Finding Hamilton Cycles in Random Graphs With Few Queries*

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ABSTRACT: We introduce a new setting of algorithmic problems in random graphs, studying the minimum number of queries one needs to ask about the adjacency between pairs of vertices of $\mathcal{G}(n, p)$ in order to typically find a subgraph possessing a given target property. We show that if $p \geq \frac{\ln n + \ln \ln n + \omega(1)}{n}$, then one can find a Hamilton cycle with high probability after exposing $(1 + o(1))n$ edges. Our result is tight in both p and the number of exposed edges. © 2016 Wiley Periodicals, Inc.   Random Struct. Alg., 49, 635–668, 2016

1. INTRODUCTION

Random Graphs is definitely one of the most popular areas in modern Combinatorics, also having a tremendous amount of applications in different scientific fields such as Networks, Algorithms, Communication, Physics, Life Sciences and more. Ever since its introduction, the *binomial random graph model* has been one of the main objects of study in probabilistic combinatorics. Given a positive integer n and a real number $p \in [0, 1]$, the binomial random graph $\mathcal{G}(n, p)$ is a probability space whose ground set consists of all labeled graphs on the vertex set $[n]$. We can describe the probability distribution of $G \sim \mathcal{G}(n, p)$ by saying that

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each pair of elements of $[n]$ forms an edge in G independently with probability p. For more details about random graphs the reader is referred to the excellent books of Bollobás [3] and of Janson, Łuczak and Ruciński [11].

Due to the importance and visibility of the subject of Random Graphs, and also due to its practical connections and the fact that random discrete spaces are frequently used to model real world phenomena, it is only natural to study the algorithmic aspects of random graphs. The reader is advised to consult an excellent survey of Frieze and McDiarmid on the subject [10], providing an extensive coverage of the variety of problems and results in Algorithmic Theory of Random Graphs. In this paper we present an apparently new and interesting setting for algorithmic type questions about random graphs.

Usually, questions considered in random graphs have the following generic form: given some *monotone increasing graph property* $\mathcal{P}$ (that is, a property of graphs that cannot be violated by adding edges) and a function $p = p(n) \in [0,1]$, determine whether a graph $G \sim \mathcal{G}(n,p)$ satisfies $\mathcal{P}$ *with high probability* (whp) (that is, with probability tending to 1 as n tends to infinity). In order to solve questions of this type, one should show that after asking for all possible pairs (i,j) of distinct elements of $[n]$ the question "is $(i,j) \in E(G)$?" and getting a positive answer with probability $p(n)$ independently, whp the graph G obtained from all positive answers possesses $\mathcal{P}$. Here we propose a different task. Given p such that a graph $G \sim \mathcal{G}(n,p)$ whp satisfies $\mathcal{P}$, we want to devise an algorithm, probably an adaptive one, that asks typically as few queries "is $(i,j) \in E(G)$?" as possible, and yet the positive answers reveal us a graph which possesses $\mathcal{P}$. We refer to such an algorithm as *an adaptive algorithm interacting with the probability space* $\mathcal{G}(n,p)$. For example, consider the case where $\mathcal{P}$ is the property "containing a Hamilton cycle" (i.e. a cycle passing through all the vertices of the graph). In this case we aim to find an algorithm that will adaptively query as few pairs as possible, yet a sufficient amount to get whp a Hamilton cycle between the positive answers. It is important to remark that we are not concerned here with the time of computation required for the algorithm to locate a target structure (thus essentially assuming that the algorithm has unbounded computational power), but we make the algorithm pay for the number of queries it asks, or for the amount of communication with the random oracle generating the random graph. Therefore, in this sense this setting is reminiscent of such branches of Computer Science as Communication Complexity and Property Testing.

In general, given a monotone property $\mathcal{P}$, what can we expect? If all n-vertex graphs belonging to $\mathcal{P}$ have at least m edges, then the algorithm should get at least m positive answers to hit the target property with the required absolute certainty. This means that the obvious lower bound in this case is $(1 + o(1))m/p$ queries, and therefore, whp one would have $(1+o(1))m$ positive answers. Continuing with our example of Hamiltonicity, this lower bound translates to $(1 + o(1))n$ positive answers. This should serve as a natural benchmark for algorithms of such type. Of course, the above described framework is very general and can be fed with any monotone property $\mathcal{P}$, thus producing a variety of interesting questions.

Here is a very simple illustration of our model. Let us choose the target property to be connectedness (i.e., the existence of a spanning tree) in $G \sim \mathcal{G}(n,p)$). Suppose the edge probability $p(n)$ is chosen to be above the threshold for connectedness, which is known to be equal to $p(n) = \frac{\ln n + \omega(1)}{n}$. In this case the minimum number of positive answers to the algorithm's queries is obviously $n - 1$. An adaptive algorithm discovering a spanning tree after $n - 1$ positive answers is very simple: start with $T = v$, where $v \in [n]$ is an arbitrary vertex, and at each step query in an arbitrary order previously non-queried pairs between the current tree T and the vertices outside of T until the first such edge $(u,w) \in G$ has been found, then update T by appending the edge (u,w). Assuming the input graph G is

connected, the algorithm clearly creates a spanning tree of G after exactly $n-1$ positive answers.

In this paper we focus on the property of Hamiltonicity, which is one of the most central notions in graph theory, and has been intensively studied by numerous researchers. The earlier results on Hamiltonicity of random graphs were proved by Korshunov [13] and by Pósa [16] in 1976. Building on these ideas, Bollobás [4], and Komlós and Szemerédi [12] independently showed that for $p \geq \frac{\ln n + \ln \ln n + \omega(1)}{n}$, a graph $G \sim \mathcal{G}(n, p)$ is whp Hamiltonian. This range of p cannot be further improved since if $p \leq \frac{\ln n + \ln \ln n - \omega(1)}{n}$, then whp a graph $G \sim \mathcal{G}(n, p)$ has a vertex of degree at most 1, and such a graph is trivially non-Hamiltonian.

In the following theorem, which is the main result of this paper, we verify what the reader may have suspected: $(1 + o(1))n$ positive answers (and thus, $(1 + o(1))n/p$ queries) are enough to create a graph which contains a Hamilton cycle, for every $p \geq \frac{\ln n + \ln \ln n + \omega(1)}{n}$.

Theorem 1.   *Let $p = p(n) \geq \frac{\ln n + \ln \ln n + \omega(1)}{n}$. Then there exists an adaptive algorithm, interacting with the probability space $\mathcal{G}(n, p)$, which whp finds a Hamilton cycle after getting $(1 + o(1))n$ positive answers.*

Note that Theorem 1 is asymptotically optimal in both the edge probability p and the number of positive answers we get. We remark that if we allow (say) $3n$ positive answers, and replace $p(n) \geq \frac{\ln n + \ln \ln n + \omega(1)}{n}$ with $p(n) \geq (1 + \varepsilon) \ln n/n$, then the result follows quite easily from a result of Bohman and Frieze [2] by effectively embedding some other random space, like a 3-out, in $\mathcal{G}(n, p)$ and accessing it in few queries. Moreover, if our goal is to find a Hamilton cycle after having $(1 + o(1))n$ positive answers but we are willing to weaken the assumption to $p = \omega(\log n/n)$, then in fact the problem becomes much easier. In what follows, as an illustrative example, we provide a sketch of a proof for this statement.

Proposition 1.1.   *Let $p = \omega(\log n/n)$. Then there exists an adaptive algorithm, interacting with the probability space $\mathcal{G}(n, p)$, which whp finds a Hamilton cycle after getting $(1+o(1))n$ positive answers.*

Proof (sketch).   Let $p = f(n) \log n/n$, where $f := f(n)$ is an arbitrary function tending to infinity with n, and set $q_1 = (1 - \varepsilon)p$ (where, say, $\varepsilon = f^{-1/3}$) and q_2 to be the unique solution to $1 - p = (1 - q_1)(1 - q_2)^4$. Our proof consists of two phases, where in Phase 1 we find a "long" path, and in Phase 2 we close it into a Hamilton cycle.

Phase 1 In this phase we construct a path P of length $t := n - m$ where $m = n/\sqrt{f}$, while exposing edges in an "online fashion" with edge probability q_1, after exposing (successfully) exactly t edges. At each time step $0 \leq \ell \leq t - 1$ of this phase we try to extend a current path $P_\ell := v_0 \ldots v_\ell$ by exposing an edge of the form $v_\ell u$, where $u \in [n] \setminus V(P_\ell)$. Note that the probability to fail in step ℓ is $(1 - q_1)^{n-(\ell+1)} \leq e^{-q_1 m} = o(1/n)$, and therefore, by applying the union bound, whp the algorithm does not terminate unsuccessfully during this phase.

Phase 2 In this phase we want to turn the path $P := v_0 \ldots v_t$ obtained in Phase 1 into a Hamilton cycle. To this end, we define an auxiliary directed graph D, based on a subgraph of G, and show that a *directed* Hamilton cycle in this graph exists whp, and that such a cycle corresponds to a Hamilton cycle of G. Moreover, we show that D contains $O(km) = o(n)$ edges. This will complete the proof.

Let $U := ([n] \setminus V(P)) \cup \{v_P\}$ and set $V(D) = U$. Let us choose (say) $k := 100$ (actually, any $k \geq 2$ will work, but for a large k the proof of the tool we base our argument on

becomes relatively simple). We choose the *arcs* (directed edges) of D according to the following procedure:

- For every $v \in U \setminus \{v_P\}$, we define

$$\mathrm{Out}(v) = (U - v) \cup \{v_0\}$$

  and

$$\mathrm{In}(v) = (U - v) \cup \{v_t\}.$$

  Now, in G, iteratively expose edges of the form vu, $u \in \mathrm{Out}(v)$, with probability q_2, independently at random, according to a *random* ordering of $\mathrm{Out}(v)$, until you have *exactly k successes*. Let $x_1, x_2, \ldots, x_k$ denote these successes, and add vx_i, $1 \le i \le k$ as arcs to $E(D)$ (if one of the x_i's is v_0, then add the arc vv_P to $E(D)$). Do the same for edges of the form uv, $u \in \mathrm{In}(v)$, where an arc of the form $v_t v$ translates to the arc $v_P v$.
- For v_P, let $\mathrm{Out}(v_P) = \mathrm{In}(v_P) := U \setminus \{v_P\}$. Expose (in G) edges of the form $v_t u$, $u \in \mathrm{Out}(v_P)$ with probability q_2, independently at random, according to a random ordering of $\mathrm{Out}(v_P)$, until having exactly k successes. For each success $v_t u$, add an arc $v_P u$ to $E(D)$. Do the same for $\mathrm{In}(v_P)$, where now one exposes edges of the form uv_0, $u \in \mathrm{In}(v_P)$. For each edge of the form uv_0 add the arc uv_P to $E(D)$.

It is relatively easy to show that the probability for not having k successes for at least one of the vertices is $o(1)$, and that the choices are made independently and uniformly. Moreover, the number of successively exposed edges is clearly $O(km) = o(n)$. Therefore, D can be seen as a directed graph obtained by the following procedure: each vertex picks exactly k in- and k out-neighbors, independently, uniformly at random. This model is known as $D_{k-in,k-out}$ and whp contains a directed Hamilton cycle (see the main result of [6], or the less complicated one in [5]). Clearly, by replacing the vertex v_P by the path P itself, such a Hamilton cycle corresponds to a Hamilton cycle of G. Moreover, note that throughout the algorithm (both phases), each edge has been queried at most once with probability q_1 (in Phase 1) and at most 4 times with probability q_2 (in Phase 2), and therefore the resulting graph naturally couples as a subgraph of $G \sim \mathcal{G}(n,p)$. It's not hard to see that if we were to expose with full probability p the edges that have been exposed by this procedure with any non-zero probability, the number of additional successively exposed edges would be $o(n)$, because of our choice s of q_1 and q_2. This completes the proof of this sketch. ∎

2. NOTATION

Most of the notation used in this paper is fairly standard. Given a natural number k and a set S, we use $[k]$ to denote the set $\{1, 2, \ldots, k\}$ and $\binom{S}{k}$ to the denote the collection of subsets of S of size k.

Given a graph G we use $V(G)$ to denote the set of vertices of G. Moreover, given a subset E of the edges of G we shall oftentimes abuse notation and refer to the subgraph of G formed by these edges simply by E (with vertex set $V(G)$ unless stated otherwise).

Given a subset $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by the vertices in S, i.e. the graph with vertex set S whose edges are the ones of G between vertices in S. Furthermore, we use $e_G(S)$ to denote the number of edges of the graph $G[S]$.

Given a vertex $v \in V(G)$ and a subset $S \subseteq V(G)$, we use $N_G(v)$ to denote the set of neighbours of v in the graph G, $N_G(S) := \cup_{v \in S} N_G(v)$ to denote the set of neighbours of vertices in S in the graph G and $N_G(v, S) := N_G(v) \cap S$ to denote the set of neighbours of v in the graph G which lie in the set S. Moreover, $d_G(v) := |N_G(v)|$ denotes the degree of v in the graph G, $d_G(v, S) := |N_G(v, S)|$ denotes the number of neighbours of v in the graph G which lie in the set S, $\Delta(G) := \max_{v \in V(G)} d_G(v)$ denotes the maximum degree of the graph G and finally $\delta(G) := \min_{v \in V(G)} d_G(v)$ denotes the minimum degree of the graph G.

A subgraph P of the graph G is called a *path* if $V(P) = \{v_1, \ldots, v_\ell\}$ and the edges of P are $v_1 v_2$, $v_2 v_3$, $\ldots$, $v_{\ell-1} v_\ell$. We shall oftentimes refer to P simply by $v_1 v_2 \ldots v_\ell$. We say that such a path P has *length* $\ell - 1$ (number of edges) and *size* ℓ (number of vertices). We say that P is a *Hamilton path* (in the graph G) if it has size $|V(G)|$. Furthermore, a subgraph C of the graph G is called a *cycle* if $V(C) = \{v_1, \ldots, v_\ell\}$ and the edges of P are $v_1 v_2$, $v_2 v_3$, $\ldots$, $v_{\ell-1} v_\ell$ and $v_\ell v_1$. As for paths, we shall oftentimes refer to C simply by $v_1 v_2 \ldots v_\ell$. We say that such a cycle has *length* ℓ (number of edges) and *size* ℓ (number of vertices). We say that C is a *Hamilton cycle* (in the graph G) if it has size $|V(G)|$. A *trail* of length t in G between two vertices x and y is a sequence of vertices $x = v_0, v_1, \ldots, v_t = y$ such that $\{v_0 v_1, v_1 v_2, \ldots, v_{t-1} v_t\}$ is a set of distinct t edges of G.

3. AUXILIARY RESULTS

3.1. Probabilistic Tools

We need to employ standard bounds on large deviations of random variables. We mostly use the following well-known bound on the lower and upper tails of the Binomial distribution due to Chernoff (see e.g. [1, 11]).

Lemma 3.1. *Let $X \sim \text{Bin}(n, p)$ and let $\mu = \mathbb{E}[X]$. Then*

- $\Pr[X \leq (1 - a)\mu] < e^{-a^2 \mu/2}$ *for every $a > 0$;*
- $\Pr[X \geq (1 + a)\mu] < e^{-a^2 \mu/3}$ *for every $0 < a < 3/2$.*

The following is a trivial yet useful bound.

Lemma 3.2. *Let $X \sim \text{Bin}(n, p)$ and $k \in \mathbb{N}$. Then the following holds:*

$$\Pr(X \geq k) \leq \left(\frac{enp}{k}\right)^k.$$

Proof. $\Pr(X \geq k) \leq \binom{n}{k} p^k \leq \left(\frac{enp}{k}\right)^k.$ ∎

3.2. Properties of Random Graphs

We start with the following natural definition for k-pseudorandomness of graphs.

Definition 3.3. *A graph G is called k-pseudorandom if $e_G(A, B) > 0$ for every two disjoint sets A and B of size at least k.*

Lemma 3.4. *Let $k = k(n)$ be an integer such that $3 \ln n \leq k \leq \frac{n}{8}$ and let $\frac{3 \ln(n/k)}{k} \leq p \leq 1$. Then whp a graph $G \sim \mathcal{G}(n, p)$ is k-pseudorandom.*

Proof.   If G is not k-pseudorandom then there exist two disjoint sets S and T with $|S| = |T| = k$ and no edge between them. Note that the probability that there is no edge between a given such pair $\{S, T\}$ is $(1 - p)^{k^2}$ and there are at most $\binom{n}{k}^2$ such pairs. Thus, applying the union bound over all pairs of disjoint sets S, T of size k we obtain that the probability that G is not k-pseudorandom is at most

$$\binom{n}{k}^2 (1 - p)^{k^2} \leq \left(\frac{en}{k}\right)^{2k} e^{-pk^2} = \left(\frac{e^2 n^2 e^{-pk}}{k^2}\right)^k \leq \left(\frac{e^2 k}{n}\right)^k = o(1).$$

Thus, we conclude that whp a graph $G \sim \mathcal{G}(n, p)$ is k-pseudorandom as claimed.   ∎

In the following two lemmas we state a few properties of a typical random graph which will be used extensively throughout the paper.

