# $Q$-CURVATURE FLOW ON $S^4$

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## 1. Introduction

Let $M$ be a closed four-manifold (compact without boundary) with metric $g$. If $Ric_g$ denotes the Ricci Tensor of $(M, g)$ and $R_g$ the scalar curvature, the $Q$-curvature $Q_g$ of $M$ is defined by the expression

$$Q_g = -\frac{1}{12}\left(\Delta_g R_g - R_g^2 + 3|Ric_g|^2\right). \tag{1}$$

(The above definition is not universally adopted and in other texts may differ by a factor 2.)

Similar to the Gauss curvature on a surface, the $Q$-curvature on a four-manifold is related to a conformally invariant operator and its integral gives information on the topology of the manifold.

Indeed, if $\Sigma$ is a closed surface with metric $g_0$ and Gauss curvature $K_0$, given a conformal metric $g = e^{2w}g_0$ on $\Sigma$, the Laplace-Beltrami operator transforms according to the rule

$$\Delta = \Delta_g = e^{-2w}\Delta_0, \tag{2}$$

where $\Delta_0 = \Delta_{g_0}$. Throughout, we use the analysts' sign convention (so that $\Delta$ is negative definite). The Gauss equation

$$-\Delta w + K_0 = K_g e^{2w}. \tag{3}$$

then relates the Gauss curvature $K_g$ of the metric $g$ to the curvature $K_0$ of $g_0$ and implies the invariance of the total scalar curvature integral

$$\int_\Sigma K_g\, d\mu_g = \int_\Sigma K_g e^{2w} d\mu_{g_0} = \int_\Sigma K_0\, d\mu_{g_0}$$

under conformal changes of the metric; in fact, by the Gauss-Bonnet formula

$$\int_\Sigma K_g\, d\mu_g = 2\pi\chi(\Sigma), \tag{4}$$

the total scalar curvature is already determined by the topology of the surface.

In complete analogy with (3), as shown by Branson-Ørsted [7], in four space dimensions the $Q$-curvature of a metric $g = e^{2w}g_0$ is related to the $Q$-curvature $Q_0$ of the background metric $g_0$ via the equation

$$P_{g_0} w + 2Q_0 = 2Q_g e^{4w}, \tag{5}$$

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where $P_{g_0}$ is the Paneitz operator in the metric $g_0$, introduced in [32]. For any given $g$ the operator $P_g$ acts on a smooth function $\varphi$ on $M$ via

$$P_g(\varphi) = \Delta_g^2 \varphi - div\left( (\frac{2}{3} R_g g - 2Ric_g) d\varphi \right). \tag{6}$$

Similar to (2), the Paneitz operator is conformally invariant in the sense that

$$P_g = e^{-4w} P_{g_0} \tag{7}$$

for any conformal metric $g = e^{2w} g_0$. Finally, if we denote as $W$ the Weyl tensor of $M$, then, similar to (4) there holds

$$\int_M \left( Q_g + \frac{|W|^2}{8} \right) d\mu_g = 4\pi^2 \chi(M); \tag{8}$$

see [5]. In particular, on a locally conformally flat manifold where $W \equiv 0$ we obtain the exact analogue of (4).

As in the two-dimensional case of equation (3), in the context of equation (5) it is natural to ask for a solution to the *uniformization problem*, that is, to ask whether on a given 4-manifold $(M, g_0)$ there exists a conformal metric of constant $Q$-curvature. On the other hand, given a smooth function $f$ on $M$, one may ask whether there exists a conformal metric having $f$ as its $Q$-curvature. In the case of (3) for conformal metrics on $S^2$, this is the famous *Nirenberg's problem*, studied, for instance, in [13], [14], [15], [30], [37].

A partial affirmative answer to the first question is given by Chang-Yang [16] under the condition $k_0 = \int_M Q_0 \, d\mu_0 < 8\pi^2$ and assuming that $P_g$ is a positive operator whose kernel only consists of the constant functions. In view of a result of Gursky [25] the latter hypothesis is satisfied whenever $k_0 > 0$ and provided $(M, g_0)$ is of positive Yamabe type; see also [17], Theorem 3.1 and the Remarks following. The same result was later rederived by Brendle [8] via a flow approach, again in the "subcritical" case when $k_0 < 8\pi^2$, which by Gursky's work [25] precisely rules out the case when $(M, g_0)$ is conformal to the standard sphere. In fact, $\int_{S^4} Q_{g_{S^4}} d\mu_{g_{S^4}} = 8\pi^2$, and blow-up phenomena may occur. The result of Chang-Yang has been extended recently by Djadli-Malchiodi [21] to the case in which $P_g$ has no kernel and $k_0$ is not a positive integer multiple of $8\pi^2$. Again, the last condition is used to rule out blow-up phenomena, see [29].

As for the analogue of Nirenberg's problem on $S^4$, since $Q_{g_{S^4}} \equiv 3$, by (5) this is equivalent to finding a solution $u$ of the equation

$$P_{g_{S^4}} u + 6 = 2 f e^{4u} \tag{9}$$

for a given function $f$ on $S^4$, where $P_{g_{S^4}} = \Delta_{g_{S^4}}^2 - 2\Delta_{g_{S^4}}$. The problem is variational; solutions can be characterized as critical points of the functional

$$E_f(u) = \fint_{S^4} \left( u P_{g_{S^4}} u + 4 Q_{g_{S^4}} u \right) d\mu_{g_{S^4}} - 3 \log \left( \fint_{S^4} f e^{4u} d\mu_{g_{S^4}} \right) \tag{10}$$

on $H^2(S^4)$, where $\fint_{S^4} \varphi d\mu_{g_{S^4}}$ denotes the average of any function $\varphi$ on $S^4$. However, as is common in geometric problems, this functional fails to satisfy standard compactness conditions like the Palais-Smale condition.

Furthermore, the Gauss-Bonnet-Chern formula (8) and identities of Kazdan-Warner [26] type impose obstructions to the existence of solutions of (9). In fact, since the Weyl tensor vanishes for any metric conformal to $g_{S^4}$, equation (8) takes the form

$$\int_{S^4} Q_g \, d\mu_g = 8\pi^2 \tag{11}$$

and there cannot be a solution of (9) when $f \leq 0$. Moreover, upon integrating by parts as in [16], p. 205, one obtains the identity

$$\int_{S^4} \langle \nabla Q_g, \nabla x_i \rangle_{g_{S^4}} \, d\mu_g = 0, \quad 1 \leq i \leq 5, \tag{12}$$

where $(x_i)_{i=1,\dots,5}$ are the restrictions of the coordinate functions of $\mathbb{R}^5$ to $S^4$. It follows that no function of the form $f = \psi \circ x_i$, where $\psi$ is monotone on $[-1,1]$, can be the $Q$-curvature of a conformal metric $g$ on $S^4$. Thus the study of (9) is rather delicate.

Recently, Wei-Xu, [38] (see also [9]), by combining blow-up analysis and degree theory, proved the following result which is the counterpart of results obtained by Chang-Yang [15] and Chang-Liu [13] for equation (3). A similar statement is given in [2] for the scalar curvature problem on $S^3$.

**Theorem 1.1.** *Suppose $f : S^4 \to \mathbb{R}$ is positive with only non-degenerate critical points with Morse indices $ind(f,p)$ and such that $\Delta_{g_{S^4}} f(p) \neq 0$ at any such point $p$. In addition, assume that*

$$\sum_{\nabla f(p)=0, \Delta_{g_{S^4}} f(p)<0} (-1)^{ind(f,p)} \neq 1. \tag{13}$$

*Then there exists a solution of* (9).

The aim of this paper is to analyze equation (9) by means of the $Q$-curvature flow and to extend the results of Brendle [8] for this flow to the critical case when $(M, g_0)$ is conformal to the standard sphere. Our approach will be similar to the treatment of the two-dimensional Hamilton-Ricci flow in [36] and the flow approach to Nirenberg's problem developed in [37]. In particular, in Theorem 4.1 we again shall see the fundamental role that the Kazdan-Warner identity (12) plays in preventing blow-up and in proving exponential convergence of the flow when $f \equiv$ const.. A similar result was independently obtained by Brendle [11] by a different method.

On the other hand, for a prescribed curvature function $f \neq$ const. we argue by contradiction, and show, thereby closely following [37], that whenever (9) does not admit a solution then, as $t \to \infty$, conformal metrics evolving under the $Q$-curvature flow concentrate in a nearly spherical shape around points $p(t) \in S^4$, $t > 0$, whose motion follows a pseudo-gradient flow for $f$. A detailed analysis, moreover, shows that the flow $(p(t))_{t>0}$ converges to a critical point $p$ of $f$ where $\Delta_{g_{S^4}} f(p) < 0$, which is the reason why only such critical points contribute in Theorem 1.1. Theorem 1.1 then may be deduced from Morse theory; in fact, Theorem 1.1 is a special case of the following result, which has counterparts in [34] and in [28] for the scalar curvature problem on $S^n$.

**Theorem 1.2.** *Suppose $f : S^4 \to \mathbb{R}$ is positive with only non-degenerate critical points with Morse indices $ind(f,p)$ and such that $\Delta_{g_{S^4}} f(p) \neq 0$ at any such point $p$. Let*

$$m_i = \#\{p \in S^4; \nabla f(p) = 0, \Delta_{g_{S^4}} f(p) < 0, ind(f,p) = 4 - i\}. \tag{14}$$

*Then, if there is no solution with coefficients* $k_i \geq 0$ *to the system of equations*

$$m_0 = 1 + k_0,\ m_i = k_{i-1} + k_i,\ 1 \leq i \leq 4,\ k_4 = 0, \tag{15}$$

*there exists a solution of* (9).

Observe that in contrast to the usual applications of Morse theory we will be dealing with a flow that increases (rather than decreases) the value of $f$; therefore the index of a critical point $p$ of $f$ (counting the number of negative eigenvalues of the Hessian) has to be substituted by its complement $i = 4 - ind(f,p) = ind(-f,p)$. Obviously, this distinction is of no consequence for the statement of Theorem 1.1.

The problem of prescribed $Q$-curvature has been studied also in dimensions greater than four by Brendle [8], still in the "subcritical" case. For the "critical" case of prescribed $Q$-curvature on $S^n$, in [22] and [23] via a finite-dimensional reduction and blow-up analysis certain perturbative results are obtained in any dimension, and non-perturbative results in dimensions 5 and 6. Similar to Brendle's work [8], it should not be difficult to extend the present flow approach to higher-dimensions and - possibly - further improve these results.

The analogue of the uniformization problem for scalar curvature in dimensions greater than 2 is the Yamabe problem. In [35] and, finally, [10] convergence of the Yamabe flow is established for arbitrary initial data in dimensions 3, 4 and 5, thereby heavily relying on the positive mass theorem of general relativity; see also [24]. It should be of considerable interest to obtain analogous results for the $Q$-curvature flow in dimensions greater than four, as well.

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## 2. The flow

Let $g_{S^4}$ be the standard metric on $S^4$. Any conformal metric $g$ can be written as $g = e^{2u} g_{S^4}$ for some smooth function $u$ on $S^4$. Let $f \in C^\infty(S^4)$ be a given positive function. For $g_0 = e^{2u_0} g_{S^4}$ satisfying

$$vol(S^4, g_0) = \int_{S^4} d\mu_{g_0} = \int_{S^4} d\mu_{g_{S^4}} = \frac{8}{3}\pi^2, \tag{16}$$

we define an evolution of conformal metrics $g(t)$, $t \geq 0$, whose $Q$-curvatures approximate (a positive constant multiple of) the prescribed curvature function $f$.

To define the flow, let $g(t)$ be given by $g(t) = e^{2u(t)} g_{S^4}$, where $u$ solves the equation

$$u_t = \frac{du}{dt} = \alpha f - Q, \tag{17}$$

with initial data $u(0) = u_0$. Here $Q = Q_g$ denotes the $Q$-curvature of $g = g(t)$, given by

$$Q = \frac{1}{2} e^{-4u} (P_{g_{S^4}} u + 6) \quad \text{on } S^4, \tag{18}$$

and $\alpha$ is chosen in such a way that

$$\alpha \int_{S^4} f \, d\mu = \int_{S^4} Q \, d\mu = 8\pi^2 \tag{19}$$

for all $t \geq 0$, where $d\mu = d\mu_g = e^{4u} d\mu_{g_{S^4}}$. Lemma 6.2 in [36] may easily be adapted to this setting to show, together with the work of Brendle [8], that the initial value problem (17), (19) has a unique, global, smooth solution on an arbitrary closed 4-manifold; see Lemma 3.8 below.

In view of (19) we have

$$\frac{d}{dt}\left(\int_{S^4} d\mu\right) = 4\int_{S^4} u_t d\mu = 4\int_{S^4} (\alpha f - Q) d\mu = 0,$$

and the initial normalization (16) implies the identity

$$vol(S^4, g) = \int_{S^4} d\mu = \frac{8}{3}\pi^2 \tag{20}$$

for all $t \geq 0$.

Moreover, the functional $E_f$ is non-increasing along the flow defined by (17), (19). Indeed, let

$$\bar{u} = \fint_{S^4} u \, d\mu_{g_{S^4}} = \frac{3}{8\pi^2}\int_{S^4} u \, d\mu_{g_{S^4}}$$

denote the mean value of $u$, etc., and let

$$E(u) = \fint_{S^4} \left(uP_{g_{S^4}} u + 4Q_{g_{S^4}} u\right) d\mu_{g_{S^4}} = \fint_{S^4} (|\Delta u|^2_{g_{S^4}} + 2|\nabla u|^2_{g_{S^4}} + 12u) d\mu_{g_{S^4}}$$

so that

$$E_f(u) = E(u) - 3\log\left(\fint_{S^4} f e^{4u} d\mu_{g_{S^4}}\right).$$

**Lemma 2.1.** *Let $u$ be a smooth solution of (16) - (19). Then one has*

$$\frac{d}{dt}E_f(u) = -4\fint_{S^4} |\alpha f - Q|^2 \, d\mu \leq 0.$$

**Proof.** Using (17) - (19), we obtain

$$\begin{aligned}
\frac{d}{dt}E_f(u) &= 2\fint_{S^4} (P_{g_{S^4}} u + 2Q_{g_{S^4}}) u_t \, d\mu_{g_{S^4}} - 12\fint_{S^4} f u_t \, d\mu / \fint_{S^4} f \, d\mu \\
&= 4\fint_{S^4} (Q - \alpha f) u_t \, d\mu = -4\fint_{S^4} |\alpha f - Q|^2 \, d\mu.
\end{aligned}$$

□

Recall the following analogue of Onofri's [31] inequality, due to Beckner [3]. The present interpretation appears as formula (4.1") in the work of Chang-Yang [16], who also give an instructive new proof of Beckner's inequality.

**Proposition 2.2.** *For any $u \in H^2(S^4)$ there holds*

$$\log\left(\fint_{S^4} e^{4u} d\mu_{g_{S^4}}\right) \leq \frac{1}{3}E(u).$$

Thus, for a metric which satisfies (20), we have the uniform lower bound

$$E_f(u) \geq -3\log(\max_{S^4} f)$$

and Lemma 2.1 implies the estimate

$$\int_0^\infty \int_{S^4} |\alpha f - Q|^2 \, d\mu \, dt \leq \frac{2}{3}\pi^2 (E_f(u_0) + 3\log(\max_{S^4} f)) < \infty. \tag{21}$$

In particular, for a suitable sequence $t_l \to \infty$ with corresponding metrics $g_l = g(t_l)$, we obtain

$$\int_{S^4} |Q_l - \alpha(t_l) f|^2 \, d\mu_{g_l} \to 0 \quad (l \to \infty). \tag{22}$$

Therefore, if $g_l$ converges to a metric $g_\infty$ with $Q$-curvature $Q_\infty$, then $Q_\infty = \alpha_\infty f$. In particular, we encounter this situation when $f \equiv$ const.; see Section 4.

On the other hand, if $g_l$ does not converge along any sequence $t_l \to \infty$, we observe a concentration behavior similar to [37], whose precise analysis in Section 5 will lead us to the proof of Theorem 1.2.

**Notation:** All norms we use are taken with respect to the standard spherical metric $g_{S^4}$, unless otherwise specified. For brevity in the following we let $\Delta_{S^4} = \Delta_{g_{S^4}}$, etc. The letter $C$ represents a generic constant which may vary from line to line (and also within the same line), occasionally numbered for clarity.

