Notes on the octonians

Dietmar A. Salamon and Thomas Walpuski
ETH Zürich

27 April 2010

1 Introduction

In these notes we give an exposition of the structures in linear algebra that underly Donaldson–Thomas theory [2, 3] and calibrated geometry [4, 5]. No claim is made to originality. All the results and ideas described here (except perhaps Theorem 7.4) can be found in the existing literature, notably in the beautiful paper [4] by Harvey and Lawson. Perhaps these notes might be a useful introduction for students who wish to enter the subject.

Our emphasis is on characterizing the relevant algebraic structures - such as cross products, triple cross products, associator and coassociator brackets, associative, coassocitative, and Cayley calibrations and subspaces - by their intrinsic properties rather than by the existence of isomorphisms to the standard structures on the octonians and the imaginary octonians, although both descriptions are of course equivalent.

Section 2 deals with cross products and their associative calibrations. It contains a proof that they exist only in dimensions 0, 1, 3, and 7. In Section 3 we discuss nondegenerate 3-forms on 7-dimensional vector spaces (associative calibrations) and explain how they give rise to unique compatible inner products. Additional structures such as associative and coassociative subspaces and the associator and coassociator brackets are discussed in Section 4. These structures are relevant for understanding G_2-structures on 7-manifolds and the Chern–Simons functional in Donaldson–Thomas theory.

The corresponding Floer theory has as its counterpart in linear algebra the product with the real line. This leads to the structure of a normed algebra which only exists in dimensions 1, 2, 4, and 8, corresponding to the reals, the complex numbers, the quaternions, and the octonians. These structures are discussed in Section 5. Going from Floer theory to an intrinsic theory for Donaldson-type invariants of 8-dimensional Spin(7)-manifolds corresponds to dropping the space–time splitting. The algebraic counterpart of this reduction is to eliminate the choice of the unit (as well as the product). What's left of the algebraic structures is the triple cross product and its Cayley calibration - a suitable 4-form on an 8-dimensional Hilbert space. These structures are discussed in Section 6. Section 7 characterizes those 4-forms on 8-dimensional vector spaces (the Cayley-forms) that give rise to (unique) compatible inner products and hence to triple cross products. The relevant structure groups G_2 (in dimension 7) and Spin(7) (in dimension 8) are discussed in Sections 8 and 9. In Section 10 we examine spin structures in dimensions 7 and 8. Section 11 relates SU(3) and SU(4) structures to cross products and triple cross products and Section 12 gives a brief introduction to the basic setting of Donaldson–Thomas theory.

2 Cross products

We assume throughout that V is a finite dimensional real Hilbert space.

Definition 2.1. *A skew-symmetric bilinear map*

$$V \times V \to V : (u, v) \mapsto u \times v \tag{1}$$

is called a **cross product** *if it satisfies*

$$\langle u \times v, u \rangle = \langle u \times v, v \rangle = 0, \tag{2}$$

$$|u \times v|^2 = |u|^2 \, |v|^2 - \langle u, v \rangle^2 \tag{3}$$

for all $u, v \in V$.

A bilinear map (1) that satisfies (3) also satisfies $u \times u = 0$ for all $u \in V$ and hence is necessarily skew-symmetric.

Theorem 2.2. *V admits a cross product if and only if its dimension is either* 0*,* 1*,* 3*, or* 7*. In dimensions* 0 *and* 1 *the cross product vanishes, in dimension* 3 *it is unique up to sign and determined by an orientation of V, and in dimension* 7 *it is unique up to orthogonal isomorphism.*

Lemma 2.3. *If* (1) *is a skew-symmetric bilinear map then the following are equivalent.*

(i) *Equation* (2) *holds for all $u, v \in V$.*

(ii) *For all $u, v, w \in V$ we have*

$$\langle u \times v, w \rangle = \langle u, v \times w \rangle. \tag{4}$$

(iii) *The map* $\phi : V^3 \to \mathbb{R}$*, defined by*

$$\phi(u, v, w) := \langle u \times v, w \rangle, \tag{5}$$

is an alternating 3*-form (called the* **associative calibration** *of* V*).*

Proof. Let (1) be a skew-symmetric bilinear map. Assume that it satisfies (2). Then, for all $u, v, w \in V$, we have

$$\begin{aligned} 0 &= \langle v \times (u + w), u + w \rangle \\ &= \langle v \times w, u \rangle + \langle v \times u, w \rangle \\ &= \langle u, v \times w \rangle - \langle u \times v, w \rangle. \end{aligned}$$

This proves (4). Now assume (4) and let ϕ be defined by (5). Then, by skew-symmetry, we have $\phi(u, v, w) + \phi(v, u, w) = 0$ for all u, v, w and, by (4), we have $\phi(u, v, w) = \phi(v, w, u)$ for all u, v, w. Hence ϕ is an alternating 3-form. Thus we have proved that (i) implies (ii) implies (iii). That (iii) implies (i) is obvious. This proves the lemma. □

Lemma 2.4. *Let* (1) *be a skew-symmetric bilinear map that satisfies* (2)*. Then the following are equivalent.*

(i) *The bilinear map* (1) *satisfies* (3)*.*

(ii) *If* u *and* w *are orthonormal then* $|u \times w| = 1$*.*

(iii) *If* $|u| = 1$ *and* w *is orthogonal to* u *then* $u \times (u \times w) = -w$*.*

(iv) *For all* $u, w \in V$ *we have* $u \times (u \times w) = \langle u, w \rangle u - |u|^2 w$*.*

(v) *For all* $u, v, w \in V$ *we have*

$$u \times (v \times w) + v \times (u \times w) = \langle u, w \rangle v + \langle v, w \rangle u - 2\langle u, v \rangle w. \tag{6}$$

Proof. That (i) implies (ii) is obvious.

We prove that (ii) implies (iii). Fix a vector $u \in V$ with $|u| = 1$ and define the linear map $A : V \to V$ by

$$Aw := u \times w.$$

Then, by skew-symmetry and (4), A is skew-adjoint and, by (2), it preserves the subspace $W := u^\perp$. Hence the restriction of A^2 to $W := u^\perp$ is self-adjoint and, by (ii), it satisfies

$$\langle w, A^2 w \rangle = - |u \times w|^2 = - |w|^2$$

for $w \in W$. Hence the restriction of A^2 to W is equal to minus the identity. This proves that (ii) implies (iii).

We prove that (iii) implies (iv). Fix a vector $u \in V$ and define $A : V \to V$ by $Aw := u \times w$ as above. By (iii) we have $A^2 w = -|u|^2 w$ whenever w is orthogonal to u. Since $A^2 u = 0$ this implies (iv).

Assertion (v) follows from (iv) by replacing u with $u + v$. To prove that (v) implies (i), set $w = v$ in (6) and take the inner product with u. Then

$$|u \times v|^2 = \langle u, u \times (v \times v) + v \times (u \times v)\rangle = |u|^2 |v|^2 - \langle u, v\rangle^2.$$

Here the first equation follows from (4) and the second equation follows from (6) with $w = v$. This proves the lemma. $\square$

Lemma 2.5. *Assume* $\dim V = 3$.

(i) *A cross product on* V *determines a unique orientation such that* $u, v, u\times v$ *form a positive basis for every pair of linearly independent vectors* $u, v \in V$.

(ii) *If* (1) *is a cross product on* V *then the* 3*-form* ϕ *given by* (5) *is the volume form associated to the inner product and the orientation in (i).*

(iii) *If* (1) *is a cross product on* V *then*

$$(u \times v) \times w = \langle u, w\rangle v - \langle v, w\rangle u \tag{7}$$

for all $u, v, w \in V$.

(iv) *Fix an orientation on* V *and denote by* $\phi \in \Lambda^3 V^*$ *the associated volume form. Then* (5) *determines a cross product on* V.

Proof. Assertion (i) follows from the fact that the space of pairs of linearly independent vectors in V is connected (whenever $\dim V \neq 2$). Assertion (ii) follows from the fact that, if u, v are orthonormal, then $u, v, u \times v$ form a positive orthonormal basis and $\phi(u, v, u \times v) = |u \times v|^2 = 1$.

We prove (iii). If u and v are linearly dependent, then both sides of (7) vanish. Hence we may assume that u and v are linearly independent or, equivalently, that $u \times v \neq 0$. Since $(u \times v) \times w$ is orthogonal to $u \times v$, by equation (4), and V has dimension 3, it follows that $(u \times v) \times w$ must be a linear combination of u and v. The formula (7) follows by taking the inner products with u and v, and using Lemma 2.4 (iv).

