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TURÁN NUMBER OF BIPARTITE GRAPHS WITH NO $K_{t,t}$

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ABSTRACT. The extremal number of a graph H, denoted by $\mathrm{ex}(n, H)$, is the maximum number of edges in a graph on n vertices that does not contain H. The celebrated Kővári-Sós-Turán theorem says that for a complete bipartite graph with parts of size $t \leq s$ the extremal number is $\mathrm{ex}(K_{s,t}) = O(n^{2-1/t})$. It is also known that this bound is sharp if $s > (t-1)!$. In this paper, we prove that if H is a bipartite graph such that all vertices in one of its parts have degree at most t but H contains no copy of $K_{t,t}$, then $\mathrm{ex}(n, H) = o(n^{2-1/t})$. This verifies a conjecture of Conlon, Janzer, and Lee.

1. INTRODUCTION

Let H be a graph. The extremal number of H, denoted by $\mathrm{ex}(n, H)$, is the maximum number of edges in a graph on n vertices that does not contain H. By the classical Erdős-Stone-Simonovits theorem [9, 10], we have $\mathrm{ex}(n, H) = (1 - \frac{1}{\chi(H)-1} + o(1))\binom{n}{2}$, where $\chi(H)$ is the chromatic number of H. Therefore, the order of $\mathrm{ex}(n, H)$ is known, unless H is a bipartite graph. One of the major open problems in extremal graph theory is to understand the function $\mathrm{ex}(n, H)$ for bipartite graphs. The history of such results began in 1954 with the Kővári-Sós-Turán theorem [19], which tells us that if $K_{s,t}$ is the complete bipartite graph with vertex classes of size $s \geq t$, then $\mathrm{ex}(n, K_{s,t}) = O(n^{2-1/t})$. This result was substantially extended by Füredi [11] and Alon, Krivelevich, and Sudakov [1].

Theorem 1. *Let H be a bipartite graph such that every vertex in one of its parts has degree at most t. Then* $\mathrm{ex}(n, H) = O(n^{2-1/t})$.

It is known (see [2, 18]) that $\mathrm{ex}(n, K_{s,t}) = \Theta(n^{2-1/t})$ if $s > (t-1)!$. Moreover, it is believed that $\mathrm{ex}(n, K_{t,t}) = \Theta(n^{2-1/t})$ as well. This shows that in general if H contains large complete bipartite subgraphs the above theorem is tight. Thus, it is natural to ask what happens when the forbidden graph H is $K_{t,t}$-free. For $t = 2$ this question was considered in 1988 by Erdős [8], who conjectured that if H is a subgraph of a subdivision of another graph, then there exists $\mu > 0$ such that $\mathrm{ex}(n, H) = O(n^{3/2-\mu})$. A subdivison of a graph Γ is obtained by replacing edges of Γ by internally vertex disjoint paths of length two. By definition, if H is a subgraph of a subdivision, then it is bipartite, has no $K_{2,2}$, and all the vertices in one of its parts have degree at most two. The conjecture of Erdős was recently confirmed by Conlon and Lee [6] and in a stronger form by Janzer [14]. Conlon and Lee [6] further proposed the following more general conjecture.

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Conjecture 2. *For an integer $t \geq 2$, let H be a $K_{t,t}$-free bipartite graph such that every vertex in one of the vertex classes of H has degree at most t. Then there is $\mu > 0$ such that* $\mathrm{ex}(n, H) = O(n^{2-1/t-\mu})$.

Despite recent progress on this topic (see, e.g., [5, 6, 12, 14–17]), this problem remains open for $t \geq 3$. Moreover the following weaker form of the above conjecture, proposed by Conlon, Janzer, and Lee [5], was open as well.

Conjecture 3. *Let $t \geq 2$ be an integer and let H be a $K_{2,t}$-free bipartite graph such that every vertex in one of the parts of H has degree at most t. Then* $\mathrm{ex}(n, H) = o(n^{2-1/t})$.