Lemma 3.5.   *Let $p = p(n) \in (0, 1)$, let $c > 1$ be a constant and let $C = C(n) \geq 6 \ln(np \ln n)$. Then whp a graph $G \sim \mathcal{G}(n, p)$ is such that the following holds:*

(P1) $\Delta(G) \leq 4np$, *provided* $p \geq \frac{\ln n}{n}$.

(P2) $e_G(X) < c|X|$ *for any subset* $X \subseteq V(G)$ *of size at most* $\left(\frac{1}{\ln n} \cdot \frac{(2c)^c}{e^{c+1} np^c}\right)^{\frac{1}{c-1}}$.

(P3) $e_G(X) < C|X|$ *for any subset* $X \subseteq V(G)$ *of size at most* $\frac{C}{2p}$.

Proof.   For (P1), note that for a vertex $v \in V(G)$ we have $d_G(v) \sim \text{Bin}(n - 1, p)$ and so by Lemma 3.2

$$\Pr\left[d_G(v) \geq 4np\right] \leq \left(\frac{e(n-1)p}{4np}\right)^{4np} \leq \left(\frac{e}{4}\right)^{4 \ln n} = o\left(\frac{1}{n}\right).$$

Thus, by applying the union bound over all vertices of G we see that the probability that there exists a vertex $v \in V(G)$ with $d_G(v) \geq 4np$ is $o(1)$, settling (P1).

For (P2), note that for a fixed $X \subseteq V(G)$ of size $|X| = x$ one has $e_G(X) \sim \text{Bin}\left(\binom{x}{2}, p\right)$. Therefore, by Lemma 3.2 we obtain

$$\Pr\left[e_G(X) \geq cx\right] \leq \left(\frac{ex^2 p}{2cx}\right)^{cx} = \left(\frac{exp}{2c}\right)^{cx}.$$

Applying the union bound over all subsets of $V(G)$ of size at most $t = \left(\frac{1}{\ln n} \cdot \frac{(2c)^c}{e^{c+1} np^c}\right)^{\frac{1}{c-1}}$ we see that the probability that there exists a set X of size $x \leq t$ with $e_G(X) \geq cx$ is upper bounded by

$$\sum_{x=1}^{t} \binom{n}{x} \left(\frac{exp}{2c}\right)^{cx} \leq \sum_{x=1}^{t} \left(\frac{en}{x}\right)^x \left(\frac{exp}{2c}\right)^{cx} = \sum_{x=1}^{t} \left[\frac{en}{x} \cdot \left(\frac{exp}{2c}\right)^c\right]^x$$

$$\leq \sum_{x=1}^{t} \left(\frac{e^{c+1} np^c t^{c-1}}{(2c)^c}\right)^x = o(1),$$

since $\frac{e^{c+1} np^c t^{c-1}}{(2c)^c} = \frac{1}{\ln n}$. This settles (P2).

For (P3) we proceed in a similar way as with (P2). By (P2), taking $c = 2$, we know that whp all sets $X \subseteq V(G)$ of size at most $\frac{1}{2np^2 \ln n}$ satisfy $e_G(X) < 2|X| \leq C|X|$. Thus, we just need to show that the probability that there exists a set X of size $\frac{1}{2np^2 \ln n} \leq x \leq \frac{C}{2p}$ with $e_G(X) \geq Cx$ is $o(1)$. Indeed, proceeding as above, we see that this probability is at most

$$\sum_{x=\frac{1}{2np^2 \ln n}}^{\frac{C}{2p}} \left[\frac{en}{x} \cdot \left(\frac{exp}{2C} \right)^C \right]^x \leq \sum_{x=1}^{\infty} \left[2e(np)^2 \ln n \left(\frac{e}{4} \right)^C \right]^x \leq \sum_{x=1}^{\infty} \left(\frac{2e}{\ln n} \right)^x = o(1).$$

since $C \geq 6 \ln(np \ln n)$ and $\left(\frac{e}{4} \right)^6 < e^{-2}$. This settles (P3). ∎

Lemma 3.6. *Let $w := w(n)$ be such that $w \to \infty$ as $n \to \infty$, let $p := p(n)$ be such that $\frac{\ln n + \ln \ln n + w}{n} \leq p \leq \frac{10 \ln n}{n}$, let $G \sim \mathcal{G}(n, p)$ and let C be a fixed positive integer. Then whp all of the following hold:*

- (P1) *G has minimum degree at least 2.*
- (P2) *there are no two cycles in G of length at most $\frac{\ln n}{4 \ln(np)}$ sharing exactly one vertex.*
- (P3) *between any two vertices $u, v \in V(G)$ there are at most 3 trails of length at most C.*

Proof. First we prove that whp (P1) holds. Note that for a fixed vertex $v \in V(G)$ we have

$$\Pr[d_G(v) < 2] = (1-p)^{n-1} + (n-1)p(1-p)^{n-2}$$
$$\leq 2e^{-np} + 2npe^{-np} \leq \frac{2e^{-w}}{n \ln n} + 20 \ln n \cdot \frac{e^{-w}}{n \ln n} = o\left(\frac{1}{n} \right)$$

where in the second inequality we used the fact that $\ln n + \ln \ln n + w \leq np \leq 10 \ln n$, and in the last step we used the fact that $w \to \infty$. Thus, taking the union bound over all vertices in $V(G)$ we obtain

$$\Pr[\delta(G) < 2] = \Pr[\exists v \in V(G) \text{ such that } d_G(v) < 2] \leq n \cdot o\left(\frac{1}{n} \right) = o(1),$$

implying that (P1) holds whp as claimed.

Next we show that (P2) also holds whp. Note that if $\mathcal{C}_1$ and $\mathcal{C}_2$ are cycles in K_n of lengths l_1, l_2, respectively, sharing exactly one vertex then $|V(\mathcal{C}_1) \cup V(\mathcal{C}_2)| = l_1 + l_2 - 1$ and $|E(\mathcal{C}_1) \cup E(\mathcal{C}_2)| = l_1 + l_2$. Thus, using the union bound, we see that the probability that in G we have two cycles of lengths l_1 and l_2 which share exactly one vertex is at most $n^{l_1+l_2-1}p^{l_1+l_2} = \frac{(np)^{l_1+l_2}}{n}$. Moreover, letting $k := \frac{\ln n}{4 \ln(np)}$ and taking the union bound over all pairs of cycles of lengths at most k which share exactly one vertex, we see that the probability of (P2) not holding is at most

$$\sum_{l_1, l_2 = 1}^{k} \frac{(np)^{l_1+l_2}}{n} \leq \frac{k^2(np)^{2k}}{n} = o(1),$$

since $(np)^{2k} = \sqrt{n}$. Thus, we conclude that (P2) holds whp as claimed.

Finally we show that whp (*P*3) holds. Let $u, v \in V(G)$ and let $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4$ be any four distinct trails between u and v in K_n, each of length at most C. A moment's thought reveals that

$$|V(\mathcal{W}_1) \cup V(\mathcal{W}_2) \cup V(\mathcal{W}_3) \cup V(\mathcal{W}_4)| < |E(\mathcal{W}_1) \cup E(\mathcal{W}_2) \cup E(\mathcal{W}_3) \cup E(\mathcal{W}_4)| \le 4C.$$

Thus, using the union bound we see that the probability of (*P*3) not holding is at most

$$\sum_{l=1}^{4C} n^{l-1} p^l = \sum_{l=1}^{4C} \frac{(np)^l}{n} \le \frac{4C(10 \ln n)^{4C}}{n} = o(1).$$

We conclude that (*P*3) holds whp, completing the proof of the lemma. ∎

3.3. Properties of Graphs

The following simple lemma can be found, e.g., in Chapter 1 of [7].

Lemma 3.7. *Let $G = (V, E)$ be a graph. There exists $S \subseteq V$ such that $G[S]$ is a connected graph of minimum degree at least $|E|/|V|$.*

The next lemma follows from a simple application of Hall's Theorem (see e.g. the exercises of Chapter 2 in [7]).

Lemma 3.8. *Let $G = (V, E)$ be a bipartite graph with bipartition $V = A \cup B$, and let k be a natural number. Suppose that for every $I \subseteq A$ we have $|N(I)| \ge k|I|$. Then for every $i \in A$ there exists a subset $J_i \subseteq N(i)$ of size $|J_i| = k$ such that all the sets $(J_i)_{i \in A}$ are disjoint.*

A routine way to turn a non-Hamiltonian graph H that satisfies some expansion properties into a Hamiltonian graph is by using boosters. A *booster* is a non-edge e of H such that the addition of e to H creates a path which is longer than a longest path of H, or turns H into a Hamiltonian graph. In order to turn H into a Hamiltonian graph, we start by adding a booster e of H. If the new graph $H \cup \{e\}$ is not Hamiltonian then one can continue by adding a booster of the new graph. Note that after at most $|V(H)|$ successive steps the process must terminate and we end up with a Hamiltonian graph. The main point using this method is that it is well-known (for example, see [3]) that a non-Hamiltonian connected graph H with "good" expansion properties has many boosters. In the proof of Theorem 1 we use a similar notion of boosters, known as *e-boosters*.

Given a graph H and a pair $e \in \binom{V(H)}{2}$, consider a path P of $H \cup \{e\}$ of maximal length which contains e as an edge. A non-edge f of H is called an *e-booster* if $H \cup \{e, f\}$ contains a path Q which passes through e and which is longer than P or if $H \cup \{e, f\}$ contains a Hamilton cycle that uses e. The following lemma appears in [8] and shows that every connected and non-Hamiltonian graph G satisfying certain expansion properties has many *e-boosters* for every possible e.

Lemma 3.9. *Let H be a connected graph for which $|N_H(X) \setminus X| \ge 2|X| + 2$ holds for every subset $X \subseteq V(H)$ of size $|X| \le k$. Then, for every pair $e \in \binom{V(H)}{2}$ such that $H \cup \{e\}$ does not contain a Hamilton cycle which uses the edge e, the number of e-boosters for H is at least $\frac{1}{2}(k + 1)^2$.*

4. PROOF OF THEOREM 1

In this section we prove Theorem 1. In our proof we present a randomised algorithm which successively queries (about adjacency) carefully selected pairs of vertices in $\mathcal{G}(n, p)$, where $p \geq \frac{\ln n + \ln \ln n + w(1)}{n}$. We then show that whp the algorithm terminates by revealing a Hamilton cycle after only $n + o(n)$ positive answers. The algorithm is divided into five different phases, labeled I, II, III, IV and V.

We remark that we may and will assume throughout the proof that $p \leq \frac{10 \ln n}{n}$. Indeed, if $p > \frac{10 \ln n}{n}$ then we can use the algorithm $\mathrm{Alg}(p')$ which queries pairs of vertices with probability $p' = \frac{10 \ln n}{n}$ with a slight modification to obtain an algorithm $\mathrm{Alg}(p)$ which queries pairs of vertices with probability p. When a pair of vertices is queried by $\mathrm{Alg}(p')$, do it in two stages: first query it with probability $\frac{p'}{p}$ to decide whether $\mathrm{Alg}(p)$ should query this pair of vertices as well; if the answer is positive, then query it a second time with probability p. A pair of vertices which is queried by $\mathrm{Alg}(p')$ is considered to be an edge if and only if the answer to both questions is positive, and so this happens with probability $p' = \frac{p'}{p} \cdot p$. However, in the algorithm $\mathrm{Alg}(p)$ pairs of vertices are queried about adjacency with probability p. Finally, note crucially that the edges which are revealed by the algorithm $\mathrm{Alg}(p)$ a re exactly the same as the ones which are revealed by the algorithm $\mathrm{Alg}(p')$ and so, if the latter whp finds a Hamilton cycle after only $n + o(n)$ positive answers then so does the former.

In order to simplify notation in the proof, we work in the following setting. Throughout the algorithm we maintain a tripartition $R \cup W \cup B$ of the edge set of the complete graph with vertex set $V = [n]$. Edges in R, W, B are called respectively *red*, *white* and *blue*. A red edge represents an edge which has been queried successfully (and thus belongs to the exposed graph G), a white edge represents an edge which has not yet been queried and a blue edge represents an edge which has been queried unsuccessfully. During the algorithm we *recolour* some white edges. *Recolouring* a white edge means that with probability p we recolour it red (i.e., we move it from the set W to the set R), and otherwise we recolour it blue (i.e. we move it from the set W to the set B). All the recolourings are considered independent. At any point during the algorithm, the *red graph* (respectively, *white graph* and *blue grap h*) refers to the graph with vertex set V and edge set R (respectively, W and B). Moreover, if $u, v \in V$ then we say that v is a *red neighbour* (respectively, *white neighbour* and *blue neighbour*) of u if $uv \in R$ (respectively, $uv \in W$ and $uv \in B$). The algorithm starts with the tripartition $(R, W, B) = \left(\emptyset, \binom{V}{2}, \emptyset\right)$ and whp it ends with the red graph containing a Hamilton cycle while having only $n + o(n)$ edges.

During the algorithm, if R denotes the set of red edges at a certain point, and if at that point it is verified that any of the events below does not hold then we stop the algorithm:

N.1 We have $\Delta(R) \leq 40 \ln n$.
N.2 If none of the edges incident to a given vertex $v \in V$ are white (i.e. they were already recoloured before) then v has at least 2 red neighbours.
N.3 There are no two cycles in R of length at most $(\ln n)^{0.9}$ sharing exactly one vertex.
N.4 Between any two vertices $u, v \in V$ there are at most three trails of length at most 6 in R.

We remark that all of these events hold whp by $(P1)$ of Lemma 3.5 and by Lemma 3.6, and so we can assume throughout the proof that these properties always hold.

In Phases I–IV we consider a finite number of properties concerning the tripartition (R, W, B), which we need for later phases. These properties will be labeled according to the phase in which they are considered, in order to make it easier for the reader to find them in a later reference. For example, II.1(b) will be used to denote property 1(b) of Phase II. In each phase, we show that all the properties considered hold whp and so we may assume that they hold for later phases.

4.1. Outline of the Algorithm

Phase I: In this phase we use a modified version of the well known graph search algorithm Depth First Search (see e.g. [17]). Starting from a complete graph with all edges white, we use this algorithm to find a "long" red path $\mathcal{P}$ by recolouring red at most $n - 1$ white edges. Afterwards by recolouring red one more white edge between "short" initial and final segments of the path $\mathcal{P}$, we create a red cycle $\mathcal{C}_1$ of size $n - \Theta(n(\ln n)^{-0.45})$. This is done whilst ensuring some technical conditions needed for later phases.

Phase II: Let $\mathcal{U} := V \setminus V(\mathcal{C}_1)$. In this phase, starting from the partition $V = V(\mathcal{C}_1) \cup \mathcal{U}$, we recolour a random subset of white edges in $\mathcal{U}$ and partition $\mathcal{U}$ into three sets EXP_1, SMALL_1 and TINY. The set EXP_1 will be such that the minimum red degree inside it is $\Omega\left((\ln n)^{0.4}\right)$ and $|\mathcal{U} \setminus \mathrm{EXP}_1| = o(n/\ln n)$. Later, we recolour all the white edges between $\mathcal{U} \setminus \mathrm{EXP}_1$ and $V(\mathcal{C}_1)$. A partition $\mathcal{U} \setminus \mathrm{EXP}_1 = \mathrm{SMALL}_1 \cup \mathrm{TINY}$ is then obtained by letting SMALL_1 be the set of all vertices of "large" red degree into a "large" subset $\mathcal{M}_1$ of $V(\mathcal{C}_1)$. Finally, we recolour all the white edges inside $\mathcal{U}$ touching vertices of TINY. All of this is achieved by recolouring red only $o(n)$ edges during this phase.

Phase III: The goal of this phase is to "swallow" the vertices of TINY one at a time into the red cycle $\mathcal{C}_1$ obtained in Phase I. This is achieved by creating at each time a larger red cycle that contains a new vertex of TINY and some vertices of EXP_1, until a red cycle $\mathcal{C}_2$ such that $V(\mathcal{C}_1) \cup \mathrm{TINY} \subseteq V(\mathcal{C}_2)$ is obtained. We ensure that whp only $o(n)$ edges are recoloured red during this phase. At the end of this phase we get a partition of the vertex set $V = V(\mathcal{C}_2) \cup \mathrm{EXP}_2 \cup \mathrm{SMALL}_2$ where $\mathrm{EXP}_2 \subseteq \mathrm{EXP}_1$ is a "good expander" and $\mathrm{SMALL}_2 \subseteq \mathrm{SMALL}_1$ is a set of vertices of "large" degree into a "large" subset $\mathcal{M}_2 \subseteq \mathcal{M}_1$ of $V(\mathcal{C}_2)$.

Phase IV: The goal of this phase is to "swallow" the vertices of SMALL_2 one at a time into the red cycle $\mathcal{C}_2$ obtained in Phase III. This is achieved by creating at each time a larger red cycle that contains a new vertex of SMALL_2, until a red cycle $\mathcal{C}_3$ such that $V(\mathcal{C}_2) \cup \mathrm{SMALL}_2 \subseteq V(\mathcal{C}_3)$ is obtained. We ensure that whp only $o(n)$ edges are recoloured red during this phase. Moreover, at the end we get a partition of the vertex set $V = V(\mathcal{C}_3) \cup \mathrm{EXP}_3$ where $\mathrm{EXP}_3 \subseteq \mathrm{EXP}_2$ is a "good expander".

