Heinz Hopf (1894–1971)

Günther Frei and Urs Stammbach

(published in I.M. James: *History of topology*, Elsevier 1999, p. 991-1008)

Introduction

Without doubt, Heinz Hopf was one of the most distinguished mathematicians of the twentieth century. His work is closely linked with the emergence of algebraic topology; it is most decisively thanks to his early works that this area established itself as a new and important branch of mathematics. His œuvre has influenced profoundly the evolution not only of topology but of a large part of mathematics. But Heinz Hopf was not only a gifted researcher: he was also an excellent teacher and a personality of the highest integrity. At the same time, he effervesced with charm and subtle humour. In the obituary that appeared in the organ of the IMU, Henri Cartan describes Heinz Hopf:[1]

> Ceux qui l'ont connu n'oublieront jamais sa finesse et sa douceur, alliées à une grande fermeté du caractère. Ils n'oublieront non plus le professeur ou le conférencier: Hopf n'avait pas besoin d'élever la voix pour se faire écouter; la précision de son langage ne l'empêchait pas, bien au contraire, d'éveiller l'intuition chez son auditeur; à partir de quelques constatations simples, de caractère élémentaire, il posait des problèmes neufs et les regardait sous leur différents aspects: analytique, géométrique, algébrique.

Youth (1894–1913)

Heinz Hopf's ancestors belonged to a respected and prosperous family of hop traders in Nuremberg.[2] The great–grandfather, Löb Hopf, came from Ühlfeld, a small town in Upper Franconia. He moved with his family to Nuremberg in 1852, where he was one of the first Jews to be able to acquire citizenship. The grandfather, Stephan Hopf (1826–1893), became respectably wealthy as a hop wholesaler and played politically important roles in Nuremberg as *Kommerzienrat*, *Magistratsrat*, and *Landrat*. The father, Wilhelm Hopf (1861–1942), first learned brewing in Flensburg and then, in 1887, moved to Breslau after he had quarreled with his father and had had his inheritance paid out. There he joined Heinrich Kirchner's brewery. Only one year later, thanks to his considerable inheritance, he became sole owner. On May 28th, 1892, he married Elisabeth Kirchner, the elder one of Heinrich Kirchner's two daughters. In 1895, Wilhelm Hopf adopted his wife's Protestantism. In their happy marriage they had two children, Hedwig Hopf (1893–1953) and Heinz Hopf.

Heinz Hopf was born on November 19th, 1894, in Gräbschen near Breslau. Father Hopf had had a villa-like house built in the Gründerzeit [3] style, surrounded by a large garden. In 1901, Heinz entered Dr. Karl Mittelhaus's higher boys' school and, from 1904, went to the König-Wilhelm-Gymnasium in Breslau. His mathematical talent was soon recognized and found an active supporter in his teacher Bruchmann. His Abitur certificate from May 13th, 1913, says:[4]

> Mathematik: Er hat für den Gegenstand, besonders nach der algebraischen Seite hin, eine nicht gewöhnliche Begabung gezeigt.*

In the other subjects his marks were not as good. It is possible that Hopf neglected his homework at times and preferred doing sports. Throughout his life he loved sports activities. Though not tall, he was of a tough and strong constitution. In his childhood his favourite sports were swimming and tennis. Later he made regular swimming outings, rambling and mountaineering, in winter often with skis. An extended daily walk was a necessity to him, as to his father before him.

Student Period (1913–1925)

In April 1913, after his Abitur, Hopf matriculated for mathematical studies at the Silesian Friedrich Wilhelms University in Breslau.[5] He attended lectures by *Adolf Kneser*, *Erhard Schmidt*, and *Rudolf Sturm* as well as by *Max Dehn* and *Ernst Steinitz* who worked at the Breslau Polytechnic at that time. Besides mathematics, Hopf also attended lectures in physics, philosophy, and psychology, subjects in which he was also interested after his studies.

One year later already, the outbreak of World War I interrupted Hopf's studies. Hopf, following the common war enthusiasm at that time, volunteered for military service. For a long time during the next four years, he fought on the Western Front as lieutenant of reserves. He was wounded twice during the war, and in 1918 he was awarded the Iron Cross (first class).

A short holiday from service in June 1917 was, according to Hopf himself, the decisive turning-point in his mathematical career. During this holiday, he attended a lecture course on set theory by Erhard Schmidt. At that time, Schmidt was treating Brouwer's theorem on the invariance of dimension under topological maps and presented the proof Brouwer had given in 1911 using the mapping degree. Hopf tells in his memoirs:[6]

> Ich war fasziniert; diese Faszination — durch die Kraft der Methode des Abbildungsgrades — hat mich nicht wieder verlassen, sondern große Teile meiner Produktion entscheidend beeinflusst. Und wenn ich heute den Gründen für diese Wirkung nachgehe, so sehe ich besonders zweierlei: erstens die Eindringlichkeit und mitreißende Begeisterung des Vortrages von Erhard Schmidt, und zweitens meine eigene gesteigerte Aufnahmefähigkeit während einer vierzehntägigen Unterbrechung eines langjährigen Militärdienstes.*

After the end of the war, in December 1918, Hopf was discharged from military service and resumed his interrupted studies at the University of Breslau. However, in the meantime, Erhard Schmidt had been appointed the successor of Hermann Amandus Schwarz in Berlin. This may have been the reason for Hopf not continuing his studies in Breslau. In autumn 1919, he changed to the University of Heidelberg. The reason for this choice can be simply assumed to be due to his sister who had begun studying law there a year already. Besides lectures on philosophy and psychology, Hopf attended only a few mathematical lecture courses by *Oskar Perron* and *Paul Gustav Stäckel* and furthermore a mathematical seminar.

Already in autumn 1920, Hopf decided to follow his teacher Erhard Schmidt from Breslau and to continue his studies in Berlin. This step was extraordinarily significant for his development. Since the time of

*Mathematics: He has shown an extraordinary gift in this topic, especially in the algebraic direction.

*I was fascinated; this fascination — of the power of the method of the mapping degree — has not left me since, but has influenced great parts of my production. And when I look for the cause of this effect, I see particularly two things: firstly, Schmidt's vividness and enthusiasm in his talk, and secondly my own increased receptivity during a fortnight off many years of military service.

Kummer, Kronecker, and Weierstrass, Berlin had been one of the leading universities in mathematics in Germany. In his scientific interests, he followed mainly Erhard Schmidt, whom he owed many ideas. Their personal relationship was based on high mutual esteem; however it was part of Schmidt's nature to maintain a certain distance. Scientifically as well as personally, Hopf was also close to the algebraist Issai Schur. Hopf attended lectures on set theory, on differential equations, and on complex analysis by Schmidt, and on number theory, elliptic functions, and invariant theory by Schur. Hopf learned much about the newest developments in topology from Schmidt, in particular about Brouwer's work and about Schmidt's own work on the Jordan curve theorem. Schmidt also made his assistant Feigl give a lecture course on Poincaré's work on *Analysis situs*.

