Extremal and Probabilistic Combinatorics

January 23rd – January 26th at UCLA

Schedule

Wednesday, January 23

09:20 – 09:30	*Opening remarks*	
09:30 – 10:00	Dhruv Mubayi	*Quasirandom hypergraphs*
10:05 – 10:35	Yufei Zhao	*Sparse regularity and counting in pseudorandom graphs*
10:35 – 11:15	*Coffee*	
11:15 – 11:45	Asaf Shapira	*Exact bounds for some hypergraph saturation problems*
11:50 – 12:20	Po-Shen Loh	*Computing with voting trees*
12:20 – 14:30	*Lunch*	
14:30 – 15:00	Joel Spencer	*Six standard deviations still suffice*
15:05 – 15:35	Wenying Gan	*Erdős-Rademacher problems in extremal set theory*
15:35 – 16:15	*Coffee*	
16:15 – 16:45	Eyal Lubetzky	*Cores of random graphs are born Hamiltonian*
16:50 – 17:20	Boris Bukh	*Generalized Erdős-Szekeres theorem*

Thursday, January 24

09:30 – 10:00	John Lenz	*Spectral Turán-type hypergraph problems*
10:05 – 10:35	David Conlon	*Ramsey-type results for semi-algebraic relations*
10:35 – 11:15	*Coffee*	
11:15 – 11:45	David Gamarnik	*Hardness results for local algorithms in sparse random graphs*
11:50 – 12:20	Zoltán Füredi	*Turán-type hypergraph problems: partial trees and linear cycles*

Friday, January 25

09:30 – 10:00	Jacques Verstraete	*Coupon colourings of graphs*
10:05 – 10:35	Andrey Grinshpun	*The Erdős-Hajnal conjecture for rainbow triangles*
10:35 – 11:15	*Coffee*	
11:15 – 11:45	Hao Huang	*Densities of cliques and independent sets in graphs*
11:50 – 12:20	Dan Hefetz	*Optimal covers of random graphs with Hamilton cycles*
12:20 – 14:30	*Lunch*	
14:30 – 15:00	Alan Frieze	*Probabilistic analysis of multi-dimensional assignment problems*
15:05 – 15:35	Choongbum Lee	*Ramsey numbers of cubes versus cliques*
15:35 – 16:15	*Coffee*	
16:15 – 16:45	Fan Chung Graham	*Braess paradox in expander graphs*

Saturday, January 26

09:20 – 09:50	Sasha Kostochka	*Ramsey numbers for uniform hypergraphs: triangles versus cliques*
09:55 – 10:25	Jie Ma	*Discrepancy of random graphs and hypergraphs*
10:25 – 11:05	*Coffee*	
11:05 – 11:35	Michael Krivelevich	*Biased games on random boards*
11:40 – 12:10	Hoi Nguyen	*Random matrices: law of the determinant*
12:10 – 14:10	*Lunch*	
14:10 – 14:40	Jozsef Balogh	*Phase transition in Ramsey-Turán theory*
14:45 – 15:15	Jozsef Solymosi	*Locally sparse triple systems*
15:15 – 15:45	*Coffee*	
15:45 – 16:15	Jacob Fox	*Graph removal lemma*

Talks

Phase transition in Ramsey-Turán theory
Jozsef Balogh
University of Illinois at Urbana-Champaign

Denote by $RT(n, L, f(n))$ the maximum number of edges of an L-free graph on independence number at most $f(n)$. This concept was defined by Erdős and Sós in 1970. In this talk I will survey some of the recent progress on studying $RT(n, L, f(n))$ and some related questions.

The newer results are partially joint with Hu, Lenz and Simonovits.

Generalized Erdős-Szekeres theorem
Boris Bukh
Carnegie Mellon University

We introduce a generalization of Erdős-Szekeres theorem, and show how it can be applied to construct a strengthening of weak ε-net. We also discuss the decision problem for Erdős-Szekeres-type for arbitrary semialgebraic predicates.