Phase V: In this phase we create a Hamilton cycle in the red graph by merging the red cycle $\mathcal{C}_3$ with the set EXP_3. We start by recolouring red $\Theta(1)$ white edges in EXP_3 in order to make the red graph in EXP_3 become connected. Afterwards, we consider two adjacent vertices v, w in the red cycle $\mathcal{C}_3$ which have large white degree onto EXP_3. By recolouring edges between v, w and EXP_3 we then find $x, y \in \mathrm{EXP}_3$ such that vx and wy are coloured red. Finally, using the fact that the red graph in EXP_3 is a connected expander we recolour red at most $|\mathrm{EXP}_3|$ edges in order to find a red Hamilton path from x to y inside the set EXP_3. This path together with the red path $\mathcal{C}_3 \setminus \{vw\}$ and the red edges vx and wy then provides the desired red Hamilton cycle in V. All of this is achieved by recolouring red only $o(n)$ edges during this phase.

4.2. Phase I

The algorithm for this phase is divided into two stages. In Stage 1 we use a randomised version of the Depth First Search exploration algorithm to obtain a "long" red path $\mathcal{P}$. In Stage 2 of the algorithm we use the red path $\mathcal{P}$ to find a red cycle $\mathcal{C}_1$ of size $n - \Theta(n(\ln n)^{-0.45})$, by recolouring red exactly one white edge between an initial and a final interval of $\mathcal{P}$.

(Stage 1) In this stage we run a (slightly modified version of) DFS algorithm on the vertex set $V = [n]$. Recall that DFS is an algorithm to explore all the connected components of a graph $G = (V, E)$, while finding a spanning tree of each of them in the following way. It maintains a tripartition (C, A, U) of the vertex set V, letting C be the set of vertices whose exploration is complete, U be the set of unvisited vertices and $A = V \setminus (C \cup U)$ (the vertices which are "active"), where the vertices of A are kept in a stack (last in first out). It starts with $C = A = \emptyset$ and $U = V$ and runs until $A \cup U = \emptyset$. In each round of the algorithm, if $A \neq \emptyset$, then it identifies the last vertex $a \in A$, and starts to query U for neighbours of a, according to the natural ordering on them. If such a neighbour exists, let $u \in U$ be the first such neighbour, then the algorithm moves u from U to A. Otherwise, the algorithm moves a from A to C. If $A = \emptyset$, then the algorithm moves the first (according to the natural ordering) vertex in U to A.

The following properties of DFS will be relevant for us and follow immediately from its description.

- (*O*1) At any point during the algorithm, it is true that all the pairs between C and U have been queried, and none of them are edges of G.
- (*O*2) Throughout the algorithm, the explored graph is a forest.
- (*O*3) At each round of the algorithm exactly one vertex moves, either from U to A or from A to C.
- (*O*4) The set A always spans a path.

For our purposes, we run DFS on a random input in the following way. At each round of the algorithm let e be the relevant pair waiting to be queried. We first decide with probability $q = (\ln n)^{-1/2}$ whether we want to recolour this pair. If yes, we recolour the pair e red with probability p (and consider it as an edge for DFS) and blue otherwise (and consider it as a non-edge for DFS). If no, we consider e to be a non-edge for DFS. All these actions happen independently at random. We stop this algorithm as soon as $|C| = |U|$ (and not when $A \cup U = \emptyset$), with the set A spanning a red path $\mathcal{P} = a_1 a_2 \ldots a_{|A|}$ of size $|A| = n - |C| - |U|$. Claim 4.1 below ensures that whp at the end of the routine one has $k := n(\ln n)^{-0.45} \geq |C| = |U|$, so that $|A| \geq n - 2k$. We assume henceforth that this holds and we proceed to Stage 2.

(End of Stage 1)

(Stage 2) Consider the intervals $I_1 := \{a_1, a_2, \ldots, a_k\}$ and $I_2 := \{a_{n-3k+1}, a_{n-3k+2}, \ldots, a_{n-2k}\}$ of $\mathcal{P}$ and let D be the set of white edges between I_1 and I_2. Following a fixed ordering of the set D, at each step first decide with probability q whether this edge should be recoloured. If yes, then recolour it red with probability p (and blue otherwise). All these actions are taken independently. The stage is completed successfully when the first red edge from D is discovered. Claim 4.1 below ensures that whp there will be such an edge. Assuming this, let $a_i a_j \in D$ be this edge, where $a_i \in I_1$ and $a_j \in I_2$. The algorithm terminates by setting $\mathcal{C}_1$

to be the red cycle formed by the vertices $a_i, a_{i+1}, \ldots, a_j$ with the red edges of $\mathcal{P}$ together with the red edge $a_i a_j$.

(End of Stage 2)

In the next claim we prove that some properties which are assumed in the algorithm hold whp.

Claim 4.1.   *The following properties hold whp:*

(i) *At the end of Stage 1 we have $|C| = |U| \le k := n(\ln n)^{-0.45}$.*
(ii) *During Stage 2, at least one edge in D is recoloured red.*

Proof of Claim 4.1.   In order to prove (*i*), note that during the algorithm, each pair which has been queried has been recoloured red with probability $pq \ge n(\ln n)^{-1/2}$, independently at random. Moreover, it follows from (*O*1) that none of the pairs between C and U have been coloured red. Note that by exploring all the edges of the graph we obtain a graph which is distributed as $\mathcal{G}(n, pq)$. Now, since

$$ pq \ge \frac{(\ln n)^{0.5}}{n} \ge \frac{1.35(\ln n)^{0.45} \ln \ln n}{n} = \frac{3 \ln (n/k)}{k} $$

it follows from Lemma 3.4 that this graph is whp k-pseudorandom and therefore, unless $|C| = |U| \le k$, there must be red edges between these sets.

Property (*ii*) follows from a similar argument.   ∎

Assuming that the properties of Claim 4.1 hold, denoting by R_1, W_1 and B_1 the sets of red, white and blue edges, respectively, at the end of this phase's algorithm, we show that whp the following technical conditions hold:

I.1 the graph R_1 contains a cycle $\mathcal{C}_1$ of size t, where $2n(\ln n)^{-0.45} < n-t \le 4n(\ln n)^{-0.45}$.
I.2 at most n white edges are recoloured red during this phase, i.e. $|R_1| \le n$.
I.3 for every $v \in V$ we have $d_{W_1}(v, V(\mathcal{C}_1)) \ge n - 5n(\ln n)^{-0.45} = (1 - o(1))n$.
I.4 letting $\mathcal{U} := V \setminus V(\mathcal{C}_1)$, we have $d_{R_1 \cup B_1}(v, \mathcal{U}) \le 4n(\ln n)^{-0.5} = o(|\mathcal{U}|)$ for every $v \in V$.

Claim 4.2.   *Properties I.1–I.4 hold (whp).*

Proof of Claim 4.2.   Note that the red cycle $\mathcal{C}_1$ formed by the vertices $a_i, a_{i+1}, \ldots, a_j$ obtained at the end of the algorithm has size $t := j - i + 1$. Moreover, assuming Claim 4.1, since $1 \le i \le k$ and $n - 3k + 1 \le j \le n - 2k$ we obtain that $n - 4k < n - 4k + 2 \le j - i + 1 = t \le n - 2k$. This settles I.1.

Since a forest in a graph of order n has less than n edges, it is clear by (O2) that in Stage 1 of the algorithm less than n edges are recoloured red. Moreover, since in Stage 2 only one edge is recoloured red, it follows that in the whole phase at most n edges are recoloured red, settling I.2.

Next, note that $R_1 \cup B_1$ can be seen as part of a graph distributed as $\mathcal{G}(n, q)$. Thus, by (P1) of Lemma 3.5 it follows that I.4 holds whp. Furthermore, by I.1 and I.4 we have that for every $v \in V$

$$d_{W_1}(v, V(\mathcal{C}_1)) \geq |V(\mathcal{C}_1)| - 1 - d_{R_1 \cup B_1}(v, V(\mathcal{C}_1))$$
$$\geq n - 4n(\ln n)^{-0.45} - 1 - \Delta(R_1 \cup B_1) \geq n - 5n(\ln n)^{-0.45}$$

which settles I.3. ∎

We have shown that whp at the end of this phase all the properties I.1-I.4 hold. We shall assume henceforth that they hold for the sets R_1, W_1 and B_1 obtained after this Phase.

4.3. Phase II

In this phase we partition $\mathcal{U}$ into three sets $\mathcal{U} = \text{EXP}_1 \cup \text{SMALL}_1 \cup \text{TINY}$ as described in the outline. The algorithm for this phase is divided into the following three stages.

(Stage 1) Let F be a subset of $W_1[\mathcal{U}]$ obtained by independently adding each edge in $W_1[\mathcal{U}]$ to F with probability $q' := 6(\ln n)^{-0.15}$. Claim 4.3 below ensures that whp $\frac{2}{3}|\mathcal{U}|q' \leq \delta(F) \leq \Delta(F) \leq \frac{4}{3}|\mathcal{U}|q'$. Assuming this, recolour all the edges in F.

Taking the set F at random serves two purposes. Firstly, it ensures that not too many edges are recoloured red in this phase. Secondly, it leaves a certain amount of randomness for the edges in $W_1[\mathcal{U}] \setminus F$, which will be used in later phases.

(End of Stage 1)

(Stage 2) Let R^1 denote the set of red edges after Stage 1 and set $T_0 := \{v \in \mathcal{U} : d_{R^1 \setminus R_1}(v, \mathcal{U}) < \frac{1}{3}|\mathcal{U}|pq'\}$. Claim 4.3 ensures that whp $|T_0| \leq ne^{-(\ln n)^{0.4}}$. Assuming this, starting with $i = 0$, as long as there exists a vertex $v \in \mathcal{U} \setminus T_i$ having at least 3 red neighbours in T_i, choose such a vertex v and set $T_{i+1} := T_i \cup \{v\}$. Let T_f be the last set obtained in this process. Claim 4.3 below shows that whp $f \leq |T_0|$. Assuming that, define $\text{EXP}_1 := \mathcal{U} \setminus T_f$.

Note that every vertex $v \in \text{EXP}_1$ has at most two red neighbours in T_f. Thus, by I.1 for every $v \in \text{EXP}_1$ we have $d_{R^1 \setminus R_1}(v, \text{EXP}_1) \geq \frac{1}{3}|\mathcal{U}|pq' - 2 \geq 3(\ln n)^{0.4}$.

(End of Stage 2)

(Stage 3) Let $P_1, \ldots, P_m$ be vertex disjoint subpaths of the red cycle $\mathcal{C}_1$, each of size 100, where $m = \left\lfloor \frac{|V(\mathcal{C}_1)|}{100} \right\rfloor$, and set $\mathcal{M}_1$ to be the union of all the vertices in the subpaths $P_1, \ldots, P_m$ which are not endpoints. These paths will be used in later phases for technical reasons to ensure that certain vertices are not neighbours on the red cycle $\mathcal{C}_1$.

Next, recolour all the white edges between T_f and $V(\mathcal{C}_1)$, set SMALL_1 to be the set of all vertices in T_f with at least $(\ln n)^{0.5}$ red neighbours in $\mathcal{M}_1$ and set $\text{TINY} := T_f \setminus \text{SMALL}_1$. The algorithm for this phase ends by recolouring all the edges in $W_1[\mathcal{U}] \setminus F$ touching at least one vertex in TINY.

(End of Stage 3)

In the next claim we prove that some properties which are assumed in the algorithm hold whp.

Claim 4.3. *All of the following properties hold whp:*

(i) *In Stage 1 one has $\frac{2}{3}|\mathcal{U}|q' \leq \delta(F) \leq \Delta(F) \leq \frac{4}{3}|\mathcal{U}|q'$.*
(ii) *In Stage 2 one has $|T_0| \leq ne^{-(\ln n)^{0.4}}$ and $f \leq |T_0|$.*

Proof. First we prove that (*i*) holds whp. For estimating $\delta(F)$, note that by I.4 we have $d_{W_1}(v, \mathcal{U}) = (1 - o(1))|\mathcal{U}|$ for every $v \in \mathcal{U}$. Thus, by Chernoff's inequality we conclude that for a vertex $v \in \mathcal{U}$ we have

$$\Pr\left[d_F(v) < \frac{2}{3}|\mathcal{U}|q'\right] \leq e^{-\Theta(|\mathcal{U}|q')} = e^{-\Theta(n(\ln n)^{-0.6})}.$$

Now, by applying the union bound over all vertices of $v \in \mathcal{U}$, it follows that whp $\delta(F) \geq \frac{2}{3}|\mathcal{U}|q'$. In a similar way, we can also conclude that whp $\Delta(F) \leq \frac{4}{3}|\mathcal{U}|q'$. This settles (*i*).

Assuming (*i*), we show next that whp $|T_0| \leq ne^{-(\ln n)^{0.4}}$. Indeed, by Chernoff's inequality we see that for every $v \in \mathcal{U}$:

$$\Pr\left[v \in T_0\right] = \Pr\left[d_{R^1 \setminus R_1}(v, \mathcal{U}) < \frac{1}{3}|\mathcal{U}|pq'\right] \leq \Pr\left[\text{Bin}\left(\frac{2}{3}|\mathcal{U}|q', p\right) \leq \frac{1}{3}|\mathcal{U}|pq'\right] \leq e^{-(\ln n)^{0.4}},$$

where in the last inequality we used the fact that $|\mathcal{U}|pq' \geq 12(\ln n)^{0.4}$ by I.1. Therefore, the expected value of $|T_0|$ is at most $|\mathcal{U}|e^{-(\ln n)^{0.4}}$. Hence, using Markov's inequality we obtain that whp $|T_0| \leq ne^{-(\ln n)^{0.4}}$, as desired.

Finally, we show that if $|T_0| \leq ne^{-(\ln n)^{0.4}}$ then whp $f \leq |T_0|$. Suppose that $f > |T_0|$. Then, note that the set $T_{|T_0|}$ contains precisely $2|T_0|$ vertices and induces at least $3|T_0| = 1.5|T_{|T_0|}|$ red edges, since for every $i \leq |T_0|$ the vertex in $T_i \setminus T_{i-1}$ has at least 3 red neighbours in T_{i-1}. By (P2) of Lemma 3.5 (with $c = 1.5$), since $p \leq \frac{10\ln n}{n}$, we know that whp every subset of vertices X of size

$$|X| \leq \left(\frac{1}{\ln n} \cdot \frac{3^{1.5}}{e^{2.5}np^{1.5}}\right)^2 \leq \frac{3^3}{e^5 \cdot 10^3} \cdot \frac{n}{(\ln n)^5}$$

induces less than $1.5|X|$ red edges. Since $|T_{|T_0|}| = 2|T_0| \leq 2ne^{-(\ln n)^{0.4}} = o(n(\ln n)^{-5})$ it follows that whp $f \leq |T_0|$. Thus, we conclude that whp (*ii*) holds as claimed. ∎

Assuming that the properties of Claim 4.3 hold, denoting by R_2, W_2 and B_2 the sets of red, white and blue edges at the end of this phase, we show that whp the following technical conditions hold:

II.1 Properties of the set EXP_1:
   (a) $|\text{EXP}_1| \geq \left(1 - \frac{1}{(\ln n)^3}\right)|\mathcal{U}| \geq (2 - o(1))n(\ln n)^{-0.45}$.
   (b) for every $v \in \text{EXP}_1$ we have $d_{R_2 \setminus R_1}(v, \text{EXP}_1) \geq 3(\ln n)^{0.4}$.
   (c) for every set $U \subseteq \text{EXP}_1$ of size $|U| \geq \left(1 - \frac{1}{\ln n}\right)|\text{EXP}_1|$:
      i. if $S \subseteq U$ is a set such that $(R_2 \setminus R_1)[S]$ has minimum degree at least $(\ln n)^{0.4}$ then for any set $X \subseteq S$ of size $|X| \leq \frac{1}{6000}n(\ln n)^{-0.45}$ we have $|N_{R_2[S]}(X) \cup X| \geq 5|X|$.
      ii. there is a set $S \subseteq U$ of size $|S| \geq \frac{1}{240}n(\ln n)^{-0.45}$ such that $R_2[S]$ has diameter at most $2\ln n$.
   (c) Let $\mathcal{H}$ be the collection of all connected graphs H whose vertex set $V(H)$ is a subset of EXP_1 and whose edge set is of the form $K_1 \cup K_2$ where $K_1 = R_2[V(H)]$ is such that $|N_{K_1}(X) \cup X| \geq 5|X|$ for every $X \subseteq V(H)$ of size $|X| \leq \frac{1}{6000}n(\ln n)^{-0.45}$ and $K_2 \subseteq \binom{V(H)}{2}$ is a set of size $|K_2| \leq |V(H)| + 24000$. Then whp for every $H \in \mathcal{H}$ and every $e = xy \in \binom{V(H)}{2}$, if the graph $H \cup \{e\}$ does not

contain a Hamilton cycle which uses the edge e, then the number of e-boosters for H in the set $W_2[V(H)]$ is at least $10^{-8}n^2(\ln n)^{-0.9}$.

