PARTIAL REGULARITY FOR HARMONIC MAPS, AND RELATED PROBLEMS

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Abstract. Via gauge theory, we give a new proof of partial regularity for harmonic maps in dimensions $m \geq 3$ into arbitrary targets. This proof avoids the use of adapted frames and permits to consider targets of "minimal" C^2 regularity. The proof we present moreover extends to a large class of elliptic systems.

1. Introduction

In [10], the first author presented a new approach to the regularity result of Hélein [6] for weakly harmonic maps in dimension $m = 2$ where he succeeded in writing the harmonic map system in the form of a conservation law whose constituents satisfied elliptic equations with a Jacobian structure to which Wente's [12] regularity results could be applied.

Consider for instance a harmonic map $u = (u^1, \ldots, u^n) \in H^1(B; \mathbb{R}^n)$ from a ball $B^m = B \subset \mathbb{R}^m$ to a hypersurface $N \subset \mathbb{R}^n$ with normal ν. In this case the harmonic map equation may be written in the form

$$-\Delta u^i = w^i \nabla w^j \cdot \nabla u^j = (w^i \nabla w^j - w^j \nabla w^i) \cdot \nabla u^j,\ 1 \leq i \leq n, \tag{1}$$

where $w = \nu \circ u$. The key idea then is to identify the anti-symmetry of the 1-form

$$\Omega^{ij} = (w^i dw^j - w^j dw^i),\ 1 \leq i, j \leq n, \tag{2}$$

as the essential structure of equation (1).

Interpreting $\Omega \in L^2(B; so(n) \otimes \wedge^1 \mathbb{R}^n)$ as a connection in the $SO(n)$-bundle u^*TN and following Uhlenbeck's approach to the existence of Coulomb gauges [11], if $m = 2$, one succeeds in finding $P \in H^1(B; SO(n))$ and $\xi \in H^1(B)$ such that

$$P^{-1}dP + P^{-1}\Omega P = *d\xi, \tag{3}$$

where $*$ is the Hodge dual. Further algebraic manipulations then yield the existence of matrix-valued $A, B \in H^1(B)$ with

$$||dist(A, SO(n))||_{L^\infty} \leq C||\Omega||_{L^2} \tag{4}$$

such that (1) may be written as

$$div(A\nabla u + B\nabla^\perp u) = 0, \tag{5}$$

where $\nabla^\perp = *d$. By Hodge decomposition one then obtains E and D in $W^{1,2}(B)$ such that

$$A\,\nabla u = \nabla D + \nabla^\perp E. \tag{6}$$

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From (5) we see that D and E satisfy the equations

$$\begin{aligned} -\Delta D &= -div(A\,\nabla u) = \nabla B \cdot \nabla^{\perp} u, \\ -\Delta E &= curl(A\,\nabla u) = \nabla^{\perp} A \cdot \nabla u, \end{aligned} \tag{7}$$

which exhibit the desired Jacobian structure. The results in [3] then imply that $D, E \in W^{2,1}_{loc}(B)$. Provided that we restrict our attention to a domain where $||\Omega||_{L^2}$ is sufficiently small, from (4) we conclude that $\nabla u = A^{-1}(\nabla D + \nabla^{\perp} E) \in W^{1,1}_{loc}(B)$ and $u \in W^{2,1}_{loc}(B) \hookrightarrow C^0(B)$, which implies full regularity.

In dimensions $m \geq 3$, the harmonic map equation is super-critical in the Sobolev space $H^1(B;\mathbb{R}^n)$ and no regularity result, not even a partial one, can be expected. In fact, in [9] the first author constructed examples of weak solutions to (1) in $H^1(B;S^2)$ for $m \geq 3$ which are *nowhere* continuous.

Under the further assumption that the solution u lies in the homogeneous Morrey space $L^{2,m-2}_1$, which sometimes also is denoted as $M^{1,2}_2$, with

$$\|u\|^2_{L^{2,m-2}_1} = \sup_{x\in B\,,\,r>0} \left(\frac{1}{r^{m-2}} \int_{B_r(x)\cap B} |\nabla u|^2 \right) < +\infty, \quad , \tag{8}$$

the harmonic map equation (1) becomes critical. More generally, this is true for any elliptic system with a nonlinearity growing quadratically in the gradient (see [5]). Assumption (8) is natural in the context of harmonic maps; in fact, it is a direct consequence of the geometric stationarity assumption described in Section 2. Observe that in dimension $m = 2$ assumption (8) corresponds exactly to the assumption of finite energy and it therefore appears as the natural extension of the finite energy condition to higher dimensions.

