

Compatible Hamilton Cycles in Random Graphs*

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Received 2 October 2014; accepted 26 August 2015

Published online 15 February 2016 in Wiley Online Library (wileyonlinelibrary.com).

DOI 10.1002/rsa.20636

ABSTRACT: A graph is Hamiltonian if it contains a cycle passing through every vertex. One of the cornerstone results in the theory of random graphs asserts that for edge probability $p \gg \frac{\log n}{n}$, the random graph $G(n, p)$ is asymptotically almost surely Hamiltonian. We obtain the following strengthening of this result. Given a graph $G = (V, E)$, an *incompatibility system* \mathcal{F} over G is a family $\mathcal{F} = \{F_v\}_{v \in V}$ where for every $v \in V$, the set F_v is a set of unordered pairs $F_v \subseteq \{\{e, e'\} : e \neq e' \in E, e \cap e' = \{v\}\}$. An incompatibility system is Δ -*bounded* if for every vertex v and an edge e incident to v , there are at most Δ pairs in F_v containing e . We say that a cycle C in G is *compatible* with \mathcal{F} if every pair of incident edges e, e' of C satisfies $\{e, e'\} \notin F_v$. This notion is partly motivated by a concept of transition systems defined by Kotzig in 1968, and can be used as a quantitative measure of robustness of graph properties. We prove that there is a constant $\mu > 0$ such that the random graph $G = G(n, p)$ with $p(n) \gg \frac{\log n}{n}$ is asymptotically almost surely such that for any μnp -bounded incompatibility system \mathcal{F} over G , there is a Hamilton cycle in G compatible with \mathcal{F} . We also prove that for larger edge probabilities $p(n) \gg \frac{\log^8 n}{n}$, the parameter μ can be taken to be any constant smaller than $1 - \frac{1}{\sqrt{2}}$. These results imply in particular that typically in $G(n, p)$ for $p \gg \frac{\log n}{n}$, for any edge-coloring in which each color appears at most μnp times at each vertex, there exists a properly colored Hamilton cycle. Furthermore, our proof can be easily modified to show that for any edge-coloring of such a random graph in which each color appears on at most μnp edges, there exists a Hamilton cycle in which all edges have distinct colors (i.e., a rainbow Hamilton cycle). © 2016 Wiley Periodicals, Inc. *Random Struct. Alg.*, 49, 533–557, 2016

Keywords: random graphs; Hamilton cycles; incompatibility systems; rainbow cycles

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*Supported by USA-Israel BSF; 2010115 (M.K.), Israel Science Foundation; 912/12 (M.K.), NSF; DMS-136232 (C.L.), SNSF; 200021-149111 (B.S.), USA-Israel BSF (B.S.)

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1. INTRODUCTION

A *Hamilton cycle* in a graph G is a cycle passing through each vertex of G , and a graph is *Hamiltonian* if it contains a Hamilton cycle. Hamiltonicity, named after Sir Rowan Hamilton who studied it in the 1850s, is an important and extensively studied concept in graph theory. It is well known that deciding Hamiltonicity is an NP-complete problem and thus one does not expect a simple sufficient and necessary condition for Hamiltonicity. Hence the study of Hamiltonicity has been concerned with looking for simple sufficient conditions implying Hamiltonicity. One of the most important results in this direction is Dirac's theorem asserting that all n -vertex graphs, $n \geq 3$, of minimum degree at least $\frac{n}{2}$ contains a Hamilton cycle. Sufficient conditions for Hamiltonicity often provide good indication towards similar results for more general graphs. For example, Pósa and Seymour's conjecture on the existence of powers of Hamilton cycles, and Bollobás and Komlós's conjecture on the existence of general spanning subgraphs of small bandwidth, are both (highly non-trivial) generalizations of Dirac's theorem (both conjectures have been settled, see [8, 18]).

A *binomial random graph* $G(n, p)$ is a probability space of graphs on n vertices where each pair of vertices form an edge independently with probability p . With some abuse of terminology, we will use $G(n, p)$ to denote both the probability space and a random graph drawn from it. We say that $G(n, p)$ possesses a graph property \mathcal{P} *asymptotically almost surely* (or *a.a.s.* in short) if the probability that $G(n, p)$ has \mathcal{P} tends to 1 as n tends to infinity. Early results on Hamiltonicity of random graphs were proved by Pósa [27], and Korshunov [20]. Improving on these results, Bollobás [4], and Komlós and Szemerédi [19] proved that if $p \geq (\log n + \log \log n + \omega(n))/n$ for any function $\omega(n)$ that goes to infinity together with n , then $G(n, p)$ is a.a.s. Hamiltonian. The range of p cannot be improved, since if $p \leq (\log n + \log \log n - \omega(n))/n$, then $G(n, p)$ a.a.s. has a vertex of degree at most one. Hamiltonicity of random graphs has been studied in great depth, and there are many beautiful results on the topic.

Recently there has been increasing interest in the study of robustness of graph properties, aiming to strengthen classical results in extremal and probabilistic combinatorics. For example, consider the property of being Hamiltonian. By Dirac's theorem, we know that all n -vertex graphs of minimum degree at least $\frac{n}{2}$ (which we refer to as *Dirac graphs*) are Hamiltonian. To measure the robustness of this theorem, we can ask questions such as: "How many Hamilton cycles must a Dirac graph contain?", "What is the critical bias of the Maker-Breaker Hamiltonicity game played on a Dirac graph?", or "When does a random subgraph of a Dirac graph typically contain a Hamilton cycle?" (see [10, 22]). Note that an answer to each question above in some sense defines a measure of robustness of a Dirac graph with respect to Hamiltonicity. Moreover, Dirac's theorem itself can be considered as measuring robustness of Hamiltonicity of complete graphs, where we measure the maximum number of edges one can delete from each vertex of the complete graph while maintaining Hamiltonicity (see [30] for further discussion).

In this paper, we are interested in yet another type of robustness measure, and study the robustness of Hamiltonicity with respect to this measure.

Definition 1.1. Let $G = (V, E)$ be a graph.

- i. An incompatibility system \mathcal{F} over G is a family $\mathcal{F} = \{F_v\}_{v \in V}$ where for every $v \in V$, the set F_v is a set of unordered pairs $F_v \subseteq \{\{e, e'\} : e \neq e' \in E, e \cap e' = \{v\}\}$.

- ii. If $\{e, e'\} \in F_v$ for some edges e, e' and vertex v , then we say that e and e' are incompatible in \mathcal{F} . Otherwise, they are compatible in \mathcal{F} . A subgraph $H \subseteq G$ is compatible in \mathcal{F} , if all its pairs of edges e and e' are compatible.
- iii. For a positive integer Δ , an incompatibility system \mathcal{F} is Δ -bounded if for each vertex $v \in V$ and an edge e incident to v , there are at most Δ other edges e' incident to v that are incompatible with e .

The definition is motivated by two concepts in graph theory. First, it generalizes *transition systems* introduced by Kotzig [21] in 1968, where a transition system is a 1-bounded incompatibility system. Kotzig’s work was motivated by a problem of Nash-Williams on cycle covering of Eulerian graphs (see, e.g. Section 8.7 of [7]).

Incompatibility systems and compatible Hamilton cycles also generalize the concept of properly colored Hamilton cycles in edge-colored graphs. The problem of finding properly colored Hamilton cycles in edge-colored graph was first introduced by Daykin [11]. He asked if there exists a constant μ such that for large enough n , there exists a properly colored Hamilton cycle in every edge-coloring of a complete graph K_n where each vertex has at most μn edges incident to it of the same color (we refer to such coloring as a μn -bounded edge coloring). Daykin’s question has been answered independently by Bollobás and Erdős [6] with $\mu = 1/69$, and by Chen and Daykin [9] with $\mu = 1/17$. Bollobás and Erdős further conjectured that all $(\lfloor \frac{n}{2} \rfloor - 1)$ -bounded edge coloring of K_n admits a properly colored Hamilton cycle. After subsequent improvements by Shearer [29] and by Alon and Gutin [2], Lo [24] recently settled the conjecture asymptotically, proving that for any positive ε , every $(\frac{1}{2} - \varepsilon)n$ -bounded edge coloring of $E(K_n)$ admits a properly colored Hamilton cycle.

Note that a μn -bounded edge coloring naturally defines μn -bounded incompatibility systems, and thus the question mentioned above can be considered as a special case of the problem of finding compatible Hamilton cycles. However, in general, the restrictions introduced by incompatibility systems need not come from edge-colorings of graphs, and thus the results on properly colored Hamilton cycles do not necessarily generalize easily to incompatibility systems.

In this paper, we study compatible Hamilton cycles in random graphs. We present two results.

Theorem 1.2. *There exists a positive real μ such that for $p \gg \frac{\log n}{n}$, the graph $G = G(n, p)$ a.a.s. has the following property. For every μnp -bounded incompatibility system defined over G , there exists a compatible Hamilton cycle.*

Our result can be seen as an answer to a generalized version of Daykin’s question. In fact, we generalize it in two directions. First, we replace properly colored Hamilton cycles by compatible Hamilton cycles, and second, we replace the complete graph by random graphs $G(n, p)$ for $p \gg \frac{\log n}{n}$ (note that for $p = 1$, the graph $G(n, 1)$ is K_n with probability 1). Since $G(n, p)$ a.a.s. has no Hamilton cycles for $p \ll \frac{\log n}{n}$, we can conclude that $\frac{\log n}{n}$ is a “threshold function” for having such constant μ . The constant μ we obtain in Theorem 1.2 is very small, and our second result improves this constant for denser random graphs.

Theorem 1.3. *For all positive reals ε , if $p \gg \frac{\log^8 n}{n}$, then the graph $G = G(n, p)$ a.a.s. has the following property. For every $(1 - \frac{1}{\sqrt{2}} - \varepsilon)np$ -bounded incompatibility system defined over G , there exists a compatible Hamilton cycle.*

In an edge-colored graph, we say that a subgraph is *rainbow* if all its edges have distinct colors. There is a vast literature on the branch of Ramsey theory where one seeks rainbow subgraphs in edge-colored graphs. Note that one can easily avoid rainbow copies by using a single color for all edges, and hence in order to find a rainbow subgraph one usually imposes some restrictions on the distribution of colors. In this context, Erdős, Simonovits and Sós [13] and Rado [28] developed anti-Ramsey theory where one attempts to determine the maximum number of colors that can be used to color the edges of the complete graph without creating a rainbow copy of a fixed graph. In a different direction, one can try to find a rainbow copy of a target graph by imposing global conditions on the coloring of the host graph. For a real Δ , we say that an edge-coloring of G is *globally Δ -bounded* if each color appears at most Δ times on the edges of G . In 1982, Erdős, Nešetřil and Rödl [12] initiated the study of the problem of finding rainbow subgraphs in a globally Δ -bounded coloring of graphs. One very natural question of this type is to find sufficient conditions for the existence of a rainbow Hamilton cycle in any globally Δ -bounded coloring. Substantially improving on an earlier result of Hahn and Thomassen [16], Albert, Frieze and Reed [1] proved the existence of a constant $\mu > 0$ for which every globally μn -bounded coloring of K_n (for large enough n) admits a rainbow Hamilton cycle. In fact, they proved a stronger statement asserting that for all graphs Γ with vertex set $E(K_n)$ (the edge set of the complete graph) and maximum degree at most μn , there exists a Hamilton cycle in K_n which is also an independent set in Γ .

