi\rangle} = \langle T, \bar{\psi}\rangle$ for all $\psi \in \bigwedge_c^{m-p,m-p}(\mathbf{C}^m)$. It is important to be aware that these currents correspond to real $2m-2p$-currents as defined at the beginning of this chapter, i.e. that a (p,p)-current means a real codimension $2p$-current in $\mathbf{R}^{2m}$. Furthermore, a real (p,p)-current is called positive in case for all $\eta \in \bigwedge_c^{m-p,0}(\mathbf{C}^m)$,

$$i^{p(p-1)/2}\langle T, \eta \wedge \bar{\eta}\rangle \geq 0 \ .$$

The notion of a closed real (p,p)-current remains the same, i.e. $\partial T = 0$. One can then immediately check that for any subset $U \subset \mathbf{C}^m$, $T \llcorner U$ is calibrated by the closed (p,p)-form given by $\frac{1}{p!}\omega^p$ which implies that a monotonicity formula holds for T. Historically, positive (p,p)-currents were introduced by P. Lelong as an outcome of his study of plurisubharmonic functions (an extension of the usual subharmonic functions) in [50]. He also proved the monotonicity formula in this case leading to the notion of 'Lelong numbers' which, in geometric measure theory terms, are precisely the densities of the measure $\|T\|$ associated with T. For an introduction to the study of positive (p,p)-cycles and their connection with plurisubharmonic functions we refer to the book by P. Lelong [51], whereas applications of such currents to complex analysis are discussed in chapter 3 of the book [31] by P. Griffiths and J. Harris.
Since in this setting tangent cones exist and are calibrated by $\frac{1}{p!}\omega^p$ it is a natural question to ask for their uniqueness. Furthermore, since at a point where the 2-density of $\|T\|$ equals 0 any tangent cone has to have mass equal to 0, the only problems could arise at points of positive 2-density. In [37], R. Harvey had conjectured that uniqueness should hold at all points in the support of T, which was answered negatively by C. Kiselman in [47]. There he constructed a counter-example by reducing to plurisubharmonic functions and showing the existence of a plurisubharmonic function with at least two functions tangent to it. In fact, he could prove a much stronger result, in that he precisely described which subsets M of $\mathrm{PSH}(\mathbf{C}^m)$, denoting the set of plurisubharmonic functions on $\mathbf{C}^m$, can be obtained as the set of tangent cones of some plurisubharmonic functions. In particular, he proved that M is closed and connected in the topology induced by $L^1_{loc}(\mathbf{C}^m)$ and gave

examples of plurisubharmonic functions admitting a whole curve of tangent functions in $\mathrm{PSH}(\mathbf{C}^m)$. Further results in this direction were obtained by M. Blel, J.-P. Demailly and M. Mouzali in [14] as well as by M. Blel in [12] and [13].
Recall that the original goal was to obtain regularity results and in particular estimates on the size of the singular sets of calibrated currents. Despite the fact that tangent cones of (p,p)-currents are not unique, Y.-T. Siu could prove a strong regularity result in [80] completely avoiding the use of tangent cones (non-uniqueness had not been known at the time). Given a closed positive (p,p)-current T on an open subset $\Omega \subset \mathbf{C}^m$ and a positive number $c > 0$, denote the set of points where the $2p$-density of $\|T\|$ (the Lelong number) is at least c by $E_c \subset \Omega$. Then Y.-T. Siu showed that E_c is an analytic subvariety of (complex) codimension at least p in Ω.

Using this last result as a guideline, we would like to end this section by showing that a similar result on almost complex manifolds could have interesting applications. In their paper [70], T. Rivière and G. Tian proved that the singular set of a weakly approximable J-holomorphic map $u : (M^4, J) \longrightarrow \mathbf{CP}^1$ from a 4-dimensional almost complex manifold into $\mathbf{CP}^1$ consists of isolated points (see chapter 2 for the precise definitions). There they showed local existence of a 2-form ω on $B^4 \subset M^4$ which is compatible with J and of comass 1. Defining a $(1,1)$-current C on B^4 by $\langle C, \psi\rangle := \int_{B^4} u^*\omega_{\mathbf{CP}^1} \wedge \psi$ for $\psi \in \bigwedge_0^2(B^4)$, they showed that this C is a finite mass 2-cycle calibrated by ω.
From now on we divert from their approach. Since the map u is weakly and stationary harmonic (see chapter 2 sections 2.2 and 2.3 or chapter 3 section 3.1 for details), it satisfies a monotonicity formula and the ϵ_0-regularity theorem shows that u is smooth aside from a singular set $\operatorname{sing} u := \{x \in B^4 \,|\, \Theta_u(x) \geq \epsilon_0 > 0\}$ which has 2-Hausdorff measure equal to 0. The density of u at each point is the same as the density of C so that the singular set $\operatorname{sing} u$ corresponds precisely to the set E_{ϵ_0} in the above paragraph. Supposing one can show an analogous version of the theorem by Y.-T. Siu in this case, we would conclude that the singular set of u were a smoothly immersed J-holomorphic curve. Since the 2-Hausdorff measure of $\operatorname{sing} u$ vanishes, it would follow that the singular set contains only finitely many isolated points in B^4.
This would give an alternative approach to the problem studied by T. Rivière and G. Tian in [70], but could also be applicable in other contexts where the singular set is a calibrated 2-current such as anti self-dual instantons on 6-dimensional manifolds (see the survey paper by G. Tian [87]) etc.

This suggests to study the following open problem:

Open Problem 1.5 *Let C be a normal 2-cycle in $\mathbf{R}^m$ which is calibrated by a smooth 2-form ω on $\mathbf{R}^m$ with comass 1. For any $c > 0$ consider the closed subset $E_c := \{x \in \mathbf{R}^m \,|\, \Theta(\|C\|, x) \geq c\}$. Can E_c be identified with a smoothly immersed 2-dimensional submanifold of $\mathbf{R}^m$?*

Naturally there are many higher dimensional extensions of this question, but finding a positive answer would already be extremely interesting even in the case when ω is a symplectic 2-form on $\mathbf{R}^4$, since to some extend this could serve as the model situation.

Chapter 2

J-holomorphic maps

Abstract

The aim of this chapter is to prove regularity results for J-holomorphic maps from an almost complex manifold into a tamed symplectic manifold. First we derive a second order elliptic equation and a monotonicity formula for them. Next we prove an ϵ-regularity result for J-holomorphic maps avoiding the use of moving frames. Under further topological assumptions on the target, using a slicing argument we then prove a rate of convergence for them and uniqueness of tangent maps is deduced. Finally, we outline how this assumption may be removed for the general case.

2.1 Introduction

In this chapter we obtain regularity results for weakly J-holomorphic maps between manifolds and begin by describing the geometric setting. Let (M, J_M) and (N, J_N) be closed (compact without boundary) almost complex manifolds, where M has dimension $2m$. We consider N to be given an almost Hermitian structure (N, J_N, h) and to be isometrically embedded in $\mathbf{R}^{2n}$ for some n. Furthermore, we assume that on N there exists a closed 2-form ω_N such that $\omega^{1,1} > 0$ and such that there is $C_1 > 1$ with

$$C_1^{-1}|X|_h|Y|_h \leq |\omega_N(X, Y)| \leq C_1|X|_h|Y|_h \ ,$$

for all $y \in N$ and all $X, Y \in T_yN$. This means that we assume the target manifold to be a tamed symplectic manifold. In addition, let g be a smooth metric on M such that $g(\cdot, J_M\cdot)$ defines a 2-form ω_M on M which need not be closed. For simplicity, one can always think of M and N as being closed almost Kähler manifolds
We can then define the following Sobolev space of maps from M into N by

$$W^{1,2}(M, N) := \{u \in W^{k,p}(M, \mathbf{R}^{2n}) \,|\, u(x) \in N \text{ for a.e. } x \in M\} \ .$$

A map $u \in W^{1,2}(M, N)$ is called J-holomorphic between M and N if it preserves their almost complex structures, i.e.

$$du_x(J_M(x)X) = J_N(u(x))du_x(X) \quad \text{for a.e. } x \in M \text{ and all } X \in T_xM \ ,$$

which is a non-linear first order equation for the map u. In addition, we will assume that the map u is locally approximable on M which means that for any ball $\bar{B} \subset M$ there exists $u_n \in C^\infty(M,N)$ such that $u_n \to u$ strongly in $W^{1,2}(B,N)$. Consequently, any such map in particular satisfies

$$d(u^*\omega_N) = 0$$

in the sense of distributions (see the paper [36] by F. Hang and F. Lin for details). Throughout this chapter we will assume our J-holomorphic maps to be locally approximable, although it is not clear whether such an assumption is really necessary, as it is conjectured that all weakly J-holomorphic maps are locally approximable.
Since for $m \geq 2$ the canonical projection $\pi : B_1 \subset \mathbf{C}^m \longrightarrow \mathbf{CP}^{m-1}$ is in fact a locally approximable J-holomorphic map which is in $W^{1,2}(B_1, \mathbf{CP}^{m-1})$, having a singularity at the origin, it is clear that we cannot expect J-holomorphic maps to be everywhere smooth. The aim is therefore to describe the singular set of these maps which is defined as

$$\operatorname{sing} u := M \setminus \{x \in M \,|\, u \in C^\infty(B,N) \text{ for some ball } B \subset M \text{ containing } x\} \ .$$

It is conjectured that the singular set of J-holomorphic maps satisfies

$$\mathcal{H}^{2m-4}(\operatorname{sing} u) < \infty \ .$$

As a first result in this direction, in [70] T. Rivière and G. Tian showed this to be true in case M is an almost complex manifold of real dimension 4 and the target is an algebraic variety in some $\mathbf{CP}^N$. Following their work and the paper [54] by F. Lin, C.-Y. Wang in [92] obtained, under additional assumptions, that $\mathcal{H}^{2m-2}(\operatorname{sing} u) = 0$. The aim of the first part of this chapter is to give a more direct proof of this last fact in the general setting above.
The theorems we want to prove for J-holomorphic maps are of a local nature, so we restrict ourselves to the case where M is a smooth bounded open subset Ω of $\mathbf{R}^{2m}$. However, it is important to note that we do not assume the almost complex structure to be the standard one. For a map $u \in W^{1,2}(\Omega, N)$ and a ball of radius $r > 0$ centred at $x_0 \in \Omega$, denoted by $B_r(x_0)$, we define the rescaled Dirichlet energy $E(u,r,x_0)$ of u on $B_r(x_0)$ to be

$$E(u,r,x_0) := \frac{1}{r^{2m-2}} \int_{B_r(x_0)} |\nabla u|^2 \, dx \ .$$

The first result we obtain is an ϵ-regularity theorem which we state as follows:

Theorem 2.1 *Let (N, J_N, ω_N) be a closed, tamed symplectic manifold, isometrically embedded in $\mathbf{R}^{2n}$, and let $\Omega \subset \mathbf{R}^{2m}$ be a smooth bounded open domain. Let $u \in W^{1,2}(\Omega, N)$ be a locally approximable J-holomorphic map. Then there exists $\epsilon > 0$ depending only on Ω and (N, J_N, ω_N) but independent of u, such that for each point $x_0 \in \Omega$ for which there exists some $r > 0$ with*

$$E(u,r,x_0) \leq \epsilon$$

we have

$$u \in C^\infty(B_{\frac{r}{4}}(x_0), N) \ .$$

The proof of Theorem 2.1 is similar to the proof of the analogous theorem for stationary harmonic map and one can to some extend view this result as a consequence of the second order properties of J-holomorphic maps. For a discussion of the regularity theory of (stationary) harmonic maps we refer the reader to the introduction of chapter 3. We first derive two second order elliptic equations for u. The first one is shown to imply that there exists a function $f \in \mathcal{H}^1(B_{\frac{3r_0}{4}}(x_0))$ (the Hardy space on $B_{\frac{3r_0}{4}}(x_0)$) such that $\tilde{\Delta}u = f$, where $\tilde{\Delta}$ is an invertible perturbation of the usual Laplace operator Δ — this is obtained without the use of moving frames (see chapter 3 for further references). The second equation is used to prove an almost monotonicity formula for the rescaled Dirichlet energy. Combining these two facts with the well-known duality of Hardy space and BMO (see for instance the book [81] by E. Stein for details) we deduce a Morrey decay rate for $E(u, r, x_0)$, from which the result follows by an iteration and bootstrapping argument.
As an immediate consequence of this Theorem and the almost monotonicity formula, we obtain the following

Corollary 2.2 *Let (N, J_N, ω_N) and $u \in W^{1,2}(\Omega, N)$ be as in Theorem 2.1. Then u is smooth outside a closed singular set $sing\, u$ with $\mathcal{H}^{2m-2}(sing\, u) = 0$.*

For our next result we need an extra assumption on the topology of N. Let $H_2(N)$ denote the second homology group of N. A homology class $B \in H_2(N)$ is called spherical if it is in the image of the Hurewicz homomorphism $H : \pi_2(N) \longrightarrow H_2(N)$. From now on we assume that N does not have any spherical homology classes which in particular implies that N does not support any pseudo-holomorphic sphere, i.e. a non-constant smooth J-holomorphic map $\phi : (S^2, j) \longrightarrow (N, J_N)$. We would like to point out that this assumption seems to be technical and in general unnecessary and refer the reader to section 2.8 for a possible bypass.
The proof of Theorem 2.1 above produces a (Morrey) decay rate for $E(u, r, x_0)$ provided $x_0 \notin \operatorname{sing} u$. From the (almost) monotonicity formula for $E(u, r, x_0)$ we deduce that at all points $x_0 \in \Omega$ the density of u at x_0 exists which is defined by

$$\Theta_u(x_0) := \lim_{r \to 0} E(u, r, x_0) \ .$$

Thus, the ϵ-regularity theorem immediately implies that $\operatorname{sing} u = \{x \in \Omega \,|\, \Theta_u(x) \neq 0\}$, so that the above decay rate shows how fast $E(u, r, x_0)$ decays to the density at x_0 — provided $x_0 \notin \operatorname{sing} u$. It would of course be interesting to understand what happens at points in the singular set, i.e. when $x_0 \in \operatorname{sing} u$. Under the additional topological assumption that there are no pseudo-holomorphic spheres in N, we were able to obtain such a rate of convergence even at singular points of u.

Theorem 2.3 *Let (N, J_N, ω_N) be a closed, tamed symplectic manifold, isometrically embedded in $\mathbf{R}^{2n}$ and assume that there are no pseudo-holomorphic spheres in N. Furthermore, let $\Omega \subset \mathbf{R}^{2m}$ be a smooth bounded open domain. Let $u \in W^{1,2}(\Omega, N)$ be a locally approximable J-holomorphic map. Then there exist $r_0 > 0$, $\gamma \in (0, 1)$ and $C > 0$ all independent of u such that for any $x_0 \in \Omega$ and any $0 < r \leq r_0$*

$$\frac{1}{r^{2m-2}} \int_{B_r(x_0)} |\nabla u|^2 \, dx - \Theta_u(x_0) \leq C r^\gamma \ .$$

The main idea of the proof is to use the projection $\pi : \mathbf{C}^m \setminus \{0\} \longrightarrow \mathbf{CP}^{m-1}$ already used in the previous chapter. The monotonicity formula together with the co-area formula shows that for almost every $p \in \mathbf{CP}^{m-1}$ the restriction of u to $\pi^{-1}(p)$ is a J-holomorphic map with finite energy. Since N does not support pseudo-holomorphic spheres, these maps satisfy a common decay rate which upon integration over all planes $p \in \mathbf{CP}^{m-1}$ yields the desired rate of convergence.

Naturally, this result should be compared with Theorem 1.2 in the previous chapter and in fact has similar consequences. For a locally approximable J-holomorphic map $u(x)$, for any point $x_0 \in \Omega$ and for any radius $0 < r \leq r_0$ we define the rescaled map $u_{r,x_0}(x) := u(x_0 + rx)$. The monotonicity formula for $E(u, r, x_o)$ then implies $\|u_{r,x_0}\|_{W^{1,2}} \leq E(u, 1, x_0) < \infty$, i.e. that the Sobolev norm is bounded independent of $r > 0$, so that for any sequence $r_i \longrightarrow 0$ there is a subsequence r_i' such that u_{r_i',x_0} converges weakly in $W^{1,2}(\Omega, N)$. Following an argument similar to the one in the paper [52] by J. Li and G. Tian for stationary harmonic maps, one can deduce existence of tangent maps. However, since there are no pseudo-holomorphic spheres in N we can apply Theorem D in [92] by C. Wang, which is an adaptation of Theorem D in [54] by F. Lin, to conclude that the above weak convergence is actually a strong one. Having obtained a tangent map it is then natural to investigate its dependence on the subsequence chosen, i.e. whether the tangent map is unique, and the next Theorem gives a positive answer to this question.

Theorem 2.4 *Let Ω, N and u be as in Theorem 2.3. Then at all points $x_0 \in \Omega$ the tangent map to u is unique. In particular,*

$$\lim_{r \to 0} u(x_0 + rx)$$

exists strongly in $W^{1,2}(\Omega, N)$.

In the context of energy minimising harmonic maps, the questions of rates of convergence and uniqueness of tangent maps were studied before . In the paper [77] L. Simon gave positive answers to both questions in case the tangent map has an isolated singularity at 0 and the target manifold is real-analytic. B. White actually showed that the latter assumption is crucial by producing a counterexample in [95] when the target manifold is merely smooth. Furthermore, rates of convergence were studied by D. Adams and L. Simon in [3] and by R. Gulliver and B. White in [33], showing that a rate of convergence does not always hold for stationary harmonic maps.

The chapter is organised as follows. In section 2.2 we derive a second order elliptic equation for J-holomorphic maps which is followed by a proof of an almost monotonicity formula in section 2.3. The necessary Morrey decay estimates are proved in section 2.4 and the proof of Theorem 2.1 is completed in section 2.5. Section 2.6 is devoted to the proof of Theorem 2.3 and the proof of Theorem 2.4 is given in section 2.7. In the final section 2.8 we explain a possible extension of Theorem 2.3 as a future research problem.

Acknowledgement: The results in sections 2.6 and 2.7 of this chapter are the outcome of joint work with my advisor Tristan Rivière [64].

2.2 J-holomorphic maps

In this section we will show that J-holomorphic maps satisfy a strictly elliptic second order equation. We will assume that the target manifold N is given a Hermitian metric (which is always possible) and then is isometrically embedded in $\mathbf{R}^{2n}$ for some $n \in \mathbf{N}$. Using this isometric embedding we can push-forward the almost complex structure on N to an almost complex structure J_N on $TN \subset T\mathbf{R}^{2n}$. For $y \in N \subset \mathbf{R}^{2n}$, $J_N(y)$ can be extended to a map $\tilde{J_N}(y)$ acting on all of $T_y\mathbf{R}^{2n}$ by setting $\tilde{J_N}(y)v := J_N(y)v_1$ where $v \in Ty\mathbf{R}^{2n}$ and $v_1 \in T_yN$ is chosen so that $v = v_1 + v_2$ with $v_2 \in T_yN^\perp$ (see the paper [92] by C.-Y. Wang for details of this construction). Note that this map $\tilde{J_N}$ is not an almost complex structure anymore (since $\tilde{J_N}^2 \neq -Id$), but the map u satisfies the equation

$$du_x(J_\Omega(x)v) = \tilde{J_N}(u(x))du_x(v)$$

for almost every $x \in \Omega$ and all $v \in \mathbf{R}^{2m}$.
Since our result is local we can in fact assume that the domain is a ball $B_{2r_0}(0) \subset \Omega$ for some $r_0 > 0$. Furthermore, through a change of coordinates we can arrange for $J_\Omega(0)$ to coincide with the standard complex structure J_0 on $\mathbf{R}^{2m}$ yet we do not assume that J_Ω to be J_0 on all of $B_{2r_0}(0)$.
We also need the following definition of Hardy space on Ω (see the book [81] by E. Stein as a reference):

Definition 2.1 *Given an open subset $\Omega \subset \mathbf{R}^{2m}$, define $\mathcal{H}^1(\Omega)$ to be the set of restrictions of functions in $\mathcal{H}^1(\mathbf{R}^{2m})$ (the usual Hardy space on $\mathbf{R}^{2m}$) to Ω. The norm for a function $f \in \mathcal{H}^1(\Omega)$ is defined as the infimum of the norm of all possible extensions $\tilde{f} \in \mathcal{H}^1(\mathbf{R}^{2m})$ of f.*

Using the above setting we will prove the following Lemma:

Lemma 2.5 *Let $u \in W^{1,2}(B_{2r_0}, N)$ be a J-holomorphic map. Then in the above setting we have that there exists $f \in \mathcal{H}^1(B_{\frac{3r_0}{2}}(0), \mathbf{R}^{2n})$ so that on $B_{\frac{3r_0}{2}}(0)$*

$$\tilde{\Delta} u = f \ ,$$

where $\tilde{\Delta}$ is a strictly elliptic, invertible second order operator on $B_{2r_0}(0)$.

Proof. At $x \in B_{2r_0}$ we choose an orthonormal frame $\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}, \frac{\partial}{\partial x_{m+1}}, \dots, \frac{\partial}{\partial x_{2m}}\right\}$ such that for $1 \le j \le m$ we have $\frac{\partial}{\partial x_{m+j}} = J_0 \frac{\partial}{\partial x_j}$. Since u is a J-holomorphic map it satisfies

$$du_x\big(J_\Omega(x)\tfrac{\partial}{\partial x_j}\big) = \tilde{J_N}(u(x))du_x\big(\tfrac{\partial}{\partial x_j}\big)$$

and hence by construction of the coordinates

$$du_x\big([J_\Omega(x) - J_0]\tfrac{\partial}{\partial x_j}\big) + du_x\big(\tfrac{\partial}{\partial x_{j+m}}\big) = \tilde{J_N}(u(x))du_x\big(\tfrac{\partial}{\partial x_j}\big) \ .$$

Writing this expression in the above coordinates yields

$$\sum_{k=1}^{2m} \frac{\partial u^\alpha}{\partial x_k} a_{jk}(x) + \frac{\partial u^\alpha}{\partial x_{j+m}} = \sum_{\beta=1}^{2n} b(u(x))_{\alpha\beta} \frac{\partial u^\beta}{\partial x_j} \ ,$$

where we set $a_{jk}(x) := ([J_\Omega(x) - J_0])_{jk}$ and $b_{\alpha\beta}(u(x)) := (\tilde{J_N}(u(x)))_{\alpha\beta}$. Similarly we get

$$\sum_{k=1}^{2m} \frac{\partial u^\alpha}{\partial x_k} a_{(j+m)k}(x) - \frac{\partial u^\alpha}{\partial x_j} = \sum_{\beta=1}^{2n} b(u(x))_{\alpha\beta} \frac{\partial u^\beta}{\partial x_{j+m}} \ .$$

Taking one more distributional derivative of these expressions with respect to $\frac{\partial}{\partial x_{j+m}}$ and $\frac{\partial}{\partial x_j}$ respectively we can show that

$$\begin{aligned} \frac{\partial}{\partial x_{j+m}} \left(\sum_{k=1}^{2m} \frac{\partial u^\alpha}{\partial x_k} a_{jk}(x) + \frac{\partial u^\alpha}{\partial x_{j+m}} \right) &= \sum_{\beta=1}^{2n} \left[\frac{\partial}{\partial x_{j+m}} \big(b(u(x))_{\alpha\beta} \big) \frac{\partial u^\beta}{\partial x_j} \right. \\ &\quad \left. + b(u(x))_{\alpha\beta} \frac{\partial}{\partial x_{j+m}} \frac{\partial u^\beta}{\partial x_j} \right] \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial x_j} \left(\sum_{k=1}^{2m} \frac{\partial u^\alpha}{\partial x_k} a_{(j+m)k}(x) - \frac{\partial u^\alpha}{\partial x_j} \right) &= \sum_{\beta=1}^{2n} \left[\frac{\partial}{\partial x_j} \big(b(u(x))_{\alpha\beta} \big) \frac{\partial u^\beta}{\partial x_{j+m}} \right. \\ &\quad \left. + b(u(x))_{\alpha\beta} \frac{\partial}{\partial x_j} \frac{\partial u^\beta}{\partial x_{j+m}} \right] . \end{aligned}$$

To justify this computation, we need to show that for the distributional derivative in the x_{j+m}-direction we have

$$\frac{\partial}{\partial x_{j+m}} \sum_{\beta=1}^{2n} b(u(x))_{\alpha\beta} \frac{\partial u^\beta}{\partial x_j} = \sum_{\beta=1}^{2n} \frac{\partial b(u(x))_{\alpha\beta}}{\partial x_{j+m}} \frac{\partial u^\beta}{\partial x_j} + b(u(x))_{\alpha\beta} \frac{\partial}{\partial x_{j+m}} \frac{\partial u^\beta}{\partial x_j}$$

(and an analogous identity for the x_j-direction). Note that this would be immediate if the $b(u(x))_{\alpha\beta}$ were smooth coefficients. For $b(u(x))$ in the present context, this can be seen as a map from the domain $B_{2r_0}(0)$ into the space of $2n \times 2n$-matrices which we identify with $\mathbf{R}^{4n^2}$. This map is the composition of the maps $u \in W^{1,2}(B_{2r_0}(0), N)$ and $\tilde{J_N}$ which is at least $C^2(N, \mathbf{R}^{4n^2})$ on the compact set N, and hence $b(u(x))$ lies in $W^{1,2}(B_{2r_0}(0), \mathbf{R}^{4n^2})$. Furthermore, since $\mathbf{R}^{4n^2}$ is a linear space, $b(u(x))$ can be strongly approximated in $W^{1,2}(B_{2r_0}(0), \mathbf{R}^{4n^2})$ by smooth maps $b_t(u(x)) \in C^\infty(B_{2r_0}(0), \mathbf{R}^{4n^2})$. As mentioned above for each $b_t(u(x))$ the identity is direct and since we have strong convergence we can pass to the limit on both sides to obtain it for $b(u(x))$ as well, which implies that the above equations hold true in the sense of distributions.
Since distributional derivatives commute, we can now subtract and sum over j, which then yields

$$\tilde{\Delta} u = \sum_{j=1}^{m} \sum_{\beta=1}^{2m} \left[\frac{\partial}{\partial x_{j+n}} \big(b(u(x))_{\alpha\beta} \big) \frac{\partial u^\beta}{\partial x_j} - \frac{\partial}{\partial x_j} \big(b(u(x))_{\alpha\beta} \big) \frac{\partial u^\beta}{\partial x_{j+m}} \right] , \tag{2.1}$$

where we define $\tilde{\Delta}u$ by

$$\begin{aligned}\tilde{\Delta}u &= \sum_{j=1}^{m}\left[\frac{\partial}{\partial x_{j+m}}\left(\sum_{k=1}^{2m}(a_{jk}(x)+\delta_{(j+m)k})\frac{\partial u^{\alpha}}{\partial x_k}\right)\right.\\ &\quad\left.+\frac{\partial}{\partial x_j}\left(\sum_{k=1}^{2m}(-a_{(j+m)k}(x)+\delta_{jk})\frac{\partial u^{\alpha}}{\partial x_k}\right)\right] .\end{aligned}$$

From this definition of $\tilde{\Delta}$ we see that it coincides with Δ at $x=0$, since $a_{jk}(0)=0$, and in fact that $\tilde{\Delta}$ is arbitrarily close to Δ on all of B_{2r_0} provided r_0 is chosen small enough. Assuming $r_0>0$ to be small, $\tilde{\Delta}$ is therefore strictly elliptic and invertible.
The fact that the right hand side of equation (2.1) is in Hardy space now follows from the work [20] by R. Coifman, P.-L. Lions, Y. Meyer and S. Semmes since it is made up of sums of the form $\{\frac{\partial f}{\partial x_k}\frac{\partial g}{\partial x_l}-\frac{\partial f}{\partial x_l}\frac{\partial g}{\partial x_k}\}$ for some $f,g\in W^{1,2}(B_{2r_0})$, i.e. has a Jacobian structure.□

2.3 Monotonicity formula

The aim of this section is to derive another second order equation for J-holomorphic maps and to deduce an almost monotonicity formula for $E(u,r,x_0)$ from it, provided that r is smaller than some fixed r_0 independent of u. We will continue to use the setting of the previous section and begin by showing that a locally approximable J-holomorphic map is almost a critical point of the Dirichlet energy $D(u):=\int_{B_{2r_0}(0)}|\nabla u|^2\,dx$ for perturbations in the domain $B_{2r_0}(0)$. Given an arbitrary smooth 1-parameter family of diffeomorphisms of $B_{2r_0}(0)$, denoted by Ψ_t for $t\in(-\epsilon,\epsilon)$, we want to compute $\frac{d}{dt}\big|_{t=0}D(u\circ\Psi_t)$. First note that one can easily verify the following alternative expression for the energy:

$$D(u)=\frac{1}{2}\int_B|\nabla u+J_N\circ\nabla u\circ J_0|^2\,dx+\int_B\langle J_N\circ\nabla u,\nabla u\circ J_0\rangle\,dx\ .$$

