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List colouring of graphs with at most $\big(2-o(1)\big)\chi$ vertices

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Abstract

Ohba has conjectured [9] that if the graph G has $2\chi(G)+1$ or fewer vertices then the list chromatic number and chromatic number of G are equal. In this paper we prove that this conjecture is asymptotically correct. More precisely we obtain that for any $0<\epsilon<1$, there exist an $n_0=n_0(\epsilon)$ such that the list chromatic number of G equals its chromatic number, provided

$$n_0 \leq |V(G)| \leq (2-\epsilon)\chi(G).$$

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1. Introduction

Recently, a host of important results on graph colouring have been obtained via the probabilistic method. The first author presented an invited lecture at the 2002 International Congress of Mathematicians surveying a number of these results. The recent monograph [8] provides a more in depth survey of the topic. This paper presents one example of a result proven using the method.

An instance of List Colouring consists of a graph G and a list $L(v)$ of colours for each vertex v of G. We are asked to determine if there is an *acceptable* colouring of G, that is a colouring in which each vertex receives a colour from its list, and no edge has both its endpoints coloured with the same colour. The *list-chromatic number* of G, denoted $\chi^l(G)$ is the minimum integer k such that for every assignment of a

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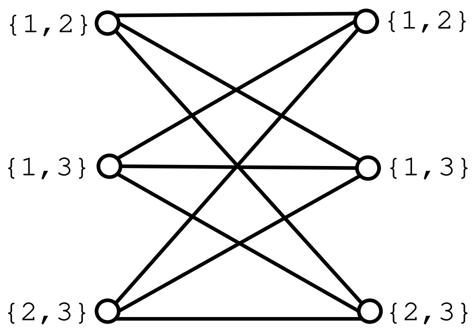

Figure 1: A bipartite graph with list chromatic number three

list $L(v)$ of size at least k to every vertex v of G, there exist an acceptable colouring of G. The list-chromatic number was introduced by Vizing [11], and independently Erdös et al. [3]. This parameter has received a considerable amount of attention in recent years (see, e.g. [4], [1]).

Clearly, by definition, $\chi^l(G) \geq \chi(G)$ because $\chi(G) = k$ precisely if an acceptable colouring exists when each L_v is $\{1, ..., k\}$. However, the converse inequality is not true, e.g. $\chi^l(K_{3,3}) = 3$ as can be easily verified by considering Figure 1. In fact, there are bipartite graphs with arbitrarily high chromatic number (indeed even for bipartite G, $\chi^l(G)$ is bounded from below by a function of the minimum degree which goes to infinity, see [1]). This shows that the gap between $\chi(G)$ and $\chi^l(G)$ can be arbitrarily large. Moreover it shows that $\chi^l(G)$ can not be bound by any function of the chromatic number of G. This gives rise to the following intriguing question in the theory of graph colourings: Find conditions which guarantee the equality of the chromatic and list-chromatic numbers.

There are many conjectures hypothesizing conditions on G which imply that $\chi(G) = \chi^l(G)$. Probably, the most famous of these is the List Colouring Conjecture (see [4]) which states that this is true if G is a line graph. One interesting example of a graph with $\chi = \chi^l$ was obtained in the original paper of Erdös et al. [3]. They proved that if G is complete k-partite graph with each part of size two then $\chi(G) = \chi^l(G) = k$. It took nearly twenty years until Ohba [9] noticed that this example is actually part of much larger phenomenon. He conjectured (cf. [9]) that $\chi(G) = \chi^l(G)$ provided $|V(G)| \leq 2\chi(G) + 1$. This conjecture if it is correct is best possible. Indeed, let G be a complete k-partite graph with $k - 1$ parts of size 2 and one part of size 4. Then the number of vertices of G is $2k+2$, the chromatic number is k and it was proved in [9] that list-chromatic number of G is at least $k + 1 > k$.

In his original paper Ohba obtained that $\chi^l(G) = \chi(G)$ for all graphs G with $|V(G)| \leq \chi(G) + \sqrt{2\chi(G)}$. His conjecture was settled for some other special cases in [2]. Recently the result of Ohba was substantially improved by the authors of this paper. In [10] they proved that Ohba's conjecture is true for all graphs G with at most $\frac{5}{3}\chi(G) - \frac{4}{3}$ vertices. In this paper we want to improve this result for large graphs and prove that the conjecture is asymptotically correct. More precisely we obtain the following theorem.

Theorem 1 *For any $0 < \epsilon < 1$, there exist an $n_0 = n_0(\epsilon)$ such that $\chi^l(G) = \chi(G)$ provided $n_0 \leq |V(G)| \leq (2 - \epsilon)\chi(G)$.*

The rest of this paper is organized as follows. In the next section we describe the main steps in the proof of Theorem 1. More precisely, we present our key lemma and show how to deduce from it the assertion of the theorem. We will prove this lemma using probabilistic arguments. In Sections 3 and 4 we discuss the main ideas we are going to use in the proof. We present the details of the proof in Section 5. Finally, the last section of the paper contains some concluding remarks.

2. The key lemma

In this section we present the main steps in the proof of Theorem 1. First we need the following lemma from [10] whose short proof we include here for the sake of completeness.

Lemma 2 *For any integer t, if $\chi^l(G) > t$ then there exist a set of lists $L(v), v \in V(G)$ for which there is no acceptable colouring such that each list has at least t elements and the set $\mathcal{A} = \cup_{v \in V(G)} L(v)$ has size less than $|V(G)|$.*

Proof. Assume $\chi^l(G) > t$ and choose a set of lists $L(v), v \in V(G)$ for which there is no acceptable colouring, in which each list has size at least t and which minimizes $|\mathcal{A}|$.

Now, if $|\mathcal{A}| < |V(G)|$ then we are done. So, we can assume the contrary. We consider the bipartite graph H with bipartition $(\mathcal{A}, V(G))$ and an edge between c and v precisely if $c \in L(v)$. If there is a matching of size $|V|$ in H then this matching saturates V and points out an acceptable colouring for the List Colouring instance in which no colour is used more than once. Since, there is no such acceptable colouring, no such matching exists. Thus there must be a smallest subset B of $\mathcal{A}$ which is not the set of endpoints of a matching in this graph and this set must have at most $|V|$ elements. Clearly, B contains at least two vertices. Now, by the minimality of B there is a matching M in H of size $|B| - 1$ whose endpoints in $\mathcal{A}$ are in B. Further, classical results in matching theory (see e.g. Theorem 1.1.3 of [6]) tell us that if W is the set of endpoints of M in V then for $v \notin W$, we have $L(v) \cap B = \emptyset$.

