Minmax Methods in the Calculus of Variations of Curves and Surfaces

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I Lecture 4

Critical and Sub-critical Curvature Energies of Surfaces

In this lecture we shall develop our understanding of potential **"smoothing lagrangians"** in order to implement the **viscosity approach** for the study of **minmax procedures for surfaces**.

I.1 Elementary Notions related to the Immersions of Surfaces in $\mathbb{R}^3$.

I.1.1 The 2 first Fundamental Forms.

We shall mostly restrict to immersions of surfaces in the 3 dimensional euclidian space. Σ^2 will denote a two dimensional manifold that we assume to be oriented and closed. The main objects of this lecture are immersions $\vec{\Phi}$ of Σ^2 into $\mathbb{R}^3$. It is usual to assume the immersion to be at least C^2 in order to define the curvature tensors associated to such an immersion, however we shall need weak notions of immersions and we shall first assume that the immersion is $W^{2,p}_{imm}(\Sigma^2,\mathbb{R}^3)$ for some $p>2$.

For such a $W^{2,p}$ immersion $\vec{\Phi}$ we define the **1st fundamental form**, the metric given by

$$\forall\ X,Y\ \in T_p\Sigma \qquad g_{\vec{\Phi}}(X,Y) := d_X\vec{\Phi}\cdot d_Y\vec{\Phi}$$

Assuming that $\vec{\Phi}$ is an immersion amounts to say that for any smooth reference metric g_0 on Σ there exists a constant $C_{\vec{\Phi}}>1$ such that

$$C_{\vec{\Phi}}^{-1}\ (g_0) \le (g_{\vec{\Phi}}) \le C_{\vec{\Phi}}^{-1}\ (g_0)$$

where the inequalities have to be understood in the sense of symmetric positive bilinear forms on $T\Sigma$. When there is no ambiguity about the

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underlying immersion we simply write $\vec{X} := d_X\vec{\Phi}$, the "realization" of X in $\mathbb{R}^3$ by $\vec{\Phi}$.

The associated **volume form** $dvol_{g_{\vec{\Phi}}}$ is given in a local chart by

$$dvol_{g_{\vec{\Phi}}} := \sqrt{g_{11}\,g_{22}-g_{12}^2}\;dx_1\wedge dx_2 \quad \text{where} \quad g_{ij} := \partial_{x_i}\vec{\Phi}\cdot\partial_{x_j}\vec{\Phi} \quad .$$

For such an immersion we define the **Gauss map** $\vec{n}$ given in local positive charts[1] by

$$\vec{n}_{\vec{\Phi}} := \frac{\partial_{x_1}\vec{\Phi}\wedge\partial_{x_2}\vec{\Phi}}{|\partial_{x_1}\vec{\Phi}\wedge\partial_{x_2}\vec{\Phi}|} \tag{I.1}$$

Observe that, if $\vec{\Phi}\in W^{2,p}_{imm}(\Sigma^2,\mathbb{R}^3)$ the Gauss Map $\vec{n}_{\vec{\Phi}}\in W^{1,p}(\Sigma,S^2)$. We can define the **2nd fundamental form** of the immersion $\vec{\Phi}$ using the Gauss map

$$\vec{\mathbb{I}}_{\vec{\Phi}}(X,Y) = -\,d_X\vec{n}\cdot\vec{Y}\;\vec{n}$$

Under the assumption that $\vec{\Phi}\in W^{2,p}_{imm}(\Sigma^2,\mathbb{R}^3)$ we have that $\vec{\mathbb{I}}_{\vec{\Phi}}\in L^p(\Sigma)$. If we extend locally X and Y, since we have $\vec{n}\cdot\vec{Y}=0$ we deduce, denoting by P^T the projection onto the tangent plane of $\vec{\Phi}_*T\Sigma^2$ a we havend P^N the projection onto the normal direction

$$\vec{\mathbb{I}}_{\vec{\Phi}}(X,Y) = \;d_X\vec{Y}\cdot\vec{n}_{\vec{\Phi}}\;\vec{n}_{\vec{\Phi}} = P^N(d_X\vec{Y}) = d_X\vec{Y}-P^T(d_X\vec{Y}) = \nabla^{g_{\mathbb{R}^3}}_{\vec{X}}\vec{Y}-\nabla^{g_{\vec{\Phi}}}_{X}Y$$

which is clearly symmetric since the two Levi-Civita connections $\nabla^{g_{\mathbb{R}^3}}$ and $\nabla^{g_{\vec{\Phi}}}$ are both torsion free. We introduce the **mean curvature vector**

$$\vec{H}_{\vec{\Phi}} := \frac{1}{2}\mathrm{Tr}\left[g^{-1}\mathbb{I}\right] = \frac{1}{2}\sum_{ij} g^{ij}\vec{\mathbb{I}}_{ij}$$

where (g^{ij}) is the inverse matrix to $g_{ij} := \partial_{x_i}\vec{\Phi}\cdot\partial_{x_j}\vec{\Phi}$. Writing $\vec{H} = H_{\vec{\Phi}}\,\vec{n}_{\vec{\Phi}}$ we can identify $H_{\vec{\Phi}}$ with the average of the two eigenvalues κ_1 and κ_2 of $\mathbb{I} = \vec{n}_{\vec{\Phi}}\cdot\vec{\mathbb{I}}_{\vec{\Phi}}$ which is symmetric and hence diagonalizable in an orthonormal basis of the tangent plane to the immersion :

$$H_{\vec{\Phi}} := \frac{\kappa_1+\kappa_2}{2}$$

The **Gauss curvature** is given by

$$K := \kappa_1\,\kappa_2$$

Since $|\vec{\mathbb{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}} = g^{kl}\,g^{ij}\,\mathbb{I}_{ik}\mathbb{I}_{lj} = \kappa_1^2+\kappa_2^2$, the **Gauss Identity** says

$$|d\vec{n}|^2_{g_{\vec{\Phi}}} = |\vec{\mathbb{I}}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}} = 4\,H^2-2\,K \quad . \tag{I.2}$$

[1] We are using the identification of 2−vectors with vectors for the canonical metric on $\mathbb{R}^3$.

I.1.2 The Willmore Energy

Recall the **Gauss Bonnet Theorem**

$$\int_\Sigma K = 2\,\pi\,\chi(\Sigma) = 4\pi(1-\text{genus}(\Sigma))$$

where $\chi(\Sigma)$ is the **Euler characteristic** of Σ. Then, modulo a topological term, the so called **Willmore energy**

$$W(\vec{\Phi}) = \int_\Sigma H^2\ dvol_{g_{\vec{\Phi}}}$$

is equal to the homogeneous $\dot{W}^{1,2}$ norm of the Gauss map. The Willmore energy, which is zero exactly on minimal surfaces, has been extensively studied in particular in conformal geometry. This is due to the following property proved in [7].