It turns out that the proof technique used in proving Theorem 1.2 can be easily modified to give the following result, that extends the above to random graphs.

Theorem 1.4. *There exists a constant $\mu > 0$ such that for $p \gg \frac{\log n}{n}$, the random graph $G = G(n, p)$ a.a.s. has the following property. Every globally μnp -bounded coloring of G contains a rainbow Hamilton cycle.*

Theorem 1.4 is best possible up to the constant μ since one can forbid all rainbow Hamilton cycles in a globally $(1 + o(1))np$ -bounded coloring by simply coloring all edges incident to some fixed vertex with the same color.

The proof of the three theorems will be given in the following sections. In Section 2, we prove Theorems 1.2 and 1.4. Then in Section 3, we prove Theorem 1.3.

Notation. A graph $G = (V, E)$ is given by a pair of its vertex set $V = V(G)$ and edge set $E = E(G)$. For a set X , let $N(X)$ be the set of vertices incident to some vertex in X . For a pair of disjoint vertex sets X and Y , let $E(X, Y) = \{(x, y) \mid x \in X, y \in Y, \{x, y\} \in E\}$, and define $e(X, Y) = |E(X, Y)|$. We define the *length* of a path as its number of edges. When there are several graphs under consideration, to avoid ambiguity, we use subscripts such as $N_G(X)$ to indicate the graph that we are currently interested in.

Throughout the paper, we tacitly assume that the number of vertices n of the graph is large enough whenever necessary. We also omit floor and ceiling signs whenever they are not crucial. All logarithms are natural.

2. PROOF OF THEOREMS 1.2 AND 1.4

To prove Theorem 1.2, we find a compatible Hamilton cycle by first finding a compatible subgraph that is also a good expander graph.

Definition 2.1. For positive reals k and r , a graph R is a (k, r) -expander if all sets $X \subseteq V(R)$ of size at most $|X| \leq k$ satisfy $|N(X) \setminus X| \geq r|X|$.

Once we find an expander subgraph, we construct a Hamilton cycle by using Pósa’s rotation-extension technique, which is a powerful tool exploiting the expansion property of the graph. The following definition captures the key concept that we will utilize.

Definition 2.2. Given a graph R and a path P defined over the same vertex set, we say that an edge $\{v, w\}$ is a booster for the pair (P, R) if there exists a path of length $|P| - 1$ in the graph $P \cup R$ whose two endpoints are v and w .

The following lemma is a well-known tool that is central to many applications of the Pósa’s rotation-extension technique (see, e.g., Lemma 8.5 of [5]).

Lemma 2.3. Suppose that $R \subseteq K_n$ is a $(k, 2)$ -expander and $P \subseteq K_n$ is a path that is of maximum length in the graph $P \cup R$. Then K_n contains at least $\frac{(k+1)^2}{2}$ boosters for the pair (P, R) .

2.1. Proof of Theorem 1.2

In this subsection, we state our main lemmas without proof and prove Theorem 1.2 using these lemmas. The proofs of the lemmas will be given in the next subsection.

Lemma 2.4. There exist positive constants μ and d such that if $p \gg \frac{\log n}{n}$, then $G = G(n, p)$ a.a.s. has the following property. For every μnp -bounded incompatibility system \mathcal{F} over G , there exists a subgraph $R \subseteq G$ with the following properties:

- i. R is compatible with \mathcal{F} ,
- ii. R is an $(\frac{n}{4}, 2)$ -expander, and
- iii. $|E(R)| \leq dn$.

The previous lemma will be used for “rotating” paths, while our next lemma will be used for “extending” cycles.

Lemma 2.5. For a positive constant d , if $p \gg \frac{\log n}{n}$, then a.a.s. in $G = G(n, p)$, each pair of subgraphs (P, R) satisfying the conditions below has at least $\frac{1}{64}n^2p$ boosters relative to it:

- i. R is an $(\frac{n}{4}, 2)$ -expander with $|E(R)| \leq dn$, and
- ii. P is a longest path in $P \cup R$.

Theorem 1.2 easily follows from the two lemmas above.

Proof of Theorem 1.2. Let μ and d be constants coming from Lemma 2.4. We assume that $\mu \leq \frac{1}{256(d+1)}$ by reducing its value if necessary. Suppose that an instance G of $G(n, p)$ that satisfies the conclusions of Lemmas 2.4 and 2.5 is given. Let $R \subseteq G$ be an $(\frac{n}{4}, 2)$ -expander whose existence is guaranteed by Lemma 2.4.

Given an incompatibility system \mathcal{F} over G , let $P \subseteq G$ be a path of maximum length among all paths satisfying the following two conditions: (i) $P \cup R$ is compatible with \mathcal{F} , and

(ii) P is a longest path in $P \cup R$. Note that we are maximizing over a non-empty collection, since a longest path in R meets the criteria.

By Lemma 2.5, the graph G contains at least $\frac{1}{64}n^2p$ boosters for the pair (P, R) . Among these boosters, we would like to find a booster e such that $P \cup R \cup \{e\}$ is compatible with \mathcal{F} . Towards this end, for each edge $e' = \{u, v\} \in E(P \cup R)$ we forbid to use the edges incompatible with e' as boosters. Since \mathcal{F} is μnp -bounded, each edge of $P \cup R$ forbids at most $2\mu np$ other edges. Furthermore, since the number of edges in $P \cup R$ is at most $n + |E(R)| \leq (d + 1)n$, the total number of edges forbidden is at most

$$(n + |E(R)|) \cdot 2\mu np \leq 2(d + 1)\mu n^2p < \frac{1}{64}n^2p,$$

which is less than the number of boosters. Therefore, we can find a booster e such that $P \cup R \cup \{e\}$ is still compatible with \mathcal{F} .

Since e is a booster for (P, R) , we see that there exists a cycle C of length $|P|$ in $P \cup R \cup \{e\}$. This cycle is compatible with \mathcal{F} , since it is a subgraph of a graph compatible with \mathcal{F} . Thus if C is a Hamilton cycle, then we are done. Otherwise since all $(\frac{n}{4}, 2)$ -expanders are connected, there exists a vertex $v \notin V(C)$ and an edge $e' \in E(R)$ connecting v to C . By using this edge, we can extend the cycle C to a path in $P \cup R \cup \{e\}$ that is longer than P . Thus if we define P' as the longest path in $P \cup R \cup \{e\}$, then since $P' \cup R \subseteq P \cup R \cup \{e\}$, we see that $P' \cup R$ is compatible with \mathcal{F} , and P' is a longest path in $P' \cup R$. This contradicts the fact that P is chosen as a path of maximum length subject to these conditions, and shows that C is a Hamilton cycle. ■

2.2. Proof of Lemmas

We first state two well-known results in probabilistic combinatorics. The first theorem is a form of Chernoff's inequality as appears in [26, Theorem 2.3].

Theorem 2.6. *Let $X \sim Bi(n, p)$, where $Bi(n, p)$ denotes the binomial random variable with parameters n and p . For any $s \leq \frac{1}{2}np$ and $t \geq 2np$, we have*

$$\mathbf{P}(X \leq s) \leq e^{-s/4} \quad \text{and} \quad \mathbf{P}(X \geq t) \leq e^{-3t/16}.$$

Moreover, for all $0 < \varepsilon < \frac{1}{2}$ we have,

$$\mathbf{P}(|X - np| > \varepsilon np) \leq e^{-\Omega(\varepsilon^2 np)}.$$

The second theorem is the standard local lemma (see, e.g., [3]).

Theorem 2.7. *Let A_1, A_2, \dots, A_n be events in an arbitrary probability space. A directed graph $D = (V, E)$ on the set of vertices $V = [n]$ is called a dependency digraph for the events A_1, \dots, A_n if for each $i \in [n]$, the event A_i is mutually independent of all the events $\{A_j : (i, j) \notin E\}$. Suppose that $D = (V, E)$ is a dependency digraph for the above events and suppose that there are real numbers x_1, \dots, x_n such that $0 \leq x_i < 1$ and $\mathbf{P}(A_i) \leq x_i \prod_{(i,j) \in E} (1 - x_j)$ for all $i \in [n]$. Then $\mathbf{P}(\bigcap_{i=1}^n \bar{A}_i) \geq \prod_{i=1}^n (1 - x_i)$. In particular, with positive probability no event A_i holds.*

The following lemma establishes several properties of $G(n, p)$ that we need.

Lemma 2.8. *If $p \gg \frac{\log n}{n}$, then $G(n, p)$ a.a.s. satisfies the following properties.*

- i. *all degrees are $(1 + o(1))np$,*
- ii. *for all sets X of size $|X| < (np^4)^{-1/3}$, we have $e(X) \leq 8|X|$,*
- iii. *for all sets X of size $|X| = t$ for $(np^4)^{-1/3} \leq t \leq n$, we have $e(X) \leq t^2 p \cdot \left(\frac{n}{t}\right)^{1/2}$, and*
- iv. *for disjoint sets X and Y satisfying $|X||Y|p \gg n$, we have $e(X, Y) \geq \frac{1}{2}|X||Y|p$.*

Proof. We omit the proofs of Properties (i) and (iv), since they follow easily from direct applications of Chernoff’s inequality together with the union bound.

The probability of a fixed set X of size t to violate Property (ii) is at most

$$\binom{\binom{t}{2}}{8t} p^{8t} \leq \left(\frac{etp}{8}\right)^{8t} < \left(\left(\frac{e}{8}\right)^8 \left(\frac{t}{n}\right)^2\right)^t.$$

Hence by the union bound, the probability of Property (ii) being violated is at most

$$\sum_{t=1}^{(np^4)^{-1/3}} \binom{n}{t} \cdot \left(\left(\frac{e}{8}\right)^8 \left(\frac{t}{n}\right)^2\right)^t < \sum_{t=1}^{n/(\log n)^{4/3}} \left(e \cdot \left(\frac{e}{8}\right)^8 \left(\frac{t}{n}\right)\right)^t = o(1).$$

Similarly, the probability of a fixed set X of size t to violate Property (iii) is, by Chernoff’s inequality, at most $e^{-c \cdot t^2 p(n/t)^{1/2}}$ for some positive constant c . The function

$$ctp \left(\frac{n}{t}\right)^{1/2} - 2 \log \left(\frac{en}{t}\right)$$

is increasing for $t > 0$ and for $t = (np^4)^{-1/3}$ equals $c(np)^{1/3} - O(\log np) > 0$. Hence, for $t \geq (np^4)^{-1/3}$, we have that

$$ct^2 p \left(\frac{n}{t}\right)^{1/2} \geq 2t \log \left(\frac{en}{t}\right).$$

Thus by the union bound, the probability of Property (iii) being violated is at most

$$\sum_{t=(np^4)^{-1/3}}^n \binom{n}{t} e^{-c t^2 p(n/t)^{1/2}} \leq \sum_{t=(np^4)^{-1/3}}^n e^{t \log(\frac{en}{t}) - c t^2 p(n/t)^{1/2}} = o(1). \quad \blacksquare$$

We first prove Lemma 2.4 which we restate here for the reader’s convenience. The proof is based on a straightforward application of local lemma, but is rather lengthy.