To simplify the notation we write $u_t:=u\circ\Psi_t$ for the perturbation of u. Note that for any map u_t in the second term above we can use $\langle J_N\circ\nabla u_t,\nabla u_t\circ J_0\rangle\cong\langle\omega_0,u_t^*\omega_N\rangle$, where ω_N is the symplectic structure on the target. For smooth perturbations of the domain, we claim that the second term in the above expression satisfies

$$\frac{d}{dt}\bigg|_{t=0}\int_{B_{2r_0}(0)}\langle\omega_0,u_t^*\omega_N\rangle\,dx=0\ .$$

To see this note that on $B_{2r_0}(0)$ there exists a smooth 1-form ϕ such that $d\phi=\omega_0$ and that $u_t^*\omega_N=\Psi_t^*u^*\omega_N$. As u is locally approximable, i.e. in particular $d(u^*\omega_N)=u^*d\omega_N=0$, we deduce that $d(u_t^*\omega_N)=\Psi_t^*d(u^*\omega_N)=0$. Using Stoke's theorem and the fact that u_t

has compact support in $B_{2r_0}(0)$ we get the claim from

$$\begin{aligned}\int_{B_{2r_0}(0)} \langle \omega_0, u_t^*\omega_N\rangle\, dx &= \int_{B_{2r_0}(0)} \langle d\phi, u_t^*\omega_N\rangle\, dx\\ &= C\int_{B_{2r_0}(0)} d(\phi\wedge\omega_0^{2m-2})\wedge u_t^*\omega_N\, dx\\ &= C\int_{B_{2r_0}(0)} d(\phi\wedge\omega_0^{2m-2}\wedge u_t^*\omega_N)\, dx\\ &= 0\ .\end{aligned}$$

Thus it remains to compute the derivative for the first term. We will show that since J_Ω is close to J_0 in $B_{2r_0}(0)$, u is close to being J_0-holomorphic. Note that

$$\nabla u + J_N\circ\nabla u\circ J_0 = \nabla u + J_N\circ\nabla u\circ J_\Omega + J_N\circ\nabla u\circ(J_\Omega - J_0)$$

so that for u_t, using the fact that J_N is compatible with h, the metric on N, this means that

$$\begin{aligned}\int_{B_{2r_0}(0)} |\nabla u_t + J_N\circ\nabla u_t\circ J_0|^2\, dx &= \int_{B_{2r_0}(0)} |\nabla u_t + J_N\circ\nabla u_t\circ J_\Omega|^2\, dx &(2.2)\\ &+\ 2\int_{B_{2r_0}(0)} \langle\nabla u_t, J_N\circ\nabla u_t\circ(J_\Omega - J_0)\rangle\, dx &(2.3)\\ &+\ \int_{B_{2r_0}(0)} \langle\nabla u_t\circ(J_0+J), \nabla u_t\circ(J_\Omega - J_0)\rangle\, dx &(2.4)\end{aligned}$$

For the first term on the right-hand side (2.2) note that since u is J-holomorphic we have $|\nabla u + J_N\circ\nabla u\circ J_0|^2\, dx = 0$ pointwise a.e. in $B_{2r_0}(0)$, i.e. that $\int_{B_{2r_0}(0)} |\nabla u + J_N\circ\nabla u\circ J_0|^2 = 0$. Furthermore,

$$\int_{B_{2r_0}(0)} |\nabla u_t + J_N\circ\nabla u_t\circ J_\Omega|^2\, dx \geq \int_{B_{2r_0}(0)} |\nabla u + J_N\circ\nabla u\circ J_\Omega|^2\, dx = 0\ ,$$

which implies that

$$\frac{d}{dt}\bigg|_{t=0}\int_{B_{2r_0}(0)} |\nabla u_t + J_N\circ\nabla u_t\circ J_\Omega|^2\, dx = 0\ .$$

For the remaining terms (2.3) and (2.4) we work in local coordinates. Since any 1-parameter group Ψ_t is generated by a vector field ξ having compact support in $B_{2r_0}(0)$, we can write $u_t(x) = u(x+t\xi)$. Writing $(J_\Omega - J_0)(x) = [a_{kl}](x)$ and $J_N(u(x)) = [b_{\alpha\beta}](u(x))$ as before, we get the following expression for the derivative of the term in (2.3):

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0}\int_{B_{2r_0}(0)} \langle\nabla u_t, J_N\circ\nabla u_t\circ(J_\Omega - J_0)\rangle\, dx &= \int_{B_{2r_0}(0)} (-\operatorname{div}\xi)\frac{\partial u^\alpha}{\partial x_k}\frac{\partial u^\beta}{\partial x_l}[a_{kl}][b_{\beta\alpha}]\, dx\\ &\quad +2\int_{B_{2r_0}(0)} \frac{\partial u^\alpha}{\partial x_k}\frac{\partial u^\beta}{\partial x_{l'}}\frac{\partial \xi^{l'}}{\partial x_l}[a_{kl}][b_{\beta\alpha}]\, dx\ ,\end{aligned}$$

where we sum over repeated indices. For the last term (2.4) we write $(J_\Omega + J_0)(x) = ([a_{kl}](x) + 2[\tilde{\delta}_{kl}])$, where $[\tilde{\delta}_{kl}]$ denotes the matrix for J_0 in the standard coordinates on $B_{2r_0}(0)$. A direct computation shows that

$$
\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} \int_{B_{2r_0}(0)} \langle \nabla u_t \circ (J_0 + J_\Omega), \nabla u_t \circ (J_\Omega - J_0)\rangle \, dx \\
= \int_{B_{2r_0}(0)} (-\operatorname{div}\xi) \frac{\partial u^\alpha}{\partial x_{k'}} \frac{\partial u^\alpha}{\partial x_{l'}} [a_{k'l}]([a_{l'k}] + 2[\tilde{\delta}_{l'k}]) \, dx \\
2\int_{B_{2r_0}(0)} \frac{\partial u^\alpha}{\partial x_{k'}} \frac{\partial u^\alpha}{\partial x_{l''}} \frac{\partial \xi^{l''}}{\partial x_{l'}} [a_{k'l}]([a_{l'k}] + 2[\tilde{\delta}_{l'k}]) \, dx \ .
\end{aligned}
$$

Combining the above steps we arrive at the following equation for u, which is valid for all $\xi \in C_c^\infty(B_{2r_0}(0), \mathbf{R}^{2m})$:

$$
\begin{aligned}
\int_{B_{2r_0}(0)} \sum_{i,j} \left(|\nabla u|^2 \delta_{ij} - 2\left\langle \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j} \right\rangle \right) \frac{\partial \xi^j}{\partial x_i} \, dx \ = \\
= \ -\frac{1}{2}\int_{B_{2r_0}(0)} \frac{\partial u^\alpha}{\partial x_k} \frac{\partial u^\beta}{\partial x_l} [a_{kl}][b_{\beta\alpha}] \delta_{ij} \frac{\partial \xi^j}{\partial x_i} \, dx \\
+ \int_{B_{2r_0}(0)} \frac{\partial u^\alpha}{\partial x_k} \frac{\partial u^\beta}{\partial x_j} [a_{ki}][b_{\beta\alpha}] \frac{\partial \xi^j}{\partial x_i} \, dx \\
-\frac{1}{2}\int_{B_{2r_0}(0)} \frac{\partial u^\alpha}{\partial x_{k'}} \frac{\partial u^\alpha}{\partial x_{l'}} [a_{k'l}]([a_{l'k}] + 2[\tilde{\delta}_{l'k}]) \delta_{ij} \frac{\partial \xi^j}{\partial x_i} \, dx \\
+ \int_{B_{2r_0}(0)} \frac{\partial u^\alpha}{\partial x_{k'}} \frac{\partial u^\alpha}{\partial x_j} [a_{k'l}]([a_{ik}] + 2[\tilde{\delta}_{ik}]) \frac{\partial \xi^j}{\partial x_i} \, dx \ . \quad (2.5)
\end{aligned}
$$

Note that this equation is very similar to the equation one obtains for stationary harmonic maps where the right-hand side vanishes identically, whereas in our case the right-hand side is $O(r)\|\nabla u\|_{L^2}^2$. It is clear that for small enough radii (depending only on the C^2-norm of J_Ω) the second order operator involved is again only a small perturbation of the Laplacian, and hence elliptic.

We now use this equation (2.5) by testing it with a special test function ξ to prove an almost monotonicity formula for $E(u, r, x_0)$. Precisely, we have the following Proposition, where from now on we use the notation $R = |x - x_0|$ for the distance between x and x_0, and $\frac{\partial u}{\partial R}$ for the derivative of u in the direction of R.

Proposition 2.6 *Let $u : B_{2r_0}(0) \longrightarrow N$ be a locally approximable J-holomorphic map in $W^{1,2}(B_{2r_0}(0), N)$. Provided $r_0 > 0$ was chosen sufficiently small, there exists a constant $C > 0$ independent of u such that for all $x_0 \in B_{2r_0}(0)$ and all $0 < s < r \leq r_0$ with $\overline{B}_r(x_0) \subset B_{2r_0}(0)$*

$$
\frac{e^{Cr} - r}{r^{2m-2}} \int_{B_r(x)} |\nabla u|^2 \, dx - \frac{e^{Cs} - s}{s^{2m-2}} \int_{B_s(x)} |\nabla u|^2 \, dx \geq 2 \int_{B_r \setminus B_s(x_0)} R^{2-2m} \left| \frac{\partial u}{\partial R} \right|^2 \, dx \ .
$$

Proof. The proof will follow the proof for stationary harmonic maps given in the book [79] by L. Simon. Given $x_0 \in B_{2r_0}(0)$ define ρ_0 to be a radius such that $\overline{B}_{\rho_0}(x_0) \subset B_{2r_0}(0)$. We use the fact that if $\int a_j \frac{\partial \xi}{\partial x_j} = 0$ for all $\xi \in C_c^\infty(B_{\rho_0}(x_0))$, then for a.e. $\rho \in (0, \rho_0)$ we have $\int_{B_\rho(x_0)} a_j \frac{\partial \xi}{\partial x_j} = \int_{\partial B_\rho(x_0)} \eta \cdot a\xi$. Using this, we test the above equation (2.5) for u with ξ defined by $\xi^j(x) = x^j - x_0^j$ (so that $\frac{\partial \xi^j}{\partial x_i} = \delta_{ij}$) which yields the following identity for u:

$$
\begin{aligned}
&(2-2m)\rho^{1-2m} \int_{B_\rho(x_0)} |\nabla u|^2 \, dx + \rho^{2-2m} \int_{\partial B_\rho} |\nabla u|^2 \, dx \\
&= \rho^{1-2m} \int_{B_\rho(x_0)} A(x) |\nabla u|^2 \, dx - \rho^{2-2m} \int_{\partial B_\rho(x_0)} B(x) |\nabla u|^2 \, dx + 2 \int_{\partial B_\rho(x_0)} R^{2-2m} \left| \frac{\partial u}{\partial R} \right|^2 dx \ .
\end{aligned}
$$

Here $\int_{B_\rho(x_0)} A(x)|\nabla u|^2 \, dx$ and $\int_{\partial B_\rho(x_0)} B(x)|\nabla u|^2 \, dx$ denote terms which can be bounded by $C\rho \int_{B_\rho(x_0)} |\nabla u|^2 \, dx$ and $C\rho \int_{\partial B_\rho(x_0)} |\nabla u|^2 \, dx$ respectively. This leads to the inequalities

$$
\begin{aligned}
&\left| \frac{d}{d\rho} \left(\frac{1}{\rho^{2m-2}} \int_{B_\rho(x_0)} |\nabla u|^2 \, dx - 2 \int_{B_\rho} R^{2-2m} \left| \frac{\partial u}{\partial R} \right|^2 dx \right) \right| \\
&\qquad \leq C \frac{1}{\rho^{2m-2}} \int_{B_\rho(x_0)} |\nabla u|^2 \, dx + C \frac{1}{\rho^{2m-3}} \int_{\partial B_\rho(x_0)} |\nabla u|^2 \, dx \\
&\qquad \leq C \frac{1}{\rho^{2m-2}} \int_{B_\rho(x_0)} |\nabla u|^2 \, dx + C \frac{d}{d\rho} \left(\rho^{2m-3} \int_{B_\rho(x_0)} |\nabla u|^2 \, dx \right) . \qquad (2.6)
\end{aligned}
$$

Using this estimate we conclude that

$$
\begin{aligned}
\frac{d}{d\rho} \left(\frac{e^{C\rho}}{\rho^{2m-2}} \int_{B_\rho(x_0)} |\nabla u|^2 \, dx \right) &= e^{C\rho} \frac{d}{d\rho} \left(\frac{1}{\rho^{2m-2}} \int_{B_\rho(x_0)} |\nabla u|^2 \, dx \right) + \frac{Ce^{C\rho}}{\rho^{2m-2}} \int_{B_\rho(x_0)} |\nabla u|^2 \, dx \\
&\geq 2 \frac{d}{d\rho} \left(\int_{B_\rho(x_0)} R^{2-2m} \left| \frac{\partial u}{\partial R} \right|^2 dx \right) + \frac{d}{d\rho} \left(\frac{1}{\rho^{2m-3}} \int_{B_\rho(x_0)} |\nabla u|^2 \, dx \right) ,
\end{aligned}
$$

and integration from s to r yields the desired result. □

Remark 2.1 *From equation (2.6) in the proof of the monotonicity formula we also get the estimate*

$$
\begin{aligned}
\left| \frac{d}{d\rho} \left(\frac{1}{\rho^{2m-2}} \int_{B_\rho(x_0)} |\nabla u|^2 \, dx \right) \right| &\leq C \frac{1}{\rho^{2m-2}} \int_{B_\rho(x_0)} |\nabla u|^2 \, dx + C \frac{1}{\rho^{2m-3}} \int_{\partial B_\rho} |\nabla u|^2 \, dx \\
&\qquad + 2 \int_{\partial B_\rho(x_0)} R^{2-2m} \left| \frac{\partial u}{\partial R} \right|^2 dx \\
&\leq C \frac{1}{\rho^{2m-2}} \int_{B_\rho(x_0)} |\nabla u|^2 \, dx + C\rho \left| \frac{d}{d\rho} \left(\frac{1}{\rho^{2m-2}} \int_{B_\rho(x_0)} |\nabla u|^2 \, dx \right) \right| \\
&\qquad + 2 \int_{\partial B_\rho(x_0)} R^{2-2m} \left| \frac{\partial u}{\partial R} \right|^2 dx \ .
\end{aligned}
$$

Therefore for ρ so small that $\frac{1}{2} \leq 1 - C\rho$ we obtain

$$
\frac{d}{d\rho} \left(\frac{1}{\rho^{2m-2}} \int_{B_\rho(x_0)} |\nabla u|^2 \, dx \right) \leq C \frac{1}{\rho^{2m-2}} \int_{B_\rho(x_0)} |\nabla u|^2 \, dx + C \int_{\partial B_\rho(x_0)} R^{2-2m} \left| \frac{\partial u}{\partial R} \right|^2 dx \ ,
$$

i.e. that

$$\frac{d}{d\rho}\left(\frac{e^{-C\rho}}{\rho^{2m-2}}\int_{B_\rho(x_0)}|\nabla u|^2\,dx\right)\leq C\int_{\partial B_\rho(x_0)}R^{2-2m}\left|\frac{\partial u}{\partial R}\right|^2 dx\ .$$

Integrating this inequality from s to r yields the useful estimate

$$\frac{e^{-Cr}}{r^{2m-2}}\int_{B_r(x_0)}|\nabla u|^2\,dx-\frac{e^{-Cs}}{s^{2m-2}}\int_{B_s(x_0)}|\nabla u|^2\,dx\leq C\int_{B_r\setminus B_s(x_0)}R^{2-2m}\left|\frac{\partial u}{\partial R}\right|^2 dx\ . \tag{2.7}$$

2.4 Morrey decay estimates

We will deduce Theorem 2.1 from the following Morrey decay estimate (compare this with the proof of Theorem 3.1 in chapter 3):

Proposition 2.7 *Let $u\in W^{1,2}(B_{2r_0}(0),N)$ be a locally approximable J-holomorphic map as above. There are constants $\epsilon>0$ and $\tau\in(0,1)$ such that for all points $x_0\in B_{r_0}(0)$ for which there exists some $r>0$ with $\overline{B}_r(x)\subset B_{r_0}(0)$,*

$$E(u,r,x_0)\leq\epsilon$$

implies

$$E(u,\tau r,x_0)\leq\tfrac{3}{4}E(u,r,x_0)\ . \tag{2.8}$$

Proof. Since each component of u lies in $W^{1,2}(B_{2r_0}(0))$ and satisfies

$$\tilde{\Delta}u=f$$

for some $f\in\mathcal{H}^1(B_{\frac{3r_0}{2}}(0))$, elliptic regularity theory shows that $u\in W^{2,1}(B_{r_0}(0))$.
Then for any $r>0$ with $\overline{B}_r(x)\subset B_{r_0}(0)$ the Dirichlet problem

$$\begin{cases}\tilde{\Delta}v=0 & \text{in } B_r(x_0)\\ v=u & \text{on } \partial B_r(x_0)\end{cases}$$

has a unique solution $v\in W^{2,1}(B_r(x_0))$, which we will call the almost harmonic extension of u on $B_r(x_0)$. Like in the case of harmonic extensions, for such a solution we have a decay estimate from the book [28] by M. Giaquinta for all $0<\rho\leq r$

$$\int_{B_\rho(x)}|\nabla v|^2\,dx\leq C\left(\frac{\rho}{r}\right)^n\int_{B_r(x)}|\nabla v|^2\,dx+C\omega^2(r)\int_{B_r(x)}|\nabla v|^2\ ,dx\ ,$$

where $\omega(r)$ is the modulus of continuity of $\tilde{\Delta}$ on $B_r(x_0)$. From the previous sections we know that by choosing r sufficiently small we can make $C\omega^2(r)$ as small as we wish.

First we deduce that for $\rho \leq \frac{r}{2}$

$$\begin{aligned}
\int_{B_\rho(x_0)} |\nabla u|^2\,dx &\leq C\int_{B_\rho(x_0)} |\nabla v|^2\,dx + C\int_{B_\rho(x_0)} |\nabla(u-v)|^2\,dx \\
&\leq C\Big(\frac{2\rho}{r}\Big)^n \int_{B_{\frac{r}{2}}(x_0)} |\nabla v|^2\,dx + C\omega^2\left(\frac{r}{2}\right)\int_{B_{\frac{r}{2}}(x_0)} |\nabla v|^2\,dx \\
&\quad + C\int_{B_{\frac{r}{2}}(x_0)} |\nabla(u-v)|^2\,dx \\
&\leq C\Big(\Big(\frac{\rho}{r}\Big)^n + \omega^2(r)\Big)\int_{B_r(x_0)} |\nabla u|^2\,dx \\
&\quad + C\Big(1+\Big(\frac{\rho}{r}\Big)^n + \omega^2(r)\Big)\int_{B_{\frac{r}{2}}(x_0)} |\nabla(u-v)|^2\,dx
\end{aligned}$$

for different constants C.
Next we apply the Hardy-BMO duality (see the book [81] by E. Stein for details) together with the Calderon-Zygmund inequality in the special case where $\tilde{\Delta} u \in \mathcal{H}^1(B_r(x_0))$ to obtain

$$\begin{aligned}
\int_{B_\rho(x_0)} |\nabla u|^2\,dx &\leq C\Big(\Big(\frac{\rho}{r}\Big)^n + \omega^2(r)\Big)\int_{B_r(x_0)} |\nabla u|^2\,dx \\
&\quad + C\Big(1+\Big(\frac{\rho}{r}\Big)^n + \omega^2(r)\Big)[u-v]_{BMO(B_{\frac{r}{2}}(x_0))}\|\tilde{\Delta}(u-v)\|_{\mathcal{H}^1(B_{\frac{r}{2}}(x_0))} \\
&\leq C\Big(\Big(\frac{\rho}{r}\Big)^n + \omega^2(r)\Big)\int_{B_r(x_0)} |\nabla u|^2\,dx \\
&\quad + C\Big(1+\Big(\frac{\rho}{r}\Big)^n + \omega^2(r)\Big)[u-v]_{BMO(B_{\frac{r}{2}}(x_0))}\int_{B_r(x_0)} |\nabla u|^2\,dx\ ,
\end{aligned}$$

since v is an almost harmonic extension. We will show that $[u-v]_{BMO(B_{\frac{r}{2}}(x_0))} \leq \epsilon$ in a separate lemma below. However, once this is established the above inequalities show that

$$\begin{aligned}
\frac{1}{\rho^{n-2}}\int_{B_\rho(x_0)} |\nabla u|^2\,dx &\leq C\Big(\frac{\rho}{r}\Big)^2 \frac{1}{r^{n-2}}\int_{B_r(x_0)} |\nabla u|^2\,dx \\
&\quad + C\omega^2(r)\Big(\frac{r}{\rho}\Big)^{n-2}\frac{1}{r^{n-2}}\int_{B_r(x_0)} |\nabla u|^2\,dx \\
&\quad + C\epsilon(1+\omega^2(r))\Big(\frac{r}{\rho}\Big)^{n-2}\frac{1}{r^{n-2}}\int_{B_r(x_0)} |\nabla u|^2\,dx \\
&\quad + C\epsilon\Big(\frac{\rho}{r}\Big)^2\frac{1}{r^{n-2}}\int_{B_r(x_0)} |\nabla u|^2\,dx\ . \qquad (2.9)
\end{aligned}$$

It remains to show how the constants $\tau > 0$ and $\epsilon > 0$ have to be chosen in order to finish the proof of the proposition. First we want $\tau := \frac{\rho}{r}$ to be so that $\sqrt{C}\frac{\rho}{r} \leq \frac{1}{4}$, i.e. set τ equal to $1/4\sqrt{C}$. Next assume that r_0 was chosen so small that $\omega^2(r) \leq \frac{\tau^{n-2}}{4C}$ for all $0 < r \leq r_0$. We finish the proof, apart from the remaining estimate on $[u-v]_{BMO(B_{\frac{r}{2}}(x_0))}$, by choosing $\epsilon > 0$ so small that the following inequalities are both satisfied: $\epsilon \leq \frac{\tau^{n-2}}{8C(1+\omega^2(r))}$

and $\epsilon \le \frac{1}{8C\tau^2}$. To complete the proof insert τ and ϵ in estimate (2.9) to deduce the desired estimate (2.8). □

Before we can prove the remaining estimate on the BMO-norm we need the Lemma below (compare this with Lemma III.6 in [55] by Y. Meyer and T. Rivière):

Lemma 2.8 *Let $s_0 > 0$ and let $u \in W^{2,1}(B_{2s_0}(x_0))$ with $\tilde{\Delta} u \in \mathcal{H}^1(B_{2s_0}(x_0))$. There exists a constant $C > 0$ independent of u such that*

$$\begin{aligned}\sup_{B_s(x)\subset B_{s_0}(x_0)} \frac{1}{s^{n-1}} \int_{B_s(x)} |\nabla u|\, dx \;&\le\; C \sup_{B_s(x)\subset B_{2s_0}(x_0)} \frac{1}{s^{n-2}} \|\tilde{\Delta} u\|_{\mathcal{H}^1(B_s(x))} \\ &\quad + C\frac{1}{(2s_0)^{n-1}} \int_{B_{2s_0}(x_0)} |\nabla u|\, dx\ .\end{aligned}$$

Proof. Let $B_s(x) \subset B_{2s_0}(x_0)$ and let $\rho \le s$. We consider the almost harmonic extension $v \in W^{2,1}$ solving

$$\begin{cases} \tilde{\Delta} v = 0 & \text{in } B_s(x) \\ v = u & \text{on } \partial B_s(x)\ , \end{cases}$$

which as before implies that for all $0 < \rho \le s$

$$\int_{B_\rho(x)} |\nabla v|\, dx \le C\Big(\frac{\rho}{s}\Big)^n \int_{B_s(x)} |\nabla v|\, dx + C\omega(s) \int_{B_s(x)} |\nabla v|\, dx\ .$$

Setting $w = u - v$ we have

$$\begin{cases} \tilde{\Delta} w = \tilde{\Delta} u & \text{in } B_s(x) \\ w = 0 & \text{on } \partial B_s(x)\ . \end{cases}$$

Then we use Poincaré's inequality and the Calderon-Zygmund inequality to deduce

$$\begin{aligned}\int_{B_s(x)} |\nabla w|\, dx \;&\le\; Cs^p \int_{B_s(x)} |\nabla^2 w|\, dx \\ &\le\; Cs \int_{B_s(x)} |\tilde{\Delta} w|\, dx \\ &=\; Cs \int_{B_s(x)} |\tilde{\Delta} u|\, dx\ .\end{aligned}$$

Combining the above estimates we derive for u

$$\begin{aligned}\int_{B_\rho(x)} |\nabla u|\, dx \;&\le\; C\Big(\Big(\frac{\rho}{s}\Big)^n + \omega(s)\Big) \int_{B_s(x)} |\nabla v|\, dx + C\int_{B_\rho(x)} |\nabla w|\, dx \\ &\le\; C\Big(\Big(\frac{\rho}{s}\Big)^n + \omega(s)\Big) \int_{B_s(x)} |\nabla u|\, dx + C\int_{B_s(x)} |\nabla w|\, dx \\ &\le\; C\Big(\Big(\frac{\rho}{s}\Big)^n + \omega(s)\Big) \int_{B_s(x)} |\nabla u|\, dx + Cs\int_{B_s(x)} |\tilde{\Delta} u|\, dx\ .\end{aligned}$$

To simplify the notation we set $T(\rho) := \rho^{1-n} \int_{B_\rho(x)} |\nabla u|\, dx$ and the above inequality becomes

$$T(\rho) \le C\Big(\frac{\rho}{s}\Big) T(s) + C\omega(s)\Big(\frac{s}{\rho}\Big)^{n-1} \frac{1}{s^{n-1}} \int_{B_s(x)} |\nabla u|\, dx + C\Big(\frac{s}{\rho}\Big)^{n-1} \frac{1}{s^{n-2}} \int_{B_s(x)} |\tilde{\Delta} u|\, dx\ .$$

Choosing $\tau := \frac{\rho}{s}$ so that $C\tau \leq \frac{1}{2}$ we have

$$T(\tau s) \leq \frac{1}{2}T(s) + C\frac{\omega(s)}{s^{n-1}} \int_{B_s(x)} |\nabla u|\, dx + C\frac{1}{s^{n-2}} \int_{B_s(x)} |\Delta u|\, dx\ .$$

To obtain the result we set $i = \left\lfloor \frac{\log s}{\log \tau} \right\rfloor$, iterate the above identity between $s = 1$ and $s = \tau^i$ and note that provided $s_0 > 0$ was chosen sufficiently small the term involving $\omega(s)$ can be absorbed in the left-hand side. □

Now we are in a position to prove the desired BMO-estimate to complete the proof of Proposition 2.7.

Lemma 2.9 *Let u be a locally approximable J-holomorphic map satisfying the assumptions of proposition 2.7, and let v be its almost harmonic extension on $B_r(x_0)$. Then we have*

$$[u - v]_{BMO(B_{\frac{r}{2}}(x_0))} \leq C\epsilon\ .$$

Proof. Since v is the almost harmonic extension of u on $B_r(x_0)$, setting $w = u - v$ we have

$$\begin{cases} \tilde{\Delta} w = \tilde{\Delta} u & \text{in } B_r(x_0) \\ w = 0 & \text{on } \partial B_r(x_0)\ , \end{cases}$$

and that $w \in W^{2,1}(B_r(x_0))$ so that we can apply Lemma 2.8 to w with $2s = r$. This, together with Poincaré's inequality, gives

$$\begin{aligned} [w]_{BMO(B_{\frac{r}{2}}(x_0))} &\leq \sup_{B_\rho(x) \subset B_{\frac{r}{2}}(x_0)} \frac{1}{\rho^{n-1}} \int_{B_\rho(x)} |\nabla w|\, dx \\ &\leq C \sup_{B_\rho(x) \subset B_r(x_0)} \frac{1}{\rho^{n-2}} \int_{B_\rho(x)} |\tilde{\Delta} w|\, dx \\ &\quad + C\frac{1}{r^{n-1}} \int_{B_r(x_0)} |\nabla w|\, dx \\ &\leq C \sup_{B_\rho(x) \subset B_{\frac{r}{2}}(x_0)} \frac{1}{\rho^{n-2}} \int_{B_\rho(x)} |\nabla u|^2\, dx \\ &\quad + C\frac{1}{(2r)^{n-1}} \int_{B_{2r}(x_0)} |\nabla w|\, dx\ , \end{aligned}$$

because of the equations for u and w.