II.2 Properties of the set SMALL_1:

    (a) $|\text{SMALL}_1| \leq 2ne^{-(\ln n)^{0.4}}$.

    (b) $d_{R_2}(u, \mathcal{M}_1) \geq (\ln n)^{0.5}$ for every vertex $u \in \text{SMALL}_1$.

II.3 Properties of the set TINY:

    (a) $|\text{TINY}| \leq n^{0.04}$.

    (b) the event "there is no red path of size at most 1000 between any two vertices of TINY after recolouring all the edges in W_2" holds whp.

    (c) $d_{R_2}(u) \geq 2$ for every vertex $u \in \text{TINY}$.

II.4 Only $o(n)$ edges in W_1 are recoloured red during this phase.

It is clear that properties II.1(a), II.1(b), II.2(a) and II.2(b) follow immediately from the algorithm. Moreover, note that at the end of this phase all the edges touching vertices in TINY are either red or blue. Thus, since the event N.2 holds whp it follows that property II.3(c) also holds whp. The next few claims ensure that the remaining properties all hold whp.

Claim 4.4. *Property II.1(c) holds whp.*

Proof of Claim 4.4. Consider the following two events:

    (E1) $e_{R_2 \setminus R_1}(S) < C(\ln n)^{0.4}|S|$ for any set $S \subseteq \text{EXP}_1$ of size $|S| \leq \frac{C}{120}n(\ln n)^{-0.45}$, for $C \in \left\{\frac{1}{10}, \frac{1}{2}\right\}$.

    (E2) $e_{R_2}(X, Y) > 0$ for every pair of disjoint sets $X, Y \subseteq \mathcal{U}$ each of size at least $\frac{1}{6000}n(\ln n)^{-0.45}$.

We shall prove that these two events all hold whp and then that II.1(c) holds whenever N.1, II.1(b), (E1) and (E2) all hold.

The event (E1) holds whp according to (P3) of Lemma 3.5 (with C in the Lemma being $C(\ln n)^{0.4}$ here) since in Stage 1 every edge in $W_1[\mathcal{U}]$ is recoloured red independently with probability $pq' \leq \frac{60(\ln n)^{0.85}}{n}$. Next we show that the event (E2) also holds whp. By I.4, after Phase I we have $e_{W_1}(X, Y) \geq (1 - o(1))s^2$ for any pair of disjoint sets $X, Y \subseteq \mathcal{U}$ each of size at least $s := \frac{1}{6000}n(\ln n)^{-0.45}$. Thus, by applying Chernoff's bound and the union bound we obtain

$$\Pr\left[\exists \text{ such sets } X, Y \text{ with } e_{R_2}(X, Y) = 0\right] \leq \binom{n}{s}^2 (1 - pq')^{(1-o(1))s^2} \leq \left(\frac{en}{s}\right)^{2s} e^{-(1-o(1))pq's^2}$$

$$\leq e^{O\left(n(\ln n)^{-0.45}\ln\ln n\right)} e^{-\Omega\left(n(\ln n)^{-0.05}\right)} = o(1).$$

implying that (E2) holds whp as desired.

Suppose now that N.1, II.1(b), (E1) and (E2) all hold. Let $U \subseteq \text{EXP}_1$ be a subset of size $|U| \geq \left(1 - \frac{1}{\ln n}\right)|\text{EXP}_1|$ and denote $R_2 \setminus R_1$ simply by R'.

Suppose $S \subseteq U$ is such that $R'[S]$ has minimum degree at least $(\ln n)^{0.4}$ and that there exists a set $X \subseteq S$ of size $|X| \leq \frac{1}{6000}n(\ln n)^{-0.45}$ such that $|N_{R'[S]}(X) \cup X| < 5|X|$. Since $R'[S]$ has minimum degree at least $(\ln n)^{0.4}$ it follows that

$$e_{R'}(N_{R'[S]}(X) \cup X) \geq \frac{1}{2}(\ln n)^{0.4}|X| \geq \frac{1}{10}(\ln n)^{0.4}|N_{R'[S]}(X) \cup X|.$$

Thus, by (E1) we have $|N_{R'[S]}(X) \cup X| \geq \frac{1}{1200}n(\ln n)^{-0.45}$, which leads to $|X| > \frac{1}{6000}n(\ln n)^{-0.45}$, contradicting our choice of X. We conclude that for every set $X \subseteq S$ of size $|X| \leq \frac{1}{6000}n(\ln n)^{-0.45}$ we have $|N_{R_2[S]}(X) \cup X| \geq |N_{R'[S]}(X) \cup X| \geq 5|X|$. This settles i. of II.1(c).

Observe now that by N.1 and II.1(b) we have

$$e_{R'}(U) \geq e_{R'}(\text{EXP}_1) - \Delta(R') \cdot |\text{EXP}_1 \setminus U|$$
$$\geq \frac{3}{2}(\ln n)^{0.4}|\text{EXP}_1| - 40 \ln n \cdot \frac{|\text{EXP}_1|}{\ln n} \geq (\ln n)^{0.4}|U|.$$

Thus, by Lemma 3.7 there exists a set $S \subseteq U$ such that $R'[S]$ is a connected graph with minimum degree at least $(\ln n)^{0.4}$. In particular, we have

$$e_{R'}(S) \geq \frac{1}{2}(\ln n)^{0.4}|S|$$

and so by (E1) it follows that $|S| \geq \frac{1}{240}n(\ln n)^{-0.45}$.

For $z \in S$, set $N^0(z) := \{z\}$ and, for $i \geq 1$, define $N^i(z) := N_{R_2[S]}(N^{i-1}(z)) \cup N^{i-1}(z)$. Note crucially that every vertex in $N^i(z)$ is at distance at most i of z in $R_2[S]$ and that, by the above, we have $|N^i(z)| \geq 5|N^{i-1}(z)|$, provided $|N^{i-1}(z)| \leq \frac{1}{6000}n(\ln n)^{-0.45}$. Thus, if we take $\ell := \log_5\left(\frac{1}{6000}n(\ln n)^{-0.45}\right)$, we see that $|N^\ell(z)| \geq \frac{1}{6000}n(\ln n)^{-0.45}$ for any $z \in S$. Now, let $x, y \in S$ be distinct. Note that if $N^\ell(x) \cap N^\ell(y) \neq \emptyset$ then x is at distance at most 2ℓ from y in $R_2[S]$. If instead we have $N^\ell(x) \cap N^\ell(y) = \emptyset$ then by (E2) there is at least one edge in R_2 between $N^\ell(x)$ and $N^\ell(y)$, implying that x and y are at distance at most $2\ell + 1$ in $R_2[S]$. Since $2\ell + 1 < 2 \ln n$, this settles ii. of II.1(c). We conclude that II.1(c) holds whp as claimed. ∎

Claim 4.5. *Property II.1(d) holds whp.*

Proof. Note first that by Lemma 3.9, given such an $H \in \mathcal{H}$ and $e \in \binom{V(H)}{2}$, the number of e-boosters for H in $\binom{V(H)}{2}$ is at least $\frac{1}{72 \cdot 10^6}n^2(\ln n)^{-0.9}$. Moreover, since e-boosters of H are not edges of H and since all the edges in $R_2[V(H)]$ are edges of H, it follows that every e-booster for H is either in $B_2[V(H)]$ or in $W_2[V(H)]$. Note that by properties I.4 and I.1 we have

$$e_{B_1}(V(H)) \leq \frac{1}{2} \cdot 4n(\ln n)^{-0.5} \cdot |\mathcal{U}| \leq 8n^2(\ln n)^{-0.95} = o\left(n^2(\ln n)^{-0.9}\right).$$

Furthermore, observe that for every e-booster of H which is not in $B_1[V(H)]$, the probability that it is in $W_2[V(H)]$ is at least the probability that it does not belong to F and so at least $1 - q' = 1 - o(1)$. Moreover, it is clear that the latter events are independent. Thus, the probability that less than $10^{-8}n^2(\ln n)^{-0.9}$ e-boosters for H are in the set $W_2[V(H)]$ is at most

$$\Pr\left[\text{Bin}\left((1 - o(1))\frac{1}{72 \cdot 10^6}n^2(\ln n)^{-0.9}, 1 - o(1)\right) < (1 - o(1))10^{-8}n^2(\ln n)^{-0.9}\right]$$
$$\leq e^{-\Theta(n^2(\ln n)^{-0.9})}$$

by Chernoff's bound (Lemma 3.1).

Suppose now that $|R_2| \leq 20n \ln n$. Assuming this, it is clear that any graph in $\mathcal{H}$ has at most $20n \ln n + n$ edges and so

$$|\mathcal{H}| \leq \sum_{i=0}^{20n \ln n+n} \binom{n^2}{i} \leq (20n \ln n + n + 1) \binom{n^2}{20n \ln n + n} \leq e^{100n(\ln n)^2}.$$

Thus, using the union bound we see that the probability that, for some $H \in \mathcal{H}$ and some $e = xy \in \binom{V(H)}{2}$ such that the graph $H \cup \{e\}$ does not contain a Hamilton cycle which uses the edge e, the number of e-boosters for H in the set $W_2[V(H)]$ is less than $10^{-8}n^2(\ln n)^{-0.9}$, conditioning on the fact that $|R_2| \leq 20n \ln n$, is at most

$$e^{100n(\ln n)^2} n^2 e^{-\Theta(n^2(\ln n)^{-0.9})} = o(1).$$

Since whp we have $|R_2| \leq 20n \ln n$ according to N.1, the claim follows.   ∎

Claim 4.6.   *Properties II.3(a) and II.3(b) hold whp.*

Proof of Claim 4.6.   Note, by the definition of the set $\mathcal{M}_1$, that $|\mathcal{M}_1| \geq 0.98|V(\mathcal{C}_1)| - 100$ and recall that by I.3, at the end of Phase I, for every $v \in V$ we have $d_{W_1}(v, V(\mathcal{C}_1)) \geq n - 5n(\ln n)^{-0.45}$. Therefore, for every $v \in V$ we have (say) $d_{W_1}(v, \mathcal{M}_1) \geq 0.97n$. Note that for a vertex $v \in T_f$ we have $d_{R_2 \setminus R_1}(v, \mathcal{M}_1) \sim \text{Bin}(d_{W_1}(v, \mathcal{M}_1), p)$. Thus, since $\ln n \leq np \leq 10 \ln n$, it follows that for any vertex $v \in T_f$:

$$\Pr[v \in \text{TINY}] \leq \Pr[\text{Bin}(0.97n, p) \leq (\ln n)^{0.5}]$$

$$\leq \sum_{i=0}^{(\ln n)^{0.5}} \binom{n}{i} p^i (1-p)^{0.97n-i} \leq (1-p)^{0.97n} \sum_{i=0}^{(\ln n)^{0.5}} \left(\frac{enp}{i(1-p)}\right)^i$$

$$\leq e^{-0.97 \ln n}((\ln n)^{0.5} + 1)(2enp)^{(\ln n)^{0.5}} \leq n^{-0.96},$$

and therefore $\mathbb{E}[|\text{TINY}|] \leq |T_f| n^{-0.96} \leq 2n^{0.04} e^{-(\ln n)^{0.4}}$ (recall that whp $|T_f| \leq 2ne^{-(\ln n)^{0.4}}$). Therefore, we conclude by Markov's inequality that whp $|\text{TINY}| \leq n^{0.04}$, settling II.3(a).

Now we show that II.3(b) also holds whp. Let R^2 and W^2 denote the sets of red and white edges after Stage 2. We assume throughout that $\Delta(R^2) \leq 40 \ln n$ (which holds whp by N.1). Suppose that in Stage 3 instead of only recolouring all the edges in W^2 between T_f and $V(\mathcal{C}_1)$ we had decided to recolour all the edges in W^2. Let R denote the set of red edges after this recolouring process. We would like to stress at this point that the edges in $R^2 \subseteq R$ are considered to be fixed and only the edges in W^2 are regarded as being randomly and independently assigned to R with probability p.

Let $\mathcal{P}$ be the set of all paths in K_n of size at most 1000. For each $P \in \mathcal{P}$ consider the indicator random variable X_P of the event that P is a path in the graph formed by the edges in R. Finally, let $X = \sum_{P \in \mathcal{P}} X_P$ denote the total number of paths in $\mathcal{P}$ which are paths in R. Note that for each $v \in V$, the number of paths in $\mathcal{P}$ starting with v which are also paths in R is at most $\Delta(R) + \Delta(R)^2 + \ldots + \Delta(R)^{1000} \leq 1000 \cdot \Delta(R)^{1000}$. Thus, it is clear that $X \leq 1000n \cdot \Delta(R)^{1000}$ and so we have

$$\mathbb{E}\left[X|\Delta(R) \leq (\ln n)^2\right] \leq 1000n(\ln n)^{2000} \leq n^{1.1}.$$

Moreover, note that as $\Delta(R^2) \leq 40 \ln n$ we have by Lemma 3.2:

$$\Pr\left[\Delta(R) > (\ln n)^2\right] \leq \Pr\left[\Delta(R \setminus R^2) > 0.5(\ln n)^2\right] \leq \sum_{v \in V} \Pr\left[d_{R \setminus R^2}(v) > 0.5(\ln n)^2\right]$$

$$\leq n \Pr\left[\text{Bin}(n, p) > 0.5(\ln n)^2\right] \leq n \left(\frac{10e \ln n}{0.5(\ln n)^2}\right)^{0.5(\ln n)^2} \leq e^{-(\ln n)^2}.$$

For every pair of vertices $\{u, v\} \subseteq T_f$ let $\mathcal{P}_{u,v} \subseteq \mathcal{P}$ be the collection of paths in K_n of size at most 1000 with endpoints u and v. For every $u, v \in T_f$ and $P \in \mathcal{P}_{u,v}$ consider now the families

$$\mathcal{A}_{u,v} := \{A \subseteq W^2 : u, v \in \text{TINY if } R = R^2 \cup A\} \text{ and } \mathcal{B}_P$$
$$:= \{B \subseteq W^2 : E(P) \subseteq R \text{ if } R = R^2 \cup B\}.$$

Observe that the families $\mathcal{A}_{u,v}$ and $\mathcal{B}_P$ are monotone decreasing and monotone increasing in the universe W^2, respectively. Furthermore, note crucially that the event "$u, v \in \text{TINY}$" is exactly the event "$R \setminus R^2 \in \mathcal{A}_{u,v}$" and that the event "$E(P) \subseteq R$" is exactly the event "$R \setminus R^2 \in \mathcal{B}_P$". Since each edge in W^2 is in $R \setminus R^2$ independently with probability p it follows from Theorem 6.3.2 of [1] that for every $u, v \in T_f$ and $P \in \mathcal{P}_{u,v}$ we have:

$$\Pr\left[u, v \in \text{TINY and } E(P) \subseteq R\right] \leq \Pr\left[u, v \in \text{TINY}\right] \cdot \Pr\left[E(P) \subseteq R\right].$$

Thus, using the union bound and the estimates above, the probability that there exist $u, v \in \text{TINY}$ and $P \in \mathcal{P}_{u,v}$ for which $E(P) \subseteq R$ is at most

$$\sum_{\{u,v\} \subseteq T_f} \sum_{P \in \mathcal{P}_{u,v}} \Pr\left[u, v \in \text{TINY and } E(P) \subseteq R\right]$$

$$\leq \sum_{\{u,v\} \subseteq T_f} \sum_{P \in \mathcal{P}_{u,v}} \Pr\left[u, v \in \text{TINY}\right] \cdot \Pr\left[E(P) \subseteq R\right]$$

$$\leq \sum_{\{u,v\} \subseteq T_f} \sum_{P \in \mathcal{P}_{u,v}} \left(n^{-0.96}\right)^2 \cdot \left(\Pr\left[E(P) \subseteq R | \Delta(R) \leq (\ln n)^2\right] + \Pr\left[\Delta(R) > (\ln n)^2\right]\right)$$

$$\leq n^{-1.92} \sum_{\{u,v\} \subseteq T_f} \sum_{P \in \mathcal{P}_{u,v}} \left(\mathbb{E}\left[X_P | \Delta(R) \leq (\ln n)^2\right] + e^{-(\ln n)^2}\right)$$

$$\leq n^{-1.92} \left(\mathbb{E}\left[X | \Delta(R) \leq (\ln n)^2\right] + e^{-(\ln n)^2} |\mathcal{P}|\right)$$

$$\leq n^{-1.92} \left(n^{1.1} + e^{-(\ln n)^2} 1000 n^{1000}\right) = o(1),$$

where in the second inequality we used the fact that the events $u \in \text{TINY}$ and $v \in \text{TINY}$ are independent. This settles II.3(b). ∎

Claim 4.7. *Property II.4 holds whp.*

Proof of Claim 4.7. In this phase edges are recoloured once in Stage 1 and in two instances in Stage 3. We shall bound the number of edges recoloured red in these three instances.