Strengthening the assumption that $u \in H^1(B;\mathbb{R}^n)$ by assuming (8), one might then hope to be able to extend the approach described above to the case $m \geq 3$. However, in order to achieve (4), in dimension $m = 2$ one crucially uses that by Wente's result mentioned above the solution $\psi \in H^1(B^2; \wedge^2\mathbb{R}^n)$ of the equation

$$\begin{cases} \Delta\psi = df \wedge dg & \text{in } B^m \\ \\ \psi = 0 & \text{on } \partial B^m \end{cases} \tag{9}$$

for given f and g in $H^1(B^2)$ belongs to L^∞. Unfortunately, this result does not extend to $m \geq 3$ when we replace the assumption $f, g \in H^1(B^m)$ by the condition that f and g belong to the Morrey space $L^{2,m-2}_1(B^m)$. Indeed, for $m = 3$, letting $f = \frac{x_1}{|x|}$ and choosing $g = \frac{x_2}{|x|}$, we have $f, g \in L^{2,1}_1(B^3)$ and equation (9) admits a unique solution $\psi \in L^{2,1}_1(B^3)$, but $\psi \notin L^\infty$. Thus the L^∞-bound (4) does not seem to be available in dimension larger than 2 and the approach outlined above seems to fail for this reason.

However, as we presently explain, (1) - (3) in combination with standard techniques of elliptic regularity theory already suffice to conclude partial regularity, directly. In fact, via the gauge transformation P, from (1) we obtain the equation

$$-div(P^{-1}\nabla u) = (P^{-1}\nabla P + P^{-1}\Omega P) \cdot P^{-1}\nabla u = *d\xi \cdot P^{-1} du, \tag{10}$$

where the right hand side already has the structure of a Jacobian – up to the harmless (bounded) factor P^{-1}. Also observe that ∇u may be recovered from the term $P^{-1}\nabla u$ without any difficulty.

More generally, partial regularity results can be obtained for a large class of elliptic systems with quadratic growth that can be cast in the form

$$-\Delta u = \Omega \cdot \nabla u \qquad \text{in } B \tag{11}$$

already considered in [10]. (In coordinates, equation (11) simply reads $-\Delta u^i = \Omega^{ij} \cdot \nabla u^j$.)

Theorem 1.1. *For every $m \in \mathbb{N}$ there exists $\varepsilon(m) > 0$ such that for every $\Omega \in L^2(B^m, so(n) \otimes \wedge^1\mathbb{R}^m)$ and for every weak solution $u \in H^1(B^m, \mathbb{R}^n)$ of equation* (11)*, satisfying the Morrey growth assumption*

$$\sup_{x\in B,\, r>0} \left(\frac{1}{r^{m-2}} \int_{B_r(x)\cap B} (|\nabla u|^2 + |\Omega|^2)\, dx \right) < \varepsilon(m) \quad , \tag{12}$$

we have that u is locally Hölder continuous in $B = B(m)$ with exponent $0 < \alpha = \alpha(m) < 1$.

The previous result is optimal, as shown by the standard example of the weakly harmonic map $u \colon B^3 \to S^2 \hookrightarrow \mathbb{R}^3$ with $u(x) = x/|x|$. We have $u \in H^1(B^3, \mathbb{R}^3)$ and, letting $\Omega = (\Omega^{ij}) := (u^i du^j - u^j du^i) \in L^2(B^m, so(n) \otimes \wedge^1\mathbb{R}^m)$, we see that u weakly satisfies the equation (11) and the condition

$$\sup_{x\in B\,,\, r>0} \left(\frac{1}{r^{m-2}} \int_{B_r(x)\cap B} (|\nabla u|^2 + |\Omega|^2)\, dx \right) < +\infty \quad . \tag{13}$$

The map u, however, is not continuous at the origin.

2. Stationary harmonic maps

For a smooth, closed, oriented k-dimensional submanifold $N \subset \mathbb{R}^n$ and a ball $B \subset \mathbb{R}^m$ let

$$H^1(B; N) = \{u \in H^1(B; \mathbb{R}^n);\ u(x) \in N \text{ for almost every } x \in B\}. \tag{14}$$

Recall that a map $u \in H^1(B; N)$ is *stationary* if u is critical for the energy

$$E(u) = \int_B |\nabla u|^2\, dx$$

both with respect to variations of the map u and with respect to variations in the domain.