We prove (iv). Assume that the bilinear map (1) is defined by (5), where ϕ is the volume form associated to an orientation of V. Then skew-symmetry and (4) follow from the fact that ϕ is a 3-form (see Lemma 2.3). If u, v are linearly independent then, by (5), we have $u \times v \neq 0$ and

$\phi(u, v, u \times v) = |u \times v|^2 > 0$. If u, v are orthonormal it follows that $u, v, u \times v$ is a positive orthogonal basis and so $\phi(u, v, u \times v) = |u \times v|$. Combining these two identities we obtain $|u \times v| = 1$ when u, v are orthonormal. Hence (3) follows from Lemma 2.4. $\square$

Example 2.6. On $\mathbb{R}^3$ the cross product associated to the standard inner product and the standard orientation is given by the familiar formula

$$u \times v = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}.$$

Example 2.7. The standard structure on $\mathbb{R}^7$ can be obtained from the basis $\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{e}, \mathbf{ei}, \mathbf{ej}, \mathbf{ek}$ where $\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{e}$ are anti-commuting generators with square minus one and $\mathbf{ij} = \mathbf{k}$. Then the cross product is given by

$$u \times v := \begin{pmatrix} u_2v_3 - u_3v_2 - u_4v_5 + u_5v_4 - u_6v_7 + u_7v_6 \\ u_3v_1 - u_1v_3 - u_4v_6 + u_6v_4 - u_7v_5 + u_5v_7 \\ u_1v_2 - u_2v_1 - u_4v_7 + u_7v_4 - u_5v_6 + u_6v_5 \\ u_1v_5 - u_5v_1 + u_2v_6 - u_6v_2 + u_3v_7 - u_7v_3 \\ -u_1v_4 + u_4v_1 - u_2v_7 + u_7v_2 + u_3v_6 - u_6v_3 \\ u_1v_7 - u_7v_1 - u_2v_4 + u_4v_2 - u_3v_5 + u_5v_3 \\ -u_1v_6 + u_6v_1 + u_2v_5 - u_5v_2 - u_3v_4 + u_4v_3 \end{pmatrix}. \tag{8}$$

With $e^{ijk} := dx_i \wedge dx_j \wedge dx_k$ the associated 3-form (5) is given by

$$\phi_0 = e^{123} - e^{145} - e^{167} - e^{246} - e^{275} - e^{347} - e^{356}.$$

The product (8) is obviously skew-symmetric. Equation (4) follows from the fact that the matrix $A(u)$ defined by

$$A(u)v := u \times v$$

is skew symmetric for every u. Indeed, we have

$$A(u) := \begin{pmatrix} 0 & -u_3 & u_2 & u_5 & -u_4 & u_7 & -u_6 \\ u_3 & 0 & -u_1 & u_6 & -u_7 & -u_4 & u_5 \\ -u_2 & u_1 & 0 & u_7 & u_6 & -u_5 & -u_4 \\ -u_5 & -u_6 & -u_7 & 0 & u_1 & u_2 & u_3 \\ u_4 & u_7 & -u_6 & -u_1 & 0 & u_3 & -u_2 \\ -u_7 & u_4 & u_5 & -u_2 & -u_3 & 0 & u_1 \\ u_6 & -u_5 & u_4 & -u_3 & u_2 & -u_1 & 0 \end{pmatrix}.$$

We leave it to the reader to verify (3) (or equivalently $|u \times v| = 1$ whenever u and v are orthonormal). See also Remark 3.5 below.

Lemma 2.8. *Let V be a be a nonzero real Hilbert space and* (1) *be a cross product on V. Let $\phi \in \Lambda^3 V^*$ be given by* (5)*. Then the following holds.*

(i) *Let $u \in V$ be a unit vector and $W_u := u^\perp$. Define $\omega_u : W_u \times W_u \to \mathbb{R}$ and $J_u : W_u \to W_u$ by*

$$\omega_u(v,w) := \langle u, v \times w\rangle, \qquad J_u v := u \times v$$

for $v, w \in W_u$. Then ω_u is a symplectic form on W_u, J_u is a complex structure compatible with ω_u, and the associated inner product is the one inherited from V. In particular, the dimension of V is odd.

(ii) *Suppose* $\dim V = 2n+1 \geq 3$. *Then there is a unique orientation of V such that the associated volume form* $\mathrm{dvol} \in \Lambda^{2n+1}V^*$ *satisfies*

$$(\iota(u)\phi)^{n-1} \wedge \phi = n!\,|u|^{n-1}\,\mathrm{dvol} \tag{9}$$

for every $u \in V$. In particular, n is odd.

Proof. We prove (i). By Lemma 2.3 the bilinear form ω_u is skew symmetric and, by Lemma 2.4, we have $J_u \circ J_u = -\mathbb{1}$. Moreover,

$$\omega_u(v, J_u w) = \langle u \times v, u \times w\rangle = -\langle v, u \times (u \times w)\rangle = \langle v, w\rangle$$

for all $v, w \in V$. Here the first equation follows from the definition of ω_u and J_u, the second follows from (4), and the last from Lemma 2.4. Thus the dimension of W_u is even and so the dimension of V is odd.

We prove (ii). The set of all bases $(u, v_1, \dots, v_{2n}) \in V^{2n+1}$, where u has norm one and $v_1, \dots, v_{2n}$ is a symplectic basis of W_u, is connected. Hence there is a unique orientation of V with respect to which every such basis is positive. Let $\mathrm{dvol} \in \Lambda^{2n+1}V^*$ be the associated volume form. To prove equation (9) assume first that $|u| = 1$ and choose an orthonormal symplectic basis $v_1, \dots, v_{2n}$ of W_u. (For example pick an orthonormal basis $v_1, v_3, \dots, v_{2n-1}$ of a Lagrangian subspace of W_u and define $v_{2k} := J_u v_{2k-1}$ for $k = 1, \dots, n$.) Now evaluate both sides of the equation on the tuple $(u, v_1, \dots, v_{2n})$. Then we obtain $n!$ on both sides. This proves (9) whenever u has norm one. The general case follows by scaling. It follows from (9) that n is odd since otherwise the left hand side changes sign when we replace u by $-u$. This proves the lemma. □

Lemma 2.9. *Let $n > 1$ be an odd integer and V be an oriented real Hilbert space of dimension $2n+1$ with volume form* $\mathrm{dvol} \in \Lambda^{2n+1}V^*$. *Let $\phi \in \Lambda^3 V^*$ be a 3-form and denote its isotropy group by*

$$\mathrm{G} := \{g \in \mathrm{Aut}(V) \,|\, g^*\phi = \phi\}.$$

If ϕ satisfies (9) *then* $\mathrm{G} \subset \mathrm{SO}(V)$.

Proof. Fix an element $g \in \mathrm{G}$. Then it follows from (9) that, for every $u \in V$, we have

$$\begin{aligned}
|gu|^{n-1} g^*\mathrm{dvol} &= \frac{1}{n!} g^* \Big((\iota(gu)\phi)^{n-1} \wedge \phi \Big) \\
&= \frac{1}{n!} \Big((g^*\iota(gu)\phi)^{n-1} \wedge g^*\phi \Big) \\
&= \frac{1}{n!} (\iota(u) g^*\phi)^{n-1} \wedge g^*\phi \\
&= \frac{1}{n!} (\iota(u)\phi)^{n-1} \wedge \phi \\
&= |u|^{n-1} \mathrm{dvol}.
\end{aligned}$$

Hence there is a constant $c > 0$ such that

$$g^*\mathrm{dvol} = c^{-1}\mathrm{dvol}, \qquad |gu|^{n-1} = c\,|u|^{n-1}$$

for every $u \in V$. Since $n > 1$, this gives $|gu| = c^{\frac{1}{n-1}} |u|$ for $u \in V$ and hence

$$g^*\mathrm{dvol} = c^{\frac{2n+1}{n-1}} \mathrm{dvol} = c^{\frac{3n}{n-1}} g^*\mathrm{dvol}.$$

Thus $c = 1$ and this proves the lemma. □

Proof of Theorem 2.2. Assume $\dim V > 1$, let (1) be a cross product on V, and define $\phi : V \times V \times V \to \mathbb{R}$ by (5). By Lemma 2.3, we have $\phi \in \Lambda^3 V^*$. By Lemma 2.8 (i), the dimension of V is odd. By Lemma 2.9, we have $\dim V = 4n + 3$ for some integer $n \geq 0$. In particular $\dim V \neq 5$.

We prove that $\dim V \leq 7$. Define $A : V \to \mathrm{End}(V)$ by

$$A(u)v := u \times v.$$

Then it follows from Lemma 2.4 that

$$A(u)u = 0, \qquad A(u)^2 = uu^* - |u|^2 \mathbb{1}.$$

Define $\gamma : V \to \mathrm{End}(\mathbb{R} \oplus V)$ by

$$\gamma(u) := \begin{pmatrix} 0 & -u^* \\ u & A(u) \end{pmatrix}, \tag{10}$$

where $u^* : V \to \mathbb{R}$ denotes the linear functional $v \mapsto \langle u, v \rangle$. Then

$$\gamma(u)^* + \gamma(u) = 0, \qquad \gamma(u)^*\gamma(u) = 2\,|u|^2\, \mathbb{1} \tag{11}$$

for every $u \in V$. Here the first equation follows from the fact that $A(u)$ is skew-adjoint for every u and the last equation follows by direct calculation. This implies that γ extends to a linear map from the Clifford algebra $C(V)$ to $\mathrm{End}(\mathbb{R} \oplus V)$. Moreover, the restriction of this extension to the Clifford algebra of any even dimensional subspace of V is injective (see eg. [6, Proposition 4.13]). Hence

$$2^{2n} \leq (2n+2)^2.$$

This implies $n \leq 3$ and therefore $\dim V = 2n+1 \leq 7$.

Thus we have proved that the dimension of V is either 0, 1, 3, or 7. That the cross product vanishes in dimension 0 and 1 is obvious. That it is uniquely determined by the orientation of V in dimension 3 follows from Lemma 2.5. The last assertion is restated and proved in Theorem 3.2 below. ☐

Remark 2.10. Let V be a nonzero real Hilbert space that admits a 3-form ϕ whose isotropy subgroup G is contained in $\mathrm{SO}(V)$. Then

$$\dim \mathrm{Aut}(V) - \dim \Lambda^3 V^* \leq \dim \mathrm{G} \leq \dim \mathrm{SO}(V).$$

Hence $\dim V \geq 7$ as otherwise $\dim \mathrm{SO}(V) < \dim \mathrm{Aut}(V) - \dim \Lambda^3 V^*$. This gives another proof for the nonexistence of cross products in dimension 5.