In Stage 1 we recoloured all the edges of F, of which there are at most $\frac{1}{2}\Delta(F)\cdot|\mathcal{U}|$. Since by the algorithm we have whp that $\Delta(F)\leq\frac{4}{3}|\mathcal{U}|q'$ we conclude using Chernoff's bounds and I.1 that whp the number of edges recoloured red in Stage 1 is at most

$$\Delta(F)\cdot|\mathcal{U}|\cdot p\leq\frac{4}{3}|\mathcal{U}|q'\cdot|\mathcal{U}|p=O\left(\frac{n}{(\ln n)^{0.05}}\right)=o(n).$$

In Stage 3 we recoloured all the white edges between T_f and $V(\mathcal{C}_1)$ and all the white edges touching vertices of TINY. Since, from the algorithm, $|T_f|\leq 2|T_0|\leq 2ne^{-(\ln n)^{0.4}}$, we conclude from Chernoff's bounds that whp the number of edges between T_f and $V(\mathcal{C}_1)$ which are recoloured red is at most

$$2|T_f|\cdot|V(\mathcal{C}_1)|\cdot p\leq 4ne^{-(\ln n)^{0.4}}\cdot n\cdot p=O\left(\frac{n\ln n}{e^{(\ln n)^{0.4}}}\right)=o(n).$$

Finally, since there are at most $|\text{TINY}|n$ edges touching vertices of TINY, we can use Chernoff's bounds together with II.3(a) to conclude that whp the number of edges touching vertices of TINY which are recoloured red is at most

$$2|\text{TINY}|\cdot n\cdot p=O\left(n^{0.04}\ln n\right)=o(n).$$

Thus, whp $o(n)$ edges in W_1 are recoloured red in this phase, proving the claim. ∎

We have shown that whp at the end of this phase all of the properties II.1–II.4 hold. We shall assume henceforth that all these properties hold for the sets R_2, W_2 and B_2.

4.4. Phase III

In this phase, we want to find a red cycle $\mathcal{C}_2$ containing $\text{TINY}\cup V(\mathcal{C}_1)$ as described in the outline. Recall that in Stage 3 of Phase II we nearly decomposed the red cycle $\mathcal{C}_1$ into $m=\left\lfloor\frac{|V(\mathcal{C}_1)|}{100}\right\rfloor$ red paths $P_1,\ldots,P_m$, needed for technical reasons for later phases. We ensure in this phase that the red cycle $\mathcal{C}_2$ will be such that most of these paths are also paths in $\mathcal{C}_2$. Concretely, we obtain a set $J\subseteq[m]$ of size $|J|\geq(1-o(1))m$ such that all the paths $(P_j)_{j\in J}$ are paths in the red cycle $\mathcal{C}_2$. At the end of this phase we get a partition of the vertex set $V=V(\mathcal{C}_2)\cup\text{EXP}_2\cup\text{SMALL}_2$ where $\text{EXP}_2\subseteq\text{EXP}_1$ is a "good expander" and $\text{SMALL}_2\subseteq\text{SMALL}_1$ is such that every vertex $v\in\text{SMALL}_2$ has "large" red degree onto the set $\mathcal{M}_2$ which is the union of all the vertices of the paths $(P_j)_{j\in J}$ which are not endpoints.

The algorithm for this phase is divided into $t=|\text{TINY}|$ parts. For each $i\in[t]$ we define during Part i the following sets:

- R^i and W^i which denote, respectively, the sets of all edges which are coloured red and white at the end of Part i.
- $\text{EXP}^i\subseteq\text{EXP}_1$, $\text{SMALL}^i\subseteq\text{SMALL}_1$ and $\mathcal{U}^i\subseteq\mathcal{U}$, the latter being the union of EXP^i, SMALL^i and $t-i$ vertices of TINY.

During Part i, we recolour "some" edges in W^{i-1} in order to obtain a red cycle $\mathcal{C}^i$ (i.e. consisting solely of edges in R^i) such that $V(\mathcal{C}^i)=V\setminus\mathcal{U}^i$ contains $V(\mathcal{C}_1)$, i vertices from TINY and "few" vertices from $\mathcal{U}$. During this phase's algorithm we keep track of which

red paths $(P_j)_{j \in [m]}$ are also paths in the red cycle $\mathcal{C}_2$. To this end, we use the function $j : V \to \{0, 1, \ldots, m\}$ defined as:

$$j(v) := \begin{cases} j & \text{if } v \in V(P_j) \text{ for } j \in [m] \\ 0 & \text{if } v \notin \bigcup_{j=1}^{m} V(P_j) \end{cases}.$$

During Part i we maintain a set $J^i \subseteq [m]$ such that for every $j \in J^i$ the path P_j is a path in the cycle $\mathcal{C}^i$. The algorithm for this phase is as follows:

Algorithm: Fix an enumeration $x_1, x_2, \ldots, x_t$ of the vertices in TINY, where $t = |\text{TINY}|$, and set $\mathcal{C}^0 := \mathcal{C}_1, \mathcal{U}^0 := \mathcal{U}, \text{EXP}^0 := \text{EXP}_1, \text{SMALL}^0 := \text{SMALL}_1, R^0 := R_2, W^0 := W_2$ and $J^0 := [m]$. For $i = 1, 2, \ldots, t$ execute the following routine which shows how to add x_i to the red cycle $\mathcal{C}^{i-1}$:

Routine: Recall from II.3(c) that $d_{R_2}(x_i) \geq 2$. Thus, since $R_2 \subseteq R^{i-1}$, exactly one of the following holds:

(a) $d_{R^{i-1}}(x_i, V(\mathcal{C}^{i-1})) \geq 1$ and $d_{R^{i-1}}(x_i, \mathcal{U}^{i-1}) \geq 1$.
(b) $d_{R^{i-1}}(x_i, V(\mathcal{C}^{i-1})) \geq 2$ and $d_{R^{i-1}}(x_i, \mathcal{U}^{i-1}) = 0$.
(c) $d_{R^{i-1}}(x_i, V(\mathcal{C}^{i-1})) = 0$ and $d_{R^{i-1}}(x_i, \mathcal{U}^{i-1}) \geq 2$.

We proceed depending on which of the cases above holds. For each of these cases we consider two red neighbours of x_i and depending on whether they lie in $V(\mathcal{C}^{i-1})$ or in $\mathcal{U}^{i-1}$ we use them in a certain way to incorporate x_i into the red cycle $\mathcal{C}^{i-1}$. For the sake of simplicity, we only describe here how to proceed if (a) holds. However, we stress that this case contains all the ideas necessary for treating the other two cases. Essentially, for case (b) (resp. (c)) the two red neighbours of x_i considered should be treated as the red neighbour of x_i in case (a) which lies in $V(\mathcal{C}^{i-1})$ (resp. $\mathcal{U}^{i-1}$). If case (a) holds, proceed as follows:

Fix a cyclic enumeration $v_1 v_2 \ldots v_\ell$ of the vertices in the red cycle $\mathcal{C}^{i-1}$, where $\ell = |V(\mathcal{C}^{i-1})|$ (indices considered modulo ℓ), and let $z_1^i \in V(\mathcal{C}^{i-1})$ and $z_2^i \in \mathcal{U}^{i-1}$ be two red neighbours of x_i. Without loss of generality we assume that $z_1^i = v_\ell$.

Set $\text{BAD}_i := \{v \in \text{EXP}^{i-1} : d_{R_2}(v, \text{EXP}^{i-1}) < 3(\ln n)^{0.4}\}$ to be the set of all vertices in EXP^{i-1} with "low" red degree inside EXP^{i-1} and set $\text{GOOD}_i := \text{EXP}^{i-1} \setminus (\text{BAD}_i \cup N_{R_2}(\text{BAD}_i) \cup \{z_2^i\})$. Let $S_i \subseteq \text{GOOD}_i$ be a subset of size at least $\frac{1}{240} n (\ln n)^{-0.45}$ such that $R^{i-1}[S_i]$ is a connected graph of diameter at most $2 \ln n$. Claim 4.8 below ensures that whp such a set S_i exists and we assume this henceforth.

For each $v_j \in V(\mathcal{C}^{i-1})$ define $s^+(v_j) := v_{j+1}$ and $s^-(v_j) := v_{j-1}$ to be the "successor" and "predecessor" of v_j in the cycle $V(\mathcal{C}^{i-1})$ (notice that $s^+(z_1^i) = s^+(v_\ell) = v_1$). Recolour all the edges in W^{i-1} between $\{s^+(z_1^i), z_2^i\}$ and $V(\mathcal{C}_1)$ and, letting R_1^i and W_1^i denote, respectively, the sets of red and white edges at this point, consider the sets $A_1^i := N_{R_1^i}(s^+(z_1^i), V(\mathcal{C}_1)) \setminus \{z_1^i\}$ and $A_2^i := N_{R_1^i}(z_2^i, V(\mathcal{C}_1)) \setminus \{z_1^i, s^+(z_1^i)\}$. Note that for any two vertices $v_a \in A_1^i$ and $v_b \in A_2^i$ we have a red path

$$P(v_a, v_b) := \begin{cases} v_{a-1} v_{a-2} \ldots v_2 v_1 v_a v_{a+1} \ldots v_{b-1} v_b z_2^i x_i v_l v_{l-1} \ldots v_{b+1} & \text{if } 1 < a \leq b < l \\ v_{a-1} v_{a-2} \ldots v_{b+1} v_b z_2^i x_i v_l v_{l-1} \ldots v_{a+1} v_a v_1 v_2 \ldots v_{b-1} & \text{if } 1 < b < a < l \end{cases}$$

from $s^-(v_a) = v_{a-1}$ to either $s^+(v_b) = v_{b+1}$ or $s^-(v_b) = v_{b-1}$ such that $V(P(v_a, v_b)) = V(\mathcal{C}^{i-1}) \cup \{x_i, z_2^i\}$. Define $B_1^i := \{s^-(v) : v \in A_1^i\}$ to be the set of possible initial vertices of these paths.

Recolour all the edges in W_1^i between vertices in B_1^i and S_i and, letting R_2^i and W_2^i denote, respectively, the sets of red and white edges at this point, let $y_1^i \in B_1^i$ and $u_1^i \in S_i$ be such that $y_1^i u_1^i \in R_2^i$. Claim 4.8 ensures that whp such vertices exist and we assume this henceforth. Define now B_2^i to be the set of possible final vertices of the paths $P(s^+(y_1^i), v)$ where $v \in A_2^i$.

Recolour all the edges in W_2^i between vertices in B_2^i and S_i and, letting R^i and W^i denote, respectively, the sets of red and white edges at this point, let $y_2^i \in B_2^i$ and $u_2^i \in S_i$ be such that $y_2^i u_2^i \in R^i$. Claim 4.8 ensures that whp such vertices exist and we assume this henceforth. Moreover, let $s(y_2^i) \in \{s^+(y_2^i), s^-(y_2^i)\}$ be the vertex of $V(\mathcal{C}^{i-1})$ such that y_2^i is the final vertex of the red path $P(s^+(y_1^i), s(y_2^i))$.

Let $P(u_1^i, u_2^i)$ be a path inside S_i from u_1^i to u_2^i of length at most $2 \ln n$ consisting solely of edges in R^i (such a path exists by the choice of S_i) and set $\mathcal{C}^i$ to be the red cycle formed by joining the red paths $P(s^+(y_1^i), s(y_2^i))$ and $P(u_1^i, u_2^i)$ with the red edges $y_1^i u_1^i$ and $y_2^i u_2^i$. Furthermore, set $J^i := J^{i-1} \setminus (\{j(z_1^i)\} \cup \{j(y_1^i)\} \cup \{j(y_2^i)\})$ to be the set of indices obtained by deleting the indices of the paths we "broke" during this routine, and note that every path P_j with $j \in J^i$ is still a subpath of the red cycle $\mathcal{C}^i$ (provided it was also a subpath of $\mathcal{C}^{i-1}$). Finally, set $\text{EXP}^i := \text{EXP}^{i-1} \setminus V(\mathcal{C}^i)$, $\text{SMALL}^i := \text{SMALL}^{i-1} \setminus V(\mathcal{C}^i)$ and $\mathcal{U}^i := \mathcal{U}^{i-1} \setminus V(\mathcal{C}^i)$.

(End of Routine)

To end the algorithm set $\text{EXP}_2 := \text{EXP}^t$, $\text{SMALL}_2 := \text{SMALL}^t$, $\mathcal{C}_2 := \mathcal{C}^t$, $J := J^t$ and $\mathcal{M}_2$ to be the union of all the inner vertices of the paths $(P_j)_{j \in J}$.

(End of Algorithm)

We make a few observations about the procedure above which will be important for later:

(O1) For every $i \in [t]$ we have $|\text{EXP}^{i-1} \setminus \text{EXP}^i| \leq 2 \ln n + 3$. This is because in Part i we have $\text{EXP}^{i-1} \setminus \text{EXP}^i \subseteq V(P(u_1^i, u_2^i)) \cup \{z_1^i, z_2^i\}$ where $P(u_1^i, u_2^i)$ is a path of size at most $2 \ln n + 1$ (we might need to remove not just z_2^i but also z_1^i from EXP^{i-1} if case (c) in the algorithm holds). Moreover, since by II.1(b) we have $d_{R_2}(v, \text{EXP}_1) \geq 3(\ln n)^{0.4}$ for every $v \in \text{EXP}_1$, every vertex $v \in \text{BAD}_i$ must have at least one red neighbour in $\text{EXP}_1 \setminus \text{EXP}^{i-1}$. Thus, we have $|\text{BAD}_i| \leq |\text{EXP}_1 \setminus \text{EXP}^{i-1}| \cdot \Delta(R_2) \leq (2 \ln n + 3)(i - 1) \cdot \Delta(R_2)$.

(O2) For every $i \in [t]$ and every $j \in \{1, 2\}$ we have $d_{R^i}(v_j^i, V(\mathcal{C}_1)) - 2 \leq |B_j^i| \leq d_{R^i}(v_j^i, V(\mathcal{C}_1))$ for some vertex v_j^i which is at distance at most 2 from x_i in R_2. For example, if case (a) holds in Part i then we have $v_1^i = s^+(z_1^i)$ and $v_2^i = z_2^i$. Moreover, for every $i \in [t]$ and every vertex $v \in B_1^i \cup B_2^i$ there is a path in R^i of length at most 4 from x_i to v. This follows immediately from the definition of the sets B_j^i.

(O3) We have $N_{W^{i-1}}(v, \text{EXP}^{i-1}) = N_{W_1}(v, \text{EXP}^{i-1})$ for every $v \in V(\mathcal{C}_1) \setminus \left(\bigcup_{j=1}^{i-1} \left(B_1^j \cup B_2^j \right) \right)$ and every $i \in [t + 1]$. This is because between Phase I and Part i of this phase's algorithm the only edges that are recoloured between $V(\mathcal{C}_1)$ and EXP^{i-1} touch vertices of $\bigcup_{j=1}^{i-1} \left(B_1^j \cup B_2^j \right)$.

Also, for the rest of this phase, we shall assume that the following event occurs:

(E1) for every $i \in [t]$ there is no path of size at most 1000 consisting solely of edges in R^i between any two vertices in TINY.

Note that this event occurs whp as indicated in II.3(b). In the next claim we prove that some properties which are assumed in the algorithm hold whp.

Claim 4.8. *All of the following properties hold whp:*

(i) *For any $i \in [t]$ there always exists a set $S_i \subseteq GOOD_i$ of size at least $\frac{1}{240}n(\ln n)^{-0.45}$ such that $R^{i-1}[S_i]$ is a connected graph of diameter at most $2 \ln n$.*

(ii) *For any $i \in [t]$, in Part i, after recolouring all the edges in W^{i-1} between the sets S_i and $B_1^i \cup B_2^i$, there exist $y_1^i \in B_1^i$, $y_2^i \in B_2^i$ and $u_1^i, u_2^i \in S_i$ such that $y_j^i u_j^i \in R^i$ for $j \in \{1, 2\}$.*

Proof of Claim 4.8. We start by proving that (i) holds whp. Assuming that $\Delta(R_2) \leq 40 \ln n$ (which holds whp after Phase II, by N.1), we have by (O1), II.1(a) and II.3(a) that

$$|GOOD_i| \geq |EXP^{i-1}| - |BAD_i \cup N_{R_2}(BAD_i)| - 2$$
$$\geq |EXP_1| - (2 \ln n + 3) \cdot (i - 1) - |BAD_i| \cdot (1 + 40 \ln n) - 2$$
$$\geq \left(1 - \frac{1}{\ln n}\right) |EXP_1|.$$

We remark that the -2 after the first inequality is necessary if case (c) holds (for case (a) one only needs -1). Thus, by II.1(c) there is always a set $S_i \subseteq GOOD_i$ with the desired properties.

Next we show that (ii) holds whp. By (O2) we know that for every $i \in [t]$ and every $j \in \{1, 2\}$ we have $|B_j^i| \geq d_{R^i}(v_j^i, V(\mathcal{C}_1)) - 2$ for some vertex v_j^i which is at distance at most 2 from x_i in R_2. We claim that whp $d_{R^i}(v_j^i, V(\mathcal{C}_1)) \geq (\ln n)^{0.5}$ for every $i \in [t]$ and $j \in \{1, 2\}$.

