Symplectic Topology
Example Sheet 10

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Let (M,ω) be a closed symplectic manifold and let

$$H:[0,1]\times M\to\mathbb{R}$$

be a smooth function. For $0\le t\le 1$ define $H_t:M\to\mathbb{R}$ by

$$H_t(p):=H(t,p)$$

and let $\{\phi_t\}_{0\le t\le 1}$ be the isotopy generated by H via

$$\partial_t\phi_t=X_t\circ\phi_t,\qquad \phi_0=\mathrm{id},\qquad \iota(X_t)\omega=dH_t. \tag{1}$$

We have seen in the lecture course that ϕ_t is a symplectomorphism of (M,ω) for every t. The time-1 map is called the **Hamiltonian symplectomorphism generated by** H and will be denoted by

$$\phi_H:=\phi_1.$$

The set of Hamiltonian symplectomorphisms of (M,ω) will be denoted by

$$\mathrm{Ham}(M,\omega):=\left\{\phi_H\,|\,H\in C^\infty([0,1]\times M)\right\}.$$

Note that for every $\phi\in\mathrm{Ham}(M,\omega)$ there are many Hamiltonian functions $H:[0,1]\times M\to\mathbb{R}$ such that $\phi_H=\phi$.

Exercise 10.1. Prove that $\mathrm{Ham}(M,\omega)$ is a subgroup of $\mathrm{Diff}(M,\omega)$.

Exercise 10.2. The **Hofer norm** of a function $H : [0,1]\times M \to \mathbb{R}$ is defined by

$$\|H\| := \int_0^1 \left(\max_M H_t - \min_M H_t\right)\,dt$$

The **Hofer distance** of two symplectomorphisms $\phi_0, \phi_1 \in \mathrm{Ham}(M,\omega)$ is the real number

$$d(\phi_0,\phi_1) := \inf_{\phi_H=\phi_1\circ\phi_0^{-1}} \|H\|\,,$$

where the infimum is taken over all smooth functions $H : [0,1]\times M \to \mathbb{R}$ that generate the symplectomorphism $\phi_1\circ\phi_0^{-1} = \phi_H$. Prove that the Hofer distance $d : \mathrm{Ham}(M,\omega)\times\mathrm{Ham}(M,\omega) \to [0,\infty)$ satisfies the following axioms.

(Symmetry) For all $\phi_0, \phi_1 \in \mathrm{Ham}(M,\omega)$

$$d(\phi_0,\phi_1) = d(\phi_1,\phi_0).$$

(Triangle Inequality) For all $\phi_0, \phi_1, \phi_2 \in \mathrm{Ham}(M,\omega)$

$$d(\phi_0,\phi_2) \le d(\phi_0,\phi_1) + d(\phi_1,\phi_2).$$

(Invariance) For all $\phi_0, \phi_1, \psi \in \mathrm{Ham}(M,\omega)$

$$d(\phi_0\circ\psi,\phi_1\circ\psi) = d(\psi\circ\phi_0,\psi\circ\phi_1) = d(\phi_0,\phi_1)$$

(Conjugacy) For all $\phi_0, \phi_1 \in \mathrm{Ham}(M,\omega)$ and all $\psi \in \mathrm{Diff}(M,\omega)$

$$d(\psi^{-1}\circ\phi_0\circ\psi,\psi^{-1}\circ\phi_1\circ\psi) = d(\phi_0,\phi_1)$$

Exercise 10.3. For $H : [0,1]\times M \to \mathbb{R}$ define

$$\|H\|' := \int_0^1 \sqrt{\int_M H_t^2 \frac{\omega^n}{n!}}\,dt.$$

For $\phi_0, \phi_1 \in \mathrm{Ham}(M,\omega)$ define

$$d'(\phi_0,\phi_1) := \inf_{\phi_H=\phi_1\circ\phi_0^{-1}} \|H\|'\,,$$

where the infimum is taken over all smooth functions $H : [0,1]\times M \to \mathbb{R}$ that satisfy $\phi_H = \phi_1\circ\phi_0^{-1}$ and $\int_M H_t\omega^n = 0$ for all t. Prove that the function $d' : \mathrm{Ham}(M,\omega)\times\mathrm{Ham}(M,\omega) \to \mathbb{R}$ also satisfies the axioms listed in Exercisee 10.2. (**Note:** It turns out that the Hofer distance is always nondegenerate, however, a theorem by Eliashberg and Polterovich asserts that the function d' vanishes on every symplectic manifold.)