## 3. Asymptotics

In this section we describe the asymptotic behavior of the solution $u(t)$ of (17) and the corresponding metric $g(t)$ when $t \to \infty$.

### 3.1. Normalized flow.

Similarly to [14], Proposition 2.2, or [36], p. 260 f. for the case of $S^2$, given a smooth family of metrics $t \mapsto g(t) = e^{2u(t)} g_{S^4}$ there exists a smooth family of conformal diffeomorphisms $t \mapsto \Phi = \Phi(t)\colon S^4 \to S^4$, normalized with respect to rotations of $S^4$, so that, letting $h = \Phi^* g$, we have

$$\int_{S^4} x \, d\mu_h = 0 \text{ for all } t. \tag{23}$$

Here the points of $S^4$ are identified with their image in $\mathbb{R}^5$ through the standard immersion. We can write $h$ as $h = e^{2v} g_{S^4}$, where

$$v = u \circ \Phi + \frac{1}{4}\log(\det(d\Phi)). \tag{24}$$

By conformal invariance there holds

$$E(v) = E(u); \tag{25}$$

see for instance [38], Lemma 2.2. Moreover by (24) we have

$$vol(S^4, h) = \int_{S^4} e^{4v} d\mu_{S^4} = \int_{S^4} e^{4u} d\mu_{S^4} = vol(S^4, g) \quad \text{ for all } t \geq 0. \tag{26}$$

Therefore, if $u(t)$ solves (17), (19) with initial data satisfying (16), from (20), Lemma 2.1, and Proposition 2.2 we deduce the uniform energy bounds

$$0 \leq E(v) = E(u) = E_f(u) + 3\log\left(\fint_{S^4} f \, d\mu\right) \leq E_f(u_0) + 3\log\left(\max_{S^4} f\right) \tag{27}$$

for $u$ and $v$.

The bounds on $v$ can be improved by using the following result of Wei-Xu [38], Theorem 2.6, which extends a previous theorem of Aubin [1] to higher dimensions. A similar result was obtained by Brendle [9], Proposition 2.2; Brendle's result, however, requires the a-priori bound [9], Lemma 2.1, which is not available here.

**Proposition 3.1.** *There exists a constants $a < 1/3$ such that for every $v \in H^2(S^4)$ with corresponding metric $h = e^{2v}g_{S^4}$ satisfying* (23) *there holds*

$$\log\left(\fint_{S^4} e^{4v} d\mu_{S^4}\right) \le a \fint_{S^4} vP_{S^4}v \, d\mu_{S^4} + 4\fint_{S^4} v \, d\mu_{S^4}.$$

Consequently, the normalized family $(v(t))_{t>0}$ is bounded in $H^2(S^4)$.

**Lemma 3.2.** *For a smooth solution $u$ of* (17), (19), (20) *and the corresponding normalized flow there holds*

$$\sup_t \|v(t)\|_{H^2(S^4)} \le C. \tag{28}$$

*Moreover, for any $\sigma \in \mathbb{R}$ we have*

$$\sup_t \int_{S^4} e^{4\sigma v} \, d\mu_{S^4} \le C(\sigma). \tag{29}$$

*The bounds in* (28) *and* (29) *depend only on $E(v)$.*

**Proof.** From Proposition 3.1 we obtain

$$a\fint_{S^4} vP_{S^4}v \, d\mu_{S^4} + 4\fint_{S^4} v \, d\mu_{S^4} \ge 0. \tag{30}$$

Hence, similar to [14], Corollary 2.2, we can estimate

$$\begin{aligned}
&(\frac{1}{3}-a)\fint_{S^4}\left(|\Delta_{S^4}v|^2 + 2|\nabla v|^2_{S^4}\right) d\mu_{S^4} = (\frac{1}{3}-a)\fint_{S^4} vP_{S^4}v \, d\mu_{S^4} \\
&\quad = \fint_{S^4}\left(\frac{1}{3}vP_{S^4}v + 4v\right) d\mu_{S^4} - \left(a\fint_{S^4} vP_{S^4}v \, d\mu_{S^4} + 4\fint_{S^4} v \, d\mu_{S^4}\right) \\
&\quad \le \frac{1}{3}E(v) \le C.
\end{aligned} \tag{31}$$

Moreover, from Jensen's inequality and (26) it follows that

$$4\overline{v} := \fint_{S^4} 4v \, d\mu_{S^4} \le \log\left(\fint_{S^4} e^{4v} \, d\mu_{S^4}\right) = 0,$$

whereas from (30) we get

$$C \le \left(a\fint_{S^4} vP_{S^4}v d\mu_{S^4} + 4\fint_{S^4} v d\mu_{S^4}\right) = aE(v) + (4-12a)\overline{v}.$$

Hence from (27) we conclude

$$|\overline{v}| \le C. \tag{32}$$

Finally, from (31), (32) and Poincaré's inequality it follows that

$$||v||_{L^2(S^4)} \le ||v-\overline{v}||_{L^2(S^4)} + ||\overline{v}||_{L^2(S^4)} \le C||\nabla v||_{L^2(S^4)} + C \le C.$$

This implies (28). Finally, (29) is a consequence of Proposition 2.2, applied to the function $\sigma v$, and (28). □

From (24) one finds

$$v_t = u_t \circ \Phi + \frac{1}{4} e^{-4v} div_{S^4}(\xi e^{4v}), \tag{33}$$

where $\xi = (d\Phi)^{-1} \frac{d\Phi}{dt}$. We can bound $\xi$ in terms of $u_t$, as follows.

Differentiating (23) with respect to $t$ and using (33), we find

$$\begin{aligned} 0 = \frac{d}{dt} \left( \int_{S^4} x \, d\mu_h \right) &= 4 \int_{S^4} x \, v_t \, d\mu_h \\ &= 4 \int_{S^4} x \, u_t \circ \Phi \, d\mu_h + \int_{S^4} x \, div_{S^4}(\xi e^{4v}) \, d\mu_{S^4} \\ &= 4 \int_{S^4} x \, u_t \circ \Phi \, d\mu_h - \int_{S^4} \xi \, d\mu_h. \end{aligned} \tag{34}$$

Let $G = \text{Möb}^+(4)$ denote the finite-dimensional Lie group of oriented conformal diffeomorphisms of $S^4$. We identify an element $\xi \in T_{id}G$ with the vector field $\frac{d}{dt}\Phi(t)|_{t=0}$ on $S^4$, where $\Phi(t) = \exp_{id}(t\xi)$ is the family of conformal diffeomorphisms generated by $\xi$. We then define the map

$$X : T_{id}G \ni \xi \mapsto \int_{S^4} \xi d\mu_h \in \mathbb{R}^5.$$

By reasoning as in [14], Appendix, or [36], p. 260, one finds that for all metrics $h$ which are uniformly equivalent to $g_{S^4}$, $X$ has full rank. Moreover, $X$ is invertible with bounded inverse on the subspace generating normalized conformal diffeomorphisms. In particular, from (34) we obtain the uniform estimate

$$||\xi||^2_{L^\infty} \le C \int_{S^4} |u_t \circ \Phi|^2 \, d\mu_h = C \int_{S^4} |\alpha f - Q|^2 \, d\mu. \tag{35}$$

As explained in [36], p. 260, the constant $C$ in (35) is uniform for all normalized metrics $h$ satisfying (29) with uniform constants $C = C(\sigma)$ for $|\sigma| = 2$ . By Lemma 3.2 and (27) this, in particular, is the case for all metrics $h$ arising from smooth solutions $u$ of (17), (19), (20) satisfying a uniform bound $E_f(u_0) \le \beta_0$ for some $\beta_0 \in \mathbb{R}$.

3.2. **Global existence.** From Lemma 2.1 and (35) we obtain the following estimate, which is similar to [36], Lemma 6.2, or [37], Lemma 3.3.

**Lemma 3.3.** *For any $T > 0$ and any smooth solution $u$ of* (16) - (19) *there holds*

$$4 \sup_{0 \le t < T} \int_{S^4} |u(t)| \, d\mu_{g_{S^4}} \le \sup_{0 \le t < T} \int_{S^4} e^{|4u(t)|} \, d\mu_{g_{S^4}} < \infty.$$

**Proof.** The argument is completely analogous to the proof of [37], Lemma 3.3. □

Combining Lemma 3.3 with (27), we obtain a locally uniform bound for $u(t)$ in $H^2$. Global existence of the flow then follows as in the work of Brendle [8].

3.3. **Curvature evolution.** We now derive the evolution equations for the $Q$-curvature and the $L^2$-norm of the flow speed. From (17) and (18) we obtain

$$\begin{aligned} Q_t = \frac{dQ}{dt} &= \frac{1}{2}\frac{d}{dt}\left(e^{-4u}(P_{S^4}u + 2Q_{S^4})\right) \\ &= -4u_tQ + \frac{1}{2}Pu_t = -4Q(\alpha f - Q) + \frac{1}{2}P(\alpha f - Q), \end{aligned} \tag{36}$$

where $P = P_g = e^{-4u}P_{S^4}$. Differentiating (19) in time, we find

$$\alpha_t \int_{S^4} f d\mu = 4\alpha \int_{S^4} (Q - \alpha f) f d\mu. \tag{37}$$

Hence (36) also yields

$$\begin{aligned} &\frac{d}{dt}\left(\int_{S^4} |Q - \alpha f|^2 d\mu\right) = \int_{S^4} (2(Q - \alpha f)(Q_t - \alpha_t f) - 4(Q - \alpha f)^3) d\mu \\ &= -\int_{S^4} (Q - \alpha f)P(Q - \alpha f)\, d\mu + 8\int_{S^4} Q(Q - \alpha f)^2\, d\mu \\ &\quad - 4\int_{S^4} (Q - \alpha f)^3 d\mu - 4\pi^2(\alpha_t/\alpha)^2, \end{aligned} \tag{38}$$

where

$$\int_{S^4} (Q - \alpha f)P(Q - \alpha f)\, d\mu = \int_{S^4} \left(|\Delta_{S^4}(Q - \alpha f)|^2 + 2|\nabla(Q - \alpha f)|^2_{S^4}\right)\, d\mu_{S^4}.$$

3.4. **Curvature decay.** In this subsection we prove that the decay in (22) is indeed uniform as $t \to \infty$.

**Lemma 3.4.** *For a smooth solution $u$ of* (16) - (19) *there holds*

$$\int_{S^2} |Q - \alpha f|^2\, d\mu \to 0 \qquad \text{as } t \to \infty.$$

**Proof.** Let $v(t)$ be the normalized flow introduced in Section 3.1, and let $h = e^{2v}g_{S^4}$. Also denote as $\Phi = \Phi(t)$ the Möbius transformations on $S^4$ such that $h = \Phi^* g$. Note that $Q_h = Q_g \circ \Phi$. We also let $f_\Phi = f \circ \Phi$.

By the geometric invariance of the Paneitz operator, from (38) we derive

$$\begin{aligned} &\frac{d}{dt}\left(\int_{S^4} |Q_h - \alpha f_\Phi|^2 d\mu_h\right) + 4\pi^2(\alpha_t/\alpha)^2 \\ &= -\int_{S^4} (Q_h - \alpha f_\Phi)P_h(Q_h - \alpha f_\Phi)\, d\mu_h + 8\int_{S^4} Q_h(Q_h - \alpha f_\Phi)^2 d\mu_h \\ &\quad - 4\int_{S^4} (Q_h - \alpha f_\Phi)^3 d\mu_h \\ &= -\int_{S^4} \left(|\Delta_{S^4}(Q_h - \alpha f_\Phi)|^2 + 2|\nabla(Q_h - \alpha f_\Phi)|^2_{S^4}\right)\, d\mu_{S^4} \\ &\quad + 8\alpha\int_{S^4} f_\Phi(Q_h - \alpha f_\Phi)^2 d\mu_h + 4\int_{S^4} (Q_h - \alpha f_\Phi)^3 d\mu_h. \end{aligned} \tag{39}$$

Given $\varepsilon_0 > 0$, by (22) there exist arbitrarily large numbers $t_0$ such that

$$\int_{S^4} |Q_h - \alpha f_\Phi|^2 d\mu_h = \int_{S^4} |Q - \alpha f|^2 d\mu < \varepsilon_0 \quad \text{at } t = t_0. \tag{40}$$

Fix such a number $t_0$ and choose $t_1 \geq t_0$ such that

$$\sup_{t_0 \leq t < t_1} \int_{S^4} |Q_h - \alpha f_\Phi|^2 d\mu_h \leq 2\varepsilon_0. \tag{41}$$

From (19) it follows that

$$\frac{3}{\max_{S^4} f} \leq \alpha(t) = 3 \left( \fint_{S^4} f \, d\mu \right)^{-1} \leq \frac{3}{\min_{S^4} f}. \tag{42}$$

Hence for $t_0 \leq t < t_1$ we find

$$\begin{aligned} ||Q_h||_{L^2(S^4,h)} &\leq ||Q_h - \alpha f_\Phi||_{L^2(S^4,h)} + \alpha ||f_\Phi||_{L^2(S^4,h)} \\ &\leq \sqrt{2\varepsilon_0} + \sqrt{24}\pi \frac{\max_{S^2} f}{\min_{S^2} f} = C(f). \end{aligned}$$

From (29) we deduce that

$$P_{S^4} v + 6 = 2 Q_h e^{4v}$$

is bounded in $L^p(S^4)$ for any $p < 2$. Standard elliptic theory then yields that $v(t)$ is bounded in $W^{4,p}(S^2)$ for any $p < 2$ and hence also in $L^\infty(S^4)$. Therefore we may improve the previous result and conclude that, in fact, $v(t)$ is bounded in $H^4(S^4)$. In particular, with a uniform constant $C$ we have

$$C^{-1} g_{S^4} \leq h \leq C g_{S^4}. \tag{43}$$

and from the Sobolev's embedding $H^1(S^4) \hookrightarrow L^4(S^4)$, we obtain

(44)

$$\begin{aligned} \int_{S^4} |Q_h - \alpha f_\Phi|^3 d\mu_h &\leq C ||Q_h - \alpha f_\Phi||_{L^2(S^4,h)} ||Q_h - \alpha f_\Phi||^2_{L^4(S^4)} \\ &\leq C ||Q_h - \alpha f_\Phi||_{L^2(S^4,h)} ||Q_h - \alpha f_\Phi||^2_{H^1(S^4)} \\ &\leq C ||Q_h - \alpha f_\Phi||_{L^2(S^4,h)} \int_{S^4} (|\nabla (Q_h - \alpha f_\Phi)|^2_{S^4} + |Q_h - \alpha f_\Phi|^2) d\mu_{S^4} \\ &\leq C_0 ||Q_h - \alpha f_\Phi||_{L^2(S^4,h)} \left( \int_{S^4} |\nabla (Q_h - \alpha f_\Phi)|^2_{S^4} \, d\mu_{S^4} + \int_{S^4} |Q_h - \alpha f_\Phi|^2 d\mu_h \right). \end{aligned}$$

If we now choose $\varepsilon_0 > 0$ so that $8\varepsilon_0 C_0^2 \leq 1$, from (39) and (44) for $t_0 \leq t \leq t_1$ we conclude

$$\begin{aligned} &\frac{d}{dt} \left( \int_{S^4} |Q_h - \alpha f_\Phi|^2 d\mu_h \right) \\ &\quad \leq (8\alpha \max_{S^2} f + 2) \int_{S^4} |Q_h - \alpha f_\Phi|^2 d\mu_h \leq C_1 \int_{S^4} |Q_h - \alpha f_\Phi|^2 d\mu_h, \end{aligned} \tag{45}$$

where $C_1 = 2 + 24 \max_{S^2} f / \min_{S^4} f$. The same estimate of course, also holds in the original coordinates for the functions $u$ and $f$ instead of $v$ and $f_\Phi$. Integrating from $t_0$ to $t$, for any $t \in [t_0, t_1[$ we then obtain

$$\int_{S^4} |Q - \alpha f|^2 d\mu \leq \int_{S^4} |Q - \alpha f|^2 d\mu|_{t=t_0} + C_1 \int_{t_0}^\infty \int_{S^4} |Q - \alpha f|^2 d\mu dt. \tag{46}$$

By (21), the right hand side is smaller than $2\varepsilon_0$ if $t_0$ is sufficiently large and satisfies (40). Then (41) and hence also (46) will be valid for every $t \geq t_0$. Finally, letting

$t_0 \to \infty$ suitably, we obtain

$$\limsup_{t\to\infty} \int_{S^4} |Q - \alpha f|^2 d\mu$$
$$\leq \liminf_{t_0\to\infty} \left( \int_{S^4} |Q - \alpha f|^2 d\mu|_{t=t_0} + C_1 \int_{t_0}^{\infty} \int_{S^4} |Q - \alpha f|^2 d\mu dt \right) = 0,$$

proving the assertion. □

3.5. **Concentration-compactness.** The next result, which can be proved as in [8], Proposition 1.4, gives a first characterization of the asymptotic behavior of sequences of functions $(u_l)_l$ whose $Q$-curvatures, given by equation (18), are bounded in $L^2(S^4, g_l)$.