Note that $v_j^i \notin TINY$ as otherwise v_j^i and x_i would be two vertices in TINY at distance at most 2 in R^i, contradicting (E1). Moreover, recall from II.2(b) that if $v_j^i \in SMALL_1$ then $d_{R^i}(v, V(\mathcal{C}_1)) \geq (\ln n)^{0.5}$. If $v_j^i \in EXP_1 \cup V(\mathcal{C}_1)$ then recall from I.3 that $d_{W_1}(v_j^i, V(\mathcal{C}_1)) = (1 - o(1))n$ and note that $d_{W_2}(v_j^i, V(\mathcal{C}_1)) = d_{W_1}(v_j^i, V(\mathcal{C}_1))$ as no edges between the sets $EXP_1 \cup V(\mathcal{C}_1)$ and $V(\mathcal{C}_1)$ were recoloured during Phase II. Thus, using the union bound, Lemma 3.1 and II.3(a) we see that the probability that $d_{R^i}(v_j^i, V(\mathcal{C}_1)) < (\ln n)^{0.5}$ for some $i \in [t]$ and $j \in \{1, 2\}$ is at most

$$2t \cdot \Pr\left[Bin((1 - o(1))n, p) < (\ln n)^{0.5}\right] \leq 2n^{0.04} \cdot e^{-\left(\frac{1}{2} - o(1)\right) \ln n} = o(1),$$

as claimed. Hence, whp $|B_j^i| \geq (\ln n)^{0.5} - 2$ for every $i \in [t]$ and $j \in \{1, 2\}$. We assume this hereafter.

Note that for $i \neq i'$ we have $(B_1^i \cup B_2^i) \cap (B_1^{i'} \cup B_2^{i'}) = \emptyset$ since otherwise (O2) would imply that there is a path in R^i of length at most 8 between x_i and $x_{i'}$, contradicting (E1). Thus, by (O3) and I.4, we see that $d_{W^{i-1}}(v, S_i) = d_{W_1}(v, S_i) = (1 - o(1))|S_i|$ for every $v \in B_1^i \cup B_2^i$ and every $i \in [t]$.

Now, for each $i \in [t]$ and $j \in \{1, 2\}$, let $C_j^i \subseteq B_j^i$ be a subset of size at least $\left(\frac{1}{2} - o(1)\right)|B_j^i| \geq \frac{1}{3}(\ln n)^{0.5}$ such that $C_1^i \cap C_2^i = \emptyset$. We then see that for any $i \in [t]$ and $j \in \{1, 2\}$ the probability that there is no edge in R^i between C_j^i and S_i is at most:

$$\Pr\left[Bin\left((1 - o(1))|S_i| \cdot |C_j^i|, p\right) = 0\right] = (1 - p)^{(1 - o(1))|S_i| \cdot |C_j^i|}$$
$$\leq e^{-\frac{\ln n}{n} \cdot \frac{1 - o(1)}{240} n(\ln n)^{-0.45} \cdot \frac{1}{3}(\ln n)^{0.5}} \leq \frac{1}{n}.$$

Using II.3(a) and the union bound we see that the probability that for some $i \in [t]$ and $j \in \{1, 2\}$ there is no edge in R^i between B_j^i and S_i is at most $2t \cdot \frac{1}{n} = o(1)$. This shows that (*ii*) holds whp. ∎

Assuming that the properties of Claim 4.8 hold, denoting by R_3, W_3 and B_3 the sets of red, white and blue edges at the end of this phase's algorithm, we show that whp the following technical conditions hold:

III.1 Properties of EXP_2:
    (a) $|\mathrm{EXP}_2| \geq \left(1 - \frac{1}{(\ln n)^2}\right) |\mathcal{U}| \geq (2 - o(1))n(\ln n)^{-0.45}$.
    (b) for every $v \in \mathrm{EXP}_2$ we have $d_{R_3 \setminus R_1}(v, \mathrm{EXP}_2) \geq 2(\ln n)^{0.4}$.
III.2 Properties of SMALL_2:
    (a) $|\mathrm{SMALL}_2| \leq |\mathrm{SMALL}_1| \leq 2ne^{-(\ln n)^{0.4}}$.
    (b) for every $v \in \mathrm{SMALL}_2$ we have $d_{R_3}(v, \mathcal{M}_2) \geq (\ln n)^{0.5} - 400$.
    (c) for every $v \in \mathrm{SMALL}_2$ we have $N_{W_3}(u, \mathrm{EXP}_2) \neq N_{W_1}(u, \mathrm{EXP}_2)$ for at most 100 vertices $u \in V(\mathcal{C}_1)$ which are at distance at most 2 from v in R_3.
III.3 All the paths $(P_j)_{j \in J}$ are paths in the red cycle $\mathcal{C}_2$.
III.4 In this phase only $o(n)$ edges of W_2 are recoloured red.

Note that property III.2(a) is just a consequence of the fact that $\mathrm{SMALL}_2 \subseteq \mathrm{SMALL}_1$ together with II.2(a). Moreover, property III.3 follows immediately from the algorithm. Indeed, $E(\mathcal{C}^{i-1}) \setminus E(\mathcal{C}^i)$ consists of at most 4 edges (only 3 edges if case (a) holds but 4 if case (b) holds) and, from the definition of J^i, for each such edge e we remove $j(v)$ from J^{i-1} for one vertex $v \in e$. Thus, it is clear that P_j is still a path in the red cycle $\mathcal{C}^i$ for every $j \in J^i$. The next few claims show that the remaining properties all hold whp, assuming that the properties of Claim 4.8 hold.

Claim 4.9. *Properties III.1(a) and III.1(b) hold whp.*

Proof of Claim 4.9. It follows from (O1), I.1, II.1(a) and II.3(a) that:

$$|\mathrm{EXP}_2| = |\mathrm{EXP}_1| - \sum_{i=1}^{t} |\mathrm{EXP}^{i-1} \setminus \mathrm{EXP}^i| \geq \left(1 - \frac{1}{(\ln n)^3}\right) |\mathcal{U}| - t \cdot (2\ln n + 3)$$

$$\geq \left(1 - \frac{1}{(\ln n)^2}\right) |\mathcal{U}|$$

and so this shows that III.1(a) holds.

We now show that III.1(b) also holds whp. Suppose $d_{R_3 \setminus R_1}(v, \mathrm{EXP}_2) < 2(\ln n)^{0.4}$ for some $v \in \mathrm{EXP}_2$ and let $i \in [t]$ be the largest possible integer such that $d_{R_2 \setminus R_1}(v, \mathrm{EXP}^{i-1}) \geq 3(\ln n)^{0.4}$ (this is well defined by II.1(b)). Note that since $R_3[\mathcal{U}] = R_2[\mathcal{U}]$ we have

$$d_{R_3 \setminus R_1}(v, \mathrm{EXP}_2) = d_{R_2 \setminus R_1}(v, \mathrm{EXP}^{i-1}) - \sum_{j=i}^{t} d_{R_2 \setminus R_1}(v, \mathrm{EXP}^{j-1} \setminus \mathrm{EXP}^j).$$

Recall that $\mathrm{EXP}^{j-1} \setminus \mathrm{EXP}^j \subseteq V(P(u_1^j, u_2^j)) \cup \{z_1^j, z_2^j\} \subseteq \mathrm{GOOD}_j \cup \{z_1^j, z_2^j\}$ where z_1^j and z_2^j are neighbours of x_j in R_2. Moreover, note that by the choice of i we have $v \in \mathrm{BAD}_j$ for

every $j > i$. Thus, since $\text{GOOD}_j \cap N_{R_2}(\text{BAD}_j) = \emptyset$ we get that

$$d_{R_3 \setminus R_1}(v, \text{EXP}_2) \geq d_{R_2 \setminus R_1}(v, \text{EXP}^{i-1}) - d_{R_2}(v, V(P(u_1^i, u_2^i))) - \sum_{j=i}^{t} d_{R_2}(v, \{z_1^j, z_2^j\}).$$

Observe now that if $d_{R_2}(v, \{z_1^j, z_2^j\}) > 0$ and $d_{R_2}(v, \{z_1^{j'}, z_2^{j'}\}) > 0$ for $j \neq j'$ then there would exist a path in R_3 of length at most 4 between x_j and $x_{j'}$, contradicting (E1). Thus, we can conclude that

$$d_{R_3 \setminus R_1}(v, \text{EXP}_2) \geq d_{R_2 \setminus R_1}(v, \text{EXP}^{i-1}) - d_{R_2}(v, V(P(u_1^i, u_2^i))) - 2$$
$$\geq 3(\ln n)^{0.4} - d_{R_2}(v, V(P(u_1^i, u_2^i))) - 2.$$

Since we assumed that $d_{R_3 \setminus R_1}(v, \text{EXP}_2) < 2(\ln n)^{0.4}$, we must have that

$$d_{R_2}(v, V(P(u_1^i, u_2^i))) > (\ln n)^{0.4} - 2.$$

It is easy to see that this implies the existence of two cycles of length $O((\ln n)^{0.6})$ sharing only the vertex v in the graph R_2 since the red path $P(u_1^i, u_2^i)$ has length at most $2 \ln n$. However, if N.3 holds then this does not happen. Since N.3 holds whp we conclude that III.1(b) holds whp as desired.　　　　　　　　　　　　　　　　　　　　　　　　■

Claim 4.10.　*Properties III.2(b) and III.2(c) hold whp.*

Proof of Claim 4.10.　First we show that whp III.2(b) holds. Note that for every $v \in \text{SMALL}_2$ we have

$$d_{R_3}(v, \mathcal{M}_2) \geq d_{R_3}(v, \mathcal{M}_1) - \sum_{i=1}^{t} \sum_{j \in J^{i-1} \setminus J^i} d_{R_3}(v, V(P_j))$$

since $\mathcal{M}_1 \setminus \left(\bigcup_{i=1}^{t} \bigcup_{j \in J^{i-1} \setminus J^i} V(P_j) \right) \subseteq \mathcal{M}_2$. Also, since $|J^{i-1} \setminus J^i| \leq 4$ and since $|V(P_j)| = 100$ we have that for every $i \in [t]$

$$\sum_{j \in J^{i-1} \setminus J^i} d_{R_3}(v, V(P_j)) \leq 400.$$

Observe now that if for $i \neq i', j \in J^{i-1} \setminus J^i$ and $j' \in J^{i'-1} \setminus J^{i'}$ we have $d_{R_3}(v, V(P_j)) > 0$ and $d_{R_3}(v, V(P_{j'})) > 0$ then there is a path in R_3 of size at most $2 \cdot 102 + 2 + 1 = 207$ between x_i and $x_{i'}$. Indeed, this follows from the fact that every vertex in $V(P_j)$ is at distance at most $3 + (|V(P_j)| - 1) = 102$ of x_i in R^i and similarly that every vertex in $V(P_{j'})$ is at distance at most 102 of $x_{i'}$ in $R^{i'}$. However, if the event (E1) holds then there does not exist such a path. Since (E1) holds whp we conclude that whp

$$d_{R_3}(v, \mathcal{M}_2) \geq d_{R_3}(v, \mathcal{M}_1) - 400,$$

and so by II.2(b) we see that III.2(b) holds.

Now we show that whp III.2(c) holds. Note that by (O3) it is enough to show that for every $v \in \text{SMALL}_2$ there are at most 100 vertices $u \in \bigcup_{j=1}^{t}(B_1^j \cup B_2^j)$ which are at distance

at most 2 from v in R_3. Assume this is not the case for a vertex $v \in \mathrm{SMALL}_2$. Note first, if for some $j \neq j'$ the sets $B_1^j \cup B_2^j$ and $B_1^{j'} \cup B_2^{j'}$ each contain a vertex which is at distance at most 2 from v in R_3, then by (O2) it follows that there is a red path from x_j to $x_{j'}$ of length at most $4 + 2 + 2 + 4 = 12$ in R_3. However, this cannot occur if the event (E1) holds. Since (E1) holds whp we conclude that whp for every $v \in \mathrm{SMALL}_2$ there is at most one $j \in [t]$ for which $B_1^j \cup B_2^j$ contains a vertex which is at distance at most 2 from v in R_3. Moreover, if for some $j \in [t]$ the set $B_1^j \cup B_2^j$ contains at least 100 vertices which are at distance at most 2 from v in R_3 then by (O2) one can find at least four trails of length at most $4 + 2 = 6$ between v and x_j. However, this cannot occur if N.4 holds. Since N.4 holds whp we conclude that whp III.2(c) holds. ∎

Claim 4.11. *Property III.4 holds whp.*

Proof of Claim 4.11. Recall that in Part i of the algorithm we recolour edges in two situations. In the first situation we recolour edges between two vertices (if case (a) holds then these vertices are $s^+(z_1^i)$ and z_2^i) and $V(\mathcal{C}_1)$. In the second situation we recolour edges between vertices in $B_1^i \cup B_2^i$ and S_i. From the algorithm it is easy to see that the number of edges in the first situation which are recoloured red is at most $|B_1^i| + |B_2^i| + 4$. For each $i \in [t]$ and $j \in \{1, 2\}$ let $E_1^{i,j}$ be the event that $|B_j^i| > (\ln n)^{1.1}$ and let $E_2^{i,j}$ be the event that $e_{R^i \setminus R^{i-1}}(B_j^i, S_i) > 80(\ln n)^{1.65}$. Notice that if none of these events hold then clearly at most $(2(\ln n)^{1.1} + 4) \cdot t + 80(\ln n)^{1.65} \cdot 2t = o(n)$ edges are recoloured red in this phase. Note that $\bigwedge_{i \in [t], j \in \{1,2\}} \overline{E_1^{i,j}}$ holds whp by (O2) and N.1. Moreover, using the fact that $S_i \subseteq \mathcal{U}$ and that $|\mathcal{U}| \leq 4n(\ln n)^{-0.45}$ by I.1, we have by Chernoff (Lemma 3.1) that

$$
\begin{aligned}
\Pr\left[E_2^{i,j} \wedge \overline{E_1^{i,j}}\right] &\leq \Pr\left[E_2^{i,j} \mid \overline{E_1^{i,j}}\right] \leq \Pr\left[\mathrm{Bin}\left((\ln n)^{1.1} \cdot 4n(\ln n)^{-0.45}, p\right) > 80(\ln n)^{1.65}\right] \\
&\leq \Pr\left[\mathrm{Bin}\left(4n(\ln n)^{0.65}, \frac{10 \ln n}{n}\right) > 80(\ln n)^{1.65}\right] \leq e^{-\frac{40}{3}(\ln n)^{1.65}}.
\end{aligned}
$$

Thus, using the union bound and II.3(a), we see that the probability that the number of edges recoloured red during this phase is larger than $(2(\ln n)^{1.1} + 4) \cdot t + 80(\ln n)^{1.65} \cdot 2t$ is at most

$$
\Pr\left[\bigvee_{i \in [t], j \in \{1,2\}} E_1^{i,j}\right] + \sum_{i \in [t], j \in \{1,2\}} \Pr\left[E_2^{i,j} \wedge \overline{E_1^{i,j}}\right] \leq o(1) + 2t \cdot e^{-\frac{40}{3}(\ln n)^{1.65}} = o(1)
$$

This, together with II.3(a), shows that whp III.4 holds. ∎

We have shown that whp at the end of this phase all of the properties III.1-III.4 hold. We shall assume henceforth that all these properties hold for the sets R_3, W_3 and B_3.

4.5. Phase IV

In this phase, we want to recolour some edges in W_3 in order to find a red cycle $\mathcal{C}_3$ containing $\mathrm{SMALL}_2 \cup V(\mathcal{C}_2)$ in a such a way that $\mathrm{EXP}_3 = V \setminus V(\mathcal{C}_3)$ is a "good expander" as described in the outline. The algorithm for this phase is similar in spirit to the one of Phase III. It is divided into three stages. In Stage 1 we define notation and sets that will be useful for us throughout the algorithm. Stage 2 is the main stage of the algorithm and is divided into

$s = |\text{SMALL}_2|$ parts. For each $i \in [s]$ we denote by R^i and W^i, respectively, the sets of all edges which are coloured red and white at the end of Part i. During Part i, we recolour "some" edges in W^{i-1} in order to obtain a red cycle $\mathcal{C}^i$ (i.e. consisting solely of edges in R^i) such that $V(\mathcal{C}^i)$ contains $V(\mathcal{C}_2)$, i vertices from SMALL_2 and the vertex set of a "small" red path in EXP_2. Finally, in Stage 3 we define sets needed for later phases. The algorithm for this phase is as follows:

Algorithm: Fix an enumeration $y_1, y_2, \ldots, y_s$ of the vertices in SMALL_2, where $s = |\text{SMALL}_2|$, set $\mathcal{C}^0 := \mathcal{C}_2$, $\text{EXP}^0 := \text{EXP}_2$, $R^0 := R_3$ and $W^0 := W_3$ and for each $j \in J$ let M_j denote the set of inner vertices of the path P_j.