It follows that u is weakly harmonic; that is, u satisfies the equation

$$-\Delta u = A(u)(\nabla u, \nabla u) = \sum_{l=1}^{n-k} \sum_{\alpha=1}^{m} \nu_l \langle d\nu_l \partial_\alpha u, \partial_\alpha u\rangle) = \sum_{l=1}^{n-k} w_l \langle \nabla w_l, \nabla u\rangle \tag{15}$$

in the sense of distributions, where A is the second fundamental form of N, defined locally via an orthonormal frame field ν_l, $1 \le l \le n-k$ for the normal bundle to N. Again we denote as $w_l = \nu_l \circ u$ the corresponding unit normal vector field along the map u, and we denote as $\langle \cdot, \cdot \rangle$ the Euclidean inner product.

Moreover, as a consequence of the stationarity condition with respect to variations in the domain we have the monotonicity estimate

$$r^{2-m}\int_{B_r(x_0)}|\nabla u|^2\,dx \le R^{2-m}\int_{B_R(x_0)}|\nabla u|^2\,dx \tag{16}$$

for all balls $B_R(x_0)\subset B$ and all $r\le R$.

The following result was obtained by Evans [4] and Bethuel [1]. Note that their approach in general requires the target manifold N^k to be of class C^5; see [6], Theorem 4.3.1 and Remark 4.3.2. As a corollary to our main result Theorem 1.1, however, we now easily obtain the following generalization of their result to manifolds of class C^2.

Theorem 2.1. *Let $N^k\subset\mathbb{R}^n$ be a closed submanifold of class C^2. Let $m\ge 3$ and suppose $u\in H^1(B^m;N)$ is a stationary harmonic map. There exists a constant $\varepsilon_0>0$ depending only on N with the following property. Whenever on some ball $B_R(x_0)\subset B$ there holds*

$$R^{2-m}\int_{B_R(x_0)}|\nabla u|^2\,dx<\varepsilon_0, \tag{17}$$

then u is Hölder continuous (and hence as smooth as permitted by the regularity of N) on $B_{R/2}(x_0)$. In particular, u is regular in B away from a singular set S with $\mathcal{H}^{m-2}(S)=0$.

Proof. As in (1), equation (15) equivalently may be written in the form

$$-\Delta u^i=\Omega^{ij}\cdot\nabla u^j, \tag{18}$$

where $\Omega\in L^2(B;so(n)\times\wedge^1\mathbb{R}^n)$ in view of our assumption on N, with components locally given by

$$\Omega^{ij}=\Omega^{ij}_\alpha dx^\alpha=\sum_{l=1}^{n-k}(w^i_l dw^j_l-w^j_l dw^i_l),\ 1\le i,j\le n. \tag{19}$$

Note that (16) and (17) imply that Ω belongs to the Morrey space $L^{2,m-2}(B)$ with

$$\begin{aligned}||\Omega||^2_{L^{2,m-2}}&=\sup_{x_0\in B} r^{2-m}\int_{B_r(x_0)\cap B}|\Omega|^2\,dx\\ &\le C\sup_{x_0\in B} r^{2-m}\int_{B_r(x_0)\cap B}|\nabla u|^2\,dx\le C\varepsilon_0.\end{aligned} \tag{20}$$

The result now is an immediate consequence of Theorem 1.1. □

3. Proof of Theorem 1.1

We may assume that condition (12) is satisfied on $B=B_1(0)$. As in (3), we obtain the existence of a suitable gauge transformation Φ, transforming Ω into Coulomb gauge by applying the following lemma. The bound (12) also yields corresponding estimates for P and ξ.

Lemma 3.1. *Suppose that condition* (12) *is satisfied on* B. *There exists* $P \in H^1(B; SO(n))$ *and* $\xi \in H^1(B, so(n) \otimes \wedge^{m-2}\mathbb{R}^m))$ *such that*

$$P^{-1}dP + P^{-1}\Omega P = *d\xi \text{ on } B,\ d(*\xi) = 0 \text{ on } B,\ \text{and } \xi = 0 \text{ on } \partial B\,. \tag{21}$$

Moreover, dP *and* $d\xi$ *belong to* $L^{2,m-2}(B)$ *with*

$$||dP||^2_{L^{2,m-2}} + ||d\xi||^2_{L^{2,m-2}} \le C(||\Omega||^2_{L^{2,m-2}} + ||du||^2_{L^{2,m-2}}) \le C\varepsilon(m). \tag{22}$$

The proof of this lemma will be given in the next section.