In Schmidt's seminar, Hopf gave talks on the Clifford surface and the Clifford–Klein space problem in the winter semester of 1921/22. The topic he treated for his dissertation under Schmidt's supervision during the following years was in this area. In the first part of his doctoral thesis, Hopf proved the theorem that a simply connected complete Riemannian 3-manifold of constant curvature is globally isometric to either the Euclidean, the spherical or the hyperbolic space. The connection between local and global phenomena that emerges here also preoccupied Hopf in many of his later works. In the second part Hopf treated the relation between the *curvatura integra* of closed hyper-surfaces M in $\mathbb{R}^{n+1}$, defined as the degree of the Gauss normal map, and the indices of the zeroes of tangent vector fields on M. Hopf proved that, independently of the vector field, the sum s of the indices of the zeroes is nought for n odd, and twice the *curvatura integra* for n even, therefore in particular even. Hopf published the results of his dissertation in two separate papers in the *Mathematische Annalen* [1], [2]. He obtained his degree in February 1925. Schmidt closed his report with the remark:[7]

> Die Kühnheit der Fragestellungen verdient ebensoviel Bewunderung wie die überraschenden Ergebnisse der Lösungen. Das Schönste der Arbeit bildet aber doch die Methode der Beweisführung, die, was bei Arbeiten in diesem Gebiet besonders selten ist, abstrakt und in jedem Schritte kontrollierbar vorgeht und kraft der Abstraktion in gleich hohem Maße Reichtum der anschaulich-geometrischen Fantasie an den Tag legt.*

For the thesis Schmidt pleaded for the rare predicate *eximium*. In the final result — Hopf was examined in mathematics by Schmidt and Bieberbach, in physics by Planck, and in philosophy by Wertheimer — Hopf got the predicate *summa cum laude*.

The period as a Privatdozent (1925 – 1931)

Immediately after his doctorate, Hopf turned to his *Habilitation*. On Schmidt's advice he intensively studied Brouwer's publications — in his memoirs he remarks:[8] *that was tough work* — and Hadamard's paper *Note sur quelques applications sur l'indice de Kronecker*. From this emerged the two papers *Abbildungsklassen n-dimensionaler Mannigfaltigkeiten* [5] (mapping classes of n-dimensional manifolds) and *Vektorfelder n-dimensionaler Mannigfaltigkeiten* [6] (vector fields on n-dimensional manifolds), which Hopf submitted as his Habilitation thesis. Hopf could already talk on these results during the annual conference of the *Deutsche Mathematiker-Vereinigung* in September 1925, only half a year after his doctorate. In the second of these papers, the famous theorem appears which says that the sum of the indices of the singularities of a vector field on a closed orientable manifold is an invariant of the manifold, namely the Euler characteristic. The first proof of this had been given by Lefschetz a short time before; Hopf presented a new proof based on a complicated induction argument on the dimension.

*The boldness of the questions deserves as much admiration as the surprising results of the solutions. But the most beautiful thing in his thesis is the method of proving, which is, particularly rarely found in works in that area, abstract and comprehensible in every step, and which, due to the abstractness, shows equally clearly the richness of the concrete geometrical imagination.

His Habilitation took place in autumn 1926. In his reference[9], Schmidt stated that according to him, Hopf should be seen as *already standing in the first rank of German mathematicians.*

Hopf spent the academic year which lies between doctorate and Habilitation in Göttingen. The university of Göttingen was a most active centre of mathematical research of international prestige at that time. Besides *David Hilbert*, *Richard Courant*, *Carl Runge* and others, also a number of prominent Privatdozenten worked there, among them *Paul Bernays* and *Emmy Noether*, and many important mathematicians from all over the world came as long- or short-term guests.

In his memoirs, Hopf begins his description of the Göttingen year as follows:[10]

> Mein wichtigstes Erlebnis in Göttingen war es, dass ich dort Paul Alexandroff traf. Aus diesem Zusammentreffen wurde bald eine Freundschaft; nicht nur Topologie, und nicht nur Mathematik wurden diskutiert; es war eine glückliche und auch eine sehr fröhliche Zeit, die nicht auf Göttingen beschränkt war, sondern sich auf vielen gemeinsamen Reisen fort setzte.*

A deep friendship started here, lasting until Hopf's death.

In every year since 1923 Alexandrov had been a guest in Göttingen. Though a little younger than Hopf, he was already regarded as one of the leaders in point-set topology. Just at that time he began to apply algebraic methods to set-theoretic questions. One of the tools developed for that purpose was to associate with a covering of a topological space its nerve, i.e. a simplicial complex describing the combinatorics of the covering. The nerve can be viewed as an abstract algebraic approximation of the space, and by means of the notions of algebraic topology, results on the topological space itself can be derived. In the following years this notion would be applied extensively and led to a great number of new and interesting results in point-set topology.

Hopf commented on Alexandrov's idea of nerves in his memoirs:[11]

> Sie war der erste erfolgreiche Versuch, algebraische Betrachtungen in die mengentheoretische Topologie einzuführen — sehr zum Missfallen mancher Verfechter der „Reinheit der Methode“ [...] Mich selbst hat damals die Erkenntnis, eine wie große Rolle die Algebra in den topologischen Problemen spielt, in entscheidender und bleibender Weise beeinflusst.†

Their common interests brought Alexandrov and Hopf together from the beginning, and thanks to Alexandrov, Hopf was warmly received in the Göttingen circle around Courant, Hilbert, and Emmy Noether.

Another important idea concerning the link between topology and algebra emerging then for the first time was due to Emmy Noether. Alexandrov tells in his autobiographic notes[12] how Emmy Noether explained the idea of Betti groups of a complex after a dinner at Brouwer's house in Blaricum in December 1925. She suggested introducing the factor group of cycles modulo boundaries and replacing the complicated numerical study of Betti numbers by the algebraic investigation of these groups. The idea was adopted at once, in particular by Vietoris, Alexandrov, and Hopf, and soon became popular in algebraic topology. It not only made it possible to give concise and simple definitions of the basic notions of algebraic topology but also prompted a wholly new view of the methods of algebraic topology. This shows up very clearly in the example of Hopf's paper *Eine Verallgemeinerung der Euler–Poincaréschen Formel* [12] which appeared in 1928. Here, for the first time Hopf explicitly uses homology groups. He shows how the

---

*My most important experience in Göttingen was to meet Paul Alexandrov. This meeting soon became friendship; not only topology, not only mathematics was discussed; it was a fortunate and also a very happy time, not restricted to Göttingen but continued on many joint journeys.

†This was the first successful attempt to introduce the algebraic study of point-set topology — much to the dislike of some supporters of the "Pureness of method". [...] I was influenced in a decisive and persistent manner by the insight into the importance of the role of algebra in topological problems.