Let $(J_i)_{i \in [s]}$ be disjoint subsets of J of size $10^3 (\ln n)^{0.45}$ such that $d_{R_3}(y_i, M_j) > 0$ for every $j \in J_i$. Claim 4.12 ensures that such sets exist and we assume that henceforth. For each $j \in J_i$ let $m_j \in M_j$ be such that $y_i m_j \in R_3$. For $i = 1, 2, \ldots, s$ execute the following routine:

Routine: Fix a cyclic orientation of the vertices in the red cycle $\mathcal{C}^{i-1}$ and denote for each $v \in V(\mathcal{C}^{i-1})$ by $s^+(v)$ and $s^-(v)$ the successor and predecessor of v in the cycle $\mathcal{C}^{i-1}$ according to this orientation, respectively.

Set $\text{BAD}_i := \{v \in \text{EXP}^{i-1} : d_{R_3}(v, \text{EXP}^{i-1}) < 2(\ln n)^{0.4}\}$, $\text{GOOD}_i := \text{EXP}^{i-1} \setminus (\text{BAD}_i \cup N_{R_3}(\text{BAD}_i))$ and let $S_i \subseteq \text{GOOD}_i$ be a subset of size at least $\frac{1}{240} n (\ln n)^{-0.45}$ such that $R^{i-1}[S_i]$ is a connected graph of diameter at most $2 \ln n$. Claim 4.12 ensures that whp such a set S_i always exists and we assume that henceforth.

Define $A_i := \{s^+(m_j) : j \in J_i\}$ and recolour all the edges in W^{i-1} between the set A_i and S_i. Letting R^i and W^i to be respectively the sets of red and white edges at this point, let $u_1^i, u_2^i \in S_i$ and $j_1^i, j_2^i \in J_i$ be distinct indices such that $s^+(m_{j_1^i}) u_1^i \in R^i$ and $s^+(m_{j_2^i}) u_2^i \in R^i$. Claim 4.12 ensures that whp such vertices u_1^i, u_2^i and distinct indices j_1^i, j_2^i always exist and we assume that hereafter.

Set

$$Q_1^i := s^+(m_{j_1^i}) \ldots s^-(m_{j_2^i}) m_{j_2^i} y_i m_{j_1^i} s^-(m_{j_1^i}) \ldots s^+(m_{j_2^i})$$

to be the red path from $s^+(m_{j_1^i})$ to $s^+(m_{j_2^i})$ which contains y_i and is obtained from $\mathcal{C}^{i-1}$ by deleting the red edges $m_{j_1^i} s^+(m_{j_1^i})$ and $m_{j_2^i} s^+(m_{j_2^i})$ and adding the red edges $m_{j_2^i} y_i$ and $y_i m_{j_1^i}$. Let Q_2^i be a path inside S_i from u_1^i to u_2^i of length at most $2 \ln n$ consisting solely of edges in R^i (such a path exists by the choice of S_i). Set $\mathcal{C}^i$ to be the red cycle formed by joining the red paths Q_1^i and Q_2^i with the red edges $s^+(m_{j_1^i}) u_1^i$ and $s^+(m_{j_2^i}) u_2^i$. To end the routine, set $\text{EXP}^i := \text{EXP}^{i-1} \setminus V(\mathcal{C}^i)$.

(End of Routine)

To finish the algorithm set $\text{EXP}_3 := \text{EXP}^s$ and $\mathcal{C}_3 := \mathcal{C}^s$.

(End of Algorithm)

We make a few observations about the procedure above which will be useful for later:

(O1) For every $i \in [s]$ we have $|\text{EXP}^{i-1} \setminus \text{EXP}^i| \leq 2 \ln n + 1$ since $\text{EXP}^{i-1} \setminus \text{EXP}^i = V(Q_2^i)$ which has size at most $2 \ln n + 1$. Moreover, since by III.1(b) we have $d_{R_3}(v, \text{EXP}_2) \geq 2(\ln n)^{0.4}$ for every $v \in \text{EXP}_2$, every vertex in BAD_i must have at least one red neighbour in $\text{EXP}_2 \setminus \text{EXP}^{i-1}$. Thus, we have $|\text{BAD}_i| \leq |\text{EXP}_2 \setminus \text{EXP}^{i-1}| \cdot \Delta(R_3) \leq (2 \ln n + 1)(i - 1) \cdot \Delta(R_3)$.

(O2) For every $i \in [s+1]$ we have $N_{W^{i-1}}(v, \text{EXP}^{i-1}) = N_{W_3}(v, \text{EXP}^{i-1})$ provided $v \in V(\mathcal{C}_2) \setminus \left(\bigcup_{k=1}^{i-1} A_k \right)$. Indeed, this holds since before Part i of the routine we only recolour edges between the sets A_k and EXP^{k-1} for $k \in [i-1]$.

(O3) For every $i \in [s+1]$ and every $j \in J \setminus \left(\bigcup_{k=1}^{i-1} \{j_1^k, j_2^k\} \right)$ the path P_j is still a path in the red cycle $\mathcal{C}^{i-1}$. Indeed, this follows by induction on i using the facts that $m_{j_1^k} \in M_{j_1^k}, m_{j_2^k} \in M_{j_2^k}$ and that the sets J_k are disjoint for $k \in [i-1]$.

(O4) The sets $(A_i)_{i \in [s]}$ are disjoint and $N_{W^{i-1}}(v, \text{EXP}^{i-1}) = N_{W_3}(v, \text{EXP}^{i-1})$ for every $v \in A_i$ and every $i \in [s]$. Indeed, (O3) together with the fact that $m_j \in M_j$ for every $j \in J_i$ ensures that $A_i \subseteq \bigcup_{j \in J_i} V(P_j)$ for every $i \in [s]$. Since the sets $(J_i)_{i \in [s]}$ are disjoint, the first observation follows. The second observation follows from the first one together with (O2).

In the next claim we prove that some properties which are assumed in the algorithm hold whp.

Claim 4.12. *All of the following properties hold whp:*

(i) *There exist disjoint subsets $(J_i)_{i \in [s]}$ of J of size $10^3 (\ln n)^{0.45}$ such that $d_{R_3}(y_i, M_j) > 0$ for every $j \in J_i$.*

(ii) *For every $i \in [s]$ there exists $S_i \subseteq GOOD_i$ of size at least $\frac{1}{240} n (\ln n)^{-0.45}$ such that $R^{i-1}[S_i]$ is a connected graph of diameter at most $2 \ln n$.*

(iii) *For every $i \in [s]$, after recolouring the edges in W^{i-1} between the sets A_i and S_i, there exist $u_1^i, u_2^i \in S_i$ and distinct indices $j_1^i, j_2^i \in J_i$ such that $s^+(m_{j_1^i}) u_1^i \in R^i$ and $s^+(m_{j_2^i}) u_2^i \in R^i$.*

Proof of Claim 4.12. We show first that whp (i) holds. Let G_{aux} be the bipartite graph with parts $[s]$ and J and edge set $\{ij : i \in [s], j \in J, d_{R_3}(y_i, M_j) > 0\}$. We want to show that whp there are disjoint subsets $(J_i)_{i \in [s]}$ of J of size $10^3 (\ln n)^{0.45}$ such that $J_i \subseteq N_{G_{\text{aux}}}(i)$ for every $i \in [s]$. In light of Lemma 3.8 it suffices to show that whp $|N_{G_{\text{aux}}}(I)| \geq 10^3 (\ln n)^{0.45} |I|$ for every $I \subseteq [s]$. With this in mind, suppose that $I \subseteq [s]$ is such that $|N_{G_{\text{aux}}}(I)| < 10^3 (\ln n)^{0.45} |I|$, and set $X := \{y_i : i \in I\}$ and $Y := \bigcup_{j \in N_{G_{\text{aux}}}(I)} M_j$. Note that we have

$$|X \cup Y| = |X| + 98|N_{G_{\text{aux}}}(I)| < 10^5 (\ln n)^{0.45} |X|$$

and, using III.2(b) and the fact that $N_{R_3}(X) \cap \mathcal{M}_2 \subseteq Y$, we also have

$$e_{R_3}(X \cup Y) \geq ((\ln n)^{0.5} - 400)|X| > \frac{(\ln n)^{0.5} - 400}{10^5 (\ln n)^{0.45}} |X \cup Y| > \frac{(\ln n)^{0.05}}{10^6} |X \cup Y|.$$

By (P3) of Lemma 3.5 we see then that whp $|X \cup Y| > \frac{1}{10^6} (\ln n)^{0.05} \cdot \frac{1}{20} n (\ln n)^{-1} = \frac{1}{2 \times 10^7} n (\ln n)^{-0.95}$. But in that case we have

$$|\text{SMALL}_2| \geq |X| > \frac{|X \cup Y|}{10^5 (\ln n)^{0.45}} > \frac{n}{2 \times 10^{12} (\ln n)^{1.4}}$$

which contradicts III.2(a). We conclude that whp (i) holds.

Next we show that whp (*ii*) also holds. Indeed, assuming that $\Delta(R_3) \leq 40 \ln n$ (which holds whp after Phase III, by N.1), we have by (O1), III.1(a) and III.2(a) that

$$
\begin{aligned}
|\mathrm{GOOD}_i| &\geq |\mathrm{EXP}^{i-1}| - |\mathrm{BAD}_i \cup N_{R_3}(\mathrm{BAD}_i)| \\
&\geq |\mathrm{EXP}_2| - (2 \ln n + 1) \cdot (i - 1) - |\mathrm{BAD}_i| \cdot (1 + 40 \ln n) \\
&\geq \left(1 - \frac{1}{\ln n}\right) |\mathrm{EXP}_1|.
\end{aligned}
$$

Thus, by II.1(c) there is always a set $S_i \subseteq \mathrm{GOOD}_i$ with the desired properties.

Finally, we show that whp (*iii*) also holds. By (O4), for every $i \in [s]$ we have $d_{W^{i-1}}(u, S_i) = d_{W_3}(u, S_i)$ for every $u \in A_i$. Moreover, since every vertex $u \in A_i$ is at distance at most 2 in R_3 from $y_i \in \mathrm{SMALL}_2$, it follows from III.2(c) and I.4 that $d_{W^{i-1}}(u, S_i) = d_{W_1}(u, S_i) = (1 - o(1))|S_i|$ for all but at most 100 vertices $u \in A_i$. Let D_i be the set of these "bad" vertices. For $j \in \{1, 2\}$ and $i \in [s]$ let $B_j^i \subseteq A_i \setminus D_i$ be sets of size at least $400(\ln n)^{0.45}$ such that $B_1^i \cap B_2^i = \emptyset$ (this is possible since $|A_i| = |J_i| = 10^3(\ln n)^{0.45}$ and $|D_i| \leq 100$). Thus, for any $i \in [s]$ and $j \in \{1, 2\}$ the probability that there is no edge in R^i between B_j^i and S_i is at most:

$$
\Pr\left[\mathrm{Bin}\left((1 - o(1))|S_i| \cdot |B_j^i|, p\right) = 0\right] = (1 - p)^{(1 - o(1))|S_i| \cdot |B_j^i|}
$$

$$
\leq e^{-\frac{\ln n}{n} \cdot \frac{1 - o(1)}{240} n (\ln n)^{-0.45} \cdot 400 (\ln n)^{0.45}} \leq \frac{1}{n}.
$$

Using III.2(a) and the union bound we see that the probability that for some $i \in [s]$ and $j \in \{1, 2\}$ there is no edge in R^i between B_j^i and S_i is at most $2s \cdot \frac{1}{n} = o(1)$. Since the sets B_1^i and B_2^i were chosen to be disjoint, we conclude that whp (*iii*) holds. This completes the proof of the claim. ∎

Assuming that the properties of Claim 4.12 hold, denoting by R_4, W_4 and B_4 the sets of red, white and blue edges at the end of this phase's algorithm, we show that whp the following technical conditions hold:

IV.1 Properties of EXP_3:
  (a) $|\mathrm{EXP}_3| \geq \left(1 - \frac{1}{\ln n}\right) |\mathcal{U}| \geq (2 - o(1))n(\ln n)^{-0.45}$.
  (b) for every $v \in \mathrm{EXP}_3$ we have $d_{R_4 \setminus R_1}(v, \mathrm{EXP}_3) \geq (\ln n)^{0.4}$.
IV.2 In this phase only $o(n)$ edges of W_3 are recoloured red.

The next few claims show that properties IV.1-IV.2 all hold whp, assuming that the properties of Claim 4.12 hold.

Claim 4.13.   *Properties IV.1(a) and IV.1(b) hold whp.*

Proof of Claim 4.13.   Note that by (O1), III.1(a) and III.2(a) we have:

$$
\begin{aligned}
|\mathrm{EXP}_3| &= |\mathrm{EXP}_2| - \sum_{i=1}^{s} |\mathrm{EXP}^{i-1} \setminus \mathrm{EXP}^i| \\
&\geq \left(1 - \frac{1}{(\ln n)^2}\right) |\mathcal{U}| - s \cdot (2 \ln n + 1) = \left(1 - \frac{1}{\ln n}\right) |\mathcal{U}|
\end{aligned}
$$

and so this together with I.1 shows that IV.1(a) holds.

We now show that IV.1(b) holds whp. Suppose $d_{R_4 \setminus R_1}(v, \mathrm{EXP}_3) < (\ln n)^{0.4}$ for some $v \in \mathrm{EXP}_3$ and let $i \in [t]$ be the largest possible integer such that $d_{R_3 \setminus R_1}(v, \mathrm{EXP}^{i-1}) \geq 2(\ln n)^{0.4}$ (this is well defined by III.1(b)). Note that by the choice of i we have $v \in \mathrm{BAD}_j$ for every $j > i$. Moreover, since $R_4[\mathcal{U}] = R_3[\mathcal{U}]$, $\mathrm{EXP}^{j-1} \setminus \mathrm{EXP}^j = V(Q_2^j) \subseteq \mathrm{GOOD}_j$ and $\mathrm{GOOD}_j \cap N_{R_3}(\mathrm{BAD}_j) = \emptyset$ for any $j > i$ we have

$$d_{R_4 \setminus R_1}(v, \mathrm{EXP}_3) = d_{R_3 \setminus R_1}(v, \mathrm{EXP}^{i-1}) - \sum_{j=i}^{s} d_{R_3 \setminus R_1}(v, V(Q_2^j)) \geq 2(\ln n)^{0.4} - d_{R_3}(v, V(Q_2^i)).$$

Thus, since we assume that $d_{R_4 \setminus R_1}(v, \mathrm{EXP}_3) < (\ln n)^{0.4}$, we must have that $d_{R_3}(v, V(Q_2^i)) > (\ln n)^{0.4}$. It is easy to see that this implies that there exist two cycles of length $O((\ln n)^{0.6})$ sharing only the vertex v in the graph R_3 (since Q_2^i is a path in R_3 of length at most $2 \ln n + 1$). However, if N.3 holds then this does not happen. Since N.3 holds whp we conclude that IV.1(b) holds whp as desired. ∎

Claim 4.14. *Property IV.2 holds whp.*

Proof of Claim 4.14. Recall from the algorithm that in this phase the only edges recoloured are the ones in W_3 between the sets A_i and S_i for each $i \in [s]$. Since for each $i \in [s]$ we have $|A_i| = |J_i| = 10^3 (\ln n)^{0.45}$ and $|S_i| \leq |\mathcal{U}| \leq 4n(\ln n)^{-0.45}$ by I.1, the total number of edges which are recoloured in this phase is at most $s \cdot 4 \cdot 10^3 n \leq 8 \cdot 10^3 n^2 e^{-(\ln n)^{0.4}}$, by III.2(a). Thus, using Chernoff (Lemma 3.1) and the fact that $np \leq 10 \ln n$ we see that the probability that more than $1.6 \cdot 10^5 n e^{-(\ln n)^{0.4}} \ln n$ edges in W_3 are recoloured red in this phase is at most:

$$\Pr\left[\mathrm{Bin}\left(8 \cdot 10^3 n^2 e^{-(\ln n)^{0.4}}, p \right) > 1.6 \cdot 10^5 n e^{-(\ln n)^{0.4}} \ln n \right] < e^{-\frac{1}{3} \cdot 8 \cdot 10^4 n e^{-(\ln n)^{0.4}} \ln n} = o(1).$$

Since $1.6 \cdot 10^5 n e^{-(\ln n)^{0.4}} \ln n = o(n)$, this concludes the proof of the claim. ∎

We have shown that whp at the end of this phase all of the properties IV.1-IV.2 hold. We shall assume henceforth that all these properties hold for the sets R_4, W_4 and B_4.

4.6. Phase V

In this phase we create a Hamilton cycle in the red graph by merging the red cycle $\mathcal{C}_3$ with the set EXP_3, by recolouring red $o(n)$ edges. To this end, note first that $R_4[\mathrm{EXP}_3]$ has a bounded number of connected components. Indeed, suppose C is a connected component of $R_4[\mathrm{EXP}_3]$. It follows from properties IV.1(a), IV.1(b), i. of II.1(c) and the fact that $R_4[\mathrm{EXP}_3] = R_2[\mathrm{EXP}_3]$, that the set C has size $|C| > \frac{1}{6000} n (\ln n)^{-0.45}$ (since $N_{R_2[S]}(C) \subseteq C$). However, since by I.1, the set EXP_3 has size $|\mathrm{EXP}_3| \leq 4n(\ln n)^{-0.45}$, we can conclude that the graph $R_4[\mathrm{EXP}_3]$ has at most 24000 connected components. By recolouring red less than 24000 white edges in EXP_3 we can then whp make the red graph in EXP_3 connected.