The last term on the right hand side is estimated using Poincaré's inequality and the Calderon-Zygmund inequality which yields

$$\begin{aligned} C\frac{1}{r^{n-1}} \int_{B_r(x_0)} |\nabla w|\, dx &\leq C\frac{1}{r^{n-2}} \int_{B_r(x_0)} |\tilde{\Delta} w|\, dx \\ &\leq C\frac{1}{r^{n-2}} \int_{B_r(x_0)} |\nabla u|^2\, dx \\ &\leq C\epsilon\ . \end{aligned}$$

To finish the proof note that, provided $r_0 > 0$ is chosen sufficiently small, for all $0 < r \leq r_0$ and for all $x \in B_{\frac{r}{2}}(x_0) \subset B_{r_0}(x_0)$ we have $B_r(x) \subset B_{r_0}(x) \subset B_{2r_0}(x_0)$. From this and the monotonicity formula for u we deduce that for all $0 < \rho \leq r \leq r_0$

$$E(u, \rho, x) \leq E(u, r, x) \leq E(u, r_0, x) \leq CE(u, 2r_0, x_0) \ .$$

Since we can assume ϵ to be less than 1, this implies

$$[w]_{BMO(B_{\frac{r}{2}}(x_0))} \leq C\epsilon$$

and the desired estimate for $u - v$ is proved. □

2.5 ϵ-regularity

We will now show how Theorem 2.1 can be deduced from Proposition 2.7 and also give a proof of Corollary 2.2. To apply Proposition 2.7 to Theorem 2.1 first assume that $x_0 \in B_{r_0}(x_0)$ such that for some $0 < r \leq r_0$

$$E(u, r, x_0) \leq \frac{\epsilon}{C2^{2m-2}} \ ,$$

where $C > 0$ is the constant coming from the almost monotonicity formula, and in particular, the Proposition can be applied at x_0. Given any point $y \in B_{\frac{r}{2}}(x_0)$ we have $B_{\frac{r}{2}}(y) \subset B_r(x_0)$ and thus $E\left(u, \frac{r}{2}, y\right) \leq 2^{2m-2}E(u, r, x_0)$. The monotonicity formula then implies that for all $0 < \rho \leq \frac{r}{2}$

$$E(u, \rho, y) \leq CE\left(u, \frac{r}{2}, y\right) \leq C2^{2m-2}E(u, r, x) \leq \epsilon$$

by our definition of r and the Proposition can therefore also be applied at any $y \in B_{\frac{r}{2}}(x_0)$. Fixing $0 < \rho \leq \frac{r}{2}$ there exists $i \in \mathbf{N}$ such that $\tau^{i+1}\frac{r}{2} \leq \rho \leq \tau^i\frac{r}{2}$ and set $\gamma := \frac{\log 3 - \log 4}{2\log\tau} > 0$ so that $\left(\frac{3}{4}\right)^i \leq (\tau^i)^\gamma$. Therefore, iterating estimate (2.8) in Proposition 2.7 we obtain

$$\begin{aligned} E(u, \rho, y) &\leq \tau^{2-2m}E\left(u, \tau^i\frac{r}{2}, y\right) \\ &\leq \tau^{2-2m}\left(\frac{3}{4}\right)^i E\left(u, \frac{r}{2}, y\right) \\ &\leq \tau^{2-2m}\left(\tau^i\right)^\gamma C2^{2m-2}E(u, r, x_0) \\ &\leq C\rho^\gamma \ . \end{aligned}$$

Since this holds true for all $B_\rho(y) \subset B_r(x_0)$ we can use Morrey's Dirichlet growth theorem (see for instance Theorem 3.5.2 in the book [28] by M. Giaquinta for a reference) to conclude that u is in the Hölder space $u \in C^{0,\frac{\gamma}{2}}(B_{\frac{r}{2}}(x_0))$. From an elliptic bootstrapping argument we deduce that u is in fact smooth in $B_{\frac{r}{4}}(x_0)$ proving Theorem 2.1.

The proof of Corollary 2.2 now follows from Corollary 3.2.3 in the book [96] by W. Ziemer by defining

$$\operatorname{sing} u := \{x \in B_{2r_0}(0) \mid \lim_{r\to 0} E(u, r, x) > 0\}$$

so that $\mathcal{H}^{2m-2}(\operatorname{sing} u) = 0$ as ∇u is in $L^2(B_{2r_0}(0))$. Since $\operatorname{sing} u$ is the complement of the set of points $x_0 \in B_{2r_0}(0)$ for which Proposition 2.7 applies for some radius $0 < r \leq r_0$ and by the above argument the latter set is open, we deduce that $\operatorname{sing} u$ is a closed subset of $B_{2r_0}(0)$ proving the Corollary.

2.6 Rate of convergence

In this section we prove Theorem 2.3. The argument will be based on slicing by pre-images of the map $\pi : \mathbf{R}^{2m} \setminus \{0\} \longrightarrow \mathbf{CP}^{m-1}$ constructed in appendix D which we already used for the proof of Theorem 1.2. Recall that the argument is local so that we can assume to work in the case where the domain is $B_{2r_0}(0) \subset \mathbf{R}^{2m}$ equipped with an arbitrary complex structure J. Fix $x_0 \in B_{2r_0}(0)$ and let $r > 0$ be sufficiently small so that $B_r(x_0) \subset B_{2r_0}(0)$ and the monotonicity formula given by Proposition 2.6 applies. Taking the limit as $s \longrightarrow 0$ we deduce that for all such radii $r > 0$ we have

$$\int_{B_r(x_0)} R^{2-2m} \left|\frac{\partial u}{\partial R}\right|^2 dx \leq C \frac{e^{Cr}}{r^{2m-2}} \int_{B_r(x_0)} |\nabla u|^2 \, dx - \Theta_u(x_0) < \infty \; . \tag{2.10}$$

Through a change of coordinates we can always assume that in the new coordinates the point x_0 becomes the origin and the complex structure J is the standard complex structure there, i.e. that $J(0) = J_0$. Furthermore, J satisfies $\|J(r) - J_0\|_{C^2} = O(r)$ on all of $B_r(0)$. Using these two facts we now slice by J-holomorphic curves passing through $x_0 = 0$ parametrised by points in $\mathbf{CP}^{m-1}$. The curves are given as the intersection of pre-images $\pi^{-1}(p)$ for $p \in \mathbf{CP}^{m-1}$ with $B_r(0)$ and in fact form a singular foliation of $B_r(0)$ (see appendix D). It is important to note that the tangent plane to each J-holomorphic curve $\pi^{-1}(p)$ is spanned by two vectors X and JX, where near the origin X is only a small perturbation of $\frac{\partial}{\partial R}$. More precisely, we have that any tangent plane is spanned by a vector X for which we have

$$X = \frac{\partial}{\partial R} + O(r)\frac{\partial}{\partial x_i} \; ,$$

i.e. for a map u we get that

$$u \cdot X = \frac{\partial u}{\partial R} + O(r)\frac{\partial u}{\partial x_i} \; .$$

The aim is to show that for almost every $p \in \mathbf{CP}^{m-1}$ the restriction of u to $\pi^{-1}(p)$ is a J-holomorphic curve in (N, J_N). To see this we will first use the monotonicity formula to show that for the J-holomorphic map u

$$\int_{B_r(0)} R^{2-2m} |u \cdot X|^2 \, dx < \infty \; . \tag{2.11}$$

Note that from the properties of X we get the estimate

$$\begin{aligned} \int_0^r \int_{\partial B_\rho(0)} R^{2-2m} |u \cdot X|^2 \, d\partial B_\rho \, d\rho \; &\leq \; \int_0^r \int_{\partial B_\rho(0)} \left[R^{2-2m} \left|\frac{\partial u}{\partial R}\right|^2 + C R^{3-2m} |\nabla u|^2 \right] d\partial B_\rho \, d\rho \\ &= \int_{B_r(0)} R^{2-2m} \left|\frac{\partial u}{\partial R}\right|^2 dx + C \int_{B_r(0)} R^{3-2m} |\nabla u|^2 \, dx \; . \end{aligned}$$

Because of bound (2.10), the bound on the first term follows directly from the monotonicity formula, whence it remains to bound the second one. This can be done through and integration by parts and applying the monotonicity formula to each term:

$$\begin{aligned}\int_0^r \rho^{3-2m} \int_{\partial B_\rho(0)} |\nabla u|^2 \, d\partial B_\rho \, d\rho \; &= \\ &= \left[\rho^{3-2m} \int_{B_\rho} |\nabla u|^2 \, dx\right]_0^r + (2m-3) \int_0^r \rho^{2-2m} \int_{B_\rho} |\nabla u|^2 \, dx \, d\rho \\ &\leq Cr\left[\sup_{0<\rho<1} \frac{1}{\rho^{2m-2}} \int_{B_\rho} |\nabla u|^2 \, dx\right] \\ &\quad - \lim_{\rho\to 0}\left(\rho\left[\sup_{0<\rho<1} \frac{1}{\rho^{2m-2}} \int_{B_\rho} |\nabla u|^2 \, dx\right]\right) < \infty \, .\end{aligned}$$

One of the properties of $\pi : B_r(0) \setminus \{0\} \longrightarrow \mathbf{CP}^{m-1}$ is that the norm of its gradient satisfies $O(\frac{1}{r})$. More precisely, there is a constant $C > 0$ such that

$$C - Cr \leq R^{2m-2}|J_{2m-2}\pi| \leq C + Cr \, ,$$

where $J_{2m-2}\pi$ denotes the $(2m-2)$-Jacobian of π, and consequently, $1 \leq \frac{C+Cr}{C-Cr} \leq 1+O(r)$. Therefore we can apply the co-area formula to the slicing by π to obtain

$$\int_{\mathbf{CP}^{m-1}} \int_{\pi^{-1}(p)\cap B_r(0)} \frac{R^{2-2m}|u \cdot X|^2}{|J_{2m-2}\pi|} \, d\sigma_p \, dp = \int_{B_r(0)} R^{2-2m}|u \cdot X|^2 \, dx < \infty \, ,$$

which is what we wanted to show since by Fubini's theorem it follows that for almost every $p \in \mathbf{CP}^{m-1}$ we have

$$\int_{\pi^{-1}(p)\cap B_r(0)} |u \cdot X|^2 \, d\sigma_p \leq C \int_{\pi^{-1}(p)\cap B_r(0)} \frac{R^{2-2m}|u \cdot X|^2}{|J_{2m-2}\pi|} \, d\sigma_p < \infty \, . \tag{2.12}$$

Since the metric h on the target is tamed by the target complex structure J_N, we have that $|u \cdot JX|^2 \cong |J_N \circ \nabla u(X)|^2 \cong |\nabla u(X)|^2$ and hence from the above argument we also deduce that for almost every $p \in \mathbf{CP}^{m-1}$

$$\int_{\pi^{-1}(p)\cap B_r(0)} |u \cdot JX|^2 \, d\sigma_p < \infty \, . \tag{2.13}$$

Next we want to show how (2.12) and (2.13) imply that $\int_{\pi^{-1}(p)\cap B_r(0)} |\nabla u|^2 \, d\sigma_p < \infty$ for almost every $p \in \mathbf{CP}^{m-1}$. To see this first note that as X and JX span the tangent space to $\pi^{-1}(p)$

$$\nabla u|_{\pi^{-1}(p)} = (\nabla u \cdot X)X + (\nabla u \cdot JX)JX$$

from which we conclude that

$$|u \cdot X|^2 \leq \frac{1}{2-Cr}|\nabla u|_{\pi^{-1}(p)}|^2 \leq \frac{2+Cr}{2-Cr}|u \cdot X|^2 \tag{2.14}$$

for some constant $C > 0$. Therefore we obtain that for r small enough $\int_{\pi^{-1}(p)\cap B_r(0)} |\nabla u|^2 \, d\sigma_p$ is bounded by some constant times the integrals in the X and JX directions which in turn are bounded by estimates (2.12) and (2.13). Hence we have shown that for almost every $p \in \mathbf{CP}^{m-1}$ the restriction $u|_{\pi^{-1}(p)}$ is a J-holomorphic map in $W^{1,2}(\pi^{-1}(p) \cap B_r(0), N)$. Since $(\pi^{-1}(p), J)$ is a Riemann surface for each $p \in \mathbf{CP}^{m-1}$, we get that $u|_{\pi^{-1}(p)}$ is smooth. Furthermore, as the target N does not admit any pseudo-holomorphic spheres, there exists $\delta \in (0,1)$, independent of u, such that for almost every $p \in \mathbf{CP}^{m-1}$

$$\int_{\pi^{-1}(p)\cap B_r(0)} |\nabla u|^2 \, d\sigma_p < \delta \int_{\pi^{-1}(p)\cap B_{2r}(0)} |\nabla u|^2 \, d\sigma_p \ . \tag{2.15}$$

Together with equation (2.14) this gives the following estimate

$$\begin{aligned} \int_{\pi^{-1}(p)\cap B_r(0)} |u \cdot X|^2 \, d\sigma_p &\leq \frac{1}{2-Cr} \int_{\pi^{-1}(p)\cap B_r(0)} |\nabla u|^2 \, d\sigma_p \\ &< \frac{\delta}{2-Cr} \int_{\pi^{-1}(p)\cap B_{2r}(0)} |\nabla u|^2 \, d\sigma_p \\ &\leq \delta \frac{2+Cr}{2-Cr} \int_{\pi^{-1}(p)\cap B_{2r}(0)} |u \cdot X|^2 \, d\sigma_p \ , \end{aligned}$$

where we can assume to have taken $r > 0$ so small that $\tilde{\delta} := \delta \frac{2+Cr}{2-Cr} < 1$. Integrating both sides over $\mathbf{CP}^{m-1}$ we then obtain

$$\int_{\mathbf{CP}^{m-1}} \int_{\pi^{-1}(p)\cap B_r(0)} |u \cdot X|^2 \, d\sigma_p \, dp < \tilde{\delta} \int_{\mathbf{CP}^{m-1}} \int_{\pi^{-1}(p)\cap B_{2r}(0)} |u \cdot X|^2 \, d\sigma_p \, dp \ . \tag{2.16}$$

Applying the co-area formula again we obtain the estimate

$$\begin{aligned} \int_{B_r(0)} |u \cdot X|^2 |J_{2m-2}\pi| \, dx &= \int_{\mathbf{CP}^{m-1}} \int_{\pi^{-1}(p)\cap B_r(0)} |u \cdot X|^2 \, d\sigma_p \, dp \\ &< \tilde{\delta} \int_{\mathbf{CP}^{m-1}} \int_{\pi^{-1}(p)\cap B_{2r}(0)} |u \cdot X|^2 \, d\sigma_p \, dp \\ &= \tilde{\delta} \int_{B_{2r}(0)} |u \cdot X|^2 |J_{2m-2}\pi| \, dx \ . \end{aligned} \tag{2.17}$$

Provided $r > 0$ was chosen sufficiently small, estimate (2.17) gives

$$\int_{B_r(0)} R^{2-2m} |u \cdot X|^2 \, dx < \tilde{\delta} \int_{B_{2r}(0)} R^{2-2m} |u \cdot X|^2 \, dx \ .$$

An iteration argument then implies that there exist $C > 0$ and $\gamma \in (0,1)$ such that, for $r > 0$ sufficiently small,

$$\int_{B_r(0)} R^{2-2m} |u \cdot X|^2 \, dx \leq C_1 r^\gamma \ .$$

From this we will deduce a rate of convergence for $\int_{B_r(0)} R^{2-2m} \left|\frac{\partial u}{\partial R}\right|^2 dx$. Note that from $X = \frac{\partial}{\partial R} + O(r)\frac{\partial}{\partial x_i}$ we deduce

$$\begin{aligned} \int_{B_r(0)} R^{2-2m} \left|\frac{\partial u}{\partial R}\right|^2 dx &\leq \int_{B_r(0)} R^{2-2m} |u \cdot X|^2 \, dx \leq C r^\gamma + C \int_{B_r(0)} R^{3-2m} |\nabla u|^2 \, dx \\ &\leq C r^\gamma + C r \sup_{0<\rho<1} \frac{1}{\rho^{2m-2}} \int_{B_\rho(0)} |\nabla u|^2 \, dx \\ &\leq C(1 + \|\nabla u\|^2_{L^2(B_{2r_0}(0))}) r^\gamma = C r^\gamma \ , \end{aligned}$$

where we used the monotonicity formula for the last estimate.
To finish the proof we will use the remark after the proof of the monotonicity formula. From estimate (2.7) we get that

$$\frac{e^{-Cr}}{r^{2m-2}}\int_{B_r(x_0)}|\nabla u|^2\,dx-\Theta_u(x_0)\leq C\int_{B_r(x_0)}R^{2-2m}\left|\frac{\partial u}{\partial R}\right|^2 dx\leq Cr^\gamma$$

which for r small enough so that $e^{2Cr}\leq 2$ becomes

$$\frac{e^{Cr}}{r^{2m-2}}\int_{B_r(x_0)}|\nabla u|^2\,dx-e^{2Cr}\Theta_u(x_0)\leq 2Cr^\gamma\ .$$

Combining these estimates we get

$$\begin{aligned}\frac{e^{Cr}-r}{r^{2m-2}}\int_{B_r(x_0)}|\nabla u|^2\,dx-\Theta_u(x_0) &\leq 2Cr^\gamma+\left(e^{2Cr}-1\right)\Theta_u(x_0)\\ &\leq Cr^\gamma+C\|\nabla u\|^2_{L^2(B_{2r_0}(0))}r\\ &\leq Cr^\gamma\ ,\end{aligned}$$

which completes the proof of Theorem 2.3.

2.7 Uniqueness of tangent maps

In this section we show how Theorem 2.3 implies the uniqueness of tangent maps at all $x_0\in M$. If $x_0\notin\operatorname{sing}u$, then for all sequences of radii $r_i\longrightarrow 0$ we have that $E(u,r_i,x_0)\longrightarrow 0$ as $i\longrightarrow\infty$. Furthermore, also $\|u_{r,x_0}\|_{W^{1,2}}\longrightarrow 0$, showing that 0 is the only tangent map regardless of our choice of sequence of radii.
Now fix $x_0\in M$, possibly in $\operatorname{sing}u$, and without loss of generality we can assume that we have chosen coordinates so that $x_0=0$. From the proof of Theorem 2.3 we know that there are constants $C>0$ and $\gamma>0$ such that

$$\int_{B_r(x_0)}R^{2-2m}\left|\frac{\partial u}{\partial R}\right|^2 dx\leq Cr^\gamma\ ,$$

where we absorbed the term $\left(1+\|\nabla u\|^2_{L^2(B_{2r_0}(0))}\right)$ in the constant C. Setting $u_{r,0}(x):=u(rx)$ we note that $\int_{B_1(0)}R^{2-2m}\left|\frac{\partial u_{r,0}}{\partial R}\right|^2dx=\int_{B_r(0)}R^{2-2m}\left|\frac{\partial u}{\partial R}\right|^2dx$. Thus

$$\int_{B_1(0)}R^{2-2m}\left|\frac{\partial u_{r,0}}{\partial R}\right|^2 dx\leq Cr^\gamma\ .$$

From this we deduce that for any $0<\sigma<\tau$ sufficiently small we get

$$\begin{aligned}\|u_{\tau,0}-u_{\sigma,0}\|_{L^2(S^{2m-1})} &\leq \int_\sigma^\tau\left\|\frac{\partial u_{r,0}}{\partial r}\right\|_{L^2(S^{2m-1})}dr\\ &\leq \left(\int_\sigma^\tau r^{1-\frac{\gamma}{2}}\left\|\frac{\partial u_{r,0}}{\partial r}\right\|^2_{L^2(S^{2m-1})}dr\right)^{\frac12}\left(\int_\sigma^\tau r^{-(1-\frac{\gamma}{2})}dr\right)^{\frac12}\\ &\leq C\tau^{\frac{\gamma}{4}}\ ,\end{aligned}$$

where the fact that $\int_\sigma^\tau r^{1-\frac{\gamma}{2}} \left\| \frac{\partial u_{r,0}}{\partial r} \right\|^2_{L^2(S^{2m-1})} dr \leq C$ follows from an integration by parts. If we now take a subsequence $\sigma_j \to 0$ as $j \to \infty$ such that $u_{\sigma_j,0}$ converges to some tangent map $u_{\infty,0}$ weakly in $W^{1,2}$ but strongly in $L^2(S^{2m-1})$, then by the triangle inequality and the above estimates for any other sequence $\tau_j \to 0$ we have $u_{\tau_j,0} \to u_{\infty,0}$ strongly in $L^2(S^{2m-1})$. Hence $u_{\tau_j,0}$ converges weakly to $u_{\infty,0}$ in $W^{1,2}$ and therefore $u_{\infty,0}$ is the unique tangent map. Consequently, Theorem 2.4 is proved.

2.8 Outlook

In the statement of Theorems 2.3 and 2.4, the restriction on the topology of the target manifold not to admit any pseudo-holomorphic spheres seems artificial. In fact, these Theorems are conjectured to hold true in general. The only step where we used this assumption in the proof was for deriving a common decay rate for the restriction of u to almost every J-holomorphic curve $\pi^{-1}(p)$. More precisely, it would suffice to derive a uniform bound for the energy of $u|_{\pi^{-1}(p)\cap B_r(0)}$ independent of u and p. Define a map $f : \mathbf{CP}^{m-1} \longrightarrow \mathbf{R}$ by setting

$$f(p) := \int_{\pi^{-1}(p)\cap B_r(0)} |\nabla u|^2 \, d\sigma_p$$

which by estimate (2.14) lies in $L^1(\mathbf{CP}^{m-1})$. It would be important to know further regularity properties of f and more specifically, if f is subharmonic on $\mathbf{CP}^{m-1}$. If that were true, f would satisfy a Harnack inequality on $\mathbf{CP}^{m-1}$ implying an L^∞-bound on f only in terms of its L^1-norm. Since the latter can be made arbitrarily small by taking $r > 0$ to be sufficiently small, the Harnack estimate on f, i.e. the energy of $u|_{\pi^{-1}(p)\cap B_r(0)}$, would imply that the ϵ-regularity Theorem 2.1 could be applied on almost every slice $\pi^{-1}(p) \cap B_r(0)$ independent of p. The proof of Theorem 2.3 could then be carried out as before, which is why we suggest the following open problem:

Open Problem 2.1 *Let $f : \mathbf{CP}^{m-1} \longrightarrow \mathbf{R}$ be defined as above. Is it true that f is subharmonic on $\mathbf{CP}^{m-1}$ and satisfies a Harnack inequality ?*

The method used in the proof of Theorem 2.3 might have applications for obtaining rates of convergence even in other areas such as anti self-dual instanton (Yang-Mills) equations and, more generally, whenever there is a suitable slice function reducing the problem to a dimension where regularity results are already known.

Chapter 3

Polyharmonic maps

Abstract

In this chapter we investigate regularity questions for polyharmonic maps. Starting from the analysis of the structure of the Euler-Lagrange equation we use a new sharp Gagliardo-Nirenberg interpolation inequality to obtain regularity results for polyharmonic maps. The same technique is then applied to general k-th order elliptic systems with critical growth.

3.1 Introduction

We begin this chapter with a brief overview of the development of the regularity theory for both harmonic and biharmonic maps. Given a closed (domain) manifold M of dimension m and a closed (target) manifold N of dimension n we can assume N to be embedded in $\mathbf{R}^N$ for some $N \in \mathbf{N}$ sufficiently large using Nash's embedding theorem in [61]. For $k \in \mathbf{N}$ and $1 \leq p \leq \infty$ we then define the Sobolev spaces of maps from M into N by

$$W^{k,p}(M,N) := \big\{ u \in W^{k,p}(M, \mathbf{R}^N) \ : \ u(x) \in N \text{ for a.e. } x \in M \big\}$$

with the topology induced from the linear Sobolev space $W^{k,p}(M, \mathbf{R}^N)$.
The Dirichlet energy of a map $u \in W^{1,2}(M,N)$ is defined by

$$D(u) := \int_M |\nabla u|^2 \, dx$$

and we will consider three kinds of critical points of this functional on $W^{1,2}(M,N)$. The critical points in $W^{1,2}(M,N)$ for compactly supported variations in the target manifold N are called *weakly harmonic maps* (compare with Definition 3.1 below). If $u \in W^{1,2}(M,N)$ is weakly harmonic and in addition a critical point for compactly supported variations in the domain manifold M we call u a *stationary harmonic map* – again compare with Definition 3.2 below. When $u \in W^{1,2}(M,N)$ satisfies $D(u) \leq D(v)$ whenever compared with $v \in W^{1,2}(M,N)$ with $u - v \in W^{1,2}_0(M,N)$, we say that u is (Dirichlet) *energy minimising*. Again we refer to the discussion following the analogous definition for polyharmonic maps (Definition 3.3) later on in this section.

The study of the regularity properties of these critical points has been an ongoing search for several decades. Since energy minimising harmonic maps are in some sense the mildest the first regularity results were obtained for them. In 1948, C.B. Morrey [59] showed smoothness for every energy minimising harmonic map $u \in W^{1,2}(M,N)$ provided the dimension of the target manifold M was equal to $m = 2$. Afterwards the regularity problem remained open for a long time until (still for the case when $m = 2$) M. Grüter [32] proved smoothness of conformal weakly harmonic maps followed by an extension to stationary harmonic maps by R. Schoen [72]. Finally, F. Hélein in [40], [41] and [42] showed that every weakly harmonic map in the case $m = 2$ is smooth.
For higher dimensional domain manifolds the situation is more complicated as the prominent example of the radial projection from B^3 into S^2 shows. This is a weakly harmonic map with an isolated singularity at the origin. The first results in the case of $m \geq 3$ were obtained by R. Schoen and K. Uhlenbeck [73]. They showed that if $u \in W^{1,2}(M,N)$ is energy minimising, then u is smooth except for a closed singular set $S \subset M$ of Hausdorff dimension $dim_{\mathcal{H}}(S) \leq m-3$. For the case $m = 3$ they could strengthen the result to S consisting only of finitely many points. Later on H. Brezis, J.-M. Coron and E. Lieb [16] proved that the above mentioned radial projection is actually an energy minimising map thus showing the optimality of the result by R. Schoen and K. Uhlenbeck [73]. For a different proof of the minimality of the radial projection valid also in higher dimensions see F. Lin [53].
Using observations made by F. Hélein in [40], L.C. Evans [23] could show partial regularity for stationary harmonic maps into spheres, which means smoothness of such a $u \in W^{1,2}(M,S^N)$ except for a closed singular set S with $\mathcal{H}^{m-2}(S) = 0$ — note the difference to the case of energy minimisers. This was later generalised by F. Bethuel [11] to include arbitrary target manifolds. The optimality of this result is still unknown.
For weakly harmonic maps in dimensions $m \geq 3$ there is no hope for analogous results since T. Rivière [66] proved existence of everywhere discontinuous weakly harmonic maps from B^3 into S^2.
For a detailed overview of many of the results mentioned above and the techniques involved in proving them we refer the reader to the books by L. Simon [79] for energy minimising maps and by F. Hélein [43] for weakly and stationary harmonic maps. One reason for the intensity of the study of the regularity of harmonic maps is the fact that they are one of the simplest (geometric) variational problems where many of the so interesting features, later on also discovered in other settings, appeared for the first time. Indeed, similar phenomena show up for Yang-Mills connections, Ginzburg-Landau vortices, Willmore surfaces and many other important problems in a wide range from differential geometry to theoretical physics and applied mathematics. Another possible extension is to consider critical points for energies involving higher derivatives of the maps, which leads to the study of biharmonic and even polyharmonic maps.

For biharmonic maps we consider critical points of the Hessian energy of a map $u \in W^{2,2}(M,N)$ defined by

$$B(u) := \int_M |\Delta u|^2 \, dx \; .$$

As for harmonic maps we assign the notions *weakly biharmonic*, *stationary biharmonic* and *energy minimising biharmonic map* to the corresponding critical points of B – again see Definitions 3.1-3.3 for precise statements.

Following the path set out for harmonic maps S.-Y.A. Chang, L. Wang and P. Yang in [18] began their study of (extrinsic) weakly biharmonic maps. For weakly biharmonic maps into the sphere they could show smoothness for $m \leq 4$ and prove partial regularity for stationary biharmonic maps when $m \geq 5$, i.e. that $u \in W^{2,2}(M, N)$ is smooth except for a singular set S of Hausdorff measure $\mathcal{H}^{m-4}(S) = 0$. C.-Y. Wang later generalised these results for arbitrary target manifolds in [90] and [91]. For a different proof of smoothness of weakly biharmonic maps see also the papers by P. Strzelecki [84] and more recently T. Lamm and T. Rivière [49] — see also the following paragraphs.