Let x be any vertex in $G - W$ and replace $L(v)$ by $L(x)$ for every vertex $v \in W$. This yields a new List Colouring Problem in which the total number of colours in all lists is smaller than $|\mathcal{A}|$ (since all the new list are disjoint from B). Therefore by the minimality of our original choice, there exist an acceptable colouring of G for this new Lists Colouring instance. In particular this implies that we can obtain an acceptable colouring of $G - W$ for the original lists $L(v)$. Since no colour in B is used in this colouring, using the colouring of W pointed out by M yields an extension of this colouring to a colouring of G in which no colour of B appears more than once. This contradicts our assumption that there is no acceptable colouring for this instance and proves the lemma. □

Proof of Theorem 1. Let $0 < \epsilon < 1$ be a fixed constant and let G be a graph satisfying $|V(G)| < (2-\epsilon)\chi(G)$. We assume that $\chi^l(G) > \chi(G)$ and obtain a contradiction. Since adding an edge between vertices in different colour classes in an optimal colouring of G does not change $\chi(G)$ and can only increase $\chi^l(G)$, we will assume that G is complete $\chi(G)$-partite graph. Thus G has a unique partition into $\chi(G)$ stable sets. We refer to these stable sets as parts rather than colour classes so as to avoid confusion with the colours used in our acceptable colouring of G.

Now by Lemma 2, if $\chi^l(G) > \chi(G)$ then there is an instance of List Colouring on G for which no acceptable colouring exists, in which each list has length at least $\chi(G)$ and such that the size of the union of all lists $L(v)$ is less than $|V(G)|$. This means that in an acceptable colouring at least one colour must be used on more than one vertex. Fortunately, it also implies that for every non-singleton part U there is at least one colour which appears on $L(v)$ for more than half the vertices of U (since each $L(v)$ contains more than half the colours).

Our proof approach is simple. For each non-singleton part U, we choose some colour c_U and colour with c_U all the vertices of U whose list contains c_U (thus we must insist that all the c_U are distinct). We complete the colouring by finding a bijection between the vertices not yet coloured and the colours not yet used so that each such colour is in the list of the vertex with which it is matched. This yields an acceptable colouring in which for each part U there is at most one colour c_U used on more than one vertex of U.

To begin, we consider the case when there is some part U such that some colour appears on all the vertices of U. We show that we can reduce to a smaller problem by using any such colour for c_U. Iteratively repeating this process yields a graph where no such U exists and hence, in particular, there are no parts of size two.

Our choices for the remaining c_U are discussed in the proof of the key Lemma 3 which consists of the analysis of a probabilistic procedure for choosing the remaining c_U. Unfortunately, before discussing this procedure we need to deal with some technical details.

So, to begin we show that we can assume that $\cap_{v\in U} L(v)$ is empty for all parts U of size bigger than 1 in the partition of G. To see this, let U be a part of size at least 2 such that $\cap_{v\in U} L(v) \neq \emptyset$. Then the graph $G-U$ has chromatic number $\chi(G)-1$ and at most $|V(G)|-2$ vertices and therefore also satisfies

$$|V(G-U)| \leq |V(G)|-2 \leq (2-\epsilon)\chi(G)-2 = (2-\epsilon)(\chi(G)-1)-\epsilon < (2-\epsilon)\chi(G-U).$$

Note that it also satisfies $\chi^l(G-U) > \chi(G-U) = \chi(G)-1$ since otherwise we can obtain an acceptable colouring of G from the lists $L(v)$. Indeed, let c be a colour in $\cap_{v\in U} L(v)$. Since $\chi^l(G-U) = \chi(G)-1$, we know there is an acceptable colouring of $G-U$ from the lists $L(v)-c$. Colouring all vertices in U with c we obtain an extension of this colouring to an acceptable colouring of G from the original lists, a contradiction. Therefore we will consider the graph $G-U$ instead of G and continue this process until we obtain a graph G' and an instance of List Colouring on G' with the following properties.

- G' is $\chi(G')$-partite graph which satisfies $|V(G')| < (2-\epsilon)\chi(G')$.
- Each list $L'(v)$ has length at least $\chi(G')$ and there is no acceptable colouring of G' from $L'(v)$.
- The size of the union of all lists is less than $|V(G')|$.
- $\cap_{v\in U} L'(v)$ is empty for all parts of size bigger than 1 in the partition of G'.

Since the size of the lists is $\chi(G') > |V(G')|/2$ we obtain that $L'(x)\cap L'(y)$ is non empty for any two vertices $\{x,y\}$ in G'. In particular this implies that in the partition of G' there are no parts of size two. Note that the original graph G has at most $|V(G)|/2$ parts of size ≥ 2 and each time we removed such a part the chromatic number of the remaining graph decreased by one. Therefore we decrease chromatic number of G by at most $|V(G)|/2$ and hence the remaining graph G' should have at least

$$|V(G')| \geq \chi(G') \geq \chi(G) - \frac{|V(G)|}{2} \geq \frac{|V(G)|}{2-\epsilon} - \frac{|V(G)|}{2} \geq \frac{\epsilon|V(G)|}{4}$$

vertices. So by choosing an appropriate bound on the size of $|V(G)|$ we can make $|V(G')|$ arbitrarily large. This completes our discussion of parts U for which some colour is in $L(v)$ for all vertices v of U. We turn now to the technical details necessary before we present the rest of the ideas needed in the proof.

Let X be the set of all the vertices in the singleton classes in the partition of G'. Pick m to be a sufficiently large integer constant $m = m(\epsilon)$ and let t be an integer which satisfies

$$\frac{t+1}{m} \leq \frac{|X|}{\chi(G')} \leq \frac{t+2}{m}. \tag{2.1}$$

Since in the partition of G' there are no parts of size two, we obtain that $|X| + 3(\chi(G') - |X|) \leq |V(G')| < (2-\epsilon)\chi(G')$. This implies that $|X| \geq (1+\epsilon)\chi(G')/2$ and that $m/2 < t \leq m-2$.