Braess paradox in expander graphs
Fan Chung Graham
University of California: San Diego

Braess paradox is the counter intuitive scenario that, without lessening demand, closing roads can improve traffic flow. In contrast to what one might expect, we show that Braess paradox is ubiquitous in expander graphs.

This is joint work with Stephen Young and Wenbo Zhao.

Ramsey-type results for semi-algebraic relations
David Conlon
University of Oxford

A k-ary semi-algebraic relation E on $\mathbb{R}^d$ is a subset of $\mathbb{R}^{kd}$, the set of k-tuples of points in $\mathbb{R}^d$, which is determined by a finite number of polynomial equations and inequalities in kd real variables. The description complexity of such a relation is at most t if the number of polynomials and their degrees are all bounded by t. A set $A \subset \mathbb{R}^d$ is called homogeneous if all or none of the k-tuples from A satisfy E. A large number of geometric Ramsey-type problems and results can be formulated as questions about finding large homogeneous subsets of sets in $\mathbb{R}^d$ equipped with semi-algebraic relations.

In this talk we study Ramsey numbers for k-ary semi-algebraic relations of bounded complexity and give matching upper and lower bounds, showing that they grow as a tower of height $k-1$. This improves on a direct application of Ramsey's theorem by one exponential and extends a result of Alon, Pach, Pinchasi, Radoičić, and Sharir, who proved this for $k = 2$. We apply our results to obtain new estimates for some geometric Ramsey-type problems relating to order types and one-sided sets of hyperplanes. We also study the off-diagonal case, achieving some partial results.

Joint work with J. Fox, J. Pach, B. Sudakov and A. Suk.

Graph removal lemma
Jacob Fox
Massachusetts Institute of Technology

The graph removal lemma states that every graph on n vertices with $o(n^h)$ copies of a fixed graph H on h vertices can be made H-free by removing $o(n^2)$ edges. I will present a new short proof giving the improved quantitative estimate for the graph removal lemma.

Joint work with David Conlon.

Probabilistic analysis of multi-dimensional assignment problems

Alan Frieze
Carnegie Mellon University

We consider two generalisations of the standard (2-dimensional) assignment problem. We assume that the cost coefficients are independent exponential random variables with mean one. We describe polynomial time algorithms that w.h.p. find better value solutions than previously obtained.

Joint work with Greg Sorkin.

Turán-type hypergraph problems: partial trees and linear cycles

Zoltán Füredi
Rényi Institute of Mathematics

A linear cycle $\mathbf{C}_\ell^{(k)}$ is a family of k-sets $\{F_1, \dots, F_\ell\}$ such that $|F_i \cap F_{i+1}| = 1$ (for $1 \le i < \ell$), $|F_\ell \cap F_1| = 1$ and there are no other intersections. We can represent the hyperedges by intervals along a cycle. With an intensive use of the delta-system method we prove that for $t > 0$, $k \ge 5$ and sufficiently large n, $(n > n_0(k,t))$, if $\mathcal{F}$ is an n-vertex k-uniform family with

$$|\mathcal{F}| > \binom{n-1}{k-1} + \binom{n-2}{k-1} + \cdots + \binom{n-t}{k-1},$$

then it contains a linear cycle of length $2t+1$. The only extremal family consists of all edges meeting a given t-set. We also determine the even case, $\mathbf{ex}_k(n, \mathbf{C}_{2t}^{(k)})$, exactly.

This is joint work with Tao Jiang.

Hardness results for local algorithms in sparse random graphs

David Gamarnik
Massachusetts Institute of Technology

We discuss algorithmic hardness of solving combinatorial optimization problems on sparse graphs by means of local algorithms. Recently a particular framework for local algorithms was proposed based on the concept of i.i.d. factors. In particular was conjectured by Hatami, Lovasz and Szegedy that such an algorithm should exist for the problem of finding a largest independent set in a random regular graph. We disprove this conjecture by showing that no local algorithm is capable of producing an independent set larger that some multiplicative factor of the optimal. Our approach is based on a powerful clustering phenomena discovered by statistical physicists in the context of spin glass theory, and recently confirmed by rigorous methods. To the best of our knowledge, our result is the first direct application of the spin glass theory methods to the area of algorithmic hardness.