**Lemma 3.5.** *Let $(u_l)_l$ be a sequence of smooth functions on $S^4$ with associated metrics $g_l = e^{2u_l} g_{S^4}$, $l \in \mathbb{N}$. Suppose that $vol(S^4, g_l) = \frac{8}{3}\pi^2$ and $||Q_{g_l}||_{L^2(S^4,g_l)} \leq C$ for some uniform constant $C$. Then, either a subsequence $(u_l)_l$ is bounded in $H^4(S^4)$, or for every sequence $l \to \infty$ we can find a subsequence (relabelled) and points $p_1, ..., p_I \in S^4$ such that for every $r > 0$ and any $i \in \{1, ..., I\}$ there holds*

$$\liminf_{l\to\infty} \int_{B_r(p_i)} |Q_l| d\mu_l \geq 4\pi^2, \tag{47}$$

*where $\mu_l = \mu_{g_l}$, $Q_l = Q_{g_l}$. In the latter case a subsequence $(u_l)_l$ converges in $H^4_{loc}(S^4 \setminus \{p_1\}, \ldots, \{p_I\})$.*

As in [37], Lemma 3.5, the previous result may be sharpened considerably if we assume $L^2$-convergence of the $Q$-curvatures associated with $(u_l)_l$ to some smooth limit function $Q_\infty > 0$. By Lemma 3.4, this assumption is satisfied by any sequence $(u(t_l))_l$, where $u(t)$ solves (17), (19) and where $t_l \to \infty$ as $l \to \infty$.

**Lemma 3.6.** *Let $(u_l)$ be as in Lemma 3.5. In addition, suppose that we have $||Q_{g_l} - Q_\infty||_{L^2(S^4,g_l)} \to 0$ as $l \to \infty$ for some smooth positive function $Q_\infty$ on $S^4$. Also let $h_l = \Phi_l^* g_l = e^{2v_l} g_{S^4}$ be the associated sequence of normalized metrics as in Section 3.1. Then, up to selection of a subsequence, either i) $u_l \to u_\infty$ in $H^4(S^4)$, where $g_\infty = e^{2u_\infty} g_{S^4}$ has $Q$-curvature $Q_\infty$, or ii) there exists $p \in S^4$ such that*

$$d\mu_{g_l} \rightharpoonup \frac{8}{3}\pi^2 \delta_p \quad \text{as } l \to \infty \tag{48}$$

*weakly in the sense of measures, and*

$$h_l \to g_{S^4} \text{ in } H^4(S^4), \quad Q_{h_l} \to 3 \text{ in } L^2(S^4).$$

*In the latter case, $\Phi_l$ converges weakly in $H^2(S^4)$ to the constant map $\Phi_\infty \equiv p$.*

**Proof.** We can apply Lemma 3.5. If $(u_l)_l$ is bounded in $H^4(S^4)$, the metrics $g_l$ are uniformly equivalent to the standard one and $Q_l \to Q_\infty$ in $L^2(S^4)$. From elliptic regularity results and (18), we get convergence $u_l \to u_\infty$, $g_l \to g_\infty$ in $H^4(S^4)$.

If, on the other hand, concentration occurs in the sense of Lemma 3.5, we now show that this leads to the behavior described in *ii*). Choose $p_l \in S^4$ and $r_l > 0$ such that

$$\sup_{p\in S^4} \int_{B_{r_l}(p)} |Q_l| d\mu_l = \int_{B_{r_l}(p_l)} |Q_l| d\mu_l = 2\pi^2.$$

By (47) we have $r_l \to 0$; in addition, we may assume that $p_l \to p$ as $l \to \infty$.

Let $\tilde{\Phi}_l \colon S^4 \to S^4$ be conformal diffeomorphisms mapping the upper hemisphere $S^4_+ = \{x_5 > 0\}$ into $B_{r_l}(p_l)$ and taking the equatorial sphere $\partial S^4_+$ to $\partial B_{r_l}(p_l)$. Consider the sequence of functions $\tilde{u}_l : S^4 \to \mathbb{R}$ defined by

$$\tilde{u}_l = u_l \circ \tilde{\Phi}_l + \frac{1}{4}\log(\det(d\tilde{\Phi}_l)). \tag{49}$$

Then these functions solve the equation

$$P_{S^4}\tilde{u}_l + 2Q_0 = 2\tilde{Q}_l e^{4\tilde{u}_l},$$

where $\tilde{Q}_l = Q_l \circ \tilde{\Phi}_l$.

By the last statement of Lemma 3.5 and our choice of $p_l, r_l$, we find that $\tilde{u}_l \to \tilde{u}_\infty$ in $H^4_{loc}(S^4 \setminus \{S\})$, where $S$ is the south pole of $S^4$. In addition, there holds $\tilde{Q}_l \to Q_\infty(p)$ almost everywhere as $l \to \infty$. Using now the map

$$\Psi(z) = \frac{1}{1+|z|^2}(2z, 1-|z|^2), \quad z \in \mathbb{R}^4,$$

and defining

$$\hat{u}_l = \tilde{u}_l \circ \Psi + \frac{1}{4}\log(\det(d\Psi)),$$

we obtain a sequence $\hat{u}_l : \mathbb{R}^4 \to \mathbb{R}$ converging in $H^4_{loc}(\mathbb{R}^4)$ to a function $\hat{u}_\infty$ which solves the equation

$$P_{\mathbb{R}^4}\hat{u}_\infty = \Delta^2_{\mathbb{R}^4}\hat{u}_\infty = 2Q_\infty(p)e^{4\hat{u}_\infty} \quad \text{in } \mathbb{R}^4. \tag{50}$$

Moreover, by lower semicontinuity, $\hat{u}_\infty$ satisfies

$$\int_{\mathbb{R}^4} e^{4\hat{u}_\infty}\,dx \le \liminf_{l\to\infty}\int_{\mathbb{R}^4} e^{4\hat{u}_l}\,dx = \frac{8}{3}\pi^2. \tag{51}$$

The solutions of (50)-(51) have been classified in [27] and only the following two possibilities occur. Either $\hat{u}_\infty$ is of the form

$$\hat{u}_\infty = \log\frac{2\lambda}{1+\lambda^2|x-x_0|^2} - \frac{1}{4}\log\left(\frac{1}{3}Q_\infty(p)\right) \tag{52}$$

for some $\lambda > 0, x_0 \in \mathbb{R}^4$, or for some $a > 0$ there holds

$$-\Delta_{\mathbb{R}^4}\hat{u}_\infty(x) \to a \text{ as } |x| \to \infty. \tag{53}$$

By the maximum principle for $-\Delta_{\mathbb{R}^4}$ and (50), in the latter case we even have $-\Delta_{\mathbb{R}^4}\hat{u}_\infty(x) \ge a$ everywhere on $\mathbb{R}^4$. Following the method in [33], we rule out (53).

In fact, assuming (53), for any fixed number $L$ and sufficiently large $l$ we have

$$\int_{B_L(0;\mathbb{R}^4)} (-\Delta_{\mathbb{R}^4}\hat{u}_l)\,dx \ge \frac{\omega_3}{8}aL^4, \tag{54}$$

where $\omega_3$ is the 3-dimensional volume of the standard $S^3$. Scaling back to $S^4$ (recall that the dilation factor is $r_l$), with a uniform constant $C_0 > 0$ we obtain

$$\int_{B_{Lr_l}(p_l;S^4)} (-\Delta_{S^4}u_l)\,d\mu_{S^4} \ge C_0 a r_l^2 L^4, \tag{55}$$

provided $l \ge l_0(L)$.

On the other hand, $(-\Delta_{S^4} u_l)$ satisfies the equation

$$-\Delta_{S^4}(-\Delta_{S^4} u_l) + 2(-\Delta_{S^4} u_l) = 2\left(Q_l e^{4u_l} - Q_{S^4}\right) \qquad \text{on } S^4. \tag{56}$$

If we denote as $G(\cdot,\cdot)$ the Green's function of the operator $(-\Delta_{S^4}+2)$ on $S^4$, by the maximum principle we have

$$(-\Delta_{S^4} u_l)(x) \le 2\int_{S^4} G(x,y) Q_l(y) e^{4u_l}(y) d\mu_{S^4}(y)$$

for almost every $x \in S^4$. Since $G$ has the asymptotic growth

$$G(x,y) \sim \frac{1}{2\omega_3}\frac{1}{|x-y|^2} \qquad \text{for } |x-y| \to 0, \tag{57}$$

by Fubini's theorem for any $p \in S^4$ and any $r > 0$ we find

$$\begin{aligned}
&\int_{B_r(p;S^4)} (-\Delta_{S^4} u_l)(x) d\mu_{S^4}(x) \\
&\quad \le C\int_{B_r(p;S^4)} d\mu_{S^4}(x) \int_{S^4} Q_l(y) e^{4u_l}(y) \frac{1}{|x-y|^2} d\mu_{S^4}(y) \\
&\quad \le C\int_{S^4} Q_l(y) e^{4u_l}(y) \left(\int_{B_r(y;S^4)} \frac{1}{|x-y|^2} d\mu_{S^4}(x)\right) d\mu_{S^4}(y) \\
&\quad \le Cr^2 \|Q_l e^{4u_l}\|_{L^1(S^4,g_{S^4})} = Cr^2\|Q_l\|_{L^1(S^4,g_l)} \le Cr^2 \|Q_l\|_{L^2(S^4,g_l)}.
\end{aligned}$$

In view of the uniform upper bound on $\|Q_l\|_{L^2(S^4,g_l)}$ implied by our hypotheses, upon applying the last inequality with $p = p_l$ and $r = Lr_l$ we then obtain the estimate

$$\int_{B_{Lr_l}(p_l;S^4)} (-\Delta_{S^4} u_l) d\mu_{S^4} \le C_1 r_l^2 L^2$$

with a uniform constant $C_1$, which contradicts (55) when $L$ is sufficiently large.

Hence (52) holds and $\hat{u}_\infty$ arises from the stereographic projection of $S^4$ onto $\mathbb{R}^4$, with

$$\int_{\mathbb{R}^4} Q_\infty(p) e^{4\hat{u}_\infty} d\mu_{\mathbb{R}^4} = 8\pi^2.$$

From the positivity of $Q_\infty$ we have

$$|Q_l| \le Q_l + 2|Q_l - Q_\infty|. \tag{58}$$

Recalling our assumption that

$$\|Q_l - Q_\infty\|_{L^2(S^4,g_l)} \to 0,$$

then with error $o(1) \to 0$ as $l \to \infty$ for any $r > 0$ in view of (11) we find

$$\begin{aligned}
8\pi^2 &= \int_{\mathbb{R}^4} Q_\infty(p) e^{4\hat{u}_\infty} dz \le \int_{B_r(p)} Q_\infty \, d\mu_l + o(1) \\
&\le \int_{B_r(p)} (|Q_l| + |Q_l - Q_\infty|) \, d\mu_l + o(1) \le \int_{S^4} Q_l \, d\mu_l + o(1) = 8\pi^2 + o(1).
\end{aligned}$$

It follows from (58) that $p$ is the only concentration point of the sequence $(g_l)_l$ and that

$$d\mu_l \rightharpoonup \frac{8}{3}\pi^2\delta_p, \quad Q_\infty(p) d\mu_l \rightharpoonup 8\pi^2\delta_p \qquad \text{as } l \to \infty, \tag{59}$$

proving the assertion (48).

In addition, it follows that $\Phi_l$ converges almost everywhere to the constant map $\Phi_\infty \equiv p$. Since $\Phi_l$ is a conformal diffeomorphism, we have

$$\int_{S^4} \left(|\Delta_{S^4}\Phi_l|^2 + 2|\nabla\Phi_l|^2_{S^4}\right) d\mu_{S^4} = C$$

for all $l \in \mathbb{N}$ and some constant $C$. Indeed, $\Phi_l$ may be written as the composition $\Psi \circ \delta_l \circ \pi$ of the stereographic projection $\pi$ from some point, which we may take to be the north pole of the sphere, a dilation $\delta_l$ in $\mathbb{R}^4$ and the map $\Psi = \pi^{-1}$ introduced earlier. The previous bound then follows from writing

$$\begin{aligned}\int_{S^4} \left(|\Delta_{S^4}\Phi_l|^2 + 2|\nabla_{S^4}\Phi_l|^2\right) d\mu_{S^4} &= \int_{S^4} \Phi_l P_{S^4}\Phi_l d\mu_{S^4} \\ = \int_{\mathbb{R}^4} |\Delta_{\mathbb{R}^4}(\Psi\circ\delta_l)|^2 dx &= \int_{\mathbb{R}^4} |\Delta_{\mathbb{R}^4}\Psi|^2 dx < \infty,\end{aligned}$$

using conformal and dilation invariance.

Hence a subsequence of $(\Phi_l)_l$ converges weakly in $H^2(S^4)$ to $\Phi_\infty \equiv p$. By (59), as $l \to \infty$ we also have

$$||Q_\infty \circ \Phi_l - Q_\infty(p)||_{L^2(S^4,h_l)} = ||Q_\infty - Q_\infty(p)||_{L^2(S^4,g_l)} \to 0,$$

and, consequently,

$$\begin{aligned}||Q_{h_l} - Q_\infty(p)||_{L^2(S^4,h_l)} &= ||Q_{h_l} - Q_\infty\circ\Phi_l||_{L^2(S^4,h_l)} + o(1) \\ &= ||Q_l - Q_\infty||_{L^2(S^4,g_l)} + o(1) \to 0.\end{aligned}$$

Hence we can apply our previous reasoning to the sequence $v_l$ and the corresponding metrics $h_l$. Since $v_l$ satisfies the condition (23), concentration in the sense of (59) is impossible. Therefore a subsequence $v_l \to v_\infty$, $h_l \to h_\infty$ in $H^4(S^4)$ as $l \to \infty$, where $h_\infty = e^{4v_\infty} g_{S^4}$ has $Q$-curvature $Q_\infty(p) = 3$. Finally, since all metrics $h_l$ satisfy (23), this is true also for $h_\infty$, which then must coincide with $g_{S^4}$. The proof is complete. □

From Lemmas 3.4 and 3.6 we can get a neat characterization of the asymptotic behavior of the flow $(u(t))$ in the case of divergence. For $t \geq 0$ let

$$S = S(t) = \fint_{S^4} x\, d\mu = \fint_{S^4} \Phi\, d\mu_h$$

be the center of mass of the metric $g(t)$. For $h$ near $g_{S^4}$ the point $S$ approximately is given by

$$p = p(t) = \fint_{S^4} \Phi\, d\mu_{S^4}.$$

For convenience, we extend $f(p) = f(p/|p|)$ for $p \in B_1(0;\mathbb{R}^5)$, $|p| \geq 1/2$.