Proposition I.1. *For every conformal transformation Ψ of $\mathbb{R}^3\cup\{\infty\}$ and any immersion $\vec{\Phi}$ of Σ such that $\vec{\Phi}(\Sigma)\cap\Psi^{-1}(\{\infty\})=\emptyset$ one has*

$$W(\Psi\circ\vec{\Phi}) = W(\vec{\Phi})$$

□

The absolute minimizers among immersions of closed surfaces are characterized by the following theorem whose proof can be found in [8]

Theorem I.1. *For any immersion $\vec{\Phi}$ of a closed oriented 2-dimensional manifold Σ^2 one has*

$$W(\vec{\Phi}) \ge 4\pi$$

with equality if and only if $\Sigma\simeq S^2$ and $\vec{\Phi}(S^2)$ is a canonical round sphere. □

A much more involved results asserts that this bound is higher if one assumes that the genus of the surface is non zero.

Theorem I.2. [Marques, Neves 2011] *For any immersion $\vec{\Phi}$ of a closed oriented 2-dimensional manifold Σ^2 of non zero genus into $\mathbb{R}^3$ one has*

$$W(\vec{\Phi}) \ge 2\pi^2$$

with equality if and only if $\vec{\Phi}(\Sigma^2)$ is the image of the Clifford torus given by $1/\sqrt{2}\ (S^1\times S^1)\subset S^3\subset\mathbb{C}^2$ by the composition of the stereographic projection with a conformal transformation of $\mathbb{R}^3\cup\{\infty\}$. □

Remark I.1. *This question was left open by T.Willmore in his famous 1965 paper [11]. The so called* **Willmore conjecture** *is still open in higher codimension.* $\square$

Isothermal charts are special coordinates that are going to play a central in the study of the variations of surfaces.

Definition I.1. *Let $\vec{\Phi}\colon \Sigma \to \mathbb{R}^3$ be an immersion of Σ. A chart $\psi\colon D^2 \to \Sigma$ is called* isothermal *or* conformal *for $\vec{\Phi}$, if*

$$\begin{cases} \langle \partial_{x_1}(\vec{\Phi}\circ\psi), \partial_{x_2}(\vec{\Phi}\circ\psi)\rangle = 0 \quad \text{in } D^2 \\ \\ |\partial_{x_1}(\vec{\Phi}\circ\psi)| = |\partial_{x_2}(\vec{\Phi}\circ\psi)| \quad\quad \text{in } D^2. \end{cases} \tag{I.3}$$

Here, $\langle \partial_{x_1}(\vec{\Phi}\circ\psi), \partial_{x_2}(\vec{\Phi}\circ\psi)\rangle$ denotes the usual inner product in $\mathbb{R}^3$. $\square$

The following result is very specific to the two dimensional case. A proof using **moving frames** can be found in [7]

Theorem I.3. *Let $\vec{\Phi} \in W^{2,p}_{imm}(\Sigma, \mathbb{R}^3)$ and $p > 2$ then about every point in Σ there exist a conformal chart.* $\square$

In conformal charts many objects defined in the previous subsection take an easier form, which we want to explore now.

Let $\psi\colon D^2 \to \Sigma$ be an isothermal chart for the immersion $\vec{\Phi}\colon \Sigma \to \mathbb{R}^m$.

The first fundamental form in the coordinates provided by ψ is

$$\psi^* g_{\vec{\Phi}} = e^{2\lambda}(dx_1^2 + dx_2^2),$$

where $e^{\lambda} = |\partial_{x_1}(\vec{\Phi}\circ\psi)| = |\partial_{x_2}(\vec{\Phi}\circ\psi)|$. The volume element is given by

$$dvol_g := e^{2\lambda}\, dx_1 \wedge dx_2 \quad .$$

Moreover, for the second fundamental form, we have

$$\vec{\mathbb{I}}_{ij} := \vec{\mathbb{I}}(\partial_{x_i}, \partial_{x_j}) = P^N\left(\partial_{x_j}\partial_{x_i}(\vec{\Phi}\circ\psi)\right). \tag{I.4}$$

Note that

$$
\begin{aligned}
&\langle \partial^2_{x_1}(\vec{\Phi}\circ\psi), \partial_{x_1}(\vec{\Phi}\circ\psi)\rangle \\
&= \partial_{x_1} \underbrace{\langle \partial_{x_1}(\vec{\Phi}\circ\psi), \partial_{x_1}(\vec{\Phi}\circ\psi)\rangle}_{=e^{2\lambda}} - \langle \partial_{x_1}(\vec{\Phi}\circ\psi), \partial^2_{x_1}(\vec{\Phi}\circ\psi)\rangle \\
&= \frac{1}{2}\partial_{x_1}\left(e^{2\lambda}\right) = \frac{1}{2}\partial_{x_1}\langle \partial_{x_2}(\vec{\Phi}\circ\psi), \partial_{x_2}(\vec{\Phi}\circ\psi)\rangle \qquad (\mathrm{I}.5)\\
&= \langle \partial_{x_1}\partial_{x_2}(\vec{\Phi}\circ\psi), \partial_{x_2}(\vec{\Phi}\circ\psi)\rangle \\
&= \partial_{x_2} \underbrace{\langle \partial_{x_1}(\vec{\Phi}\circ\psi), \partial_{x_2}(\vec{\Phi}\circ\psi)\rangle}_{=0} - \langle \partial^2_{x_2}(\vec{\Phi}\circ\psi), \partial_{x_1}(\vec{\Phi}\circ\psi)\rangle \\
&= -\langle \partial^2_{x_2}(\vec{\Phi}\circ\psi), \partial_{x_1}(\vec{\Phi}\circ\psi)\rangle .
\end{aligned}
$$

Similarly, one obtains that

$$
\langle \partial^2_{x_1}(\vec{\Phi}\circ\psi), \partial_{x_2}(\vec{\Phi}\circ\psi)\rangle = -\langle \partial^2_{x_2}(\vec{\Phi}\circ\psi), \partial_{x_2}(\vec{\Phi}\circ\psi)\rangle . \tag{I.6}
$$

(I.4), (I.5) and (I.6) together imply that

$$
\vec{\mathbb{I}}_{11} + \vec{\mathbb{I}}_{22} = P^N\left(\Delta(\vec{\Phi}\circ\psi)\right) = \Delta(\vec{\Phi}\circ\psi), \tag{I.7}
$$

where $\Delta = \partial^2_{x_1} + \partial^2_{x_2}$ denotes the negative flat Laplacian.