3 Associative calibrations

Definition 3.1. *Let V be a real vector space. A 3-form $\phi \in \Lambda^3 V^*$ is called* **nondegenerate** *if, for every pair of linearly independent vectors $u, v \in V$, there is a vector $w \in V$ such that $\phi(u, v, w) \neq 0$. An inner product on V is called* **compatible** *with ϕ if the map* (1) *defined by* (5) *is a cross product.*

Theorem 3.2. *Let V be a 7-dimensional real vector space and $\phi, \phi' \in \Lambda^3 V^*$. Then the following holds.*

(i) *ϕ is nondegenerate if and only if it admits a compatible inner product.*

(ii) *The inner product in (i), if it exists, is uniquely determined by ϕ.*

(iii) *If ϕ, ϕ' are nondegenerate there is a $g \in \mathrm{Aut}(V)$ such that $g^*\phi = \phi'$.*

Remark 3.3. If $\dim V = 3$ then $\phi \in \Lambda^3 V^*$ is nondegenerate if and only if it is nonzero. If $\phi \neq 0$ then, by Lemma 2.5, an inner product on V is compatible with ϕ if and only if ϕ is the associated volume form with respect to some orientation, i.e. $\phi(u, v, w) = \pm 1$ for every orthonormal basis u, v, w of V. Thus assertion (i) of Theorem 3.2 continues to hold in dimension three. However, assertion (ii) is specific to dimension seven.

Lemma 3.4. *Let V be a 7-dimensional real Hilbert space and $\phi \in \Lambda^3 V^*$. Then the following are equivalent.*

(i) *ϕ is compatible with the inner product.*

(ii) *There is an orientation on V such that the associated volume form* $\mathrm{dvol} \in \Lambda^7 V^*$ *satisfies*

$$\iota(u)\phi \wedge \iota(v)\phi \wedge \phi = 6\langle u, v\rangle \mathrm{dvol} \tag{12}$$

for all $u, v \in V$.

Each of these conditions implies that ϕ is nondegenerate. Moreover, the orientation in (ii), if it exists, is uniquely determined by ϕ.

Remark 3.5. It is convenient to use equation (12) to verify that the bilinear map in Example 2.7 satisfies (3). In fact, it suffices to check (12) for every pair of standard basis vectors. Care must be taken. There are examples of 3-forms ϕ on $V = \mathbb{R}^7$ for which the quadratic form

$$V \times V \to \Lambda^7 V^* : (u, v) \mapsto \iota(u)\phi \wedge \iota(v)\phi \wedge \phi$$

has signature $(3, 4)$. One such example can be obtained from the 3-form ϕ_0 in Example 2.7 by changing the minus signs to plus.

Proof of Lemma 3.4. If (i) holds then, by Lemma 2.8 (ii), there is a unique orientation on V such that the associated volume form satisfies

$$\iota(u)\phi \wedge \iota(u)\phi \wedge \phi = 6\,|u|^2\,\mathrm{dvol}$$

for every $u \in V$. Applying this identity to $u+v$ and $u-v$ and taking the difference we obtain (12). Moreover, if $u, v \in V$ are linearly independent then $\phi(u, v, u \times v) = |u \times v|^2 = |u|^2\,|v|^2 - \langle u, v\rangle^2 \neq 0$. Hence ϕ is nondegenerate. This shows that (i) implies (ii) and nondegeneracy.

Conversely, assume (ii). We prove that ϕ is nondegenerate. Let $u, v \in V$ be linearly independent. Then $u \neq 0$ and hence, by (12), the 7-form

$$\sigma := \iota(u)\phi \wedge \iota(u)\phi \wedge \phi = 6\,|u|^2\,\mathrm{dvol} \in \Lambda^7 V^*$$

is nonzero. Choose a basis $v_1, \ldots, v_7$ of V with $v_1 = u$ and $v_2 = v$. Evaluating σ on this basis we obtain that one of the terms $\phi(u, v, v_j)$ with $j \geq 3$ must be nonzero. Hence ϕ is nondegenerate as claimed.

Now define the bilinear map $V \times V \to V : (u, v) \mapsto u \times v$ by (5). This map is skew-symmetric and, by Lemma 2.3, it satisfies (2). We must prove that it also satisfies (3). By Lemma 2.4, it suffices to show

$$|u| = 1, \;\; \langle u, v\rangle = 0 \qquad \Longrightarrow \qquad |u \times v| = |v|\,. \tag{13}$$

We prove this in five steps. Throughout we fix a unit vector $u \in V$.

Step 1. *Define the linear map* $A : V \to V$ *by* $Av := u \times v$. *Then* A *is skew-adjoint and its kernel is spanned by* u.

That A is skew-adjoint follows from the identity $\langle Av, w\rangle = \phi(u, v, w)$. That its kernel is spanned by u follows from the fact that ϕ is nondegenerate.

Step 2. *Let* A *be as in Step 1. Then there are positive constants* $\lambda_1, \lambda_2, \lambda_3$ *and an orthonormal basis* $v_1, w_1, v_2, w_2, v_3, w_3$ *of* $u^\perp$ *such that* $Av_j = \lambda_j w_j$ *and* $Aw_j = -\lambda_j v_j$ *for* $j = 1, 2, 3$.

By Step 1, there is a constant $\lambda > 0$ and a vector $v \in u^\perp$ such that

$$A^2 v = -\lambda^2 v, \qquad |v| = 1.$$

Denote $w := \lambda^{-1} Av$. Then $Av = \lambda w$, $Aw = -\lambda v$, w is orthogonal to v, and

$$|w|^2 = \lambda^{-2}\langle Av, Av\rangle = -\lambda^{-2}\langle v, A^2 v\rangle = |v|^2 = 1.$$

Moreover, the orthogonal complement of u, v, w is invariant under A. Hence Step 2 follows by induction.

Step 3. *Let* λ_i *be as in Step 2. Then* $\lambda_1\lambda_2\lambda_3 = 1$.

Let A be as in Step 1, denote $W := u^\perp$, and define $\omega : W \times W \to \mathbb{R}$ by

$$\omega(v, w) := \langle Av, w\rangle = \phi(u, v, w)$$

for $v, w \in W$. Then, by Step 1, $\omega \in \Lambda^2 W^*$ is a symplectic form. Moreover, $\omega(v_i, w_i) = \langle Av_i, w_i\rangle = \lambda_i$ for $i = 1, 2, 3$ while $\omega(v_i, w_j) = 0$ for $i \neq j$ and $\omega(v_i, v_j) = \omega(w_i, w_j) = 0$ for all i and j. Hence

$$\begin{aligned} \lambda_1\lambda_2\lambda_3 &= \frac{1}{6}\omega^3(v_1, w_1, v_2, w_2, v_3, w_3) \\ &= \mathrm{dvol}(u, v_1, w_1, v_2, w_2, v_3, w_3). \end{aligned}$$

Here the first equation follows from Step 2 and the definition of ω and the second equation follows from (12) with $u = v$ and $|u| = 1$. Since the vectors $u, v_1, w_1, v_2, w_2, v_3, w_3$ form an orthonormal basis of V the last expression must be plus or minus one. Since it is positive, Step 3 follows.

Step 4. *Denote*

$$\mathrm{G} := \{g \in \mathrm{Aut}(V) \,|\, g^*\phi = \phi\}, \qquad \mathrm{H} := \{g \in \mathrm{G} \,|\, gu = u\}. \qquad (14)$$

Then $\dim \mathrm{G} \geq 14$ *and* $\dim \mathrm{H} \geq 8$.

Since $\dim \mathrm{Aut}(V) = 49$ and $\dim \Lambda^3 V^* = 35$, the isotropy subgroup G of ϕ has dimension at least 14. Moreover, by Lemma 2.9, G acts on the sphere $S := \{v \in V \mid |v| = 1\}$ which has dimension 6. Thus the isotropy subgroup H of u under this action has dimension $\dim \mathrm{H} \geq \dim \mathrm{G} - \dim S \geq 14 - 6 = 8$. This proves Step 2.

Step 5. *Let λ_i be as in Step 2. Then $\lambda_1 = \lambda_2 = \lambda_3 = 1$.*

By definition of A in Step 1 and H in Step 4, we have $\langle Agv, gw\rangle = \langle Av, w\rangle$ for all $g \in \mathrm{H}$ and all $v, w \in V$. Moreover, $\mathrm{H} \subset \mathrm{SO}(V)$, by Lemma 2.9. Hence

$$g \in \mathrm{H} \qquad \Longrightarrow \qquad gA = Ag. \tag{15}$$

Now suppose that the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are not all equal. Without loss of generality, we may assume $\lambda_1 \notin \{\lambda_2, \lambda_3\}$. Then, by (15), the subspaces $W_1 := \mathrm{span}\{v_1, w_1\}$ and $W_{23} := \mathrm{span}\{v_2, w_2, v_3, w_3\}$ are preserved by each element $g \in \mathrm{H}$. Thus $\mathrm{H} \subset \mathrm{O}(W_1) \times \mathrm{O}(W_{23})$. Since $\dim \mathrm{O}(W_1) = 1$ and $\dim \mathrm{O}(W_{23}) = 6$, this implies $\dim \mathrm{H} \leq 7$ in contradiction to Step 4. Thus we have proved that $\lambda_1 = \lambda_2 = \lambda_3$ and, by Step 3, this implies $\lambda_j = 1$ for every j. This proves Step 5.