Similarly to Erdős, one can also formulate this conjecture as a question on extremal numbers of subdivisions. For a hypergraph $\mathcal{H}$, the *subdivision* of $\mathcal{H}$ is the bipartite graph $\mathcal{H}'$ whose two vertex classes are $V(\mathcal{H})$ and $E(\mathcal{H})$, and $v \in V(\mathcal{H})$ and $e \in E(\mathcal{H})$ are joined by an edge if $v \in e$. Then Conjecture 3 is equivalent to asking whether $\mathrm{ex}(n, \mathcal{H}') = o(n^{2-1/t})$ for a subdivision $\mathcal{H}'$ of a t-uniform hypergraph. In [5], this conjecture is proved in the special case $\mathcal{H}$ is a linear hypergraph (that is, any two edges of $\mathcal{H}$ intersect in at most one vertex), which corresponds to the case in which the bipartite graph H is $K_{2,2}$-free. Also, it is mentioned in [6] and [5] that Conjecture 3 holds in case H is the subdivision of the complete t-uniform hypergraph with $t+1$ vertices or the subdivision of a t-partite t-uniform hypergraph.

In this paper we prove Conjecture 3 in a very strong form, showing already that Conjecture 2 holds with the same upper bound.

Theorem 4. *Let $t \geq 2$ be an integer. Let H be a $K_{t,t}$-free bipartite graph such that every vertex in one of the parts of H has degree at most t. Then* $\mathrm{ex}(n, H) = o(n^{2-1/t})$.

2. The extremal number of $K_{t,t}$-free bipartite graphs

2.1. **Preliminaries.** In this section, we introduce our notation (which is mostly conventional), and state a few technical lemmas to prepare the proof of Theorem 4. We omit floors and ceilings whenever they are not crucial.

If k is a positive integer and X is a set, $X^{(k)}$ denotes the family of k element subsets of X. If G is a graph, $V(G)$ and $E(G)$ are the vertex set and edge set of G, respectively, and $v(G) = |V(G)|$, $e(G) = |E(G)|$. If $S \subset V(G)$, then $N_G(S)$ denotes the *common neighborhood* of S, that is, the set of vertices that are joined to every element of S by an edge. If $S = \{x\}$, we write simply $N_G(x)$ instead of $N_G(S)$. The degree of a vertex $x \in V(G)$ in G is $d_G(x) = |N_G(x)|$. The complete t-uniform hypergraph on k vertices is denoted by $K_k^{(t)}$.

In the proof of our main theorem, we use the following technical lemma, which can be found as Lemma 2.2 in [5]. Here, a graph G is K-*almost regular* if the maximum degree of G is at most K-times the minimum degree.

Lemma 5. *Let $c, \alpha > 0$ such that $\alpha < 1$. Let n be a positive integer that is sufficiently large with respect to c and α. Let G be a graph on n vertices such that $e(G) \geq cn^{1+\alpha}$. Then G contains a K-almost regular subgraph G' on $m \geq n^{\frac{\alpha-\alpha^2}{4+4\alpha}}$ vertices such that $e(G') \geq \frac{2c}{5} m^{1+\alpha}$ and $K = 20 \cdot 2^{\frac{1}{\epsilon^2}+1}$.*

More precisely, we need the following immediate consequence of the above result.

Lemma 6. *Let $0 < c < 10^{-4}$ and $\frac{1}{2} \leq \alpha < 1$. Let n be a positive integer that is sufficiently large with respect to c and α. Let G be a graph on n vertices such that $e(G) \geq cn^{1+\alpha}$. Then G contains a bipartite subgraph G', both of whose vertex classes have size $m \geq \frac{1}{2}n^{\frac{\alpha-\alpha^2}{4+4\alpha}}$, $e(G') \geq \frac{c}{10}m^{1+\alpha}$, and the maximum degree of G' is less than m^α.*

Proof. By the previous lemma, G contains a subgraph G_0 such that G_0 has $m_0 \geq n^{\frac{\alpha-\alpha^2}{4+4\alpha}}$ vertices, $e(G_0) \geq \frac{2c}{5}m_0^{1+\alpha}$, and G_0 is K-almost regular. Since $\alpha > 1/2$ we have $K < 1000$. By randomly sampling the edges of G_0 with probability $p = \frac{2cm_0^{1+\alpha}}{5e(G_0)}$ and using standard concentration arguments, we can find a subgraph G_0' of G_0 such that G_0' is $2K$-almost regular and $\frac{4c}{5}m_0^{1+\alpha} \geq e(G_0') \geq \frac{c}{5}m_0^{1+\alpha}$.