From expression of the intrinsic negative Laplace-Beltrami operator Δ_g in coordinates

$$
\Delta_{g_{\vec{\Phi}}}\phi = \sqrt{det(g^{kl})}\sum_{i,j=1}^{2}\frac{\partial}{\partial x_i}\left(\frac{g^{ij}}{\sqrt{det(g^{kl})}}\frac{\partial\phi}{\partial x_j}\right)
$$

the following relation follows immediately:

$$
\Delta_g = e^{-2\lambda}\Delta. \tag{I.8}
$$

Using (I.7) and (I.8), the mean curvature vector takes the following form:

$$
\vec{H} = \frac{1}{2}\mathrm{Tr}(g^{-1}\,\vec{\mathbb{I}}) = \frac{e^{-2\lambda}}{2}\sum_{i,j=1}^{2}\delta^{ij}\,\vec{\mathbb{I}}_{ij} = \frac{e^{-2\lambda}}{2}\Delta(\vec{\Phi}\circ\psi) = \frac{1}{2}\Delta_g(\vec{\Phi}\circ\psi). \tag{I.9}
$$

The Willmore functional has therefore the expression

$$
W(\vec{\Phi}) = \frac{1}{4}\int_\Sigma |\Delta_g\vec{\Phi}|^2\, dvol_g. \tag{I.10}
$$

I.1.3 The Riemann Surface associated to an Immersion.

Assume $\vec{\Phi}$ is a conformal immersion of the disc D^2. Denote $e^\lambda := |\partial_{x_1}\vec{\Phi}| = |\partial_{x_2}\vec{\Phi}|$. We consider the following **orthonormal frame**

$$(\vec{e}_1, \vec{e}_2) = e^{-\lambda}\,(\partial_{x_1}\vec{\Phi}, \partial_{x_2}\vec{\Phi})$$

A short computation gives

$$\vec{e}_1 \cdot \nabla \vec{e}_2 = -\nabla^\perp \lambda \tag{I.11}$$

where $\nabla^\perp \cdot = (-\partial_{x_2}\cdot\ , \partial_{x_1}\cdot)$. Hence

$$\begin{aligned} -\,\Delta\lambda &= -\operatorname{div}\left(\vec{e}_1 \cdot \nabla^\perp \vec{e}_2\right) = -\,\nabla\vec{e}_1 \cdot \nabla^\perp \vec{e}_2 \\ &= \partial_{x_1}\vec{e}_1 \cdot \partial_{x_2}\vec{e}_2 - \partial_{x_2}\vec{e}_1 \cdot \partial_{x_1}\vec{e}_2 \end{aligned} \tag{I.12}$$

Observe that

$$\partial_{x_1}\vec{e}_1 \cdot \partial_{x_2}\vec{e}_2 - \partial_{x_2}\vec{e}_1 \cdot \partial_{x_1}\vec{e}_2 = P^N(\partial_{x_1}\vec{e}_1) \cdot P^N(\partial_{x_2}\vec{e}_2) - P^N(\partial_{x_2}\vec{e}_1) \cdot P^N(\partial_{x_1}\vec{e}_2)$$

and $P^N(\partial_{x_i}\vec{e_j}) = e^{-\lambda}\ P^N(\partial_{x_i}\partial_{x_j}\vec{\Phi}) = e^{-\lambda}\ \mathbb{I}_{ij}$. Recall that

$$K = \det(g^{-1}\mathbb{I}) = e^{-4\lambda}\ \det(\mathbb{I})$$

Hence we have obtained the **Liouville Equation**

$$-\ \Delta\lambda = e^{2\lambda}\,K \tag{I.13}$$

More generally we have the following result.

Proposition I.2. *Let $\vec{\Phi}\colon \Sigma \to \mathbb{R}^3$ be a smooth immersion and $g := \vec{\Phi}^* g_{\mathbb{R}^3}$ the induced first fundamental form. Let h be a conformally equivalent metric on Σ satisfying*

$$g = e^{2\alpha} h.$$

Then

$$-\,\Delta_h \alpha = e^{2\alpha} K_g - K_h, \tag{I.14}$$

where K_g and K_h are the Gauss curvatures of (Σ, g) and (Σ, h) respectively.

Proof of Proposition I.2. We have seen in theorem I.3 that there exist isothermal coordinates for $\vec{\Phi}$, i.e. locally g is of the form

$$g = e^{2\lambda}(dx_1^2 + dx_2^2). \tag{I.15}$$

Hence, we have

$$h = e^{2\sigma}(dx_1^2 + dx_2^2), \tag{I.16}$$

where $\sigma := \lambda - \alpha$.

Then we can apply (I.13) to (I.15) and (I.16) respectively and obtain

$$K_g = -e^{-2\lambda}\Delta\lambda \tag{I.17}$$

and

$$K_h = -e^{-2(\lambda-\alpha)}\Delta(\lambda - \alpha) \tag{I.18}$$

Combining (I.17) and (I.18) yields

$$K_h = e^{2\alpha}\left(K_g + e^{-2\lambda}\Delta\alpha\right) = e^{2\alpha}K_g + \Delta_h\alpha,$$

where we used (I.8) in the last identity. This finishes the proof. □

The following result, whose proof is a way beyond the scope of these lectures and can be found for instance in [10], is central in surface theory and is one of the variants of the so called **uniformization theorem** that we apply to the special case of metrics induced by an immersion into $\mathbb{R}^3$.

Theorem I.4. *Let Σ be a closed oriented two dimensional surface. Then there exists a conformally equivalent Riemannian metric h on Σ,*

$$g = e^{2\alpha}h,$$

with

- *constant (sectional) curvature K_h and*
- *unit volume: $vol_h(\Sigma) = 1$.*

Such h is unique if the genus of the surface is larger or equal than one. For $\Sigma \simeq S^2$ it is unique modulo the pull-back by elements of the Möbius group of conformal transformations of the sphere given by the maps of the form

$$\Psi_{\vec{a}}(z) = (1 - |\vec{a}|^2)\frac{z - \vec{a}}{|z - \vec{a}|^2} - \vec{a}$$

where $\vec{a} \in B^3$

Moreover, we have

$$K_h \begin{cases} < 0 \\ = 0 \\ > 0 \end{cases} \quad \textit{if and only if } genus(\Sigma) \begin{cases} \geq 2 \\ = 1 \\ = 0 \end{cases} \quad .$$

□

I.2 Weak Closure of $W^{2,q}-$Immersions.