Set $\mathcal{A} = \cup_{v\in G'} L'(v)$. Let H be a bipartite graph with bipartition $(X,\mathcal{A})$ and an edge between c and v precisely when $c \in L'(v)$. Note that the degree of every vertex from X in H is at least $\chi(G') \geq |V(G')|/2 > |\mathcal{A}|/2$. Therefore by well known results on Zarankiewicz's problem (see, e.g., [5], Problem 10.37), H contains a complete bipartite graph with t vertices in X and m vertices in $\mathcal{A}$. Denote the set of vertices from X and $\mathcal{A}$ by $\mathcal{S}_1$ and $\mathcal{C}_1$ respectively and remove them from H. Note that the bound on Zarankiewicz's problem guarantees that we will continue to find a copy of the complete bipartite graph $K_{t,m}$ in H until the minimal degree of a vertex in X is $o(\chi(G')) = o(|\mathcal{A}|)$. Thus in the end we obtain at least $k = (1-o(1))\chi(G')/m$ disjoint sets of colours $\mathcal{C}_1,\dots,\mathcal{C}_k$ and also k disjoint sets of singleton partition classes $\mathcal{S}_1,\dots,\mathcal{S}_k$, such that $\mathcal{C}_i \subset L'(s)$ for every vertex $s\in\mathcal{S}_i$. Denote by $\mathcal{C} = \cup_i \mathcal{C}_i$, by $\mathcal{S} = \cup_i \mathcal{S}_i$ and let C and S be the sizes of $\mathcal{C}$ and $\mathcal{S}$ respectively. Now using (2.1) we can obtain the following inequalities

$$\begin{aligned}|X| - S &= |X| - kt \le \frac{t+2}{m}\chi(G') - (1+o(1))\frac{t}{m}\chi(G') \\ &= (1+o(1))\frac{2}{m}\chi(G') = (2+o(1))k < 3k\end{aligned}$$

and

$$|X| - S = |X| - kt \ge \frac{t+1}{m}\chi(G') - (1+o(1))\frac{t}{m}\chi(G') = (1+o(1))\frac{\chi(G')}{m}.$$

In the above discussion and in particular in the last two inequalities we used that m and t are constants but $|V(G')|$ (and thus also $\chi(G')$) tends to infinity.

Let W be the union of some set of $r = \chi(G') - |\mathcal{C}|$ singleton partition classes which do not belong to $\mathcal{S}$. Such a set W exists, since the number of singleton partition classes outside $\mathcal{S}$ is at least $(1+o(1))\chi(G')/m \gg r = \chi(G') - km = o(\chi(G'))$. Note that we can obtain an acceptable colouring of W with the lists $L'(v) - \mathcal{C}$ greedily, since the size of $L'(v) - \mathcal{C}$ is equal to r. Let T be the set of r colours used to colour W in one such acceptable colouring. Denote by $G'' = G' - W$ and let $L''(v) = L'(v) - T$ for every vertex $v \in G''$. Then to finish the proof it is enough to show the existence of an acceptable colouring of the G'' from the set of lists $L''(v)$.

By definition, we have that $\chi(G'') = \chi(G') - r = |\mathcal{C}| = C$ and G'' is a complete C partite graph. The number of vertices of G'' satisfies

$$|V(G'')| = |V(G')| - r < |V(G')| < (2-\epsilon)\chi(G') = (1+o(1))(2-\epsilon)\chi(G'').$$

So by choosing $\delta = \epsilon/2$ we obtain that $|V(G'')| < (2-\delta)\chi(G'')$. This together with above discussion implies that G'' satisfies all the condition (1–5) of the next lemma. This lemma guarantees the existence of an acceptable colouring of G'' from the set of lists $L''(v)$ and completes the proof of Theorem 1.

Lemma 3 *Let $0 < \delta < 1$ be a constant and let C, S, k, m, t and n be integers with $m > 6/\delta$, $C = km$, $S = kt$ and $n < (2-\delta)C$. Suppose, in addition, that m is fixed and n (and hence C) is a sufficiently large function of m. Let G be a complete C-partite graph on n vertices and let $L(v)$ be the set of lists of colours of size C one for each vertex v of G such that the following holds.*

1. *$\cap_{v \in U} L(v) = \emptyset$ for any part U of size bigger than one in the partition of G.*
2. *G contains a set of vertices $\mathcal{S}$ of size S such that the vertices in $\mathcal{S}$ form parts of size one in the partition of G. The set $\mathcal{S}$ is partitioned into k parts $\mathcal{S}_1, \dots, \mathcal{S}_k$ each of size t.*
3. *G contains no parts of size two and at most $3k$ singleton parts which do not belong to $\mathcal{S}$.*
4. *There exist a set of colours $\mathcal{C}$ of size C and its partition $\mathcal{C}_1, \dots, \mathcal{C}_k$ into k sets of size m. Such that $\mathcal{C}_i \subset L(s)$ for every vertex $s \in \mathcal{S}_i$. In particular, for any subset of $\mathcal{C}_i$ of size t there exist an acceptable colouring of the vertices of $\mathcal{S}_i$ which uses the colours in this subset.*

5. The total number of colours in the union of all the lists $L(v)$ is less than n.

Then there exist an acceptable colouring of G from the lists $L(v)$.

We finish this section with discussion of the proof of Lemma 3. We postpone all the details to the subsequent sections of the paper.
Proof Overview. The proof proceeds as follows:

(I) We choose a random partition of each $\mathcal{C}_i$ into two subsets $\mathcal{A}_i$ of size t and $\mathcal{B}_i$ of size $m-t$ where these choices are made uniformly and independently.
(II) We use the colours in $\mathcal{A}_i$ to colour the vertices of $\mathcal{S}_i$ which is possible by Condition 4 of the lemma.
(III) We choose a (random) bijection between $\mathcal{B}=\cup_{i=1}^{k}\mathcal{B}_i$ and the parts of G not in $\mathcal{S}$ in such a way that, for each part U not in $\mathcal{S}$, U is equally likely to correspond to each colour $c\in\mathcal{B}$. We denote by c_U be the colour corresponding to U.
(IV) For each part U not in $\mathcal{S}$ we colour every vertex v of U for which $c_U\in L(v)$ with the colour c_U.
(V) We match the set V' of vertices not yet assigned a colour with the set of colours not yet used (i.e those colours not in $\mathcal{C}$) so that every vertex is matched with a colour on its list. We colour each vertex of V' with the colour with which it is matched.