By Steps 2 and 5 we have $A^2 v = -v$ for every $v \in u^\perp$. Hence, by Step 1, $|Av|^2 = -\langle v, A^2 v\rangle = |v|^2$ for every $v \in u^\perp$. By definition of A, this proves (13) and the lemma. □

Proof of Theorem 3.2 (i) and (ii). The "if" part of (i) is the last assertion of Lemma 3.4. To prove (ii) and the "only if" part of (i) we assume that ϕ is nondegenerate. Then, for every nonzero vector $u \in V$, the restriction of the 2-form $\iota(u)\phi \in \Lambda^2 V^*$ to $u^\perp$ is a symplectic form. Namely, if $v \in u^\perp$ is nonzero, then u, v are linearly independent and hence there is a vector $w \in V$ such that $\phi(u, v, w) \neq 0$; the vector w can be chosen orthogonal to u. This implies that the restriction of the 6-form $(\iota(u)\phi)^3 \in \Lambda^6 V^*$ to $u^\perp$ is nonzero for every nonzero vector $u \in V$. Hence the 7-form $\iota(u)\phi \wedge \iota(u)\phi \wedge \phi \in \Lambda^7 V^*$ is nonzero for every nonzero vector $u \in V$. Since $V \setminus \{0\}$ is connected, there is a unique orientation of V such that $\iota(u)\phi \wedge \iota(u)\phi \wedge \phi$ is a positive volume form on V for every $u \in V \setminus \{0\}$. Fix a volume form $\sigma \in \Lambda^7 V^*$ compatible with this orientation. Then the bilinear form

$$V \times V \to \mathbb{R} : (u, v) \mapsto \frac{\iota(u)\phi \wedge \iota(u)\phi \wedge \phi}{\sigma} =: g(u, v)$$

is an inner product. Define $\mu > 0$ by $\sigma = \mu \mathrm{dvol}_g$. Replacing σ by $\widetilde{\sigma} := \lambda^2 \sigma$ we get $\widetilde{g} = \lambda^{-2} g$ and $\mathrm{dvol}_{\widetilde{g}} = \lambda^{-7} \mathrm{dvol}_g$. Thus

$$\widetilde{\sigma} = \lambda^2 \sigma = \lambda^2 \mu \mathrm{dvol}_g = \lambda^9 \mu \mathrm{dvol}_{\widetilde{g}}.$$

With $\lambda := (6/\mu)^{1/9}$ we get $\widetilde{\sigma} = 6\mathrm{dvol}_{\widetilde{g}}$. Thus we have proved that there is a unique orientation and inner product on V such that ϕ satisfies (12). Hence the assertion follows from Lemma 3.4. This proves (ii). ☐

Remark 3.6. Here is a sketch of another uniqueness proof for assertion (ii) in Theorem 3.2. Suppose V is a 7-dimensional real Hilbert space, $\phi \in \Lambda^3 V^*$ is nondegenerate and compatible with the inner product, the map (1) is defined by (5), and $A : V \to \mathrm{End}(V)$ is defined by $A(u)v := u \times v$. Suppose $Q = Q^* : V \to V$ is positive definite and the inner product $(u, v) \mapsto \langle u, Qv\rangle$ is also compatible with ϕ. Then the associated cross product is given by $(u, v) \mapsto u \times_Q v = Q^{-1}A(u)v$. Then, for every $u \in V$, we have

$$A(Qu) = Q^{-1}A(u)Q^{-1}.$$

(Verify this first for the eigenvectors of Q.) It follows that, for any two eigenvalues λ, μ of Q with eigenvectors u, v also $(\lambda\mu)^{-1}$ must be an eigenvalue of Q with eigenvector $u \times v$ and that the multiplicity $m((\lambda\mu)^{-1})$ is bounded below by $m(\lambda)$ when $\lambda \neq \mu$ and by $m(\lambda) - 1$ when $\lambda = \mu$. By considering the extremal eigenvalues $\lambda_{\min}$ and $\lambda_{\max}$ of Q one then finds that they have multiplicities at most 2 (if they are distinct) and that the only other eigenvalue is $\lambda := (\lambda_{\min}\lambda_{\max})^{-1}$ with multiplicity at least 3. This leads to a contradiction unless $Q = \mathbb{1}$.

Lemma 3.7. *Let V be a 7-dimensional real Hilbert space equipped with a cross product $V \times V \to V : (u, v) \to u \times v$. If u and v are orthonormal and $w := u \times v$ then $v \times w = u$ and $w \times u = v$.*

Proof. This follows immediately from equation (6) in Lemma 2.4. ☐

Proof of Theorem 3.2 (iii). Let $\phi_0 : \mathbb{R}^7 \times \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}$ be the 3-form in Example 2.7 and let $\phi \in \Lambda^3 V^*$ be a nondegenerate 3-form. Let V be equipped with the compatible inner product of Theorem 3.2 and denote by $V \times V \to V : (u, v) \mapsto u \times v$ the associated cross product. Let $e_1, e_2 \in V$ be orthonormal and define

$$e_3 := e_1 \times e_2.$$

Let $e_4 \in V$ be any unit vector orthogonal to e_1, e_2, e_3 and define

$$e_5 := -e_1 \times e_4.$$

Then e_5 has norm one and is orthogonal to e_1, e_2, e_3, e_4. For e_1 and e_4 this follows from the definition and (4). For e_3 we observe

$$\langle e_3, e_5\rangle = -\langle e_1 \times e_2, e_1 \times e_4\rangle = \langle e_2, e_1 \times (e_1 \times e_4)\rangle = -\langle e_2, e_4\rangle = 0.$$

Here the last but one equation follows from Lemma 2.4. For e_2 the argument is similar; since $e_2 = e_3 \times e_1$, by Lemma 3.7, and $\langle e_3, e_4\rangle = 0$, we obtain $\langle e_2, e_5\rangle = 0$. Now let e_6 be a unit vector orthogonal to $e_1, \dots, e_5$ and define

$$e_7 := -e_1 \times e_6.$$

As before we have that e_7 has norm one and is orthogonal to $e_1, \dots, e_6$. Thus the vectors $e_1, \dots, e_7$ form an orthonormal basis of V and it follows from Lemma 3.7 that they satisfy the same relations as the standard basis of $\mathbb{R}^7$ in Example 2.7. Hence the map

$$\mathbb{R}^7 \xrightarrow{g} V : x = (x_1, \dots, x_7) \mapsto \sum_{i=1}^{7} x_i e_i$$

is a Hilbert space isometry and it satisfies $g^*\phi = \phi_0$. This proves Theorem 3.2 (and the last assertion of Theorem 2.2). □

4 The associator and coassociator brackets

We assume throughout that V is a 7-dimensional real Hilbert space, that $\phi \in \Lambda^3 V^*$ is a nondegenerate 3-form compatible with the inner product, and (1) is the cross product given by (5). It follows from (6) that the expression $(u \times v) \times w$ is alternating on any triple of pairwise orthogonal vectors $u, v, w \in V$. Hence it extends uniquely to an alternating 3-form $V^3 \to V : (u, v, w) \mapsto [u, v, w]$ called the **associator bracket**. An explicit formula for this 3-form is

$$[u, v, w] := (u \times v) \times w + \langle v, w\rangle u - \langle u, w\rangle v. \tag{16}$$

The associator bracket can also be expressed in the form

$$[u, v, w] = \frac{1}{3}\Big((u \times v) \times w + (v \times w) \times u + (w \times u) \times v\Big). \tag{17}$$

Remark 4.1. If V is any Hilbert space with a skew-symmetric bilinear form (1), then the associator bracket (16) is alternating iff (6) holds. Indeed, skew-symmetry of the associator bracket in the first two arguments is obvious, and the identity

$$\begin{aligned}[u, v, w] + [u, w, v] &= w \times (v \times u) + v \times (w \times u) \\ &\quad - \langle u, w\rangle v - \langle u, v\rangle w + 2\langle v, w\rangle u\end{aligned}$$

shows that skew-symmetry in the last two arguments is equivalent to (6). By Lemma 2.5, the associator bracket vanishes in dimension three.

The square of the volume of the 3-dimensional parallelepiped spanned by $u, v, w \in V$ will be denoted by

$$|u \wedge v \wedge w|^2 := \det \begin{pmatrix} |u|^2 & \langle u, v \rangle & \langle u, w \rangle \\ \langle v, u \rangle & |v|^2 & \langle v, w \rangle \\ \langle w, u \rangle & \langle w, v \rangle & |w|^2 \end{pmatrix}.$$

Lemma 4.2. *For all $u, v, w \in V$ we have*

$$\phi(u, v, w)^2 + |[u, v, w]|^2 = |u \wedge v \wedge w|^2. \tag{18}$$

Proof. If w is orthogonal to u and v we have

$$\begin{aligned} |[u, v, w]|^2 &= |(u \times v) \times w|^2 \\ &= |u \times v|^2 |w|^2 - \langle u, v \times w \rangle^2 \\ &= |u \wedge v \wedge w|^2 - \phi(u, v, w)^2. \end{aligned}$$

Here the first equation follows from the definition of the associator bracket and orthogonality, the second equation follows from (3), and the last equation follows from (3) and orthogonality, as well as (5). The general case can be reduced to the orthogonal case by Gram-Schmidt. □

Definition 4.3. *A 3-dimensional subspace $\Lambda \subset V$ is called* **associative** *if $[u, v, w] = 0$ for all $u, v, w \in \Lambda$.*

Lemma 4.4. *Let $\Lambda \subset V$ be a 3-dimensional linear subspace. Then the following are equivalent.*

(i) *Λ is associative.*

(ii) *If u, v, w is an orthonormal basis of Λ then $\phi(u, v, w) = \pm 1$.*

(iii) *If $u, v \in \Lambda$ then $u \times v \in \Lambda$.*

(iv) *If $u \in \Lambda^\perp$ and $v \in \Lambda$ then $u \times v \in \Lambda^\perp$.*

(v) *If $u, v \in \Lambda^\perp$ then $u \times v \in \Lambda$.*

Moreover, if $u, v \in V$ are linearly independent then the subspace spanned by $u, v, u \times v$ is associative.

Proof. That (i) is equivalent to (ii) follows from Lemma 4.2.

We prove that (i) is equivalent to (iii). That the associator bracket vanishes on a 3-dimensional subspace that is invariant under the cross product follows from Lemma 2.5 (iii). Conversely suppose that the associator bracket vanishes on Λ. Let $u, v \in \Lambda$ be linearly independent and let $w \in \Lambda$ be a nonzero vector orthogonal to u and v. Then, by Lemma 4.2, we have

$$\langle u \times v, w \rangle^2 = \phi(u, v, w)^2 = |u \wedge v \wedge w|^2 = |u \times v|^2 \, |w|^2$$

and hence $u \times v$ is a real multipe of w. Thus $u \times v \in \Lambda$.