Recall that a function $f \in L^1(B)$ belongs to the space $BMO(B)$ if

$$[f]_{BMO} = \sup_{x_0 \in B,\, r>0} \left(⨍_{B_r(x_0)\cap B} |f - \bar{f}_{x_0,r}|\, dx \right) < \infty,$$

where

$$\bar{f}_{x_0,r} = ⨍_{B_r(x_0)\cap B} f\, dx$$

denotes the average of f over $B_r(x_0) \cap B$, and so on. By Poincaré's inequality, moreover, for $1 \le p \le m$ any function $f \in W^{1,p}(B)$ with $df \in L^{p,m-p}(B)$ belongs to $BMO(B)$ and there holds

$$[f]^p_{BMO} \le C||df||^p_{L^{p,m-p}} = \sup_{x_0 \in B,\, r>0} \left(r^{p-m} \int_{B_r(x_0)\cap B} |df|^p\, dx \right).$$

Applying the gauge transformation P^{-1} to ∇u and observing the identity $dP^{-1} = -P^{-1}dPP^{-1}$, from (11) we obtain the equation (10), that is

$$-div(P^{-1}\nabla u) = (P^{-1}\nabla P + P^{-1}\Omega P)\cdot P^{-1}\nabla u = *d\xi \cdot P^{-1}du. \tag{23}$$

Fix a smooth cut-off function $\tau \in C^\infty_0(B)$ such that $0 \le \tau \le 1$, $\tau = 1$ on $B_{1/2}(0)$. Multiplying (23) by τ, we obtain the equation

$$-div(P^{-1}\nabla(u\tau)) = *d\xi \cdot P^{-1}d(u\tau) - e, \tag{24}$$

with "error" term

$$e = div(P^{-1}u\nabla\tau)) + P^{-1}\nabla u \cdot \nabla\tau + *d\xi \cdot P^{-1}ud\tau. \tag{25}$$

Since $u \in H^1 \cap L^{2,m-2}_1(B)$, we have $u \in L^p(B)$ for every $p < \infty$. Therefore, a direct application of Hölder's inequality tells us that $u \in L^{p,m-\delta}$ for every $p < \infty$ and for every $\delta > 0$. Using this last observation and the fact that $d\xi$ is in $L^{2,m-2}$, we conclude that

$$\forall s \in [1,2[\quad \forall \delta > 0 \quad : \quad \|e\|_{L^{s,m-s-\delta}} < \infty \quad . \tag{26}$$

We claim that $v = u\tau$ is Hölder continuous in B, provided the bound (12) holds with $\varepsilon(m) > 0$ sufficiently small.

Let $B_R(x_0) \subset B$ and let

$$P^{-1}dv = df + *dg + h \tag{27}$$

be the Hodge decomposition of $P^{-1}dv$ on $B_R(x_0)$, where $f \in H^1_0(B_R(x_0))$ and where g is a co-closed $m-2$-form of class $H^1(B_R(x_0))$ whose restriction to the boundary ∂B also vanishes, and with a harmonic 1-form $h \in L^2(B_R(x_0))$; see [7]

Corollary 10.5.1, p.236, for the Hodge decomposition of forms in Sobolev Spaces. Similar to (7) we have the equations

$$\begin{aligned}
&-\Delta f = -div(P^{-1}\nabla v) = *d\xi \cdot P^{-1}dv - e,\\
&-\Delta g = *d(P^{-1}dv) = *(dP^{-1}\wedge dv).
\end{aligned} \tag{28}$$

Fix a number $1 < p < m/(m-1)$ and let $q > m$ be the conjugate exponent with $1/p + 1/q = 1$. Since $f = 0$ on $\partial B_R(x_0)$, by duality we have

$$||df||_{L^p} \le C \sup_{\varphi \in W_0^{1,q}(B_R(x_0));||\varphi||_{W^{1,q}}\le 1} \int_{B_R(x_0)} df \cdot d\varphi \, dx. \tag{29}$$

Here and in the following computations all norms refer to the domain $B_R(x_0)$. Note that $W_0^{1,q}(B_R(x_0)) \hookrightarrow C^{1-m/q}(B_R(x_0))$ and for all $\varphi \in W_0^{1,q}(B_R(x_0))$ with $||\varphi||_{W^{1,q}} \le 1$ there holds

$$||\varphi||_{L^\infty} \le CR^{1-m/q}||\varphi||_{W^{1,q}} \le CR^{1-m/q}, \ ||d\varphi||_{L^2} \le CR^{m/2-m/q}. \tag{30}$$