If we successfully complete this five step process, we have an acceptable colouring as every colour not in $\mathcal{B}$ appears on at most one vertex, and every colour in $\mathcal{B}$ appears only on a subset of some part, and hence on an independent set of G.

To prove that we can find the colouring in this fashion, we need to describe and analyze our method for choosing the random bijection between the parts and the colours in $\mathcal{C}$ made in Steps I–IV, in order to show that (with positive probability) we can complete the colouring by finding the desired matching in Step V.

A key tool will be Hall's Theorem which states that in a bipartite graph with bipartition (A,B), we can find a matching M such that every vertex of A is the endpoint of an edge of M provided there is no subset X of A such that setting $N(X)=\cup\{N(x)|x\in X\}$ we have $|N(X)|<|X|$.

We remark that although Steps I–III are presented as though they are separate processes performed sequentially, in the more complicated case of our analysis we will need to interleave these processes by first choosing some of the $\mathcal{B}_i$, then choosing the parts with which these colours will be matched, and finally completing Step I and then Step III.

To determine if we can find the desired matching in Step V, we will need to examine the sets $L'(v)=L(v)-\mathcal{C}$ for the vertices of V'. Let H be a bipartite graph with bipartition $\big(V',\cup_v L'(v)\big)$ and an edge between c and v precisely when $c\in L'(v)$. For each vertex v, we let the weight of v, denoted $w(v)$, be $\frac{1}{|L'(v)|}$. For any set S of vertices we use $W(S)$ to denote the sum of the weights of the vertices in S.

This definition of weight is motivated by the following immediate consequence of Hall's Theorem:

Observation 4 *If we cannot find the desired matching in Step 5 then there exist a subset X of V' such that $W(X) > 1$. In this case $W(V') > 1$ as well.*

Proof. By Hall's Theorem there exist a subset X of V' such that $|N(X)| < |X|$. Then, we obtain that

$$W(V') \geq W(X) = \sum_{x \in X} \frac{1}{|L'(v)|} = \sum_{x \in X} \frac{1}{|N(x)|} \geq \sum_{x \in X} \frac{1}{|N(X)|} \geq \frac{|X|}{|N(X)|} > 1. \quad \square$$

Thus an analysis of the random parameter $W(V')$ will be crucial to the proof of the lemma. In the next section, by computing the expected value of the parameter, we show that the lemma holds if $n \leq C + S$. In later sections, we complete the proof using a more complicated analysis along the same lines.

3. The expected value of $W(V')$

For each part U which is not in $\mathcal{S}$, our choices in Steps I and III guarantee that each colour of $\mathcal{C}$ is equally likely to be c_U. Thus, for each vertex v in such a part, the probability that v is in V', i.e., $c_U \notin L(v)$, is $1 - \frac{|L(v) \cap \mathcal{C}|}{C}$. Since $|L(v)| = C$, this is $\frac{|L'(v)|}{C}$. So, we have:

$$\mathbf{E}\big(W(V')\big) = \sum_{v \in V - \mathcal{S}} w(v) \mathbf{Pr}(v \in V') = \sum_{v \in V - \mathcal{S}} \frac{1}{L'(v)} \frac{L'(v)}{C} = \frac{n - S}{C}. \qquad (3.1)$$

So, if $n \leq S + C$, then this expected value is less than or equal to one. Since the probability that a random variable exceeds its expected value is less than one, this implies that we can make the choices in Steps I–IV so that $W(V') \leq 1$, and hence by Observation 4, the desired matching can be found in Step V.

Analyzing the behaviour of the (random) weights of various subsets of V' will allow us extend our proof technique to handle larger values of n. In doing so, the following definitions and observations will prove useful.

We let A be the number of non-singleton parts. Then by Condition 3 of the lemma, A is at least $C - S - 3k$. Since each non singleton colour class has at least three vertices and the total number of classes is C, we obtain $(2-\delta)C \geq n \geq C+2A$, i.e., $A \leq \frac{(1-\delta)C}{2}$. On the other hand, the analysis above shows that we can assume that $n > S+C$ and hence that $S \leq (1-\delta)C$. Thus, $A \geq \delta C - 3k = \delta C - \frac{3}{m}C \geq \frac{\delta C}{2}$. Both these bounds on A will be useful in our analysis. Note also that

$$m - t = \frac{C - S}{k} \geq \frac{C - (1-\delta)C}{k} = \delta \frac{C}{k} = \delta m.$$

4. Completing the proof : The idea

Let H be a bipartite graph with bipartition $\big(V', \cup_v L'(v)\big)$ and an edge between c and v precisely when $c \in L'(v)$. Our first step will be to check Hall's criterion for a fixed subset of colours K in $\cup_v L'(v)$ and show that the expected number of vertices in $\{v|v \in V', L'(v) \subseteq K\}$ is less than $|K|$.

To begin, we note that for any such v, $w(v) \geq \frac{1}{|K|}$. Therefore, defining the set S_K to be $S_K = \{v|v \in V', L'(v) \subseteq K\}$, we have that

$$\begin{aligned}\mathbf{E}\big(|S_K|\big) &= \sum_{v\in V-\mathcal{S},\, L'(v)\subseteq K} \mathbf{Pr}(v \in V') \leq |K| \sum_{v\in V-\mathcal{S},\, L'(v)\subseteq K} w(v)\mathbf{Pr}(v \in V') \\ &\leq |K| \sum_{v\in V-\mathcal{S}} w(v)\mathbf{Pr}(v\in V') = |K|\mathbf{E}\big(W(V')\big) = |K|\frac{n-S}{C}\end{aligned}$$

which, since $n \leq (2-\delta)C \leq S+A+3k+(1-\delta)C$, is at most $|K|(1+\frac{A+3k}{C}-\delta)$. On the other hand, this estimate is not good enough to guarantee Hall's criterion, since it still can be greater than $|K|$.