As motivation for the approach set out for polyharmonic maps below we would like to focus on some of the aspects common to both the regularity theory for harmonic but also for biharmonic maps. One common feature is the fact that all the improvements were made after observing small gains in regularity coming from the algebraic structure of the quantities involved. For instance, when we consider harmonic maps into spheres S^n the harmonic map equation becomes

$$\Delta u = u|\nabla u|^2, \tag{3.1}$$

where the right-hand side is a-priori only in L^1. This prevents the direct use of standard regularity estimates from harmonic analysis since the Laplace operator is not invertible there. However, J. Shatah discovered the conservation law

$$\div(u^i \nabla u^j - u^j \nabla u^i) = 0 \quad \text{for all } 1 \leq i, j \leq m$$

for weakly harmonic $u \in W^{1,2}(M, S^n)$. Later on, based on observations by H. Wente [93], F. Hélein [42] used this to prove that the Laplace operator is actually invertible on the right-hand side of (3.1) with the result that u is continuous. By the results of S. Hildebrandt and K.-O. Widman [44], O. Ladyzhenskaya and N. Ural'tseva [48] and C.B. Morrey [59] u is therefore smooth. See also the book by F. Hélein [43] for details.
In the case of target manifolds without symmetry, another important tool for proving (partial) regularity for harmonic and biharmonic maps is the technique of moving frames. This was introduced for harmonic maps in two dimensions by F. Hélein [42] and later on applied to stationary harmonic maps by F. Bethuel [11]. For an application to (stationary) biharmonic maps see C.-Y. Wang in [90] and [91].
Only very recently, in [67], T. Rivière succeeded to rephrase the harmonic map system as a conservation law when $m = 2$, allowing him (amongst other results) to give a direct proof of regularity of weakly harmonic maps in two dimensions avoiding the use of moving frames. Prompted by [69], where T. Rivière and M. Struwe developed a related gauge-theoretic approach to prove partial regularity in higher dimensions. Moreover, this new approach allows the authors to reduce Hélein's C^5-assumption on the target manifold to C^2, which seems to be the natural assumption in order to ensure that the second fundamental form is well defined. Finally, T. Lamm and T. Rivière in [49] could show smoothness for weakly biharmonic maps in four dimensions avoiding moving frames, and M. Struwe [82] proves partial regularity for stationary biharmonic maps in higher dimensions again via gauge theory.

Strengthening the natural hypotheses for the regularity of a stationary biharmonic map u slightly, by assuming some higher integrability of the leading order derivative, we show here similar partial regularity results for biharmonic maps without using moving frames. Our method is not restricted to this fourth order problem and provides regularity results for polyharmonic maps. Therefore, we now switch to the more general setting of polyharmonic maps below.

More precisely, for $k \in \mathbf{N}$ and $u \in W^{k,2}(\Omega, N)$, we consider the k-harmonic energy functional

$$E(u) = \int_\Omega |\nabla^k u|^2 \, dx \ .$$

Define the BMO space and the Morrey spaces $M^{p,\lambda}$ for the domain Ω as

$$BMO(\Omega) := \{u \in L^1(\Omega) : [u]_{BMO(\Omega)} := \sup_{B_r \subset \mathbf{R}^m} \{r^{-m} \int_{B_r \cap \Omega} |u - \overline{u}_{B_r \cap \Omega}| \, dx\} < \infty\}$$

and

$$M^{p,\lambda}(\Omega) := \{u \in L^p(\Omega) : [u]^p_{M^{p,\lambda}(\Omega)} := \sup_{B_r \subset \mathbf{R}^m} \{r^{\lambda - m} \int_{B_r \cap \Omega} |u|^p \, dx\} < \infty\} \ ,$$

where $\overline{u}_{B_r} := \fint_{B_r} u \, dx$.

Definition 3.1 *A map $u \in W^{k,2}(\Omega, N)$ is called weakly k-harmonic if u is a critical point of the k-harmonic energy functional with respect to compactly supported variations on N, that is, if for all $\xi \in C_0^\infty(\Omega, \mathbf{R}^N)$ we have*

$$\left.\frac{d}{dt}\right|_{t=0} E(\pi(u + t\xi)) = 0 \ ,$$

where π denotes the nearest point projection onto N.

Definition 3.2 *A weakly k-harmonic map in $W^{k,2}(\Omega, N)$ is called stationary k-harmonic if, in addition, u is a critical point of the k-harmonic energy $E(\cdot)$ with respect to compactly supported variations on the domain manifold, i.e. if*

$$\left.\frac{d}{dt}\right|_{t=0} E\left(u \circ (id + t\xi)\right) = 0 \text{ for all } \xi \in C_0^\infty(\Omega, \mathbf{R}^m) \ , \tag{3.2}$$

where id denotes the identity map.

Remark 3.1 *(Stationary) 1-harmonic maps are (stationary) harmonic maps. Observing that $|\nabla^2 u|^2$ and $|\Delta u|^2$ only differ by a divergence term, we conclude that the (stationary) 2-harmonic maps are precisely the (stationary) biharmonic maps.*

Definition 3.3 *A map $u \in W^{k,2}(\Omega, N)$ is called k-energy minimising if for all $v \in W^{k,2}(\Omega, N)$ with $u - v \in W_0^{k,2}(\Omega, N)$*

$$E(u) \leq E(v) \ .$$

Our main result then reads

Theorem 3.1 *For $p > 1$ and $2kp \leq m$, let $u \in W^{k,2p}(\Omega, N)$ be weakly k-harmonic. There exists $\epsilon > 0$, such that for each point $x_0 \in \Omega$ for which there exists some $r_0 > 0$ with*

$$\sum_{\mu=1}^{k} [\nabla^\mu u]_{M^{\frac{2k}{\mu},2k}(B_{r_0}(x_0))} \leq \epsilon \ ,$$

we have

$$u \in C^\infty(B_{\frac{r_0}{4}}(x_0), N) \ .$$

Remark 3.2 *For $2kp > m$, Sobolev's embedding theorem implies that every map u in $W^{k,2p}(\Omega, N)$ is Hölder-continuous. Smoothness then follows at once from elliptic bootstrapping arguments.*

In view of the monotonicity formulae for stationary harmonic and biharmonic maps, the condition on the Morrey estimate is, in these cases, satisfied almost everywhere. More precisely, we deduce the following

Corollary 3.2 *For $k \in \{1, 2\}$ and $p > 1$, let $u \in W^{k,2p}(\Omega, N)$ be stationary k-harmonic. Then, u is smooth outside a closed set S with $\mathcal{H}^{m-2kp}(S) = 0$.*

Conceivably, a monotonicity formula allowing to guarantee the Morrey norm condition in Theorem 3.1 will also hold for $k \geq 3$.

The proof of Theorem 3.1 is based on Morrey decay estimates for the rescaled polyharmonic energy. We apply an interpolation inequality by Y. Meyer and T. Rivière in [55] and by P. Strzelecki in [84] (see also our appendix A) in order to bound the $W^{k,2p}$-norm by the BMO- and $W^{2k,p}$-norms.
The idea of proving ϵ-regularity results using improved interpolation inequalities first appeared in the paper [55] by Y. Meyer and T. Rivière in the context of Yang-Mills fields and was also used in the paper [68] by T. Rivière and P. Strzelecki for more general elliptic systems.
Of course, the critical case $p = 1$ would be the natural exponent for the present problem. Moreover, for $k = 1$ and $k = 2$, Corollary 3.2 directly follows from F. Bethuel [11] respectively C.-Y. Wang [91] and Poincaré's inequality. However, our proof is more direct and avoids the moving frame technique.

We want to remark that polyharmonic maps have already been studied by A. Gastel in [27], where he considered the polyharmonic map heat flow in the critical dimension.

For another application of the techniques used to prove Theorem 3.1 we consider slightly different $2k$-th order elliptic systems. This time we are interested in nonlinear elliptic systems of the type

$$\Delta^k u = Q(x, u, \ldots, \nabla^{k-1} u) \ , \tag{3.3}$$

where $k \geq 2$ and the nonlinearity in Q satisfies

$$|Q(x, u, \ldots, \nabla^{k-1}u)| \leq C \sum_{j=1}^{k-1} |\nabla^j u|^{\frac{2k}{j}}$$

for some constant $C > 0$ independent of u but without further assumptions on the structure of Q. We will consider solutions of (3.3) which are a priori in $W^{k,2} \cap L^\infty(\Omega, N)$. Then we have the following result

Theorem 3.3 *Let $u \in W^{k,2}(\Omega, N)$ be weak solution to equation (3.3). There exists $\epsilon > 0$ independent of u, such that for each point $x_0 \in \Omega$ for which there exists some $r_0 > 0$ with*

$$\sum_{\mu=1}^{k} [\nabla^\mu u]_{M^{\frac{2k}{\mu},2k}(B_{r_0}(x_0))} \leq \epsilon \ ,$$

we have

$$u \in C^\infty(B_{\frac{r_0}{4}}(x_0), N) \ .$$

This result was recently conjectured by P. Strzelecki and A. Zatorska-Goldstein in [85], where they obtained this result for $k = 2$. The above system is of interest as from the outset the right hand side again is in L^1 but has at first no higher integrability. Because of the much simpler structure of the non-linearity when compared with the one for polyharmonic maps, this time we are in fact able to derive higher integrability of the right-hand side of (3.3) from a reverse Hölder's inequality — see Lemma 3.12. In the proof of this inequality again the sharp Gagliardo-Nirenberg inequality has to be used.

The chapter is organised as follows. Section 3.2 gives the Euler-Lagrange equation for polyharmonic maps. In 3.3 we prove the Morrey decay estimates. In 3.4 we deduce Theorem 3.1 and in 3.5, we conclude Corollary 3.2. Finally, a proof of Theorem 3.3 is given in section 3.6.
A proof of the sharp Gagliardo-Nirenberg inequality is given in appendix A.

Acknowledgement: Sections 3.2–3.5 are based on the joint work with Gilles Angelsberg in [9].

3.2 Euler-Lagrange equation for polyharmonic maps

In this section we derive the geometric form of the Euler-Lagrange equation for weakly polyharmonic maps and analyse its structure. We consider, for $\delta > 0$ sufficiently small, the tubular neighbourhood V_δ of N in $\mathbf{R}^N$ and the smooth nearest point projection $\Pi_N : V_\delta \longrightarrow N$. For $p \in N$, $P(p) := \nabla\Pi(p)$ is the orthonormal projection onto the tangent space T_pN. The orthonormal projection onto the normal space will be denoted by $P^\perp$. Recall that $P + P^\perp = \mathrm{id}$. Then, we have

Lemma 3.4 (Euler-Lagrange) *If $u \in W^{k,2}(\Omega, N)$ is weakly k-harmonic, then it satisfies*

$$P(u)(\Delta^k u) = 0 \tag{3.4}$$

in the sense of distributions.

Proof. For $\xi \in C_0^\infty(\Omega, \mathbf{R}^N)$, we compute

$$\begin{aligned} 0 &= \frac{d}{dt}\bigg|_{t=0} \int_\Omega |\nabla^k(\Pi_N \circ (u+t\xi))|^2\,dx \\ &= 2\int_\Omega \nabla^k u \nabla^k(P(u)(\xi))\,dx\ . \end{aligned}$$

□

Remark 3.3 *For a weakly polyharmonic map u in $C^\infty(\Omega, N) \cap W^{k,2}(\Omega, N)$ and $\xi \in C_0^\infty(\Omega, \mathbf{R}^N)$, with $\xi(x)$ parallel to $T_{u(x)}N$ for all $x \in \Omega$, we have $P(u)(\xi) = \xi$. The proof of Lemma 3.4 then shows that*

$$\Delta^k u \perp T_u N$$

in the sense of distributions.

In order to formulate the following lemma, we introduce the l-divergence $\nabla^{(l)}\cdot$ as follows. We define $\nabla^{(1)} \cdot u := \nabla \cdot u$ and $\nabla^{(l)} \cdot u := \nabla \cdot \left(\nabla^{(l-1)} \cdot u\right)$ for $l \geq 2$.

Lemma 3.5 *If $u \in W^{k,2}(\Omega, N)$ is weakly k-harmonic, then there exist f and g_{jl} with*

$$\Delta^k u = f + \sum_{\substack{j,l\geq 0 \\ 1\leq 2j+l\leq k}} \nabla^{(l)} \cdot \Delta^j g_{jl}\ , \tag{3.5}$$

where

$$\begin{aligned} |f| &\leq C\sum_{\lambda\in\Lambda}\prod_{\mu=0}^{k} |\nabla^\mu u|^{\gamma_{\lambda,\mu}} \quad \textit{with } \sum_\mu \mu\gamma_{\lambda,\mu} = 2k \textit{ for every } \lambda\in\Lambda\ , \\ |g_{jl}| &\leq C\sum_{\lambda\in\Lambda}\prod_{\mu=0}^{k} |\nabla^\mu u|^{\gamma_{\lambda,\mu}} \quad \textit{with } \sum_\mu \mu\gamma_{\lambda,\mu} = 2k-(2j+l) \textit{ for every } \lambda\in\Lambda\ , \end{aligned}$$

with Λ consisting of finitely many indices and $\gamma_{\lambda,\mu} \geq 0$ for every $\lambda \in \Lambda$ and $0 \leq \mu \leq k$.

Remark 3.4 *Note that the representations in Lemma 3.5 are not unique. See for example Remark 3.5.*

Proof. We observe that

$$\Delta^k(a\cdot b) = \sum_{\substack{0\leq i,j,q\leq k \\ i+j+q=k}} c^k_{ijq}\Delta^i\nabla^q a \cdot \Delta^j \nabla^q b$$

where c^k_{ijq} are positive integers. In particular, we have $c^k_{k00} = c^k_{0k0} = 1$. Combining this with equation (3.4) shows that u satisfies

$$\begin{aligned} 0 &= P(u)(\Delta^k u) \\ &= \Delta^{k-1}(P(u)(\Delta u)) - \sum_{\substack{0\leq i,j,q\leq k-1 \\ i+j+q=k-1 \\ (i,j,q)\neq(0,k-1,0)}} c^{k-1}_{ijq}\Delta^i\nabla^q(P(u))\Delta^{j+1}\nabla^q u \end{aligned}$$

in the sense of distributions. Let $A(\cdot)(\cdot,\cdot)$ denote the second fundamental form of N in $\mathbf{R}^N$ and use the property $P(u)(\Delta u) = \Delta u + A(u)(\nabla u, \nabla u)$ to derive

$$\Delta^k u = \sum_{\substack{i,j,q \\ i+j+q=k-1 \\ (i,j,q)\neq(0,k-1,0)}} c_{ijq}^{k-1} \Delta^i \nabla^q (P(u)) \Delta^{j+1} \nabla^q u - \Delta^{k-1}(A(u)(\nabla u, \nabla u)) \; . \tag{3.6}$$

First we consider the case when k is even and analyse the Euler-Lagrange equation (3.6) term by term. It suffices to show that every term in (3.6) can be written in the desired form.

For i, q such that $i + \frac{q}{2} = \frac{k}{2}$, we have that $c_{ijq}^{k-1} \Delta^i \nabla^q (P(u)) \Delta^{j+1} \nabla^q u$ is of the form f. Here we used the fact that

$$|\nabla^\beta P(u)| \leq C \sum_{\lambda \in \Lambda} \prod_{\mu=0}^{\beta} |\nabla^\mu u|^{\gamma_{\lambda,\mu}} \text{ with } \sum_\mu \mu \gamma_{\lambda,\mu} = \beta \text{ for every } \beta \leq k \text{ and } \lambda \in \Lambda \; . \tag{3.7}$$

Indeed, the chain rule gives $\nabla(P(u)) = \nabla P(u) \nabla u$ and $\nabla^2(P(u)) = \nabla^2 P(u) \nabla u \nabla u + \nabla P(u) \nabla^2 u$. We infer estimate (3.7) by iterating this computation and taking the smoothness of the nearest point projection into account.

For i, q such that $i + \frac{q}{2} > \frac{k}{2}$, we compute

$$\begin{aligned} &\Delta^i \nabla^q (P(u)) \Delta^{j+1} \nabla^q u \\ &= \nabla \cdot (\Delta^{i-1} \nabla^{q+1} (P(u)) \Delta^{j+1} \nabla^q u) - \Delta^{i-1} \nabla^{q+1} (P(u)) \Delta^{j+1} \nabla^{q+1} u \end{aligned}$$

and/or

$$\begin{aligned} &\Delta^i \nabla^q (P(u)) \Delta^{j+1} \nabla^q u \\ &= \nabla \cdot (\Delta^i \nabla^{q-1} (P(u)) \Delta^{j+1} \nabla^q u) - \Delta^i \nabla^{q-1} (P(u)) \Delta^{j+2} \nabla^{q-1} u \; . \end{aligned}$$

Iterating these computations, we get with estimate (3.7) that $c_{ijq}^{k-1} \Delta^i \nabla^q (P(u)) \Delta^{j+1} \nabla^q u$ is of the form

$$f + \sum_{\substack{\tilde{j}, l \geq 0 \\ 1 \leq 2\tilde{j}+l \leq k}} \nabla^{(l)} \cdot \Delta^{\tilde{j}} g_{\tilde{j}l}$$

whenever $i + \frac{q}{2} > \frac{k}{2}$. The terms for i, q such that $i + \frac{q}{2} < \frac{k}{2}$ are estimated similarly. Moreover, we have

$$\Delta^{k-1}(A(u)(\nabla u, \nabla u)) = \nabla \cdot \Delta^\gamma g_{\gamma 1}, \text{with } \gamma = \frac{k}{2} - 1 \; ,$$

completing the case when k is even.

For k odd we distinguish between the three cases $i+\frac{q}{2}=\frac{k+1}{2}$, $i+\frac{q}{2}>\frac{k+1}{2}$ and $i+\frac{q}{2}<\frac{k+1}{2}$, and proceed similarly to the case when k is even. Moreover, we get

$$\Delta^{k-1}(A(u)(\nabla u,\nabla u))=\begin{cases} f & \text{for } k=1 \\ \Delta^{\gamma} g_{\gamma 0}, \quad \text{with } \gamma=\frac{k-1}{2} & \text{for } k\geq 3 \text{ odd}. \end{cases}$$

This completes the proof. $\square$

Remark 3.5 *Observe that harmonic maps ($k=1$) satisfy*

$$\Delta u=-A(u)(\nabla u,\nabla u) \text{ in } \mathcal{D}' .$$

Thus, the harmonic map equation is of the form $\Delta u=f$ with

$$f=-A(u)(\nabla u,\nabla u)\leq C|\nabla u|^2.$$

Weakly biharmonic maps ($k=2$) satisfy

$$\begin{aligned}\Delta^2 u &= \Delta P(u)\Delta u+2\nabla P(u)\Delta\nabla u-\Delta(A(u)(\nabla u,\nabla u)) \\ &= -\Delta P(u)\Delta u+\nabla\cdot(2\nabla P(u)\Delta u-\nabla(A(u)(\nabla u,\nabla u))) \text{ in } \mathcal{D}' ,\end{aligned}$$

i.e. the biharmonic map equation is of the form $\Delta^2 u=f+\nabla\cdot g_{01}$ with

$$f=-\Delta P(u)\Delta u\leq C|\nabla^2 u|^2$$

and

$$g_{01}=2\nabla P(u)\Delta u-\nabla(A(u)(\nabla u,\nabla u))\leq C|\nabla^2 u||\nabla u| .$$

However, we could also write the biharmonic map equation as

$$\Delta^2 u=-\Delta P(u)\Delta u+\nabla\cdot(2\nabla P(u)\Delta u)+\Delta(-A(u)(\nabla u,\nabla u)) \text{ in } \mathcal{D}',$$

i.e. it is also of the form $\Delta^2 u=f+\nabla\cdot g_{01}+\Delta g_{10}$ with

$$f=-\Delta P(u)\Delta u\leq C|\nabla^2 u|^2 ,$$

$$g_{01}=2\nabla P(u)\Delta u\leq C|\nabla^2 u||\nabla u|$$

and

$$g_{10}=-A(u)(\nabla u,\nabla u)\leq C|\nabla u|^2 .$$

This illustrates the non-uniqueness of the representation mentioned in Remark 3.4.

3.3 Morrey decay estimates

We will deduce Theorem 3.1 from the following

Proposition 3.6 *For $p > 1$, let $u \in W^{k,2p}(\Omega, N)$ be weakly k-harmonic. There exist $\epsilon > 0$ and $\tau \in (0,1)$ such that for each point $y_0 \in \Omega$ for which there exists a radius $r_0 > 0$ with*

$$\sum_{\mu=1}^{k} [\nabla^\mu u]_{M^{\frac{2k}{\mu},2k}(B_{r_0}(y_0))} \leq \epsilon \ ,$$

we have

$$(\tau r)^{2kp-m} \sum_{\mu=1}^{k} \int_{B_{\tau r}(x_0)} |\nabla^\mu u|^{\frac{2kp}{\mu}}\, dx \leq \frac{3}{4} r^{2kp-m} \sum_{\mu=1}^{k} \int_{B_r(x_0)} |\nabla^\mu u|^{\frac{2kp}{\mu}}\, dx \ , \tag{3.8}$$

for all $x_0 \in B_{r_0}(y_0)$, $0 < 4r < dist(x_0, \partial B_{r_0}(y_0))$.

Proof. We consider the k-harmonic extension v of u, defined as the unique solution to the following Dirichlet problem:

$$\begin{cases} \Delta^k v = 0 & \text{in } B_r(x_0) \\ \frac{\partial^l v}{\partial \nu^l} = \frac{\partial^l u}{\partial \nu^l} & \text{on } \partial B_r(x_0) \ , \end{cases}$$

for $0 \leq l \leq k-1$, where ν denotes the unit normal vector to $\partial B_r(x_0)$. According to Lemma B.4, we have

$$\int_{B_\rho(x_0)} |\nabla^\mu v|^{\frac{2kp}{\mu}}\, dx \leq C \left(\frac{\rho}{r}\right)^m \sum_{\lambda=1}^{k} \int_{B_{\frac{r}{4}}(x_0)} |\nabla^\lambda v|^{\frac{2kp}{\lambda}}\, dx \tag{3.9}$$

for $0 < \rho \leq \frac{r}{4}$ and $1 \leq \mu \leq k$. It follows that

$$\begin{aligned} &\sum_{\mu=1}^{k} \int_{B_\rho(x_0)} |\nabla^\mu u|^{\frac{2kp}{\mu}}\, dx \\ &\leq C \sum_{\mu=1}^{k} \int_{B_\rho(x_0)} |\nabla^\mu v|^{\frac{2kp}{\mu}}\, dx + C \sum_{\mu=1}^{k} \int_{B_\rho(x_0)} |\nabla^\mu (u-v)|^{\frac{2kp}{\mu}}\, dx \\ &\leq C \left(\frac{\rho}{r}\right)^m \sum_{\mu=1}^{k} \int_{B_{\frac{r}{4}}(x_0)} |\nabla^\mu v|^{\frac{2kp}{\mu}}\, dx + C \sum_{\mu=1}^{k} \int_{B_{\frac{r}{4}}(x_0)} |\nabla^\mu (u-v)|^{\frac{2kp}{\mu}}\, dx \\ &\leq C \left(\frac{\rho}{r}\right)^m \sum_{\mu=1}^{k} \int_{B_r(x_0)} |\nabla^\mu u|^{\frac{2kp}{\mu}}\, dx + C \sum_{\mu=1}^{k} \int_{B_{\frac{r}{4}}(x_0)} |\nabla^\mu (u-v)|^{\frac{2kp}{\mu}}\, dx \ . \end{aligned} \tag{3.10}$$

In view of Lemma 3.5, we introduce the auxiliary maps u_f and $u_{g_{jl}}$ for all $j, l \geq 0$ such that $1 \leq 2j + l \leq k$ as the solutions to the Dirichlet problems

$$\begin{aligned} \Delta^k u_f &= f && \text{with} \quad u_f - u \in W_0^{k,2p}(B_r(x_0)) \ , \\ \Delta^k u_{g_{jl}} &= \nabla^{(l)} \cdot \Delta^j g_{jl} && \text{with} \quad u_{g_{jl}} \in W_0^{k,2p}(B_r(x_0)) \ , \end{aligned}$$

where f and g_{jl} satisfy (3.5). Observe that the uniqueness of the Dirichlet problem implies

$$u = u_f + \sum_{\substack{j,l \geq 0 \\ 1 \leq 2j+l \leq k}} u_{g_{jl}} \ . \tag{3.11}$$

Moreover, $u_f - v \in W_0^{k,2p}(B_r(x_0))$ satisfies

$$\Delta^k(u_f - v) = f \ .$$

Lemma 3.5, Hölder's inequality and Nirenberg's interpolation inequality (A.1) give

$$\begin{aligned}
\|f\|_{L^p(B_r(x_0))} &\leq C \left\| \sum_{\lambda\in\Lambda} \prod_{\mu=0}^{k} |\nabla^\mu u|^{\gamma_{\lambda,\mu}} \right\|_{L^p(B_r(x_0))} \\
&\leq C \sum_{\lambda\in\Lambda} \prod_{\mu=0}^{k} \|\nabla^\mu u\|^{\gamma_{\lambda,\mu}}_{L^{\frac{2kp}{\mu}}(B_r(x_0))} \\
&\leq C \sum_{\lambda\in\Lambda} \prod_{\mu=0}^{k} \|u\|^{\gamma_{\lambda,\mu}(1-\frac{\mu}{k})}_{L^\infty(B_r(x_0))} \|u\|^{\frac{\mu\gamma_{\lambda,\mu}}{k}}_{W^{k,2p}(B_r(x_0))} \\
&\leq C\|u\|^2_{W^{k,2p}(B_r(x_0))} < \infty \ ,
\end{aligned}$$

and

$$\begin{aligned}
\|g_{jl}\|_{L^{r_{jl}}(B_r(x_0))} &\leq C \left\| \sum_{\lambda\in\Lambda} \prod_{\mu=0}^{k} |\nabla^\mu u|^{\gamma_{\lambda,\mu}} \right\|_{L^{r_{jl}}(B_r(x_0))} \\
&\leq C \sum_{\lambda\in\Lambda} \prod_{\mu=0}^{k} \|\nabla^\mu u\|^{\gamma_{\lambda,\mu}}_{L^{\frac{2kp}{\mu}}(B_r(x_0))} \\
&\leq C \sum_{\lambda\in\Lambda} \prod_{\mu=0}^{k} \|u\|^{\gamma_{\lambda,\mu}(1-\frac{\mu}{k})}_{L^\infty(B_r(x_0))} \|u\|^{\frac{\mu\gamma_{\lambda,\mu}}{k}}_{W^{k,2p}(B_r(x_0))} \\
&\leq C\|u\|^{\eta_{jl}}_{W^{k,2p}(B_r(x_0))} < \infty \ ,
\end{aligned} \tag{3.12}$$

where

$$r_{jl} = \frac{2kp}{2k-(2j+l)} \quad \text{and} \quad 1 \leq \eta_{jl} := \frac{2k-(2j+l)}{k} < 2 \ .$$

Thus, Corollary B.2 and Lemma B.7 imply that

$$u_f - v \in W^{2k,p}(B_r(x_0)), \quad u_{g_{jl}} \in W^{2k-(2j+l),r_{jl}}(B_r(x_0)) \ ,$$

and

$$\begin{aligned}
\|\nabla^{2k}(u_f - v)\|_{L^p(B_r(x_0))} &\leq C\|f\|_{L^p(B_r(x_0))} \ , \\
\|\nabla^{2k-(2j+l)} u_{g_{jl}}\|_{L^{r_{jl}}(B_r(x_0))} &\leq C\|g_{jl}\|_{L^{r_{jl}}(B_r(x_0))} \ .
\end{aligned} \tag{3.13}$$