We prove that (iii) is equivalent to (iv). First assume (iii) and let $u \in \Lambda$, $v \in \Lambda^\perp$. Then, by (iii), we have $w \times u \in \Lambda$ for every $w \in \Lambda$. Hence $\langle w, u \times v \rangle = \langle w \times u, v \rangle = 0$ for every $w \in \Lambda$ and so $u \times v \in \Lambda^\perp$. Conversely assume (iv) and let $u, v \in \Lambda$. Then, by (iii), we have $w \times u \in \Lambda^\perp$ for every $w \in \Lambda^\perp$. Hence $\langle w, u \times v \rangle = \langle w \times u, v \rangle = 0$ for every $w \in \Lambda^\perp$. This implies $u \times v \in \Lambda$. Thus we have proved that (iii) is equivalent to (iv).

We prove that (iv) is equivalent to (v). Fix a unit vector $u \in \Lambda^\perp$ and define the endomorphism $J : u^\perp \to u^\perp$ by $Jv := u \times v$. By Lemma 2.4 this is an isomorphism with inverse $-J$. Condition (iv) asserts that J maps Λ to $\Lambda^\perp \cap u^\perp$ while condition (v) asserts that J maps $\Lambda^\perp \cap u^\perp$ to Λ. Since both are 3-dimensional subspaces of $u^\perp$ these two assertions are equivalent. This proves that (iv) is equivalent to (v).

If u and v are linearly independent then $u \times v \neq 0$, by (3), and $u \times v$ is orthogonal to u and v, by (2). Hence the subspace Λ spanned by $u, v, u \times v$ is 3-dimensional. That it is invariant under the cross product follows from assertion (iv) in Lemma 2.4. Hence Λ is associative. □

Lemma 4.5. *The map* $\psi : V^4 \to \mathbb{R}$ *defined by*

$$\begin{aligned} \psi(u, v, w, x) &:= \langle [u, v, w], x \rangle \\ &= \frac{1}{3}\Big(\phi(u \times v, w, x) + \phi(v \times w, u, x) + \phi(w \times u, v, x) \Big) \end{aligned} \tag{19}$$

is an alternating 4*-form (the* **coassociative calibration** *of* V*). Moreover,* $\psi = *\phi$ *where* $* : \Lambda^k V^* \to \Lambda^{7-k} V^*$ *denotes the Hodge* $*$*-operator associated to the inner product and the orientation in Lemma 3.4.*

Remark 4.6. The standard associative calibration on $\mathbb{R}^7$ is

$$\phi_0 = e^{123} - e^{145} - e^{167} - e^{246} + e^{257} - e^{347} - e^{356} \tag{20}$$

(see Example 2.7). The corresponding coassociative calibration is

$$\psi_0 = -e^{1247} - e^{1256} + e^{1346} - e^{1357} - e^{2345} - e^{2367} + e^{4567}. \tag{21}$$

Remark 4.7. Let $u \in V$ and denote by $A_u \in \mathfrak{so}(V)$ the skew-adjoint endomorphism given by $A_u v := u \times v$. Then equation (19) can be expressed in the form

$$\mathcal{L}_{A_u}\phi := \phi(A_u\cdot,\cdot,\cdot) + \phi(\cdot,A_u\cdot,\cdot) + \phi(\cdot,\cdot,A_u\cdot) = 3\iota(u)\psi. \tag{22}$$

Here $\mathcal{L}_\xi\phi$ denotes the infinitesimal action of $\xi \in \mathfrak{so}(V)$ on ϕ. Using the formula $\psi = *\phi$ and equation (24) below we obtain

$$\mathcal{L}_{A_u}\psi = *_V(3\iota(u)\psi) = -3u^* \wedge \phi. \tag{23}$$

Remark 4.8. Let $V \to V^* : u \mapsto u^* := \langle u, \cdot\rangle$ be the isomorphism induced by the inner product. Then, for $\alpha \in \Lambda^k V^*$ and $u \in V$, we have

$$*\iota(u)\alpha = (-1)^{k-1} u^* \wedge *\alpha. \tag{24}$$

This holds on any finite dimensional oriented Hilbert space.

Proof of Lemma 4.5. It follows from Remark 4.1 that ψ is alternating in the first three arguments. To prove that $\psi \in \Lambda^4 V^*$ we compute

$$\begin{array}{rcl} \psi(u,v,w,x) & = & \langle (u\times v)\times w + \langle v,w\rangle u - \langle u,w\rangle v, x\rangle \\ & = & \langle u\times v, w\times x\rangle + \langle v,w\rangle\langle u,x\rangle - \langle u,w\rangle\langle v,x\rangle. \end{array}$$

Swapping x and w as well as u and v gives the same expression. Thus

$$\psi(u,v,w,x) = \psi(v,u,x,w) = -\psi(u,v,x,w).$$

This shows that $\psi \in \Lambda^4 V^*$ as claimed. To prove the second assertion we observe the following.

Claim. *If u, v, w, x are orthonormal and $u\times v = w\times x$ then $\psi(u,v,w,x) = 1$.*

This claim follows directly from the definition of ψ and of the associator bracket in (16) and (19). Now, by Theorem 3.2, we can restrict attention to the standard structures on $\mathbb{R}^7$. Thus $\phi = \phi_0$ is given by (20) and this 3-form is compatible with the standard inner product on $\mathbb{R}^7$. We have the product rule $e_i \times e_j = e_k$ whenever the term e^{ijk} or one of its cyclic permutations shows up in this sum, and the claim shows that we have a summand $\varepsilon e^{ijk\ell}$ in $\psi = \psi_0$ whenever $e_i \times e_j = \varepsilon e_k \times e_\ell$ with $\varepsilon \in \{\pm 1\}$. Hence ψ_0 is given by (21). Term by term inspection shows that $\psi_0 = *\phi_0$. This proves the lemma. □

Lemma 4.9. *For all* $u, v, w, x \in V$ *we have*

$$\begin{aligned} [u,v,w,x] &:= \phi(u,v,w)x - \phi(x,u,v)w + \phi(w,x,u)v - \phi(v,w,x)u \\ &= \frac{1}{3}\Big(-[u,v,w] \times x + [x,u,v] \times w - [w,x,u] \times v + [v,w,x] \times u\Big). \end{aligned} \tag{25}$$

The resulting multi-linear map $V^4 \to V : (u,v,w,x) \mapsto [u,v,w,x]$ *is alternating and is called the* **coassociator bracket** *on* V.

Proof. Define the alternating multi-linear map $\tau : V^4 \to V$ by

$$\begin{aligned} \tau(u,v,w,x) := 3\Big(&\phi(u,v,w)x - \phi(x,u,v)w + \phi(w,x,u)v - \phi(v,w,x)u\Big) \\ &+ [u,v,w] \times x - [x,u,v] \times w + [w,x,u] \times v - [v,w,x] \times u. \end{aligned}$$

We must prove that τ vanishes. The proof has three steps.

Step 1. $\tau(u,v,w,x)$ *is orthogonal to* u, v, w, x *for all* $u, v, w, x \in V$.

It suffices to assume that u, v, w, x are pairwise orthogonal. Under this assumption we have $[u,v,w] = (u \times v) \times w$ and similarly for $[x,v,w]$ etc. Hence

$$\begin{aligned} \langle \tau(u,v,w,x), x\rangle &= 3\,|x|^2\,\phi(u,v,w) - \langle [u,v,x], w \times x\rangle \\ &\quad - \langle [w,u,x], v \times x\rangle - \langle [v,w,x], u \times x\rangle \\ &= 3\,|x|^2\,\phi(u,v,w) - \langle (u \times v) \times x, w \times x\rangle \\ &\quad - \langle (w \times u) \times x, v \times x\rangle - \langle (v \times w) \times x, u \times x\rangle \\ &= 0. \end{aligned}$$

Here the last step uses the identity (4) and the fact that $x \times (u \times x) = |x|^2 u$ whenever u is orthogonal to x. Thus $\tau(u,v,w,x)$ is orthogonal to x. Since τ is alternating this proves Step 1.

Step 2. $\tau(u,v,u \times v,x) = 0$ *for all* $u, v, x \in V$.

It suffices to assume that u, v are orthonormal and that x is orthogonal to u, v, and $w := u \times v$. Then $v \times w = u$, $w \times u = v$, $\phi(u,v,w) = 1$, and $\phi(x,v,w) = \phi(x,w,u) = \phi(x,u,v) = 0$. Moreover, $[u,v,w] = 0$ and

$$[x,v,w] = [v,w,x] = (v \times w) \times x = u \times x, \qquad [x,v,w] \times u = x,$$

and similarly $[x,w,u] = [x,u,v] = x$. This implies that $\tau(u,v,w,x) = 0$.

Step 3. $\tau(u,v,w,x) = 0$ *for all* $u, v, w, x \in V$.

By the alternating property we may assume that u and v are orthonormal. Using the alternating property again and Step 2 we may assume that w is a unit vector orthogonal to $u, v, u \times v$ and that x is a unit vector orthogonal to u, v, w and $v \times w, w \times u, u \times v$. This implies that

$$\phi(u, v, w) = \phi(x, v, w) = \phi(x, w, u) = \phi(x, u, v) = 0.$$

Hence the vectors $x \times u, x \times v, x \times w$ form a basis of the orthogonal complement of the space spanned by u, v, w, x. Each of these vectors is orthogonal to $\tau(u, v, w, x)$ and hence it follows from Step 1 that $\tau(u, v, w, x) = 0$. This proves the lemma. $\square$

The square of the volume of the 4-dimensional parallelepiped spanned by $u, v, w, x \in V$ will be denoted by

$$|u \wedge v \wedge w \wedge x|^2 := \det \begin{pmatrix} |u|^2 & \langle u, v \rangle & \langle u, w \rangle & \langle u, x \rangle \\ \langle v, u \rangle & |v|^2 & \langle v, w \rangle & \langle v, x \rangle \\ \langle w, u \rangle & \langle w, v \rangle & |w|^2 & \langle w, x \rangle \\ \langle x, u \rangle & \langle x, v \rangle & \langle x, w \rangle & |x|^2 \end{pmatrix}.$$

Lemma 4.10. *For all $u, v, w, x \in V$ we have*

$$\psi(u, v, w, x)^2 + |[u, v, w, x]|^2 = |u \wedge v \wedge w \wedge x|^2 . \tag{26}$$

Proof. **Step 1.** *If u, v, w, x are orthogonal then*

$$\begin{aligned} \psi(u, v, w, x)^2 &= \langle u \times v, w \times x \rangle^2, \\ |[u, v, w, x]|^2 &= \langle u \times v, w \rangle^2 |x|^2 + \langle u \times v, x \rangle^2 |w|^2 \\ &\quad + \langle u, w \times x \rangle^2 |v|^2 + \langle v, w \times x \rangle^2 |u|^2 , \\ |u \wedge v \wedge w \wedge x|^2 &= |u|^2 |v|^2 |w|^2 |x|^2 . \end{aligned}$$

The first equation follows from (16) and (19), using (4). The other two equations follow immediately from the definitions.