Euler–Poincaré formula, interpreted in this new framework, can be generalised easily to yield a simple and lucid proof of the Lefschetz fixed point formula.

Alexandrov and Hopf, soon later joined by Otto Neugebauer, formed a closely linked group of friends in Göttingen, and they called themselves a *two-dimensional simplex*. They spent a lot of their spare time together on walks or in the attendant Klie's swimming pool at the river Leine. That was where often the whole mathematical department of Göttingen met, together with the guests who were present. Alexandrov writes in his Memories of Heinz Hopf:[13] *many a discussion, mathematical and non-mathematical, took place* there, and *many mathematical ideas were born there*. In the semester vacation, Alexandrov, Hopf, and Neugebauer made several major journeys, for example to Brittany in France, to the Pyrenees and to Corsica after the end of the summer semester 1926, and later, in May 1927, to Upper Bavaria, after the summer semester 1927 to Dauphiné, to Cassis near Marseille and to Portofino on the Italian Riviera.

Following the first journey to France, Hopf returned to Berlin in the autumn of 1926. During the following winter semester he gave a course on *combinatorial topology* which encompassed many of the most recent results. Hopf's student Erika Pannwitz compiled this into a script. He regularly informed Alexandrov in Moscow about the contents of his lectures, who in turn discussed the new results in his circle. During that time, Hopf was thoroughly occupied with the analysis of the mapping degree and the question of how far the mapping degree determines the homotopy class of a map between manifolds. The two resulting papers [11], [14] appeared in the Mathematische Annalen in 1928 and 1929, respectively.

Hopf and Alexandrov spent the academic year 1927/28 together at Princeton University on a Rockefeller fellowship. In his final report on this stay Hopf says that he went to lectures by Lefschetz and Alexander on *Analysis Situs* and that, on the Princeton mathematicians' request, he also gave a number of talks himself on his own works and those of other European mathematicians. He continues:[14]

> Jedoch erblicke ich in diesen äußeren Ereignissen keineswegs den wichtigsten Teil meines Princetoner Aufenthaltes. Diesen sehe ich vielmehr in den häufigen zwanglosen Gesprächen mit [den] Professor[en] Alexander, Lefschetz und Veblen, sowie mit Professor P. Alexandroff aus Moskau, mit dem ich in Princeton täglich zusammen war und alle frisch empfangenen wissenschaftlichen Eindrücke und Gedanken sofort gründlich durchsprach.*

During his time at Princeton, Hopf worked primarily on the homology of manifolds. The discovery of the intersection ring of a manifold goes back to that time. Hopf showed that the homology of a manifold becomes a ring when one views the intersection of two cycles as a product. This *intersection ring* behaves contravariantly — this was completely surprising to Hopf — in that a map between manifolds corresponds to the so-called inverse homomorphism between the intersection rings. Only a few years later could this contravariant behaviour be explained completely with the introduction of cohomology: the intersection ring can be identified with the cohomology ring of a manifold by means of Poincaré duality. Hopf's paper [16], where he expands the theory of the intersection ring, appeared in the Journal für reine und angewandte Mathematik in 1930.

Having returned from Princeton, Hopf and Alexandrov again spent the summer of 1928 in Göttingen. During that time, Courant proposed that they should write a book on topology for the Springer-Verlag series *Grundlehren der mathematischen Wissenschaften*. They agreed but did not suspect that this joint work should take up so much of their time during the following seven years. They outlined a comprehensive exposition of the whole area of point-set and algebraic topology. For this extended programme a single volume would certainly not suffice, as they soon realized. They planned a second and later even a third volume, but only the first one was finished. It was published in 1935. The difficulties of that time and

*But I do not regard these circumstantial events as the most important part of my Princeton stay, but much more the frequent informal talks with Professors Alexander, Lefschetz, and Veblen, as well as with Professor P. Alexandrov from Moscow whom I met daily in Princeton and with whom I discussed all the freshly absorbed scientific impressions and thought thoroughly.

eventually the outbreak of World War II contributed to the discontinuation of the project. Also it is clear that the very rapid development of algebraic topology in the 30s would have made the task very difficult, even in ideal circumstances.

In October 1928, Hopf married *Anja von Mickwitz* (1891–1967). Anja von Mickwitz came from a German–Baltic family of pastors blessed with many children.[15] She had trained in St. Petersburg to become a teacher. After the First World War she moved to North Germany and later worked as a private teacher in Berlin. After the wedding the couple spent a few days in Hopf's parents' holiday home in Hain in the Sudeten Mountains. It was there where Hopf often retired for rambling and skiing.

In December 1929 Hopf was offered by Princeton University an assistant professorship, but he turned it down. In the autumn of the following year, the Eidgenössische Technische Hochschule in Zürich asked in a diplomatically worded letter whether Hopf would accept an offer to succeed Hermann Weyl. This inquiry was in part induced by a statement by Issai Schur who had written about Hopf to Zürich:[16]

> Hopf ist ein ganz vorzüglicher Dozent, ein Mathematiker von starkem Temperament und starker Wirkung, ein Muster seiner Disziplin, der auch auf anderen Gebieten vorzüglich geschult ist. [...] Was seine Art, seine Bildung und liebenswürdiges Wesen betrifft, wünsche ich Ihnen keinen besseren Kollegen.*

After a short consultation with Courant, Hopf replied:[17]

> [...] eine Berufung in die Schweiz nach der schönen Stadt Zürich würde mich sehr locken und ehren, zumal auf einen so berühmten Lehrstuhl. Ich erkläre mich daher grundsätzlich bereit, eine eventuelle Wahl anzunehmen.†

While Hopf was waiting for a reply from Zürich, he received another offer from Freiburg i. Br., where Lothar Heffter's chair was vacant. But Hopf maintained his decision for Zürich, and before the end of the year he was elected Full Professor for Mathematics at the Eidgenössische Technische Hochschule. In the beginning of April 1931 he took up his new position.

Zürich before World War II (1931 – 1939)

Only a few days before he wrote his acceptance in the autumn of 1930, Hopf had finished his manuscript *Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche* (*On the maps from a three-dimensional sphere to the two-dimensional sphere*) in his parents' holiday home in the Sudeten Mountains and submitted it to the Mathematische Annalen. We will take a deeper look into this work because it is particularly typical for Hopf's methods of working and thinking; it is illustrated beautifully by Eckmann's words:[18]

> [Hopf hat] mit sicherem Instinkt tiefe Probleme ausgewählt und reifen lassen, um dann jeweils in einem Wurf eine Lösung zu geben, in der neue Gedanken und Methoden zu Tage traten.‡

*Hopf is an excellent lecturer, a mathematician of strong temperament and strong influence, a leading example in his discipline, and he is also well-educated in other subjects. [...] I cannot wish you a better colleague in respect to his manners, his education and his sympathetic nature.