Joint work with Madhu Sudan.

Erdős-Rademacher problems in extremal set theory

Wenying Gan
University of California: Los Angeles

The Erdős-Rademacher Theorem is a quantitative strengthening of the celebrated Mantel's Theorem, answering the following question: how many triangles must a graph with n vertices and $\left\lfloor \frac{n^2}{4} \right\rfloor + t$ edges have? One can ask similar questions for any extremal problem, namely how many forbidden configurations must appear once the size of the structure exceeds the extremal threshold. So far this has only been studied in the setting of graph and hypergraph theory. In this talk we will address this question for two central results in Extremal Set Theory - the Erdős-Ko-Rado Theorem and Sperner's Theorem.

This is joint work with Shagnik Das and Benny Sudakov.

The Erdős-Hajnal conjecture for rainbow triangles

Andrey Grinshpun
Massachusetts Institute of Technology

We prove that every 3-coloring of the edges of the complete graph on n vertices without a rainbow triangle contains a set of vertices so that the edges between those vertices use only two colors and the set has size $\Omega(n^{1/3} \log^2 n)$, and this bound is tight up to a constant factor. This verifies a conjecture of Hajnal which is a case of the multicolor generalization of the Erdős-Hajnal conjecture. For fixed positive integers s and r with $1 < s < r$, we prove that every r-coloring of the edges of the complete graph on n vertices without a rainbow triangle contains a set of polynomial size which uses at most s colors, and the polynomial bound we give is tight up to a constant factor. The proof of the lower bound utilizes Gallai's classification of edge-colorings of the complete graph which are rainbow-triangle free. The proof of the upper bound uses the Erdős-Szekeres lower bound on Ramsey numbers by considering lexicographic products of 2-edge-colorings of complete graphs without large monochromatic cliques.

Optimal covers of random graphs with Hamilton cycles

Dan Hefetz
University of Birmingham

We prove that if $\frac{\log^{117} n}{n} \le p \le 1 - n^{-1/8}$, then asymptotically almost surely the edges of $G(n,p)$ can be covered by $\lceil \Delta(G(n,p))/2 \rceil$ Hamilton cycles. This is clearly best possible and improves an approximate result of Glebov, Krivelevich and Szabó, which holds for $p \ge n^{-1+\varepsilon}$.

Based on joint work with Daniela Kuhn, John Lapinskas and Deryk Osthus.

Densities of cliques and independent sets in graphs

Hao Huang
Institute of Advanced Study, Princeton

Many problems in extremal combinatorics can be stated as: for given graphs H_1 and H_2, if the number of induced copies of H_1 in a graph G is known, what is the maximum or minimum number of copies of H_2 in G? Turán proved that the maximal edge density of K_r-free graph is attained by an $(r-1)$-partite graph. Kruskal and Katona found that cliques, among all graphs, maximize the number of induced copies of K_s when the number of induced copies of K_r is fixed and $r < s$. In this talk, using the technique of shifting borrowed from extremal set theory and some powerful analytical methods, we prove an analogue of Kruskal-Katona theorem: the K_s-density of a graph with fixed $\overline{K}_r$-density is maximized when it is either a clique, or the complement of a clique. Using these results, we show that all the 3-local profiles (the vector (p_0, p_1, p_2, p_3) with p_i being the probability that three distinct random vertices in the graph span exactly i edges) of triangle-free graphs can be realized by bipartite graphs.

Joint work with Linial, Naves, Peled and Sudakov.