Lemma. *There exist positive constants μ and d such that if $p \gg \frac{\log n}{n}$, then $G = G(n, p)$ a.a.s. has the following property. For every μnp -bounded incompatibility system \mathcal{F} over G , there exists a subgraph $R \subseteq G$ with the following properties:*

- i. *R is compatible with \mathcal{F} ,*
- ii. *R is an $(\frac{n}{4}, 2)$ -expander, and*
- iii. *$|E(R)| \leq dn$.*

Proof. Throughout the proof, let $c_0 = e, C_1 = C_2 = \frac{1}{2}, \alpha = \frac{1}{2} \left(\frac{1}{20e}\right)^2, d = 10(20e)^2$, and $\mu = \frac{1}{25c_0 d^2} < 10^{-11}$. Condition on $G = G(n, p)$ satisfying the events of Lemma

2.8. Suppose that we are given a μnp -bounded incompatibility system \mathcal{F} over G . For all $v \in V(G)$, independently (with repetition) choose d random edges in G incident to v , and let $F(v)$ be the set of chosen edges. Let R be the graph whose edge set is $\bigcup_v F(v)$. We claim that R has the properties listed above with positive probability. Note that Property (iii) trivially holds.

Let $t_0 = \frac{1}{3}(np^4)^{-1/3}$, $t_1 = \alpha n$, and $t_2 = n/4$ for some constant α to be chosen later. There are three types of events that we consider. First are events considering compatibility of edges. For a pair of edges e_1 and e_2 , if $e_1 \neq e_2$, then let $A(e_1, e_2)$ be the event that both edges e_1 and e_2 are in R , and if $e_1 = e_2 = e$, then let $A(e, e)$ be the event that the edge e is chosen in two different trials. Define

$$\mathcal{A} = \{A(e_1, e_2) : e_1, e_2 \text{ are incompatible, or } e_1 = e_2\}.$$

Second are events considering expansion of small sets. For a set W , let $B(W)$ be the event that $e_R(W) \geq \frac{d}{3}|W|$, and define, for $t_0 \leq t \leq t_1$,

$$\mathcal{B}_t = \{B(W) : |W| = 3t\}.$$

Third are events considering expansion of large sets. For a pair of disjoint subsets X and Y , let $C(X, Y)$ be the event that $e_R(X, Y) = 0$, and define, for $t_1 \leq t \leq t_2$,

$$\mathcal{C}_t = \{C(X, Y) : X \cap Y = \emptyset, |X| = t, |Y| = n - 3t\}.$$

We first prove that Properties (i) and (ii) hold if none of the events in $\mathcal{A}, \mathcal{B}_t$, and \mathcal{C}_t happen. Property (i) obviously holds if none of the events in \mathcal{A} happens. Note that not having the events $A(e, e)$ for all edges e implies the fact that we obtain distinct edges at each trial. Hence each set X has at least $d|X|$ distinct edges incident to it, and in particular, we have $|E(R)| = dn$. For Property (ii), consider a set X of size $|X| = t$ and assume that $|N_R(X) \setminus X| < 2|X|$. Let W be a superset of $X \cup N_R(X)$ of size exactly $3|X|$. By the fact mentioned above, we see that $e_R(W) \geq d|X| > \frac{d}{3}|W|$.

1. If $t < t_0$, then since $d > 24$, this contradicts the fact that $G(n, p)$ has $e(W) \leq 8|W|$,
2. if $t_0 \leq t \leq t_1$, then it contradicts the event $B(W)$, and
3. if $t_1 \leq t \leq n/4$, then we have $e(X, V \setminus W) = 0$ and it contradicts the event $C(X, V \setminus W)$.

Hence in all three cases we arrive at a contradiction. Therefore if none of the events in $\mathcal{A}, \mathcal{B}_t$, and \mathcal{C}_t holds, then we obtain all the claimed properties (i), (ii), and (iii).

We will use the local lemma to prove that with positive probability none of the events in $\mathcal{A}, \mathcal{B}_t$, and \mathcal{C}_t holds. Towards this end, we define the dependency graph Γ with vertex set $\mathcal{V} = \mathcal{A} \cup \bigcup_{t=t_0}^{t_1} \mathcal{B}_t \cup \bigcup_{t=t_1}^{t_2} \mathcal{C}_t$. Note that the graph R is determined by the outcome of n events $\{F(v)\}_{v \in V(G)}$ (recall that $F(v)$ is a set of d random edges incident to v). We let $V_1, V_2 \in \mathcal{V}$ be adjacent in Γ if there exists a vertex $v \in V(G)$ such that both V_1 and V_2 are dependent on the outcome of $F(v)$, i.e., if there exist edges e_1 and e_2 both incident to v such that V_1 depends on e_1 and V_2 depends on e_2 . This graph can be used as the dependency digraph in the local lemma since $F(v)$ and $F(w)$ are independent for all distinct vertices $v, w \in V(G)$.

In order to apply the local lemma, for each event in \mathcal{V} , we estimate its probability and the degree of the corresponding vertex in Γ . In all cases, for the dependency with events in \mathcal{B}_t and \mathcal{C}_t , we use the crude bounds $|\mathcal{B}_t|$ and $|\mathcal{C}_t|$.

Family \mathcal{A} : For a fixed pair of intersecting edges e_1 and e_2 , to bound the probability of the event $A(e_1, e_2)$, first, for each e_1 and e_2 , select from which vertex and on which trial that edge was chosen (at most $2d$ choices for each edge), and second, compute the probability that e_1 and e_2 were chosen at those trials (probability at most $\left(\frac{1+o(1)}{np}\right)^2$). This gives

$$\mathbf{P}(A(e_1, e_2)) \leq (1 + o(1)) \left(\frac{2d}{np}\right)^2.$$

Define $x = c_0 \left(\frac{2d}{np}\right)^2$ for some constant $c_0 \geq 1$ to be chosen later (this parameter will be used in the local lemma). To compute the number of neighbors of $A(e_1, e_2)$ in Γ in the set \mathcal{A} , note that $A(e_1, e_2)$ is adjacent to $A(f_1, f_2)$ if and only if some two edges e_i and f_j intersect. There are at most three vertices in $e_1 \cup e_2$, each vertex has degree $(1 + o(1))np$ in $G(n, p)$, and each edge has at most $2\mu np$ other edges incompatible with it. Therefore the number of neighbors of $A(e_1, e_2)$ in \mathcal{A} is at most

$$3 \cdot (1 + o(1))np \cdot 2\mu np = 6\mu(np)^2(1 + o(1)).$$

Family \mathcal{B}_t : Assume that $t_0 \leq t \leq t_1$ and consider a fixed set W of size $|W| = 3t$. To bound the probability of the event $B(W)$, first, choose $\frac{d}{3}|W| = dt$ edges among the edges of $G(n, p)$ in W ($\binom{e(W)}{dt}$ choices), second, for each chosen edge, select from which vertex and on which trial that edge was chosen (at most $(2d)^{dt}$ choices altogether), third, compute the probability that each choice became the particular edge of interest (probability at most $\left(\frac{1+o(1)}{np}\right)^{dt}$). This gives

$$\mathbf{P}(B(W)) \leq \binom{e_G(W)}{dt} \cdot (2d)^{dt} \cdot \left(\frac{1 + o(1)}{np}\right)^{dt} \leq \left(\frac{e \cdot e_G(W)}{td} \cdot \frac{(2 + o(1))d}{np}\right)^{dt},$$

which by the assumption that $e_G(W) \leq 9t^2p \cdot \left(\frac{n}{3t}\right)^{1/2}$ (coming from Lemma 2.8 (iii)) gives

$$\mathbf{P}(B(W)) \leq \left(20e \cdot \left(\frac{t}{n}\right)^{1/2}\right)^{dt}.$$

Define $y_t = e^{C_1 t} \left(20e \cdot \left(\frac{t}{n}\right)^{1/2}\right)^{dt}$ for some positive constant C_1 , and for later usage, note that

$$\sum_{t=t_0}^{t_1} |\mathcal{B}_t| \cdot y_t = \sum_{t=t_0}^{t_1} \binom{n}{3t} \cdot e^{C_1 t} \left(20e \cdot \left(\frac{t}{n}\right)^{1/2}\right)^{dt} \leq \sum_{t=t_0}^{t_1} \left(e^{C_1} \left(\frac{en}{3t}\right)^3 \cdot (20e)^d \cdot \left(\frac{t}{n}\right)^{d/2}\right)^t = o(1), \tag{1}$$

since α, d , and C_1 satisfy $e^{C_1} \left(\frac{e}{3\alpha}\right)^3 \cdot (20e)^d \cdot \alpha^{d/2} < 1$.

To compute the number of neighbors of $B(W)$ in \mathcal{A} in Γ , note that the event $B(W)$ is adjacent to $A(e_1, e_2)$ if e_1 or e_2 intersect W . Therefore, the number of neighbors as above is at most

$$|W| \cdot (1 + o(1))np \cdot \mu np = (1 + o(1))3\mu n^2 p^2.$$

Family \mathcal{C}_t : Assume that $t_1 \leq t \leq n/4$ and consider a fixed pair of sets X and Y of sizes $|X| = t$ and $|Y| = n - 3t$. For each vertex $v \in X$, let $d_G(v, Y)$ be the number of neighbors of v in Y in $G(n, p)$. Then the probability that $e_R(X, Y) = 0$ is

$$\mathbf{P}(C(X, Y)) = \prod_{v \in X} \left(1 - \frac{d_G(v, Y)}{(1 + o(1))np} \right)^d \leq e^{-(1+o(1))de_G(X,Y)/(np)} \leq e^{-dt(n-3t)/(3n)},$$

where we used the assumption that $e_G(X, Y) \geq \frac{t(n-3t)}{2}p$. Define $z_t = e^{C_2 n} e^{-dt(n-3t)/(3n)}$ for some positive constant C_2 , and for later usage, note that

$$\sum_{t=t_1}^{t_2} |\mathcal{C}_t| \cdot z_t \leq \sum_{t=t_1}^{t_2} 2^{2n} \cdot e^{C_2 n} \cdot e^{-dt(n-3t)/(3n)} = o(1), \tag{2}$$

since $d\alpha(1 - 3\alpha) > 3(C_2 + 2)$ and $d/16 > 3(C_2 + 2)$. For the degree of $C(X, Y)$ in \mathcal{A} in Γ , we use the crude bound

$$|\mathcal{A}| \leq n \cdot (1 + o(1))np \cdot \mu np = (1 + o(1))\mu n^3 p^2.$$