†[...] A call to Switzerland, to the beautiful city of Zürich, could indeed tempt and honour me, particularly to such a famous chair. I therefore declare that I am in principle willing to accept such a choice.

‡Hopf selected deep problems with an unerring instinct and let them mature. Then he presented in one piece a solution that showed new thoughts and methods.

Since Brouwer, the theory of the mapping degree had developed from the theory of maps between spheres of the same dimension. In 1925, Hopf was able to prove that the homotopy class is characterized by the mapping degree. Along these lines it seemed natural also to study maps between spheres of different dimensions. At that time nothing was known about this apart from the simple fact that all continuous maps $f : \mathbb{S}^n \longrightarrow \mathbb{S}^m$ with $n < m$ are contractible to a point. Since all maps between spheres of different dimensions induce the zero homomorphism in the homology groups, they cannot be distinguished homologically. Hopf considered the simplest case of maps from the three-dimensional to the two-dimensional sphere. To tackle this problem, Hopf introduced a new invariant which was later named after him. Hopf defined it as the linking number of the pre-images of two different points of $\mathbb{S}^2$ in $\mathbb{S}^3$. In an involved proof with the aid of simplicial approximation, he could show that this linking number is independent of the choice of the two points and that it is an invariant of the homotopy class. To exhibit a *topologically essential* map it was therefore sufficient to construct a map with a non-vanishing Hopf invariant. Due to his knowledge of classical projective geometry, Hopf could describe such a map — nowadays known as the Hopf fibration: he embeds $\mathbb{S}^3$ as the unit sphere into the four-dimensional space $\mathbb{R}^4$. Then he regards $\mathbb{R}^4$ as $\mathbb{C}^2$, maps a point P of $\mathbb{S}^3$ to the line that joins P with 0 and interprets it as a point in $P^1(\mathbb{C})$. Finally, he uses that $P^1(\mathbb{C})$ is homeomorphic to $\mathbb{S}^2$. This map can be described in a simple and completely explicit way using coordinates. But it is more difficult to prove that it is essential, i.e. not homotopic to the trivial map. Hopf derived this using his invariant.

Looking at the explicit form of this map, it can easily be inferred that the pre-image of any point in $\mathbb{S}^2$ is a great circle in $\mathbb{S}^3$. Hopf explains the fact that the linking number of any two of these great circles is ± 1 as follows — this quotation at the same time illustrates Hopf's graphic and clear formulation:[19]

> Eine dreidimensionale und eine zweidimensionale Ebene durch den Mittelpunkt der $\mathbb{S}^3$ schneiden sich, wenn die letztere nicht ganz in der ersteren liegt, in einer Geraden durch den Mittelpunkt; dies bedeutet, wenn man zu den Schnitten mit der $\mathbb{S}^3$ übergeht: eine zweidimensionale Großkugel und ein Großkreis schneiden sich, wenn der Kreis nicht auf der Kugel verläuft, in zwei zueinander diametralen Punkten; folglich wird die Hälfte H einer Großkugel von jedem Großkreis, der fremd zu dem Rand von H ist und daher nicht auf der Großkugel verläuft, stets in genau einem Punkt geschnitten; da es zu jedem Großkreis (unendlich viele) von ihm berandete Hälften von Großkugeln gibt, folgt hieraus: je zwei zueinander fremde Großkreise der $\mathbb{S}^3$ sind miteinander verschlungen, und zwar ist ihre Verschlingungszahl ± 1.*

The last statement follows from the fact that the linking number of the two great circles is equal to the intersection number of one great-circle with the great hemisphere bounded by the other.

Hopf would generalise the methods and results of this work to maps between spheres of higher dimensions a few years later (in 1935). Surprisingly, a connection to the theory of real algebras showed up here.

The result described above invoked several important lines of development in algebraic topology; it stimulated algebraic topology frequently and for years, and prompted further developments. Examples that should be mentioned are the homotopy groups (Hurewicz 1935), in particular those of spheres, the notion of fibration (Seifert 1932), the conclusion of the study of the Hopf invariant one maps (Adams 1958/60), and their various relations to the theory of real algebras.

Almost at the same time as the work on maps of spheres, he wrote with his Berlin student Willi Rinow the joint work *"Über den Begriff der vollständigen differentialgeometrischen Fläche"* (On the notion of

*A three-dimensional and a two-dimensional plane through the center of $\mathbb{S}^3$ intersect in a line through the center unless the latter lies completely in the former; this means when passing to the intersection with $\mathbb{S}^3$: a two-dimensional great sphere and a great circle intersect in two antipodal points unless the circle lies inside the sphere; therefore the hemisphere H of the great sphere intersects every great circle which is disjoint from the boundary of H and which is therefore not part of the great sphere, in precisely one point; since for every great circle there are (infinitly many) great hemispheres bounded by it, it follows: any two disjoint great circles in $\mathbb{S}^3$ are intertwined; their linking number is ± 1.

complete differential geometric surfaces), which appeared in Commentarii Mathematici Helvetici in 1931 [20]. Here they prove the equivalence of different definitions of completeness. In particular it is proved that completeness in the sense of point-set topology is equivalent to the property that on a geodesic ray starting at any point one can go arbitrarily far (*auf jedem geodätischen Strahl [...] [kann man] von jedem Punkt aus jede Strecke abtragen*). Here again the fascination is apparent which Hopf felt for the link between local and global properties.

From the 4th to 12th of September, the International Congress of Mathematicians took place in Zürich. Hopf was one of the organizers, being a member of an executive committee of five. At the congress itself Hopf talked about results he had achieved together with his Berlin student Erika Pannwitz. Soon later the paper *Über stetige Deformationen von Komplexen in sich* (on continuous deformations of complexes into themselves) appeared in the Mathematische Annalen [25]. The question was here which complexes can be deformed into proper subcomplexes of themselves.

Alexandrov took the occasion of the International Congress for a longer stay in Zürich. This provided a welcome opportunity to pursue the book project further in direct cooperation. Until now, Hopf and Alexandrov had been posting each other the manuscripts for correction and criticism. Now much could simply be settled in direct discussions. They did not have another such opportunity before September 1935, when Hopf participated in the *Erste Internationale Konferenz über Topologie* in Moscow, run by Alexandrov.[20] Almost all important topologists of that time were present. In the talks, a number of new ideas and results were presented for the first time. Alexander, Gordon, and Kolmogorov, for example, talked about their independently obtained results on cohomology. A surprising fact — also for Hopf — was that a product could be defined for cohomology classes of arbitrary complexes and spaces, which gave cohomology a ring structure. Hopf had thought that such a product structure — as he had given for homology in his definition of the intersection ring — could only exist for manifolds, due to the local Euclidicity.