Step 2. *Equation* (26) *holds when u, v, w, x are orthogonal and, in addition, w and x are orthogonal to $u \times v$.*

If $u \times v = 0$ both sides of equation (26) vanish. Otherwise, it follows from the assumptions and Lemma 4.4 that $w \times x$ is a linear combination of the vectors u, v, $u \times v$. Hence the assertion follows from Step 1.

Step 3. *Equation* (26) *holds when u, v, w, x are orthogonal.*

Suppose, in addition, that w and x are orthogonal to $u \times v$ and replace x by $x_\lambda := x + \lambda u \times v$ for $\lambda \in \mathbb{R}$. Then $\psi(u, v, w, x_\lambda)$ is independent of λ and

$$|[u, v, w, x_\lambda]|^2 = |[u, v, w, x]|^2 + \lambda^2 |u|^2 |v|^2 |w|^2 |u \times v|^2 .$$

Hence it follows from Step 2 that (26) holds when u, v, w, x are orthogonal and, in addition, w is orthogonal to $u \times v$. This condition can be achieved by rotating the pair (w, x). This proves Step 3.

Step 4. *Equation* (26) *holds always.*

The general case follows from the orthogonal case via Gram-Schmidt, because both sides of equation (26) remain unchanged if we add to any of the four vectors a multiple of any of the other three. This proves the lemma. □

Definition 4.11. *A* 4*-dimensional subspace* $H \subset V$ *is called* **coassociative** *if* $[u, v, w, x] = 0$ *for all* $u, v, w, x \in H$*.*

Lemma 4.12. *Let* $H \subset V$ *be a* 4*-dimensional linear subspace. Then the following are equivalent.*

(i) H *is coassociative.*

(ii) *If* u, v, w, x *is an orthonormal basis of* H *then* $\psi(u, v, w, x) = \pm 1$*.*

(iii) *For all* $u, v, w \in H$ *we have* $\phi(u, v, w) = 0$*.*

(iv) *If* $u, v \in H$ *then* $u \times v \in H^\perp$*.*

(v) *If* $u \in H$ *and* $v \in H^\perp$ *then* $u \times v \in H$*.*

(vi) *If* $u, v \in H^\perp$ *then* $u \times v \in H^\perp$*.*

(vii) *The orthogonal complement* $H^\perp$ *is associative.*

Proof. That (i) is equivalent to (ii) follows from Lemma 4.10.

We prove that (i) is equivalent to (iii). That (iii) implies (i) is obvious by definition of the coassociator bracket in (25). Conversely, assume (i) and choose a basis u, v, w, x of H. Then $[u, v, w, x] = 0$ and hence, by (25), we have $\phi(u, v, w) = \phi(x, v, w) = \phi(x, w, u) = \phi(x, u, v) = 0$. This implies (iii).

We prove that (iii) is equivalent to (iv). If (iii) holds and $u, v \in H$ then $\langle u \times v, w \rangle = \phi(u, v, w) = 0$ for every $w \in H$ and hence $u \times v \in H^\perp$. Conversely, if (iv) holds and $u, v \in H$ then $u \times v \in H^\perp$ and hence $\phi(u, v, w) = \langle u \times v, w \rangle = 0$ for all $w \in H$.

Thus we have proved that (i), (ii), (iii), (iv) are equivalent. That assertions (iv), (v), (vi), (vii) are equivalent was proved in Lemma 4.4. □

5 Normed algebras

Definition 5.1. *A* **normed algebra** *consists of a finite dimensional real Hilbert space W, a bilinear map*

$$W \times W \to W : (u, v) \mapsto uv,$$

(called the **product***), and a unit vector $1 \in W$ (called the* **unit***), satisfying*

$$1u = u1 = u$$

and

$$|uv| = |u|\,|v| \tag{27}$$

for all $u, v \in W$.

When W is a normed algebra it is convenient to identify the real numbers with a subspace of W via multiplication with the unit 1. Thus, for $u \in W$ and $\lambda \in \mathbb{R}$, we write $u + \lambda$ instead of $u + \lambda 1$. Define an involution $W \to W : u \mapsto \bar{u}$ (called **conjugation**) by $\bar{1} := 1$ and $\bar{u} := -u$ for $u \in 1^\perp$. Thus

$$\bar{u} := 2\langle u, 1\rangle - u. \tag{28}$$

We think of $\mathbb{R} \subset W$ as the real part of W and of its orthogonal complement as the imaginary part. The real and imaginary parts of $u \in W$ will be denoted by $\mathrm{Re}\, u := \langle u, 1\rangle$ and $\mathrm{Im}\, u := u - \langle u, 1\rangle$.

Theorem 5.2. *Normed algebras and vector spaces with cross products are related as follows.*

(i) *If W is a normed algebra then $V := 1^\perp$ is equipped with a cross product $V \times V \to V : (u, v) \mapsto u \times v$ defined by*

$$u \times v := uv + \langle u, v\rangle \tag{29}$$

for $u, v \in 1^\perp$.

(ii) *If V is a finite dimensional Hilbert space equipped with a cross product then $W := \mathbb{R} \oplus V$ is a normed algebra with*

$$uv := u_0 v_0 - \langle u_1, v_1\rangle + u_0 v_1 + v_0 u_1 + u_1 \times v_1 \tag{30}$$

for $u = u_0 + u_1, v = v_0 + v_1 \in \mathbb{R} \oplus V$. Here we identify a real number λ with the pair $(\lambda, 0) \in \mathbb{R} \oplus V$ and a vector $v \in V$ with the pair $(0, v) \in \mathbb{R} \oplus V$.

These constructions are inverses of each other. In particular, a normed algebra has dimension 1, 2, 4, or 8 and is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$.

Lemma 5.3. *Let W be a normed algebra. Then the following holds.*
(i) *For all $u, v, w \in W$ we have*

$$\langle uv, w\rangle = \langle v, \bar{u}w\rangle, \qquad \langle uv, w\rangle = \langle u, w\bar{v}\rangle. \tag{31}$$

(ii) *For all $u, v \in W$ we have*

$$u\bar{u} = |u|^2, \qquad u\bar{v} + v\bar{u} = 2\langle u, v\rangle. \tag{32}$$

(iii) *For all $u, v \in W$ we have*

$$\langle u, v\rangle = \langle \bar{u}, \bar{v}\rangle, \qquad \overline{uv} = \bar{v}\bar{u}. \tag{33}$$

(iv) *For all $u, v, w \in W$ we have*

$$u(\bar{v}w) + v(\bar{u}w) = 2\langle u, v\rangle w, \qquad (u\bar{v})w + (u\bar{w})v = 2\langle v, w\rangle u \tag{34}$$

Proof. We prove (i). The first equation in (31) is obvious when u is a real multiple of 1. Hence it suffices to assume that u is orthogonal to 1. Expanding the identities $|uv + uw|^2 = |u|^2 |v + w|^2$ and $|uv + wv|^2 = |u + w|^2 |v|^2$ we obtain the equations

$$\langle uv, uw\rangle = |u|^2 \langle v, w\rangle, \qquad \langle uv, wv\rangle = \langle u, w\rangle |v|^2. \tag{35}$$

If u is orthogonal to 1 the first equation in (35) gives

$$\langle uv, w\rangle + \langle v, uw\rangle = \langle (1 + u)v, (1 + u)w\rangle - (1 + |u|^2)\langle v, w\rangle = 0.$$

Since $\bar{u} = -u$ for $u \in 1^\perp$ this proves the first equation in (31). The proof of the second equation is similar.

We prove (ii). Using the second equation in (31) with $v = \bar{u}$ we obtain $\langle u\bar{u}, w\rangle = \langle u, wu\rangle = \langle 1, w\rangle |u|^2$. Here we have used the second equation in (35). This implies $u\bar{u} = |u|^2$ for every $u \in W$. Replacing u by $u + v$ gives $u\bar{v} + v\bar{u} = 2\langle u, v\rangle$. This proves (32).

We prove (iii). That conjugation is an isometry follows immediately from the definition. Using (32) with v replaced by $\bar{v}$ we obtain

$$\bar{v}\bar{u} = 2\langle u, \bar{v}\rangle - uv = 2\langle uv, 1\rangle - uv = \overline{uv}.$$

Here the second equation follows from (31). This proves (33).