To improve on this bound, we use the fact that no colour c appears on the list of all the vertices of any non-singleton part of G. Note that $k = C/m$, $m > 6/\delta$, the number of non-singleton parts is A and the total number of vertices is at most $n \leq (2-\delta)C \leq S+A+3k+(1-\delta)C$. This altogether implies that for every c,

$$\begin{aligned}\mathbf{E}\Big(W\big(V' \cap \{v|c \in L'(v)\}\big)\Big) &= \mathbf{E}\big(W(V')\big) - \sum_{v\in V-\mathcal{S},\, c\notin L'(v)} w(v)\mathbf{Pr}(v \in V') \\ &= \mathbf{E}\big(W(V')\big) - \sum_{v\in V-\mathcal{S},\, c\notin L'(v)} \frac{1}{C} \leq \frac{n-S}{C} - \frac{A}{C} \\ &= \frac{n-S-A}{C} \leq \frac{(1-\delta)C+3k}{C} \\ &= 1-\delta+\frac{3}{m} \leq 1-\frac{\delta}{2}. \end{aligned} \tag{4.1}$$

Applying this fact for the c in K allows us to improve our bound on $\mathbf{E}\big(|S_K|\big)$. Specifically, we note that summing this bound over all the colours c in K

$$\mathbf{E}\Big(\sum_{c\in K} W\big(V' \cap \{v|c \in L'(v)\}\big)\Big) = \sum_{c\in K} \mathbf{E}\Big(W\big(V' \cap \{v|c\in L'(v)\}\big)\Big) \leq \left(1-\frac{\delta}{2}\right)|K|.$$

Now, each vertex v of S_K contributes $w(v) = 1/|L'(v)|$ to exactly $|L'(v)|$ terms in the first sum in this equation, so its total contribution to the sum is 1. I.e., we have:

$$\mathbf{E}\big(|S_K|\big) \leq \left(1-\frac{\delta}{2}\right)|K|.$$

So, we don't expect any particular set K of colours to provide an obstruction to finding the desired matching in the bipartite graph H in Step V. However, we need to handle all the K at once. In order to do so, we would like to prove that for each K, the size of S_K is highly concentrated around its expected value and hence is greater than $|K|$ only with exponentially small probability. As above, rather than focusing on all the K we actually consider, for each colour c, the weight of the subset V'_c of V' consisting of those v with c on $L'(v)$. There are two major difficulties which complicate our approach.

- some of the parts U can be very large making it impossible for us to obtain the desired concentration results directly (e.g., there could be a part of size exceeding $\frac{n}{3}$).
- If $L'(v)$ is very small then $w(v) = 1/|L'(v)|$ is large and putting v into V' can have a significant effect on the weight of the various V'_c. This makes proving a concentration result directly impossible.

In order to deal with these problems, we proceed as follows:

(A) We colour the "big" parts first, ignoring concentration in our computation and focusing only on the expected weight of the subset of V' intersecting the big parts. We note that by considering the expected overall weight and not focusing on a specific V'_c, we only lose a factor of $\frac{1}{C}$ per part. We will define big parts so that there are $o(1)$ of them, and hence the total loss will not be significant.

(B) We treat v with $|L'(v)|$ small separately using an expected value argument to bound the weight of the vertices in this set.

5. Completing the proof: The details

In this section we will complete the proof of Lemma 3 using the ideas which have already been discussed above. We choose an integer b so that

$$\frac{\delta^2 C}{40} \leq b(m-t) \leq \frac{\delta^2 C}{20}$$

which is possible because $m \leq \frac{\delta^2 C}{40}$ (this holds, since m and δ are fixed but C tends to infinity) and $m - t > 0$ (in fact it exceeds δm as we remarked at the end of Section 3). We call the largest $b(m-t)$ parts in our partition of G *big*, and the others *small*. Let Big be the union of the vertex sets of the big parts. We will need the following lemma.

Lemma 5 *Every small non-singleton partition class contains at least two v which satisfy:*

$$|L'(v)| > \frac{\delta^3}{80} C.$$

Proof. Let U be a small non-singleton colour class. We already mentioned that every colour of $\mathcal{C}$ is missed by a vertex of U so $\sum_{v \in U} |L'(v)| = \sum_{v \in U} |\mathcal{C} - L(v)| \geq |\mathcal{C}| = C$. Now, since there are less than $n < (2-\delta)C$ colours in total, every $L(v)$ must contain at least δC colours in $\mathcal{C}$ and so the largest $L'(v) = L(v) - \mathcal{C}$ in U has at most $(1-\delta)C$ elements. Thus, the sum of $|L'(v)|$ over the remaining vertices of U is at least δC.

Since there are at least $\frac{\delta^2 C}{40}$ big colour classes, the largest small colour class has at most $\frac{40n}{\delta^2 C} < \frac{80}{\delta^2}$ vertices. So, the second largest $L'(v)$ has size at least $\delta C \cdot (\frac{80}{\delta^2})^{-1}$. This is the desired result. $\square$

With this auxiliary result in hand, we can now complete the proof. We proceed as follows:

First Process: We randomly choose b of the $\mathcal{C}_i$ and a partition of each of these into subsets $\mathcal{A}_i$ of size t and $\mathcal{B}_i$ of size $m-t$ where these choices are all made independently and uniformly. We then choose a uniformly random bijection between the $b(m-t)$ colours in the union of these $\mathcal{B}_i$ and the big parts.

Second Process: We chose a partition of each remaining $\mathcal{C}_i$ into $\mathcal{A}_i$ and $\mathcal{B}_i$ where again these choices are uniform, independent, and independent of all the earlier choices. We then choose a uniformly random bijection between the colours in these $\mathcal{B}_i$ and the small parts.

Denote by c_U the colour which is assigned by the above bijection to the partition class U. Use the colours in $\mathcal{A}_i$ to colour the vertices of $\mathcal{S}_i$ and for each part U not in $\mathcal{S}$ colour every vertex v of U for which $c_U \in L(v)$ with the colour c_U. Let V' be a set of vertices not yet assigned a colour. We set $V'' = V' - Big$ and $V''' = V' \cap Big$.

Note that V''' is determined by our choices in the first process. So, using a computation similar to that in (3.1) we obtain

$$\mathbf{E}\big(W(V''')\big) = \sum_{v\in Big} w(v)\mathbf{Pr}(v\in V''') = \sum_{v\in Big}\frac{1}{L'(v)}\frac{L'(v)}{C} = \frac{|Big|}{C}.$$

Furthermore, by the definition of expectation, there exist at least one set of choices for the first process such that $W(V''') \le \mathbf{E}(W(V''')) = \frac{|Big|}{C}$. We condition on any such set of choices which ensures that this inequality holds. We use **CP** and **CE** for the conditional probability of an event and conditional expectation of a variable for the second process, given this set of choices.