To apply the local lemma, we must verify the following three inequalities (for appropriate choices of t as determined by the sets W , X , and Y):

$$\begin{aligned} \mathbf{P}(A(e_1, e_2)) &\leq x \cdot (1 - x)^{(1+o(1))6\mu(np)^2} \cdot \left(\prod_{t=t_0}^{t_1} (1 - y_t)^{|\mathcal{B}_t|} \right) \cdot \left(\prod_{t=t_1}^{n/4} (1 - z_t)^{|\mathcal{C}_t|} \right), \\ \mathbf{P}(B(W)) &\leq y_t \cdot (1 - x)^{(1+o(1))3\mu t^2 p^2} \cdot \left(\prod_{t=t_0}^{t_1} (1 - y_t)^{|\mathcal{B}_t|} \right) \cdot \left(\prod_{t=t_1}^{n/4} (1 - z_t)^{|\mathcal{C}_t|} \right), \\ \mathbf{P}(C(X, Y)) &\leq z_t \cdot (1 - x)^{(1+o(1))\mu n^3 p^2} \cdot \left(\prod_{t=t_0}^{t_1} (1 - y_t)^{|\mathcal{B}_t|} \right) \cdot \left(\prod_{t=t_1}^{n/4} (1 - z_t)^{|\mathcal{C}_t|} \right), \end{aligned}$$

By (1) and (2), we know that

$$\left(\prod_{t=t_0}^{t_1} (1 - y_t)^{|\mathcal{B}_t|} \right) \cdot \left(\prod_{t=t_1}^{n/4} (1 - z_t)^{|\mathcal{C}_t|} \right) = 1 - o(1).$$

Recall that $y_t \geq e^{C_1 t} \mathbf{P}(B(W))$ and $z_t \geq e^{C_2 n} \mathbf{P}(C(X, Y))$. Thus to have the above three inequalities, it suffices to prove that

$$\begin{aligned} 1 &\leq (1 + o(1))c_0 \cdot (1 - x)^{(1+o(1))6\mu(np)^2}, \\ \forall t_0 \leq t \leq t_1, 1 &\leq (1 + o(1))e^{C_1 t} (1 - x)^{(1+o(1))3\mu t^2 p^2}, \\ \forall t_1 \leq t \leq t_2, 1 &\leq (1 + o(1))e^{C_2 n} (1 - x)^{(1+o(1))\mu n^3 p^2}. \end{aligned}$$

Note that $1 - x = e^{-(1+o(1))x}$ and $x = c_0 \cdot \left(\frac{2d}{np}\right)^2$. Since $C_1 \geq (1 + o(1))12\mu d^2 c_0$ and $C_2 \geq (1 + o(1))4\mu d^2 c_0$, the second and the third inequalities hold. Also, the first inequality holds since the parameters are chosen so that

$$c_0 \cdot (1 - x)^{(1+o(1))6\mu(np)^2} = c_0 \cdot e^{-(1+o(1))24c_0 d^2 \mu} > 1. \quad \blacksquare$$

We now prove Lemma 2.5 (restated here).

Lemma. For a positive constant d , if $p \gg \frac{\log n}{n}$, then a.a.s. in $G = G(n, p)$, each pair of subgraphs (P, R) satisfying the conditions below has at least $\frac{1}{64}n^2p$ boosters relative to it:

- i. R is an $(\frac{n}{4}, 2)$ -expander with $|E(R)| \leq dn$, and
- ii. P is a longest path in $P \cup R$.

Proof. To prove the lemma, we first fix a pair (P, R) satisfying the conditions given above, and estimate the probability that $G(n, p)$ contains enough boosters for the pair.

Since R is an $(\frac{n}{4}, 2)$ -expander, Lemma 2.3 implies that K_n contains at least $\frac{n^2}{32}$ boosters for the pair (P, R) . It thus follows that the expected number of boosters for (P, R) in $G(n, p)$ is at least $\frac{1}{32}n^2p$. By Chernoff's inequality, with probability at least $1 - e^{-\Omega(n^2p)}$, we have at least $\frac{1}{64}n^2p$ boosters for (P, R) in $G(n, p)$.

We use this estimate on the probability together with the union bound to prove the lemma. The total number of paths of all possible length is at most $n \cdot n! \leq e^{n \log n}$, and the total number of graphs R that we must consider is at most

$$\binom{n^2}{dn} \leq e^{dn \log n}.$$

Since $p \gg \frac{\log n}{n}$, we obtain our conclusion by taking the union bound. ■

To prove Theorem 1.4, given a globally μnp -bounded edge coloring of $G(n, p)$, call a pair of edges *compatible* if they are of different color, and a subgraph $H \subseteq G$ *compatible* if it is rainbow. Re-define the events $A(e_1, e_2)$ accordingly. One can easily check that the proof given in this section establishes Theorem 1.4 after slightly changing the method used in estimating the degree in the dependency graph. We omit the straightforward details.

3. PROOF OF THEOREM 1.3

In this section we present the proof of our second result which is based on several ideas. First we use a strategy from [2] to transform the problem of finding a compatible Hamilton cycle into a problem of finding a directed Hamilton cycle in an appropriately defined auxiliary graph. This strategy requires a 'well-behaved' perfect matching in our graph, which will be taken using a 'nibble method'. Finally to complete the proof we use recent resilience-type results on Hamiltonicity of random directed graphs proved in [17] and [14].

Before we delve into the (rather technical) details of the proof, let us provide a brief outline of our argument. Let $G = G(n, p)$ with $p \gg \frac{\log^8 n}{n}$. Assume a μnp -bounded incomparability system \mathcal{F} over G is given, and our aim is to find a Hamilton cycle in G compatible with \mathcal{F} . Assume for simplicity n is even. Let $V = A \cup B$ be a random equipartition of $V(G)$ with $|A| = |B| = m = \frac{n}{2}$, and let M be a randomly chosen perfect matching between A and B in G . Let $M = \{e_1, \dots, e_{n/2}\}$, with $e_i = (a_i, b_i)$, $a_i \in A, b_i \in B$. We construct a Hamilton cycle by further adding $n/2$ edges to M , while obeying compatibility. Define an auxiliary directed graph $D_G(M)$ as follows: its vertices are the edges of M , and (e_i, e_j) is a directed edge of $D_G(M)$ if $\{b_i, a_j\} \in E(G)$. Since the edges of $D_G(M)$ are in one-to-one

correspondence with the edges of G between A and B outside M , we may consider $D_G(M)$ as a random directed graph on m vertices with edge probability p . Observe that a directed Hamilton cycle in $D_G(M)$ translates into a Hamilton cycle in G in an obvious way. To obtain a Hamilton cycle compatible with \mathcal{F} through this correspondence, we remove some edges from $D_G(M)$. Consider an edge e_i of M (in its capacity as a vertex of $D_G(M)$). Let us see which directed edges (e_i, e_j) leaving e_i in $D_G(M)$ need to be deleted. Those are edges for which $\{b_i, a_j\}$ is incompatible with \mathcal{F} , and the number of such edges should be at most (about) μmp . In addition, we need to delete (e_i, e_j) for which $\{b_i, a_j\}$ is compatible with e_i (about $(1 - \mu)$ proportion of edges) but incompatible with e_j – and the proportion of such edges should be about μ . We expect these heuristic estimates to hold due to our random choice of M . Assuming these estimates, altogether we need to delete from $D_G(M)$ about $\mu mp + (1 - \mu)\mu mp$ edges leaving e_i , and a similar amount of edges entering e_i . As mentioned above, the graph $D_G(M)$ is basically a random directed graph on m vertices with edge probability p . At this stage, we invoke a recent result of Ferber et al. ([14]; Theorem 3.7 below), stating that if $p \gg \frac{\log^8 n}{n}$ then a random directed graph $D = D(n, p)$ is a.a.s. such that every subgraph of D of minimum in- and out-degrees at least $(\frac{1}{2} + \varepsilon)np$ contains a directed Hamilton cycle. In order to be able to apply this theorem to $D_G(M)$ we need to estimate from above the deleted in- and out-degrees at every vertex e_i of $D_G(M)$ – as we indicated above, and then to require that $\mu mp + (1 - \mu)\mu mp \leq (\frac{1}{2} - \varepsilon)mp$. So essentially we need to solve: $\mu + (1 - \mu)\mu = \frac{1}{2}$ – which gives us $\mu = 1 - \frac{1}{\sqrt{2}}$. It should be mentioned that this approach borrows some ideas from the argument of Alon and Gutin [2], who also arrived at the same magical constant $1 - \frac{1}{\sqrt{2}}$ (but for the simpler case of the complete graph to start with).

We start with the following two definitions.

Definition 3.1. A perfect matching on n vertices over a partition $A \cup B$ (for even n) or $A \cup B \cup \{v_*\}$ (for odd n), with $|A| = |B| = \lfloor n/2 \rfloor$, is a collection $\{(a_i, b_i)\}_{i=1}^{\lfloor n/2 \rfloor}$ of disjoint pairs of vertices with $a_i \in A$ and $b_i \in B$ with the following property:

- i. if n is even, then $\{a_i, b_i\}$ is an edge for every i , and
- ii. if n is odd, then $\{a_i, b_i\}$ is an edge for $i = 1, 2, \dots, (n-3)/2$, and $(a_{(n-1)/2}, v_*, b_{(n-1)/2})$ is a path of length 2.

In most cases, for a given vertex a_i , the edge incident to a_i in the perfect matching is $\{a_i, b_i\}$. However, when n is odd and $i = (n-1)/2$, the edge incident to $a_{(n-1)/2}$ is $\{a_{(n-1)/2}, v_*\}$. This distinction is made for technical reasons and we recommend the reader to assume that n is even for the first time reading.

Definition 3.2. Let G be a graph and let M be a perfect matching $\{e_i\}_{i=1}^{\lfloor n/2 \rfloor}$ for $e_i = (a_i, b_i)$ (and $e_{\lfloor n/2 \rfloor} = (a_{\lfloor n/2 \rfloor}, v_*, b_{\lfloor n/2 \rfloor})$ if n is odd).

- i. Define $D_G(M)$ as the directed graph over the vertex set $\{e_1, e_2, \dots, e_{\lfloor n/2 \rfloor}\}$, where there is a directed edge from e_i to e_j if and only if $\{b_i, a_j\} \in E(G)$.
- ii. For a given incompatibility system \mathcal{F} over G , define $D_G(M; \mathcal{F})$ as the subgraph of $D_G(M)$ obtained by the following process: remove the directed edge (e_i, e_j) whenever $\{b_i, a_j\}$ is incompatible with the edge of M incident to b_i , or with the edge of M incident to a_j . Moreover, if n is odd and $\{a_{\lfloor n/2 \rfloor}, v_*\}$ and $\{v_*, b_{\lfloor n/2 \rfloor}\}$ are not compatible, then remove all edges in $D_G(M)$ incident to the vertex $e_{\lfloor n/2 \rfloor}$.

We may also write $D(M)$ or $D(M; \mathcal{F})$, when G is clear from the context. The following proposition explains the motivation behind the definition given above.

Proposition 3.3. *Let G be a graph with an incompatibility system \mathcal{F} , and let M be a perfect matching in G . If $D_G(M; \mathcal{F})$ contains a directed Hamilton cycle, then G contains a Hamilton cycle compatible with \mathcal{F} .*

Proof. Let M be given as e_1, e_2, \dots, e_k , and without loss of generality let $(e_1, e_2, \dots, e_k, e_1)$ be the Hamilton cycle in $D_G(M; \mathcal{F})$.