We prove (iv). For all $u, w \in W$ we have

$$\langle u(\bar{u}w), w\rangle = |\bar{u}w|^2 = |\bar{u}|^2 |w|^2 = |u|^2 |w|^2 \tag{36}$$

Since the operator $w \mapsto u(\bar{u}w)$ is self-adjoint, by (31), this shows that $u(\bar{u}w) = |u|^2 w$ for all $u, w \in W$. Replacing u by $u + v$ we obtain the first equation in (34). The proof of the second equation is similar. □

Proof of Theorem 5.2. Let W be a normed algebra. It follows from (31) that $\langle u,v\rangle = -\langle uv,1\rangle$ and hence $u\times v := uv + \langle u,v\rangle \in 1^\perp$ for all $u,v\in 1^\perp$. We write an element of W as $u = u_0 + u_1$ with $u_0 := \langle u,1\rangle \in \mathbb{R}$ and $u_1 := u - \langle u,1\rangle \in V = 1^\perp$. For $u,v\in W$ we compute

$$\begin{aligned}
|u|^2\,|v|^2 - |uv|^2 &= \left(u_0^2 + |u_1|^2\right)\left(v_0^2 + |v_1|^2\right) - (u_0v_0 - \langle u_1,v_1\rangle)^2 \\
&\quad - |u_0v_1 + v_0u_1 + u_1\times v_1|^2 \\
&= u_0^2\,|v_1|^2 + v_0^2\,|u_1|^2 + 2u_0v_0\langle u_1,v_1\rangle + |u_1|^2\,|v_1|^2 - \langle u_1,v_1\rangle^2 \\
&\quad - |u_0v_1 + v_0u_1|^2 - |u_1\times v_1|^2 - 2\langle u_0v_1 + v_0u_1, u_1\times v_1\rangle \\
&= |u_1|^2\,|v_1|^2 - \langle u_1,v_1\rangle^2 - |u_1\times v_1|^2 \\
&\quad -\ 2u_0\langle v_1, u_1\times v_1\rangle - 2v_0\langle u_1, u_1\times v_1\rangle.
\end{aligned}$$

The right hand side vanishes for all u and v if and only if the product on V satisfies (2) and (3). Hence (29) defines a cross product on V and the product can obviously be recovered from the cross product via (30). Conversely, the same argument shows that, if V is equipped with a cross product, the formula (30) defines a normed algebra structure on $W := \mathbb{R}\oplus V$. Moreover, by Theorem 2.2, V has dimension 0, 1, 3, or 7. This proves the theorem. □

Remark 5.4. If W is a normed algebra and the cross product on $V := 1^\perp$ is defined by (29) then the commutator of two element $u,v\in W$ is given by

$$[u,v] := uv - vu = 2u_1\times v_1. \tag{37}$$

In particular, the product on W is commutative in dimensions 1 and 2 and is not commutative in dimensions 4 and 8.

Remark 5.5. Let W be a normed algebra of dimension 4 or 8. Then $V := 1^\perp$ has a natural orientation determined by Lemma 2.5 or Lemma 3.4, respectively, in dimensions 3 and 7. We orient W as $\mathbb{R}\oplus V$.

Remark 5.6. If W is a normed algebra and the cross product on $V := 1^\perp$ is defined by (29) then the associator bracket on V is related to the product on W by

$$(uv)w - u(vw) = 2[u_1,v_1,w_1] \tag{38}$$

for all $u,v,w\in W$. Thus W is an associative algebra in dimensions $1,2,4$ and is not associative in dimension 8. The formula (38) is the reason for the term *associator bracket*. Many authors actually define the associator bracket as the left hand side of equation (38) (see for example [4]).

To prove (38), we observe that the associator bracket on V can be written in the form

$$\begin{aligned} 2[u,v,w] &= 2(u\times v)\times w + 2\langle v,w\rangle u - 2\langle u,w\rangle v \\ &= (u\times v)\times w - u\times(v\times w) + \langle v,w\rangle u - \langle u,v\rangle w \end{aligned} \tag{39}$$

for $u,v,w\in V$. Here the first equation follows from (16) and the second from (6). For $u,v,w\in V$ we compute

$$\begin{aligned} (uv)w - u(vw) &= (-\langle u,v\rangle + u\times v)w - u(-\langle v,w\rangle + v\times w) \\ &= (u\times v)\times w - u\times(v\times w) - \langle u,v\rangle w + \langle v,w\rangle u \\ &= 2[u,v,w]. \end{aligned}$$

Here the first equation follows from the definition of the cross product in (29), the second equation follows by applying (29) again and using (4), and the last equation follows from (39). Now, if any of the factors u,v,w is a real number, the term on the left vanishes. Hence real parts can be added to the vectors without changing the expression.

Theorem 5.7. *Let W be an 8-dimensional normed algebra.*
(i) *The map $W^3\to W:(u,v,w)\mapsto u\times v\times w$ defined by*

$$u\times v\times w := \frac{1}{2}\big((u\bar{v})w - (w\bar{v})u\big) \tag{40}$$

(called the **triple cross product** *of W) is alternating and satisfies*

$$\langle x, u\times v\times w\rangle + \langle u\times v\times x, w\rangle = 0, \tag{41}$$

$$|u\times v\times w| = |u\wedge v\wedge w|, \tag{42}$$

for all $u,v,w,x\in W$ and

$$\langle e\times u\times v, e\times w\times x\rangle = -|e|^2\langle u\times v\times w, x\rangle \tag{43}$$

whenever $e,u,v,w,x\in W$ are orthonormal.
(ii) *The map $\Phi: W^4\to\mathbb{R}$ defined by*

$$\Phi(x,u,v,w) := \langle x, u\times v\times w\rangle$$

(called the **Cayley calibration** *of W) is an alternating 4-form. Moreover, Φ is self-dual, i.e.*

$$\Phi = *\Phi, \tag{44}$$

where $*:\Lambda^k W^*\to\Lambda^{8-k}W^*$ denotes the Hodge $*$-operator associated to the inner product and the orientation of Remark 5.5.

(iii) *Let* $V := 1^\perp$ *with the cross product defined by* (29) *and the associator bracket* $[\cdot,\cdot,\cdot]$ *defined by* (16)*. Let* $\phi \in \Lambda^3 V^*$ *and* $\psi \in \Lambda^4 V^*$ *be the associative and coassociative calibrations of* V *defined by* (5) *and* (19)*, respectively. Then the triple cross product* (40) *of* $u, v, w \in W$ *can be expressed as*

$$\begin{aligned} u \times v \times w = \phi(u_1, v_1, w_1) - [u_1, v_1, w_1] \\ - u_0(v_1 \times w_1) - v_0(w_1 \times u_1) - w_0(u_1 \times v_1) \end{aligned} \tag{45}$$

and the Cayley calibration is given by

$$\Phi = 1^* \wedge \phi + \psi. \tag{46}$$

(iv) *For all* $u, v \in W$ *we have*

$$uv = u \times 1 \times v + \langle u, 1\rangle v + \langle v, 1\rangle u - \langle u, v\rangle. \tag{47}$$

Remark 5.8. There is a choice involved in the definition of the triple cross product in (40). An alternative formula is

$$(u, v, w) \mapsto \frac{1}{2}\big(u(\bar{v}w) - w(\bar{v}u)\big).$$

This map also satisfies (41) and (42). However, it satisfies (43) with the minus sign changed to plus and the resulting Cayley calibration is given by $\Phi = 1^* \wedge \phi - \psi$ and is anti-self-dual. Equation (47) remains again unchanged.

Proof of Theorem 5.7. We prove that the formulas (40) and (45) agree. By (37), we have

$$\bar{v}w - w\bar{v} = -2v_1 \times w_1, \quad uw - wu = -2w_1 \times u_1, \quad u\bar{v} - \bar{v}u = -2u_1 \times v_1,$$

for all $u, v, w \in W$. Multiplying these expressions by u_0, v_0, w_0, respectively, we obtain (twice) the last three expressions on the right in (45). Thus it suffices to assume $u, v, w \in V$. Then we obtain

$$\begin{aligned} 2u \times v \times w &= (u\bar{v})w - (w\bar{v})u \\ &= -(uv)w + (wv)u \\ &= -(-\langle u, v\rangle + u \times v)w + (-\langle w, v\rangle + w \times v)u \\ &= \langle u \times v, w\rangle + \langle u, v\rangle w - (u \times v) \times w \\ &\quad -\langle w \times v, u\rangle - \langle w, v\rangle u + (w \times v) \times u \\ &= 2\phi(u, v, w) - 2[u, v, w]. \end{aligned}$$

Here the third and fourth equations follow from (29), and the last follows from (5) and (39). This proves that the formulas (40) and (45) agree.

We prove (i). By (45) we have

$$\begin{aligned}\langle x, u\times v\times w\rangle = x_0\phi(u_1,v_1,w_1) + \psi(x_1,u_1,v_1,w_1)\\ - u_0\phi(x_1,v_1,w_1) - v_0\phi(x_1,w_1,u_1) - w_0\phi(x_1,u_1,v_1)\end{aligned} \tag{48}$$

for $x,u,v,w\in W$. Here we have used $\phi(u_1,v_1,w_1)=\langle u_1, v_1\times w_1\rangle$ and

$$-\langle x_1,[u_1,v_1,w_1]\rangle = -\psi(u_1,v_1,w_1,x_1) = \psi(x_1,u_1,v_1,w_1).$$

It follows from the alternating properties of ϕ and ψ that the right hand side of (48) is an alternating 4-form. Hence the map (40) is alternating and satisfies (41). For $u,v,w\in V=1^\perp$ equation (42) follows from Lemma 4.2. In general, if $u,v,w\in W$ are pairwise orthogonal, it follows from (32) and (34) that

$$(u\bar{v})w = -(u\bar{w})v = (w\bar{u})v = -(w\bar{v})u.$$

This shows that

$$\langle u,v\rangle = \langle v,w\rangle = \langle w,u\rangle = 0 \qquad\Longrightarrow\qquad u\times v\times w = u(\bar{v}w) \tag{49}$$

and hence, by (27), we have $|u\times v\times w| = |u\wedge v\wedge w|$ in the orthogonal case. This equation continues to hold in general by Gram-Schmidt. This proves that the triple cross product satisfies (42).