Let $\mathcal{C}'$ be the union of the set of colours in the $\mathcal{C}_i$ which were chosen in the first process. At the end of Section 3 we proved that $m-t$ is at least δm. Therefore $|\mathcal{C}'| = mb = \frac{m}{m-t}b(m-t) \le \delta^{-1}\cdot\frac{\delta^2 C}{20} = \frac{\delta C}{20}$. Hence, we have that for every v in a small part which is not in $\mathcal{S}$,

$$\begin{aligned}\mathbf{CP}(v\in V'') &\le \frac{|L'(v)|}{C-|\mathcal{C}'|} \le \frac{|L'(v)|}{C}\left(1+\frac{|\mathcal{C}'|}{C-|\mathcal{C}'|}\right) \le \frac{|L'(v)|}{C}\left(1+\frac{\delta/20}{1-\delta/20}\right)\\ &\le \left(1+\frac{\delta}{10}\right)\frac{|L'(v)|}{C} = \left(1+\frac{\delta}{10}\right)\mathbf{Pr}(v\in V'').\end{aligned}$$

Clearly, this implies that for every subset X of the set of vertices $V-\mathcal{S}-Big$ we have

$$\mathbf{CE}\big(W(V''\cap X)\big) \le \left(1+\frac{\delta}{10}\right)\mathbf{E}\big(W(V''\cap X)\big).$$

In particular, for every colour c

$$\mathbf{CE}\Big(W\big(V''\cap\{v\,|\,c\in L'(v)\}\big)\Big) \le \left(1+\frac{\delta}{10}\right)\mathbf{E}\Big(W\big(V''\cap\{v\,|\,c\in L'(v)\}\big)\Big) \quad (5.1)$$

and also

$$\mathbf{CE}\left(W\Big(V''\cap\Big\{v\,\big|\,|L'(v)|<\frac{n}{\sqrt{\log n}}\Big\}\Big)\right) \le \quad (5.2)$$

$$\leq \left(1+\frac{\delta}{10}\right)\mathbf{E}\left(W\Big(V''\cap\Big\{v\,\big|\,|L'(v)|<\frac{n}{\sqrt{\log n}}\Big\}\Big)\right)$$

Before we proceed with the proof, we need the following lemma.

Lemma 6 *For every color c the probability that*

$$W\left(V''\cap\Big\{v\,\big|\,c\in L'(v),|L'(v)|\geq\frac{n}{\log n}\Big\}\right)$$
$$>\mathbf{CE}\left(W\Big(V''\cap\Big\{v\,\big|\,c\in L'(v),|L'(v)|\geq\frac{n}{\log n}\Big\}\Big)\right)+\frac{\delta}{20}$$

is $o(n^{-1})$.

Proof. To prove the lemma we need the following variant of a standard large deviation inequality for martingales. Since the proof of this inequality is essentially the same as other proofs which already appeared in the literature (see, e.g., Section 3 of the survey [7]), we will omit it here.

Given a finite set $\{1,2,\ldots,r\}$, let S_r denotes the set of all $r!$ permutations or linear orders on this set. Let $\mathbf{X}=(X_1,\ldots,X_l)$ be a family of independent random variables, where the random variable X_j takes values in a finite set Ω_j. Thus $\mathbf{X}$ takes values in the set $\Omega=\prod_j\Omega_j$. Let $\pi\in S_r$ be a random permutation independent from $\mathbf{X}$. Suppose that the non-negative real-valued function $h:\Omega\times S_r\to\mathbb{R}$ satisfies the following two conditions for every $(\mathbf{x},\pi)$.

- For every j, changing the value of a coordinate x_j can change the value of $h(\mathbf{x},\pi)$ by at most d.
- Swapping any two elements in permutation π can change the value of $h(\mathbf{x},\pi)$ by at most d.

Denote by $\mathbf{E}h$ the expected value of h. Then for every $t\geq 0$ we have that

$$\mathbf{Pr}\Big(|h-\mathbf{E}h|>t\Big)\leq e^{-\Omega\left(\frac{t^2}{(r+l)d^2}\right)}.$$

Now fix a color c and define the function h to be

$$h(\mathbf{x},\pi)=W\left(V''\cap\Big\{v\,\big|\,c\in L'(v),|L'(v)|\geq\frac{n}{\log n}\Big\}\right),$$

where $(\mathbf{x},\pi)$ corresponds to the set of random choices for the second process. More precisely, x_i is a random partition of the set $\mathcal{C}_i$ into subsets $\mathcal{A}_i$ and $\mathcal{B}_i$ and π is a random bijection between the colors in these $\mathcal{B}_i$ and the small parts of G. Since we can fix one canonical ordering of these small parts we can assume that π is just a random permutation of the set of colors which is, by definition, independent from the variables x_i.

Next, note that changing the outcome of the variable x_i, i.e., changing one particular $\mathcal{B}_i$ can only affect vertices in at most $m-t$ small parts of G. As we already mentioned in the proof of Lemma 5, each small part contains at most $80/\delta^2$

vertices. Since we considering only vertices v satisfying $|L'(v)| \ge \frac{n}{\log n}$, the weight of such a vertex is at most $w(v) = 1/|L'(v)| \le \frac{\log n}{n}$. Therefore, changing outcome of one x_i can change the value of h by at most $(m-t)\frac{80}{\delta^2}\frac{\log n}{n} = O\big(\frac{\log n}{n}\big) = d$. Similarly swapping any two colors in π can affect only vertices in two small parts of G. So again this can only change h by at most $d = O\big(\frac{\log n}{n}\big)$.