For such φ then we can estimate

$$\begin{aligned}
\int_{B_R(x_0)} df \cdot d\varphi \, dx &= -\int_{B_R(x_0)} \Delta f \varphi \, dx\\
&= \int_{B_R(x_0)} d\xi \wedge P^{-1}dv\varphi - \int_{B_R(x_0)} e\varphi \, dx = I + II
\end{aligned} \tag{31}$$

as follows. Similar to the approach introduced in [2], upon integrating by parts and using [3], Theorem II.1, we have (up to sign, which is of no importance in what follows)

$$\begin{aligned}
I &= \int_{B_R(x_0)} d\xi \wedge P^{-1}dv\varphi = \int_{B_R(x_0)} d\xi \wedge d(P^{-1}\varphi)(v - \bar v_{x_0,R})\\
&\le C||d\xi \wedge d(P^{-1}\varphi)||_{\mathcal{H}^1}[v]_{BMO} \le C||d\xi||_{L^2}||d(P^{-1}\varphi)||_{L^2}[v]_{BMO}\\
&\le C||d\xi||_{L^2}(||dP||_{L^2}||\varphi||_{L^\infty} + ||d\varphi||_{L^2})[v]_{BMO}\\
&\le CR^{m-1-m/q}||d\xi||_{L^{2,m-2}}(||dP||_{L^{2,m-2}} + ||d\varphi||_{L^q})[v]_{BMO}\\
&\le CR^{m/p-1}\varepsilon(m)[v]_{BMO},
\end{aligned} \tag{32}$$

while (25), combined with (26) and (30), for any $\delta > 0$ gives the bound

$$\begin{aligned}
II &= -\int_{B_R(x_0)} e\varphi \, dx \le ||e||_{L^1(B_R(x_0))}||\varphi||_{L^\infty}\\
&\le C_\delta R^{m-1-\delta}||e||_{L^{1,m-1-\delta}}||\varphi||_{L^\infty} \le C_\delta R^{m-m/q-\delta} = C_\delta R^{m/p-\delta}.
\end{aligned} \tag{33}$$

Hence from (28) we conclude that for every $\delta > 0$ there holds

$$||df||_{L^p} \le CR^{m/p-1}\varepsilon(m)[v]_{BMO} + C_\delta R^{m/p-\delta}. \tag{34}$$

Similarly, letting s satisfy $1/2 + 1/q + 1/s = 1$, by Hölder's inequality for an arbitrary $(m-2)$-form $\psi \in W^{1,q}(B_R(x_0), \wedge^{m-2}\mathbb{R}^m)$ vanishing on ∂B and with

$||\psi||_{W^{1,q}} \leq 1$ in view of the decomposition $-\Delta = *d*d + d*d*$ of the Hodge-Laplacian and the equation $d(*g) = 0$ we can bound (again up to sign)

$$\begin{aligned}
&\int_{B_R(x_0)} dg \cdot d\psi\, dx = -\int_{B_R(x_0)} \Delta g \cdot \psi\, dx \\
&= \int_{B_R(x_0)} dP^{-1} \wedge dv \wedge \psi = \int_{B_R(x_0)} dP^{-1} \wedge d\psi (v - \bar{v}_{x_0,R}) \\
&\leq C||dP||_{L^2}||d\psi||_{L^q}||v - \bar{v}_{x_0,R}||_{L^s} \leq CR^{m/p-1}\varepsilon(m)[v]_{BMO}.
\end{aligned} \tag{35}$$

By duality, we have

$$||dg||_{L^p} = \sup_{k\in L^q(B_R(x_0);\wedge^{m-1}\mathbb{R}^m);\ \|k\|_q \leq 1} \int_{B_R(x_0)} dg \cdot k\, dx. \tag{36}$$

Decomposing any $k \in L^q(B_R(x_0); \wedge^{m-2}\mathbb{R}^m)$ as $k = d\psi + *d\rho + \eta$ with η satisfying $d\eta = 0$, $d(*\eta) = 0$ and with $\psi = 0$ on ∂B as in [7], Corollary 10.5.1, and recalling that the restriction of g to $\partial B_R(x_0)$ vanishes, we then arrive at the estimate

$$\begin{aligned}
||dg||_{L^p} &\leq C \sup_{\psi\in W^{1,q}(B_R(x_0),\wedge^{m-2}\mathbb{R}^m);\ \|d\psi\|_q\leq 1} \int_{B_R(x_0)} dg \cdot d\psi \\
&\leq CR^{m/p-1}\varepsilon(m)[v]_{BMO}.
\end{aligned} \tag{37}$$