Ramsey numbers for uniform hypergraphs: triangles versus cliques

Sasha Kostochka
University of Illinois at Urbana-Champaign

It is known that the order of magnitude of the triangle-complete graph Ramsey numbers $R(3,t)$ is $t^2/\log t$. We consider an analogue of this problem for uniform hypergraphs. A *triangle* is a hypergraph consisting of edges e, f, g such that $|e \cap f| = |f \cap g| = |g \cap e| = 1$ and $e \cap f \cap g = \emptyset$. For all $r \ge 2$, let $R(C_3, K_t^r)$ be the smallest positive integer n such that in every red-blue coloring of the edges of the complete r-uniform hypergraph K_n^r, there exists a red triangle or a blue K_t^r. We show that there exist constants $a, b_r > 0$ such that that for all $t \ge 3$,

$$\frac{at^{\frac{3}{2}}}{(\log t)^{\frac{3}{4}}} \le R(C_3, K_t^3) \le b_3 t^{\frac{3}{2}}$$

and for $r \ge 4$

$$\frac{t^{\frac{3}{2}}}{(\log t)^{\frac{3}{4}+o(1)}} \le R(C_3, K_t^r) \le b_r t^{\frac{3}{2}}.$$

This determines up to a logarithmic factor the order of magnitude of $R(C_3, K_t^r)$.

This is joint work with Dhruv Mubayi and Jacques Verstraete.

Biased games on random boards

Michael Krivelevich
Tel Aviv University

We consider biased Maker-Breaker games played on the edge set of a random board $G(n,p)$. We prove that $G(n,p)$ is with high probability such that the critical bias $b^* = b^*(n,p)$ for the Hamiltonicity game, the perfect matching game and the k-connectivity game satisfies: $b^* = \frac{(1+o(1))np}{\ln n}$ for $p(n) \gg \frac{\log n}{n}$, and $b^* = \Theta\left(\frac{np}{\log n}\right)$ for $p(n) = \Theta\left(\frac{\log n}{n}\right)$. This settles a conjecture of Szabó and Stojakovic from 2005. Similar results are also obtained for biased Avoider-Enforcer games on random graphs.

Joint work with Asaf Ferber, Roman Glebov and Alon Naor.

Ramsey numbers of cubes versus cliques

Choongbum Lee
Massachusetts Institute of Technology

The cube graph Q_n is the skeleton of the n-dimensional cube. It is an n-regular graph on 2^n vertices. The Ramsey number $r(Q_n, K_s)$ is the minimum N such that every graph of order N contains the cube graph Q_n or an independent set of order s. Burr and Erdős in 1983 asked whether the simple lower bound $r(Q_n, K_s) >= (s-1)(2^n - 1) + 1$ is tight for s fixed and n sufficiently large. We make progress on this problem, obtaining the first upper bound which is within a constant factor of the lower bound.

Joint work with David Conlon, Jacob Fox, and Benny Sudakov

Spectral Turán-type hypergraph problems

John Lenz
University of Illinois at Chicago

Let $\lambda(G)$ be the largest adjacency eigenvalue of a graph G and $T_r(n)$ the r-partite Turán graph of order n. The spectral Turán theorem of Nikiforov states that if $\lambda(G) > \lambda(T_r(n))$, then G contains a copy of K_{r+1}. In recent work with Peter Keevash and Dhruv Mubayi, we proved a generalization of this to k-uniform hypergraphs and certain forbidden subhypergraphs F, but only for Fs where the Turán density is already known. Secondly, using similar techniques, we generalized Erdős-Ko-Rado on intersecting families to the spectral world, proving that the star has the largest eigenvalue among k-uniform, t-intersecting families.

This is joint work with Peter Keevash and Dhruv Mubayi.

Computing with voting trees

Po-Shen Loh
Carnegie Mellon University

The classical paradox of social choice theory asserts that there is no fair way to deterministically select a winner in an election among more than two candidates. One well-studied procedure for selecting a winner is to specify a complete binary tree whose leaves are labeled by the candidates, and evaluate it by running pairwise elections between the pairs of leaves, sending the winners to successive rounds of pairwise elections which ultimately terminate with a single winner. This structure is called a voting tree.