On the Banach manifold $W^{2,q}_{imm}(\Sigma,\mathbb{R}^3)$ equipped with the Finsler structure (see lecture 2 proposition **??**)

$$\|\vec{v}\|_{\vec{\Phi}}:=\left[\int_\Sigma\left[|\nabla^2\vec{v}|^2_{g_{\vec{\Phi}}}+|\nabla\vec{v}|^2_{g_{\vec{\Phi}}}+|\vec{v}|^2\right]^{q/2}\ dvol_{g_{\vec{\Phi}}}\right]^{1/q}+\|\,|\nabla\vec{v}|_{g_{\vec{\Phi}}}\,\|_{L^\infty(\Sigma)}$$

we introduce the following $q-$Energy

$$E_q(\vec{\Phi}):=\int_\Sigma\left[1+|\mathbb{I}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}\right]^p\ dvol_{g_{\vec{\Phi}}}$$

where $2\,p=q$. We have the following weak closure result

Theorem I.5. **[Langer 1985]** *Let $\vec{\Phi}_k$ be a sequence in $W^{2,q}_{imm}(\Sigma,\mathbb{R}^3)$ such that*

$$\limsup_{k\rightarrow+\infty}E_q(\vec{\Phi}_k)<+\infty$$

Then there exists a subsequence $\vec{\Phi}_{k'}$ and a sequence of $W^{2,q}-$diffeomorphisms $\Psi_{k'}$ and a sequence of constant vectors $\vec{A}_{k'}$ such that

$$\vec{\xi}_k:=\vec{\Phi}_{k'}\circ\Psi_{k'}+\vec{A}_{k'}\quad\rightharpoonup\quad\vec{\xi}_\infty\qquad\text{weakly in } W^{2,q}(\Sigma,\mathbb{R}^3)$$

moreover $\vec{\xi}_\infty\in W^{2,q}_{imm}(\Sigma,\mathbb{R}^3)$. □

The method adopted by Langer for proving theorem I.5 in [**?**] is following an "ambiant" approach where $\vec{\Phi}_k(\Sigma)$ is the central object rather than $\vec{\Phi}_k$. Due to the compact embedding of $W^{2,q}$ into C^1 for $q>2$ one can locally approximate the image $\vec{\Phi}_k(\Sigma)$ by local graphs that are going to converge weakly in $W^{2,q}$...etc. Before proving such a weak closure property one might try first to control the underlying conformal class of the sequence of $W^{2,q}-$immersions. We will prove the following result.

Theorem I.6. *Assume genus$(\Sigma)>0$. Let $\vec{\Phi}_k$ be a sequence in $W^{2,q}_{imm}(\Sigma,\mathbb{R}^3)$ such that*

$$\limsup_{k\rightarrow+\infty}\int_\Sigma\left[1+|\mathbb{I}_{\vec{\Phi}_k}|^2_{g_{\vec{\Phi}_k}}\right]^p\ dvol_{g_{\vec{\Phi}_k}}<+\infty$$

then the sequence of underlying constant curvature metric h_k of volume 1 such that $g_{\vec{\Phi}_k}=e^{2\alpha_k}\,h_k$ is pre-compact in any $C^l(\Sigma)$ topology ($l\in\mathbb{N}$) modulo the pull-back action by diffeomorphisms. □

Proof of theorem I.6. We present the proof in the case of the torus : $\Sigma\simeq T^2$. The proof for genus(Σ) does not differ much[2]. The uniformization

[2]One argues by contradiction and one uses Mumford compactification of the Moduli Spaces of conformal structures (see [2]). The estimates for the Flat tori in the T^2 case can be carried over to the analysis of the size of the **collar regions** that are assumed to be formed in order to get a contradiction (see [9])

theorem (see for instance [3]) gives that modulo the composition with diffeomorphisms (Σ, h_k) is a sequence of flat Tori of the form $\mathbb{R}^2/(\mathbb{Z}\times\tau_k\mathbb{Z})$ where $\tau = \tau_{1,k} + i\,\tau_k^2$

$$\tau_{2,k} > 0 \quad -\frac{1}{2} < \tau_{1,k} < \frac{1}{2} \quad |\tau_k| \ge 1 \quad \text{ and } \tau_{1,k} \ge 0 \quad \text{ if } |\tau_k| = 1 \quad .$$

and $h_k = e^{2\,c_k}\,[dx_1^2 + dx_2^2]$ where c_k satisfies

$$1 = \int_{\mathbb{R}^2/(\mathbb{Z}\times\tau_k\mathbb{Z})} e^{2\,c_k}$$

In order to prove the theorem we have to show that $\tau_{2,k}$ is in fact uniformly bounded. Consider in $\mathbb{R}^2/(\mathbb{Z}\times\tau_k\mathbb{Z})$ the foliation by circles given by

$$\gamma_{k,x_2}(t) := (x_2 + \tau_{1,k} + t, x_2) \qquad \forall\ x_2 \in [0, \tau_{2,k})$$

The immersion $\vec{\Phi}_k$ is conformal from $\mathbb{R}^2/(\mathbb{Z}\times\tau_k\mathbb{Z})$ into $\mathbb{R}^3$. Denote $e^{\lambda_k} := |\partial_{x_1}\vec{\Phi}_k| = |\partial_{x_2}\vec{\Phi}_k|$. The unit tangent to the image by $\vec{\Phi}_k$ of the circles $\gamma_{k,x_2}(t)$ is then given by

$$\vec{e}_{1,k} := e^{-\lambda_k}\ \partial_{x_1}\vec{\Phi}_k = e^{-\lambda_k}\,\frac{d}{dt}(\vec{\Phi}_k(\gamma_{k,x_2}(t))) \quad .$$

Fenchel theorem applied to the circle $\gamma_{k,x_2}(t)$ says

$$2\pi \le \int_{\gamma_{k,x_2}(t)} \kappa\ dl = \int_{\gamma_{k,x_2}(t)} \left|\frac{d\vec{e}_{1,k}}{dt}\right|\ dt = \int_{x_2+\tau_{1,k}}^{x_2+\tau_{1,k}+1} |\partial_{x_1}\vec{e}_{1,k}|\ dx_1$$

Observe that $|\partial_{x_1}\vec{e}_{1,k}|^2 = |\partial_{x_1}\vec{e}_{1,k}\cdot\vec{e}_{2,k}|^2 + |\partial_{x_1}\vec{n}\cdot\vec{e}_{1,k}|^2$ and since $\nabla\vec{e}_{1,k}\cdot\vec{e}_{2,k} = \nabla^\perp\lambda_k$, we have

$$2\pi \le \int_{x_2+\tau_{1,k}}^{x_2+\tau_{1,k}+1} |\nabla\lambda_k| + |\nabla\vec{n}_k| \quad .$$

Integrating for x_2 going from 0 to $\tau_{2,k}$ gives

$$2\pi\ \tau_{2,k} \le \int_{\mathbb{R}^2/(\mathbb{Z}\times\tau_k\mathbb{Z})} |\nabla\lambda_k| + |\nabla\vec{n}_k|\ dx^2 \tag{I.19}$$