If G has an even number of vertices, then by the definition of $D_G(M; \mathcal{F})$, we see that

$$(a_1, b_1, a_2, b_2, \dots, a_k, b_k, a_1)$$

is a Hamilton cycle in G . Moreover, the edge $\{a_i, b_i\}$ is compatible with both $\{b_i, a_{i+1}\}$ and $\{a_i, b_{i-1}\}$ (addition and subtraction of indices are modulo k) by the definition of $D_G(M; \mathcal{F})$. Therefore, we found a Hamilton cycle in G compatible with \mathcal{F} .

If G has an odd number of vertices, then as before, we see that $(a_1, b_1, a_2, b_2, \dots, a_k, v_*, b_k, a_1)$ is a Hamilton cycle in G . If $\{a_k, v_*\}$ and $\{v_*, b_k\}$ were not compatible, then by the definition of $D_G(M; \mathcal{F})$, the vertex e_k must be isolated in $D_G(M; \mathcal{F})$, contradicting the fact that $D_G(M; \mathcal{F})$ is Hamiltonian. Therefore, the pair is compatible. All other pairs are compatible as seen above. ■

We prove the Hamiltonicity of $D_G(M; \mathcal{F})$ by carefully choosing a perfect matching M so that it satisfies the following two properties.

Definition 3.4. *Let ε be a fixed positive real, let G be a given graph with incompatibility system \mathcal{F} , and let M be a perfect matching in G .*

- i. The pair (G, M) is ε -di-ham-resilient if every subgraph of $D_G(M)$ of minimum in- and out-degrees at least $(\frac{1}{2} + \varepsilon) \frac{\delta(G)}{2}$ contains a directed Hamilton cycle.*
- ii. The triple (G, M, \mathcal{F}) is ε -typical if $D_G(M; \mathcal{F})$ has minimum in- and out-degrees at least $(\frac{1}{2} + \varepsilon) \frac{\delta(G)}{2}$.*

The following lemma is the key ingredient of our proof, asserting the a.a.s. existence of a perfect matching in $G(n, p)$ for which (G, M) is ε -di-ham-resilient, and (G, M, \mathcal{F}) is ε -typical.

Lemma 3.5. *Let ε be a fixed positive real, and $p = \frac{\omega \log^8 n}{n}$ for some function $\omega = \omega(n) \leq \log n$ that tends to infinity. Then $G = G(n, p)$ a.a.s. has the following property: for every $(1 - \frac{1}{\sqrt{2}} - 2\varepsilon)np$ -bounded incompatibility system \mathcal{F} over G , there exists a perfect matching $M \subseteq G$ such that*

- i. (G, M) is ε -di-ham-resilient and*
- ii. (G, M, \mathcal{F}) is ε -typical.*

Next lemma allows us to restrict our attention to small values of p as in Lemma 3.5. Its proof will be given in the following subsection.

Lemma 3.6. *Suppose that p_1 and p_2 satisfying $1 \geq p_1 \geq p_2 \gg \frac{\log n}{n}$ are given. If there exist positive real numbers α and ε such that $G(n, p_2)$ a.a.s. contains a compatible Hamilton cycle for every $(\alpha + 2\varepsilon)np_2$ -bounded incompatibility system, then $G(n, p_1)$ a.a.s. contains a compatible Hamilton cycle for every $(\alpha + \varepsilon)np_1$ -bounded incompatibility systems.*

The proof of Theorem 1.3 easily follows from Lemma 3.5.

Proof of Theorem 1.3. It suffices to prove the statement for $p = \frac{\omega \log^8 n}{n}$ for some $\omega = \omega(n) \leq \log n$ that tends to infinity since larger edge probabilities can be handled by Lemma 3.6 below. Let ε be a given positive real and suppose that $G = G(n, p)$ satisfies the properties guaranteed by Lemma 3.5: for every $(1 - \frac{1}{\sqrt{2}} - 2\varepsilon)n$ -bounded incompatibility system \mathcal{F} over G , there exists a perfect matching M for which the pair (G, M) is ε -di-ham-resilient and (G, M, \mathcal{F}) is ε -typical. These two properties imply that $D_G(M; \mathcal{F})$ contains a directed Hamilton cycle, which by Proposition 3.3 implies that G contains a Hamilton cycle compatible with \mathcal{F} . ■

3.1. Preliminaries

Before proceeding to the proof of Lemma 3.5, we state some results needed for our proof. Let $D(n, p)$ be a random directed graph on n vertices, in which for every pair $i \neq j$ the edge $i \rightarrow j$ appears independently with probability p . The first theorem is a resilience-type result for Hamiltonicity of $D(n, p)$ which extends a classical result of Ghouila-Houri [15]. It was first proved by Hefetz, Steger, and Sudakov [17] (for edge probabilities $p \geq n^{-1/2+o(1)}$) and then strengthened by Ferber, Nenadov, Noever, Peter, and Skoric [14] to much smaller values of $p(n)$.

Theorem 3.7. *For all fixed positive reals ε , if $p \gg \frac{\log^8 n}{n}$ then $D(n, p)$ a.a.s. has the following property: every spanning subgraph of $D(n, p)$ of minimum in- and out-degrees at least $(\frac{1}{2} + \varepsilon)np$ contains a directed Hamilton cycle.*

We will often be considering events defined over the product of two probability spaces, and the following simple lemma will be handy.

Lemma 3.8. *Let X_1 and X_2 be two random variables, and suppose that there exists a set A such that $\mathbf{P}((X_1, X_2) \in A) = 1 - x$ for some positive real x . Let*

$$A_1 = \left\{ a \mid \mathbf{P}((X_1, X_2) \in A \mid X_1 = a) \geq 1 - \sqrt{x} \right\}.$$

Then $\mathbf{P}(X_1 \in A_1) \geq 1 - \sqrt{x}$.

Proof. Since

$$x = \mathbf{P}((X_1, X_2) \notin A) \geq \mathbf{P}(X_1 \notin A_1) \cdot \sqrt{x},$$

we have $\mathbf{P}(X_1 \notin A_1) \leq \sqrt{x}$, or equivalently $\mathbf{P}(X_1 \in A_1) \geq 1 - \sqrt{x}$. ■

We prove Lemma 3.6 which, as seen in the previous subsection, allows us to restrict our attention to sparse random graphs. For two graphs $G_1 \supseteq G_2$ and an incompatibility system \mathcal{F} defined over G_1 , we define the *incompatibility system induced by \mathcal{F} on G_2* as the

incompatibility system where two edges $e, e' \in E(G_2)$ are incompatible if and only if they are in \mathcal{F} .

Proof. Let $G_1 = G(n, p_1)$ and let G_2 be a random subgraph of G_1 obtained by retaining every edge independently with probability $\frac{p_2}{p_1}$. Note that the distribution of the subgraph G_2 is identical to that of $G(n, p_2)$.

Let \mathcal{R} be the collection of graphs that contain a compatible Hamilton cycle for every $(\alpha + 2\varepsilon)np_2$ -bounded incompatibility system. By the assumption of the lemma, we know that

$$\mathbf{P}(G_2 \in \mathcal{R}) = 1 - o(1).$$

Let \mathcal{R}_1 be the collection of graphs Γ such that $\mathbf{P}(G_2 \in \mathcal{R} \mid G_1 = \Gamma) \geq \frac{1}{2}$. By Lemma 3.8, we see that

$$\mathbf{P}(G_1 \in \mathcal{R}_1) = 1 - o(1).$$

On the other hand, for each fixed $(\alpha + \varepsilon)np_1$ -bounded incompatibility system \mathcal{F} over G_1 , by Chernoff's inequality and the union bound, with probability greater than $\frac{1}{2}$, the incompatibility system induced by \mathcal{F} on G_2 is $(\alpha + 2\varepsilon)np_2$ -bounded.

Therefore, if $G_1 \in \mathcal{R}_1$, then for every $(\alpha + \varepsilon)np_1$ -bounded incompatibility system \mathcal{F} over G_1 , there exists a subgraph $G'_2 \subseteq G_1$ such that \mathcal{F} induces an $(\alpha + 2\varepsilon)np_2$ -bounded incompatibility system over G'_2 , and $G'_2 \in \mathcal{R}$. These two properties imply that G'_2 contains a Hamilton cycle compatible with \mathcal{F} , which in turn implies that G_1 also contains such Hamilton cycle. ■

3.2. Proof of Lemma 3.5

In this subsection, we prove Lemma 3.5. The perfect matching M in the statement of the lemma will be chosen according to some random process that we denote by Φ , i.e., $M = \Phi(G)$. In fact we prove the following strengthening of Lemma 3.5.

Lemma 3.9. *If $p = \frac{\omega_n \log^8 n}{n}$ for some $\omega_n \leq \log n$ that tends to infinity, then $G = G(n, p)$ a.a.s. has the following property. For every $\left(1 - \frac{1}{\sqrt{2}} - 2\varepsilon\right)np$ -bounded incompatibility system \mathcal{F} over G , a random perfect matching $M = \Phi(G)$ satisfies each of the following properties with probability $1 - o(1)$,*

- i. (G, M) is ε -di-ham-resilient, and
- ii. (G, M, \mathcal{F}) is ε -typical.

Note that Lemma 3.5 immediately follows from Lemma 3.9, since the latter implies the a.a.s. existence of a particular instance of M for which both Properties (i) and (ii) hold.

Throughout the section, we assume that ε is a given fixed positive real (we may assume that ε is small enough by decreasing its value if necessary), and let $\delta = e^{-22\varepsilon^{-1} \ln \varepsilon^{-1}}$. Given $G = G(n, p)$, we construct a perfect matching $\Phi(G)$ by the following algorithm (we first give a description for even n).

1. Take a random bipartite subgraph H of G by choosing a uniform bisection $A \cup B$ and then taking each edge crossing the bisection independently with probability $\frac{\varepsilon}{4}$. Initialize $H_0 := H, A_0 := A, B_0 := B$.

2. Repeat the following steps $T = \frac{\ln(4/\varepsilon)}{-\ln(1-\delta)} \approx \frac{\ln(4/\varepsilon)}{\delta}$ times (start from $i = 0$).
 - 2.1. Given a bipartite graph H_i with bipartition $A_i \cup B_i$ satisfying $n_i := |A_i| = |B_i|$ and $m_i := e(H_i)$, choose each edge of H_i independently with probability $\frac{\delta n_i}{m_i}$ to form a set of edges $M_i^{(0)}$.
 - 2.2. Let $M_i \subseteq M_i^{(0)}$ be the set of edges incident to no other edges in $M_i^{(0)}$.
 - 2.3. Remove the vertices incident to the edges in M_i from H_i to obtain H_{i+1} .
3. Take an arbitrary perfect matching M_T in the remaining graph H_T and define $\Phi(G)$ as the union of the matchings M_0, \dots, M_T .