We remark that the only place where we need $p > 1$ is to ensure the first estimate for $u_f - v$.
We have

$$\int_{B_{\frac{r}{4}}(x_0)} |\nabla^{\tilde\mu}(u-v)|^{\frac{2kp}{\tilde\mu}}\,dx \leq C\int_{B_{\frac{r}{4}}(x_0)} |\nabla^{\tilde\mu}(u_f-v)|^{\frac{2kp}{\tilde\mu}}\,dx + C\sum_{\substack{j,l\geq 0\\ 1\leq 2j+l\leq k}} \int_{B_{\frac{r}{4}}(x_0)} |\nabla^{\tilde\mu} u_{g_{jl}}|^{\frac{2kp}{\tilde\mu}}\,dx \tag{3.14}$$

for $1 \leq \tilde{\mu} \leq k$. We apply the Gagliardo-Nirenberg interpolation inequality (A.2), Lemma B.3, estimates (3.13) and Lemma 3.5 to obtain

$$\begin{aligned}
\|\nabla^{\tilde{\mu}}(u_f - v)\|_{L^{\frac{2kp}{\tilde{\mu}}}(B_{\frac{r}{4}}(x_0))} &\leq C\,[u_f - v]^{1-\frac{\tilde{\mu}}{2k}}_{BMO(B_{\frac{r}{4}}(x_0))}\,\|u_f - v\|^{\frac{\tilde{\mu}}{2k}}_{W^{2k,p}(B_{\frac{r}{4}}(x_0))} \\
&\leq C\,[u_f - v]^{1-\frac{\tilde{\mu}}{2k}}_{BMO(B_{\frac{r}{4}}(x_0))}\,\|\nabla^{2k}(u_f - v)\|^{\frac{\tilde{\mu}}{2k}}_{L^p(B_r(x_0))} \\
&\leq C\,[u_f - v]^{1-\frac{\tilde{\mu}}{2k}}_{BMO(B_{\frac{r}{4}}(x_0))}\,\|f\|^{\frac{\tilde{\mu}}{2k}}_{L^p(B_r(x_0))} \\
&\leq C\,[u_f - v]^{1-\frac{\tilde{\mu}}{2k}}_{BMO(B_{\frac{r}{4}}(x_0))}\,\left\|\sum_{\lambda\in\Lambda}\prod_{\mu=0}^{k}|\nabla^{\mu}u|^{\gamma_{\lambda,\mu}}\right\|^{\frac{\tilde{\mu}}{2k}}_{L^p(B_r(x_0))},
\end{aligned}$$

with $\sum_{\mu}\mu\gamma_{\lambda,\mu} = 2k$ for every $\lambda \in \Lambda$. Next we use Hölder's inequality and Young's inequality to deduce

$$\begin{aligned}
\left\|\sum_{\lambda\in\Lambda}\prod_{\mu=0}^{k}|\nabla^{\mu}u|^{\gamma_{\lambda,\mu}}\right\|_{L^p(B_r(x_0))} &\leq \sum_{\lambda\in\Lambda}\prod_{\mu=0}^{k}\|\nabla^{\mu}u\|^{\gamma_{\lambda,\mu}}_{L^{\frac{2kp}{\mu}}(B_r(x_0))} \\
&\leq C\sum_{\mu=1}^{k}\|\nabla^{\mu}u\|^{\frac{2k}{\mu}}_{L^{\frac{2kp}{\mu}}(B_r(x_0))},
\end{aligned}$$

where we remark that $\sum_{\mu=1}^{k}\frac{\mu\gamma_{\lambda,\mu}}{2k} = 1$ and the L^{∞}-norm of u enters into the constant C. Combining the above estimates we obtain

$$\int_{B_{\frac{r}{4}}(x_0)}|\nabla^{\tilde{\mu}}(u_f - v)|^{\frac{2kp}{\tilde{\mu}}}\,dx \leq C\,[u_f - v]^{\left(1-\frac{\tilde{\mu}}{2k}\right)\frac{2kp}{\tilde{\mu}}}_{BMO(B_{\frac{r}{4}}(x_0))}\sum_{\mu=1}^{k}\int_{B_r(x_0)}|\nabla^{\mu}u|^{\frac{2kp}{\mu}}\,dx\ . \tag{3.15}$$

For the second term in (3.14), as before, the Gagliardo-Nirenberg interpolation inequality (A.2) gives

$$\|\nabla^{\tilde{\mu}}u_{g_{jl}}\|_{L^{\frac{2kp}{\tilde{\mu}}}(B_{\frac{r}{4}}(x_0))} \leq C\,\left[u_{g_{jl}}\right]^{1-\frac{\tilde{\mu}}{k}\theta_{jl}}_{BMO(B_{\frac{r}{4}}(x_0))}\,\|u_{g_{jl}}\|^{\frac{\tilde{\mu}}{k}\theta_{jl}}_{W^{2k-(2j+l),r_{jl}}(B_{\frac{r}{4}}(x_0))}, \tag{3.16}$$

where

$$\frac{1}{2} < \theta_{jl} := \frac{k}{2k-(2j+l)} = \eta_{jl}^{-1} \leq 1$$

for all $j, l \geq 0$ such that $1 \leq 2j + l \leq k$. Furthermore, we again apply Lemma B.3 with $\mu = 2k-(2j+l)$, estimates (3.13), Lemma 3.5, Hölder's inequality and Young's inequality to get

$$\begin{aligned}
\|u_{g_{jl}}\|_{W^{2k-(2j+l),r_{jl}}(B_r(x_0))} &\le \|\nabla^{2k-(2j+l)}u_{g_{jl}}\|_{L^{r_{jl}}(B_r(x_0))} \\
&\le C\|g_{jl}\|_{L^{r_{jl}}(B_r(x_0))} \\
&\le C\left\|\sum_{\lambda\in\Lambda}\prod_{\mu=0}^{k}|\nabla^\mu u|^{r_{jl}\gamma_{\lambda,\mu}}\right\|_{L^1(B_r(x_0))}^{\frac{1}{r_{jl}}} \\
&\le C\sum_{\lambda\in\Lambda}\prod_{\mu=0}^{k}\||\nabla^\mu u|^{r_{jl}\gamma_{\lambda,\mu}}\|_{L^{\frac{2kp}{r_{jl}\mu\gamma_{\lambda,\mu}}}(B_r(x_0))}^{\frac{1}{r_{jl}}} \\
&\le C\sum_{\lambda\in\Lambda}\prod_{\mu=0}^{k}\|\nabla^\mu u\|_{L^{\frac{2kp}{\mu}}(B_r(x_0))}^{\gamma_{\lambda,\mu}} \\
&\le C\sum_{\mu=1}^{k}\|\nabla^\mu u\|_{L^{\frac{2kp}{\mu}}(B_r(x_0))}^{\frac{k\eta_{jl}}{\mu}}\ , \qquad (3.17)
\end{aligned}$$

where

$$\eta_{jl} := \frac{2k-(2j+l)}{k} = \theta_{jl}^{-1}.$$

Combining (3.16) and (3.17) gives

$$\int_{B_{\frac{r}{4}}(x_0)}|\nabla^{\tilde\mu}u_{g_{jl}}|^{\frac{2kp}{\tilde\mu}}\,dx \le C\left[u_{g_{jl}}\right]_{BMO(B_{\frac{r}{4}}(x_0))}^{\left(1-\frac{\tilde\mu}{k}\theta_{jl}\right)\frac{2kp}{\tilde\mu}}\sum_{\mu=1}^{k}\int_{B_r(x_0)}|\nabla^\mu u|^{\frac{2kp}{\mu}}\,dx\ . \qquad (3.18)$$

Thus, we conclude with (3.14), (3.15) and (3.18) that

$$\begin{aligned}
&\int_{B_{\frac{r}{4}}(x_0)}|\nabla^{\tilde\mu}(u-v)|^{\frac{2kp}{\tilde\mu}}\,dx \qquad (3.19)\\
&\le C\left([u_f-v]_{BMO(B_{\frac{r}{4}}(x_0))}^{\left(1-\frac{\tilde\mu}{2k}\right)\frac{2kp}{\tilde\mu}}+\sum_{j,l}\left[u_{g_{jl}}\right]_{BMO(B_{\frac{r}{4}}(x_0))}^{\left(1-\frac{\tilde\mu}{k}\theta_{jl}\right)\frac{2kp}{\tilde\mu}}\right)\sum_{\mu=1}^{k}\int_{B_r(x_0)}|\nabla^\mu u|^{\frac{2kp}{\mu}}\,dx\ .
\end{aligned}$$

From Lemma 3.8 below we infer

$$[u_f-v]_{BMO(B_{\frac{r}{4}}(x_0))}^{\left(1-\frac{\tilde\mu}{2k}\right)\frac{2kp}{\tilde\mu}}+\sum_{j,l}\left[u_{g_{jl}}\right]_{BMO(B_{\frac{r}{4}}(x_0))}^{\left(1-\frac{\tilde\mu}{k}\theta_{jl}\right)\frac{2kp}{\tilde\mu}}\le C\epsilon^\beta \qquad (3.20)$$

for some $\beta > 0$, all $1 \le \tilde\mu \le k$ and all $j, l \ge 0$ such that $1 \le 2j + l \le k$. Then we can combine this with inequalities (3.10) and (3.19) into

$$\rho^{2kp-m}\sum_{\mu=1}^{k}\int_{B_\rho(x_0)}|\nabla^\mu u|^{\frac{2kp}{\mu}}\,dx \le C\left(\left(\frac{\rho}{r}\right)^{2kp}+\epsilon^\beta\left(\frac{\rho}{r}\right)^{2kp-m}\right)r^{2kp-m}\sum_{\mu=1}^{k}\int_{B_r(x_0)}|\nabla^\mu u|^{\frac{2kp}{\mu}}\,dx.$$

We conclude the proof of this proposition by setting $\tau := \frac{\rho}{\tau}$ equal to $(2C)^{-\frac{1}{2kp}}$ and choosing $\epsilon > 0$ sufficiently small so that $C\epsilon^\beta\tau^{2kp-m} \le \frac{1}{4}$. □

Now it remains to show (3.20). In a first step, we prove the subsequent lemma.

Lemma 3.7 *There exists a constant $C > 1$ such that*

$$\sum_{\mu=1}^{k} \left[\nabla^\mu u_{g_{jl}}\right]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_r(x_0))} \le C \sum_{\mu=1}^{k} \left[\nabla^\mu u\right]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_{4r}(x_0))} + Cr^{2k-m} \sum_{\mu=1}^{k} \int_{B_r(x_0)} |\nabla^\mu u_{g_{jl}}|^{\frac{2k}{\mu}}\, dx \ . \tag{3.21}$$

Proof. For $B_s(x) \subset B_{4r}(x_0)$, we consider the k-harmonic extension v_{jl} of $u_{g_{jl}}$ on $B_s(x)$, where $u_{g_{jl}}$ is set to zero outside $B_r(x_0)$. Then $w_{jl} := u_{g_{jl}} - v_{jl} \in W_0^{k,2}(B_s(x))$ satisfies

$$\Delta^k w_{jl} = \nabla^{(l)} \cdot \Delta^j \tilde{g}_{jl} \ ,$$

where $\tilde{g}_{jl} := \chi_{B_r(x_0)} g_{jl}$. Analogously to (3.12) we conclude that $w_{jl} \in W^{2k-(2j+l),r_{jl}}(B_s(x))$, and similarly to (3.17) we get

$$\|w_{jl}\|_{W^{2k-(2j+l),r_{jl}}(B_s(x))} \le C \sum_{\mu=1}^{k} \|\nabla^\mu u\|^{\frac{k\eta_{jl}}{\mu}}_{L^{\frac{2k}{\mu}}(B_s(x))} \ .$$

Here and henceforth in the proof of Lemma 3.7, we set $p = 1$. Observe that the second estimate in (3.13) is still valid in this case.
Applying the Gagliardo-Nirenberg inequality (A.2) and the preceding estimate gives

$$\begin{aligned}
\int_{B_{\frac{s}{4}}(x)} |\nabla^\lambda w_{jl}|^{\frac{2k}{\lambda}}\, dx &\le C\, [w_{jl}]^{2k\left(\frac{1}{\lambda} - \frac{1}{2k-(2j+l)}\right)}_{BMO(B_{\frac{s}{4}}(x))} \sum_{\mu=0}^{2k-(2j+l)} \int_{B_{\frac{s}{4}}(x)} |\nabla^\mu w_{jl}|^{r_{jl}}\, dx \\
&\le C\, [w_{jl}]^{2k\left(\frac{1}{\lambda} - \frac{1}{2k-(2j+l)}\right)}_{BMO(B_{\frac{s}{4}}(x))} \sum_{\mu=1}^{k} \int_{B_s(x)} |\nabla^\mu u|^{\frac{2k}{\mu}}\, dx
\end{aligned} \tag{3.22}$$

for $1 \le \lambda \le k$. As v_{jl} is the k-harmonic extension of $u_{g_{jl}}$, we obtain with Hölder's inequality, Poincaré's inequality and Lemma B.6

$$\begin{aligned}
[w_{jl}]^{2k}_{BMO(B_{\frac{s}{4}}(x))} &\le \sup_{B \subset B_{\frac{s}{2}}(x)} \left\{ \fint_B |w_{jl} - \overline{w_{jl}}_B|\, dx \right\}^{2k} \\
&\le C\, [\nabla w_{jl}]^{2k}_{M^{2,2}(B_{\frac{s}{2}}(x))} \\
&\le C\, \left[\nabla u_{g_{jl}}\right]^{2k}_{M^{2,2}(B_{\frac{s}{2}}(x))} + C\, [\nabla v_{jl}]^{2k}_{M^{2,2}(B_{\frac{s}{2}}(x))} \\
&\le C\, \left[\nabla u_{g_{jl}}\right]^{2k}_{M^{2,2}(B_{\frac{s}{2}}(x))} + C \left(\sum_{\mu=1}^{k} \int_{B_s(x)} |\nabla^\mu u_{g_{jl}}|^{\frac{2k}{\mu}}\, dx \right)^{\mu} .
\end{aligned} \tag{3.23}$$

Arguing with the help of Poincaré's inequality and (3.12) we show that

$$\|u_{g_{jl}}\|_{W^{k,2}(B_r(x_0))} \le C \|\nabla^k u_{g_{jl}}\|_{L^2(B_r(x_0))} \le C \|g_{jl}\|_{L^2(B_r(x_0))} \le C \|u\|_{W^{k,2}(B_r(x_0))} \le C\epsilon \le 1 \ ,$$

provided $\epsilon > 0$ is sufficiently small. Using this we can omit the exponent μ in the last term of (3.23) and estimate

$$[w_{jl}]^{2k}_{BMO(B_{\frac{s}{4}}(x))} \le C \sum_{\mu=1}^{k} \left[\nabla^\mu u_{g_{jl}}\right]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_s(x))} \ . \tag{3.24}$$

Combining estimates (3.22) and (3.24) with Young's inequality yields

$$\begin{aligned}\int_{B_{\frac{s}{4}}(x)} |\nabla^\lambda w_{jl}|^{\frac{2k}{\lambda}}\,dx &\le C\left(\sum_{\mu=1}^k \left[\nabla^\mu u_{g_{jl}}\right]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_s(x))}\right)^{\left(\frac{1}{\lambda}-\frac{1}{2k-(2j+l)}\right)} \sum_{\mu=1}^k \int_{B_s(x)} |\nabla^\mu u|^{\frac{2k}{\mu}}\,dx \\ &\le C\gamma \sum_{\mu=1}^k \left[\nabla^\mu u_{g_{jl}}\right]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_s(x))} + C(\gamma)\sum_{\mu=1}^k \|\nabla^\mu u\|^{\frac{2k\overline{C}(\lambda)}{\mu}}_{L^{\frac{2k}{\mu}}(B_s(x))}\,, \qquad (3.25)\end{aligned}$$

with $\gamma > 0$ and $\overline{C}(\lambda) > 1$ for $1 \le 2j+l,\ \lambda \le k$. With a rescaling argument this gives

$$\int_{B_{\frac{s}{4}}(x)} |\nabla^\lambda w_{jl}|^{\frac{2k}{\lambda}}\,dx \le C s^{m-2k}\gamma \sum_{\mu=1}^k \left[\nabla^\mu u_{g_{jl}}\right]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_s(x))} + C(\gamma)\sum_{\mu=1}^k \int_{B_s(x)} |\nabla^\mu u|^{\frac{2k}{\mu}}\,dx\,, \qquad (3.26)$$

where the constants are now independent of s and λ. Here we also used the fact that $\|\nabla^\mu u\|_{L^{\frac{2k}{\mu}}(B_{4r}(x_0))} < 1$ for $1 \le \mu \le k$, provided $\epsilon > 0$ is sufficiently small. Combining estimate (3.26) with Lemma B.4, we estimate for $0 < \rho \le \frac{s}{4}$ as in (3.10)

$$\begin{aligned}\sum_{\mu=1}^k \int_{B_\rho(x)} |\nabla^\mu u_{g_{jl}}|^{\frac{2k}{\mu}}\,dx &\le C\left(\frac{\rho}{s}\right)^m \sum_{\mu=1}^k \int_{B_{\frac{s}{4}}(x)} |\nabla^\mu u_{g_{jl}}|^{\frac{2k}{\mu}}\,dx + C\sum_{\mu=1}^k \int_{B_{\frac{s}{4}}(x)} |\nabla^\mu w_{jl}|^{\frac{2k}{\mu}}\,dx \\ &\le C\left(\frac{\rho}{s}\right)^m \sum_{\mu=1}^k \int_{B_s(x)} |\nabla^\mu u_{g_{jl}}|^{\frac{2k}{\mu}}\,dx + C(\gamma)\sum_{\mu=1}^k \int_{B_s(x)} |\nabla^\mu u|^{\frac{2k}{\mu}}\,dx \\ &\quad + C s^{m-2k}\gamma \sum_{\mu=1}^k \left[\nabla^\mu u_{g_{jl}}\right]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_s(x))}\,. \qquad (3.27)\end{aligned}$$

The proof of the lemma is completed by the following iteration argument. To simplify notation, we define $T(\rho) := \rho^{2k-m}\sum_{\mu=1}^k \int_{B_\rho(x)} |\nabla^\mu u_{g_{jl}}|^{\frac{2k}{\mu}}\,dx$ so that the above estimate becomes

$$\begin{aligned}T(\rho) &\le C\left(\frac{\rho}{s}\right)^{2k} T(s) + C(\gamma)\left(\frac{\rho}{s}\right)^{2k-m} s^{2k-m}\sum_{\mu=1}^k \int_{B_s(x)} |\nabla^\mu u|^{\frac{2k}{\mu}}\,dx \\ &\quad + C\gamma\left(\frac{\rho}{s}\right)^{2k-m} \sum_{\mu=1}^k \left[\nabla^\mu u_{g_{jl}}\right]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_s(x))}\,.\end{aligned}$$

Next we choose $\tau := \frac{\rho}{s}$ sufficiently small such that $C\tau^{2k} \le \frac{1}{2}$, which implies

$$\begin{aligned}T(\tau s) &\le \frac{1}{2}T(s) + C(\gamma)\tau^{2k-m}s^{2k-m}\sum_{\mu=1}^k \int_{B_s(x)} |\nabla^\mu u|^{\frac{2k}{\mu}}\,dx \\ &\quad + C\gamma\tau^{2k-m} \sum_{\mu=1}^k \left[\nabla^\mu u_{g_{jl}}\right]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_s(x))}\,. \qquad (3.28)\end{aligned}$$

Now we consider $B_\sigma(x) \subset B_{2r}(x_0)$. There exists $i \in \mathbf{N}$ (simply set $i = \left\lfloor \log \frac{\sigma}{2r} / \log \tau \right\rfloor$) with $\tau^{i+1}2r \leq \sigma \leq \tau^i 2r$ and note that therefore $(\tau^i 2r)^{2k-m} \leq \sigma^{2k-m} \leq (\tau^{(i+1)}2r)^{(2k-m)}$. We estimate

$$\begin{aligned} T(\sigma) &= \sigma^{2k-m} \sum_{\mu=1}^{k} \int_{B_\sigma(x)} |\nabla^\mu u_{g_{jl}}|^{\frac{2k}{\mu}}\, dx \\ &\leq \left(\tau^{(i+1)}2r\right)^{(2k-m)} \sum_{\mu=1}^{k} \int_{B_{\tau^i 2r}(x)} |\nabla^\mu u_{g_{jl}}|^{\frac{2k}{\mu}}\, dx \\ &\leq CT(\tau^i 2r)\ . \end{aligned} \tag{3.29}$$

Furthermore, estimate (3.28) gives

$$\begin{aligned} T(\tau^i 2r) &\leq \frac{1}{2} T(\tau^{i-1}2r) + C(\gamma)\,(\tau 2r)^{2k-m}\,\tau^{(i-1)(2k-m)} \sum_{\mu=1}^{k} \int_{B_{\tau^{i-1}2r}(x)} |\nabla^\mu u|^{\frac{2k}{\mu}}\, dx \\ &\quad + C\gamma\tau^{2k-m} \sum_{\mu=1}^{k} \left[\nabla^\mu u_{g_{jl}}\right]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_{\tau^{i-1}2r}(x))} \\ &\leq \frac{1}{2} T(\tau^{i-1}2r) + C(\gamma)\tau^{2k-m} \sum_{\mu=1}^{k} [\nabla^\mu u]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_{4r}(x_0))} \\ &\quad + C\gamma\tau^{2k-m} \sum_{\mu=1}^{k} \left[\nabla^\mu u_{g_{jl}}\right]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_r(x_0))}\ , \end{aligned}$$

where in the last term we used the fact that $u_{g_{jl}}$ is supported in $B_r(x_0)$ allowing us to reduce to the Morrey norm on $B_r(x_0)$. Setting

$$S := C(\gamma)\tau^{2k-m} \sum_{\mu=1}^{k} [\nabla^\mu u]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_{4r}(x_0))} + C\gamma\tau^{2k-m} \sum_{\mu=1}^{k} \left[\nabla^\mu u_{g_{jl}}\right]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_r(x_0))}$$

the above becomes

$$T(\tau^i 2r) \leq \frac{1}{2} T(\tau^{i-1}2r) + S(r)$$

which after iteration yields

$$\begin{aligned} T(\tau^i 2r) &\leq T(2r) + \sum_{\tilde\mu=1}^{i} \frac{1}{2^{\tilde\mu}} S \\ &\leq T(2r) + S\ . \end{aligned}$$

Combining this with (3.29) gives

$$\begin{aligned} T(\sigma) &\leq C(\gamma) \sum_{\mu=1}^{k} [\nabla^\mu u]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_{4r}(x_0))} + Cr^{2k-m} \sum_{\mu=1}^{k} \int_{B_r(x_0)} |\nabla^\mu u_{g_{jl}}|^{\frac{2k}{\mu}}\, dx \\ &\quad + C\gamma \sum_{\mu=1}^{k} \left[\nabla^\mu u_{g_{jl}}\right]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_r(x_0))}\ . \end{aligned}$$

The desired result now follows by taking the supremum over all such balls $B_\sigma(x)$, and choosing $\gamma > 0$ sufficiently small to absorb the last term on the right-hand side. $\square$

Now we are able to complete the proof of Proposition 3.6 with the following

Lemma 3.8 *We have*

$$[u_f - v]_{BMO(B_{\frac{r}{4}}(x_0))} + \sum_{j,l} \left[u_{g_{jl}}\right]_{BMO(B_{\frac{r}{4}}(x_0))} \le C\epsilon^\beta$$

for some $\beta > 0$ and all $j, l \ge 0$ such that $1 \le 2j + l \le k$.

Proof. Similarly to (3.22) and (3.25) we estimate

$$\int_{B_r(x_0)} |\nabla^\lambda u_{g_{jl}}|^{\frac{2k}{\lambda}}\, dx \le C\gamma \left[u_{g_{jl}}\right]^{2k}_{BMO(B_r(x_0))} + C(\gamma) \sum_{\mu=1}^{k} \|\nabla^\mu u\|^{\frac{2k}{\mu}}_{L^{\frac{2k}{\mu}}(B_r(x_0))} \tag{3.30}$$

for $1 \le \lambda \le k$ and every $\gamma > 0$. Lemma 3.7, estimate (3.30) and Poincaré's inequality give

$$\begin{aligned}
\left[\nabla u_{g_{jl}}\right]^{2k}_{M^{2k,2k}(B_r(x_0))} &\le C\gamma \left[u_{g_{jl}}\right]^{2k}_{BMO(B_r(x_0))} + C(\gamma) \sum_{\mu=1}^{k} \left[\nabla^\mu u\right]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_{4r}(x_0))} \\
&\le C\gamma \left[u_{g_{jl}}\right]^{2k}_{BMO(\mathbf{R}^m)} + C(\gamma) \sum_{\mu=1}^{k} \left[\nabla^\mu u\right]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_{4r}(x_0))} \\
&\le C\gamma \left[\nabla u_{g_{jl}}\right]^{2k}_{M^{2k,2k}(B_r(x_0))} + C(\gamma) \sum_{\mu=1}^{k} \left[\nabla^\mu u\right]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_{4r}(x_0))}
\end{aligned}$$

which for $\gamma > 0$ sufficiently small implies

$$\left[\nabla u_{g_{jl}}\right]^{2k}_{M^{2k,2k}(B_r(x_0))} \le C \sum_{\mu=1}^{k} \left[\nabla^\mu u\right]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_{4r}(x_0))} .$$

Applying Hölder's inequality and Poincaré's inequality together with the above estimate we infer

$$\begin{aligned}
\left[u_{g_{jl}}\right]^{2k}_{BMO(B_{\frac{r}{4}}(x_0))} &\le C \left[\nabla u_{g_{jl}}\right]^{2k}_{M^{2k,2k}(B_r(x_0))} \\
&\le C \sum_{\mu=1}^{k} \left[\nabla^\mu u\right]^{\frac{2k}{\mu}}_{M^{\frac{2k}{\mu},2k}(B_{4r}(x_0))} \\
&\le C\epsilon^\beta
\end{aligned}$$

for some $\beta > 0$. From (3.11), we deduce

$$[u_f - v]_{BMO(B_{\frac{r}{4}}(x_0))} \le C \sum_{j,l \ge 0} \left[u_{g_{jl}}\right]_{BMO(B_{\frac{r}{4}}(x_0))} + C\,[u - v]_{BMO(B_{\frac{r}{4}}(x_0))} . \tag{3.31}$$

Hölder's inequality and Poincaré's inequality imply

$$\begin{aligned}
[u-v]^{2k}_{BMO(B_{\frac{r}{4}}(x_0))} &\le C\,[\nabla(u-v)]^{2k}_{M^{2,2}(B_{\frac{r}{2}}(x_0))} \\
&\le C\left([\nabla u]^{2k}_{M^{2,2}(B_{\frac{r}{2}}(x_0))} + [\nabla v]^{2k}_{M^{2,2}(B_{\frac{r}{2}}(x_0))}\right). \qquad (3.32)
\end{aligned}$$

As v is the k-harmonic extension of u, Lemma B.6 and Hölder's inequality yield

$$\begin{aligned}
[\nabla u]^{2k}_{M^{2,2}(B_{\frac{r}{2}}(x_0))} + [\nabla v]^{2k}_{M^{2,2}(B_{\frac{r}{2}}(x_0))} &\le C\sum_{\mu=1}^{k}[\nabla^\mu u]^{2k}_{M^{2,2\mu}(B_{2r}(x_0))} \\
&\le C\sum_{\mu=1}^{k}[\nabla^\mu u]^{2k}_{M^{\frac{2k}{\mu},2k}(B_{2r}(x_0))} \\
&\le C\epsilon^\beta \qquad (3.33)
\end{aligned}$$

for some $\beta > 0$. Estimate (3.20) now follows from (3.31)-(3.33). This completes the proof. □

3.4 Proof of Theorem 3.1

Theorem 3.1 now follows from Proposition 3.6 (compare with section 2.5 in chapter 2). Consider $x_0 \in B_{\frac{r_0}{4}}(y_0)$. An iteration of estimate (3.8) implies

$$\left(\frac{\tau^i r_0}{4}\right)^{2kp-m}\sum_{\mu=1}^{k}\int_{B_{\frac{\tau^i r_0}{4}}(x_0)}|\nabla^\mu u|^{\frac{2kp}{\mu}}\,dx \le \left(\frac{3}{4}\right)^i\left(\frac{r_0}{4}\right)^{2kp-m}\sum_{\mu=1}^{k}\int_{B_{\frac{r_0}{4}}(x_0)}|\nabla^\mu u|^{\frac{2kp}{\mu}}\,dx \qquad (3.34)$$

for all i. For $0<\rho\le\frac{r_0}{4}$, choose $i\in\mathbf{N}$ such that $\tau^{i+1}\frac{r_0}{4}\le\rho\le\tau^i\frac{r_0}{4}$, set $\gamma=\frac{\log 3-\log 4}{2\log\tau}>0$ and observe that $\left(\frac{3}{4}\right)^i\le(\tau^i)^\gamma$. Thus, inequality (3.34) implies

$$\begin{aligned}
\rho^{2kp-m}\sum_{\mu=1}^{k}\int_{B_\rho(x_0)}|\nabla^\mu u|^{\frac{2kp}{\mu}}\,dx &\le \tau^{2kp-m}\left(\frac{\tau^i r_0}{4}\right)^{2kp-m}\sum_{\mu=1}^{k}\int_{B_{\frac{\tau^i r_0}{4}}(x_0)}|\nabla^\mu u|^{\frac{2kp}{\mu}}\,dx \\
&\le \tau^{2kp-m}\left(\frac{r_0}{4}\right)^{2kp-m}\left(\frac{3}{4}\right)^i\sum_{\mu=1}^{k}\int_{B_{\frac{r_0}{4}}(x_0)}|\nabla^\mu u|^{\frac{2kp}{\mu}}\,dx \\
&\le \left(\frac{\tau r_0}{4}\right)^{2kp-m}(\tau^i)^\gamma\sum_{\mu=1}^{k}\int_{B_{r_0}(y_0)}|\nabla^\mu u|^{\frac{2kp}{\mu}}\,dx \\
&\le \left(\left(\frac{\tau r_0}{4}\right)^{2kp-m-\gamma}\sum_{\mu=1}^{k}\int_{B_{r_0}(y_0)}|\nabla^\mu u|^{\frac{2kp}{\mu}}\,dx\right)\rho^\gamma \\
&\le C\rho^\gamma\,.
\end{aligned}$$

for all $B_\rho(x_0)\subset B_{\frac{r_0}{4}}(y_0)$. Hence from Morrey's Dirichlet growth theorem, Theorem 3.5.2 in [60] by C.B. Morrey, we conclude that $u\in C^{0,\frac{\gamma}{p}}$ in a neighbourhood of y_0. The smoothness of u near y_0 follows now from elliptic bootstrapping arguments. This completes the proof of Theorem 3.1.