This implies using Hölder for an $s \in (0,1)$ we are going to fix later

$$\begin{aligned} 2\pi\ \tau_{2,k} \le \tau_{2,k}^{(1-s)/2}\ &\left(\int_{\mathbb{R}^2/(\mathbb{Z}\times\tau_k\mathbb{Z})} |\nabla\lambda_k|^2\ e^{-2\,s\,\lambda_k}\ dx^2\right)^{1/2} \left(\int_{\mathbb{R}^2/(\mathbb{Z}\times\tau_k\mathbb{Z})} e^{2\lambda_k}\ dx^2\right)^{s/2}\\ &+\tau_{2,k}^{1/2}\ \left(\int_{\mathbb{R}^2/(\mathbb{Z}\times\tau_k\mathbb{Z})} |\nabla\vec{n}_k|^2\ \ dx^2\right)^{1/2} \end{aligned} \tag{I.20}$$

On $\mathbb{R}^2/(\mathbb{Z}\times\tau_k\mathbb{Z})$ the Liouville equation reads

$$-\Delta\lambda_k = e^{2\lambda_k}\,K_{\vec{\Phi}_k}$$

We multiply the equation by $-\,e^{-2\,s\,\lambda_k}$ and we integrate by parts. This gives for $s=1-2/q$

$$
\begin{aligned}
2\,s\int_{\mathbb{R}^2/(\mathbb{Z}\times\tau_k\mathbb{Z})} e^{-2\,s\lambda_k}\ |\nabla\lambda_k|^2\ dx^2 &= -\int_{\mathbb{R}^2/(\mathbb{Z}\times\tau_k\mathbb{Z})} e^{2(1-s)\lambda_k}\ K_{\vec{\Phi}_k}\\
&\le \left[\int_{\mathbb{R}^2/(\mathbb{Z}\times\tau_k\mathbb{Z})} |\mathbb{I}_{\vec{\Phi}_k}|^q_{g_{\vec{\Phi}_k}}\ dvol_{g_{\vec{\Phi}_k}}\right]^{2/q}\ \left[\int_{\mathbb{R}^2/(\mathbb{Z}\times\tau_k\mathbb{Z})} dx\right]^s\\
&\le C\,\tau_{2,k}^s\ \left[\int_{\mathbb{R}^2/(\mathbb{Z}\times\tau_k\mathbb{Z})} |\mathbb{I}_{\vec{\Phi}_k}|^q_{g_{\vec{\Phi}_k}}\ dvol_{g_{\vec{\Phi}_k}}\right]^{2/q}
\end{aligned}
\tag{I.21}
$$

Due to the conformal invariance of the Dirichlet energy we have

$$
\begin{aligned}
\int_{\mathbb{R}^2/(\mathbb{Z}\times\tau_k\mathbb{Z})} |\nabla\vec{n}_k|^2\ \ dx^2 &= \int_{\mathbb{R}^2/(\mathbb{Z}\times\tau_k\mathbb{Z})}\ |\mathbb{I}_{\vec{\Phi}_k}|^2_{g_{\vec{\Phi}_k}}\ dvol_{g_{\vec{\Phi}_k}}\\
&\le \left[\int_{\mathbb{R}^2/(\mathbb{Z}\times\tau_k\mathbb{Z})} |\mathbb{I}_{\vec{\Phi}_k}|^q_{g_{\vec{\Phi}_k}}\ dvol_{g_{\vec{\Phi}_k}}\right]^{2/q}\ \left[\int_{\mathbb{R}^2/(\mathbb{Z}\times\tau_k\mathbb{Z})}\ dvol_{\vec{\Phi}_k}\right]^s
\end{aligned}
\tag{I.22}
$$

Combining (I.20), (I.21) and (I.22) we get the existence of a constant C_q depending only on $q\in(2,+\infty)$ such that

$$
\sqrt{\tau_{2,k}}\le C_q\ \left[\int_{\mathbb{R}^2/(\mathbb{Z}\times\tau_k\mathbb{Z})} |\mathbb{I}_{\vec{\Phi}_k}|^q_{g_{\vec{\Phi}_k}}\ dvol_{g_{\vec{\Phi}_k}}\right]^{1/q}\left[\int_{\mathbb{R}^2/(\mathbb{Z}\times\tau_k\mathbb{Z})}\ dvol_{\vec{\Phi}_k}\right]^{1/2-1/q}
$$

we deduce that $\tau_{2,k}$ is uniformly bounded which implies the theorem I.6. $\Box$

One could wonder whether the previous result extends to the case when one only assumes that the trace of the second fundamental form is bounded : The answer to this question is negative.

Proposition I.3. *Assume genus* $(\Sigma)>0$. *There exists a sequence* $\vec{\Phi}_k$ *in* $W^{2,q}_{imm}(\Sigma,\mathbb{R}^3)$ *such that*

$$
\limsup_{k\to+\infty} H_p(\vec{\Phi}_k):=\int_\Sigma\left[1+|\vec{H}_{\vec{\Phi}_k}|^2_{g_{\vec{\Phi}_k}}\right]^{q/2}\ dvol_{g_{\vec{\Phi}_k}}<+\infty
$$

and the underlying conformal class of $g_{\vec{\Phi}_k}$ *has no pre-compact subsequence in the* **Moduli Space** $\mathfrak{M}(\Sigma)$ *of* Σ. *In other words no subsequence of the constant scalar curvature metric of volume 1* h_k *such that* $g_{\vec{\Phi}_k}=e^{2\alpha_k}\,h_k$ *converge.* $\Box$.

Proof of proposition I.3. The glueing of two opposite hemispheres by a tiny catenoid gives a surface with uniformly bounded mean curvature and with boundary given by two fixed circles. It suffices to join the two circles by a fixed cylinder in order to produce a family of tori with degenerating conformal class but with uniformly bounded area and mean curvature. □

However if one adds to the H_p energy the Dirichlet energy of the α_k factor in (I.14) there is again a **weak closure property**.