Much research has investigated which functions on tournaments are computable in this way. Fischer, Procaccia, and Samorodnitsky quantitatively studied the computability of the Copeland rule, which returns a vertex of maximum out-degree in the given tournament. The best previously known voting tree could only guarantee a returned out-degree at least logarithmic in the number of candidates N. Our work finds three constructions, the first of which substantially improves this guarantee to $\sqrt{N}$. The other two demonstrate the richness of the voting tree universe, with a tree that resists manipulation, and a tree which implements arithmetic modulo three.

Joint work with Jennifer Iglesias and Nate Ince.

Cores of random graphs are born Hamiltonian

Eyal Lubetzky
Microsoft Research

Let (G_t) be the random graph process and let τ_k denote the minimum time t such that the k-core of G_t is nonempty. For any fixed $k \geq 3$ the k-core is known to emerge via a discontinuous phase transition, where at time $t = \tau_k$ its size jumps from 0 to being linear in the number of vertices with high probability. It is believed that for every $k \geq 3$ the core is Hamiltonian upon creation w.h.p., yet even the asymptotic threshold for the Hamiltonicity of the k-core in $\mathcal{G}(n,p)$ was unknown for any k. We show here that for every fixed $k \geq 15$ the k-core of G_t is w.h.p. Hamiltonian for all $t \geq \tau_k$, i.e., immediately as the k-core appears and indefinitely afterwards. Moreover, we prove that for large enough fixed k the k-core in fact contains $\lfloor (k-3)/2 \rfloor$ edge-disjoint Hamilton cycles w.h.p. for all $t \geq \tau_k$.

Joint work with Michael Krivelevich and Benny Sudakov.

Discrepancy of random graphs and hypergraphs

Jie Ma
University of California: Los Angeles

Let G and H be two k-uniform hypergraphs over the same vertex set V with $|V| = n$. Define the discrepancy of H to be $\mathrm{disc}(H) = \max_{S \subseteq V(H)} \left| e(H[S]) - \rho_H \binom{|S|}{k} \right|$, where ρ_H is the edge-density of H. The discrepancy can be viewed as a measure of how uniformly the edges of H are distributed among the vertices. This important concept is widely adopted in many branches of combinatorics and has been the subject of extensive research, some of which will be mentioned in this talk. In a recent paper, Bollobás and Scott defined the discrepancy of G with respect to H as

$$\mathrm{disc}(G, H) = \max_{\pi} \left| e(G_\pi \cap H) - \rho_G \rho_H \binom{n}{k} \right|$$

over all bijections $\pi : V \to V$. This represents how far the overlap $G_\pi \cap H$ of two hypergraphs can be from its average. In some sense, the definition of $\mathrm{disc}(G, H)$ is more general than $\mathrm{disc}(H)$. They asked the following: if G and H are two random graphs distributed according to $G(n, p)$, then what is the expectation of $\mathrm{disc}(G, H)$? We fully answer this problem for general k-uniform random hypergraphs. We will also discuss other related results and problems.

Joint work with Humberto Naves and Benny Sudakov.

Quasirandom hypergraphs

Dhruv Mubayi
University of Illinois at Chicago

Since the foundational results of Thomason and Chung-Graham-Wilson on quasirandom graphs over 20 years ago, there has been a lot of effort by many researchers to extend the theory to hypergraphs. I will present some of this history, and then describe our recent results that provide such a generalization and unify much of the previous work. One key new aspect in the theory is the development of hypergraph eigenvalues.

Joint work with John Lenz.

Random matrices: law of the determinant

Hoi Nguyen
Yale University

Let A_n be an n-by-n matrix whose entries are i.i.d. Bernoulli random variables. In this talk we discuss the basic question: What is the law of the determinant of A_n?

Joint work with Van Vu.

Exact bounds for some hypergraph saturation problems

Asaf Shapira
Tel Aviv University

A bipartite graph G on vertex sets X, Y is said to be *weakly H-saturated* if one can add the edges between X and Y that are missing in G one after the other so that whenever a new edge is added, a new copy of H is created. Balogh, Bollobas, Morris and Riordan have recently raised the question of finding the minimum number of edges in an $n \times n$ bipartite graph that is weakly $K_{p,q}$-saturated. They used algebraic arguments to give a partial answer to this question, as well as to its hypergraph analogue. We settle this question, in both the graph and hypergraph cases. As a byproduct, we also get a new, multi-partite variant of the well-known Two Families theorem.