3.5 The harmonic and biharmonic cases

Here we give a derivation of Corollary 3.2. For stationary harmonic maps, i.e. $k = 1$, Corollary 3.2 follows from Theorem 3.1 and the monotonicity formula below.

Proposition 3.9 (Monotonicity formula [63]) *For $u \in W^{1,2}(B_2, N)$ stationary harmonic and $0 \le \rho \le r \le 1$, we have*

$$\rho^{2-m} \int_{B_\rho} |\nabla u|^2 \, dx \le r^{2-m} \int_{B_r} |\nabla u|^2 \, dx \ . \tag{3.35}$$

Indeed, consider $u \in W^{1,2p}(\Omega, \mathbf{R}^N)$ stationary harmonic and define the set

$$S := \left\{ x_0 \in \Omega : \limsup_{r \to 0} r^{2p-m} \int_{B_r(x_0)} |\nabla u|^{2p} \, dx \ge \gamma^p \right\} \ ,$$

with $\gamma > 0$ small. By Corollary 3.2.3 in the book by W. Ziemer [96] we have $\mathcal{H}^{m-2p}(S) = 0$. Applying Hölder's inequality, we get that for any $y_0 \in \Omega \setminus S$, there exists $R > 0$ such that

$$R^{2-m} \int_{B_R(y_0)} |\nabla u|^2 \, dx \le C \left(R^{2p-m} \int_{B_R(y_0)} |\nabla u|^{2p} \, dx \right)^{\frac{1}{p}} < \gamma.$$

Hence, the monotonicity formula (3.35) implies for $x_0 \in B_{\frac{R}{2}}(y_0)$ that

$$\begin{aligned} \rho^{2-m} \int_{B_\rho(x_0)} |\nabla u|^2 \, dx &\le C R^{2-m} \int_{B_{\frac{R}{2}}(x_0)} |\nabla u|^2 \, dx \\ &\le C \int_{B_R(y_0)} |\nabla u|^2 \, dx \\ &\le C_2 \gamma \ . \end{aligned}$$

Fix $\gamma := \frac{\epsilon}{C_2}$, where ϵ is given by Theorem 3.1, and the claim follows.

For stationary biharmonic maps, i.e. $k = 2$, we replace Proposition 3.9 by Theorem 3.10 below which was proved in [8] by G. Angelsberg.

Theorem 3.10 (Monotonicity formula) *For $K > 0$ and $u \in W^{2,2}(B_K, N)$ (extrinsically) stationary biharmonic, it holds for a.e. $0 < \rho < r \le \frac{K}{2}$*

$$r^{4-m} \int_{B_r} |\Delta u|^2 \, dx - \rho^{4-m} \int_{B_\rho} |\Delta u|^2 \, dx = P + R \ ,$$

where

$$\begin{aligned} P &= 4 \int_{B_r \setminus B_\rho} \left(\frac{(u_j + x^i u_{ij})^2}{|x|^{m-2}} + \frac{(m-2)(x^i u_i)^2}{|x|^m} \right) dx \\ R &= 2 \int_{\partial B_r \setminus \partial B_\rho} \left(-\frac{x^i u_j u_{ij}}{|x|^{m-3}} + 2 \frac{(x^i u_i)^2}{|x|^{m-1}} - 2 \frac{|\nabla u|^2}{|x|^{m-3}} \right) d\sigma \ . \end{aligned}$$

We observe that Nirenberg's interpolation inequality (A.1) implies that $\nabla u \in L^{4p}(\Omega)$, and define

$$S := \left\{ x_0 \in \Omega : \limsup_{r \to 0} r^{4p-m} \int_{B_r(x_0)} \left(|\nabla^2 u|^{2p} + |\nabla u|^{4p} \right) dx \geq \eta^p \right\}$$

with $\eta > 0$ small. By Corollary 3.2.3 in the book by W. Ziemer [96] we now have $\mathcal{H}^{m-4p}(S) = 0$. For any $y_0 \in \Omega \setminus S$, there exists $R > 0$ such that

$$R^{4-m} \int_{B_R(y_0)} (|\nabla u|^4 + |\nabla^2 u|)^2 \, dx \leq C \left(R^{4p-m} \int_{B_R(y_0)} (|\nabla u|^{4p} + |\nabla^2 u|^{2p}) \, dx \right)^{\frac{1}{p}} < \eta \, .$$

We have the following proposition by C.-Y. Wang in [91] and also M. Struwe in [82].

Proposition 3.11 *Let $u \in W^{2,2}(B_2, N)$ be stationary biharmonic and $0 < R \leq 1$. There exists $\epsilon > 0$ and $\rho_0 > 0$ sufficiently small such that if*

$$R^{4-m} \int_{B_R} (|\nabla u|^4 + |\nabla^2 u|^2) \, dx < \epsilon,$$

we have

$$[\nabla u]_{M^{4,4}(B_{\rho_0})} + [\nabla^2 u]_{M^{2,4}(B_{\rho_0})} \leq C\epsilon.$$

Thus the monotonicity formula implies the existence of $\rho_0 > 0$ and $\bar{\epsilon}_0 > 0$ such that if $R^{4-m} \int_{B_R(y_0)} (|\nabla u|^4 + |\nabla^2 u|^2) \, dx \leq \bar{\epsilon} \leq \bar{\epsilon}_0$, we have

$$\rho^{4-m} \int_{B_\rho(x_0)} (|\nabla u|^4 + |\nabla^2 u|^2) \, dx \leq C_3 \bar{\epsilon}$$

for all $B_\rho(x_0) \subset B_{\rho_0}(y_0)$. Fix $\bar{\epsilon} := \min(\frac{\epsilon}{C_3}, \bar{\epsilon}_0) > 0$ and $\eta := \bar{\epsilon} > 0$, where $\epsilon > 0$ is given by Theorem 3.1. This completes the proof of Corollary 3.2.

3.6 Applications to $2k$-th order elliptic systems

The aim of this section is to use the results from section 3.3 to prove Theorem 3.3. From the assumptions on the right-hand side of the equation we deduce that we have a structure similar to the one of the Euler-Lagrange equation for polyharmonic maps in Lemma 3.5 where this time the terms involving g_{jl} vanish. Therefore we would be able to apply our results from section 3.3 except for the lack of a priori higher integrability. In contrast to the case of polyharmonic maps, where such higher integrability had to be assumed, this time we will show a reverse Hölder inequality and deduce this higher regularity from it. More precisely, we have the following Lemma

Lemma 3.12 *Let $u \in W^{k,2}(\Omega, \mathbf{R}^N) \cap L^\infty(\Omega)$ be a solution of equation 3.3. Then there exists $\epsilon > 0$, $C > 0$ and $s > 0$ depending only on the dimensions and $\|u\|_{L^\infty(\Omega)}$ such that if u satisfies*

$$\sum_{\mu=1}^{k} [\nabla^\mu u]_{M^{\frac{2k}{\mu}, 2k}(B_{r_0}(y_0))} \leq \epsilon$$

for some $r_0 > 0$*, then*

$$\left(\fint_{B_r} \left(\sum_{\mu=1}^{k} |\nabla^\mu u|^{\frac{k}{\mu}} \right)^{2(1+s)} dx \right)^{\frac{1}{1+s}} \leq C \sum_{\mu=1}^{k} \fint_{B_{2r}} |\nabla^\mu u|^{\frac{2k}{\mu}} \, dx \tag{3.36}$$

for all balls $B_r \subset B_{4r} \subset\subset B_{r_0}(x_0)$*.*

In the course of the proof of this Lemma we will make use of the following Lemma by F. Gehring, M. Giaquinta and G. Modica stated in the book by M. Giaquinta [28] as Proposition V.1.1.

Lemma 3.13 *Let* $\Omega \subset \mathbf{R}^m$ *be a bounded open subset and let* $g \in L^p_{loc}(\Omega)$ *with* $1 < p < \infty$ *be a nonnegative function. There exists a constant* $\kappa_0 > 0$ *depending only on* m *and* p *with the following property: if for every ball* B_r *with* $B_{4r} \subset\subset \Omega$ *we have*

$$\fint_{B_r} g^p \, dx \leq b \left(\fint_{B_{4r}} g \, dx \right)^p + \kappa \fint_{B_{4r}} g^p \, dx$$

for some $b > 1$ *and* $0 < \kappa \leq \kappa_0$*, then there exists* $s_0 > 0$ *depending also on* $b > 1$ *so that* $g \in L^s_{loc}(\Omega)$ *for all* $1 < s < s_0$ *and*

$$\fint_{B_r} g^s \, dx \leq C \left(\fint_{B_{4r}} g^p \, dx \right)^{\frac{1}{p}} .$$

We are now in position to prove Lemma 3.12.

Proof. The proof is in some way similar to the proof of Lemma 3.2 in the paper by P. Strzelecki and A. Zatorska-Goldstein [85]. We begin by constructing a suitable test function for equation (3.3). Fixing a ball $B_{4r} \subset\subset \Omega$ we can assume it to be centred at $0 \in \mathbf{R}^m$. Take a cut-off function $\eta \in C_0^\infty(B_{2r})$ satisfying $0 \leq \eta \leq 1$, $\eta \equiv 1$ on B_r and $|\nabla^\mu \eta| \leq \frac{1}{r^\mu}$. We will construct the test function $\psi := \eta^k \left(u - P\left(x^{k-1}\right)\right)$ where $P\left(x^{k-1}\right)$ is a suitable polynomial of order $k-1$ in the entries of x with constant coefficients depending on u. The top coefficient p_{k-1} of $P\left(x^{k-1}\right)$ is defined to be $p_{k-1} := \overline{\nabla^k u}_{B_{2r}}$, which as before denotes the average, so that $\nabla^k \left(u - P\left(x^{k-1}\right)\right) = \nabla^k u - \overline{\nabla^k u}_{B_{2r}}$. The remaining coefficients are determined recursively by $p_{k-j} := \overline{\nabla^{k-j} u}_{B_{2r}} - \overline{\nabla^{k-j} P\left(x^{k-1}\right) - P(x^{k-j})}_{B_{2r}}$ so that this time $\nabla^{k-j}\left(u - P\left(x^{k-1}\right)\right) = \nabla^{k-j} u - \overline{\nabla^{k-j} u}_{B_{2r}} + \overline{\nabla^{k-j} P\left(x^{k-1}\right) - P\left(x^{k-j}\right)}_{B_{2r}} - P\left(x^{k-1}\right) - P(x^{k-j})$. The reason for this choice of coefficients is that it ensured the averages over B_{2r} of all derivatives of $u - P\left(x^{k-1}\right)$ to vanish, which will be of use later on.
Testing equation (3.3) with the ψ constructed above we obtain

$$\int_\Omega \nabla^k u \nabla^k \psi \, dx = \int_\Omega \psi Q(x, u, \nabla u, \ldots, \nabla^{k-1} u) \, dx \tag{3.37}$$

for which we will estimate right- and left-hand side separately. For the left-hand one we immediately compute that it equals

$$\int_{B_{2r}} \left[\eta^k |\nabla^k u|^2 + \sum_{j=1}^{k} \nabla^k u \nabla^j \eta^k \nabla^{k-j} \left(u - P\left(x^{k-1}\right)\right) \right] dx$$

which using the properties of η we bound by

$$\int_{B_{2r}} \eta^k |\nabla^k u|^2 \, dx + \sum_{j=1}^{k} \frac{1}{r^j} \int_{B_{2r}} |\nabla^k u| \left|\nabla^{k-j} \left(u - P\left(x^{k-1}\right)\right)\right| \, dx \, . \tag{3.38}$$

Next we set $p := \frac{2n}{n+1}$, $\frac{1}{p} + \frac{1}{q} = 1$ and estimate for each j the appropriate term in (3.38) applying Hölder's, Poincaré's ($j-1$-times) and Sobolev's inequalities

$$\begin{aligned} & r^{n-j} \fint_{B_{2r}} |\nabla^k u| \left|\nabla^{k-j} \left(u - P\left(x^{k-1}\right)\right)\right| \, dx \\ & \leq r^{n-j} \left(\fint_{B_{2r}} |\nabla^k u|^p \, dx\right)^{\frac{1}{p}} \left(\fint_{B_{2r}} \left|\nabla^{k-j} \left(u - P\left(x^{k-1}\right)\right)\right|^q \, dx\right)^{\frac{1}{q}} \\ & \leq C r^{n-1} \left(\fint_{B_{2r}} |\nabla^k u|^p \, dx\right)^{\frac{1}{p}} \left(\fint_{B_{2r}} \left|\nabla^{k-1} \left(u - P\left(x^{k-1}\right)\right)\right|^q \, dx\right)^{\frac{1}{q}} \\ & \leq C r^{n} \left(\fint_{B_{2r}} |\nabla^k u|^p \, dx\right)^{\frac{1}{p}} \left(\fint_{B_{2r}} \left|\nabla^{k} \left(u - P\left(x^{k-1}\right)\right)\right|^{q^*} \, dx\right)^{\frac{1}{q^*}} , \end{aligned} \tag{3.39}$$

where $q^* := \frac{nq}{n+q} = p$. Thus with the properties of $P\left(x^{k-1}\right)$we summarise the above inequalities into

$$r^{n-j} \fint_{B_{2r}} |\nabla^k u| \left|\nabla^{k-j} \left(u - P\left(x^{k-1}\right)\right)\right| \, dx \leq C r^n \left(\fint_{B_{2r}} |\nabla^k u|^p \, dx\right)^{\frac{2}{p}} . \tag{3.40}$$

Therefore together with (3.38) for the left-hand side of (3.37) this yields

$$\int_{\Omega} \nabla^k u \nabla^k \psi \, dx \geq \int_{B_r} |\nabla^k u|^2 \, dx - C r^n \left(\fint_{B_{2r}} |\nabla^k u|^{\frac{2n}{n+1}} \, dx\right)^{\frac{n+1}{n}} . \tag{3.41}$$

Next we will bound the right-hand side of (3.37) using the sharp Gagliardo-Nirenberg inequality. First we get

$$\begin{aligned} \int_{B_{2r}} \psi Q(x, u, \nabla u, \ldots, \nabla^{k-1} u) \, dx & \leq C \left(\|u\|_{L^\infty} + \epsilon\right) \sum_{\mu=1}^{k} \int_{B_{2r}} |\nabla^\mu u|^{\frac{2k}{\mu}} \, dx \\ & \leq C \left(\|u\|_{L^\infty} + \epsilon\right) \sum_{\mu=1}^{k} \left(\int_{B_{2r}} \left|\nabla^\mu \left(u - P\left(x^{k-1}\right)\right)\right|^{\frac{2k}{\mu}} \, dx + r^n \left|\overline{\nabla^\mu u}_{B_{2r}}\right|^{\frac{2k}{\mu}}\right) . \end{aligned}$$

From Theorem A.2 we estimate

$$\begin{aligned} \int_{B_{2r}} \left|\nabla^\mu \left(u - P\left(x^{k-1}\right)\right)\right|^{\frac{2k}{\mu}} \, dx & \leq C \left\|u - P\left(x^{k-1}\right)\right\|_{BMO(B_{2r})}^{2\left(\frac{2k}{\mu}-1\right)} \left\|u - P\left(x^{k-1}\right)\right\|_{W^{2k,2}(B_{2r})}^2 \\ & \leq C \left\|u - P\left(x^{k-1}\right)\right\|_{BMO(B_{2r})}^{2\left(\frac{2k}{\mu}-1\right)} \int_{B_{2r}} \left|\nabla^k u\right|^2 \, dx \qquad (3.42) \\ & \leq C \epsilon^2 \int_{B_{2r}} \left|\nabla^k u\right|^2 \, dx \, , \qquad (3.43) \end{aligned}$$

provided $\epsilon > 0$ was chosen sufficiently small. Thus, combining the above estimates for the right-hand side of (3.37) we get

$$\begin{aligned} \int_{B_{2r}} \psi Q(x, u, \nabla u, \ldots, \nabla^{k-1} u)\, dx &\leq C\epsilon^2 \int_{B_{2r}} \left|\nabla^k u\right|^2 \, dx + C \sum_{\mu=1}^{k} r^n \left|\overline{\nabla^\mu u}_{B_{2r}}\right|^{\frac{2k}{\mu}} \\ &\leq C\epsilon^2 \int_{B_{2r}} \left|\nabla^k u\right|^2 \, dx + C r^n \left(\fint_{B_{2r}} \sum_{\mu=1}^{k} \left|\nabla^\mu u\right|^{\frac{2k}{\mu} \frac{n}{n+1}} \, dx \right)^{\frac{n+1}{n}} \qquad (3.44) \end{aligned}$$

Putting both estimates for (3.37) and (3.42) together we have

$$\fint_{B_r} \sum_{\mu=1}^{k} \left|\nabla^\mu u\right|^{\frac{2k}{\mu}} dx \leq C \left(\fint_{B_{2r}} \left(\sum_{\mu=1}^{k} \left|\nabla^\mu u\right|^{\frac{2k}{\mu}} \right)^{\frac{n}{n+1}} dx \right)^{\frac{n+1}{n}} + C\epsilon^2 \fint_{B_{2r}} \sum_{\mu=1}^{k} \left|\nabla^\mu u\right|^{\frac{2k}{\mu}} dx \, ,$$

which, provided $\epsilon > 0$ is chosen possibly even smaller, implies that we can apply Lemma 3.13 with $g = \sum_{\mu=1}^{k} \left|\nabla^\mu u\right|^{\frac{2k}{\mu}}$ and $p = \frac{n+1}{n}$ to obtain the desired reverse Hölder inequality.$\square$

Having obtained higher integrability, the proof of Proposition 3.6 together with section 3.5 shows Theorem 3.3.

Appendix A

Sharp Gagliardo-Nirenberg interpolation inequalities

In this section we prove the sharp Gagliardo-Nirenberg interpolation inequality used in the proofs of Theorems 3.1 and 3.3. The following interpolation inequality was proved by L. Nirenberg in [62].

Theorem A.1 (Gagliardo-Nirenberg inequality) *For $k \in \mathbf{N}$ and $1 < q, r \leq \infty$, let $u \in \mathcal{D}^{k,r}(\mathbf{R}^m) \cap L^q(\mathbf{R}^m)$. Then, for $0 \leq j < k$, we have*

$$\|\nabla^j u\|_{L^p(\mathbf{R}^m)} \leq C \|\nabla^k u\|^a_{L^r(\mathbf{R}^m)} \|u\|^{1-a}_{L^q(\mathbf{R}^m)} \ , \tag{A.1}$$

where

$$\frac{1}{p} - \frac{j}{m} = a\left(\frac{1}{r} - \frac{k}{m}\right) + (1-a)\frac{1}{q} \ ,$$

for all

$$\frac{j}{k} \leq a \leq 1 \ .$$

The constant C is independent of u.

Remark A.1 *For a bounded domain Ω with smooth boundary, the result remains true if we add to the right-hand side of (A.1) the term $C\|u\|_{L^{\tilde{q}}(\Omega)}$ for any $\tilde{q} > 0$. The constants then also depend on the domain.*

In particular, we infer from Theorem A.1 that for $a = \frac{1}{2}$ and $q = \infty$

$$\|\nabla^j u\|^2_{L^p(\mathbf{R}^m)} \leq C \|\nabla^k u\|_{L^r(\mathbf{R}^m)} \|u\|_{L^\infty(\mathbf{R}^m)} \ ,$$

where

$$\frac{1}{p} - \frac{j}{m} = \frac{1}{2r} - \frac{k}{2m} \ .$$

However, for our purposes to prove partial regularity for polyharmonic maps in section 3.3, this is insufficient as the L^∞-norm of u cannot be made small. Therefore we need to improve the preceding inequality, where the L^∞-norm is substituted by the BMO-seminorm. Such an inequality was mentioned in [2] by D. Adams and M. Frazier. Later it was proved in [55] by Y. Meyer and T. Rivière for the case $j = 1$, $k = 2$ and $p = 4$ and, using different tools, in [84] by P. Strzelecki in the general case. For another proof using symmetrization we refer to the paper by J. Martin and M. Milman [57]. Here we state and prove the general case using the techniques of Y. Meyer and T. Rivière in [55], showing that their method can actually be extended to cover all situations.

Theorem A.2 (Gagliardo-Nirenberg type inequality) *Let Ω be a bounded domain with smooth boundary. Assume that $u \in W^{k,r}(\Omega)$ for some $r > 1$ and $1 \leq j < k$, with $j, k \in \mathbf{N}$. If $u \in BMO(\Omega)$, then $\nabla^j u \in L^p(\Omega)$ for $p := \frac{k}{j} r$ and*

$$\|\nabla^j u\|_{L^p} \leq C\, [u]_{BMO}^{1-\theta}\, \|u\|_{W^{k,r}}^{\theta}\ , \tag{A.2}$$

where $\theta := \frac{j}{k}$, for some constant $C = C(k, j, r)$.

The proof of this Theorem will be done in three steps. First, using Besov spaces, we will prove the Theorem in case $j = 1$ and $k = 2$ on $\mathbf{R}^n$. Next, we use a Whitney decomposition to extend this to arbitrary domains Ω with smooth boundary, still in the case $j = 1$, $k = 2$. Finally, a double induction argument as in [84] by P. Strzelecki shows the general result.

Step 1.
The proof will show an even sharper inequality and uses the fact that derivatives of BMO-functions are in the homogeneous Besov maximal space $\dot{B}_{\infty}^{-1,\infty}$ (see the book [1] by R. Adams and J. Fournier for details). Therefore we will first prove the following theorem on $\mathbf{R}^n$.

Theorem A.3 *Let $f \in \dot{B}_{\infty}^{-1,\infty}(\mathbf{R}^n) \cap W^{1,q}(\mathbf{R}^n)$. Then there is a constant C depending only on n and q such that*

$$\|f\|_{L^{2q}}^2 \leq C \|f\|_{\dot{B}_{\infty}^{-1,\infty}} \|\nabla f\|_{L^q}\ .$$

Proof. From the Littlewood-Paley decomposition of f (see again the book [1] by R. Adams and J. Fournier for details) we know that

$$\|f\|_{L^{2q}} \leq C \left\| \Big(\sum_j |\Delta_j f|^2 \Big)^{\frac{1}{2}} \right\|_{L^{2q}} .$$

Therefore we estimate

$$\begin{aligned}
\|f\|_{L^{2q}}^{2q} &\leq C \int_{\mathbf{R}^n} \Big(\sum_j |\Delta_j f|^2 \Big)^q dx \\
&\leq C \int_{\mathbf{R}^n} \Big(\sum_j \sum_{j'} |\Delta_j f|^2 |\Delta_{j'} f|^2 \Big)^{\frac{q}{2}} dx \\
&\leq C \|f\|_{\dot{B}_{\infty}^{-1,\infty}}^q \int_{\mathbf{R}^n} \Big(\sum_j \sum_{j'} 2^{2j} |\Delta_{j'} f|^2 \Big)^{\frac{q}{2}} dx \\
&\leq C \|f\|_{\dot{B}_{\infty}^{-1,\infty}}^q \int_{\mathbf{R}^n} \Big(\sum_j 2^{2j} |\Delta_j f|^2 \Big)^{\frac{q}{2}} dx \\
&\leq C \|f\|_{\dot{B}_{\infty}^{-1,\infty}}^q \|\nabla f\|_{L^q}^q\ ,
\end{aligned}$$

which proves the Theorem. $\square$

Applying this in the case where $f = \nabla u$, with u as in Theorem A.2, we immediately deduce inequality (A.2) for $\mathbf{R}^n$ when $j = 1$ and $k = 2$. In fact only the L^r-norm of $\nabla^2 u$ is needed on the right-hand side.

Step 2.
To apply the previous result we need the following definition:

Definition A.1 *Given an open (not necessarily bounded) subset $\Omega \subset \mathbf{R}^n$, define $\dot{B}_\infty^{-1,\infty}(\Omega)$ to be the set of restrictions of functions in $\dot{B}_\infty^{-1,\infty}(\mathbf{R}^n)$ to Ω. The norm for a function $f \in \dot{B}_\infty^{-1,\infty}(\Omega)$ is defined as the infimum of the norm of all possible extensions $\tilde{f} \in \dot{B}_\infty^{-1,\infty}(\mathbf{R}^n)$ of f.*

In case $j = 1$ and $k = 2$, the interpolation inequality follows from the next Theorem which is a consequence of Theorem A.3.

Theorem A.4 *Let $\Omega \subset \mathbf{R}^n$ be either a convex bounded open and regular subset of $\mathbf{R}^n$ or a half-space $\mathbf{R}^n_+ = \{(x_1, \dots, x_n) \ : \ x_n \geq 0\}$. There is a constant C depending only on Ω and q such that for any $f \in W^{1,q}(\Omega) \cap \dot{B}_\infty^{-1,\infty}(\Omega)$, we have*

$$\|f - m_\Omega f\|^2_{L^{2q}(\Omega)} \leq C(\Omega, q) \|\nabla f\|_{L^q(\Omega)} \|f\|_{\dot{B}_\infty^{-1,\infty}(\Omega)} \ ,$$

where $m_\Omega f$ is the average of f on Ω.

Proof. We first prove the Theorem for the half space $\mathbf{R}^n_+$ from which the result for general domains will follow using a partition of unity together with the results for $\mathbf{R}^n$ and $\mathbf{R}^n_+$.