In Moscow, Hopf himself reported on his student Eduard Stiefel's results on the question of whether there are m continuous vector fields on an n-dimensional manifold. Stiefel had introduced *characteristic classes* in his work, which could be used to answer the question. After Hopf's talk, Whitney remarked in a discussion that a part of Stiefel's results were also contained in his note *Sphere Spaces* that had just appeared. Subsequently it became common in algebraic topology to name the characteristic classes after Stiefel *and* Whitney.

Hopf had travelled to Moscow together with his wife; their plan was to spent several weeks with Alexandrov and Kolmogorov on the Crimean Peninsula after the congress. During this stay in Gaspra near Jalta, where Alexandrov had been several times in his holidays, the joint book was completed; they read the last corrections and finally edited the preface.

After his return to Switzerland, Hopf took part in a conference on topology in Geneva. Elie Cartan talked on his result that the homology of the classical compact simple Lie groups is the homology of a product of spheres of odd dimensions. Afterwards Cartan posed the question of whether this is also true for the exceptional groups and hence, because of the structure theorems, in general for all compact Lie groups.

Hopf was able to solve this problem in an utterly new way in the course of the following years. The resulting paper appeared in 1941 in the *Annals of Mathematics* [40]. It had been submitted to the *Compositio Mathematica* in August 1939, but because of the war this journal had to be discontinued. Like Elie Cartan, Hopf was not satisfied by a proof by direct verification because such a proof[21] *contained no general reasons for the truth of the theorem.* He therefore tried to determine the homology of a compact Lie group using only general properties. For this goal he introduced so-called Γ-manifolds; these are manifolds on which a continuous but not necessarily associative product is defined. So, group spaces are particular examples of Γ–manifolds. Hopf then showed that the intersection ring of a Γ–manifold is isomorphic to a product of intersection rings of spheres of odd dimension. It is essential for his proof that the product structure of the manifold induces a coproduct in homology via the inverse homomorphism.

The intersection ring therefore becomes — as it is called nowadays — a Hopf algebra. The result then follows because the algebra structure of a Hopf algebra is very restricted. In the case of a Γ–manifold one gets an exterior algebra with generators in odd dimensions.

In that way, Hopf solved the problem in unexpectedly great generality. At that time, no further examples of Γ–manifolds were known other than Lie groups and spheres of odd dimension, but Hopf had recognized the pivotal role this concept plays in the study of Lie groups. Starting with Hopf's work, the theory of H-spaces was intensively developed in algebraic topology in the following years. The insight that the existence of a coproduct in an algebra posed severe restrictions on its structure is the beginning of the theory of Hopf algebras which, as should be shown later, plays an important role not only in algebraic topology but also in many other areas.

At the end of his paper, Hopf briefly referred to the algebra structure in the homology of a Lie group introduced by Pontryagin a short time before and conjectured that the proof could also be done using the Pontryagin algebra. This was soon later proved by Hopf's student Hans Samelson.

Instead of the Hopf intersection ring, one today considers the cohomology ring which carries a Hopf algebra structure due to the product on the manifold. This point of view was already well-known at that time. But Hopf preferred — here as well as in other works — to use homology; apparently the more geometric cycles were nearer to his way of thinking than the cocycles which are better suited for computations.[22] During the first ten years of his Zürich time Hopf published, besides the voluminous book with Alexandrov, about twenty papers; several among these papers have influenced the further development of algebraic topology in a pioneering way. He achieved this in addition to all the duties of his professorship at the ETH. From the very beginning, Hopf devoted himself to extensive lecturing, comprising various areas of mathematics and also many elementary courses. His lectures were regarded — as before in Berlin — as excellent by his students, and they were known to be extraordinarily clear and gripping. He always succeeded in making his audience ask, think, and work together with him. Therefore it is not astonishing that he attracted a number of excellent students who wanted to work for a Diplom or doctorate under his supervision. In particular his PhD students always found him attentive — discussions took place in his house in Zollikon where tea and cakes were served afterwards —, and he supported them generously with ideas and consultation.

Zürich during World War II (1939–1945)

In addition to the great demands of his work, Hopf was also under great psychological stress during his first years in Zürich, due to the political situation in Germany. His parents still lived in Breslau. Being a Jew, his father was exposed to increasing pressure by the Nazis. Until 1939, Hopf could visit his parents regularly and get his own impression of the situation.[23] This made him try to get them an immigration permit for Switzerland. Although his application was approved, the planned journey had to be deferred because the father became seriously ill. The outbreak of the war made it impossible to pursue the plans later. Hopf's father died in Breslau in 1942.

In Zürich and at his place in Zollikon, Hopf, together with his wife, tried to provide aid for persecuted people from Germany. His cousin Ludwig Hopf was a regular guest in Zollikon. Ludwig Hopf had been professor at the Technische Hochschule Aachen and lost his position in 1934 because of the Nazi laws. In 1938 he managed to flee from Germany. He became lecturer at Trinity College, Dublin, but died only a few months later. After the loss of his position, Issai Schur, Hopf's former teacher in Berlin, spent some time at the ETH with a teaching post before he could emigrate to Palestine in 1939.[24] Hopf tried to help many other persecuted people financially or by supporting their cause outside Germany. For his student Hans Samelson he managed to organize a position in Princeton in July 1940, when Switzerland was already almost surrounded by the Axis occupied territory.

In Princeton, people were worrying about Heinz Hopf's future fate, and Solomon Lefschetz sent him an invitation to Princeton in November 1940. Hopf replied in his letter from January 1st, 1941:[25]

> Das ist sehr nett von Ihnen, und ich bin Ihnen für diese Anfrage sehr dankbar. [...] Aber [wir halten] es aus prinzipiellen Gründen für richtiger, das Schiff nicht zu verlassen, solange trotz des Sturmes doch noch eine Möglichkeit besteht, dass es nicht untergeht.*

Two years later, circumstances had deteriorated for Hopf in such a way that he had to apply for the Swiss citizenship.[26] Until then, he had not planned to take this step before the end of the war in order not to be considered an opportunist. But in March 1943, he received a notice that his property had been confiscated by the German authorities. Soon afterwards, the German consulate general in Zürich refused to extend his *Heimatschein*, and he was threatened with the loss of his German citizenship unless he moved back to the area of the German Reich. Hopf's plea for Swiss citizenship was approved by the *Bürgergemeindeversammlung* of Zollikon in the same year.

Whereas until the outbreak of the war there were some, albeit censored, connections with Germany, they were disrupted completely when the war began. Scientific contacts with France, Great Britain, and America were strongly restricted and even the formerly frequent correspondence with Alexandrov in Moscow ended around Christmas 1940. In spite of this isolation which Hopf found oppressing, he and his students continued to publish works of the highest standards.