Remark. If n is odd, then for each $i = 0, 1, \dots, T$, the bipartite graphs H_i will have bipartition $A_i \cup B_i$ with $|A_i| = n_i + 1$ and $|B_i| = n_i$ for all $i = 0, 1, \dots, T$. In this case, in Step 3, first choose an edge $\{v_*, a_{(n-1)/2}\}$ within A_T , and then choose $b_{(n-1)/2} \in B_T$ so that $\{v_*, a_{(n-1)/2}\}$ and $\{v_*, b_{(n-1)/2}\}$ are compatible. Afterwards, find a perfect matching between $A_T \setminus \{v_*, a_{(n-1)/2}\}$ and $B_T \setminus \{b_{(n-1)/2}\}$.

Since each M_i forms a matching, the algorithm above produces a sequence of balanced bipartite graphs H_i with vertex partition $A_i \cup B_i$ for $i = 0, 1, \dots, T$, where $H_0 = H$ and $A_0 = A, B_0 = B$. Note that the algorithm might fail to produce a perfect matching of H , as there is no guarantee on H_T containing a perfect matching in the final step. However, in Lemma 3.14 we will prove that such ‘bad event’ rarely happens.

Proof of Lemma 3.9 (i). We view the probability space generated by the pair $(G(n, p), H)$ from a slightly different perspective. Let $p_1 = \frac{\varepsilon}{4}p$, and define p_2 by $p = p_1 + p_2 - p_1p_2$. Let $G_1 = G(n, p_1)$ and $G_2 = G(n, p_2)$, and note that $G = G_1 \cup G_2$ has the same distribution as $G(n, p)$. The random algorithm Φ can equivalently be defined by first taking a random subgraph G_1 , and then applying a random algorithm Ψ , i.e. $\Phi(G) = \Psi(G_1)$. Further note that all events in the probability space $\mathcal{P}(G, G_1)$ generated by the pair of graphs G and G_1 are measurable in the probability space $\mathcal{P}(G_1, G_2)$ generated by the pair of graphs G_1 and G_2 . Therefore, since the event that we would like to study lies in the probability space $\mathcal{P}(G, G_1, \Phi)$ we may as well compute its probability in the space $\mathcal{P}(G_1, G_2, \Psi)$.

Since G_1 and G_2 are independent, conditioned on $\Psi(G_1) = M$, the graph $D_{G_2}(M)$ has the distribution of the random directed graph $D(\lfloor \frac{n}{2} \rfloor, p_2)$. We thus know by Theorem 3.7 that a.a.s. every subgraph of $D_{G_2}(M)$ of minimum in- and out-degrees at least $(\frac{1}{2} + \frac{\varepsilon}{2})np_2$ is Hamiltonian, i.e.

$$\mathbf{P}\left((G_2, \Psi(G_1)) \text{ is } \frac{\varepsilon}{2}\text{-di-ham-resilient} \mid \Psi \text{ succeeds}\right) = 1 - o(1).$$

Hence as long as Ψ outputs a perfect matching with high probability (this fact will be proved in Lemma 3.14),

$$\mathbf{P}\left((G_2, \Phi(G)) \text{ is } \frac{\varepsilon}{2}\text{-di-ham-resilient}\right) = 1 - o(1).$$

Observe that if G_1 has maximum degree at most $\frac{\varepsilon}{2}np$, then every subgraph of $D_G(M)$ of minimum in- and out-degrees at least $(\frac{1}{2} + \varepsilon)np$ contains a subgraph of $D_{G_2}(M)$ of minimum in- and out-degrees at least $(\frac{1}{2} + \frac{\varepsilon}{2})np$. Hence in this case, $(G_2, \Phi(G))$ being $\frac{\varepsilon}{2}$ -di-ham-resilient implies $(G, \Phi(G))$ being ε -di-ham resilient. Let \mathcal{E} be the event that $(G, \Phi(G))$ is ε -di-ham-resilient. Since G_1 a.a.s. has maximum degree at most $\frac{\varepsilon}{2}np$, the observations above imply $\mathbf{P}(\mathcal{E}) = 1 - o(1)$. Let \mathcal{R} be the collection of graphs Γ such

that $\mathbf{P}(\mathcal{E} \mid G = \Gamma) = 1 - o(1)$. Then by $\mathbf{P}(\mathcal{E}) = 1 - o(1)$ and Lemma 3.8, we have $\mathbf{P}(G \in \mathcal{R}) = 1 - o(1)$, thus proving the lemma. \blacksquare

It thus remains to prove Lemma 3.9 (ii). Before proceeding further, we establish some simple properties of $G(n, p)$ and H in the following two lemmas. Let

$$q = \frac{\varepsilon}{4}p.$$

Thus q is the probability that a pair of vertices in $A \times B$ forms an edge in H .

Lemma 3.10. *If $p \leq \frac{\log^9 n}{n}$, then $G(n, p)$ a.a.s. has the following property. For every fixed vertex v , there exists at most one vertex having codegree 2 with v , and all other vertices have codegree at most 1 with v .*

Proof. The claim follows immediately from the easily established (say, through the first moment method) fact that for such values of $p(n)$, the random graph $G(n, p)$ a.a.s. does not contain two cycles of length 4 sharing a vertex. \blacksquare

The following lemma involves two layers of randomness; first of $G(n, p)$ and then of the graph H . It asserts that $G(n, p)$ a.a.s. is chosen so that H (which is determined by another random event) a.a.s. has the listed properties.

Lemma 3.11. *For $p \gg \frac{\log n}{n}$ and fixed reals $\mu, \xi > 0$, $G = G(n, p)$ has the following property with probability $1 - o(1)$. For every μnp -bounded incompatibility system \mathcal{F} over G , the random graph H and the partition $A \cup B$ a.a.s. have the following properties:*

- i. In H , all vertices have degree $(1 + o(1))n_0q$, and in G , all vertices have degree $(1 + o(1))n_0p$ across the partition $A \cup B$.*
- ii. For all $A' \subseteq A$ and $B' \subseteq B$, $e_H(A', B') = |A'| |B'| q + o(n^2 q)$.*
- iii. All pairs of sets $A' \subseteq A$ and $B' \subseteq B$ of sizes $|A'| = |B'| \leq e^{-2} \xi n_0$ satisfy $e_H(A', B') \leq |A'| \cdot \xi n_0 q$.*
- iv. \mathcal{F} induces a $(\mu + \frac{\xi}{5})n_0q$ -bounded incompatibility system on H , and \mathcal{F} induces a $(\mu + \frac{\xi}{5})n_0p$ -bounded incompatibility system on the bipartite subgraph of G between A and B .*

Proof. Note that the distribution of H is identical to that of the random bipartite graph with parts of sizes $|A| = |B| = n_0$ obtained by taking each edge independently with probability q . Hence Properties (i) and (ii) follow from Chernoff’s inequality, the union bound, and Lemma 3.8. Similarly, Property (iv) follows from Chernoff’s inequality, the concentration of hypergeometric distribution, and the union bound.

To prove Property (iii), note that the probability of a fixed pair of sets A' and B' of size $|A'| = |B'| = k \leq e^{-2} \xi n_0$ to satisfy $e_H(A', B') > k \xi n_0 q$ is at most

$$\binom{k^2}{k \xi n_0 q} q^{k \xi n_0 q} \leq \left(\frac{ek}{\xi n_0} \right)^{k \xi n_0 q} \leq e^{-k \xi n_0 q} \ll n^{-3k},$$

where the last inequality follows since $p \gg \frac{\log n}{n}$ and $q = \frac{\varepsilon}{4}p$. By taking the union bound over all choices of A' and B' , we see that the probability of the existence of a pair of sets A' and B' violating (iii) is at most $\sum_{k=1}^{e^{-2} \xi n_0} \binom{n}{k}^2 n^{-3k} \ll 1$. \blacksquare

Throughout the proof, we will often use the phrase ‘condition on the outcome of Lemmas 3.10 and 3.11’, to indicate that we first condition on $G = G(n, p)$ satisfying Lemmas 3.10 and 3.11, and then given a μnp -bounded incompatibility system \mathcal{F} over G , condition on H satisfying Lemma 3.11.

For $i = 0, 1, \dots, T - 1$, let $M_i^{(1)} = M_i^{(0)} \setminus M_i$ be the set of edges that were first chosen but then removed at the i -th stage. For a set of vertices X , we use the notation $X \cap M_i$ to denote the set of vertices in X that intersect an edge in M_i (similarly define $X \cap M_i^{(0)}$ and $X \cap M_i^{(1)}$). We also use the notation $x \pm \alpha$ to denote a quantity between $x - \alpha$ and $x + \alpha$. A combination of two such estimates $x \pm \alpha = x \pm \alpha'$ means that $|\alpha'| \geq |\alpha|$, i.e., that the estimate on the right hand side is rougher than that on the left hand side. The following lemma gives estimate on the number of edges in M_i intersecting a fixed given set.

Lemma 3.12. *Condition on the outcome of Lemmas 3.10 and 3.11. For an integer i with $0 \leq i \leq T - 1$ and a positive real ξ_i satisfying $\delta \leq \xi_i \leq \frac{\epsilon}{32}$, suppose that all vertices $x \in V(H_i)$ have degree $d_{H_i}(x) = ((1 - \delta)^i \pm \xi_i)n_0q$. For a vertex $v \in V(H_i)$ and a set $X \subseteq N_{H_i}(v)$ satisfying $|X| \geq 4\delta^{-2}$,*

- i. *If Γ is a fixed subset of edges incident to X , then $|\Gamma \cap M_i^{(0)}| = (1 \pm 9\epsilon^{-1}\xi_i) \frac{\delta|\Gamma|}{(1-\delta)^i n_0q}$ with probability $1 - e^{-\Omega(|\Gamma|\delta^3/n_0q)}$.*
- ii. *$|X \cap M_i^{(1)}| \leq 5\delta^2|X|$ with probability $1 - e^{-\Omega(\delta^2|X|)}$.*
- iii. *$|X \cap M_i| = (1 \pm (15\epsilon^{-1}\xi_i + 5\delta))\delta|X|$ with probability $1 - e^{-\Omega(\delta^3|X|)}$.*

Proof. The following estimate deduced from $(1 - \delta)^i \geq (1 - \delta)^T = \frac{\epsilon}{4}$ will be repeatedly used throughout the proof:

$$(1 - \delta)^i \pm \xi_i = (1 \pm 4\epsilon^{-1}\xi_i)(1 - \delta)^i. \tag{3}$$

Since $m_i = n_i((1 - \delta)^i \pm \xi_i)n_0q$, by linearity of expectation, we have

$$\mathbb{E}[|\Gamma \cap M_i^{(0)}|] = |\Gamma| \frac{\delta n_i}{m_i} = \frac{\delta}{((1 - \delta)^i \pm \xi_i)n_0q} |\Gamma| = (1 \pm 8\epsilon^{-1}\xi_i) \frac{\delta}{(1 - \delta)^i n_0q} |\Gamma|,$$

where the last equality follows from (3). Part (i) follows by Chernoff’s inequality since $|\Gamma \cap M_i^{(0)}|$ is a sum of independent random variables (each indicating whether an edge is chosen or not).