**Lemma 3.7.** *Suppose the equation $Q_g = f$ has no solution in the conformal class of $g_{S^4}$. Suppose $u(t)$ solves* (16) - (19)*, and let $v(t)$ be the corresponding normalized flow. Then, as $t \to \infty$, one has $v(t) \to 0$, $h(t) = e^{2v(t)} g_{S^4} \to g_{S^4}$ in $H^4(S^4)$ and uniformly, and $Q_{h(t)} \to Q_{S^4} = 3$ in $L^2(S^4)$. Furthermore, $||\Phi(t) - p(t)||_{L^2(S^4)} \to 0$, and consequently $||f\circ\Phi(t) - f(p(t))||_{L^2(S^4)} \to 0$, $\alpha(t) f(p(t)) \to 3$.*

**Proof.** Assuming the contrary, for a suitable sequence $t_l \to \infty$ we have

$$\liminf_{l\to\infty}(||h(t_l) - g_{S^4}||_{H^4} + ||\Phi(t_l) - p(t_l)||_{L^2(S^4)}) > 0.$$

By Lemma 3.6, there exists a subsequence $t_l \to \infty$ such that $u(t_l) \to u_\infty$, $g_l = e^{2u(t_l)}g_{S^4} \to g_\infty = e^{2u_\infty}g_{S^4}$ in $H^4$, and $\alpha(t_l) \to \alpha$. By Lemma 3.4 then the metric $\alpha g_\infty$ has $Q$-curvature $f$, contradicting our assumption on $f$. This proves the first assertions. Finally, by (19) and Hölder's inequality we have

$$3 - \alpha f(p) = \alpha \fint_{S^2} (f \circ \Phi - f(p))\, d\mu_h \to 0 \qquad \text{as } t \to \infty, \tag{60}$$

which concludes the proof. □

Now define

$$F_2(t) = \int_{S^4} |\alpha f - Q|^2\, d\mu = \int_{S^4} |\alpha f_\Phi - Q_h|^2\, d\mu_h \tag{61}$$

and

$$\begin{aligned} G_2(t) &= \int_{S^4} (Q - \alpha f) P (Q - \alpha f)\, d\mu = \int_{S^4} (Q_h - \alpha f_\Phi) P_h (Q_h - \alpha f_\Phi)\, d\mu_h \\ &= \int_{S^4} \left( |\Delta_{S^4}(Q - \alpha f)|^2 + 2|\nabla(Q - \alpha f)|^2_{S^4} \right) d\mu_{S^4} \\ &= \int_{S^4} \left( |\Delta_{S^4}(Q_h - \alpha f_\Phi)|^2 + 2|\nabla(Q_h - \alpha f_\Phi)|^2_{S^4} \right) d\mu_{S^4}. \end{aligned} \tag{62}$$

**Lemma 3.8.** *With error $o(1) \to 0$ as $t \to \infty$ we have*

$$\frac{d}{dt} F_2 = -(1 + o(1)) G_2 + (24 + o(1)) F_2 \qquad \text{as } t \to \infty.$$

**Proof.** From (39), (44), and Lemma 3.4 we deduce

$$\begin{aligned} &\frac{d}{dt} F_2 \le -G_2 + 8\alpha \int_{S^4} f_\Phi (\alpha f_\Phi - Q)^2\, d\mu_h + o(1)(G_2 + F_2) \\ &= -(1 + o(1)) G_2 + (8\alpha + o(1)) f(p) F_2 + 8\alpha \int_{S^4} (f_\Phi - f(p))(\alpha f_\Phi - Q_h)^2\, d\mu_h. \end{aligned}$$

By Lemma 3.7 we have $\alpha f(p) \to 3$. Moreover, from the Sobolev's embedding $H^1 \hookrightarrow L^4$ and Lemma 3.7, we deduce

$$\begin{aligned} |\alpha \int_{S^4} (f_\Phi - f(p))(\alpha f_\Phi - Q_h)^2\, d\mu_h| &\le \alpha ||f_\Phi - f(p)||_{L^2} ||\alpha f_\Phi - Q_h||^2_{L^4} \\ &\le o(1) ||\alpha f_\Phi - Q_h||^2_{H^1} \le o(1)(F_2 + G_2), \end{aligned}$$

which proves the assertion. □

## 4. Uniformization: The case $f \equiv const$

We first turn our attention to the uniformization problem, that is, the case when $f \equiv$ const. With no loss of generality we may assume that $f \equiv Q_{S^4} = 3$. The condition (19) then implies that $\alpha = 1$ for all $t > 0$. Our aim is to prove the following result.

**Theorem 4.1.** *Suppose that $g_0 = e^{2u_0} g_{S^4}$ satisfies (16), and let $f \equiv 3$. Then the flow (17), (19) converges exponentially fast to a metric $g_\infty = e^{2u_\infty} g_{S^4}$ of constant $Q$-curvature $Q_\infty \equiv Q_{S^4} = 3$ in the sense that $||u(t) - u_\infty||_{H^4} \le Ce^{-\delta t}$ for some constants $C$ and $\delta > 0$.*

4.1. **Spectral decomposition.** Let $(\varphi_i^g)_{i\in\mathbb{N}_0}$ be an $L^2(S^4,g)$-orthonormal basis of eigenfunctions for $\Delta=\Delta_g$ with eigenvalues $\lambda_0^g=0<\lambda_1^g\le\lambda_2^g\le\ldots$, solving the equation

$$-\Delta\varphi_i^g=\lambda_i^g\varphi_i^g,\ i\in\mathbb{N}_0.$$

Similarly, we define $\varphi_i^h, \varphi_i=\varphi_i^{g_{S^4}}$ with corresponding eigenvalues $\lambda_i^h$ and $\lambda_i=\lambda_i^{g_{S^4}}, i\in\mathbb{N}_0$. It is well-known that

$$\lambda_0=0<\lambda_1=\lambda_2=\lambda_3=\lambda_4=\lambda_5=4<\lambda_6=10\le\ldots;$$

see for instance [4], Proposition C.I.1. Clearly there holds $\lambda_i^g=\lambda_i^h$ and Lemma 3.7 implies $\lambda_i^h\to\lambda_i$. The eigenfunctions can be chosen in such a way that $\varphi_i^h=\varphi_i^g\circ\Phi$ and $\varphi_i^h\to\varphi_i$ smoothly as $t\to\infty$, for all $i\in\mathbb{N}_0$. Moreover, if we let $\Lambda_i^g,\Lambda_i^h,\Lambda_i=\Lambda_i^{g_{S^4}}$ denote the eigenvalues of $P_g$, etc., by the results in [20] one has

$$\Lambda_i^g=(\lambda_i^g)^2+2\lambda_i^g. \tag{63}$$

In terms of $\varphi_i^g$, $\varphi_i^h$ the functions $\alpha f-Q$, $\alpha f_\Phi-Q_h$ may be decomposed as

$$\alpha f-Q=\sum_{i\ge0}\beta^i\varphi_i^g,\quad \alpha f_\Phi-Q_h=\sum_{i\ge0}\gamma^i\varphi_i^h,$$

respectively, with

$$\beta^i=\int_{S^4}(\alpha f-Q)\varphi_i^g\,d\mu_g=\int_{S^4}(\alpha f_\Phi-Q_h)\varphi_i^h\,d\mu_h=\gamma^i.$$

By the normalization (19) it follows that $\beta^0=0$. In particular, with error $o(1)\to0$ as $t\to\infty$ one finds

$$G_2=\sum_{i\ge0}\Lambda_i^g|\beta^i|^2\ge\Lambda_1^g\sum_{i\ge0}|\beta^i|^2=(\Lambda_1+o(1))F_2=(24+o(1))F_2. \tag{64}$$

Let $x=(x^1,\ldots,x^5)$ denote the coordinate functions in $\mathbb{R}^5$ restricted to $S^4$. Then we may choose $\varphi_i=\sqrt{\frac{15}{8\pi^2}}x^i$ for $i=1,\ldots,5$. Define also

$$b^i=\int_{S^4}x^i(\alpha f_\Phi-Q_h)\,d\mu_h,\qquad i=1,\ldots,5. \tag{65}$$

By Lemma 3.7, up to the factor $\sqrt{\frac{15}{8\pi^2}}$ and up to an error of order $o(1)||\alpha f-Q||_{L^2(S^4,g)}$, the two vectors $b=(b^1,\ldots,b^5)$ and $\beta=(\beta^1,\ldots,\beta^5)$ coincide. Also let $B=\sqrt{\frac{15}{8\pi^2}}b$.

4.2. **Decay of $B$.** For $f\equiv Q_{S^4}=3$, we can estimate $b^1,\ldots,b^5$ via the Kazdan-Warner identity. Indeed, using (12) together with the relation

$$-\Delta_{S^4}x^i=4x^i, \tag{66}$$

and observing that $\alpha\equiv1$, for $i=1,\ldots,5$ upon integrating by parts we find that

$$b^i=\int_{S^4}(Q_h-Q_{S^4})x^i\,d\mu_h=\int_{S^4}(Q_h-Q_{S^4})\langle\nabla x^i,\nabla v\rangle_{S^4}\,d\mu_h.$$

From Lemma 3.7 then it follows that with error $o(1) \to 0$ as $t \to \infty$

$$
\begin{aligned}
|b^i| &= \left| \int_{S^4} (Q_h - Q_{S^4}) \langle \nabla x^i, \nabla v \rangle_{S^4} \, d\mu_h \right| \\
&\leq C F_2(t)^{1/2} ||v(t)||_{H^1} = o(1) F_2(t)^{1/2}.
\end{aligned} \tag{67}
$$

**Proof of Theorem 4.1**. We divide the proof into three steps. We show first the decay of $F_2$, then the decay of $v$, and finally we control the Möbius map $\Phi(t)$.

*i*) In view of (67) we can improve (64) to obtain

$$
G_2 \geq (\Lambda_6^g + o(1)) \sum_{i \geq 0} |\beta^i|^2 = (\Lambda_6 + o(1)) F_2. \tag{68}
$$

Taking advantage of the "spectral gap" between $\Lambda_1$ and $\Lambda_6$, from Lemma 3.8 for sufficiently large $t$ we then infer the estimate

$$
\frac{d}{dt} F_2(t) < -\delta F_2(t)
$$

for some uniform constant $\delta > 0$, and it follows that

$$
F_2(t) \leq C e^{-\delta t} \tag{69}
$$

for all $t \geq 0$ with some constant $C$.

*ii*) We can now show exponential decay of $v$. Observing that

$$
\int_{S^4} (e^{4v} - 1) \, d\mu_{S^4} = \int_{S^4} d\mu_h - \int_{S^4} d\mu_{S^4} = 0
$$

and recalling the normalization condition

$$
\int_{S^4} x d\mu_h = \int_{S^4} (e^{4v} - 1) x \, d\mu_{S^4} = 0,
$$

we have an expansion

$$
e^{4v} - 1 = \sum_{i=0}^{\infty} V^i \varphi_i
$$

in terms of the basis functions $\varphi_i$, where $V^0 = \ldots = V^5 = 0$.

Also let $v = \sum_{i=0}^{\infty} v^i \varphi_i$. Note that for every $i$ on account of Lemma 3.7, and by using Sobolev's embedding $H^2(S^4) \hookrightarrow L^p(S^4)$ for every $p < \infty$ together with the estimate (29) from Lemma 3.2, we obtain

$$
4v^i = 4 \int_{S^4} v \varphi_i \, d\mu_{S^4} = \int_{S^4} (e^{4v} - 1) \varphi_i \, d\mu_{S^4} + O(||v||_{H^2}^2) = V^i + o(1) ||v||_{H^2},
$$

where $o(1) \to 0$ as $t \to \infty$. In particular, we have

$$
\sum_{i=0}^{5} |v^i|^2 \leq o(1) ||v||_{H^2}^2. \tag{70}
$$

Writing (18) as

$$
P_{S^4} v = 2(Q_h - Q_{S^4}) e^{4v} + 2 Q_{S^4} (e^{4v} - 1),
$$

from Young's inequality and the uniform boundedness of $v$ on the one hand, and by using Sobolev's embedding together with (29) on the other, for any $\varepsilon > 0$ we obtain

$$\int_{S^4} |P_{S^4} v|^2 \, d\mu_{S^4} \le C(\varepsilon) F_2 + 36(1+\varepsilon) \int_{S^4} (e^{4v} - 1)^2 \, d\mu_{S^4}$$
$$\le C(\varepsilon) e^{-\delta t} + 24^2 (1+\varepsilon) \int_{S^4} |v|^2 \, d\mu_{S^4} + o(1) ||v||^2_{H^2},$$

where $o(1) \to 0$ as $t \to \infty$. In terms of the coefficients $v^i$, this implies that

$$\sum_{i=0}^{\infty} \Lambda_i^2 |v^i|^2 \le C(\varepsilon) e^{-\delta t} + (\Lambda_1^2 (1+\varepsilon) + o(1)) \sum_{i=0}^{\infty} |v^i|^2.$$

Hence, if we choose $\varepsilon > 0$ so that

$$\Lambda_1^2 (1+\varepsilon) < \Lambda_6^2$$

and take (70) into account, we find

$$||v(t)||^2_{H^4} \le C \sum_{i=0}^{\infty} (1 + \Lambda_i^2) |v^i|^2 \le C e^{-\delta t} + o(1) ||v(t)||^2_{H^4}$$

where $o(1) \to 0$ as $t \to \infty$. Hence

$$||v(t)||^2_{H^4} \le C e^{-\delta t}, \tag{71}$$

as claimed.

*iii*) In view of (69) and (35) we have

$$||d\Phi(t)^{-1} \Phi_t(t)||^2_{T_{id}G} \le C ||\xi(t)||^2_{L^\infty} \le C F_2(t) \le C e^{-\delta t}$$

for all $t \ge 0$. Thus, we have smooth exponential convergence $\Phi(t) \to \Phi_\infty$ as $t \to \infty$. By (71) therefore, also $g(t) = (\Phi(t)^{-1})^* h(t) \to g_\infty = (\Phi_\infty^{-1})^* g_0$ and hence $u(t) \to u_\infty$ exponentially fast in $H^4$ as $t \to \infty$, where $Q_\infty \equiv Q_{S^4}$. □

## 5. Prescribed $Q$-curvature

We now focus on the analogue of Nirenberg's problem. Throughout this section we assume that the given function $f$ cannot be realized as the $Q$-curvature of a conformal metric and hence the flow $u(t)$ does not converge in $H^4(S^4)$ as $t \to \infty$. Lemma 3.7 then is applicable and it follows that $g(t)$ concentrates near points $p(t)$ of $S^4$. As in [37], their motion, and the evolution of the concentration scale, can be studied by means of the spectral decomposition introduced in Section 4.1.

5.1. **Dominance of $B$.** We first observe that when the flow $(u(t))$ fails to converge, then the coefficients $B(t)$ dominate the other Fourier coefficients, and the flow shadows the solution of an ODE in a five-dimensional space.

**Lemma 5.1.** *For $i = 1, \dots, 5$ with error $o(1) \to 0$ as $t \to \infty$ there holds*

$$\frac{dB^i}{dt} = o(1) F_2(t)^{1/2}.$$

**Proof.** The proof is analogous to the proof of Lemma 4.1 in [37] and is included here only for completeness. We may argue for $b$ instead of $B$.

From the equations (18) and (66) we deduce the identity

$$\int_{S^4} x^i Q_h \, d\mu_h = \frac{1}{2}\int_{S^4} x^i (P_{S^4} v + 6)\, d\mu_{S^4} = 12 \int_{S^4} x^i v \, d\mu_{S^4}.$$

It follows that

$$\begin{aligned}\frac{db^i}{dt} &= \frac{d}{dt}\int_{S^4} x^i(\alpha f_\Phi - Q_h)\, d\mu_h = \int_{S^4} x^i \left(\frac{d(\alpha f_\Phi)}{dt} + 4v_t(\alpha f_\Phi - 3e^{-4v})\right) d\mu_h \\ &= \alpha_t \int_{S^4} x^i f_\Phi \, d\mu_h + \alpha \int_{S^4} x^i (df_\Phi \cdot \xi + 4v_t(f_\Phi - f(p)))\, d\mu_h \\ &\quad + 4\int_{S^4} x^i v_t(\alpha f(p) - 3e^{-4v})\, d\mu_h = I + II + III.\end{aligned}$$

By (23), (37), and Lemma 3.7 the term

$$I = \alpha_t \int_{S^4} x^i f_\Phi \, d\mu_h = \alpha_t \int_{S^4} x^i (f_\Phi - f(p))\, d\mu_h$$

is bounded by

$$|I| \le C||f_\Phi - f(p)||_{L^2} F_2^{1/2} = o(1) F_2^{1/2}.$$

Similarly, upon invoking (33) to write

$$\begin{aligned}II &= \alpha \int_{S^4} x^i (df_\Phi \cdot \xi + 4v_t(f_\Phi - f(p)))\, d\mu_h \\ &= 4\alpha \int_{S^4} x^i u_t \circ \Phi (f_\Phi - f(p))\, d\mu_h + \alpha \int_{S^4} x^i \operatorname{div}_{S^4}(\xi e^{4v}(f_\Phi - f(p)))\, d\mu_{S^4} \\ &= 4\alpha \int_{S^4} x^i u_t \circ \Phi (f_\Phi - f(p))\, d\mu_h - \alpha \int_{S^4} \xi^i (f_\Phi - f(p))\, d\mu_h,\end{aligned}$$

in view of Lemma 3.7 and (35) we obtain the estimate

$$\begin{aligned}|II| &\le C||f_\Phi - f(p)||_{L^2}(||u_t \circ \Phi||_{L^2} + ||\xi||_{L^2}) \\ &\le C||f_\Phi - f(p)||_{L^2} F_2^{1/2} = o(1) F_2^{1/2}.\end{aligned}$$

Finally, again using (33), we have

$$\begin{aligned}III &= 4\int_{S^4} x^i v_t (\alpha f(p) - 3e^{-4v})\, d\mu_h \\ &= 4\int_{S^4} x^i u_t \circ \Phi(\alpha f(p) - 3e^{-4v})\, d\mu_h + \int_{S^4} x^i (\alpha f(p) - 3e^{-4v}) \operatorname{div}_{S^4}(\xi e^{4v})\, d\mu_{S^4} \\ &= 4\int_{S^4} x^i u_t \circ \Phi(\alpha f(p) - 3e^{-4v})\, d\mu_h - \int_{S^4} (\alpha f(p) - 3e^{-4v}) \xi^i \, d\mu_h \\ &\quad - 12 \int_{S^4} x^i dv \cdot \xi \, d\mu_{S^4}.\end{aligned}$$

By Lemma 3.7 and (35) also this term then may be bounded by

$$\begin{aligned}|III| &\le C(||1 - e^{-4v}||_{L^2} + |\alpha f(p) - 3|)(||u_t \circ \Phi||_{L^2} + ||\xi||_{L^2}) + C||v||_{H^1}||\xi||_{L^2} \\ &\le o(1) F_2^{1/2}.\end{aligned}$$

This concludes the proof. □

From Lemma 5.1 we now obtain the analogue of Lemma 4.2 in [37].