Consider a decomposition of $\mathbf{R}^n_+$ by Whitney cubes $\prod_{j=1}^{n-1} [p_j 2^{-i}, (p_j{+}1)2^{-i}] \times [2^{-j}, 2^{-j+1}]$. Call these cubes Q, their centres x_Q and $d_Q = 2^{-i}$ their sizes. Let ϕ be a smooth function with support in $\widetilde{Q}_0 = [-2, 2]^n$ equal to 1 on the cube $\frac{1}{2}\widetilde{Q}_0 = [-1, 1]^n$. Setting $\phi_Q(x) := \phi\big(\frac{x - x_Q}{d_Q}\big)$ we obtain that $1 = \sum_Q \phi_Q$ on $\mathbf{R}^n$. Similarly, given $\theta \in C_0^\infty\big(\frac{1}{2}\widetilde{Q}_0\big)$ verifying $\int_{\mathbf{R}^n} \theta = 1$ we consider $\theta_Q := d_Q^{-n} \theta\big(\frac{x - x_Q}{d_Q}\big)$.
In the proof we will need the following Lemma about Besov spaces (see the book [1] by R. Adams and J. Fournier for details):

Lemma A.5 *The Banach space $\dot{B}_\infty^{-1,\infty}(\mathbf{R}^n)$ is a module over the Schwartz space $\mathcal{S}(\mathbf{R}^n)$ for $n \geq 1$. More precisely, if we denote the norm on $\dot{B}_\infty^{-1,\infty}(\mathbf{R}^n)$ by $\| \cdot \|_*$, for any $g \in \mathcal{S}(\mathbf{R}^n)$ there exists C_g such that for any $R > 0$ the following holds*

$$\Big\| g\Big(\frac{x}{R}\Big) \Big\|_* \leq C \|f\|_* \ .$$

Using the above decomposition we write any $f \in \dot{W}^{1,q}(\mathbf{R}^n_+)$ as $f = v_1 + v_2$, where

$$\begin{cases} v_2 = \displaystyle\sum_Q \gamma_Q \phi_Q \\ v_1 = \displaystyle\sum_Q (f - \gamma_Q)\phi_Q \ , \end{cases}$$

with $\gamma_Q := \int_{\mathbf{R}^n} f \theta_Q dx$. Then we have the following Proposition for v_1:

Proposition A.6 *Using the above notation we have*

$$\|v_1\|_{\dot{W}^{1,q}} \leq \|f\|_{\dot{W}^{1,q}}$$

and

$$\sum_Q \int_Q |f-\gamma_Q|^q |\nabla \phi_Q|^q \, dx \leq C \|\nabla f\|_{L^q}^q \ .$$

Proof. Choosing p so that $\frac{1}{q} - \frac{1}{p} = \frac{1}{n}$, from Poincaré's inequality we deduce that

$$\Big(\int_Q |f-\gamma_Q|^p \, dx \Big)^{\frac{1}{p}} \leq C \Big(\int_Q |\nabla f|^q \, dx \Big)^{\frac{1}{q}} \ .$$

Here we modified the standard Poincaré inequality which involves the average $m_\Omega f$. This modification follows from Poincaré's inequality since

$$\Big(\int_Q |m_Q - \gamma_Q|^p \, dx \Big)^{\frac{1}{p}} = |m_Q - \gamma_Q| |Q|^{\frac{1}{p}}$$

and

$$|m_Q - \gamma_Q| \leq \frac{1}{|Q|} \int_Q |f(x) - m_Q| \, dx \leq \Big(\frac{1}{|Q|} \int_Q |f - m_Q|^p \, dx \Big)^{\frac{1}{p}} \ .$$

Thus we deduce that

$$\int_Q |f-\gamma_Q|^q \, dx \leq C d_Q^q \int_Q |\nabla f|^q \, dx \ ,$$

which implies the result. □

From this Proposition we deduce that $v_1 \in W^{1,q}(\mathbf{R}^n_+)$ and hence that $v_2 \in W^{1,q}(\mathbf{R}^n_+)$ also.
Setting $F_Q := (f-\gamma_Q)\phi_Q$, we apply Theorem A.3 to F_Q to obtain

$$\|F_Q\|_{L^{2q}}^{2q} \leq C \|\nabla F_Q\|_{L^q}^q \|F_Q\|_*^q \ .$$

From the above Proposition it follows that $\sum_Q \|\nabla F_Q\|_{L^q}^q \leq C \|\nabla f\|_{L^q}^q$. The other term can be estimated by

$$\|F_Q\|_* \leq \|f\phi_Q\|_* + |\gamma_Q| \|\phi_Q\|_*$$

where $|\gamma_Q|$ in turn is estimated by

$$|\gamma_Q| \leq \|f\|_* d_Q^{-1} \ .$$

The estimate is completed applying Lemma A.5 to $\|f\phi_Q\|_*$ since functions with compact support are in $\mathcal{S}(\mathbf{R}^n)$. Hence

$$\|v_2\|_{L^{2q}}^{2q} \leq C \|\nabla f\|_{L^q}^q \|f\|_*^q \ .$$

The aim of the rest of the proof is to establish the same inequality for v_1. Note that

$$\begin{aligned}|v_1(x)| &\leq \sum_Q |\gamma_Q(x)||\phi_Q(x)| \\ &\leq \sum_Q \frac{\|f\|_*}{d_Q}|\phi_Q(x)| \\ &\leq C\frac{\|f\|_*}{x_n} .\end{aligned}$$

The remaining inequality will follow from the next Lemma:

Lemma A.7 *Let f be a function from $\mathbf{R}^n_+$ into $\mathbf{C}$ with*

$$|f(x)| \leq \frac{m}{x_n}$$

and

$$\left(\int_{\mathbf{R}^n_+} |\nabla f|^q \, dx\right)^{\frac{1}{q}} \leq M \ ,$$

then we have

$$\|f\|_{L^{2q}} \leq C(q)\sqrt{mM} \ .$$

Proof. Let us first consider the one dimensional case where we write t for x_n. We have

$$\begin{aligned}\int_0^\infty |f|^{2q}\,dt &\leq \int_0^{(\frac{m}{M})^{\frac{q}{2q-1}}} |f|^{2q}\,dt + \int_{(\frac{m}{M})^{\frac{q}{2q-1}}}^\infty |f|^{2q}\,dt \\ &\leq \left(\frac{m}{M}\right)^{\frac{q}{2q-1}} \sup_{\left[0,(\frac{m}{M})^{\frac{q}{2q-1}}\right]} |f|^{2q} + \frac{1}{2q-1}m^{2q}\left(\frac{m}{M}\right)^{-q} .\end{aligned}$$

The first hypothesis implies that

$$\left|f\left(\left(\frac{m}{M}\right)^{\frac{q}{2q-1}}\right)\right| \leq m^{\frac{q-1}{2q-1}}M^{\frac{q}{2q-1}}$$

and

$$|f(t)-f(t')| \leq M|t'-t|^{\frac{q-1}{q}} \leq m^{\frac{q-1}{2q-1}}M^{\frac{q}{2q-1}} .$$

From this it follows that on $[0,(\frac{m}{M})^{\frac{q}{2q-1}}]$

$$\|f\|_{L^\infty} \leq 2m^{\frac{q-1}{2q-1}}M^{\frac{q}{2q-1}}$$

which completes the proof of the lemma in the one dimensional case. In higher dimensions we integrate the one dimensional inequality in $x' = (x_1, \ldots, x_n)$ and the lemma follows.□

Therefore we have proved the remaining inequality for v_1, i.e. that

$$\|v_1\|^{2q}_{L^{2q}} \leq C\|\nabla f\|^q_{L^q}\|f\|^q_* .$$

Combining the estimates for v_1 and v_2 we obtain the desired estimate for f on $\mathbf{R}^n_+$.

For a regular bounded domain in $\mathbf{R}^n$ and $f \in W^{1,q}(\Omega) \cap \dot{B}^{-1,\infty}_{\infty}(\Omega)$ we use a partition of unity and diffeomorphisms to reduce the problem to the case of the whole $\mathbf{R}^n$ or $\mathbf{R}^n_+$.□

Step 3.
Having obtained the result in case $j = 1$ and $k = 2$ we now prove the general case from a double induction argument following [84] by P. Strzelecki. We work on $\mathbf{R}^n$ first. Note that from the remark at the end of step 1, we have

$$\|\nabla u\|_{L^{2p}(\mathbf{R}^n)} \leq C\,[u]^{\frac{1}{2}}_{BMO(\mathbf{R}^n)}\,\|\nabla^2 u\|^{\frac{1}{2}}_{L^p(\mathbf{R}^n)}\ ,$$

which, assuming the conditions of Theorem A.2 on j, k, r, and θ, we will show to imply

$$\|\nabla^j u\|_{L^p(\mathbf{R}^n)} \leq C\,[u]^{1-\theta}_{BMO(\mathbf{R}^n)}\,\|\nabla^k u\|^{\theta}_{L^r(\mathbf{R}^n)}\ . \tag{A.3}$$

The general case of a smooth bounded domain Ω then follows by an extension argument. Assume estimate (A.3) to hold for fixed $1 \leq j < k$ and all $r > 1$. For the first induction step we will show that it continues to hold for $k+1$, so we let $r > 1$, define $s := \frac{(k+1)r}{j}$ and set $p := \frac{(k+1)r}{k}$. The induction hypothesis yields

$$\|\nabla^j u\|_{L^s(\mathbf{R}^n)} \leq C[u]^{1-\frac{j}{k}}_{BMO(\mathbf{R}^n)}\|\nabla^k u\|^{\frac{j}{k}}_{L^p(\mathbf{R}^n)}$$

and from the usual Gagliardo-Nirenberg inequality, estimate (A.1), applied to $\nabla^j u$ we derive

$$\|\nabla^k u\|_{L^p(\mathbf{R}^n)} \leq C\|\nabla^j u\|^{1-\theta}_{L^s(\mathbf{R}^n)}\|\nabla^k u\|^{\theta}_{L^r(\mathbf{R}^n)}\ ,$$

where $\theta = \frac{k-j}{k-j+1}$. Combining the last two estimates and cancelling a suitable power of $\|\nabla^j u\|_{L^s(\mathbf{R}^n)}$ we deduce (A.3) for $k+1$.
For the second induction step, again using (A.1), we have for $r > 1$ and $m+1 < k$ fixed

$$\|\nabla^{j+1} u\|_{L^p(\mathbf{R}^n)} \leq C\|\nabla^j u\|^{1-\theta}_{L^s(\mathbf{R}^n)}\|\nabla^k u\|^{\theta}_{L^r(\mathbf{R}^n)}\ ,$$

where this time $s := \frac{kr}{j}$, $p := \frac{kr}{j+1}$ and $\theta = \frac{1}{k-j}$. Estimating the term involving $\|\nabla^j u\|_{L^s(\mathbf{R}^n)}$ by invoking the induction hypothesis

$$\|\nabla^j u\|_{L^s(\mathbf{R}^n)} \leq C[u]^{1-\frac{j}{k}}_{BMO(\mathbf{R}^n)}\|\nabla^k u\|^{\frac{j}{k}}_{L^r(\mathbf{R}^n)}$$

inequality (A.3) again follows from combining these estimates since $(1-\theta)(1-\frac{j}{k}) = 1-\frac{j+1}{k}$ and $(1-\theta)\frac{j}{k} + \theta = \frac{j+1}{k}$. Thus the induction argument is completed and Theorem A.2 holds.

Appendix B

Linear Estimates for polyharmonic equations

Singular Integral Operators

We recall the following singular integral theorem (Theorem II.3.2) in the book [81] by E. Stein.

Theorem B.1 *Let $K : \mathbf{R}^m \longrightarrow \mathbf{R}$ be a measurable function such that*

$$|K(x)| \leq C|x|^{-m} \textit{ for } |x| > 0 \ , \tag{B.1}$$

$$\int_{|x|\geq 2|y|} |K(x-y) - K(x)|\, dx \leq C \textit{ for } |y| > 0 \tag{B.2}$$

and

$$\int_{R_1<|x|<R_2} K(x)\, dx = 0 \textit{ for all } 0 < R_1 < R_2 < \infty \ . \tag{B.3}$$

For $\epsilon > 0$, $1 < p < \infty$ and $f \in L^p(\mathbf{R}^m)$, we set

$$T_\epsilon f(x) := \int_{|y|\geq\epsilon} f(x-y)K(y)dy \ .$$

Then, we have

$$\|T_\epsilon f\|_{L^p} \leq C\|f\|_{L^p} \ ,$$

where C is independent of ϵ and f. Moreover, there exists $Tf \in L^p(\mathbf{R}^m)$ such that

$$T_\epsilon f \longrightarrow Tf \textit{ in } L^p \ ,$$

as $\epsilon \to 0$, for all $f \in L^p(\mathbf{R}^m)$.

For $m \geq 2k+1$, the fundamental solution of Δ^k on $\mathbf{R}^m$ is

$$\Gamma_k(x-y) = c|x-y|^{2k-m} \ ,$$

i.e.

$$\Delta^k\Gamma_k(x-y) = \delta(x-y) \text{ for } x, y \in \mathbf{R}^m \ .$$

The kernel $K := \nabla^{2k}\Gamma_k$ verifies the hypotheses of Theorem B.1. Thus, we conclude

Corollary B.2 *Let $f \in L^p(\Omega)$, $1 < p < \infty$, $K = \nabla^{2k}\Gamma_k$ and $u = Tf$. Then, $u \in W^{2k,p}(\Omega)$,*

$$\Delta^k u = f \ a.e.$$

and

$$\|\nabla^{2k} u\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}$$

where C depends only on n and p.

Remark B.1 *Corollary B.2 also follows from the proof of Theorem 9.9 in [30] by D. Gilbarg and N. Trudinger. The arguments can be carried over line by line to the case of general k.*

Furthermore, we have

Lemma B.3 *For $1 < p < \infty$, $\mu \in \mathbf{N} \cap (k, 2k]$ and $u \in W^{\mu,p}(\Omega) \cap W_0^{k,p}(\Omega)$, there exists a constant C (independent of u) such that*

$$\|u\|_{W^{\mu,p}(\Omega)} \leq C\|\Delta^{\frac{\mu}{2}} u\|_{L^p(\Omega)} \quad \text{for } \mu \text{ even,}$$

and

$$\|u\|_{W^{\mu,p}(\Omega)} \leq C\|\nabla\Delta^{\frac{\mu-1}{2}} u\|_{L^p(\Omega)} \quad \text{for } \mu \text{ odd.}$$

Proof. We follow the scheme of Lemma 9.17 in [30] by D. Gilbarg and N. Trudinger. We consider the case μ even. If Lemma B.3 is not true, there exists $\{u_l\}_{l\in\mathbf{N}} \subset W^{\mu,p}(\Omega) \cap W_0^{k,p}(\Omega)$ satisfying

$$\|u_l\|_{W^{\mu,p}(\Omega)} = 1 \text{ and } \|\Delta^{\frac{\mu}{2}} u_l\|_{L^p(\Omega)} \to 0 \ ,$$

as $l \to \infty$. After passing to a subsequence, we may assume the existence of $u \in W^{\mu,p}(\Omega) \cap W_0^{k,p}(\Omega)$ such that $\|u\|_{W^{\mu,p}(\Omega)} = 1$ and

$$u_l \rightharpoonup u \text{ in } W^{\mu,p}(\Omega) \ ,$$

as $l \to \infty$. Since

$$\int_\Omega g\nabla^\alpha u_l \, dx \to \int_\Omega g\nabla^\alpha u \, dx$$

for all $|\alpha| \leq \mu$ and $g \in L^{\frac{p}{p-1}}(\Omega)$, we must have

$$\int_\Omega g\Delta^{\frac{\mu}{2}} u \, dx = 0$$

for all $g \in L^{\frac{p}{p-1}}(\Omega)$. Thus, $\Delta^{\frac{\mu}{2}} u = 0$ and $u = 0$ by the uniqueness assertion, contradicting the condition $\|u\|_{W^{\mu,p}(\Omega)} = 1$. For μ odd, the proof is similar. □

Decay Lemmas

Here we prove the following decay lemmas used in the proofs of partial regularity.

Lemma B.4 *Let $u \in W^{k,2}(\Omega)$ be a weakly k-harmonic function with $\|u\|_{W^{k,p}(\Omega)} \leq 1$. For $x_0 \in \Omega$, $0 < \rho \leq r \leq dist(x_0, \partial\Omega)$ and $2 \leq p < \infty$ we have*

$$\rho^{-m} \sum_{l=1}^{k} \int_{B_\rho(x_0)} |\nabla^l u|^{\frac{kp}{l}}\, dx \leq C r^{-m} \sum_{l=1}^{k} \int_{B_r(x_0)} |\nabla^l u|^{\frac{kp}{l}}\, dx \ ,$$

where C is independent of u and ρ.

Proof. Due to the Weyl Lemma we know that u is smooth. Moreover, we have the following Cacciopoli type estimate. For all $B_R \subset \Omega$ and for all $\gamma \in \mathbf{N}$, it holds

$$\|u\|_{W^{\gamma,2}(B_{\frac{R}{2}})} \leq C(\gamma, R)\|u\|_{W^{k-1,2}(B_R)} \ . \tag{B.4}$$

To see this choose $\eta \in C_0^\infty(B_R)$ satisfying $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_{\frac{R}{2}}$. For $\psi := \eta^{k+1}u$, we obtain with

$$\int_{B_R} \nabla^k u \nabla^k \psi\, dx = 0$$

that

$$\begin{aligned}
&\int_{B_R} \eta^{k+1} |\nabla^k u|^2\, dx \\
&\qquad = -2\sum_{\alpha=2}^{k} \int_{B_R} \nabla^\alpha(\eta^{k+1})\nabla^k u \nabla^{k-\alpha} u\, dx \\
&\qquad\quad - \int_{B_R} \nabla \eta^{k+1} \nabla^{k-1} u \nabla^k u\, dx \\
&\qquad = 2\sum_{\alpha=2}^{k} \int_{B_R} (\nabla^{\alpha+1}(\eta^{k+1})\nabla^{k-\alpha} u + \nabla^\alpha(\eta^{k+1})\nabla^{k+1-\alpha} u)\nabla^{k-1} u\, dx \\
&\qquad\quad \frac{1}{2}\int_{B_R} \nabla^2 \eta^{k+1} |\nabla^{k-1} u|^2\, dx \\
&\qquad \leq C(R) \sum_{\alpha \leq k-1} \int_{B_R} |\nabla^\alpha u|^2\, dx \ .
\end{aligned}$$

Thus we have shown estimate (B.4) for $\gamma = k$. For $\gamma \geq k+1$, we observe that ∇u is k-harmonic. Repeating the preceding argument for ∇u we deduce the case $\gamma = k+1$ and by iteration we conclude (B.4) for all γ.

Now let $\rho < \frac{r}{2}$. Applying equation (B.4) to $\nabla^l u$ (which is k-harmonic) and Sobolev embedding theorem, we infer with s sufficiently large

$$\begin{aligned}
\rho^{-m} \int_{B_\rho(x_0)} |\nabla^l u|^{\frac{kp}{l}} \, dx &\leq C \sup_{x \in B_\rho(x_0)} |\nabla^l u|^{\frac{kp}{l}}(x) \\
&\leq C(r) \|\nabla^l u\|^{\frac{kp}{l}}_{W^{s,2}(B_{\frac{r}{2}})} \\
&\leq C(r) \Big(\sum_{\alpha=l}^{k-1} \int_{B_r} |\nabla^\alpha u|^2 \, dx \Big)^{\frac{kp}{2l}} \\
&\leq C(r) \sum_{\alpha=l}^{k-1} \Big(\int_{B_r} |\nabla^\alpha u|^2 \, dx \Big)^{\frac{kp}{2l}}
\end{aligned}$$

Using the fact that $\|u\|_{W^{k,p}(\Omega)} \leq 1$ this can be estimated by

$$C(r) \sum_{\alpha=l}^{k-1} \Big(\int_{B_r} |\nabla^\alpha u|^2 \, dx \Big)^{\frac{kp}{2\alpha}}$$

from which by Jensen's inequality we deduce that

$$\rho^{-m} \int_{B_\rho(x_0)} |\nabla^l u|^{\frac{kp}{l}} \, dx \leq C(r) \sum_{\alpha=l}^{k-1} \int_{B_r} |\nabla^\alpha u|^{\frac{kp}{\alpha}} \, dx \ .$$

To get the desired we apply a rescaling argument showing that $C(r) = Cr^{-m}$ and sum over all $1 \leq l \leq k$. □

Lemma B.5 *Let $u \in W^{k,2}(\Omega)$ be a weakly k-harmonic function. For $x_0 \in \Omega$, $0 < \rho \leq r \leq dist(x_0, \partial\Omega)$ and $2 \leq p < \infty$ we have*

$$\rho^{-m} \int_{B_\rho(x_0)} |\nabla^k u|^p \, dx \leq Cr^{-m} \int_{B_r(x_0)} |\nabla^k u|^p \, dx \ ,$$

where C is independent of u and ρ.

Proof. We begin by observing that we can add any polynomial of order $k-1$ to a k-harmonic function and still obtain a k-harmonic function. Therefore the estimate in (B.4) remains valid if one subtracts the average of u on B_R from u, i.e. together with Poincaré's inequality we get

$$\begin{aligned}
\int_{B_{\frac{R}{2}}} |\nabla^k u|^2 \, dx &\leq C \frac{1}{R^{2k}} \int_{B_R} |u - \overline{u}_{B_R(x_0)}|^2 \, dx + C \sum_{\alpha=1}^{k-1} \frac{1}{R^{2(k-\alpha)}} \int_{B_R} |\nabla^\alpha u|^2 \, dx \\
&\leq C \sum_{\alpha=1}^{k-1} \frac{1}{R^{2(k-\alpha)}} \int_{B_R} |\nabla^\alpha u|^2 \, dx \ .
\end{aligned}$$

For ∇u we define the polynomial $p_1(x) = \sum_{i=1}^m x_i \overline{\partial_{x_i} u}_{B_R(x_0)}$ where we note that $\nabla p_1 = \overline{\nabla u}_{B_R(x_0)}$. Since the previous inequality still remains valid for $u - p_1$ we apply Poincaré's inequality again which yields

$$\begin{aligned}
\int_{B_{\frac{R}{2}}} |\nabla^k u|^2 \, dx &\leq C \frac{1}{R^{2(k-1)}} \int_{B_R} |\nabla u - \overline{\nabla u}_{B_R(x_0)}|^2 \, dx + C \sum_{\alpha=2}^{k-1} \frac{1}{R^{2(k-\alpha)}} \int_{B_R} |\nabla^\alpha u|^2 \, dx \\
&\leq C \sum_{\alpha=2}^{k-1} \frac{1}{R^{2(k-\alpha)}} \int_{B_R} |\nabla^\alpha u|^2 \, dx \; .
\end{aligned}$$

Iterating this procedure for ∇^α for $2 \leq \alpha \leq k-1$ in the same way we note that (B.4) can actually be sharpened to only involve the term with $\nabla^{k-1} u$ on the right-hand side. Thus we deduce a better Cacciopoli-type inequality for $\nabla^k u$

$$\|\nabla^k u\|^2_{L^2(B_{\frac{R}{2}})} \leq C \frac{1}{R^2} \|\nabla^{k-1} u\|^2_{L^2(B_R)} \; ,$$

and iterating again as in the previous lemma

$$\|\nabla^k u\|_{W^{\gamma,2}(B_{\frac{R}{2}})} \leq C(\gamma, R) \|\nabla^k u\|_{L^2(B_R)} \; . \tag{B.5}$$

To deduce the decay estimate we proceed similarly to the above proof. Letting $\rho < \frac{r}{2}$ we apply Sobolev embedding, inequality (B.5) and Jensen's inequality (noting that $p \geq 2$) to get

$$\begin{aligned}
\rho^{-m} \int_{B_\rho(x_0)} |\nabla^k u|^p \, dx &\leq C \sup_{x \in B_\rho(x_0)} |\nabla^k u|^p(x) \\
&\leq C(r) \|\nabla^k u\|^p_{W^{s,2}(B_{\frac{r}{2}})} \\
&\leq C(r) \Big(\int_{B_r} |\nabla^k u|^2 \, dx \Big)^{\frac{p}{2}} \\
&\leq C(r) \int_{B_r} |\nabla^k u|^p \, dx \; .
\end{aligned}$$

The same scaling argument as before shows that $C(r) = Cr^{-m}$ and the lemma is proved.$\Box$

Lemma B.6 *For $r > 0$ we consider $u \in W^{k,2}(B_r)$ and its k-harmonic extension on B_r, i.e. v solves the Dirichlet problem*

$$\begin{cases} \Delta^k v = 0 \\ u - v \in W^{k,2}_0(B_r) \; . \end{cases}$$

We have

$$\sum_{\mu=1}^k [\nabla^\mu v]^2_{M^{2,2\mu}(B_{\frac{r}{2}})} \leq C \sum_{\mu=1}^k \int_{B_r} |\nabla^\mu u|^2 \, dx \; .$$

Proof. Observe that v satisfies the Cacciopoli type estimate (B.4). For all $\rho < \frac{r}{4}$, $x \in B_{\frac{r}{2}}$ and $s > 0$ sufficiently large, Sobolev embedding theorem and the Cacciopoli type estimate imply

$$\begin{aligned} \rho^{-m} \int_{B_\rho(x)} |\nabla^\mu v|^2 \, dx &\leq C \sup_{x \in B_\rho(x)} |\nabla^\mu v|^2 \\ &\leq C \sum_{\lambda=0}^{s} \int_{B_{\frac{r}{4}}(x)} |\nabla^\lambda v|^2 \, dx \\ &\leq C \sum_{\lambda=0}^{k} \int_{B_{\frac{r}{2}}(x)} |\nabla^\lambda v|^2 \, dx \ , \end{aligned}$$

It follows that

$$\begin{aligned} \rho^{2\mu-m} \int_{B_\rho(x)} |\nabla^\mu v|^2 \, dx &\leq C \sum_{\lambda=0}^{k} \int_{B_r} |\nabla^\lambda v|^2 \, dx \\ &\leq C \sum_{\lambda=0}^{k} \left(\int_{B_r} |\nabla^\lambda u|^2 \, dx + \int_{B_r} |\nabla^\lambda (v-u)|^2 \, dx \right) \ . \end{aligned} \tag{B.6}$$

Applying Poincaré's inequality yields

$$\int_{B_r} |\nabla^\lambda (v-u)|^2 \, dx \leq C \int_{B_r} |\nabla^k (v-u)|^2 \, dx \ . \tag{B.7}$$

As v is k-harmonic, we have

$$\int_{B_r} \nabla^k v \nabla^k (u-v) \, dx = 0 \ ,$$

i.e.

$$\int_{B_r} |\nabla^k v|^2 \, dx \leq \int_{B_r} |\nabla^k u|^2 \, dx \ . \tag{B.8}$$

Combining (B.6) - (B.8) completes the proof. □

Divergence form

This section is devoted to the proof of the following lemma.

Lemma B.7 *Consider a ball $B \subset \mathbf{R}^m$, $g \in L^r(B)$, $k > 1$ and $j, l \geq 0$ with $1 \leq 2l + j \leq k$. There exists a unique weak solution $u \in W_0^{k,2}(B, \mathbf{R}^N)$ of*

$$\Delta^k u = \nabla^{(l)} \cdot \Delta^j g$$

satisfying

$$\|\nabla^{2k-(2j+l)} u\|_{L^r(B)} \leq C \|g\|_{L^r(B)}$$

for $1 < r < \infty$.

The proof is based on results in the book [30] by D. Gilbarg and N. Trudinger and the books [28] and [29] by M. Giaquinta. In a first step we prove existence of a unique solution.

Lemma B.8 *Consider a ball $B \subset \mathbf{R}^m$ and $g \in L^2(B)$. Then there exists a unique weak solution $u \in W_0^{k,2}(B, \mathbf{R}^N)$ of*

$$\Delta^k u = \nabla^{(l)} \cdot \Delta^j g$$

for $j, l \geq 0$ with $1 \leq 2l + j \leq k$.