But then the minimum degree of G_0' is at most $\frac{8c}{5}m_0^\alpha$, so the maximum degree of G_0' is less than $4Kcm_0^\alpha$. By well-known folklore results, $V(G_0')$ can be partitioned into two sets U and V of size $m = \frac{1}{2}m_0$ such that the number of edges connecting U and V is at least $\frac{1}{2}e(G_0') \geq \frac{c}{10}m^{1+\alpha}$. Let G' be the bipartite subgraph of G_0' with vertex classes U and V. Then the maximum degree of G' is less than $4Kcm_0^\alpha < 8Kcm^\alpha < m^\alpha$. Therefore, G' satisfies the desired conditions. □

We will also use the hypergraph version of the classical Ramsey theorem [21].

Lemma 7. *Let k, t be positive integers. Then there exists $\Delta = \Delta(k,t)$ such that any two-coloring of the edges of the complete t-uniform hypergraph $K_\Delta^{(t)}$ contains a monochromatic copy of $K_k^{(t)}$.*

Finally, we will use the celebrated Hypergraph Removal Lemma, proved independently by Nagle, Rödl, Schacht [20], and Gowers [13].

Lemma 8. *Let k, t be positive integers. For every $\beta > 0$ there exists $\delta = \delta(k,t,\beta) > 0$ such that the following holds. If $\mathcal{H}$ is a t-uniform hypergraph on n vertices such that one needs to remove at least βn^t edges of $\mathcal{H}$ to make it $K_k^{(t)}$-free, then $\mathcal{H}$ contains at least δn^k copies of $K_k^{(t)}$.*

2.2. **Overview of the proof.** Despite our proof being quite short, it might help to briefly outline the main ideas.

Let H_k be the bipartite graph with vertex classes X and Y such that $|X| = k$, $|Y| = (t-1)\binom{k}{t}$, and for every t-tuple $S \in X^{(t)}$, there are exactly $t-1$ vertices in Y whose neighborhood is equal to S. Clearly, for every H there is a large enough integer k such that H is a subgraph of H_k. Therefore, it is enough to show that $\mathrm{ex}(n, H_k) = o(n^{2-1/t})$.

Let us fix an H_k-free graph G on n vertices with $\epsilon n^{2-1/t}$ edges, where we think of ϵ as a small constant. Then our goal is to show that n cannot be arbitrarily large. We first pass to a bipartite subgraph with parts V and W, where $|V|$ is of order n and $|W|$ is of order $n^{1-1/t}$. This is in contrast to a few previous papers on the same topic [5, 6, 14] which work with a bipartite subgraph G' of G in which both parts have roughly the same size. By setting the parameters correctly, the advantage of our first step is that the average size of the common neighborhood in V of the $(t-1)$-tuples of vertices from W is some large constant. Next we consider the t-uniform hypergraph $\mathcal{H}$ on W where each $e \in W^{(t)}$ is an edge if it has at least $t-1$ common neighbors in V. We color the edges of $\mathcal{H}$ by red and blue such that an edge is red if it has at least $(t-1)\binom{k}{t}$ common neighbors. One can argue that $\mathcal{H}$ cannot have a red $K_k^{(t)}$, since otherwise we can find greedily a copy of H_k. Thus, using Ramsey's theorem, we find many blue copies of $K_k^{(t)}$. We further prove that one needs to remove many hyperedges to destroy all these blue copies of $K_k^{(t)}$. Therefore we can apply the Hypergraph Removal Lemma to show that $\mathcal{H}$ must contain $\Omega(|W|^k)$ copies of $K_k^{(t)}$. Then by counting certain bad copies of $K_k^{(t)}$, we conclude that there must exist a copy R such that the common neighborhoods $N_{G'}(S)$ for $S \in E(R)$ are all pairwise disjoint. Using such R as one part of H_k we can clearly embed the other part in $\bigcup_{S\in E(R)} N_{G'}(S)$.

2.3. **The proof of Theorem 4.** In this section, we present the proof of Theorem 4. Our proof works for all $t \geq 2$, but since the case $t = 2$ is already known by [6, 14], for computational convenience we assume that $t \geq 3$.