A compilation of works from the school of algebraic topology Hopf had founded in Zürich can be found in his report *Bericht über einige neue Ergebnisse in der algebraischen Topologie* [42], which was meant to be a contribution to the Festschrift for Brouwer's sixtieth birthday in 1941 but then could only be published in 1946 because of the war. In that paper Hopf first reports on Eduard Stiefel's study of the existence of systems of continuous tangent vector fields on real projective spaces, in particular of the parallelizability of these spaces (1941); then on Beno Eckmann's results on the homotopic properties of fibred spaces from which statements about the parallelizability of spheres follow, among others (1941). He continued with an account of Werner Gysin's work on the homology of fibred spaces with fibre a sphere (1941), of Hans Samelson's work on the homology of spaces on which Lie groups act, from which a general reason for the special structure of the homology of compact Lie groups could be derived (1941), and finally of Alexandre Preissmann's results on the fundamental group of closed Riemannian manifolds of negative curvature (1942–43). His own contribution was about the question in how far the fundamental group of a connected complex determines the second Betti group. Using ideas which also appeared in the works of Samelson and Preissmann in other contexts, Hopf considered the quotient of the second Betti group of a complex with fundamental group G modulo homology classes whose cap-products with arbitrary one-dimensional cohomology classes vanish. He showed that this quotient only depended on the fundamental group G. As a conclusion he obtained that the second Betti group is completely determined by the fundamental group if every image of a two-dimensional sphere is null-homologous.[27]

In the sequel, these considerations led to the important paper *Fundamentalgruppe und zweite Bettische Gruppe* [44]. Here Hopf weakened the preconditions and investigated the quotient of the second Betti group modulo the homology classes which contain continuous images of spheres. Hopf showed that also this quotient depends only on the fundamental group. In particular, the theorem followed that the second Betti group is completely determined by the fundamental group if every image of a two-dimensional sphere is null-homotopic. Starting with a free presentation F/R of the fundamental group G, Hopf gave an explicit description of this group, namely $[F,F]\cap R/[F,R]$. The abelian group associated with G by this formula had already arisen in works of Issai Schur's on projective representations which had appeared just after the turn of the century. But Hopf does not seem to have noted this connection with the Schur multiplier in the beginning.

*That is very kind, and I am very grateful that you offered this. [...] However, for reasons of principle we consider it better not to leave the ship as long as despite of the tempest, there is a possibility that it will not sink.

The paper *Fundamentalgruppe und zweite Bettische Gruppe* is legitimately regarded to be the beginning of homological algebra. It opened the way for the definition of the homology and cohomology of a group. This step was made independently at different places shortly after the paper had become known: in the USA in the circle around Samuel Eilenberg and Saunders MacLane, in Switzerland by Heinz Hopf and Beno Eckmann and in the Netherlands by Hopf's former student Hans Freudenthal. Hopf's own paper on this topic *Über die Bettischen Gruppen, die zu einer beliebigen Gruppe gehören* [49] appeared in 1944/45. Following his work mentioned above, he had conjectured that its main result could be generalised to higher dimensions. Hurewicz had shown in the thirties that the homology groups of an aspherical connected space are completely determined by the fundamental group G. Hopf's first work contained the algebraic details of this proposition for the second homology group. In the comprehensive sequel he now showed how one can treat higher dimensions similarly. From today's point of view, one could describe his purely algebraic construction as a G-free resolution of $\mathbb{Z}$. For Hopf, it arose as the algebraic analogue of the complex of the universal covering $\tilde{X}$ of an aspherical space X with fundamental group G (whose existence was proved by Eilenberg and MacLane at the same time and independently of Hopf). The Betti groups were then defined as the homology groups of the complex which resulted from the free resolution by trivializing the G-action (tensor product with $\mathbb{Z}$ over G). Hurewicz's result mentioned above corresponds in this context to the fact that the Betti groups do not depend on the choice of a particular free resolution of $\mathbb{Z}$. By his procedure, Hopf assigned Betti groups to a given group in a purely algebraic way; so the basis for the (co-)homology theory of groups and of homological algebra was established. In the following years, this theory earned broad appreciation only slowly, possibly due to the necessary complex algebraic machinery. But gradually it became an indispensable tool in quite a large range of mathematical areas.

At the same time, Hopf occupied himself with the theory of ends of open spaces already developed in 1931 by his student Hans Freudenthal. Hopf considered spaces which are regular coverings of a compact space. He showed that there are only three possibilities: either the number of ends is one or two, or the set of ends has the cardinality of the continuum. If the (finitely generated) group G is realized as the group of deck transformations of the covering $\tilde{X}$ of a compact space X, then the number of ends of $\tilde{X}$ is an invariant of G. Hopf posed the question about the group-theoretic significance of the number of ends and solved the case of two ends completely: a group G has two ends if and only if it contains an infinite cyclic subgroup of finite index. Hopf did not succeed in characterising the other cases completely. The theory of ends was taken up again soon later by Hopf's student Ernst Specker. In the end of the sixties, the theories of ends of a group played a key role in Stallings' solution for the problem of groups of cohomological dimension one.[28]

Zürich after World War II (1945 – 1971)

After the end of the war, the interrupted scientific relations were gradually reestablished. First, Hopf tried to contact Alexandrov. The latter had come through the war safe and sound. His house in the vicinity of Moscow was slightly damaged by grenade splinters but he was able to spend the time in safety east of Moscow, although in rather primitive conditions.

In the period just after the war, Hopf tried to help his relatives and friends on the other side of the Swiss border to the best of his ability. On the one hand, the support consisted of the bare necessities of life, for the terrible shortage could only be alleviated by food parcels from foreign countries. But Hopf's assistance was also aimed at helping reestablishing mathematical life in Germany. Already in August 1946, Hopf was guest at the mathematical research institute at Oberwolfach in the Black Forest, which was founded by Wilhelm Süss after the war.

In the period from October 1946 until March 1947, Hopf went to America.[29] On the journey he first went to Paris where he spent a few days with Jean Leray and participated in a meeting of the Académie

Française. Then he boarded a ship in Le Havre for New York. After arrival he spent a few weeks in New York, the remaining time primarily in Princeton. He met many old friends for the first time after a long period, Courant, Friedrichs, Stoker, Neugebauer in New York, Veblen, Alexander, Lefschetz in Princeton. In Princeton he shared a flat with J. H. C. Whitehead.

At New York University, Hopf gave talks on *Selected Topics in Geometry* with much success. In his audience there were some young mathematicians who would become well-known later, e.g. Louis Nirenberg, Peter Lax, and Anneli Leopold (later Lax). Peter Lax worked up his notes of these lectures; they were published posthumously in volume 1000 of the Springer Lecture Notes in Mathematics, together with a lecture *Differential Geometry in the Large* Hopf gave in Stanford in 1956 written up by John W. Gray.