Theorem I.7. *Let $\vec{\Phi}_k$ be a sequence in $W^{2,q}_{imm}(\Sigma,\mathbb{R}^3)$ such that*

$$\limsup_{k\to+\infty} F_p(\vec{\Phi}_k) := \int_\Sigma \left[1+|\vec{H}_{\vec{\Phi}_k}|^2_{g_{\vec{\Phi}_k}}\right]^{q/2} + 2^{-1}\,|d\alpha_k|^2_{g_{\vec{\Phi}_k}}\ dvol_{g_{\vec{\Phi}_k}} < +\infty \quad . \tag{I.23}$$

Then there exists a subsequence $\vec{\Phi}_{k'}$ and a sequence of $W^{2,q}-$diffeomorphisms $\Psi_{k'}$ of Σ and a sequence of constant vectors $\vec{A}_{k'}$ such that

$$\vec{\xi}_k := \vec{\Phi}_{k'}\circ\Psi_{k'} + \vec{A}_{k'} \quad \rightharpoonup \quad \vec{\xi}_\infty \quad \text{weakly in } W^{2,q}(\Sigma,\mathbb{R}^3)$$

moreover $\vec{\xi}_\infty \in W^{2,q}_{imm}(\Sigma,\mathbb{R}^3)$. □

Before proving the theorem we shall make use of the following lemma.

Lemma I.1. *Assume genus$(\Sigma) > 0$. Let $\vec{\Phi}_k$ be a sequence in $W^{2,q}_{imm}(\Sigma,\mathbb{R}^3)$ such that*

$$\limsup_{k\to+\infty}\ \mathfrak{F}(\vec{\Phi}_k) := \int_\Sigma |\vec{H}_{\vec{\Phi}_k}|^2_{g_{\vec{\Phi}_k}} + 2^{-1}|d\alpha_k|^2_{g_{\vec{\Phi}_k}}\ dvol_{g_{\vec{\Phi}_k}} < +\infty$$

then the sequence of underlying constant curvature metric h_k of volume 1 such that $g_{\vec{\Phi}_k} = e^{2\alpha_k}\,h_k$ is pre-compact in any $C^l(\Sigma)$ topology ($l\in\mathbb{N}$) modulo the pull-back action by diffeomorphisms. □

Remark I.2. *In the case $\Sigma \simeq T^2$, for immersions into $\mathbb{R}^m$,the $\mathfrak{F}-$Energy has the following geometric interpretation*[3].

$$\mathfrak{F}(\vec{\Phi}) = \inf_{\vec{e}\in\mathcal{F}}\ \frac{1}{4}\int_\Sigma |d\vec{e}|^2_{g_{\vec{\Phi}}}\ dvol_{g_{\vec{\Phi}}} \quad ,$$

where $\mathcal{F}$ is the space of **orthonormal frames** *associated to the immersion, i.e. $\vec{e} = (\vec{e}_1,\vec{e}_2)$ where $\vec{e}_i \in W^{1,2}(T^2,S^{m-1})$ and for almost every x $(\vec{e}_1,\vec{e}_2)(x)$ realizes an orthonormal basis of $T_{\vec{\Phi}(x)}\vec{\Phi}(\Sigma)$. In [5] the* **Willmore conjecture** *for the Frame Energy in arbitrary co-dimension is proved. For all immersion $\vec{\Phi}$ on has*

$$\mathfrak{F}(\vec{\Phi}) \geq 2\pi^2$$

with equality if and only if $\vec{\Phi}(T^2)$ is the image of the Willmore torus by the composition of an isometry and a dilation. □

[3]The identity (I.24) below illustrates the link between $F(\vec{\Phi})$ and the Dirichlet energy of the frame given by local conformal coordinates

Proof of Lemma I.1. Again here we make the proof in the case $\Sigma \simeq T^2$. The higher genus case is similar and can be found in [9]. We can import inequality (I.19) from the proof of theorem I.6 and we write

$$
\begin{aligned}
2\pi\ \tau_{2,k} &\le \int_{\mathbb{R}^2/(\mathbb{Z}\times\tau_k\mathbb{Z})} |\nabla\lambda_k| + |\nabla\vec{n}_k|\ dx^2\\
&\le \sqrt{\tau_{2,k}}\ \sqrt{\int_{\mathbb{R}^2/(\mathbb{Z}\times\tau_k\mathbb{Z})} |\nabla\lambda_k|^2 + |\nabla\vec{n}_k|^2\ dx^2}
\end{aligned}
$$

Observe that λ is just the sum of α with a constant. Thus $\nabla\alpha = \nabla\lambda$, moreover the Gauss equation (I.2) we deduce that

$$
|\tau_k| \le C\ \mathfrak{F}(\vec{\Phi}) \quad ,
$$

which implies lemma I.1. □

Proof of theorem I.7. From the previous lemma I.1 we know that the underlying constant curvature metric of volume 1 is pre-compact in every C^l space modulo the pull-back by diffeomorphisms. So we can assume we are in a system of charts in which the metric h_k is strongly converging. We can also these charts to be conformal with respect to h_k and strongly converging with respect to a reference system of chart. In these charts we have

$$
h_k = e^{2\,\sigma_k}\ [dx_1^2 + dx_2^2] \quad \text{and} \quad g_{\vec{\Phi}_k} = e^{2\,\lambda_k}\ [dx_1^2 + dx_2^2]
$$

where μ_k is converging in every C^l space and $\alpha_k = \lambda_k - \sigma_k$. Hence we have by assumptions

$$
\limsup_{k\rightarrow+\infty} \int_{D^2} |\nabla\lambda_k|^2 dx^2 < +\infty \quad .
$$

Recall the computations (I.11) and (I.12) : for $\vec{e}_{j,k} := e^{-\lambda_k}\,\partial_{x_j}\vec{\Phi}_k$ we have that

$$
\vec{e}_{1,k}\cdot\nabla\vec{e}_{2,k} = -\nabla^\perp\lambda_k
$$

and the Liouville equation reads

$$
-\,\Delta\lambda_k =\ \partial_{x_1}\vec{e}_{1,k}\cdot\partial_{x_2}\vec{e}_{2,k} - \partial_{x_2}\vec{e}_{1,k}\cdot\partial_{x_1}\vec{e}_{2,k}
$$

Observe that

$$
\begin{aligned}
|\nabla\vec{e}_{1,k}|^2 + |\nabla\vec{e}_{2,k}|^2 &= |\nabla\vec{e}_{1,k}\cdot\vec{e}_{2,k}|^2 + |\nabla\vec{e}_{2,k}\cdot\vec{e}_{1,k}|^2\\
&\quad + |\nabla\vec{e}_{1,k}\cdot\vec{n}_{\vec{\Phi}_k}|^2 + |\nabla\vec{e}_{2,k}\cdot\vec{n}_{\vec{\Phi}_k}|^2
\end{aligned}
$$

Hence

$$
|\nabla\vec{e}_{1,k}|^2 + |\nabla\vec{e}_{2,k}|^2 = 2\,|\nabla\lambda_k|^2 + |\nabla\vec{n}|^2 \tag{I.24}
$$