Since the total number of random variables x_i and also the length of permutation π are bounded by n we have that in our case $(r+l)d^2 \le O\big(n(\frac{\log n}{n})^2\big) = O\big(\frac{\log^2 n}{n}\big)$. Therefore it follows form the above large deviation inequality that

$$\mathbf{Pr}\Big(h - \mathbf{E}h > t = \frac{\delta}{20}\Big) \le e^{-\Omega\left(\frac{t^2}{(r+l)d^2}\right)} = e^{-\Omega\left(\frac{(\delta/20)^2}{O(\log^2 n/n)}\right)} = e^{-\Omega\left(\frac{n}{\log^2 n}\right)} = o(n^{-1}).$$

This completes the proof of the lemma. □

Now, using the fact that the total number of colors is at most n, we deduce from this lemma that with probability $1-o(1)$ the following holds for every color c

$$W\left(V'' \cap \Big\{v \,\big|\, c \in L'(v), |L'(v)| \ge \frac{n}{\log n}\Big\}\right) \le \mathbf{CE}\left(W\Big(V'' \cap \Big\{v \,\big|\, c \in L'(v), |L'(v)| \ge \frac{n}{\log n}\Big\}\Big)\right) + \frac{\delta}{20}. \tag{5.3}$$

In addition, we also want to satisfy the following inequality:

$$W\left(V'' \cap \Big\{v \,\big|\, |L'(v)| < \frac{n}{\sqrt{\log n}}\Big\}\right) \le \Big(1+\frac{\delta}{10}\Big)\mathbf{CE}\left(W\Big(V'' \cap \Big\{v \,\big|\, |L'(v)| < \frac{n}{\sqrt{\log n}}\Big\}\Big)\right). \tag{5.4}$$

Since the probability that this last inequality fails is at most $\frac{1}{1+\delta/10} < 1 - o(1)$, there does indeed exist a set of random choices for the second process which satisfies simultaneously (5.3) and (5.4).

Fix any such set of choices. Then, combining the inequalities (5.2) and (5.4) together with the facts that $W(V''') \le \frac{|Big|}{C}$ and $V' = V'' \cup V'''$ we obtain that

$$\begin{aligned} W\left(V' \cap \Big\{v \,\big|\, |L'(v)| < \frac{n}{\sqrt{\log n}}\Big\}\right) &\le W\left(V'' \cap \Big\{v \,\big|\, |L'(v)| < \frac{n}{\sqrt{\log n}}\Big\}\right) + W(V''') \\ &\le \Big(1+\frac{\delta}{10}\Big)^2 \mathbf{E}\left(W\Big(V'' \cap \Big\{v \,\big|\, |L'(v)| < \frac{n}{\sqrt{\log n}}\Big\}\Big)\right) + \frac{|Big|}{C}. \end{aligned} \tag{5.5}$$

Note that, by Lemma 5, every small non-singleton partition class contains at least two vertices v such that $|L'(v)| > \frac{\delta^3}{80}C = \Omega(n) > \frac{n}{\sqrt{\log n}}$. Since the number of small non-singleton partition classes is at least $A - \frac{\delta^2}{20}C$ we obtain that

$$\left|(V - \mathcal{S} - Big) \cap \Big\{v \,\big|\, |L'(v)| < \frac{n}{\sqrt{\log n}}\Big\}\right| \le n - S - |Big| - 2\Big(A - \frac{\delta^2}{20}C\Big)$$

$$
\begin{aligned}
&\le (2-\delta)C - S - |Big| - 2A + \frac{\delta^2}{10}C \\
&= (1-\delta)C - |Big| + (C - S - A) - A + \frac{\delta^2}{10}C \\
&\le (1-\delta)C - |Big| + 3k - A + \frac{\delta^2}{10}C \\
&\le \left(1 - \frac{4}{5}\delta\right) C - |Big|.
\end{aligned}
$$

Here, in the last inequality we used that $A > \frac{\delta}{2}C > \frac{3}{m}C = 3k$ and $\delta^2 \le \delta$. Note that a similar computation as in (3.1) shows that for any subset $Y \subseteq V - \mathcal{S}$ the expectation $\mathbf{E}\big(W(V' \cap Y)\big) = \frac{|Y|}{C}$. In particular, for $Y = \big(V - \mathcal{S} - Big\big) \cap \big\{v \,\big|\, |L'(v)| < \frac{n}{\sqrt{\log n}}\big\}$ we obtain

$$
\begin{aligned}
\mathbf{E}\left(W\left(V'' \cap \left\{v \,\middle|\, |L'(v)| < \frac{n}{\sqrt{\log n}}\right\}\right)\right) &= \frac{\big|(V - \mathcal{S} - Big) \cap \{v \,|\, |L'(v)| < \frac{n}{\sqrt{\log n}}\}\big|}{C} \\
&\le 1 - \frac{4}{5}\delta - \frac{|Big|}{C}.
\end{aligned}
$$

Combining this inequality with (5.5) we have

$$
\begin{aligned}
W\left(V' \cap \left\{v \,\middle|\, |L'(v)| < \frac{n}{\sqrt{\log n}}\right\}\right) &\le \left(1 + \frac{\delta}{10}\right)^2 \left(1 - \frac{4}{5}\delta - \frac{|Big|}{C}\right) + \frac{|Big|}{C} \\
&\le \left(1 + \frac{\delta}{4}\right)\left(1 - \frac{4}{5}\delta\right) \le 1 - \frac{\delta}{2}. \qquad (5.6)
\end{aligned}
$$

This completes our analysis of the weight of vertices with short lists. We now consider the remaining vertices.

As we already mentioned, for every color c and every non-singleton part of G there is at least one vertex v in this part such that $c \notin L(v)$. Since there are at least $A - \frac{\delta^2}{20}C$ small non-singleton parts, a similar computations as in (4.1) shows for every color c that

$$
\begin{aligned}
\mathbf{E}\Big(W\big(V'' \cap \{v \,|\, c \in L'(v)\}\big)\Big) &= \frac{\big|(V - \mathcal{S} - Big) \cap \{v \,|\, c \in L'(v)\}\big|}{C} \\
&\le \frac{n - S - |Big| - (A - \frac{\delta^2}{20}C)}{C} \\
&\le \frac{(2-\delta)C - S - |Big| - A}{C} + \frac{\delta^2}{20} \\
&= (1-\delta) + \frac{C - S - A}{C} - \frac{|Big|}{C} + \frac{\delta^2}{20} \\
&\le (1-\delta) + \frac{3k}{C} - \frac{|Big|}{C} + \frac{\delta^2}{20} \\
&= (1-\delta) + \frac{3}{m} - \frac{|Big|}{C} + \frac{\delta^2}{20}
\end{aligned}
$$

$$\leq 1-\delta+\frac{\delta}{2}+\frac{\delta^2}{20}-\frac{|Big|}{C}\leq 1-\frac{2}{5}\delta-\frac{|Big|}{C}.$$