Joint work with Guy Moshkovitz.

Locally sparse triple systems

Jozsef Solymosi
University of British Columbia

A (k, l)-configuration in a 3-uniform hypergraph is a set of l edges whose union contains precisely k vertices. In this talk we consider two major problems in extremal combinatorics; What is the maximal edge density of a hypergraph containing no $(k+2,k)$-configuration, and what is the maximal edge density of a hypergraph containing no $(k+3,k)$-configuration. Erdős conjectured that for every $k>3$ there are arbitrary large Steiner triple systems containing no $(k+2,k)$-configuration. A related conjecture of Brown, Erdős, and Sós states that 3-uniform hypergraphs on n vertices with no $(k+3,k)$-configuration are sparse, they have $o(n^2)$ edges. We show that any construction proving Erdős' conjecture for large k, and any counterexample to the Brown, Erdős, Sós conjecture should come from triple systems which are far from any underlying group structure. (Note that any Steiner triple system or partial Steiner triple system defines a quasigroup in a natural way.)

Six standard deviations still suffice

Joel Spencer
New York University

A quarter century or so ago the speaker resolved a conjecture of Erdős, showing that any n sets on n vertices may be two colored so that all sets have discrepancy at most $K\sqrt{n}$, K (originally 6) an absolute constant. We discuss recent works of Bansal and of Lovett and Meka that reprove this result, giving an algorithm (long conjectured by the speaker *not* to exist!) finding the coloring. We emphasize the approach of Lovett and Meka that uses floating colors in $[-1,+1]$ which move around in a restricted Brownian motion.

Coupon colourings of graphs

Jacques Verstraete
University of California: San Diego

A *k-coupon coloring* of a graph G is a coloring of the vertices with k colors so that the neighborhood of each vertex contains a vertex of each of the k colors. The coupon chromatic number of G is the largest k for which a k-coupon coloring of G exists. The existence of a 2-coupon coloring of a graph corresponds to the so-called Property B of the hypergraph whose edges are the neighborhoods of vertices in the graph; in particular it is known that the coupon chromatic number of a 4-regular graph is always at least 2. We prove that every d-regular graph has coupon chromatic number asymptotically at least $d/(\log d)$ as $d\to\infty$, and that almost every d-regular graph has coupon chromatic number asymptotic to $d/(\log d)$ as $d\to\infty$. Explicit examples of regular graphs with such small coupon chromatic number are Paley graphs. In addition, we discuss coupon colorings of Hamming cubes, for instance if Q_d denotes the d-dimensional hypercube then the coupon chromatic number is asymptotic to d as $d\to\infty$ and exactly d when d is a power of two. Some open questions related to coding theory on coupon coloring of hypercubes are presented.

Sparse regularity and counting in pseudorandom graphs

Yufei Zhao
Massachusetts Institute of Technology

Szemerédi's regularity lemma is a fundamental tool in extremal combinatorics. However, the original version is only helpful in studying dense graphs. In the 1990s, Kohayakawa and Rödl proved an analogue of Szemerédi's regularity lemma for sparse graphs as part of a general program toward extending extremal results to sparse graphs. Many of the key applications of Szemerédi's regularity lemma use an associated counting lemma. In order to prove extensions of these results which also apply to sparse graphs, it remained a well-known open problem to prove a counting lemma in sparse graphs.

In this talk, I will discuss a new counting lemma, proved following the functional approach of Gowers, which complements the sparse regularity lemma of Kohayakawa and Rödl, allowing us to count small graphs in regular subgraphs of a sufficiently pseudorandom graph. Applications include sparse extensions of several well-known combinatorial theorems, including the removal lemmas for graphs and groups, the Erdős-Stone-Simonovits theorem and Ramsey's theorem.

Joint work with David Conlon and Jacob Fox.