Fix k such that H is contained in H_k, where H_k is the bipartite graph defined in the previous section. We prove that for every $0 < \epsilon < 10^{-4}$ if n is sufficiently large, then $\mathrm{ex}(n, H_k) \leq \epsilon n^{2-1/t}$. Let G be a graph with n vertices and at least $\epsilon n^{2-1/t}$ edges, and assume that G does not contain H_k. By Lemma 6, G has a bipartite subgraph G' with vertex classes U and V, $|U| = |V| = n' > \frac{1}{2}n^{\frac{(1-1/t)}{8t-4}}$ such that $e(G') \geq \frac{\epsilon}{10}(n')^{2-1/t}$, and the maximum degree of G' is at most $(n')^{1-1/t}$. In the rest of the proof, we shall work only with G' instead of G, so with slight abuse of notation, let $G := G'$, $n := n'$, and $\epsilon := \frac{\epsilon}{10}$. Clearly, it is enough to prove that G contains H_k if n is sufficiently large with respect to k, t, ϵ.

As the next step, we pass to an even smaller subgraph G' of G with parts of size roughly $n^{1-1/t}$ and n. This is done using the following claim.

Claim 9. Let $n^{-1/t} < p < 1$. If n is sufficiently large with respect to ϵ and t, then there exists $W \subset U$ such that $\frac{pn}{2} < |W| < 2pn$, the graph $G' = G[W \cup V]$ has at least $\frac{p}{4}e(G)$ edges, and $d_{G'}(x) < 2pn^{1-1/t}$ holds for every $x \in V$.

Proof. Pick each element of U with probability p, and let W be the set of selected vertices. Then the statement follows by standard concentration arguments. By Chernoff's inequality, with high probability we have $|d_{G'}(x) - pd_G(x)| < \frac{1}{2}pd_G(x)$ for every $x \in V$ satisfying $d_G(x) \geq n^{1/2}$. Also, with high probability, $||W| - pn| \leq \frac{1}{2}pn$. Therefore, there exists a choice for W which satisfies these inequalities. But then every $x \in V$ satisfies $d_{G'}(x) < 2pn^{1-1/t}$ as the maximum degree of G' is at most $n^{1-1/t}$. Finally, we have

$$e(G') = \sum_{x\in V} d_{G'}(x) \geq \sum_{x\in V, d_G(x)\geq n^{1/2}} \frac{1}{2}pd_G(x) \geq \frac{1}{2}p(e(G) - n \cdot n^{1/2}) \geq \frac{1}{4}pe(G)\,.$$

□

We would like to choose p such that the average size of a common neighborhood of a $(t-1)$-tuple of vertices in V is some large constant (independent of n). Let $n^{-1/t} < p < 1$, which we will specify later, and let W be a subset of U satisfying the properties described in Claim 9. Consider the sum $L = \sum_{C\in V^{(t-1)}} |N_{G'}(C)|$.

We have

$$L = \sum_{x \in W} \binom{d_{G'}(x)}{t-1} \geq |W| \binom{e(G')/|W|}{t-1} > (t-1)^{-(t-1)} e(G')^{t-1} |W|^{-(t-2)}$$
$$> (t-1)^{-(t-1)} \left(\frac{p\epsilon}{4} n^{2-1/t}\right)^{t-1} (2pn)^{-(t-2)} = \left(\frac{\epsilon}{t-1}\right)^{t-1} 2^{-3t+4} pn^{t-1+1/t},$$

where the first inequality holds by convexity.

By Ramsey's theorem (Lemma 7), there exists a positive integer $\Delta = \Delta(k,t)$ such that any red-blue coloring of the edges of the hypergraph $K_\Delta^{(t)}$ contains either a red or a blue copy of $K_k^{(t)}$. Choose p such that $L \geq 2\Delta n^{t-1}$ holds. Then by the previous calculations, we can choose $p = \alpha n^{-1/t}$, where $\alpha = 2\Delta(\frac{t-1}{\epsilon})^{t-1} 2^{3t-4}$. The important thing to notice is that $\alpha = \alpha(k,t,\epsilon)$ does not depend on n. Also, we remark that $\frac{\alpha}{2} n^{1-1/t} < |W| < 2\alpha n^{1-1/t}$, and every $x \in V$ has degree at most $2pn^{1-1/t} = 2\alpha n^{1-2/t}$ in G'.