Proof. This follows from a direct application of the Lax-Milgram theorem. On the space $W_0^{k,2}(B, \mathbf{R}^N)$ we define the functional $F(v) = \int_B g \nabla^{(l)} \cdot \Delta^j v \, dx$ for $v \in W_0^{k,2}(B, \mathbf{R}^N)$. Then by Hölder's inequality this functional is bounded on $W_0^{k,2}(B, \mathbf{R}^N)$. Also note that the bilinear form defined by $\mathcal{L}(u, v) = \int_B \nabla^k u \nabla^k v \, dx$ is again bounded by Hölder's inequality but also coercive since

$$\mathcal{L}(u, u) = \|\nabla^k u\|^2_{L^2(B)} \geq \frac{1}{C} \|u\|^2_{W_0^{k,2}(B)}$$

by Poincaré's inequality. Thus we apply the Lax-Milgram theorem (see for instance Theorem 5.8 in [30]) to deduce existence of a unique weak solution. $\square$

In a second step, using the method of difference quotient, we show

Lemma B.9 *Consider a ball $B \subset \mathbf{R}^m$ and $g \in L^2(B)$. If $u \in W_0^{k,2}(B, \mathbf{R}^N)$ is a weak solution of*

$$\Delta^k u = \nabla^{(l)} \cdot \Delta^j g$$

for $j, l \geq 0$ with $1 \leq 2l + j < k$, then

$$u \in W^{2k-(2j+l),2}(B, \mathbf{R}^N) \ .$$

Proof. We define $\left(\Delta_\nu^h u\right)(x) := \frac{1}{h}(u(x + he_\nu) - u(x))$ for $1 \leq \nu \leq m$. Choose the test function $\Delta_\nu^h \psi$ with $\psi \in W_0^{k,2}(B, \mathbf{R}^N)$. It follows that

$$\begin{aligned}(-1)^l \int_B g \nabla^l \Delta^j \Delta_\nu^{-h} \psi \, dx &= \int_B \nabla^k u \nabla^k \Delta_\nu^{-h} \psi \, dx = \int_B \nabla^k u \Delta_\nu^{-h} \nabla^k \psi \, dx \\ &= \int_B \Delta_\nu^h \nabla^k u \nabla^k \psi \, dx = \int_B \nabla^k \Delta_\nu^h u \nabla^k \psi \, dx \ . \end{aligned} \tag{B.9}$$

Now we choose $\psi = \eta^{k+1} \Delta_\nu^h u$ for arbitrary $\eta \in C_0^\infty(B)$ with $0 \leq \eta \leq 1$. We compute

$$\begin{aligned}\int_B \nabla^k \Delta_\nu^h u \nabla^k \psi \, dx &= \int_B \eta^{k+1} |\nabla^k \Delta_\nu^h u|^2 \, dx + C \sum_{\alpha=1}^k \int_B \nabla^\alpha (\eta^{k+1}) \nabla^k \Delta_\nu^h u \nabla^{k-\alpha} \Delta_\nu^h u \, dx \\ &= \int_B \eta^{k+1} |\nabla^k \Delta_\nu^h u|^2 \, dx + C \sum_{\alpha=1}^k \int_B \nabla^\alpha (\eta^{k+1}) \nabla^{k-1} \Delta_\nu^h u \nabla^{k+1-\alpha} \Delta_\nu^h u \, dx \\ &\quad + C \sum_{\alpha=1}^k \int_B \nabla^{\alpha+1} (\eta^{k+1}) \nabla^{k-1} \Delta_\nu^h u \nabla^{k-\alpha} \Delta_\nu^h u \, dx \ , \end{aligned} \tag{B.10}$$

where $-\infty < c_i < \infty$ for $1 \le i \le 3$. Moreover, we define $\eta_{h,\nu}(x) := \eta(x + he_\nu)$. We estimate

$$
\begin{aligned}
(-1)^l \int_B g\Delta_\nu^{-h}\nabla^l\Delta^j\psi\,dx &\le \int_B |g\,\nabla^l\Delta^j(\Delta_\nu^{-h}(\eta^{k+1}\Delta_\nu^h u))|\,dx \\
&= \int_B |g\,\nabla^l\Delta^j(\eta_{-h,\nu}^{k+1}\Delta_\nu^{-h}(\Delta_\nu^h u) + \Delta_\nu^{-h}(\eta^{k+1})\Delta_\nu^h u)|\,dx \\
&\le \int_B |g\,\eta_{-h,\nu}^{k+1}\nabla^l\Delta^j(\Delta_\nu^{-h}(\Delta_\nu^h u))|\,dx \\
&\quad + \sum_{\alpha=1}^{2j+l}\int_B |g||\nabla^\alpha\eta_{h,\nu}^{k+1}||\nabla^{2j+l-\alpha}\Delta_\nu^{-h}\Delta_\nu^h u|\,dx \\
&\quad + \sum_{\alpha=0}^{2j+l}\int_B |g||\nabla^\alpha\Delta_\nu^{-h}\eta^{k+1}||\nabla^{2j+l-\alpha}\Delta_\nu^h u|\,dx\ . \qquad \text{(B.11)}
\end{aligned}
$$

Combining (B.9)-(B.11) with Young's inequality for some $\epsilon > 0$ to be chosen later gives (using also that $0 \le \eta(x) \le 1$ and that $|\nabla\eta^{k+1}|^2 \le C(|\nabla\eta|)\eta^{k+1}$)

$$
\begin{aligned}
\int_B \eta^{k+1}|\nabla^k\Delta_\nu^h u|^2\,dx &\le \frac{1}{2}\int_B \eta_{h,\nu}^{k+1}|\nabla^{2j+l}\Delta_\nu^{-h}\Delta_\nu^h u|^2\,dx + C\sum_{\alpha=0}^{k-1}\int_{\operatorname{spt}\eta}|\nabla^\alpha\Delta_\nu^h u|^2\,dx \\
&\quad + C\sum_{\alpha=1}^{2j+l}\int_{\operatorname{spt}\eta_{h,\nu}}|\nabla^{2j+l-\alpha}\Delta_\nu^{-h}\Delta_\nu^h u|^2\,dx + C\int_B g^2\,dx \\
&\quad + C\epsilon\int_B \eta^{k+1}|\nabla^k\Delta_\nu^h u|^2\,dx\ , \qquad \text{(B.12)}
\end{aligned}
$$

where we can therefore absorb the last term in the left-hand side provided $\epsilon > 0$ is chosen sufficiently small. For $h > 0$ sufficiently small, we infer that the second and the third term on the right-hand side are uniformly bounded. For example, applying Lemma 8.2.1 in the book [45] by J. Jost to $\nabla^\alpha u$, we estimate

$$
\int_{\operatorname{spt}\eta}|\nabla^\alpha\Delta_\nu^h u|^2\,dx \le \int_{\operatorname{spt}\eta}|\Delta_\nu^h\nabla^\alpha u|^2\,dx \le \int_B|\nabla^{\alpha+1}u|^2\,dx \le C
$$

for $0 \le \alpha \le k-1$.
If $2j+l+1 < k$, we similarly show that the first term on the right-hand side is uniformly bounded. Otherwise $2j+l+1 = k$, and we again apply Lemma 8.2.1 in [45] twice (noting that the presence of a cut-off function does not affect the proof) to obtain

$$
\frac{1}{2}\int_B \eta_{h,\nu}^{k+1}|\nabla^{2j+l}\Delta_\nu^{-h}\Delta_\nu^h u|^2\,dx \le \frac{1}{2}\int_B \eta_{h,\nu}^{k+1}|\Delta_\nu^h\nabla^k u|^2\,dx\ .
$$

Using the continuity of η and choosing $h > 0$, $\epsilon > 0$ sufficiently small we can absorb the terms involving $\eta_{h,\nu}^{k+1}|\Delta_\nu^h\nabla^k u|^2$ in the left-hand side of (B.12). Combining the above steps shows that from (B.12) we can deduce

$$
\int_{\{x\in B:\eta(x)\equiv 1\}}|\nabla^k\Delta_\nu^h|^2\,dx \le C\ .
$$

Now we observe that for all $\Omega \subset\subset B$ there exists $\eta \in C_0^\infty(B)$ such that $0 \le \eta \le 1$ and $\eta \equiv 1$ on Ω. We conclude

$$\int_\Omega |\nabla^k \Delta_\nu^h|^2 \, dx \le C$$

for all $\Omega \subset\subset B$. Hence, from Lemma 8.2.2 in [45] we infer that $u \in W^{k+1,2}(B)$. The existence of higher derivatives now follows by induction which completes the proof. □

Thus, we look at solutions $u \in W^{2k-(2j+l),2}(B, \mathbf{R}^N) \cap W_0^{k,2}(B, \mathbf{R}^N)$ satisfying

$$(-1)^k \int_B \nabla^{2k-(2j+l)} u \nabla^{2j+l} \psi \, dx = \int_B g \nabla^l \Delta^j \psi \, dx \tag{B.13}$$

for every test function $\psi \in W_0^{2j+l}(B, \mathbf{R}^N)$.

Set $T(g) := \nabla^{2k-(2j+l)} u$. Lemma B.10 below states that T is a continuous linear operator

$$T : L^2(B) \longrightarrow L^2(B) \ ,$$

respectively

$$T : L^\infty(B) \longrightarrow \mathrm{BMO}(B) \ .$$

Thus, Stampacchia's interpolation theorem (see Theorem 4.6 in [29]) then implies that $T : L^q(B) \longrightarrow L^q(B)$ is also continuous for $2 < q < \infty$.

For $1 < q < 2$ we deduce continuity from a duality argument as follows. Let $f \in L^q$, $g \in L^p$ with $\frac{1}{q} + \frac{1}{q'} = 1$, where by density we can assume that $f, g \in C_0^\infty(B)$. Thus noting that then the fundamental solution of $\Delta^k u = \nabla^{(l)} \cdot \Delta^j g$ is given by $u_g(x) = C(-1)^l \int_B \nabla_y^l \Delta_y^j \Gamma_k(x-y) g(y) dy$ we compute

$$\begin{aligned}
\int_B (Tf)(x) g(x) \, dx &= \int_B \nabla_x^{2k-(2j+l)} u_f(x) g(x) \, dx \\
&= (-1)^{2k-(2j+l)} \int_B u_f(x) \nabla_x^{2k-(2j+l)} g(x) \, dx \\
&= (-1)^{2k-(2j+l)} C \int_B \int_B \nabla_y^l \Delta_y^j \Gamma_k(x-y) f(y) \, dy \, \nabla_x^{2k-(2j+l)} g(x) \, dx \\
&= (-1)^{2k-(2j+l)} C \int_B \int_B \nabla_y^l \Delta_y^j \Gamma_k(x-y) f(y) \nabla_x^{2k-(2j+l)} g(x) \, dx dy \\
&= C \int_B f(y) \int_B \nabla_x^{2k-(2j+l)} \nabla_y^l \Delta_y^j \Gamma_k(x-y) g(x) \, dx dy \\
&= C \int_B f(y) \int_B \nabla_y^{2k-(2j+l)} \nabla_x^l \Delta_x^j \Gamma_k(x-y) g(x) \, dx dy \\
&= C \int_B f(y) \nabla_y^{2k-(2j+l)} \int_B \nabla_x^l \Delta_x^j \Gamma_k(x-y) g(x) \, dx dy \\
&= C \int_B f(y) \nabla_y^{2k-(2j+l)} u_g(y) \, dy \\
&= C \int_B f(x) (Tg)(x) \, dx \\
&\le C \|f\|_{L^q} \|Tg\|_{L^{q'}} \ .
\end{aligned}$$

Since $q' > 2$ we can apply the above continuity result to g to obtain

$$\int_B (Tf)g \leq C\|f\|_{L^q}\|g\|_{L^{q'}} \ .$$

After taking the supremum over all $\|g\|_{L^{q'}} = 1$ by the dual characterisation of the norm this enables us to conclude that

$$\|Tf\|_{L^q} \leq C\|f\|_{L^q}$$

for $q > 1$. The proof of Lemma B.7 is thus complete.

It remains to show

Lemma B.10 *Consider a ball $B \subset \mathbf{R}^m$, $g \in L^2(B)$, $k > 1$ and $j, l > 0$ with $1 \leq 2l + j \leq k$. If $u \in W_0^{k,2}(B, \mathbf{R}^N)$ is a weak solution of*

$$\Delta^k u = \nabla^{(l)} \cdot \Delta^j g \ ,$$

then $g \in \mathcal{L}^{2,\lambda}_{loc}(B)$ implies that $\nabla^{2k-(2j+l)} u \in \mathcal{L}^{2,\lambda}_{loc}(B)$ for $0 \leq \lambda < m+2$, where $\mathcal{L}^{2,\lambda}_{loc}(B) := \left\{ u \in L^2_{loc}(B) : [u]_{\mathcal{L}^{2,\lambda}_{loc}(B)} := \sup_{B_\rho(x_0) \subset B} \rho^{-\lambda} \int_{B_\rho(x_0)} |u - \overline{u}_{B_\rho(x_0)}|^2 \, dx < \infty \right\}$ is the Campanato space.

Proof. The proof is similar to the one of Theorem 3.3 in the book [29] by M. Giaquinta. Set $\alpha = 2k - (2j + l)$. Consider $B_r(x_0) \subset B$ and let v be the α-harmonic extension of u on $B_r(x_0)$, i.e. $w := u - v \in W_0^{\alpha,2}(B_r(x_0))$ and $\Delta^\alpha v = 0$ on $B_r(x_0)$. From Poincaré's inequality and the application of Lemma B.5 we can estimate

$$\begin{aligned}
\int_{B_\rho(x_0)} |\nabla^\alpha v - \overline{\nabla^\alpha v}_{B_\rho(x_0)}|^2 \, dx &\leq C\rho^2 \int_{B_\rho(x_0)} |\nabla^{\alpha+1} v|^2 \, dx \\
&\leq C\rho^2 \left(\frac{\rho}{r}\right)^m \int_{B_{\frac{r}{2}}(x_0)} |\nabla^{\alpha+1} v|^2 \, dx \\
&\leq C \left(\frac{\rho}{r}\right)^{m+2} \int_{B_r(x_0)} |\nabla^\alpha v - \overline{\nabla^\alpha v}_{B_r(x_0)}|^2 \, dx \ . \qquad \text{(B.14)}
\end{aligned}$$

for $\rho \leq \frac{r}{2}$. On the other hand, w satisfies $(B.13)$ and $\psi = \nabla^{2k-2(2j+l)} w \in W_0^{2j+l,2}$ is an admissible test function. Thus we estimate

$$\begin{aligned}
\int_{B_r(x_0)} |\nabla^\alpha w|^2 \, dx &= (-1)^k \int_{B_r(x_0)} (g - \overline{g}_{B_r(x_0)}) \nabla^l \Delta^j w \, dx \\
&\leq \frac{1}{2} \int_{B_r(x_0)} |g - \overline{g}_{B_r(x_0)}|^2 \, dx + \frac{1}{2} \int_{B_r(x_0)} |\nabla^\alpha w|^2 \, dx \ ,
\end{aligned}$$

i.e.

$$\int_{B_r(x_0)} |\nabla^\alpha w|^2 \, dx \leq C \int_{B_r(x_0)} |g - \overline{g}_{B_r(x_0)}|^2 \, dx \leq C[g]^2_{\mathcal{L}^{2,\lambda}(B)} r^\lambda \ . \qquad \text{(B.15)}$$

Combining (B.14) and (B.15) yields

$$\begin{aligned}
&\int_{B_\rho(x_0)} |\nabla^\alpha u - \overline{\nabla^\alpha u}_{B_\rho(x_0)}|^2 \, dx \\
&\quad \le \int_{B_\rho(x_0)} |\nabla^\alpha v - \overline{\nabla^\alpha v}_{B_\rho(x_0)}|^2 \, dx + \int_{B_\rho(x_0)} |\nabla^\alpha w - \overline{\nabla^\alpha w}_{B_\rho(x_0)}|^2 \, dx \\
&\quad \le C \left(\frac{\rho}{r}\right)^{m+2} \int_{B_r(x_0)} |\nabla^\alpha u - \overline{\nabla^\alpha u}_{B_r(x_0)}|^2 \, dx + C \int_{B_r(x_0)} |\nabla^\alpha w|^2 \, dx \\
&\quad \le C \left(\frac{\rho}{r}\right)^{m+2} \int_{B_r(x_0)} |\nabla^\alpha u - \overline{\nabla^\alpha u}_{B_r(x_0)}|^2 \, dx + C[g]^2_{\mathcal{L}^{2,\lambda}(B)} r^\lambda \, .
\end{aligned}$$

The conclusion now follows as in the proof of Theorem III.2.2 in [28] by M. Giaquinta. Finally, we apply Lemma III.2.1, Theorem III.1.2 and Theorem III.1.3 in [28] to conclude.□

Appendix C

Structure results for calibrated currents

In this section we recall some structure theorems and other results proved in the fundamental paper [38] by R. Harvey and B. Lawson on calibrated geometries. The first one is a result on the structure of calibrated k-vectors.

Lemma C.1 (Lemma 7.1 in [38]) *Let ω be a constant coefficient k-form on $\mathbf{R}^m$ with comass equal to 1. Then for any unit k-vector τ with $\omega(\tau) = 1$ there are N unit simple k-vectors ξ_i with $\omega(\xi_i) = 1$ and real numbers $0 \leq \lambda_i \leq 1$ with $\sum_{i=1}^{N} \lambda_i = 1$ such that*

$$\tau = \sum_{i=1}^{N} \lambda_i \xi_i \ .$$

The structure Theorem below was used in the proof of the monotonicity formula in section 1.2.

Theorem C.2 (Theorem 5.11 in [38]) *Let $\tau = \sum_{i=1}^{N} \lambda_i \xi_i$ be the decomposition from Lemma C.1 for a unit k-vector which is calibrated by a constant form ω. Then*

$$\langle \tau, \omega^t \rangle = \sum_{i=1}^{N} \lambda_i \left| \xi_i \wedge \frac{\partial}{\partial r} \right|^2 \ .$$

For the perturbation argument in section 1.7 we used the following two results:

Theorem C.3 (Theorem 7.16 in [38]) *Let ω be a constant coefficient 2-form on $\mathbf{R}^m$ with comass equal to 1. Then there exists a subspace $\mathbf{R}^{2n} \subset \mathbf{R}^m$ of even dimension, a complex structure J on $\mathbf{R}^{2n}$ such that $\omega = dx_1 \wedge Jdx_1 + \cdots + dx_n \wedge Jdx_n$. In particular, any simple vector calibrated by ω is a J-holomorphic 2-vector in $\mathbf{R}^{2n}$.*

Lemma C.4 (Lemma 6.13 in [38]) *Let ξ be a unit simple k-vector in $\mathbf{R}^{2n}$ equipped with a complex structure J. Then there exist coordinates $x_1, Jx_1, \ldots, x_n, Jx_n$ on $\mathbf{R}^{2n}$ and angles*

$$0 \leq \theta_1 \leq \theta_2 \leq \cdots \leq \theta_{k-1} \leq \frac{\pi}{2}, \quad \theta_{k-1} \leq \theta_k \leq \pi \ ,$$

such that

$$\xi = \frac{\partial}{\partial x_1} \wedge \left(J\frac{\partial}{\partial x_1} \cos\theta_1 + \frac{\partial}{\partial x_2} \sin\theta_1 \right) \wedge \frac{\partial}{\partial x_3} \wedge \left(J\frac{\partial}{\partial x_3} \cos\theta_2 + \frac{\partial}{\partial x_4} \sin\theta_2 \right) \wedge \cdots \wedge$$
$$\wedge \frac{\partial}{\partial x_{2k-1}} \wedge \left(J\frac{\partial}{\partial x_{2k-1}} \cos\theta_k + \frac{\partial}{\partial x_{2k}} \sin\theta_k \right) .$$

Appendix D

Constructing the projection

The existence of the map π immediately follows from the Lemma below (compare with Lemma A.2 in the appendix of [70] by T. Rivière and G. Tian). From now on we denote by D^2_r the disk of radius $r > 0$ in $\mathbf{R}^2$ and by B^{2m-2}_r the ball of radius $r > 0$ in $\mathbf{R}^{2m-2}$.

Lemma D.1 *Given $\epsilon > 0$ there exists $\alpha > 0$ such that for any almost complex structure J on $D^2_2 \times B^{2m-2}_2$ verifying*

1. $J(0) = J_0$,
2. $\|J - J_0\|_{C^{l,\nu}} \leq \alpha$ *for* $l \geq 3, 0 < \nu < 1$,

and for any $p \in \mathbf{CP}^{m-1}$ such that $p = [p_1, \ldots, p_m]$ with $|p_i| \leq |p_1|$ for all $1 \leq i \leq m$, there exists an embedded J-holomorphic disk D_p in $(\mathbf{R}^{2m}, J)$ such that

1. $D_p = \{(z_1, \ldots, z_m) \in (\mathbf{R}^{2m}, J_0) \,|\, (z_1, \ldots, z_m) = (z_1, h_p(z_1))$ *for* $z_1 \in D^2_1\}$, *where* $h_p : D^2_1 \longrightarrow B^{2m-2}_2$ *is smooth* ,
2. $\left\| h_p(z_1) - \left(\frac{p_2}{p_1} z_1, \ldots, \frac{p_m}{p_1} z_1\right) \right\|_{C^{l,\nu}} \leq \epsilon$,
3. $T_0 D_p = p$.

Furthermore, given any $w = (w_1, \ldots, w_m) \in D^2_1 \times B^{2m-2}_1 \setminus \{(0, \ldots, 0)\}$ with $|w_i| \leq \frac{|w_1|}{2}$ for all $1 \leq i \leq m$, there exists a unique $p \in \mathbf{CP}^{m-1}$ with $|p_i| \leq |p_1|$ for all $1 \leq i \leq m$ such that $w \in D_p$.

Proof. The proof will follow from constructing a suitable perturbation of the J_0-holomorphic disk defined by p. In fact, given $w = (w_1, \ldots, w_m) \in D^2_2 \times B^{2m-2}_2$ and p as above, we denote the J_0-holomorphic disk defined by p passing through w by

$$D^0_{w,p} = \{(z_1, \ldots, z_m) \in (\mathbf{R}^{2m}, J_0) \,|\, (z_1, \ldots, z_m) = (z_1, h^0_{w,p}(z_1)) \text{ for } z_1 \in D^2_2\} ,$$

where the map defining this disk, namely $h^0_{w,p} : D^2_2 \longrightarrow B^{2m-2}_2$, is given by $h^0_{w,p}(z_1) = \left(\frac{p_2}{p_1}(z_1 - w_1) + w_2, \ldots, \frac{p_m}{p_1}(z_1 - w_1) + w_m\right)$. In other words, we first write $D^0_{w,p}$ as graph $H^0_{w,p}$ of $h^0_{w,p}$ over D^2_2 and now look for a small perturbation of the form $\hat{H}_{w,p}(z_1) := \big(z_1 + \mu(z_1), h^0_{w,p}(z_1) + \lambda(z_1)\big)$ with $\mu : D^2_2 \longrightarrow D^2_2$ and $\lambda : D^2_2 \longrightarrow B^{2m-2}_2$. The aim is to turn $\hat{H}_{w,p}$ into a J-holomorphic curve in $D_2^2 \times B_2^{2m-2}$, i.e. we require $\hat{H}_{w,p}$ to solve for $z_1 = x + iy$

$$\frac{\partial \hat{H}_{w,p}}{\partial y} = J(\hat{H}_{w,p})\frac{\partial \hat{H}_{w,p}}{\partial x} = i\frac{\partial \hat{H}_{w,p}}{\partial x} + (J - J_0)(\hat{H}_{w,p})\frac{\partial \hat{H}_{w,p}}{\partial x}$$

or, since $H^0_{w,p}$ is J_0-holomorphic, written alternatively as

$$\frac{\partial T_{w,p}}{\partial \bar{z}_1} = i(J - J_0)(\hat{H}_{w,p})\frac{\partial H^0_{w,p}}{\partial x} + i(J - J_0)(\hat{H}_{w,p})\frac{\partial T_{w,p}}{\partial x} \ , \tag{D.1}$$

where $\hat{H}_{w,p} = H^0_{w,p} + T_{w,p}$. To solve equation (D.1) for $T_{w,p}$ first note that the following is a well-posed elliptic problem from $C^{l,\nu}(D_1^2)$ into $C^{l+1,\nu}(D_1^2)$

$$\begin{cases} \frac{\partial u}{\partial \bar{z}_1} = F & \text{in } D_1^2 \\ u|_{\partial D_1^2} \in \operatorname{span}\{e^{-i\theta k} \,|\, k \in \mathbf{N}\} \ . \end{cases} \tag{D.2}$$

Provided $\|J - J_0\|_{C^{l,\nu}(D_2^2 \times B_2^{2m-2})}$ is sufficiently small, we can apply (D.2) to solving (D.1) by a fixed point argument in $C^{l,\nu}(D_1^2)$ and $C^{l+1,\nu}(D_1^2)$. Furthermore, the solution $T_{w,p}$ obtained this way satisfies $\|T_{w,p}\|_{C^{l+,\nu}(D_1^2)} \leq C(\alpha)$ with $C(\alpha) \longrightarrow 0$ as $\alpha \longrightarrow 0$ and there are estimates

$$\left\| \frac{\partial^{i+j} T_{w,p}}{\partial w^i \partial p^j} \right\|_{C^{l+1-i-j,\nu}(D_1^2)} \leq C(\alpha) \tag{D.3}$$

for the derivatives along the variables w, p defining $H^0_{w,p}$. Since the fact that $T_{w,p}$ is small in $C^{l+1,\nu}$ implies the same for the perturbation μ, defining the shifting of z_1 by $\mu(z_1)$ as $\zeta(z_1) := z_1 + \mu(z_1)$ we denote the composition with $\hat{H}_{w,p}$ by $\tilde{H}_{w,p} := \hat{H}_{w,p} \circ \zeta^{-1}$ so that $\tilde{H}_{w,p}(z_1) = (z_1, \tilde{h}_{w,p}(z_1))$ for some smooth function $\tilde{h}_{w,p} : D_1^2 \longrightarrow B_1^{2m-2}$.

Before we can use $\tilde{h}_{w,p}$ to finally define h_p, we need the following map which we will see is a perturbation of the identity

$$\begin{aligned} \Psi : B_1^{2m-2} \times \mathbf{CP}^{m-1} \quad &\longrightarrow \quad B_1^{2m-2} \times \mathbf{CP}^{m-1} \\ ((w_2, \ldots, w_m), q) \quad &\longmapsto \quad \left(\tilde{h}_{w,q}, \left[\frac{\partial \hat{H}_{w,q}}{\partial z_1}(\zeta^{-1}(0))\right]\right) \ , \end{aligned}$$

where as before $\left[\frac{\partial \hat{H}_{w,q}}{\partial z_1}(\zeta^{-1}(0))\right]$ denotes the point in $\mathbf{CP}^{m-1}$ defined by $\frac{\partial \hat{H}_{w,q}}{\partial z_1}(\zeta^{-1}(0))$. Since $\hat{H}_{w,q}$ and ζ are small perturbations of $H^0_{w,q}$ and respectively the identity, the map Ψ can indeed be written as $\Psi = \mathrm{id} + K$, where from (D.1) K satisfies $\|\nabla K\|_{C^{l,\nu}(B_1^{2m-2})} \leq C(\alpha)$. Since we can choose $\alpha > 0$ sufficiently small, Ψ becomes a local diffeomorphism from $B_1^{2m-2} \times U$, where $U \subset \mathbf{CP}^{m-1}$ is the open subset defined by our condition on p. Using the implicit function theorem there exist smooth functions $w_p : B_1^{2m-2} \longrightarrow B_1^{2m-2}$ and $q_p : \mathbf{CP}^{m-1} \longrightarrow \mathbf{CP}^{m-1}$ such that on $B_1^{2m-2} \times U$ we have $\Psi(w_p, q_p) = (0, p)$.

Having obtained this, we can finally define the map $h_p(z_1) := \tilde{h}_{w_p,q_p}(z_1)$ leading to the definition

$$D_p = \{(z_1, \ldots, z_m) \in (\mathbf{R}^{2m}, J_0) \,|\, (z_1, \ldots, z_m) = (z_1, h_p(z_1)) \text{ for } z_1 \in D_1^2\} \ .$$

By construction, D_p is a J-holomorphic disk passing through the origin since $\hat{H}_{w_p,q_p} \circ \zeta^{-1}(0) = \tilde{H}_{w_p,q_p}(w_p) = (0,\ldots,0)$. Furthermore, since $\hat{H}_{w_p,q_p}$ is J-holomorphic and $J(0) = J_0$, we also deduce that $p = \left[\frac{\partial \hat{H}_{w_p,q_p}}{\partial z_1}(\zeta^{-1}(0))\right]$ coincides with $T_0 D_p$ completing the proof of the first assertions of the Lemma.

To prove the last assertion, let $w = (w_1,\ldots,w_m)$ be a point with $|w_i| \leq \frac{|w_1|}{2}$ for all i. We need to show that there is a unique $p \in U \subset \mathbf{CP}^{m-1}$ such that $w \in D_p$. In other words, we look for a unique $p \in U \subset \mathbf{CP}^{m-1}$ such that $w = (w_1, \tilde{h}_{\Psi^{-1}(0,p)}(w_1))$, where we will abbreviate $\tilde{h}_{\Psi^{-1}(0,p)}$ by $\tilde{h}_p$. As above, we will use a perturbation argument to prove this. Define a map

$$\begin{array}{rcl} \Phi_w : U & \longrightarrow & \mathbf{CP}^{m-1} \\ p & \longmapsto & [w_1, \tilde{h}_p(w_1)] \ , \end{array}$$

which we claim to be a small perturbation of the identity independent of w where

$$\begin{array}{rcl} \mathrm{id} : U & \longrightarrow & U \\ p & \longmapsto & [w_1, h^0_{0,p}(w_1)] \ . \end{array}$$

Using coordinates on U given by $\phi : U \longrightarrow \mathbf{C}^{m-1}$, $\phi(p) = \phi([p_1,\ldots,p_m]) = \left(\frac{p_2}{p_1},\ldots,\frac{p_m}{p_1}\right)$, we thus have to verify that for α sufficiently small but independent of w,

$$\left\| \frac{\partial}{\partial \phi_i}(\Phi_w - \mathrm{id})(\phi(p)) \right\|_\infty = \left\| \frac{\partial}{\partial \phi_i}\left(\frac{\tilde{h}_p(w_1)}{w_1} - \frac{h^0_{0,p}(w_1)}{w_1} \right) \right\|_\infty \leq \epsilon \ .$$

Defining $r_p := \tilde{h}_p - h^0_{0,p}$ and noting that $r_p(0) = \tilde{h}_p(0) - h^0_{0,p}(0) = 0$ with the same for the first derivative, we therefore estimate

$$\|\nabla(\Phi_w - id)\|_\infty \leq \left\| \frac{1}{w_1}\left(\frac{\partial r_p(w_1)}{\partial \phi_i} - \frac{\partial r_p(0)}{\partial \phi_i} \right) \right\|_\infty \leq \left\| \nabla_{w_1} \frac{\partial r_p}{\partial \phi_i} \right\|_\infty \leq C(\alpha) \ ,$$

where the last estimate follows from estimate (D.3). Consequently, for α sufficiently small Ψ_w is a diffeomorphism onto its image which is close to U. We then define $p_w := \Psi_w^{-1}([w_1,\ldots,w_m])$ and the disk D_{p_w} defined by it is the unique disk containing w. Therefore the Lemma is proved. $\square$

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