Combining this inequality with (5.1) and (5.3) and using the fact that $V'=V''\cup V'''$ we will have that for every color c

$$W\left(V'\cap\left\{v|c\in L'(v),|L'(v)|\geq\frac{n}{\log n}\right\}\right)$$

$$\begin{aligned}
&\leq W\left(V''\cap\left\{v|c\in L'(v),|L'(v)|\geq\frac{n}{\log n}\right\}\right)+W\left(V'''\right)\\
&\leq \mathbf{CE}\left(W\left(V''\cap\left\{v|c\in L'(v),|L'(v)|\geq\frac{n}{\log n}\right\}\right)\right)+\frac{\delta}{20}+\frac{|Big|}{C}\\
&\leq \mathbf{CE}\Big(W\left(V''\cap\left\{v|c\in L'(v)\right\}\right)\Big)+\frac{\delta}{20}+\frac{|Big|}{C}\\
&\leq \left(1+\frac{\delta}{10}\right)\mathbf{E}\Big(W\left(V''\cap\left\{v|c\in L'(v)\right\}\right)\Big)+\frac{\delta}{20}+\frac{|Big|}{C}\\
&\leq \left(1+\frac{\delta}{10}\right)\left(1-\frac{2}{5}\delta-\frac{|Big|}{C}\right)+\frac{\delta}{20}+\frac{|Big|}{C}\\
&\leq \left(1+\frac{\delta}{10}\right)\left(1-\frac{2}{5}\delta\right)+\frac{\delta}{20}\leq 1-\frac{\delta}{4}.
\end{aligned}\tag{5.7}$$

Recall that H is a bipartite graph with bipartition $\big(V',\,\cup_v L'(v)\big)$ and an edge between c and v precisely when $c\in L'(v)$. Let K be any subset of colours in $\cup_v L'(v)$ and denote by $S_K=\{v|v\in V',L'(v)\subseteq K\}$. We complete the proof of the lemma by showing that the graph H satisfies Hall's condition, i.e., $|S_K|\leq|K|$ for every set S_K. Then in Step V we can match all uncoloured vertices in V' with the set of colours yet not used and and produce an acceptable coloring of G.

First, note that any set K of fewer than $\frac{n}{\sqrt{\log n}}$ colours cannot be an obstruction to the existence of the desired matching. Indeed, if $|S_K|>|K|$, then by Observation 4 we have that $W(S_K)>1$. On the other hand, for every vertex $v\in S_K$ the size of $L'(v)$ is at most $|K|<\frac{n}{\sqrt{\log n}}$. Therefore we obtain a contradiction, since by (5.6)

$$W(S_K)\leq W\left(V'\cap\left\{v\,|\,|L'(v)|<\frac{n}{\sqrt{\log n}}\right\}\right)<1-\frac{\delta}{2}.$$

Turning to larger K, we note next that the inequality (5.6) yields:

$$\left|V'\cap\left\{v\,|\,|L'(v)|\leq\frac{n}{\log n}\right\}\right|\leq W\left(V'\cap\left\{v\,|\,|L'(v)|\leq\frac{n}{\log n}\right\}\right)\left(\min_{v,\,|L'(v)|\leq\frac{n}{\log n}}w(v)\right)^{-1}$$

$$\leq\left(1-\frac{\delta}{2}\right)\left(\min_{v,\,|L'(v)|\leq\frac{n}{\log n}}\frac{1}{|L'(v)|}\right)^{-1}\leq\left(1-\frac{\delta}{2}\right)\frac{n}{\log n}<\frac{n}{\log n}.\tag{5.8}$$

Next, observe that the set of inequalities (5.7) imply that for any set of colours K

$$\begin{aligned}
\left|S_K \cap \left\{v \,\middle|\, |L'(v)| > \frac{n}{\log n}\right\}\right| &= \sum_{v \in S_K, |L'(v)| > \frac{n}{\log n}} w(v) \cdot |L'(v)| \\
&\leq \sum_{c \in K} \sum_{\{v \,|\, c \in L'(v), |L'(v)| > \frac{n}{\log n}\}} w(v) \\
&= \sum_{c \in K} W\left(V' \cap \left\{v \,\middle|\, c \in L'(v), |L'(v)| \geq \frac{n}{\log n}\right\}\right) \\
&\leq \left(1 - \frac{\delta}{4}\right)|K|.
\end{aligned}$$

This, together with the inequality (5.8) yields that any set of colours K of size at least $\frac{n}{\sqrt{\log n}}$ satisfies

$$\begin{aligned}
\left|S_K\right| &= \left|S_K \cap \left\{v \,\middle|\, |L'(v)| > \frac{n}{\log n}\right\}\right| + \left|S_K \cap \left\{v \,\middle|\, |L'(v)| \leq \frac{n}{\log n}\right\}\right| \\
&\leq \left(1 - \frac{\delta}{4}\right)|K| + \left|V' \cap \left\{v \,\middle|\, |L'(v)| \leq \frac{n}{\log n}\right\}\right| \\
&\leq \left(1 - \frac{\delta}{4}\right)|K| + \frac{n}{\log n} \; < |K|.
\end{aligned}$$

Thus we obtain that these larger K also do not violate Hall's condition and hence the desired matching of Step V does indeed exist. This completes the proof. □

6. Concluding remarks

In this paper we proved that for every $\epsilon > 0$ and for every sufficiently large graph G of order n, the list chromatic number of G equals its chromatic number, provided $n \leq (2-\epsilon)\chi(G)$. A more careful analysys of our methods yields that the value of ϵ in this result can be made as small as $O(1/\log^{\eta} n)$ for any constant $0 < \eta < 1$. Nevertheless the conjecture of Ohba remains open for graphs with $2\chi(G)$ vertices and it seems one needs new ideas to tackle this problem. Even to show that there is a constant N such that $\chi^l(G) = \chi(G)$ for every graph G with at most $2\chi(G) - N$ vertices, would be very interesting.

In conclusion we would like to propose a related problem, which was motivated by Ohba's conjecture. Let t be an integer and let G be a graph with at most $t\chi(G)$ vertices. Find the smallest constant c_t such that for any such a graph G its list chromatic is bounded by $c_t\chi(G)$. Note that Ohba's conjecture if true, implies that $c_2 = 1$. An additional intriguing question is to determine graphs with $|V(G)| \leq t\chi(G)$ and for which the ratio $\chi^l(G)/\chi(G)$ is maximal. Here the case $t = 2$ gives some indication that a complete multi-partite graph with all parts of size t may have this property.

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