Let $\mathcal{H}$ be the t-uniform hypergraph on W in which $S \in W^{(t)}$ is an edge if $|N_{G'}(S)| \geq t-1$. Color an edge $S \in E(\mathcal{H})$ red if $|N_{G'}(S)| \geq (t-1)\binom{k}{t}$, and color S blue otherwise. If $\mathcal{H}$ contains a red clique of size k, then G' contains H_k. Indeed, if $R \subset W$ spans a red clique of size k in $\mathcal{H}$, then for each $S \in R^{(t)}$ one can greedily select a set $Q_S \subset N_{G'}(S)$ of $t-1$ vertices such that Q_S and $Q_{S'}$ are disjoint if $S \neq S'$. This clearly gives a copy of H_k, a contradiction. Therefore, in what follows, we can assume that $\mathcal{H}$ does not contain a red clique of size k.

Let $C \in V^{(t-1)}$ and consider $T = N_{G'}(C)$. Let $r = \lfloor \frac{|T|}{\Delta} \rfloor > \frac{|T|}{\Delta} - 1$, and let $T_1, \ldots, T_r$ be disjoint sets of size Δ in T. Note that for $i = 1, \ldots, r$, $\mathcal{H}[T_i]$ is a clique of size Δ in $\mathcal{H}$. But $\mathcal{H}[T_i]$ does not contain a red clique of size k, so by the definition of Δ, $\mathcal{H}[T_i]$ contains a blue clique of size k. Let A_i be the vertex set of such a clique. Set $Z_C = \{A_i : i = 1, \ldots, r\}$, and let Z be the multiset $\bigcup_{C \in V^{(t-1)}} Z_C$ (that is, we count each k-tuple with multiplicity s if it appears in s of the sets Z_C for $C \in V^{(t-1)}$). Then

$$|Z| = \sum_{C \in V^{(t-1)}} |Z_C| \geq \sum_{C \in V^{(t-1)}} \left(\frac{|N_{G'}(C)|}{\Delta} - 1\right) = \frac{L}{\Delta} - \binom{n}{t-1} \geq n^{t-1}.$$

Next, we show that Z contains a large subset in which the size of the intersection of any two elements is less than t.

Claim 10. There exists a constant $\beta = \beta(k,t,\epsilon) > 0$ and $Z' \subset Z$ such that $|Z'| \geq \beta |W|^t$, and if $A, B \in Z'$, then $|A \cap B| < t$.

Proof. Let D be the graph on vertex set Z in which A and B are joined by an edge if $|A \cap B| \geq t$. Let $A \in Z$ and let $S \in A^{(t)}$. Then S is blue, so $|N_{G'}(S)| \leq (t-1)\binom{k}{t} = u$. But then there are at most $\binom{u}{t-1}$ sets $C \in V^{(t-1)}$ such that $S \subset N_{G'}(C)$. For each such C, at most one element of Z_C contains S, so in total at most $\binom{u}{t-1}$ elements of Z contain S. Hence, as A has $\binom{k}{t}$ subsets of size t, A has degree at most $d = \binom{k}{t}\binom{u}{t-1}$ in D.

But then D contains an independent set of size at least $\frac{|Z|}{d+1} \geq \frac{n^{t-1}}{d+1} > \beta |W|^t$, where $\beta = \frac{1}{(2\alpha)^t (d+1)}$. Let Z' be such an independent set. $\square$

Note that by our construction, Z' corresponds to a family of copies of $K_k^{(t)}$ in $\mathcal{H}$ such that no two copies share a hyperedge. Let M be the total number of copies of $K_k^{(t)}$ in $\mathcal{H}$.

Claim 11. There exists a constant $\gamma = \gamma(k, t, \epsilon)$ such that $M \geq \gamma n^{(t-1)k/t}$.