Using the conformal invariance of the Dirichlet energy we have

$$\int_{D^2} |\nabla \vec{n}_{\vec{\Phi}_k}|^2 \, dx^2 = \int_{x^{-1}(D^2)} |d\vec{n}|^2_{g_{\vec{\Phi}_k}} \, dvol_{g_{\vec{\Phi}_k}} \le \int_\Sigma |d\vec{n}_{\vec{\Phi}_k}|^2_{g_{\vec{\Phi}}} \, dvol_{g_{\vec{\Phi}_k}} \tag{I.25}$$

Using Gauss identity and Gauss Bonnet theorem we have

$$\int_\Sigma |d\vec{n}_{\vec{\Phi}_k}|^2_{g_{\vec{\Phi}}} \, dvol_{g_{\vec{\Phi}_k}} = 4 \int_\Sigma |\vec{H}_{\vec{\Phi}_k}|^2_{g_{\vec{\Phi}}} \, dvol_{g_{\vec{\Phi}_k}} - 4\pi \, \chi(\Sigma) \tag{I.26}$$

Hence combining (I.24), (I.25) and (I.26) we obtain

$$\limsup_{k\to+\infty} \int_{D^2} |\nabla \vec{e}_{1,k}|^2 + |\nabla \vec{e}_{2,k}|^2 \, dx^2 < +\infty \quad . \tag{I.27}$$

A mean value argument gives the existence of a radius $\rho \in [1/2, 1]$ such that

$$\int_0^{2\pi} |\nabla \lambda_k|(\rho,\theta) \, d\theta \le \sqrt{3\pi} \left[\int_{D^2_1 \setminus D^2_{1/2}} |\nabla \lambda|^2 \, dx^2\right]^{1/2}$$

Hence there exists a sequence of constants $\overline{\lambda_k}$ such that

$$\limsup_{k\to+\infty} \|\lambda_k - \overline{\lambda_k}\|_{L^\infty(\partial D^2_\rho)} < +\infty \tag{I.28}$$

We recall **Wente theorem** which is a milestone in **integrability by compensation theory** (see a proof for instance in [8]).

Theorem I.8. *Let a and b be in $W^{1,2}(D^2,\mathbb{R})$ and let φ be the $W^{1,1}$ solution of*

$$\left\{\begin{array}{l} \Delta\varphi = \partial_{x_1} a \, \partial_{x_2} b - \partial_{x_2} a \, \partial_{x_1} b \qquad \textit{in } D^2 \\[3mm] \varphi = 0 \qquad\qquad \textit{on } \partial D^2 \end{array}\right.$$

then $\varphi \in W^{1,2} \cap C^0$ and

$$\|\varphi\|_{L^\infty(D^2)} + \|\nabla\varphi\|_{L^2(D^2)} \le C \, \|\nabla a\|_{L^2(D^2)} \, \|\nabla b\|_{L^2(D^2)} \quad .$$

where C is a universal positive constant. □

Let φ_k be the solution of

$$\left\{\begin{array}{l} -\Delta\varphi_k = \partial_{x_1}\vec{e}_{1,k} \cdot \partial_{x_2}\vec{e}_{2,k} - \partial_{x_2}\vec{e}_{1,k} \cdot \partial_{x_1}\vec{e}_{2,k} \qquad \text{in } D^2_\rho \\[3mm] \varphi_k = 0 \qquad\qquad \text{on } \partial D^2_\rho \end{array}\right.$$

Because of (I.27) Wente theorem implies

$$\limsup_{k\to+\infty} \|\varphi_k\|_{L^\infty(D^2_\rho)} < +\infty$$

Since using the maximum principle we obtain

$$\limsup_{k\to+\infty} \|\lambda_k - \overline{\lambda_k} - \varphi_k\|_{L^\infty(D^2_\rho)} = \limsup_{k\to+\infty} \|\lambda_k - \overline{\lambda_k}\|_{L^\infty(\partial D^2_\rho)} < +\infty$$

Combining the two previous estimates give

$$\limsup_{k\to+\infty} \|\lambda_k - \overline{\lambda_k}\|_{L^\infty(D^2_\rho)} < +\infty$$

The covering by conformal charts is strongly converging, hence we could have taken it fine enough in such a way that all the $x^{-1}(D^2_{1/2})$ are covering Σ and we deduce the existence of a sequence of constants $\tilde{\alpha}_k$ on each connected component of $\tilde{\Sigma}$ such that

$$\limsup_{k\to+\infty} \|\alpha_k - \tilde{\alpha}_k\|_{L^\infty(\tilde{\Sigma})} < +\infty \tag{I.29}$$

The lower bound on Willmore energy given theorem I.1 implies

$$4\pi \le \int_{\tilde{\Sigma}} H^2_{\vec{\Phi}_k}\ dvol_{g_{\vec{\Phi}_k}} \le \left[\int_{\tilde{\Sigma}} H^q_{\vec{\Phi}_k}\ dvol_{g_{\vec{\Phi}_k}}\right]^{2/q}\ \left[\int_{\tilde{\Sigma}}\ dvol_{g_{\vec{\Phi}_k}}\right]^{1-2/q}$$

hence the area is bounded from above and from below, which gives

$$\limsup_{k\to+\infty} \left|\log \mathrm{Area}(\vec{\Phi}_k(\tilde{\Sigma}))\right| < +\infty \tag{I.30}$$

Therefore there exists $p_k \in \tilde{\Sigma}$ such that

$$\limsup_{k\to+\infty} |\alpha_k(p_k)| < +\infty$$

Since, by Wente theorem, $\alpha_k \in C^0(\Sigma)$, we deduce from (I.29) that $\tilde{\alpha}_k$ is uniformly bounded and hence

$$\limsup_{k\to+\infty} \|\alpha_k\|_{L^\infty(\Sigma)} < +\infty \quad .$$

Taking now the expression of the mean curvature vector $\vec{H}_{\vec{\Phi}_k}$ given by (I.9), we obtain

$$\vec{H}_{\vec{\Phi}_k} = \frac{1}{2}\Delta_{g_{\vec{\Phi}_k}}\vec{\Phi}_k = \frac{e^{-2\alpha_k}}{2}\Delta_{h_k}\vec{\Phi}_k$$

Since h_k is strongly converging (in every C^l norm) with respect to some reference metric g_0, **Calderon Zygmund theory** gives

$$\limsup_{k\to+\infty} \|\vec{\Phi}_k\|_{W^{2,q}_{g_0}(\Sigma)} < +\infty$$

The theorem follows from this control of the $W^{2,q}$ norm of $\vec{\Phi}_k$ in a system of chart in which the fixed reference metric is C^∞. $\Box$

I.3 The space of weak immersions.