**Lemma 5.2.** *In the above notation there holds*

$$F_2 \leq (1+o(1))|B|^2.$$

**Proof.** Again we include the proof for completeness. Denote as $\hat{F}_2 = \sum_{i\geq 6}|\beta^i|^2$, so that, with error $o(1) \to 0$ as $t \to \infty$,

$$F_2 = |\beta|^2 + \hat{F}_2 = (1+o(1))|B|^2 + (1+o(1))\hat{F}_2.$$

Assuming that $2|B|^2 \leq \hat{F}_2$ for sufficiently large $t$, similar to equation (68) we obtain

$$G_2 \geq \Lambda_6 \hat{F}_2 \geq \frac{1}{2}\Lambda_6 F_2 = 60 F_2$$

and from Lemma 3.8 we deduce

$$\frac{d}{dt}F_2 \leq -(36+o(1))F_2 \leq -F_2$$

for $t$ large. As in the proof of Theorem 4.1 it then follows that the flow $(g(t))$ converges exponentially fast to a metric of $Q$-curvature proportional to $f$, contradicting our hypothesis in this section.

Hence there exist arbitrarily large numbers $t_1 \geq 0$ so that $2|B(t_1)|^2 > \hat{F}_2(t_1)$. Writing

$$F_2 = (1+\delta)|B|^2,$$

for sufficiently large $t_0 \geq 0$ we have $\delta(t) > -1/2$ for all $t \geq t_0$. In addition, for sufficiently large $t_1$ as above we have $\delta(t_1) < 4$ and hence, by continuity, also $\delta(t) < 4$ for all $t$ sufficiently close to $t_1$. From Lemma 3.8 with error $o(1) \to 0$ as $t \to \infty$ we obtain

$$\begin{aligned} \frac{d}{dt}F_2 = \frac{d\delta}{dt}|B|^2 + 2(1+\delta)B \cdot \frac{dB}{dt} &\leq -\sum_{i\geq 0}(\Lambda_i - 24 + o(1))|\beta^i|^2 \\ \leq -96\hat{F}_2 + o(1)F_2 &= -\left(\frac{96\delta}{1+\delta} + o(1)\right)F_2, \end{aligned} \tag{72}$$

where we again used the fact that $\Lambda_i \geq \Lambda_6 = 120$ for $i \geq 6$. Since Lemma 5.1 implies

$$\left| B \cdot \frac{dB}{dt} \right| \leq o(1)F_2,$$

for $t$ near $t_1$ as above it follows that

$$\frac{d\delta}{dt}|B|^2 \leq -\left(\frac{96\delta}{1+\delta} + o(1)\right)F_2 = -(96\delta + o(1))|B|^2$$

and then

$$\frac{d\delta}{dt} \leq -(96\delta + o(1)).$$

In particular, for sufficiently large $t_1$ we obtain that $\delta(t) \leq 4$ for all $t \geq t_1$. The previous inequality then shows that $\delta(t) \to 0$ as $t \to \infty$, as claimed. □

5.2. **Scaled stereographic coordinates.** As in [37], for the following detailed estimates it is convenient to introduce stereographic coordinates. Let $\pi\colon S^4 \setminus \{(0,0,0,0,-1)\} \to \mathbb{R}^4$ denote the stereographic projection from the south pole, given by

$$\pi(x) = \frac{(x^1, x^2, x^3, x^4)}{1+x^5},\ x = (x^1, \dots, x^5) \in S^4. \tag{73}$$

Also denote its inverse as $\Psi\colon \mathbb{R}^4 \to S^4$, with

$$\Psi(z) = \frac{1}{1+|z|^2}(2z^1, \dots, 2z^4, 1-|z|^2),\ z = (z^1, \dots, z^4) \in \mathbb{R}^4.$$

For $q \in \mathbb{R}^4$ and $r > 0$ define the conformal map $\Psi_{q,r}\colon \mathbb{R}^4 \to S^4$

$$\Psi_{q,r} = \Psi \circ \delta_{q,r},$$

obtained by the composition of $\Psi$ with the affine linear map

$$\delta_{q,r}\colon \mathbb{R}^4 \ni z \mapsto z_{q,r} = q + rz \in \mathbb{R}^4.$$

There holds

$$\frac{\partial \Psi_{q,r}}{\partial q^i}\Big|_{q=0,r=1} = \frac{\partial \Psi}{\partial z^i} =: e_i,\ i = 1, \dots, 4,$$

and

$$\frac{\partial \Psi_{q,r}(z)}{\partial r}\Big|_{q=0,r=1} = \sum_{i=1}^4 z^i e_i(z),\ z \in \mathbb{R}^4,$$

where the vector field $e_1 = \frac{\partial \Psi}{\partial z^1}$ is given by

$$\begin{aligned} e_1(z) &= \frac{1}{(1+|z|^2)^2}(2(1-2|z^1|^2+|z|^2), -4z^1z^2, -4z^1z^3, -4z^1z^4, -4z^1) \\ &= (1+x^5-|x^1|^2, -x^1x^2, -x^1x^3, -x^1x^4, -x^1(1+x^5)), \end{aligned} \tag{74}$$

and with similar formulae for $e_2, e_3, e_4$. In particular, we have

$$\sum_{i=1}^4 z^i e_i(z) = (x^1x^5, x^2x^5, x^3x^5, x^4x^5, |x^5|^2 - 1). \tag{75}$$

For $t_0 \geq 0$ and $t \geq 0$ close to $t_0$, let

$$\Phi_{t_0}(t) = \Phi(t_0)^{-1}\Phi(t).$$

Define $q = q(t), r = r(t)$ so that

$$\Phi_{t_0}(t) \circ \Psi = \Psi_{q(t),r(t)}.$$

The vector field $\xi = (d\Phi(t_0))^{-1}\frac{d\Phi}{dt}\big|_{t=t_0} = \frac{d\Phi_{t_0}}{dt}\big|_{t=t_0}$ then has the following representation

$$\xi = \frac{d}{dt}(\Phi_{t_0} \circ \Psi)\big|_{t=t_0} = \sum_{i=1}^4 \frac{\partial \Psi_{q,r}}{\partial q^i}\frac{dq^i}{dt} + \frac{\partial \Psi_{q,r}}{\partial r}\frac{dr}{dt} = \sum_{i=1}^4 \left(\frac{dq^i}{dt} + z^i\frac{dr}{dt}\right) e_i,$$

where the derivatives of $\Psi$ are evaluated at $q = 0$, $r = 1$. Letting

$$X = \int_{S^4} \xi \, d\mu_{S^4} = (X^1, \dots, X^5) \in \mathbb{R}^5,$$

and using cancellations by oddness, from (74), (75) we obtain

$$X^i = \frac{dq^i}{dt} \int_{S^4} (1 - |x^i|^2)\, d\mu_{S^4} = \frac{32\pi^2}{15} \frac{dq^i}{dt},\ i = 1, \dots, 4; \tag{76}$$

moreover, we have

$$X^5 = -\frac{dr}{dt} \int_{S^4} (1 - |x^5|^2)\, d\mu_{S^4} = -\frac{32\pi^2}{15} \frac{dr}{dt}. \tag{77}$$

5.3. **The shadow flow.** Recall that by Lemma 3.7 the center of mass $S(t)$ of $g(t)$ is given approximately by

$$p = p(t) = \fint_{S^4} \Phi(t)\, d\mu_{S^4}.$$

Similar to [37], Section 4.2, we now relate $B$ (or $b$) to the gradient of $f$ at $\hat{p} = p/|p|$. For this we need the analogues of [37], Lemmas 4.3 - 4.8. Since the proofs may be carried over with straightforward modifications, we may be brief.

Given $t_0 \geq 0$, consider a rotation which maps $\hat{p}(t_0)$ into the north pole $N = (0, 0, 0, 0, 1)$. Then $\Phi(t_0)\colon S^4 \to S^4$ can be expressed as $\Phi(t_0) = \Psi_\varepsilon \circ \pi$ for some $\varepsilon = \varepsilon(t_0) > 0$, where $\Psi_\varepsilon(z) = \Psi(\varepsilon z) = \Psi_{0,\varepsilon}(z)$ in our previous notation. Therefore, in stereographic coordinates, $\Phi(t)$ is given by the map

$$\Phi(t) \circ \Psi = \Phi(t_0) \circ \Phi_{t_0}(t) \circ \Psi = \Psi_\varepsilon \circ \delta_{q,r}.$$

For the following lemma also recall that we extend $f$ as $f(p) = f(p/|p|)$ for $p \in S^4$ with $|p| > 1/2$.

**Lemma 5.3.** *For some uniform constant $C$ there holds*

$$||f_\Phi - f(p)||_{L^2} + ||\nabla f_\Phi||_{L^{4/3}} \leq C\varepsilon.$$

**Proof.** For $1 < s < 2$ we have

$$\begin{aligned}
\int_{S^4} |\nabla \Phi|^s_{S^4}\, d\mu_{S^4} &= \int_{\mathbb{R}^4} |\nabla \Psi_\varepsilon|^s_{\Psi^* g_{S^4}} \frac{16\, dz}{(1 + |z|^2)^4} \leq C \int_{\mathbb{R}^4} |\nabla \Psi_\varepsilon|^s_{\mathbb{R}^4} \frac{dz}{(1 + |z|^2)^{4-s}} \\
&\leq C \int_{\mathbb{R}^4} \frac{\varepsilon^s dz}{(1 + \varepsilon^2 |z|^2)^s (1 + |z|^2)^{4-s}} \\
&\leq C \int_{B_{1/\varepsilon}(0)} \frac{\varepsilon^s dz}{(1 + |z|^2)^{4-s}} + C\varepsilon^{-s} \int_{\mathbb{R}^4 \setminus B_{1/\varepsilon}(0)} \frac{dz}{|z|^8} \\
&\leq C\varepsilon^s + C\varepsilon^{4-s} \leq C\varepsilon^s.
\end{aligned}$$

Choosing $s = 4/3$ and observing that $p$ is the average of $\Phi$, by the Poincaré-Sobolev inequality then we have

$$||\Phi - p||_{L^2} \leq C ||\nabla \Phi||_{L^{4/3}} \leq C\varepsilon.$$

The claim now follows from the inequalities

$$|f_\Phi - f(p)| \leq ||\nabla f||_{L^\infty} |\Phi - p|$$

and

$$|\nabla f_\Phi| \leq ||\nabla f||_{L^\infty} |\nabla \Phi|.$$

□

**Lemma 5.4.** *For some fixed constant $C$ there holds*
$$||v||_{H^4} \le C(F_2^{1/2} + ||f_\Phi - f(p)||_{L^2}).$$

**Proof.** Expanding
$$v = \sum_{i=0}^{\infty} v^i \varphi_i$$
as in the proof of Theorem 4.1, from (70) with error $o(1) \to 0$ as $t \to \infty$ we have
$$|v^i| \le o(1)||v||_{H^2}, i = 0, \dots, 5. \tag{78}$$

We may write equation (18) in the form
$$\begin{aligned} P_{S^4} v &= 2Q_h e^{4v} - 6 \\ &= 2((Q_h - \alpha f_\Phi) + \alpha(f_\Phi - f(p)) + (\alpha f(p) - 3))e^{4v} + 6(e^{4v} - 1). \end{aligned}$$
From (60), and Young's inequality, then for any $\delta > 0$ with a constant $C(\delta)$ similar to the proof of Theorem 4.1 we find
$$\begin{aligned} \sum_{i=1}^{\infty} \Lambda_i^2 (v^i)^2 &= ||P_{S^4} v||_{L^2}^2 \\ &\le C(\delta)(||\alpha f_\Phi - Q_h||_{L^2}^2 + ||f_\Phi - f(p)||_{L^2}^2) + 36(1+\delta)||e^{4v} - 1||_{L^2}^2 \\ &= C(\delta)(||\alpha f_\Phi - Q_h||_{L^2}^2 + ||f_\Phi - f(p)||_{L^2}^2) + (\Lambda_1^2(1+\delta) + o(1)) \sum_{i=0}^{\infty} (v^i)^2. \end{aligned} \tag{79}$$
In view of (78), for small enough $\delta > 0$ the last term on the right of (79) may be absorbed on the left, and the claim follows. □

We can now relate the components of $b$ to the gradient and the Laplacian of the function $\log f$.

**Lemma 5.5.** *With error $O(\varepsilon) \le C\varepsilon$ and $O(1) \le C$ as $t \to \infty$ there holds*
$$\begin{aligned} b^i &= 4\pi^2 \varepsilon \left( \frac{\partial \log f}{\partial x^i}(p) + O(\varepsilon) \right), \ i = 1, \dots, 4; \\ b^5 &= -4\pi^2 \varepsilon^2 (\Delta_{S^4} \log f(p) + O(1)|\nabla f(p)|^2 + O(\varepsilon)). \end{aligned}$$

**Proof.** With identical reasoning as in the proof of [37], Lemma 4.5, from the equation $-\Delta_{S^4} x^i = 4x^i$ and the Kazdan-Warner identity (12) we eventually obtain
$$b^i = \alpha \int_{S^4} x^i (f_\Phi - f(p))\, d\mu_{S^4} + R_1^i = C_1 \alpha \varepsilon \frac{\partial f}{\partial x^i}(p) + R_2^i.$$
for $i = 1, \dots, 4$, with
$$C_1 = \int_{\mathbb{R}^4} \frac{16|z|^2\, dz}{(1+|z|^2)^5} = \frac{4}{3}\pi^2$$
and with error terms bounded by
$$\begin{aligned} |R_1^i| + |R_2^i| &\le C\varepsilon^2 + C\big(||v||_{H^4}^2 + ||\alpha f_\Phi - Q_h||_{L^2}^2 + ||f_\Phi - f(p)||_{L^2}^2\big) \\ &\le C\varepsilon^2 + CF_2 + C||f_\Phi - f(p)||_{L^2}^2 \le C\varepsilon^2 + C|b|^2. \end{aligned}$$
in view of Lemmas 5.2 - 5.4. Moreover, by Lemma 3.7 we have $\alpha f(p) \to 3$ as $t \to \infty$.