Proof. In order to destroy every copy of $K_k^{(t)}$ in $\mathcal{H}$, one needs to remove at least one hyperedge from every element of Z', which results in the removal of at least $\beta|W|^t$ edges. Let $\delta = \delta(k, t, \beta)$ be the constant given by the Hypergraph Removal Lemma (Lemma 8). Then $M \geq \delta|W|^k \geq \delta(\frac{\alpha}{2})^k n^{(t-1)k/t}$. Choosing $\gamma = \delta(\frac{\alpha}{2})^k$ completes the proof. □

Let us say that a copy R of $K_k^{(t)}$ in $\mathcal{H}$ is *bad* if there exist two distinct t-tuples $S, S' \in E(R)$ such that $N(S) \cap N(S') \neq \emptyset$; otherwise, say that R is *good*. Clearly, if there exists a good copy of $K_k^{(t)}$, then G' contains H_k. Indeed, if R is a good copy, then for every $S \in R^{(t)}$, let Q_S be any $(t-1)$-element subset of $N_{G'}(S)$. Then the sets Q_S for $S \in E(R)$ are pairwise disjoint, so there is a copy of H_k with vertex set $R \cup \bigcup_{S \in R^{(t)}} Q_S$. To show that there is a good copy of $K_k^{(t)}$, let us count the number of bad copies.

Claim 12. There exists a constant $\gamma' = \gamma'(k, t, \epsilon)$ such that the number of bad copies of $K_k^{(t)}$ is at most $\gamma' n^{(k(t-1)-1)/t}$.

Proof. If R is a bad copy of $K_k^{(t)}$ in $\mathcal{H}$, then there are two sets $S, S' \in E(R)$ such that $N_{G'}(S) \cap N_{G'}(S')$ is non-empty. Let $x \in N_{G'}(S) \cap N_{G'}(S')$. Then $N_{G'}(x)$ contains $S \cup S'$. This implies that $|N_{G'}(x) \cap V(R)| \geq |S \cup S'| \geq t + 1$. Therefore, summing over all the vertices $x \in V$ we have that the number of bad copies of $K_k^{(t)}$ is at most

$$\sum_{x \in V} \binom{d_{G'}(x)}{t+1} |W|^{k-t-1} \leq n(2\alpha n^{1-2/t})^{t+1}(2\alpha n^{1-1/t})^{k-t-1} = (2\alpha)^k n^{(k(t-1)-1)/t}.$$

Setting $\gamma' = (2\alpha)^k$ suffices. □

To conclude the proof, note that if n is sufficiently large as a function of k, t, ϵ, then by Claims 11 and 12 there is a good copy of $K_k^{(t)}$, implying that G' contains H_k, a contradiction.

3. Concluding remarks

Although Conjecture 2 remains open, our proof of Theorem 4 can be slightly modified to confirm the conjecture for the following general family of bipartite graphs H. If $\mathcal{H}$ is a hypergraph, define the *r-fold subdivision* of $\mathcal{H}$, denoted by $\mathcal{H}^{[r]}$, as follows: let the two vertex classes of $\mathcal{H}^{[r]}$ be $V(\mathcal{H})$ and $E = E_1 \cup \cdots \cup E_r$, where $E_1, \ldots, E_r$ are disjoint copies of $E(\mathcal{H})$ and $e \in E_i$ is joined to $v \in V(\mathcal{H})$ by an edge if $v \in e$. Theorem 4 is equivalent to the statement that if $\mathcal{H}$ is a t-uniform hypergraph, then $\mathrm{ex}(n, \mathcal{H}^{[t-1]}) = o(n^{2-1/t})$. However, in case $\mathcal{H}$ is t-partite, we can do slightly better.

Theorem 13. *Let $\mathcal{H}$ be a t-partite t-uniform hypergraph. Then there exists $\mu > 0$ such that*

$$\mathrm{ex}(n, \mathcal{H}^{[t-1]}) = O(n^{2-1/t-\mu}).$$

To prove this theorem, one can use the proof of our main result. The only difference is in the application of the Hypergraph Removal Lemma. When we need to count the copies of a t-partite hypergraph instead of $K_k^{(t)}$, we can choose δ to be a polynomial of β (whose degree depends only on the t-partite hypergraph in question). This fact follows from an approach used by Erdős [7] to bound extremal numbers of complete t-uniform t-partite hypergraphs (for an application of this technique for counting copies of such hypergraphs see, e.g., Proposition 3.6 in [4]). Therefore, one can take $\epsilon = n^{-\mu}$ with some small enough μ in order for our arguments to work. We omit further details.

Note added in proof

After this paper was published on the arXiv, David Conlon brought to our attention that some parts of our proof use similar ideas to an argument presented in the recent paper [3] (see Theorem 7).

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