Afterwards, we consider two adjacent vertices v, w in the red cycle $\mathcal{C}_3$ which have large white degree onto EXP_3. By recolouring edges between v, w and EXP_3 we can then whp find $x \neq y \in \mathrm{EXP}_3$ such that vx and wy are red edges. Finally, by recolouring red at most $|\mathrm{EXP}_3|$ edges inside EXP_3 we can find a red Hamilton path from x to y inside the set EXP_3. This path together with the red path $\mathcal{C}_3 \setminus \{vw\}$ and the red edges vx and wy then provides the desired red Hamilton cycle in V. The algorithm for this phase is as follows:

Algorithm: Let $C_1, \ldots, C_\ell$ be the connected components of the graph $R_4[\text{EXP}_3]$ where, as indicated above, we have $\ell \leq 24000$ and $|C_i| > \frac{1}{6000} n (\ln n)^{-0.45}$ for every $i \in [\ell]$. For $1 \leq i < \ell$ recolour white edges between C_i and C_{i+1} one by one until exactly one edge is recoloured red for each such i. Claim 4.15 ensures that this happens whp and we assume this henceforth. Note that this procedure makes the red graph in EXP_3 connected.

Next, let v, w be adjacent vertices of $\mathcal{C}_3$ such that $d_{W_4}(v, \text{EXP}_3) \geq \frac{2}{3} |\text{EXP}_3|$ and $d_{W_4}(w, \text{EXP}_3) \geq \frac{2}{3} |\text{EXP}_3|$. Recolour edges in W_4 between $\{v, w\}$ and EXP_3 until there exist two edges vx and wy which are red, where $x, y \in \text{EXP}_3$ are distinct vertices. Claim 4.15 ensures that whp such vertices v, w, x and y exist and we assume this henceforth. Now, set $e = xy$ and run the following routine:

Routine: Consider the graph H which is the current red graph on EXP_3. If $H \cup \{e\}$ does not contain a Hamilton cycle that uses the edge e, recolour e-boosters of H which are white edges one by one until one of them is recoloured red. Repeat this procedure until the graph $H \cup \{e\}$ considered contains a Hamilton cycle which uses the edge e.

(End of Routine)

Claim 4.15 ensures that whp this procedure is successful and we assume this henceforth. A Hamilton cycle in $H \cup \{e\}$ which uses the edge e then provides a red Hamilton path in EXP_3 from x to y. This red path together with the red path $\mathcal{C}_3 \setminus \{vw\}$ and the red edges vx and wy forms the desired red Hamilton cycle in V.

(End of Algorithm)

In the next claim we prove that some properties which are assumed in the algorithm hold whp.

Claim 4.15. *All of the following properties hold whp:*

(i) *For every $1 \leq i < \ell$, if we recolour all the white edges between C_i and C_{i+1} then at least one edge is recoloured red.*

(ii) *There exist adjacent vertices v, w in $\mathcal{C}_3$ such that $d_{W_4}(v, \text{EXP}_3) \geq \frac{2}{3} |\text{EXP}_3|$ and $d_{W_4}(w, \text{EXP}_3) \geq \frac{2}{3} |\text{EXP}_3|$. Moreover, after recolouring all the edges in W_4 between $\{v, w\}$ and EXP_3, there exist distinct vertices $x, y \in \text{EXP}_3$ such that vx and wy are red edges.*

(iii) *At any point during the routine, if H is the graph considered and if we recolour all the e-boosters of H which are white edges then there will be one which is recoloured red.*

Proof. We start by showing that (i) holds whp. Indeed, this follows from Lemma 3.4 since for every i we have $|C_i| > \frac{1}{6000} n (\ln n)^{-0.45}$.

Next we show that (ii) holds whp. Recall from I.4 that for every $u \in V(\mathcal{C}_1)$ we have $d_{W_1}(u, \mathcal{U}) = (1 - o(1)) |\mathcal{U}|$. Moreover, since $|\text{EXP}_3| = (1 - o(1)) |\mathcal{U}|$ by IV.1(a) it follows that for every $u \in V(\mathcal{C}_1)$ we have $d_{W_1}(u, \text{EXP}_3) = (1 - o(1)) |\text{EXP}_3|$. Recall now that in Phase II there were no edges recoloured between EXP_3 and $V(\mathcal{C}_1)$ and that in Phases III and IV the number of vertices $u \in V(\mathcal{C}_1)$ for which we recoloured edges touching u and vertices of EXP_3 is $o(n)$. Thus, it follows that for all but $o(n)$ vertices $u \in V(\mathcal{C}_1)$ we have $d_{W_4}(u, \text{EXP}_3) = (1 - o(1)) |\text{EXP}_3|$. Since $|V(\mathcal{C}_3) \setminus V(\mathcal{C}_1)| = o(n)$ we can find two vertices $v, w \in V(\mathcal{C}_1)$ which are adjacent in $\mathcal{C}_3$ and for which $d_{W_4}(v, \text{EXP}_3) \geq \frac{2}{3} |\text{EXP}_3|$ and $d_{W_4}(w, \text{EXP}_3) \geq \frac{2}{3} |\text{EXP}_3|$, as claimed.

Partition now the set EXP_3 into two sets A_v and A_w of size as equal as possible. If we recolour all the edges in W_4 between $\{v, w\}$ and EXP_3 then the probability that afterwards either there is no red edge vx with $x \in A_v$ or wy with $y \in A_w$ is at most

$$2 \Pr\left[\text{Bin}\left(\left(\frac{1}{6} - o(1)\right) |EXP_3|, p\right) = 0\right] = 2(1-p)^{\left(\frac{1}{6} - o(1)\right)|EXP_3|} \le e^{-\left(\frac{1}{6} - o(1)\right)|EXP_3|p} = o(1)$$

by IV.1(a). We conclude that property (ii) of the claim holds whp.

Finally we show that (iii) also holds whp. Note first that the routine can be executed at most $|EXP_3|$ times since each time the size of a longest path in the graph considered increases by at least one. Now, let H be one of the graphs considered during the routine. Note that the edge set $E(H)$ of H is of the form $E(H) = K_1 \cup K_2$ where $K_1 = R_4[EXP_3]$ and $K_2 \subseteq EXP_3$ is a set of size $|K_2| < |EXP_3| + \ell$. Indeed, $K_2 = E(H) \setminus K_1$ consists of the $\ell - 1$ red edges added in this phase connecting the connected components of K_1 and at most $|EXP_3|$ red edges added during the routine. Moreover, note that the graph H is connected since the red graph on EXP_3 is already connected when the routine is executed. Since $R_4[EXP_3] = R_2[EXP_3]$ and since the graph $(R_4 \setminus R_1)[EXP_3]$ has minimum degree at least $(\ln n)^{0.4}$ by IV.1(b), it follows from i. of II.1(c) that for any set $X \subseteq EXP_3$ of size $|X| \le \frac{1}{6000} n(\ln n)^{-0.45}$ we have $|N_H(X) \cup X| \ge 5|X|$. Thus, again using the fact that $R_4[EXP_3] = R_2[EXP_3]$ and that $\ell \le 24000$ we conclude from II.1 (d) that the number of e-boosters for H in the set $W_2[EXP_3]$ is at least $10^{-8} n^2 (\ln n)^{-0.9}$.

Recall now that in Phases III and IV no white edges inside EXP_3 were recoloured and so $W_4[EXP_3] = W_2[EXP_3]$. Moreover, as indicated above, in Phase V we recolour red less than $\ell + |EXP_3|$ white edges inside EXP_3. The probability that at least $10^{-9} n^2 (\ln n)^{-0.9}$ white edges are recoloured blue during Phase V is then at most the probability that at most $\ell + |EXP_3| - 1 \le 5n(\ln n)^{-0.45}$ white edges are recoloured red before $10^{-9} n^2 (\ln n)^{-0.9}$ white edges are recoloured blue, which is at most

$$\sum_{j=0}^{5n(\ln n)^{-0.45}} \binom{10^{-9} n^2 (\ln n)^{-0.9} + j}{j}(1-p)^{10^{-9} n^2 (\ln n)^{-0.9}} p^j$$

$$\le O(n(\ln n)^{-0.45}) \cdot \binom{O(n^2(\ln n)^{-0.9})}{5n(\ln n)^{-0.45}} \cdot p^{5n(\ln n)^{-0.45}} \cdot e^{-p \cdot \Omega(n^2(\ln n)^{-0.9})}$$

$$\le O(n(\ln n)^{-0.45}) \cdot (O((\ln n)^{0.55}))^{5n(\ln n)^{-0.45}} \cdot e^{-\Omega(n(\ln n)^{0.1})} = o(1).$$

Thus, whp less than $10^{-9} n^2 (\ln n)^{-0.9}$ white edges are recoloured blue in this phase. Thus, since the set $W_4[EXP_3]$ contains at least $10^{-8} n^2 (\ln n)^{-0.9}$ e-boosters for H we conclude that whp at any point in the routine there are at least $9 \cdot 10^{-9} n^2 (\ln n)^{-0.9}$ e-boosters for H which are white edges (at that point). The probability that none of these e-boosters is recoloured red is then at most

$$\Pr\left[\text{Bin}(9 \cdot 10^{-9} n^2 (\ln n)^{-0.9}, p) = 0\right]$$
$$= (1-p)^{9 \cdot 10^{-9} n^2 (\ln n)^{-0.9}} \le e^{-p \cdot 9 \cdot 10^{-9} n^2 (\ln n)^{-0.9}} \le e^{-9 \cdot 10^{-9} n(\ln n)^{0.1}}.$$

Thus, since the routine is executed at most $\ell + |EXP_3| \le 5n(\ln n)^{-0.45}$ many times, we conclude by the union bound that the probability that at some point in the routine all the e-boosters of H which are white edges (where H is the graph being considered at that point) are recoloured blue is at most

$$5n(\ln n)^{-0.45} \cdot e^{-9 \cdot 10^{-9} n(\ln n)^{0.1}} = o(1),$$

and so property (iii) holds whp as claimed. ∎

Assuming that the properties of Claim 4.15 hold and denoting by R_5 the set of red edges at the end of this phase's algorithm, it is clear from it that the graph R_5 contains a Hamilton cycle.

We claim now that $|R_5 \setminus R_4| < \ell + 3|\text{EXP}_3| = o(n)$. Indeed, in the beginning of this phase's algorithm exactly $\ell - 1$ edges are recoloured red in order to make the red graph in EXP_3 connected. Later, we recoloured edges between $\{v, w\}$ and EXP_3 and so at that point at most $2|\text{EXP}_3|$ edges are recoloured red. Finally, we recoloured one edge red each time the routine was executed. However, as indicated in the proof of Claim 4.15 the routine can be executed at most $|\text{EXP}_3|$ times. Thus, we conclude that less than $\ell + 3|\text{EXP}_3| = o(n)$ edges are recoloured red in this phase, as claimed.

To finish the proof of Theorem 1 we show that $|R_5| = n + o(n)$. Indeed by I.2, II.4, III.4, IV.2 and by the previous paragraph we conclude that

$$|R_5| = |R_1| + |R_5 \setminus R_1| = n + o(n)$$

as desired.

5. CONCLUDING REMARKS

In this paper we introduced a new type of problems in random graphs, where the goal is to expose a subgraph which possesses some target property $\mathcal{P}$, by asking as few queries as possible. Note that this problem is general and can be considered in any model of random structures.

Although we chose to focus on the property of Hamiltonicity, our proof method can be applied to prove analogous statements regarding other interesting properties. For example, one can show that for $p \geq \frac{\ln n + \omega(1)}{n}$, there exists an adaptive algorithm, interacting with the probability space $\mathcal{G}(n, p)$, which whp finds a matching of size $\lfloor n/2 \rfloor$ (a perfect matching) after getting $n/2 + o(n)$ positive answers.

Let us now show that one cannot get rid of the $o(n)$ term. More precisely, we show that whp at least $n + \Omega\left(\sqrt{\frac{1}{p}}\right)$ positive answers are needed in order to find a Hamilton cycle. In particular, if $p = \Theta\left(\frac{\log n}{n}\right)$ then this means that at least $\Theta\left(\sqrt{\frac{n}{\log n}}\right)$ extra positive answers are needed to find a Hamilton cycle. For this aim, let $k := k(n)$ be an integer and let G be a non-Hamiltonian graph on n vertices with exactly $n + k$ edges. Suppose that there exist a non-edge xy of G for which $G + xy$ is Hamiltonian, and observe that G contains at most two vertices of degree 1 and no isolated vertices. First, let us note that both x and y can have at most one neighbor in G which is of degree 2. Indeed, every neighbor of (say) x which has degree exactly 2 must be connected to x on the Hamilton cycle created by adding xy to G. Since xy must be an edge of this cycle, x cannot have more than one such neighbor. Second, we try to estimate the number of pairs xy for which both x and y have at most one neighbor of degree 2 in G. For this end, let A denote the set of all vertices in G of degree distinct than 2, and let $a := |A|$. Since G has exactly $n + k$ edges, and since there are at most 2 vertices of degree 1, it follows that $2 + 2(n - a) + 3(a - 2) \leq 2n + 2k$. Therefore, we have $a \leq 2k + 4$. Next, since $2(n - a) + \sum_{v \in A} d_G(v) \leq 2n + 2k$, using the estimate we obtained on a we conclude that

$$|N(A)| \leq \sum_{v \in A} d_G(v) = O(k).$$

Thus, all in all, we have $|A \cup N(A)| = O(k)$. Observe now crucially that if xy is a non-edge of G for which $G + xy$ is Hamiltonian then x and y must be in $A \cup N(A)$. Indeed, if (say) $x \notin A$ then x must have degree 2 and so at least one of its neighbours lies in A (as discussed above) and so $x \in N(A)$. Hence, there are $O(k^2)$ pairs xy for which $G + xy$ might be Hamiltonian. Suppose now that we have an adaptive algorithm, interacting with the probability space $\mathcal{G}(n, p)$, which whp finds a Hamilton cycle after getting at most $n + k + 1$ positive answers. Let G be the random graph obtained by the algorithm whose edges correspond to the positive answers until the step just before a Hamilton cycle is found. Note that by hypothesis whp G has at most $n + k$ edges and so by the reasoning above, it follows that there are at most $O(k^2)$ possible non-edges of G which can be queried to turn G into a Hamiltonian graph. However, if $k = o\left(\sqrt{\frac{1}{p}}\right)$, since $k^2 p = o(1)$, by conditioniong on what the graph G can be, we see by Markov's inequality that whp none of these pairs of vertices obtain positive answers even if we query all of them. Thus, we conclude that no such algorithm exists.

Note that even though the general setting we introduced appears to be new, there has been some previous work of this flavor in the literature, albeit inexplicit. For example, the DFS-based argument of [15] indicates that in the super-critical regime $p = \frac{1+\varepsilon}{n}$, $\Theta(\varepsilon)n$ positive answers suffice to uncover typically a connected component of size proportional to εn, and this is clearly optimal. The analysis from [15] also gives an adaptive algorithm for finding a path of length $\Theta(\varepsilon^2)n$ (which is whp the asymptotic order of magnitude of a longest path in such a random graph) after querying $\Theta(\varepsilon)n$ edges successfully. What matters here is the dependence on ε, and the above stated algorithmic bound is above the trivial lower bound by the $1/\varepsilon$ factor. In a companion paper [9] we show that this gap cannot be bridged and the algorithm from [15] is essentially (up to a $\log(1/\varepsilon)$ factor) best possible.

Another natural instance of the setting promoted in this paper is when the target property $\mathcal{P}$ is the containment of a fixed graph H. In this case, the obvious lower bound for the total number of queries needed is of order $1/p$. It appears that the form of the optimal bound for the number of queries may depend heavily on the value of p. Consider for example the case $H = K_3$. For constant p, one can just query the pairs in $\omega(1)$ pairwise disjoint triples of vertices to find w.h.p. a copy of the triangle. However, say, for the case $p = n^{-1/2}$ the right bound seems to be around $n^{3/4}$. Indeed, a simple algorithm asking a bit more than $n^{3/4}$ queries would be first to query pairs containing a fixed vertex v till $\omega(n^{1/4})$ edges touching v are found – this would take $\omega(n^{3/4})$ queries. Querying now all pairs between the other points of these edges uncovers w.h.p. an edge (u, w) closing a triangle with v. For the lower bound, one can argue that having $o(n^{1/4})$ positive answers on the board produces only $o(n^{1/2})$ pairs of vertices at distance two, and w.h.p. none of these pairs will show up in the random graph to close a desired triangle. This argument has certain similarities to the lower bound for avoidance of a given graph in Achlioptas processes given in [14]. Of course the case of triangles appears to be relatively easy, and we expect much more involved analysis for a general H.

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