Similarly, for $i = 5$, we obtain the expression

$$b^5 = \alpha \int_{S^4} x^5 (f_\Phi - f(p))\, d\mu_{S^4} + R_1^5 = -C_2 \alpha \varepsilon^2 \Delta_{S^4} f(p) + R_2^5$$

with

$$C_2 = -\int_{\mathbb{R}^4} \frac{8(1-|z|^2)|z|^2\, dz}{(1+|z|^2)^5} = \frac{4}{3}\pi^2,$$

and with error

$$|R_1^5| + |R_2^5| \le C\varepsilon^3 + C|b|^2 + C||f_\Phi - f(p)||_{L^2}^2. \tag{80}$$

As was the case in [37], again Lemma 5.3 needs to be improved in order to arrive at the desired conclusion. By using the expansion

$$\begin{aligned} f(\Psi_\varepsilon(z)) - f(p) &= df(p) \cdot d\Psi_\varepsilon(0) z + \frac{1}{2} \nabla df(p)(d\Psi_\varepsilon(0)z, d\Psi_\varepsilon(0)z) + R_\varepsilon^f(z) \\ &= 2\varepsilon\, df(p) z + 2\varepsilon^2 \nabla df(p)(z,z) + O(\varepsilon^3 |z|^3) \end{aligned} \tag{81}$$

of $f \circ \Psi_\varepsilon - f(p)$ to second order, however, as in [37] we find

$$\begin{aligned} ||f_\Phi - f(p)||_{L^2}^2 &= \int_{B_{1/\varepsilon}(0)} |f(\Psi_\varepsilon(z)) - f(p)|^2 \frac{16\, dz}{(1+|z|^2)^4} + O(\varepsilon^4) \\ &\le C\varepsilon^2 |\nabla f(p)|^2 \int_{B_{1/\varepsilon}(0)} \frac{|z|^2\, dz}{(1+|z|^2)^4} + C\varepsilon^4 |\log \varepsilon| \\ &\le C\varepsilon^2 |\nabla f(p)|^2 + C\varepsilon^4 |\log\varepsilon|. \end{aligned} \tag{82}$$

From the above expressions for the components of $b$ in particular we obtain the estimate

$$|b|^2 \le C\varepsilon^2 |\nabla f(p)|^2 + O(\varepsilon^4)(1 + |\Delta f(p)|^2),$$

which allows to bound the error terms

$$R_2^i = O(\varepsilon^2),\ 1 \le i \le 4,$$

and

$$R_2^5 = O(\varepsilon^2 |\nabla f(p)|^2) + O(\varepsilon^3),$$

respectively, as desired. □

Observe that Lemmas 5.2 and 5.5 yield the bound

$$F_2 \le C\varepsilon^2. \tag{83}$$

**Lemma 5.6.** *As $t \to \infty$ there holds*

$$b = \frac{8\pi^2}{15} \left( \frac{dq^1}{dt}, \dots, \frac{dq^4}{dt}, -\frac{dr}{dt} \right) + O(\varepsilon^2).$$

**Proof.** As in [37], proof of Lemma 4.6, equation (34) yields the identity

$$4b = 4\int_{S^4} x(\alpha f_\Phi - Q_h)\, d\mu_h = 4\int_{S^4} x u_t \circ \Phi\, d\mu_h = \int_{S^4} \xi\, d\mu_h = X + I,$$

where

$$I = \int_{S^4} \xi (e^{4v} - 1)\, d\mu_{S^4}.$$

The claim then follows from (76) - (77) and the estimate

$$|I| \leq C||v||_{L^\infty}||\xi||_{L^1} \leq C||v||_{H^4}||\xi||_{L^\infty} \leq C\left(F_2 + \|f_\Phi - f(p)\|_{L^2}^2\right) \leq C\varepsilon^2$$

implied by (35), Lemmas 5.3 and 5.4, and (83). □

The proof of the next lemma is identical with the proof of its counterpart Lemma 4.7 in [37] and may be omitted.

**Lemma 5.7.** *As $t \to \infty$ there holds*

$$\frac{dp^i}{dt} = (2 + o(1))\varepsilon \frac{dq^i}{dt},\ i = 1, 2, 3, 4,$$

*and*

$$\frac{d}{dt}(1 - |p(t)|^2) = (2 + o(1))(1 - |p(t)|^2)\frac{dr}{dt}.$$

Finally, $\varepsilon$ and $|p(t)|$ can be related in the following way.

**Lemma 5.8.** *As $t \to \infty$ there holds*

$$1 - |p(t)|^2 = (8 + o(1))\varepsilon^2.$$

**Proof.** For $t = t_0$ with the above choice of coordinates we find

$$\begin{aligned} 1 - |p|^2 = 1 - |p^5|^2 = (1 + p^5)(1 - p^5) &= (2 + o(1)) \fint_{S^4} (1 - \Phi^5(t_0))\, d\mu_{S^4} \\ &= \left(\frac{12}{\pi^2} + o(1)\right) \int_{\mathbb{R}^4} (1 - \Psi_\varepsilon^5(z)) \frac{dz}{(1 + |z|^2)^4} \\ &= \left(\frac{12}{\pi^2} + o(1)\right) \int_{\mathbb{R}^4} \frac{2|\varepsilon z|^2 dz}{(1 + |\varepsilon z|^2)(1 + |z|^2)^4} \end{aligned}$$

where $o(1) \to 0$ as $t \to \infty$.

With the estimate

$$\int_{\mathbb{R}^4 \setminus B_{1/\varepsilon(0)}} \frac{|\varepsilon z|^2\, dz}{(1 + |\varepsilon z|^2)(1 + |z|^2)^4} \leq \int_{\mathbb{R}^4 \setminus B_{1/\varepsilon(0)}} \frac{dz}{(1 + |z|^2)^4} \leq C\varepsilon^4$$

the claim follows from

$$\lim_{\varepsilon \to 0}\Big(\int_{B_{1/\varepsilon}(0)} \frac{|z|^2\, dz}{(1 + |\varepsilon z|^2)(1 + |z|^2)^4}\Big) = \lim_{\varepsilon \to 0}\Big(\int_{B_{1/\varepsilon}(0)} \frac{|z|^2\, dz}{(1 + |z|^2)^4}\Big) = \frac{1}{3}\pi^2.$$

□

We summarize the results of Lemmas 5.5-5.8 in the following Proposition. Below, we denote by $\left(\frac{dp}{dt}\right)^T$ the component of $\frac{dp}{dt}$ which is tangential to $S^4$ (at $p/|p|$).

**Proposition 5.9.** *i) As $t \to \infty$ there holds*

$$\left(\frac{dp}{dt}\right)^T = \frac{15}{2}\varepsilon^2 \left(\nabla \log f(p) + o(1)\right),$$

*and*

$$\frac{d}{dt}(1 - |p|^2) = 120\varepsilon^4(\Delta_{S^4} \log f(p) + O(1)|\nabla f(p)|^2 + o(1)),$$

*with*

$$1-|p(t)|^2=(8+o(1))\varepsilon^2.$$

*ii) As $t\to\infty$ the metrics $g(t)$ concentrate at critical points $p$ of $f$ satisfying $\Delta_{S^4}f(p)\le 0$.*

**Proof.** *i*) Without loss of generality we can assume that $\frac{p}{|p|}=(0,0,0,0,1)$. Then the first statement is a direct consequence of Lemmas 5.5 - 5.8.

*ii*) From Lemma 5.8 and *i*) we obtain

$$\left|\frac{d}{dt}(1-|p|^2)\right|\le C(1-|p|^2)^2.$$

Hence we conclude that there holds

$$1-|p(t)|^2\ge\frac{C}{t},$$

which by Lemma 5.8 implies that

$$\varepsilon^2\ge\frac{C_1}{t} \tag{84}$$

for all $t\ge 1$, with a uniform constant $C>0$. From i) we then deduce the differential inequality

$$\frac{df(p(t))}{dt}=\nabla f(p)\frac{dp}{dt}\ge\frac{C_2}{t}(|\nabla f(p)|^2+o(1)). \tag{85}$$

Since the integral of $t^{-1}$ is divergent, the flow $(p(t))_{t\ge0}$ must accumulate at a critical point $p$ of $f$. The proof of convergence of the flow and the characterization $\Delta_{S^4}f(p)\le 0$ of possible limit points $p$ now again are identical with the proof of Proposition 4.9 in [37]. □

For future reference we also note the following result whose proof is identical with that of Lemma 4.10 in [37].

**Lemma 5.10.** *As $t\to\infty$ we have*

$$E_f(u(t))\to -3\log f(p),$$

*where $p=\lim_{t\to\infty}p(t)$ is the unique limit of the shadow flow $(p(t))$ associated with $(u(t))$.*

5.4. **Existence results.** We can now derive various existence results for metrics of prescribed curvature $f$.

Following Chang-Yang, for $p\in S^4$, $0<\varepsilon<\infty$, in stereographic coordinates with the point $-p$ at infinity (so that $p$ becomes the north pole and we may continue to use our previous notation) we let $\Phi_{p,\varepsilon}=\Psi_\varepsilon\circ\pi$, so that $\Phi_{p,\varepsilon}\rightharpoonup\Phi_{p,0}\equiv p$ weakly in $H^2$ as $\varepsilon\to0$. Note that $\Phi_{p,\varepsilon}=\Phi^{-1}_{p,\varepsilon^{-1}}=\Phi_{-p,\varepsilon^{-1}}$ for all $p\in S^4$, $0<\varepsilon<\infty$. Letting

$$C^\infty_*=\{u\in C^\infty(S^4);\, g=e^{2u}g_{S^4}\text{ satisfies (16)}\},$$

in view of (26) we then obtain a map

$$j\colon S^4\times]0,\infty[\ni(p,\varepsilon)\mapsto u_{p,\varepsilon}=1/4\log\det(d\Phi_{p,\varepsilon})\in C^\infty_*.$$

Observe that $\Phi_{p,1} = id$ and hence $j(p,1) = 0$ for all $p \in S^4$. Also let $g_{p,\varepsilon} = \Phi^*_{p,\varepsilon} g_{S^4} = e^{2u_{p,\varepsilon}} g_{S^4}$ with

$$d\mu_{g_{p,\varepsilon}} \rightharpoonup \frac{8}{3}\pi^2 \delta_p \tag{86}$$

as $\varepsilon \to \infty$.

Given a map $u_0 \in C^\infty_*$, for $t \geq 0$ we let $u(t, u_0)$ be the solution of the flow (17), (19) at time $t$ for initial data $u(0) = u_0$, and we let $\Phi(t, u_0)$ be the family of normalized conformal diffeomorphisms such that (23) holds for the pull-back metric

$$h = h(t, u_0) = e^{2v} g_{S^4} = \Phi^* g,$$

where we let $g = g(t, u_0) = e^{2u} g_{S^4}$, $\Phi = \Phi(t, u_0)$, and with suitable $v = v(t, u_0)$. Also let

$$p = p(t, u_0) = \fint_{S^4} \Phi(t, u_0)\, d\mu_{S^4} \tag{87}$$

denote the approximate center of mass of $g(t, u_0)$, so that $\Phi = \Phi_{p,\varepsilon}$ for some unique number $0 < \varepsilon = \varepsilon(t, u_0) < 1$ whenever $p \neq 0$. Note that the flow continuously depends on the initial data $u_0$ in any smooth topology, and therefore also all the above related quantities continuously depend on $u_0$.

Rescale the flow $u(t, u_0)$ by letting $s = s(t)$ solve

$$\frac{ds}{dt} = \min\{1/2, \varepsilon^2(t, u_0)\},\ s(0) = 0. \tag{88}$$

Note that (84) implies that $s(t) \to \infty$ as $t \to \infty$. For $0 \leq s < \infty$ then we let $U(s, u_0) = u(t(s), u_0)$, $P(s, u_0) = p(t(s), u_0)$. In view of Proposition 5.9 for $\varepsilon < 1/2$ the rescaled flow satisfies

$$\frac{dP^i}{ds} = 15 \left( \frac{\partial \log f}{\partial x^i}(P) + o(1) \right),\ i = 1, 2, 3, 4, \tag{89}$$

and

$$\frac{d}{ds}(1 - |P|^2) = 15(1 - |P|^2)(\Delta_{S^4} \log f(P) + O(1)|\nabla f(P)|^2 + o(1)). \tag{90}$$

In the following we again denote (rescaled) time as $t$, thus freeing the use of the letter $s$ for other purposes; moreover, for $\beta \in \mathbb{R}$ denote as

$$L_\beta = \{u \in C^\infty_*;\ E_f(u) \leq \beta\}$$

the sub-level sets $E_f$.

**Proof of Theorem 1.2** Suppose by contradiction that $f$ cannot be realized as the $Q$-curvature of a conformal metric $g$ on $S^4$. As we shall presently explain, the flow (17), (19) then may be used to show that for sufficiently large $\beta_0$ the set $L_{\beta_0}$ is contractible; moreover, the flow defines a homotopy equivalence of the set $N_0 = L_{\beta_0}$ with a set whose homotopy type is that of a point $\{p_0\}$ with cells of dimension $4 - ind(f, p)$ attached for every critical point $p$ of $f$ on $S^4$ such that $\Delta_{S^4} f(p) < 0$. We then obtain the identity (see e.g. [12], Theorem 4.3)

$$\sum_{i=0}^{4} t^i m_i = 1 + (1 + t) \sum_{i=0}^{4} t^i k_i \tag{91}$$

for the Morse polynomials of $\{p_0\}$ and $N_0$ and a connection term with coefficients $k_i \geq 0$, where

$$m_i = \#\{p \in S^4; \nabla f(p) = 0, \Delta_{S^4} f(p) < 0, ind(f,p) = 4 - i\},$$

as defined in (14). Equating the coefficients in the polynomials on the left and right hand side, we obtain (15), which violates the hypothesis in Theorem 1.2 and thus leads to the desired contradiction. By forming the alternating sum of the terms in (15) - which corresponds to setting $t = -1$ in (91) - we likewise obtain the statement of Theorem 1.1.

To proceed with the details of the proof we label all critical points $p_1, \dots, p_n$ of $f$ so that $f(p_i) \leq f(p_j)$ for $1 \leq i \leq j \leq n$ and let $\beta_i = -3\log f(p_i) = \lim_{s\to 0} E_f(u_{p_i,s})$, $1 \leq i \leq n$. For notational convenience only in the following we assume that all critical levels $f(p_i)$, $1 \leq i \leq n$, are distinct. We then can find a number $\nu_0 > 0$ so that $\beta_i - \nu_0 > \beta_{i+1}$ for all $i$. Theorem 1.2 now is immediate from the following result. □

**Proposition 5.11.** *i) For any $\beta_0 > \beta_1$ the set $L_{\beta_0}$ is contractible.*

*ii) For any $0 < \nu \leq \nu_0$ and each $i$ the set $L_{\beta_i - \nu}$ is homotopy equivalent to the set $L_{\beta_{i+1}+\nu}$.*

*iii) For each critical point $p_i$ of $f$ with $\Delta_{S^4} f(p_i) > 0$ the set $L_{\beta_i+\nu_0}$ is homotopy equivalent to the set $L_{\beta_i-\nu_0}$.*

*iv) For each critical point $p_i$ of $f$ with $\Delta_{S^4} f(p_i) < 0$ the set $L_{\beta_i+\nu_0}$ is homotopy equivalent to the set $L_{\beta_i-\nu_0}$ with a sphere of dimension $4 - ind(f,p_i)$ attached.*

**Proof.** *i*) Fix $\beta_0 > \beta_1 = -3\log f(p_1)$. For $u_0 \in L_{\beta_0}$, $0 \leq s \leq 1$ then let $H(s,u_0) = (1-s)u_0 + c(s,u_0)$, where $c(s,u_0)$ is a suitable constant so that $H(s,u_0) \in C^\infty_*$. Clearly, the map $H$ defines a contraction of $L_{\beta_0}$ within $C^\infty_*$. Moreover, by Lemma 5.10 for each such $u_0$, $0 \leq s \leq 1$ there exists a minimal $T = T(s,u_0) \geq 0$ such that $E_f(U(T, H(s,u_0))) \leq \beta_0$. From Lemma 2.1 and our assumption on $f$ we conclude that $T$ depends continuously on $u_0$ and $s$. The map $K\colon (s,u_0) \mapsto U(T(s,u_0), H(s,u_0))$ then yields the desired contraction of $L_{\beta_0}$ within itself.