To prove part (ii), recall that $X \subseteq N_{H_i}(v)$ for some vertex v . Let H'_i be the subgraph of H_i obtained by removing all edges incident to v . For each vertex $x \in X$, let $\mathbf{1}_x$ be the indicator random variable of the event that (a) there exist two edges in $M_i^{(0)} \cap E(H'_i)$ incident to x , or (b) there exists a path of length two that has x as its endpoint and consists of edges in $M_i^{(0)} \cap E(H'_i)$. Let Γ_v be the set of edges of H_i incident to v , and note that

$$|X \cap M_i^{(1)}| \leq |\Gamma_v \cap M_i^{(0)}| + \sum_{x \in X} \mathbf{1}_x. \tag{4}$$

A bound on the first term can be obtained from union bound as follows:

$$\mathbf{P}\left(|\Gamma_v \cap M_i^{(0)}| > \frac{\delta^2}{2}|X|\right) \leq \sum_{k=\delta^2|X|/2}^{\infty} \binom{d_{H_i}(v)}{k} \left(\frac{\delta n_i}{m_i}\right)^k < \sum_{k=\delta^2|X|/2}^{\infty} \left(\frac{\epsilon \delta n_i d_{H_i}(v)}{m_i k}\right)^k.$$

Since $m_i = n_i((1 - \delta)^i \pm \xi_i)n_0q$ and $d_{H_i}(v) = ((1 - \delta)^i \pm \xi_i)n_0q$, it follows from the estimates given above that

$$\mathbf{P}\left(|\Gamma_v \cap M_i^{(0)}| > \frac{\delta^2}{2}|X|\right) \leq \sum_{k=\delta^2|X|/2}^{\infty} \left(\frac{2e\delta}{k}\right)^k < e^{-\Omega(\delta^2|X|)}, \tag{5}$$

where the last inequality follows since $2e\delta/k < 4e/(\delta|X|) \leq e\delta \leq \frac{1}{2}$.

For the second term on the right-hand-side of (4), even though the events $\{\mathbf{1}_x\}_{x \in X}$ are not necessarily independent, we claim that there exists a large subset $X' \subseteq X$ for which they are independent. To see this, note that since $X \subseteq N_{H_i}(v)$ for some vertex v , and $G(n, p)$ satisfies the event in Lemma 3.10, there exists at most one vertex $w \neq v$ for which $|X \cap N_{H_i}(w)| = 2$ and all other vertices $w' \neq w, v$ have $|X \cap N_{H_i}(w')| \leq 1$. Define $X' = X \setminus N_{H_i}(w)$ if such vertex w exists, and $X' = X$ otherwise (note that $|X| - |X'| \leq 2$). Note that $\mathbf{1}_x$ depends only on the set of edges in H'_i that intersect $\{x\} \cup N_{H'_i}(x)$. Since the sets $\{x\} \cup N_{H'_i}(x)$ are disjoint and H'_i is bipartite, the events $\mathbf{1}_x$ are independent for $x \in X'$ (conditioned on $G(n, p)$ satisfying the event in Lemma 3.10). For a fixed $x \in X'$, the probability of event (a) is at most

$$\binom{d_{H_i}(x)}{2} \binom{\delta n_i}{m_i}^2 \leq \frac{((1 - \delta)^i \pm \xi_i)^2 (n_0q)^2}{2} \left(\frac{\delta}{((1 - \delta)^i \pm \xi_i)n_0q}\right)^2 \leq \delta^2,$$

by (3) and $\xi_i \leq \frac{\epsilon}{32}$. Similarly, the probability of event (b) is at most

$$\sum_{y \in N_{H_i}(x)} d_{H_i}(y) \binom{\delta n_i}{m_i}^2 \leq ((1 - \delta)^i \pm \xi_i)^2 (n_0q)^2 \left(\frac{\delta}{((1 - \delta)^i \pm \xi_i)n_0q}\right)^2 \leq 2\delta^2.$$

Therefore $\mathbb{E}[\mathbf{1}_x] \leq 3\delta^2$, and by Chernoff's inequality we obtain

$$\mathbf{P}\left(\sum_{x \in X'} \mathbf{1}_x \leq 4\delta^2|X'|\right) \geq 1 - e^{-\Omega(\delta^2|X'|)}. \tag{6}$$

Since $|X| \leq |X'| + 2$ and $|X| \geq 4\delta^{-2}$, part (ii) follows from (4), (5), and (6).

To prove part (iii), let Γ_X be the set of edges incident to X , and note that $|\Gamma_X| = ((1 - \delta)^i \pm \xi_i)n_0q|X|$. By part (i), we have $|\Gamma_X \cap M_i^{(0)}| = (1 \pm 15\epsilon^{-1}\xi_i)\delta|X|$ with probability $1 - e^{-\Omega(\delta^3|X|)}$. By part (ii), we have $|X \cap M_i^{(1)}| \leq 5\delta^2|X|$ with probability $1 - e^{-\Omega(\delta^2|X|)}$. Now part (iii) follows since

$$|\Gamma_X \cap M_i^{(0)}| - |X \cap M_i^{(1)}| \leq |X \cap M_i| \leq |\Gamma_X \cap M_i^{(0)}|. \quad \blacksquare$$

In order to prove Lemma 3.9 (ii), we need to understand how the edges of $D_G(M)$ get removed in $D_G(M; \mathcal{F})$. In particular, we need to track these changes with each iteration of the random algorithm. The following technical definitions are made with this purpose in mind.

Suppose that an instance G of $G(n, p)$ and a μnp -bounded incompatibility system \mathcal{F} over G are fixed. Let $e = \{a, b\} \in E(G)$ and $e' = \{a', b'\} \in E(G)$ be two edges such that $a, a' \in A$ and $b, b' \in B$. We say that e' is *A-bad for e* if $\{a, b'\} \in E(G)$ is an edge compatible with e , but incompatible with e' . Similarly, we say that e' is *B-bad for e* if $\{b, a'\} \in E(G)$ is an edge compatible with e , but incompatible with e' .

For an edge $e = \{a, b\}$ with $a \in A$ and $b \in B$, define

$$A_e^{(G)} = \{x \in A : \{b, x\} \in E(G), \{b, x\} \text{ and } \{a, b\} \text{ are compatible}\} \quad \text{and}$$

$$B_e^{(G)} = \{y \in B : \{a, y\} \in E(G), \{a, y\} \text{ and } \{a, b\} \text{ are compatible}\}.$$

Similarly define $A_e^{(H)}$ and $B_e^{(H)}$ by considering the edges of H instead of the edges of G in the definition above.

Remark. For odd n , we need to extend the definitions above to edges whose both endpoints are in A . In this case, for an edge $e = \{a_1, a_2\}$ with $a_1, a_2 \in A$, we consider the edge as an ordered pair (a_1, a_2) to distinguish (a_1, a_2) and (a_2, a_1) , and define an edge $e' = \{a', b'\}$ (where $a' \in A$ and $b' \in B$) to be *A-bad for* (a_1, a_2) if $\{a_1, b'\} \in E(G)$ is compatible with e but incompatible with e' (there is no need to consider *B-bad* edges for e). Also, instead of having a pair of sets $A_e^{(G)}$ and $B_e^{(G)}$ as above, we define two sets $B_{(a_1, a_2)}^{(G)}$ and $B_{(a_2, a_1)}^{(G)}$ as

$$B_{(a_1, a_2)}^{(G)} = \{y \in B : \{a_1, y\} \in E(G), \{a_1, y\} \text{ and } \{a_1, a_2\} \text{ are compatible}\} \quad \text{and}$$

$$B_{(a_2, a_1)}^{(G)} = \{y \in B : \{a_2, y\} \in E(G), \{a_2, y\} \text{ and } \{a_2, a_1\} \text{ are compatible}\}.$$

Similarly define the sets $B_{(a_1, a_2)}^{(H)}$, $B_{(a_2, a_1)}^{(H)}$. As this modified definition becomes relevant only in few places, we will abuse notation and use *A-bad* edges and $B_e^{(G)}, B_e^{(H)}$ to denote both ordered pairs (a_1, a_2) and (a_2, a_2) .

Define

$$\xi_i = \delta(1 + 21\varepsilon^{-1}\delta)^i.$$

For each $i = 0, 1, \dots, T$, we say that $H_i = H[A_i \cup B_i]$ is *normal* if

- i. $n_i = |A_i| = |B_i| = \left((1 - \delta)^i \pm \xi_i\right)n_0$.
- ii. For all vertices $v \in V(H_i)$, $d_{H_i}(v) = \left((1 - \delta)^i \pm \xi_i\right)n_0q$.
- iii. For all vertices $v \in V(G)$, $|N_G(v) \cap A_i| = \left((1 - \delta)^i \pm \xi_i\right)n_0p$ and $|N_G(v) \cap B_i| = \left((1 - \delta)^i \pm \xi_i\right)n_0p$.
- iv. For all edges $e \in E(G)$, $|A_e^{(G)} \cap A_i| = \left((1 - \delta)^i \pm \xi_i\right)|A_e^{(G)}|$, and a similar estimate holds for the sets $B_e^{(G)}, A_e^{(H)}, B_e^{(H)}$.
- v. If $i \neq 0$, then for all edges $e \in E(G)$, there are at most $\left(\mu + \frac{\varepsilon}{3}\right)\delta(1 - \delta)^{i-1}|B_e^{(G)}|$ *A-bad* edges for e in M_{i-1} (a similar estimate for *B-bad* edges).

Note that since $T = \frac{\ln(4/\varepsilon)}{-\ln(1-\delta)} \leq \frac{\ln(4/\varepsilon)}{\delta}$ and $\delta = e^{-22\varepsilon^{-1} \ln \varepsilon^{-1}}$, the error parameter ξ_i satisfies

$$\xi_i \leq \xi_T \leq \delta(1 + 21\varepsilon^{-1}\delta)^T \leq \delta e^{21\varepsilon^{-1} \ln(4/\varepsilon)} \leq e^{-\varepsilon^{-1}}. \tag{7}$$

The following lemma asserts that with high probability all H_i are normal (for $i = 0, 1, \dots, T$).