From the Campanato estimates for harmonic functions, as in Giaquinta [5], proof of Theorem 2.2, p.84 f., we thus conclude that for any $r < R$ there holds

$$\begin{aligned}
\int_{B_r(x_0)} |dv|^p\, dx &\leq C\int_{B_r(x_0)} |h|^p\, dx + C\int_{B_r(x_0)} (|df|^p + |dg|^p)\, dx \\
&\leq C\left(\frac{r}{R}\right)^m \int_{B_R(x_0)} |h|^p\, dx + C\int_{B_r(x_0)} (|df|^p + |dg|^p)\, dx \\
&\leq C\left(\frac{r}{R}\right)^m \int_{B_R(x_0)} |dv|^p\, dx + C\int_{B_R(x_0)} (|df|^p + |dg|^p)\, dx \\
&\leq C\left(\frac{r}{R}\right)^m \int_{B_R(x_0)} |dv|^p\, dx + CR^{m-p}\varepsilon(m)[v]^p_{BMO} + C_\delta R^{m-\delta p},
\end{aligned} \tag{38}$$

for any $\delta > 0$. Set

$$\Phi(x_0, r) = r^{p-m}\int_{B_r(x_0)} |dv|^p\, dx$$

and define

$$\Psi(R) = \sup_{x_0\in B,\, 0<r<R} \Phi(x_0, r).$$

Then we can bound

$$\sup_{x_0\in B} [v]^p_{BMO(B_R(x_0))} \leq C\Psi(R).$$

Fixing $\delta < 1 - 1/p$, thus from (38) we have

$$\begin{aligned}
\Phi(x_0, r) &\leq C\left(\frac{r}{R}\right)^p \Phi(x_0, R) + C\left(\frac{r}{R}\right)^{p-m} \varepsilon(m)\Psi(R) + C_\delta \left(\frac{r}{R}\right)^{p-m} R^{p-\delta p} \\
&\leq C_1\left(\frac{r}{R}\right)^p \left(1 + \left(\frac{r}{R}\right)^{-m} \varepsilon(m)\right)\Psi(R) + C\left(\frac{r}{R}\right)^{p-m} R
\end{aligned} \tag{39}$$

with a uniform constant C_1. Also fixing $r/R = \gamma$ such that $C_1\gamma^{(p-1)/2} \leq 1/2$, and choosing $\varepsilon(m) = \gamma^m$, for any $R_0 < 1$ and $0 < R < R_0$ we obtain

$$\begin{aligned}\Phi(x_0, \gamma R) &\leq C_1\gamma^p(1 + \varepsilon(m)\gamma^{-m})\Psi(R) + C\gamma^{p-m}R \\ &\leq \gamma^{(p+1)/2}\Psi(R) + CR \leq \gamma^{(p+1)/2}\Psi(R_0) + CR_0.\end{aligned}$$

Upon passing to the supremum with respect to x_0 and $R < R_0$ we deduce that

$$\Psi(\gamma R_0) \leq \gamma^{(p+1)/2}\Psi(R_0) + CR_0$$

for any $R_0 \in]0, 1]$.

Finally, for any $r \in]0, \gamma]$, letting $k \in \mathbb{N}$ be such that $\gamma^{k+1} < r \leq \gamma^k$ and iterating as in Giaquinta [5], proof of Lemma 2.1, p.86, we conclude that

$$\Psi(r) \leq \Psi(\gamma^k) \leq \gamma^{k(p+1)/2}\Psi(1) + C\gamma^k(\sum_{j=1}^{\infty}\gamma^{j(p-1)/2}) \leq Cr.$$

Hence $v \in C^{1/p}(B)$ and therefore also $u \in C^{1/p}(B_{1/2}(0))$, as claimed.

4. Proof of Lemma 3.1

For the proof of Lemma 3.1 we follow [8], where Uhlenbeck's [11] construction of a local Coulomb gauge in Sobolev spaces was generalized to Morrey spaces. Due to the fact that the space $L^{2,m-2}_1$ defined earlier does not embed into C^0, the inverse mapping $P \to P^{-1}$ is not smooth from the space $L^{2,m-2}_1$ into itself. In order to avoid this difficulty, similar to [11] we first construct the local Coulomb gauge under slightly more stringent regularity assumptions.