*ii*) Let $0 < \nu \leq \nu_0$ be given. We claim there exists $T > 0$ with $U(T, L_{\beta_i-\nu}) \subset L_{\beta_{i+1}+\nu}$. Suppose by contradiction that there exist $T_k \to \infty$, $u_k \in L_{\beta_i-\nu}$ such that

$$E_f(U(T_k,u_k)) > \beta_{i+1} + \nu \text{ for all } k.$$

By Lemma 2.1 there is a sequence $t_k \in [T_k/2, T_k]$ such that $\int_{S^4} |Q_k - \alpha f| d\mu_{g_k} \to 0$ as $k \to \infty$, where $g_k = e^{2U(t_k,u_k)} d\mu_{g_{S^4}}$, $k \in \mathbb{N}$, and where $Q_k$ is the $Q$-curvature of $g_k$. In view of Lemma 3.6, for sufficiently large $k$ the metrics $g_k$ will be arbitrarily close to round metrics centered at the points $P(t_k,u_k) \in S^4$. Recalling (89) and (90), we may assume that the limit $P = \lim_{k\to\infty} P(t_k,u_k)$ exists and is a critical point of $f$. Lemma 3.6 then also yields convergence $E_f(U(t_k,u_k)) \to -3\log f(P)$ as $k \to \infty$. Since we assume that $E_f(u_k) \leq \beta_i - \nu$, by Lemma 2.1 we have $P = p_{i_0}$ for some $i_0 > i$ and hence

$$E_f(U(T_k,u_k)) \leq E_f(U(t_k,u_k)) \leq -3\log f(P) + \nu \leq \beta_{i+1} + \nu$$

for large $k$. The contradiction shows that there exists $T > 0$, as claimed. For $u_0 \in L_{\beta_i-\nu}$ then let

$$T(u_0) = \inf\{t \geq 0; E_f(U(t,u_0)) \leq \beta_{i+1} + \nu\} \leq T.$$

As in $i$), the number $T(u_0)$ continuously depends on $u_0$. The map $(t,u_0) \mapsto U(\min\{t,T(u_0)\},u_0)$ then yields the desired homotopy equivalence.

The remaining assertions $iii$) and $iv$) will be derived with the help of the following lemmas. □

**Lemma 5.12.** *There exists an absolute constant $C$ such that*

$$||v||^2_{H^2} \leq CE(v)$$

*for all $v \in H^2(S^4)$ inducing a normalized metric $h$, satisfying* (23)*and* (26)*, and provided $||v||_{H^2}$ is sufficiently small.*

**Proof.** From the spectral analysis of the operator $P_{S^4}$, as in the proof of Theorem 4.1 we obtain

$$\begin{aligned} E(v) &= \int_{S^4} (vP_{S^4}v + 12v)d\mu_{S^4} \\ &= \int_{S^4} (vP_{S^4}v + 3(e^{4v}-1) - 24v^2)d\mu_{S^4} + O(||v||^3_{H^2}) \\ &= \sum_{i=0}^{\infty}(\Lambda_i - 24)|v^i|^2 + O(||v||^3_{H^2}) \geq \sum_{i=6}^{\infty}(\Lambda_i - 24)|v^i|^2 + o(||v||^2_{H^2}) \\ &\geq \frac{\Lambda_6 - 24}{\Lambda_6 + 1}\sum_{i=0}^{\infty}(\Lambda_i + 1)|v^i|^2 + o(||v||^2_{H^2}), \end{aligned}$$

proving the claim. □

For $r_0 > 0$ and any critical point $p_i \in S^4$ of $f$ let

$$\begin{aligned} B_{r_0}(p_i) = \{u \in C^\infty_*;\ & g = e^{2u}g_{S^4} \text{ induces a normalized metric} \\ & h = \Phi^* g = e^{2v}g_{S^4} \text{ with } \Phi = \Phi_{p,s} \text{ for some } p \in S^4, \\ & 0 < s \leq 1 \text{ such that } ||v||^2_{H^2} + |p-p_i|^2 + s^2 < r_0^2\}. \end{aligned} \tag{92}$$

In the following estimates it will be convenient to use $s$, $p$, and $v$ as coordinates for $u \in B_{r_0}(p_i)$, where $g = e^{2u}g_{S^4}$ and $h = \Phi^*_{p,s}g = e^{2v}g_{S^4}$ as above. Moreover, since all critical points of $f$ by assumption are non-degenerate, we may use the Morse lemma to introduce coordinates $p = p^+ + p^-$ on $T_{p_i}S^4$ near $p_i = 0$ so that

$$f(p) = f(p_i) + |p^+|^2 - |p^-|^2, \tag{93}$$

and we may refer to these coordinates in the definition of $B_{r_0}(p_i)$ above.

**Lemma 5.13.** *For $r_0 > 0$ let $u \in B_{r_0}(p_i)$ be represented by $s$, $p$, and $v$ as above.*

*i) We have*

$$⨍_{S^4} f \circ \Phi_{p,s}\, d\mu_h = f(p) + 6s^2\Delta_{S^4}f(p) + o(1)s\|v\|_{H^2}, \tag{94}$$

*where $o(1) \to 0$ as $r_0 \to 0$.*

*ii) In addition, we may bound*

$$\left|\frac{\partial E_f(u)}{\partial s} + \frac{36 s \Delta_{S^4} f(p)}{f(p)}\right| \le C s^2 + C(s + |p - p_i|)\|v\|_{H^2}. \tag{95}$$

*iii) Likewise, for any $q \in T_p S^4$ there holds*

$$\left|\frac{\partial E_f(u)}{\partial p} \cdot q + \frac{3\, df(p) \cdot q}{f(p)}\right| \le C s(s + \|v\|_{H^2})|q|. \tag{96}$$

*iv) Finally, denoting as $\langle \cdot, \cdot \rangle$ the duality pairing of $H^2$ with its dual, with a uniform constant $c_0 > 0$ we have*

$$\langle \frac{\partial E_f(u)}{\partial v}, v \rangle \ge c_0 \|v\|_{H^2}^2 - o(1) s \|v\|_{H^2}, \tag{97}$$

*where $o(1) \to 0$ as $r_0 \to 0$.*

**Proof.** To simplify the notation, let

$$A = A(u) = \fint_{S^4} f \circ \Phi_{p,s}\, d\mu_h.$$

$i$) We have

$$A - f(p) = \fint_{S^4} (f \circ \Phi_{p,s} \; - f(p))\, d\mu_{S^4} + I,$$

where

$$I = \fint_{S^4} (f \circ \Phi_{p,s} \; - f(p))(e^{4v} - 1)\, d\mu_{S^4}.$$

Observing that $|df(p)| \le o(1) \to 0$ as $r_0 \to 0$, by (82) the error term may be bounded

$$|I| \le ||f \circ \Phi_{p,s} \; - f(p)||_{L^2} ||e^{4v} - 1||_{L^2} \le o(1) s ||v||_{H^2},$$

where $o(1) \to 0$ as $r_0 \to 0$. To proceed, we may assume that $p$ is the north pole. Upon introducing stereographic coordinates and expanding $f$ as in equation (81) in the proof of Lemma 5.5 then we obtain

$$\begin{aligned}
\frac{8\pi^2}{3}(A - f(p)) &= \int_{B_{1/s}(0)} (f(\Psi_s(z)) - f(p)) \frac{16\, dz}{(1 + |z|^2)^4} + O(s^4) + o(1) s ||v||_{H^2} \\
&= \int_{B_{1/s}(0)} \big(s df(p) z + s^2 \nabla df(p)(z,z)\big) \frac{32\, dz}{(1 + |z|^2)^4} + O(s^3) + o(1) s ||v||_{H^2} \\
&= II + O(s^3) + o(1) s ||v||_{H^2}.
\end{aligned}$$

By symmetry, the contribution from the linear term in the integral vanishes, yielding

$$\begin{aligned}
II &= s^2 \int_{B_{1/s}(0)} \nabla df(p)(z,z) \frac{32\, dz}{(1 + |z|^2)^4} \\
&= 8 s^2 \Delta_{S^4} f(p) \int_{B_{1/s}(0)} \frac{|z|^2 dz}{(1 + |z|^2)^4} = 16\pi^2 s^2 \Delta_{S^4} f(p).
\end{aligned}$$

Dividing again by the volume of $S^4$, we obtain the claim.

$ii$) Observe that by (27) we have

$$E_f(u) = E(u) - 3 \log\left(\fint_{S^4} f\, d\mu\right) = E(v) - 3 \log\left(\fint_{S^4} f \circ \Phi_{p,s}\, d\mu_h\right), \tag{98}$$

and

$$\frac{\partial E_f(u)}{\partial s} = -3A^{-1}\frac{\partial}{\partial s}\left(\fint_{S^4} f\circ\Phi_{p,s}\, d\mu_h\right).$$

For convenience, we again may identify $p$ with the north pole of $S^4$. In stereographic coordinates then for the unnormalized integral we have

$$\begin{aligned}\frac{\partial}{\partial s}\int_{S^4} f_{\Phi_{p,s}} d\mu_h &= \int_{\mathbb{R}^4}\left(\frac{\partial}{\partial s}(f(\Psi_s(z))\right)\frac{16\, e^{4v}dz}{(1+|z|^2)^4}\\ &= \int_{\mathbb{R}^4}\left(\frac{\partial}{\partial s}(f(\Psi_s(z))\right)\frac{16\, dz}{(1+|z|^2)^4} + \int_{\mathbb{R}^4}\left(\frac{\partial}{\partial s}(f(\Psi_s(z))\right)\frac{16(e^{4v}-1)dz}{(1+|z|^2)^4}\\ &= I + II.\end{aligned}$$

Expanding $f$ as in equation (81), we obtain

$$\frac{\partial}{\partial s}(f(\Psi_s(z)) = 2df(p)z + 4s\nabla df(p)(z,z) + O(s^2|z|^3).$$

By oddness then we find

$$\begin{aligned}I &= \int_{\mathbb{R}^4} df(p)z\frac{32\, dz}{(1+|z|^2)^4} + 16s\Delta_{S^4}f(p)\int_{\mathbb{R}^4}\frac{|z|^2dz}{(1+|z|^2)^4} + O(s^2)\\ &= 16s\Delta_{S^4}f(p)\int_{\mathbb{R}^4}\frac{|z|^2dz}{(1+|z|^2)^4} + O(s^2) = 32\pi^2 s\Delta_{S^4}f(p) + O(s^2).\end{aligned}$$

On the other hand, the expansion to first order

$$\frac{\partial}{\partial s}(f(\Psi_s(z)) = 2df(p)z + O(s|z|^2) = 2(df(p) - df(p_i))z + O(s|z|^2)$$

yields the uniform estimate

$$|\frac{\partial}{\partial s}(f(\Psi_s(z))| \le C|p-p_i||z| + O(s|z|^2).$$

Thus we obtain the bound

$$|II| \le C(s+|p-p_i|)\int_{\mathbb{R}^4}|e^{4v}-1|\frac{(1+|z|^2)\, dz}{(1+|z|^2)^4} \le C(s+|p-p_i|)\|v\|_{H^2},$$

and the claim follows from $i$).

$iii$) For any $q\in T_pS^4$ we may write

$$\begin{aligned}A\frac{\partial E_f(u)}{\partial p}\cdot q + 3\, df(p)\cdot q &= 3\fint_{S^4}\left(df(p) - \frac{\partial(f\circ\Phi_{p,s})}{\partial p}\right)\cdot q\, d\mu_h\\ &= 3\fint_{S^4}\left(df(p) - \frac{\partial(f\circ\Phi_{p,s})}{\partial p}\right)\cdot q\, d\mu_{S^4}\\ &\quad + 3\fint_{S^4}\left(df(p) - \frac{\partial(f\circ\Phi_{p,s})}{\partial p}\right)\cdot q(e^{4v}-1)\, d\mu_{S^4} = I + II.\end{aligned}$$

Proceeding similarly to the proof of $i$) one finds

$$|I| \le Cs^2|q|$$

and, on the other hand,

$$|II| \le Cs\|v\|_{H^2}|q|.$$

*iv*) In the above notation we have

$$\langle\frac{\partial E_f(u)}{\partial v},v\rangle=\langle\frac{\partial E(v)}{\partial v},v\rangle-12A^{-1}\fint_{S^4}f_{\Phi_s}ve^{4v}d\mu_{S^4}$$
$$=\langle\frac{\partial E(v)}{\partial v},v\rangle-12\fint_{S^4}ve^{4v}d\mu_{S^4}-I,$$

where

$$I=12A^{-1}\fint_{S^4}(f_{\Phi_s}-f(p))ve^{4v}d\mu_{S^4}+12(A^{-1}f(p)-1)\fint_{S^4}ve^{4v}d\mu_{S^4}=II+III.$$

As in the proof of Lemma 5.12, with error $o(1)\to 0$ as $r_0\to 0$ we can estimate

$$\langle\frac{\partial E(v)}{\partial v},v\rangle-12\fint_{S^4}ve^{4v}d\mu_{S^4}=\fint_{S^4}(2vP_{S^4}v+4Q_{S^4}v)\,d\mu_{S^4}-12\fint_{S^4}ve^{4v}d\mu_{S^4}$$
$$=\fint_{S^4}2vP_{S^4}v\,d\mu_{S^4}-12\fint_{S^4}v(e^{4v}-1)d\mu_{S^4}$$
$$=2\fint_{S^4}(vP_{S^4}v-24v^2)\,d\mu_{S^4}-o(1)\|v\|^2_{H^2}\ge c_0\|v\|^2_{H^2}-o(1)\|v\|^2_{H^2}$$

for some constant $c_0>0$. Moreover, similar to the proof of *i*) we can bound

$$|II|\le||f\circ\Phi_{p,s}\ -f(p)||_{L^2}||v(e^{4v}-1)||_{L^2}\le Co(1)s\|v\|_{H^2},$$

while from *i*) we find

$$|III|\le C|A-f(p)|\int_{S^4}|v|e^{4v}\,d\mu_{S^4}\le C(s^2+s\|v\|_{H^2})\|v\|_{H^2}.$$

Combining these estimates, we obtain the claim. □

**Proof of Proposition 5.11 (completed)** Given $r_0>0$, let $\nu=r_0^3>0$. In the following we will always assume that $r_0>0$ is chosen so small that $B_{r_0}(p_i)\subset(L_{\beta_i+\nu_0}\setminus L_{\beta_i-\nu_0})$ and $\nu<\nu_0$. By *ii*), for any $i$ and sufficiently large $T>0$ we have $U(T,L_{\beta_i+\nu_0})\subset L_{\beta_i+\nu}$.

Recall that by (98) for $u=(s,p,v)\in B_{r_0}(p_i)$ we have

$$E_f(u)=E(u)-3\log\left(\fint_{S^4}f\,d\mu\right)=E(v)-3\log A=E(v)+\beta_i-3I \tag{99}$$

where

$$I=\log\left(1+\frac{A-f(p_i)}{f(p_i)}\right)=\frac{A-f(p_i)}{f(p_i)}+O(|A-f(p_i)|^2).$$

Splitting

$$A-f(p_i)=(A-f(p))+(f(p)-f(p_i)),$$

from (93) and Lemma 5.13.*i*) we obtain

$$A-f(p_i)=6s^2\Delta_{S^4}f(p)+|p^+|^2-|p^-|^2+o(1)s\|v\|_{H^2}.$$

Hence we find the expansion

$$I\cdot f(p_i)=6s^2\Delta_{S^4}f(p)+|p^+|^2-|p^-|^2+o(1)(s^2+|p-p_i|^2+\|v\|^2_{H^2}). \tag{100}$$

With constants $c_1>0$, $C$ we conclude

$$E_f(u)\ge\beta_i+c_1\|v\|^2_{H^2}-C(s^2+|p-p_i|^2).$$

It follows that for $u \in L_{\beta_i+\nu} \cup B_{r_0}(p_i)$ there holds

$$\|v\|_{H^2}^2 \le C(s^2 + |p - p_i|^2 + r_0^3).$$

On the other hand, Lemmas 2.1 and 5.5 yield the bound

$$\frac{d}{dt}\Big|_{t=0} E_f(U(t,u_0)) \le -c_2 \frac{|\nabla f(p)|^2 + s^2|\Delta_{S^4} f(p)|^2}{f(p)^2} \le -c_3(s^2 + |p - p_i|^2)$$

with uniform constants $c_2, c_3 > 0$, and where $o(1) \to 0$ as $r_0 \to 0$, uniformly in $u_0 \in B_{r_0}(p_i)$. Hence for $u_0 \in B_{r_0} \setminus B_{r_0/4}(p_i)$ we have

$$\frac{d}{dt}\Big|_{t=0} E_f(U(t,u_0)) \le -c_4 r_0^2,$$

with a uniform constant $c_4 > 0$. Since the time needed for the flow $U(t,\cdot)$ to traverse the annular region $B_{r_0/2} \setminus B_{r_0/4}(p_i)$ is uniformly bounded from below as $r_0 \to 0$, for sufficiently large $T > 0$ and sufficiently small $r_0 > 0$ therefore we have $U(T, L_{\beta_i+\nu_0}) \subset L_{\beta_i-\nu} \cup B_{r_0/2}(p_i)$. Let $T(u_0) = \min\{T, \inf\{t \ge 0; E_f(U(t,u_0)) \le \beta_i - \nu\}\}$, depending continuously on $u_0$. The map $(t, u_0) \mapsto U(\min\{t, T(u_0)\}, u_0)$ then defines a homotopy equivalence of $L_{\beta_i+\nu_0}$ with a subset of $L_{\beta_i-\nu} \cup B_{r_0/2}(p_i)$.