During his stay in Princeton he received several calls and offers from American universities, among them Harvard University, the California Institute of Technology in Pasadena, and Princeton University. According to the reports Hopf sent home, Courant, whom Hopf consulted, had a very high opinion of the offer from Harvard: *only few mathematicians have such good positions.* In the sequel, Hopf could slightly improve his position at the ETH, and after thorough consideration he decided to stay in Zürich.

On the occasion of the bicentenary of the University of Princeton he was awarded with the title of honorary doctor. In a letter to his wife he tells :

> Dinge gehen vor im Mond.* [...] So werden Illusionen zerstört. Was habe ich mir doch als unschuldiger Jüngling unter einem Ehrendoktor, noch dazu von Princeton, für einen klugen und weisen Mann vorgestellt. Aber es freut mich natürlich gewaltig.†

He also tells in detail about the celebration itself in his letter and adds humorously:

> Mein 'gown' war mir zwar zu weit, aber glücklicherweise nicht zu lang, so dass ich nicht daraufgetreten bin.‡

Towards the end of his American visit Hopf made some major journeys. He visited Harvard University, Brown University and the University of North Carolina at Chapel Hill. Finally he undertook an extended journey with lectures at Toronto, Chicago, Bloomington, and Ann Arbor. In the beginning of April, he returned from New York to Zürich by aeroplane.

As before the war, Hopf made it possible for many, especially younger mathematicians, to stay in Zürich. For example, in 1948 the young Hirzebruch was hospitably welcomed at Hopf's home in Zollikon. Also Tits and Nirenberg both spent several months at the ETH in Zürich as post-doctoral visitors.

Towards the end of the forties, mathematics in Europe came to life again. Hopf was now invited frequently, often as the principal speaker at congresses and conferences. In 1947 he travelled to Paris, in 1949 to a major conference on topology in Oberwolfach, in 1950 to Brussels. On the occasion of Severi's seventieth birthday in Rome in the same year, he met Paul Alexandrov for the first time after a long period. In 1953, he was Henry Whitehead's guest in Oxford while the conference for "Young Topologists" was held there.

In the winter semester 1955/56, Hopf again went to America, this time together with his wife. The ship voyage to New York was followed by an excellently organized lecture trip across the whole American country, and during a longer stay at Stanford University, Hopf gave a lecture course on *Differential*

* *Dinge gehen vor im Mond / die das Kalb selbst nicht gewohnt / ...* (things happen in the moon that even the calf is not used to) is the beginning of the humorous poem "Mondendinge" by C. Morgenstern.

† This is how you lose your illusions. What an intelligent and wise man I imagined a honorary doctor, and in particular one from Princeton, must be when I was an innocent youth. But of course it makes me tremendously happy.

‡ Although my gown was too wide, it was fortunately not too long so that I did not step on it.

Geometry in the Large which appeared posthumously in the volume of the Lecture Notes in Mathematics mentioned above.

Thanks to his high scientific and personal reputation, Hopf was elected President of the International Mathematical Union from 1955 until 1958. Since Alexandrov worked in the executive committee too, the two old friends now met more frequently, at the international congresses in Amsterdam (1954), in Edinburgh (1958), in Stockholm (1962), and finally in Moscow (1966). When René Thom received the Fields Medal in Edinburgh, Hopf was asked to give the laudatory speech. Now the honours accumulated: the Princeton honorary doctorate was followed by five more: Freiburg i. Br. (1957), Manchester (1958), Sorbonne at Paris (1964), Brussels (1964), and Lausanne (1965). From the University of Göttingen he received the Gauß–Weber Medal in 1955, and from the Academy of Sciences of the USSR in Moscow the Lobachevsky award in 1967. In 1958, he became member of the Deutsche Akademie der Naturforscher Leopoldina in Halle. Furthermore, he was corresponding member of the Heidelberg Akademie der Wissenschaften (1949) and of the Akademie der Wissenschaften in Göttingen (1966), honorary member of the London Mathematical Society (1956), of the Schweizerische Mathematische Gesellschaft (1957), of the American Academy of Arts and Sciences (1961) as well as foreign member of the National Academy of Sciences of the USA (1957) and the Accademia Nazionale dei Lincei (1962).

On the occasion of his seventieth birthday in 1964, the *Selecta Heinz Hopf* appeared, in which the 19 most important of his over 70 articles were published and in that way made accessible to the mathematical world more easily. One year later, on 6$^{\text{th}}$ July 1965, Hopf gave his retirement lecture at the ETH as part of a major celebration.

In personal life he could not escape sorrows. In the year 1959, he had to be operated on for a stomach ulcer and had to recover at home for an extended period. Around the middle of the sixties, his wife Anja fell very ill. They had planned a journey together to the International Congress of 1966 in Moscow and to Alexandrov, but Hopf had to go alone. Anja died in February 1967. Hopf did not recover from this blow. Symptoms of a geriatric disease appeared which confined him to his house. He died in hospital on 3$^{\text{rd}}$ June 1971.

Acknowledgements.

We wish to thank the employees of the Wissenschaftshistorische Sammlungen der ETH–Bibliothek Zürich, particularly the former director, Dr. Beat Glaus, who supported us continually by word and deed; Dr. Elisabeth Ettlinger–Lachmann for many informative talks on Heinz Hopf and for allowing us to look at the letters he wrote to his wife from America; and Professor Beno Eckmann for retelling many personal memories of Heinz Hopf and for numerous comments about the development of mathematics as he had experienced it as one of Hopf's students. Finally we thank Tilman Bauer for the excellent translation of our German text into English.

Sources

Heinz Hopf's scientific papers are in the Wissenschaftshistorische Sammlungen der ETH–Bibliothek Zürich, under the reference Hs 620–622. In the same collection, under the reference Hs 160, there are copies of fifty letters by Heinz Hopf to Paul Alexandrov; the originals are kept in the Russian Academy of Sciences in Moscow. A comprehensive collection of Heinz Hopf's letters to his wife during his America visit in 1946–47 is property of Dr. Elisabeth Ettlinger–Lachmann, Heinz Hopf's niece.

Several obituaries on Heinz Hopf have appeared. We want to mention in particular:

Alexandrov, P.: Einige Erinnerungen an Heinz Hopf. Jber. Dt. Math. Verein. 78 (1976), pp. 113–125.

Behnke, H., Hirzebruch, F.: In memoriam Heinz Hopf. Math. Ann. 196 (1972), pp. 1–7.

Cartan, H.: Heinz Hopf (1894–1971). International Mathematical Union (IMU), 1972, pp. 7–10.

Eckmann, B.: Zum Gedenken an Heinz Hopf. Neue Zürcher Zeitung, June 18$^{\text{th}}$, 1971; reprint in L'enseignement mathématique 18 (1972), 105–112.

Hilton, P. J.: Heinz Hopf. Bull. London Math. Soc. 4 (1972), 202–217.

Samelson, H.: Zum wissenschaftlichen Werk von Heinz Hopf. Jber. Dt. Math. Verein. 78 (1976), 126–146.