Lemma 4.1. *There exists $\varepsilon(m, n) > 0$ and $C > 0$ such that, on $B = B^m$ for every $\alpha > 0$ and every $\Omega \in L^{2,m-2+\alpha}(B, so(n) \otimes \wedge^1\mathbb{R}^m)$ with*

$$||\Omega||^2_{L^{2,m-2}} \leq \varepsilon(n, m) \tag{40}$$

there exist $P \in L^{2,m-2+\alpha}_1(B; SO(n))$ and $\xi \in L^{2,m-2+\alpha}_1(B, so(n) \otimes \wedge^{m-2}\mathbb{R}^m)$ such that

$$P^{-1}dP + P^{-1}\Omega P = *d\xi \text{ on } B,\ d(*\xi) = 0 \text{ on } B,\ \text{and } \xi = 0 \text{ on } \partial B\,. \tag{41}$$

Moreover, dP and $d\xi$ satisfy

$$||dP||^2_{L^{2,m-2+\alpha}} + ||d\xi||^2_{L^{2,m-2+\alpha}} \leq C||\Omega||^2_{L^{2,m-2+\alpha}}\,, \tag{42}$$

and

$$||dP||^2_{L^{2,m-2}} + ||d\xi||^2_{L^{2,m-2}} \leq C||\Omega||^2_{L^{2,m-2}} \leq C\varepsilon(n, m). \tag{43}$$

Proof of Lemma 3.1. Let Ω be in $L^{2,m-2}$ and suppose that $\|\Omega\|_{L^{2,m-2}} < \varepsilon$ for some number $\varepsilon > 0$ to be fixed below. Although smooth functions are not dense in $L^{2,m-2}$, it is not difficult to show that the mollified forms $\Omega_\delta = \Omega * \chi_\delta$ obtained from Ω by convoluting Ω with a standard mollifyer satisfy the uniform estimate $\|\Omega_\delta\|_{L^{2,m-2}} \leq C\|\Omega\|_{L^{2,m-2}}$. By choosing $\varepsilon > 0$ sufficiently small, we can then achieve the uniform bound $\|\Omega_\delta\|_{L^{2,m-2}} \leq \varepsilon(m, n)$, where $\varepsilon(m, n)$ is given in Lemma 4.1, to obtain the existence of ξ_δ and P_δ satisfying (41), (42), and (43) for Ω_δ instead of Ω. The uniform bound given by (43) permits to pass to the limit $\delta \to 0$ in (41), and the assertion of Lemma 3.1 follows. $\square$

Proof of Lemma 4.1. For $\alpha > 0$ introduce the set

$$\mathcal{U}^\alpha_{\varepsilon,C} := \left\{ \begin{array}{l} \Omega \in L^{2,m-2+\alpha}(B^m, so(n));\ \|\Omega\|_{L^{2,m-2}} \leq \varepsilon,\ \text{and} \\ \\ \text{there exist } P \text{ and } \xi \text{ satisfying } (41),\ (42),\ (43) \end{array} \right\}$$

Since clearly $\Omega = 0 \in \mathcal{U}^\alpha_{\varepsilon,C}$, the set $\mathcal{U}^\alpha_{\varepsilon,C}$ is not empty. The proof therefore will be complete once we show that, for ε small enough and C large enough, $\mathcal{U}^\alpha_{\varepsilon,C}$ is both open and closed in the star-shaped and hence path-connected set

$$\mathcal{V}^\alpha_\varepsilon := \left\{ \Omega \in L^{2,m-2+\alpha}(B^m, so(n) \otimes \wedge^1\mathbb{R}^m);\ \|\Omega\|_{L^{2,m-2}} \leq \varepsilon \right\}.$$

The proof of closedness is similar to the proof of Lemma 3.1 given above. To see that $\mathcal{U}^\alpha_{\varepsilon,C}$ is open in $\mathcal{V}^\alpha_\varepsilon$, observe that for $\alpha > 0$ the space $L^{2,m-2+\alpha}_1$ embeds continuously into C^0 and the inverse mapping $P \to P^{-1}$ from the space $L^{2,m-2+\alpha}_1(B, SO(n))$ into itself is smooth. Therefore the argument of [11] can be applied to show that, for sufficiently small $\varepsilon > 0$ and sufficiently large C, for every Ω in $\mathcal{U}^\alpha_{\varepsilon,C}$ there exists $\eta_\Omega > 0$ with the property that for every $\omega \in L^{2,m-2+\alpha}$ satisfying the bound $\|\omega\|_{L^{2,m-2+\alpha}} \leq \eta_\Omega$ and $\|\Omega + \omega\|_{L^{2,m-2}} < \varepsilon$ we can find $\xi_\omega \in L^{2,m-2+\alpha}_1(B, so(n) \otimes \wedge^{m-2}\mathbb{R}^m)$ and $P_\omega \in L^{2,m-2+\alpha}_1(B, SO(n))$, respectively, satisfying (41). The openness of $\mathcal{U}^\alpha_{\varepsilon,C}$ then may be obtained as in [10] from the following lemma. This completes the proof. □