Let $(\Sigma, j, \mathrm{dvol}_\Sigma)$ be a closed Riemann surface, (M, ω) be a symplectic manifold, let $\Sigma \to \mathcal{J}(M, \omega) : z \mapsto J_z$ be a smooth family of ω-compatible almost complex structures, and let $K \in \Omega^1(\Sigma, C^\infty(M))$ be a 1-form on Σ with values in the space of smooth functions on M. Thus K assigns to each $z \in \Sigma$ a linear map

$$T_z\Sigma \to C^\infty(M) : \widehat{z} \mapsto K_{z,\widehat{z}}$$

and to each point $p \in M$ a 1-form $K(p) \in \Omega^1(\Sigma)$ via $K(p)_z(\widehat{z}) := K_{z,\widehat{z}}(p)$. Note that K can be also thought of as a 1-form on the product manifold

$$\widetilde{M} := \Sigma \times M$$

which assigns to a tangent vector $(\widehat{z}, \widehat{p}) \in T_{(z,p)}\widetilde{M} = T_z\Sigma \times T_pM \to \mathbb{R}$ the real number $K_{z,\widehat{z}}(p)$. When understood as a 1-form on $\widetilde{M}$ the differential of K will be denoted by $d^{\widetilde{M}}K \in \Omega^2(\widetilde{M})$. When understood as 1-form on Σ with values in $C^\infty(M)$, the differential of K will be denoted by $d^\Sigma K \in \Omega^2(\Sigma, C^\infty(M))$; this 2-form is defined by defined by

$$(d^\Sigma K)_z(\widehat{z}_1\widehat{z}_2)(p) := (d(K(p)))_z(\widehat{z}_1, \widehat{z}_2)$$

for $p \in M$, $z \in \Sigma$, and $\widehat{z}_1, \widehat{z}_2 \in T_z\Sigma$.

Define the function $R_K : \Sigma \times M \to \mathbb{R}$ by

$$R_K \mathrm{dvol}_\Sigma := d^\Sigma K + \tfrac{1}{2}\{K \wedge K\}.$$

Here $\frac{1}{2}\{K \wedge K\} \in \Omega^2(\Sigma, C^\infty(M))$ is the 2-form which assigns to each pair $\widehat{z}_1, \widehat{z}_2 \in T_z\Sigma$ the Poisson bracket $\{K_{z,\widehat{z}_1}, K_{z,\widehat{z}_2}\} \in C^\infty(M)$.

Denote by $\mathrm{pr}_\Sigma : \widetilde{M} \to \Sigma$ and $\mathrm{pr}_M : \widetilde{M} \to M$ the obvious projections, choose a function $\kappa : \Sigma \to (0, \infty)$, and define $\widetilde{\omega} \in \Omega^2(\widetilde{M})$ by

$$\widetilde{\omega} := \mathrm{pr}_M^*\omega - d^{\widetilde{M}}K + \mathrm{pr}_\Sigma^*(\kappa \mathrm{dvol}_\Sigma)$$

For $u : \Sigma \to M$ define

$$d_K u := du + X_K(u),$$

where $X_K(u) \in \Omega^1(\Sigma, u^*TM)$ is defined by $(X_K(u))_z(\widehat{z}) := X_{K_{z,\widehat{z}}}(u(z))$, and

$$\bar{\partial}_{J,K}(u) := (d_K u)^{0,1} := \tfrac{1}{2}(d_K u + J(u) \circ d_K u \circ j).$$

Exercise 10.4. Prove that

$$\frac{\widetilde{\omega}^{n+1}}{(n+1)!} := (\kappa - R_K)\mathrm{pr}_\Sigma^*\mathrm{dvol}_\Sigma \wedge \mathrm{pr}_M^*\frac{\omega^n}{n!}.$$

If $\bar{\partial}_{J,K}(u) = 0$ prove that

$$E_K(u) := \tfrac{1}{2}\int_\Sigma |d_K u|_J^2 \,\mathrm{dvol}_\Sigma = \int_\Sigma u^*\omega + \int_\Sigma R_K(u)\,\mathrm{dvol}_\Sigma.$$

Relate this to the energy identity for the $\widetilde{J}$-holomorphic curve $\widetilde{u} : \Sigma \to \widetilde{M}$ defined by $\widetilde{u}(z) := (z, u(z))$. Do the solutions of the perturbed equation $\bar{\partial}_{J,K}(u) = 0$ minimize the energy $E_K(u)$ in their homology class? What is the relation between $E_K(u)$ and $E(\widetilde{u})$. **Hint:** See Exercise 4.1. Carry out the calculations in local coordinates on Σ.