Depending on the sign of $\Delta_{S^4} f(p_i)$ we now proceed as follows.

*iii)* Suppose $\Delta_{S^4} f(p_i) > 0$. Given $r_0 > 0$, for $u = (s,p,v) \in B_{r_0}(p_i)$ with

$$||v||_{H^2}^2 + |p - p_i|^2 + s^2 < r_0^2,\ p - p_i = p^+ + p^-,$$

let the vector field $X(u)$ be defined as

$$X(u) = (1, r_0^{-1}(p^+ - p^-), -r_0^{-1} v).$$

Then let $G(u,r)$ solve the flow equation

$$\frac{d}{dr} G(u,r) = X(G(u,r)),\ 0 \le r \le r_0,$$

with initial value $G(u,0) = u$. Note that $X$ is transversal to the boundary of $B_{r_0}(p_i)$ and $G(u,r_0) \notin B_{r_0}(p_i)$; hence there is a first time $0 \le r = r(u) \le r_0$ such that $G(u,r) \notin B_{r_0}(p_i)$, and the map $u \mapsto r(u)$ is continuous. Defining $H(u,r) = G(u, \min\{r, r(u)\})$, we then obtain a homotopy $H\colon \overline{B_{r_0}(p_i)} \times [0, r_0] \to \overline{B_{r_0}(p_i)}$ such that

$$H(B_{r_0}(p_i), r_0) \subset \partial B_{r_0}(p_i),\ H(\cdot, r)_{|\partial B_{r_0}(p_i)} = id,\ 0 \le r \le r_0.$$

Moreover, by Lemma 5.13, letting $u_r = H(u,r)$, for $0 \le r \le r(u)$ we have

$$\begin{aligned}\frac{d}{dr} E_f(u_r) &= dE_f(u_r) \cdot X(u_r)\\ &= \frac{dE_f(u_r)}{ds} + r_0^{-1} \frac{dE_f(u_r)}{dp}(p^+ - p^-) - r_0^{-1}\langle \frac{dE_f(u_r)}{dv}, v\rangle\\ &\le -36 s \Delta_{S^4} f(p)/f(p) - r_0^{-1}(3|p - p_i|^2/f(p_i) + c_0\|v\|_{H^2}^2) + O(r_0^2),\end{aligned}$$

and it follows that

$$E_f(H(u,r_0)) \le E_f(u) \text{ for all } u \in B_{r_0}(p_i).$$

In particular, we have

$$E_f(H(u,r_0)) \le \beta_i - \nu \text{ for all } u \in L_{\beta_i-\nu}.$$

In addition, with a uniform constant $c_1 > 0$ there holds

$$E_f(H(u,r_0)) \le E_f(u) - c_1 r_0^2 \text{ for all } u \in B_{r_0/2}(p_i) \cap L_{\beta_i+\nu},$$

and the right hand side will be smaller than $\beta_i - \nu$ if $r_0 > 0$ is sufficiently small. Composing $H$ with the flow $(t, u_0) \mapsto U(\min\{t, T(u_0)\}, u_0)$, we then obtain a homotopy $K\colon L_{\beta_i+\nu_0} \times [0,1] \to L_{\beta_i+\nu_0}$ such that $K(L_{\beta_i+\nu_0}, 1) \subset L_{\beta_i-\nu}$. Moreover, by our choice of $r_0 > 0$, for $0 \le r \le 1$ the map $K(\cdot, r)$ will be the identity map on $L_{\beta_i-\nu_0}$. For each $u_0 \in L_{\beta_i-\nu}$, finally, let $T_0(u_0) = \inf\{t \ge 0; E_f(U(t,u_0)) \le \beta_i - \nu_0\}$. As in Step *ii)* the numbers $T_0(u_0)$ are uniformly bounded and depend continuously on $u_0$. Upon composing $K$ with the flow $(t, u_0) \mapsto U(\min\{t, T_0(u_0)\}, u_0)$ we then obtain the desired homotopy equivalence of $L_{\beta_i+\nu_0}$ with $L_{\beta_i-\nu_0}$.

*iv)* From (99) and (100), with the constant $c_0 > 0$ from Lemma 5.12 we conclude

$$\begin{aligned} E_f(u) - \beta_i \ge c_0\|v\|^2_{H^2} - 18s^2 f(p_i)^{-1}\Delta_{S^4} f(p) \\ + 3f(p_i)^{-1}(|p^-|^2 - |p^+|^2) + o(1)(s^2 + |p-p_i|^2 + \|v\|^2_{H^2}), \end{aligned} \tag{101}$$

where $o(1) \to 0$ as $r_0 \to 0$. Assuming that $\Delta_{S^4} f(p_i) < 0$, we deduce that there exists a number $\delta > 0$ such that there holds

$$s^2 + |p^-|^2 + \|v\|^2_{H^2} < r_0^2/4 \tag{102}$$

for any $u = (s,p,v) \in B_{r_0}(p_i) \cap L_{\beta_i+\nu}$ with $|p^+| < 2\delta r_0$, provided $r_0 > 0$ and $\nu = r_0^3$ are sufficiently small.

Let $\eta$ be the cut-off function given by $\eta = (1 - \frac{(|p^+|-\delta r_0)_+}{\delta r_0})_+$, where $a_+ = \max\{a, 0\}$ for $a \in \mathbb{R}$, and fix some number $0 < s_0 < r_0^3$. (For conceptual simplicity it would be best to choose $s_0 = 0$, which, unfortunately, is not permitted.) For $0 \le r \le 1$, $u \in B_{r_0}(p_i)$ then define $H_0(u, r) = u_r$ by letting

$$u_r = (s_r, p_r, v_r) = (r\eta s_0 + (1 - r\eta)s, p^+ + (1 - r\eta)p^-, (1 - r\eta)v)$$

to obtain a homotopy $H_0\colon \overline{B_{r_0}(p_i)} \cap L_{\beta_i+\nu} \times [0,1] \to \overline{B_{r_0}(p_i)}$ such that $H_0(\cdot, 1)$ maps the set $\{u \in B_{r_0}(p_i) \cap L_{\beta_i+\nu};\ |p^+| < \delta r_0\}$ to the set $B^+_{\delta r_0}$, where for $0 < \rho < r_0$ we denote as

$$B^+_\rho := \{u \in B_{r_0}(p_i);\ s = s_0, p^- = 0, |p^+| < \rho, v = 0\}.$$

Note that the set $B^+_{\delta r_0}$ is diffeomorphic to the unit ball of dimension $4 - ind(f, p_i)$. Moreover, letting

$$\frac{d}{dr}E_f(u_r) = \eta\left(\frac{dE_f(u_r)}{ds}(s_0 - s) - \frac{dE_f(u_r)}{dp}p^- - \langle\frac{dE_f(u_r)}{dv}, v\rangle\right) =: \eta D,$$

with the help of Lemma 5.13 for $0 \le r \le 1$ we compute

$$\begin{aligned} D \le Cs_0 s + 36s^2\Delta_{S^4} f(p)/f(p) - 3|p^-|^2/f(p_i) - c_0\|v\|^2_{H^2} \\ + o(1)(s^2 + \|v\|^2_{H^2} + |p^-|^2 + r_0^6). \end{aligned}$$

By the choice of $s_0$, for sufficiently small $r_0 > 0$ the term $D$ is negative whenever $s^2 + \|v\|^2_{H^2} + |p^-|^2 \ge r_0^4$, and $E_f(u_r) \le \beta_i + Cr_0^4 < \beta_i + \nu$, else. It follows that $H_0(\cdot, r)$ maps the set $B_{r_0}(p_i) \cap L_{\beta_i+\nu}$ into itself for all $r$. Finally, by (102) we have

$$H_0(\cdot, r)_{|\partial B_{r_0}(p_i) \cap L_{\beta_i+\nu}} = id,\ 0 \le r \le 1.$$

Now let the vector field $X_1(u)$ be defined as

$$X_1(u) = (0, p^+, 0)$$

and let $G_1(u, r)$ solve the flow equation

$$\frac{d}{dr}G_1(u, r) = X_1(G_1(u, r)),\ 0 \le r \le \delta^{-1},$$

with initial value $G_1(u,0)=u$. Again we note that $X$ is transversal to the boundary of $B_{r_0}(p_i)$; moreover, for any $u \in B_{r_0}(p_i)$ with $|p^+| \geq \delta r_0$ there holds $G_1(u,\delta^{-1}) \notin B_{r_0}(p_i)$; hence there is a first time $0 \leq r = r_1(u) \leq \delta^{-1}$ such that $G_1(u,r) \notin B_{r_0}(p_i)$, and the map $u \mapsto r_1(u)$ is continuous. We extend this map continuously to the whole ball $B_{r_0}(p_i)$ by letting $r_1(u) = \delta^{-1}$ whenever $G_1(u,r) \in B_{r_0}(p_i)$ for all $r \in [0,\delta^{-1}]$. Defining $H_1(u,r) = G_1(u, min\{r, r_1(u)\}) = u_r$, by Lemma 5.13 with a uniform constant $c_5 > 0$ for sufficiently small $r_0 > 0$ we have

$$\frac{d}{dr}E_f(u_r) = \frac{dE_f(u_r)}{dp}p^+ \leq -3|p^+|^2/f(p_i) + Cr_0^3 \leq -c_5 r_0^2,$$

if $|p^+| > \delta r_0$. Hence, letting $H$ be the composition of $H_0$ with $H_1$, for sufficiently small $r_0 > 0$ we obtain a homotopy $H\colon \overline{B_{r_0}(p_i)} \cap L_{\beta_i+\nu} \times [0,1] \to \overline{B_{r_0}(p_i)} \cap L_{\beta_i+\nu}$ such that $H(B_{r_0}(p_i) \cap L_{\beta_i+\nu}, 1) \subset B^+_{r_0} \cup L_{\beta_i-\nu}$ and

$$H(B_{r_0}(p_i),1) \subset \partial B_{r_0}(p_i) \cup B^+_{r_0},\ H(\cdot,r)_{|\partial B_{r_0}(p_i)} = id,\ 0 \leq r \leq 1.$$

The proof can now be finished as in *iii*). □

## References


[1] Th. Aubin: *Meilleures constantes dans le théorème d'inclusion de Sobolev et un théorème de Fredholm non linéaire pour la transformation conforme de la courbure scalaire*, J. Funct. Anal. 32 (1979), 148-174.

[2] A. Bahri, J. M. Coron: *The Scalar-Curvature problem on the standard three-dimensional sphere*, J. Funct. Anal. 95 (1991), 106-172.

[3] W. Beckner: *Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality*, Annals of Math. 138 (1993), 213-242.

[4] M. Berger, P. Gauduchon, E. Mazet: *Le spectre d'une varieté Riemannienne*, Lecture Notes in Mathematics 194, Springer (1971).

[5] A. L. Besse: *Einstein manifolds*, Springer-Verlag, Berlin (1987).

[6] T. P. Branson: *Differential operators canonically associated to a conformal structure*, Mathematica Scandinavica, 57-2 (1985), 293-345.

[7] T. P. Branson, B. Ørsted: *Explicit functional determinants in four dimensions*, Proc. Amer. Math. Soc. 113-3 (1991), 669-682.

[8] S. Brendle: *Global existence and convergence for a higher order flow in conformal geometry*, Annals of Math. 158 (2003), 323-343.

[9] S. Brendle: *Prescribing a higher order conformal invariant on $S^n$*, Comm. Anal. Geom. 11-5 (2003), 837-858.

[10] S. Brendle: *Convergence of the Yamabe flow*, preprint 2004.

[11] S. Brendle: *On the asymptotic behavior of the Q-curvature flow on $S^4$*, preprint (November 28, 2004).

[12] K. C. Chang: *Infinite Dimensional Morse Theory and Multiple Solutions Problems*, Birkhäuser (1993).

[13] K. C. Chang, L. J. Liu: *On Nirenberg's problem*, Intern. J. Math. 4 (1993), 35-58.

[14] S. Y. A. Chang, P. C. Yang: *Prescribing Gaussian curvature on $S^2$*, Acta Math. 159 (1987), 215-259.

[15] S. Y. A. Chang, P. C. Yang: *Conformal deformation of metrics on $S^2$*, J. Diff. Geom. 27 (1988), 256-296.

[16] S. Y. A. Chang, P. C. Yang: *Extremal metrics of zeta function determinants on 4-manifolds*, Annals of Math. 142 (1995), 171-212.

[17] S. Y. A. Chang, P. C. Yang: *On a fourth order curvature invariant*, Contemp. Math. 237 (1999), 9-28.

[18] S. Y. A. Chang, P. C. Yang: *The inequality of Moser and Trudinger and applications to conformal geometry. Dedicated to the memory of Jürgen K. Moser.* Comm. Pure Appl. Math. 56 (2003), no. 8, 1135–1150.

[19] W. Chen, C. Li: *Classification of solutions of some nonlinear equations*, Duke Math. J. 63 (1991), 615-623.

[20] Z. Djadli, E. Hebey, M. Ledoux: *Paneitz-type operators and applications*, Duke Math. J. 104-1 (2000) 129-169.

[21] Z. Djadli, A. Malchiodi: *Existence of conformal metrics with constant $Q$-curvature*, preprint (2004).

[22] Z. Djadli, A. Malchiodi, M. Ould Ahmedou: *Prescribing a fourth order conformal invariant on the standard sphere, Part I: a perturbation result*, Comm. Contemp. Math. 4 (2002), 1-34.

[23] Z. Djadli, A. Malchiodi, M. Ould Ahmedou: *Prescribing a fourth order conformal invariant on the standard sphere, Part II: blow up analysis and applications*, Annali Scuola Norm. Sup. Pisa, vol. 5 (1) (2002), 387-434.

[24] M. Grüneberg: Ph.D. thesis, Stanford (2004).

[25] M. Gursky: *The principal eigenvalue of a conformally invariant differential operator, with an application to semilinear elliptic pde*, preprint (1998).

[26] J. Kazdan, F. Warner: *Curvature functions for compact 2-manifolds*, Annals of Math. 99 (1974), 14-47.

[27] C. S. Lin: *A classification of solutions of conformally invariant fourth order equation in $\mathbb{R}^n$*, Commentari Mathematici Helveticii, 73 (1998), 206-231.

[28] A. Malchiodi: *The Scalar Curvature Problem on $S^n$: an approach via Morse Theory*, Calc. Var. 14 (2002), 429-445.

[29] A. Malchiodi: *Boundedness of Palais-Smale sequences associated to a fourth-order equation in conformal geometry*, preprint (2004).

[30] J. Moser: *On a nonlinear problem in differential geometry*, Dynamical Systems (M. Peixoto ed.), Academic Press, New York, 1973, 273-280.

[31] E. Onofri: *On the positivity of the effective action in a theory of random surfaces*, Comm. Math. Phys. 86 (1982), 321-326.

[32] S. Paneitz: *A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds*, preprint, 1983.

[33] F. Robert, M. Struwe: *Asymptotic profile for a fourth order PDE with critical exponential growth in dimension four*, to appear in Advanced Nonlinear Studies, preprint (2004).

[34] R. Schoen, D. Zhang: *Prescribed scalar curvature on the $n$-spere*, Calc. Var. 4 (1996), 1-25.

[35] H. Schwetlick, M. Struwe: *Convergence of the Yamabe flow for "large" energies*, J. Reine Angew. Math. 562 (2003), 59-100.

[36] M. Struwe: *Curvature flows on surfaces*, Annali Sc. Norm. Sup. Pisa, Ser. V, 1 (2002), 247-274.

[37] M. Struwe: *A flow approach to Nirenberg's problem*, to appear in Duke Math. J., preprint (2004)

[38] J. Wei, X. Xu: *On conformal deformations of metrics on $S^n$*, J. Funct. Anal. 157-1 (1998), 292-325.

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