Lemma 4.2. *There exists $\delta > 0$ with the following property. Suppose that for $\Omega \in \mathcal{V}^\alpha_\varepsilon$ there exist $\xi \in L^{2,m-2+\alpha}_1(B^m, so(n) \otimes \wedge^{m-2}\mathbb{R}^m)$, $P \in L^{2,m-2+\alpha}_1(B^m, SO(n))$ satisfying (41) and the estimate*

$$\|d\xi\|_{L^{2,m-2}} + \|dP\|_{L^{2,m-2}} \leq \delta \quad . \tag{44}$$

Then (42) and (43) hold for some C independent of $\Omega \in \mathcal{V}^\alpha_\varepsilon$.

Proof. In view of (41), the $(m-2)$-form ξ satisfies

$$\begin{cases} \Delta\xi = *(dP^{-1} \wedge dP) + *d(P^{-1}\Omega P) & \text{in } B, \\ \\ \xi = 0 & \text{on } \partial B \end{cases} \tag{45}$$

We decompose $\xi = u + w$ in two forms u and w solving, respectively,

$$\begin{cases} \Delta u = *(dP^{-1} \wedge dP) & \text{in } B \\ \\ u = 0 & \text{on } \partial B\ , \end{cases} \tag{46}$$

and

$$\begin{cases} \Delta w = *d(P^{-1}\Omega P) & \text{in } B \\ \\ w = 0 & \text{on } \partial B\ . \end{cases} \tag{47}$$

From [5], Theorem III.2.2, for $s \in \{2-\alpha, 2\}$ first we obtain the bound

$$\|dw\|_{L^{2,m-s}(B)} \leq C\|\Omega\|_{L^{2,m-s}(B)}\ . \tag{48}$$

Following the strategy of the proof of Theorem III.2.2 in [5], likewise for $s \in \{2-\alpha, 2\}$ we obtain that

$$\begin{aligned} \|du\|_{L^{2,m-s}(B)} &\leq C\|P\|_{BMO(B)}\|dP\|_{L^{2,m-s}(B)} \\ &\leq C\|dP\|_{L^{2,m-2}(B)}\|dP\|_{L^{2,m-s}(B)} \leq C\delta\|dP\|_{L^{2,m-s}(B)}\ . \end{aligned} \tag{49}$$

The only modification required with respect to [5], p. 85, is that we use [3], Theorem II.1, to estimate, with v as defined in [5],

$$\begin{aligned} \|d(u-v)\|^2_{L^2(B_R(x_0)\cap B)} &= -\int_{B_R(x_0)\cap B} \Delta(u-v)\wedge(u-v)\\ &= \int_{B_R(x_0)\cap B} dP^{-1}\wedge dP\wedge(u-v)\\ &= \int_{B_R(x_0)\cap B} dP^{-1}(P-\bar{P}_{x_0,R})\wedge d(u-v)\\ &\le \|d(u-v)\|_{L^2(B_R(x_0)\cap B)}\,\|P\|_{BMO(B)}\,\|dP^{-1}\|_{L^2(B_r(x_0)\cap B)}\ . \end{aligned} \tag{50}$$

Combining (48) and (49) we then conclude

$$\|d\xi\|_{L^{2,m-s}(B)} \le C\delta\ \|dP\|_{L^{2,m-s}(B)} + C\ \|\Omega\|_{L^{2,m-s}(B)} \quad . \tag{51}$$

Moreover, from (41) we have

$$\|dP\|_{L^{2,m-s}(B)} \le \|d\xi\|_{L^{2,m-s}(B)} + \|\Omega\|_{L^{2,m-s}(B)} \quad . \tag{52}$$

Putting the estimates (51) and (52) together, upon choosing $\delta > 0$ small enough we then obtain (42) and (43). The proof is complete. ☐

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