Lemma 3.13. *Conditioned on the outcome of Lemmas 3.10 and 3.11, the graph H_i is a.a.s. normal for all $i = 0, 1, \dots, T$.*

Proof. We proceed by induction on i . For $i = 0$, the statement follows immediately since we conditioned on Lemma 3.11. For $i \geq 0$, suppose that H_i is normal. For each vertex $v \in V(H_i)$, since $d_{H_i}(v) = ((1 - \delta)^i \pm \xi_i)n_0q$, by applying Lemma 3.12 (iii) with $X = N_{H_i}(v)$, we see that

$$\begin{aligned} |N_{H_i}(v) \cap M_i| &= (1 \pm (15\varepsilon^{-1}\xi_i + 5\delta))\delta((1 - \delta)^i \pm \xi_i)n_0q \\ &= (1 \pm (15\varepsilon^{-1}\xi_i + 5\delta))\delta(1 \pm 4\varepsilon^{-1}\xi_i)(1 - \delta)^i n_0q \\ &= (1 \pm 20\varepsilon^{-1}\xi_i)\delta(1 - \delta)^i n_0q \end{aligned}$$

with probability $1 - e^{-\Omega(\delta^3(1-\delta)^i n_0q)} = 1 - n^{-\omega(1)}$. Since

$$\begin{aligned} d_{H_{i+1}}(v) &= d_{H_i}(v) - |N_{H_i}(v) \cap M_i| = ((1 - \delta)^i \pm \xi_i)n_0q - (1 \pm 20\varepsilon^{-1}\xi_i)\delta(1 - \delta)^i n_0q \\ &= (1 - \delta)^{i+1} n_0q \pm (1 + 20\varepsilon^{-1}\delta)\xi_i n_0q = ((1 - \delta)^{i+1} \pm \xi_{i+1})n_0q, \end{aligned}$$

by taking the union bound over all vertices, we see that Property (ii) of H_{i+1} being normal a.a.s. holds. Properties (iii) and (iv) follow by the same argument applied to the corresponding sets. Furthermore, if Property (ii) holds, then H_{i+1} is a balanced bipartite graph with n_{i+1} vertices in each part whose number of edges is

$$m_{i+1} = n_{i+1} \left((1 - \delta)^{i+1} \pm (1 + 20\varepsilon^{-1}\delta)\xi_i \right) n_0q.$$

By Lemma 3.11 (iii), we have that n_{i+1} is linear in n . Then part (ii) of the same lemma implies that $n_{i+1} = ((1 - \delta)^{i+1} \pm \xi_{i+1})n_0$, proving Property (i).

For a fixed edge $e = \{a, b\} \in E(G)$ with $a \in A$ and $b \in B$, let $\Gamma_{e,A}$ be the set of A -bad edges for e in H_i . Each A -bad edge in H_i can be accounted for by first taking a vertex $x \in B_e^{(G)} \cap B_i$, and then counting the number of edges $\{x, y\} \in E(H_i)$ that are incompatible with $\{a, x\}$. This gives

$$|\Gamma_{e,A}| = \sum_{x \in B_e^{(G)} \cap B_i} \left(d_{H_i}(x) - |A_{\{x,a\}}^{(H)} \cap A_i| \right) = ((1 - \delta)^i \pm \xi_i) \sum_{x \in B_e^{(G)} \cap B_i} \left(n_0q - |A_{\{x,a\}}^{(H)}| \right).$$

By Lemma 3.11 (i) and (iv), we have $n_0q - |A_{\{x,a\}}^{(H)}| \leq (\mu + \frac{\varepsilon}{4})n_0q$, which in turn gives

$$|\Gamma_{e,A}| \leq ((1 - \delta)^i \pm \xi_i) \sum_{x \in B_e^{(G)} \cap B_i} \left(\mu + \frac{\varepsilon}{4} \right) n_0q = ((1 - \delta)^i \pm \xi_i)^2 \left(\mu + \frac{\varepsilon}{4} \right) n_0q |B_e^{(G)}|.$$

By Lemma 3.12 (i), with probability $1 - n^{-\omega(1)}$,

$$\begin{aligned} |\Gamma_{e,A} \cap M_i| &\leq |\Gamma_{e,A} \cap M_i^{(0)}| \leq (1 + 9\varepsilon^{-1}\xi_i) \frac{\delta |\Gamma_{e,A}|}{(1 - \delta)^i n_0q} \\ &\leq (1 + 20\varepsilon^{-1}\xi_i) \left(\mu + \frac{\varepsilon}{4} \right) \delta (1 - \delta)^i |B_e^{(G)}|. \end{aligned}$$

Equation (7) implies $(1 + 20\varepsilon^{-1}\xi_i)(\mu + \frac{\varepsilon}{4}) \leq \mu + \frac{\varepsilon}{3}$, and thus Property (v) for A -bad edges follows by taking the union bound over all edges. The conclusion for B -bad edges follows by a similar argument. ■

We show that Φ successfully terminates as long as the final iteration is normal.

Lemma 3.14. *Conditioned on the outcome of Lemmas 3.10 and 3.11, if H_T is normal, then it contains a perfect matching.*

Proof. If n is odd, then we must first choose the vertices $v_*, a_{(n-1)/2}$ from A_T and $b_{(n-1)/2}$ from B_T . Property (iii) of normality implies the existence of an edge $\{v_*, a_{(n-1)/2}\} \in E(G)$ within A_T . Then by Properties (ii) and (iv) of normality, we can find a vertex $b_{(n-1)/2} \in B_T$ for which $\{v_*, a_{(n-1)/2}\}$ and $\{v_*, b_{(n-1)/2}\} \in E(H_T)$ are compatible. Remove the vertices $v_*, a_{(n-1)/2}$ from A_T and $b_{(n-1)/2}$ from B_T and update n_T as the size of the new sets A_T and B_T .

Now consider both even and odd n , and consider the graph H_T with bipartition $A_T \cup B_T$. By Hall’s theorem it suffices to prove that

$$\forall A' \subseteq A_T, |A'| \leq \frac{1}{2}n_T \quad |N_{H_T}(A')| \geq |A'| \quad \text{and} \quad \forall B' \subseteq B_T, |B'| \leq \frac{1}{2}n_T \quad |N_{H_T}(B')| \geq |B'|.$$

Suppose that there exists a set $A' \subseteq A_T$ for which $|N_{H_T}(A')| < |A'|$, and let X be a superset of $N_{H_T}(A')$ of size exactly $|A'|$. Since H_T is normal, by Property (ii) of normality, we see that

$$e_{H_T}(A', X) = e_{H_T}(A', B_T) \geq |A'|((1 - \delta)^T - \xi_i)n_0q \geq |A'| \frac{\varepsilon}{8} n_0q.$$

By Lemma 3.11 (iii), this implies $|A'| > \frac{\varepsilon}{8e^2}n_0$. Let $Y = B_T - X$. Then $|Y| \geq \frac{1}{2}n_T$ and by definition $e_{H_T}(A', Y) = 0$. On the other hand, by Lemma 3.11 (ii), $e_{H_T}(A', Y) = |A'| |Y|q + o(n^2q) > 0$. This contradiction proves that no such set A' can exist. Similarly, we can show that no such set B' exists. ■

We conclude this section with the proof of the second part of Lemma 3.9.

Proof of Lemma 3.9 (ii). Let $\mu = 1 - \frac{1}{\sqrt{2}} - 2\varepsilon$ and condition on $G = G(n, p)$ satisfying Lemmas 3.10 and 3.11. Since $G(n, p)$ satisfies these lemmas with probability $1 - o(1)$, it suffices to prove that for every μnp -bounded incompatibility system \mathcal{F} over G , the probability that $(G, \Phi(G), \mathcal{F})$ is ε -typical is $1 - o(1)$. This will be achieved by proving that $(G, \Phi(G), \mathcal{F})$ is ε -typical if H satisfies Lemma 3.11 and all H_i are normal (note that all events have probability $1 - o(1)$).

Since H_T is normal, Lemma 3.14 implies that our algorithm produces a perfect matching $M = \Phi(G)$. By Lemma 3.11 (i), each vertex in $D_G(M)$ has in- and out-degrees $(1 + o(1))n_0p$. Take a vertex $e = \{a, b\} \in D_G(M)$. An out-neighbor $f = \{a', b'\}$ of e in $D_G(M)$ can be removed in $D_G(M; \mathcal{F})$ for two reasons: first, if $\{b, a'\}$ is an edge incompatible with e , and second, if $\{b, a'\}$ is an edge compatible with e but incompatible with f (i.e. f is a B -bad edge for e). By the definition of the set $B_e^{(G)}$, we see that there are at most $(1 + o(1))n_0p - |B_e^{(G)}|$ edges of the first type. Since all H_i are normal, by conditions (iii) and (v) of normality, the number of edges of the second type is at most

$$|N_G(a) \cap B_T| + \sum_{i=0}^{T-1} \left(\mu + \frac{\varepsilon}{3}\right) \delta(1 - \delta)^i |B_e^{(G)}| \leq ((1 - \delta)^T + \xi_T)n_0p + \left(\mu + \frac{\varepsilon}{3}\right) |B_e^{(G)}|.$$

Therefore, since $(1 - \delta)^T = \frac{\varepsilon}{4}$, the total number of out-edges removed from e is at most

$$\left((1 + o(1))n_0p - |B_e^{(G)}| \right) + \frac{\varepsilon n_0p}{3} + \left(\mu + \frac{\varepsilon}{3}\right) |B_e^{(G)}|,$$

which by $|B_e^{(G)}| \geq (1 - \mu - o(1))n_0p$ and $\mu = 1 - \frac{1}{\sqrt{2}} - 2\varepsilon$ is at most

$$\begin{aligned} & (1 + o(1))n_0p + \frac{\varepsilon n_0p}{3} + \left(-1 + \mu + \frac{\varepsilon}{3}\right)(1 - \mu - o(1))n_0p \\ & \leq \left(1 + \frac{\varepsilon}{3} - (1 - \mu)^2 + \frac{\varepsilon}{3}(1 - \mu) + o(1)\right)n_0p \leq \left(\frac{1}{2} - 2\varepsilon\right)n_0p. \end{aligned}$$

Hence the minimum out-degree of $D_G(M; \mathcal{F})$ is at least $(1 + o(1))n_0p - (\frac{1}{2} - 2\varepsilon)n_0p \geq (\frac{1}{2} + \varepsilon)n_0p$. A similar bound on the minimum in-degree of $D_G(M; \mathcal{F})$ holds. ■

4. CONCLUDING REMARKS

In this paper, we proved the existence of a positive real μ such that if $p \gg \frac{\log n}{n}$, then $G = G(n, p)$ a.a.s. has the following property. For every μnp -bounded incompatibility system \mathcal{F} defined over G , there exists a Hamilton cycle in G compatible with \mathcal{F} . The value of μ that we obtained in Theorem 1.2 is very small, but for $p \gg \frac{\log^8 n}{n}$ we improved it in Theorem 1.3 to $1 - \frac{1}{\sqrt{2}} - o(1)$ (roughly 0.29). The bound of $\frac{\log^8 n}{n}$ came from Theorem 3.7, and in fact, any improvement in the range of probability for Theorem 3.7 will immediately imply Theorem 1.3 for the extended range of probabilities. It is not clear what the best possible value of μ should be. The example of Bollobás and Erdős [6] of a $\lfloor \frac{1}{2}n \rfloor$ -bounded edge-coloring of K_n with no properly colored Hamilton cycles implies that the optimal value of μ is at most $\frac{1}{2}$, since it provides an upper bound for the case $p = 1$.

The concept of incompatibility systems seems to give an interesting new take on robustness of graphs properties. Further study of how various extremal results can be strengthened using this notion appears to be a promising direction of research. For example in a companion paper [23], we show that there exists a constant $\mu > 0$ such that for any μn -bounded system \mathcal{F} over a graph G on n vertices with minimum degree at least $n/2$, there is a compatible Hamilton cycle in G . This establishes in a very strong sense an old conjecture of Häggkvist from 1988.

ACKNOWLEDGMENTS

We thank Asaf Ferber for calling our attention to the problem of rainbow Hamilton cycles in random graphs. A major part of this work was carried out when Benny Sudakov was visiting Tel Aviv University, Israel. He would like to thank the School of Mathematical Sciences of Tel Aviv University for hospitality and for creating a stimulating research environment.

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