Notes

References to Heinz Hopf's publications are marked by square brackets; the numbering corresponds to the one in *Selecta Heinz Hopf*, Springer-Verlag, 1964.

[1] International Mathematical Union (IMU), 1972, p. 7–10.

[2] The details of the Hopfs' family tree are drawn from Arnd Müller: Geschichte der Juden in Nürnberg 1146–1945, Selbstverlag der Stadtbibliothek Nürnberg 1968. Together with much more information on the circumstances of Heinz Hopf's life, they were kindly put at our disposal by Elisabeth Ettlinger-Lachmann, Heinz Hopf's niece.

[3] In German history the period from the 1870/71 war between France and Germany until about the end of the century is known as the "Gründerzeit". These early years of German unification were marked by periods of exceptional economic growth. The architecture of the time was designed to show to the world the newly accumulated wealth of the owner.

[4] Wissenschaftshistorische Sammlungen der ETH-Bibliothek Zürich, Hs. 622:7.

[5] Wissenschaftshistorische Sammlungen der ETH-Bibliothek Zürich, Hs. 622:9–10.

[6] Hopf, H.: Einige persönliche Erinnerungen aus der Vorgeschichte der heutigen Topologie. CBRM Bruxelles (1966), 9–20. From these autobiographic notes we also extract many more remarks on Hopf's personal and mathematical development.

[7] Promotion Heinz Hopf. Gutachten und Prüfungsprotokoll. Cited in Biermann, K.–R.: Die Mathematik und ihre Dozenten an der Berliner Universität. 1810–1920. Akademie-Verlag, Berlin 1973. p. 335–338.

[8] Hopf, H.: Einige persönliche Erinnerungen aus der Vorgeschichte der heutigen Topologie, see note 6.

[9] Reference by Erhard Schmidt on Heinz Hopf's Habilitation thesis, partially quoted in Biermann, K.–R.: Die Mathematik und ihre Dozenten an der Berliner Universität. 1810–1920. Akademie-Verlag, Berlin 1973, p. 205.

[10] Hopf, H.: Einige persönliche Erinnerungen aus der Vorgeschichte der heutigen Topologie, see note 6. — There is much information on Paul Alexandrov and Heinz Hopf in: Alexandroff, P.: Einige Erinnerungen an Heinz Hopf. Jahresbericht der Deutschen Mathematiker-Vereinigung 78 (1976), 113–125 as well as in Alexandrov, P. S.: Pages from an autobiography. Russian Mathematical Surveys 35 (1980), 315–358. See also Frei, G., Stammbach, U.: Correspondence between Alexandrov and Hopf. In: Proceedings of the International Topology Conference, dedicated to P.S. Alexandrov's 100th birthday, Phasis

Publishing House, Moscow 1996, pp. xxiii–xxxviii.

[11]Hopf, H.: Einige persönliche Erinnerungen aus der Vorgeschichte der heutigen Topologie, see note 6.

[12]Alexandrov, P. S.: Pages from an autobiography. Russian Math. Surveys 34: 6 (1979), p. 324.

[13]Alexandroff, P.: Einige Erinnerungen an Heinz Hopf, Jber. Dt. Math. Verein. 78 (1976), p. 113.

[14]Bericht über die Zeit meines "Fellowships" der Internationalen Education Board vom 1. Oktober 1927 bis 1. Juni 1928. Sketch. Wissenschaftshistorische Sammlungen der ETH–Bibliothek Zürich, Hs 622:47.

[15]The information on Anja von Mickwitz was taken from a typoscript which Leopold Ettlinger put generously at our disposal.

[16]Letter by Issai Schur to George Polya from June 30th, 1930. Archiv des Schweizerischen Schulrates, Korrespondenz des Schweizerischen Schulrates, Akten. Wissenschaftshistorische Sammlungen der ETH–Bibliothek, Zürich.

[17]Letter by Heinz Hopf to Schulratspräsident Rohn from September 30th, 1930. Archiv des Schweizerischen Schulrates, Korrespondenz des Schweizerischen Schulrates, Akten. Wissenschaftshistorische Sammlungen der ETH–Bibliothek, Zürich.

[18]Eckmann, B.: Zum Gedenken an Heinz Hopf. Neue Zürcher Zeitung, June 18th, 1971; reprint in: L'enseignement mathématique 18 (1972), 105–112.

[19]The quotation is taken from [18], see also Selecta Heinz Hopf, p. 54.

[20]The source for the information on the *Erste Internationale Konferenz über Topologie* is Hopf, H.: Einige persönliche Erinnerungen aus der Vorgeschichte der heutigen Topologie, see note 6; Alexandroff, P.: Einige Erinnerungen an Heinz Hopf, Jber. Dt. Math.-Verein. 78 (1976), 113–146, as well as the correspondence between P. Alexandrov and H. Hopf: Wissenschaftshistorische Sammlungen der ETH-Bibliothek, Zürich, Hs 621:15–146 and Hs 160.

[21]The quotation is from the work [40], see also Selecta Heinz Hopf, p. 125.

[22]Cf. Hilton, P. J.: A brief, subjective history of homology and homotopy theory in this century, Math. Magazine 61 (1988), 282–291.

[23]The journeys are mentioned in Hopf's letters to Alexandrov. For Hopf's correspondence with the Swiss authorities in that respect, consult Wissenschaftshistorische Sammlungen der ETH–Bibliothek Zürich, Hs 622:43–44.

[24]Brauer, A.: Gedenkrede auf Issai Schur. In: Issai Schur Gesammelte Abhandlungen, Springer Verlag 1973. Volume I, v–xiv.

[25]Draft of a letter to Solomon Lefschetz, dated 1/41. Wissenschaftshistorische Sammlungen der ETH–Bibliothek Zürich, Hs 92:289.

[26]Cf. the correspondence in Wissenschaftshistorische Sammlungen der ETH–Bibliothek Zürich, Hs 622:45–75.

[27]Hopf had summarized these results in a note *Relations between the fundamental group and the second Betti group* in 1940 and sent it to America where they were thoroughly studied by Eilenberg and MacLane on the occasion of a topology conference at the University of Michigan in Ann Arbor. See MacLane, S.: Group extensions for 45 years, Math. Intelligencer 10 (1988), No. 2, pp. 29–35.

[28]Stallings, J.: On torsion free groups with infinitely many ends. Ann. Math. 88 (1968), 312–334.

[29]The information on Hopf's time in America stems from letters Hopf wrote to his wife from America; the three following quotations are from letters written on December 20th, 1946, January 13th, and February 22nd, 1947. We thank Elisabeth Ettlinger for allowing us to read these letters.

Günther Frei
Lützelstr. 36
CH 8634 Hombrechtikon

Urs Stammbach
Mathematik
ETH-Zentrum
CH-8092 Zürich