The limiting case $q = 2$ is more involved and requires additional work. First of all we don't have the embedding into C^1 anymore

$$W^{2,2}(\Sigma, \mathbb{R}^m) \not\hookrightarrow C^1(\Sigma, \mathbb{R}^m)$$

and in general $W^{2,2}(\Sigma, N^n)$ does not posses a Banach manifold structure. Nevertheless we can still introduce the following notion.

Definition I.2. [R. 2010] *A map $\vec{\Phi} \in W^{1,\infty}(\Sigma, \mathbb{R}^3)$ is called a* **weak Immersion** *if the rank of $d\vec{\Phi}$ is almost everywhere equal to 2 and*

i) There exists $C_{\vec{\Phi}} > 1$ such that on Σ

$$C_{\vec{\Phi}}^{-1}\ (g_0) \le\ (g_{\vec{\Phi}}) \le C_{\vec{\Phi}}\ (g_0)$$

where g_0 is a fixed reference metric[4] *on the surface Σ.*

ii) The **weak Gauss map** *$\vec{n}_{\vec{\Phi}}$ defined by (I.1) is in $W^{1,2}(\Sigma)$.*

We shall denote $\mathcal{E}_\Sigma$ the space of weak immersions of Σ. □

A striking fact about these weak immersions is the existence of an underlying <u>smooth</u> riemann surface.

Theorem I.9. [R. 2010] *Let $\vec{\Phi}$ be a weak immersion in $\mathcal{E}_\Sigma$. Then there exists $\alpha \in L^\infty \cap W^{1,2}(\Sigma, \mathbb{R})$ and a metric h of constant scalar Gauss curvature on Σ such that*

$$g_{\vec{\Phi}} = e^{2\alpha}\ h$$

□

This result is a consequence of **integrability by compensation theory** in the continuity of the works by S.Müller and V.Šverák (see [6]) as well as F.Hélein [1]. There are 3 main obstructions for extending the previous **sequential weak closure results** to the Willmore energy in the space $\mathcal{E}_\Sigma$. If we take a sequence $\vec{\Phi}_k$ such that

$$\int_\Sigma \left[1 + |\mathbb{I}_{\vec{\Phi}_k}|^2_{g_{\vec{\Phi}_k}}\right]\ dvol_{g_{\vec{\Phi}_k}} = \mathrm{Area}(\vec{\Phi}_k) + 4\, W(\vec{\Phi}_k) -\ 4\,\pi\,\chi(\Sigma) < +\infty \quad \text{(I.31)}$$

i) The Willmore energy W is in particular dilation invariant and nothing prevents that a sequence $\vec{\Phi}_k$ for which (I.31) holds can **collapse to a point**.

[4] It is straightforward to see that this condition does not depend on the initial choice of the reference metric since the surface is assumed to be closed.

ii) The Willmore energy W as well as the area are not coercive enough to control the conformal class of $(\Sigma, g_{\vec{\Phi}_k})$ and there is **no pre-compactness of the underlying riemann surface** (Σ, h_k) .

iii) Even if the sequence of surface is assumed not to collapse and have a pre-compact conformal class, **bubbling** can occur at isolated points.

These are however the 3 sole possible degeneracies. We have the following result.

Theorem I.10. **[R. 2010]** *Let $\vec{\Phi}_k$ be a sequence of weak immersions of a closed surface Σ. Assume that*

$$\limsup_{k\to+\infty} W(\vec{\Phi}_k) < +\infty$$

and that a constant Gauss curvature metric h_k of volume 1 such that

$$g_{\vec{\Phi}_k} = e^{2\,\alpha_k}\, h_k$$

is pre-compact (modulo the pull back by diffeomorphism of Σ) in C^0, then, modulo extraction of a subsequence, there exists a sequence of **bi-lipschitz diffeomorphisms** Ψ_k *of Σ and a sequence Ξ_k of bf conformal transformations of $\mathbb{R}^3 \cup \{\infty\}$ such that*

$$\vec{\xi}_k := \Xi_k \circ \vec{\Phi}_k \circ \Psi_k \quad \rightharpoonup \vec{\xi}_\infty \qquad \textit{weakly in } W^{2,2}_{loc}(\Sigma \setminus \{a_1 \cdots a_Q\}, \mathbb{R}^3)$$

where $a_1 \cdots a_Q$ are Q **blow-up points** *in Σ. Moreover $\vec{\xi}_\infty$ is a* **weak immersion** *away from the blow-ups.* □

Remark I.3. *The poof of this result is making use again of Wente integrability by compensation theorem I.8 combined with covering arguments (see a proof in [7]).* □

Remark I.4. *Within the space of conformal transformations of $\mathbb{R}^3 \cup \{\infty\}$ the use of inversion is mostly necessary in order to preserve the topology (see figure 1 below). If Σ is a 2 sphere one can simply restrict Ξ_k in the space of compositions of translations and dilations.* □

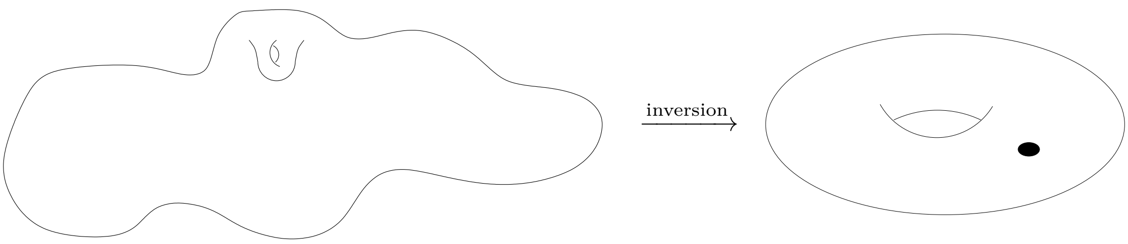

Figure 1: Loss of energy, no loss of topology.

Remark I.5. *In the absence of any control of the underlying conformal class, there is still a weak compactness result based on* **Mumford Compactification of the Moduli Spaces of Riemann Surfaces** *(see for instance [2]. In few words, there is a sequential weak closure result but only in the so called* **thick parts** *of the limiting nodal surface.* □

It remains now to confront our 2 main candidates respectively

$$E_q(\vec{\Phi}) = \int_\Sigma \left[1 + |\mathbb{I}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}\right]^{q/2} \ dvol_{g_{\vec{\Phi}_k}}$$

or

$$F_q(\vec{\Phi}) := \int_\Sigma \left[1 + |\vec{H}_{\vec{\Phi}}|^2_{g_{\vec{\Phi}}}\right]^{q/2} + 2^{-1}\,|d\alpha_k|^2_{g_{\vec{\Phi}}} \ dvol_{g_{\vec{\Phi}}}$$

for being "ideal smoothers" to the **Palais Smale Condition**. This will be one of the purposes of the next lecture.

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