We prove that the triple cross product satisfies (43). The second equation in (34) asserts that $(yz)\bar{z} = |z|^2 y$ for all $y,z\in W$. Hence, by (49), we have

$$\begin{aligned}\langle e\times u\times v, e\times w\times x\rangle &= \langle u\times v\times e, w\times x\times e\rangle\\ &= \langle (u\bar{v})e, (w\bar{x})e\rangle\\ &= \langle u\bar{v}, ((w\bar{x})e)\bar{e}\rangle\\ &= |e|^2\langle u\bar{v}, w\bar{x}\rangle\\ &= |e|^2\langle (u\bar{v})x, w\rangle\\ &= |e|^2\langle u\times v\times x, w\rangle\\ &= -|e|^2\langle x, u\times v\times w\rangle\end{aligned}$$

whenever $e,u,v,w,x\in W$ are pairwise orthogonal. Thus the triple cross product (40) satisfies (43). This proves (i).

We prove (ii) and (iii). That Φ is a 4-form follows from (i). That it satisfies equation (46) follows directly from the definition of Φ and equation (48). That Φ is self-dual with respect to the orientation of Remark 5.5 follows from (46) and Lemma 4.5. Equation (45) was proved above.

We prove (iv). By (37) and (40), we have

$$u_1 \times v_1 = \frac{1}{2}\bigl(uv - vu\bigr) = u \times 1 \times v.$$

Hence it follows from (30) that

$$\begin{aligned} uv &= u_0v_0 - \langle u_1, v_1\rangle + u_0v_1 + v_0u_1 + u_1 \times v_1 \\ &= -u_0v_0 - \langle u_1, v_1\rangle + u_0v + v_0u + u_1 \times v_1 \\ &= -\langle u, v\rangle + \langle u, 1\rangle v + \langle v, 1\rangle u + u \times 1 \times v. \end{aligned}$$

This proves (47) and the theorem. □

Example 5.9. If $W = \mathbb{R}^8 = \mathbb{R} \oplus \mathbb{R}^7$ with coordinates $x_0, x_1, \dots, x_7$ and the cross product of Example 2.7 on $\mathbb{R}^7$ then the associated Cayley calibration is given by

$$\begin{aligned} \Phi_0 =& e^{0123} - e^{0145} - e^{0167} - e^{0246} + e^{0257} - e^{0347} - e^{0356} \\ &+ e^{4567} - e^{2367} - e^{2345} - e^{1357} + e^{1346} - e^{1256} - e^{1247}. \end{aligned}$$

Thus

$$\Phi_0 \wedge \Phi_0 = 14\mathrm{dvol}.$$

(See the proof of Lemma 4.5.)

Definition 5.10. *Let W be an 8-dimensional normed algebra. The* **fourfold cross product** *on W is the alternating multi-linear map $W^4 \to W : (u, v, w, x) \mapsto u \times v \times w \times x$ defined by*

$$4x\times u\times v\times w := (u\times v\times w)\bar{x} - (v\times w\times x)\bar{u} + (w\times x\times u)\bar{v} - (x\times u\times v)\bar{w}. \quad (50)$$

Theorem 5.11. *Let W be an 8-dimensional normed algebra with triple cross product* (40)*, Cayley calibration $\Phi \in \Lambda^4W^*$, and fourfold cross product* (50)*. Then, for all $x, u, v, w \in W$, we have*

$$|x \times u \times v \times w| = |x \wedge u \wedge v \wedge w| \quad (51)$$

and

$$\begin{aligned} \mathrm{Re}(x \times u \times v \times w) &= \Phi(x, u, v, w), \\ \mathrm{Im}(x \times u \times v \times w) &= [x_1, u_1, v_1, w_1] - x_0[u_1, v_1, w_1] \\ &\quad + u_0[v_1, w_1, x_1] - v_0[w_1, x_1, u_1] + w_0[x_1, u_1, v_1], \end{aligned} \quad (52)$$

where the last five terms use the associator and coassociator brackets on $V := 1^\perp$ defined by (16) *and* (25)*. In particular,*

$$\Phi(x, u, v, w)^2 + |\mathrm{Im}(x \times u \times v \times w)|^2 = |x \wedge u \wedge v \wedge w|^2. \quad (53)$$

Proof. That the fourfold cross product is alternating is obvious from the definition and the alternating property of the triple cross product. We prove that it satisfies (51). For this it suffices to assume that u, v, w, x are pairwise orthogonal. Then $u \times v \times w = (u\bar{v})w$ and hence

$$(u \times v \times w)\bar{x} = ((u\bar{v})w)\bar{x} = -((u\bar{v})x)\bar{w} = -(u \times v \times x)\bar{w}.$$

Here we have used (32) and (34). Using the alternating property of the triple cross product we obtain that the four summands in (50) agree in the orthogonal case. Hence $x \times u \times v \times w = ((u\bar{v})w)\bar{x}$ and so equation (51) follows from (27).

We prove (52). Since $u \times 1 \times v = u_1 \times v_1$ we have

$$\begin{aligned}
1 \times u \times v \times w &= \frac{1}{4}\Big(u \times v \times w + (v_1 \times w_1)\bar{u} + (w_1 \times u_1)\bar{v} + (u_1 \times v_1)\bar{w}\Big) \\
&= \frac{1}{4}\Big(u \times v \times w + u_0(v_1 \times w_1) + v_0(w_1 \times u_1) + w_0(u_1 \times v_1)\Big) \\
&\quad + \frac{1}{4}\Big(\langle v_1 \times w_1, u_1\rangle + \langle w_1 \times u_1, v_1\rangle + \langle u_1 \times v_1, w_1\rangle\Big) \\
&\quad - \frac{1}{4}\Big((v_1 \times w_1) \times u_1 + (w_1 \times u_1) \times v_1 + (u_1 \times v_1) \times w_1\Big) \\
&= \phi(u_1, v_1, w_1) - [u_1, v_1, w_1].
\end{aligned}$$

The last equation follows from (45) and the definition of the associator bracket in (16). This proves (52) in the case $x_1 = 0$. Using the alternating property we may now assume that $x, u, v, w \in V := 1^\perp$. If x, u, v, w are orthogonal to 1 it follows from (45) that $u \times v \times w = \phi(u, v, w) - [u, v, w]$ and $\Phi(x, u, v, w) = -\langle x, [u, v, w]\rangle = \psi(x, u, v, w)$. Moreover $\bar{x} = -x$ and similarly for u, v, w. Hence

$$\begin{aligned}
&4x \times u \times v \times w \\
&= -(u \times v \times w)x + (v \times w \times x)u - (w \times x \times u)v + (x \times u \times v)w \\
&= [u, v, w]x - [v, w, x]u + [w, x, u]v - [x, u, v]w \\
&\quad - \phi(u, v, w)x + \phi(v, w, x)u - \phi(w, x, u)v + \phi(x, u, v)w \\
&= -\langle [u, v, w], x\rangle + \langle [v, w, x], u\rangle - \langle [w, x, u], v\rangle + \langle [x, u, v], w\rangle \\
&\quad + [u, v, w] \times x - [v, w, x] \times u + [w, x, u] \times v - [x, u, v] \times w \\
&\quad - \phi(u, v, w)x + \phi(v, w, x)u - \phi(w, x, u)v + \phi(x, u, v)w \\
&= -4\psi(u, v, w, x) - 4[u, v, w, x] = 4\Phi(x, u, v, w) + 4[x, u, v, w].
\end{aligned}$$

Here the last but one equation follows from Lemma 4.9. Thus we have proved (52). □

6 Triple cross products

In this section we show how to recover the normed algebra structure on W from the triple cross product. In fact we shall see that every unit vector in W can be used as a unit for the algebra structure. We assume throughout that W is a finite dimensional real Hilbert space.

Definition 6.1. *An alternating multi-linear map*

$$W \times W \times W \to W : (u, v, w) \mapsto u \times v \times w \tag{54}$$

is called a **triple cross product** *if it satisfies*

$$\langle u \times v \times w, u\rangle = \langle u \times v \times w, v\rangle = \langle u \times v \times w, w\rangle = 0, \tag{55}$$

$$|u \times v \times w| = |u \wedge v \wedge w| \tag{56}$$

for all $u, v, w \in W$.

A multi-linear map (54) that satisfies (56) also satisfies $u \times v \times w = 0$ whenever $u, v, w \in W$ are linearly dependent, and hence is necessarily alternating.

Lemma 6.2. *Let* (54) *be an alternating multi-linear map. Then* (55) *holds if and only if, for all $x, u, v, w \in W$, we have*

$$\langle x, u \times v \times w\rangle + \langle u \times v \times x, w\rangle = 0. \tag{57}$$

Proof. If (57) holds then (55) follows immediately from the alternating property of the map (54). To prove the converse, expand the expression $\langle u \times v \times (w + x), w + x\rangle$ and use (55) to obtain (57). □

Lemma 6.3. *Let* (54) *be an alternating multi-linear map satisfying* (55)*. Then* (56) *holds if and only if, for all $u, v, w \in W$, we have*

$$\begin{aligned} &u \times v \times (u \times v \times w) + |u \wedge v|^2 w \\ &= \Big(|v|^2 \langle u, w\rangle - \langle u, v\rangle\langle v, w\rangle\Big) u + \Big(|u|^2 \langle v, w\rangle - \langle v, u\rangle\langle u, w\rangle\Big) v. \end{aligned} \tag{58}$$

Proof. If (58) holds and w is orthogonal to u and v then

$$u \times v \times (u \times v \times w) = -\,|u \wedge v|^2 w.$$

Taking the inner product with w and using (57) we obtain (56) under the assumption $\langle u, w\rangle = \langle v, w\rangle = 0$. Since both sides of equation (56) remain unchanged if we add to w a linear combination of u and v, this proves that (58) implies (56). To prove the converse we assume (56). If w is orthogonal to u and v we have $|u \times v \times w|^2 = |u \wedge v|^2 |w|^2$. Replacing w by $w + x$ we obtain

$$w, x \in u^\perp \cap v^\perp \quad \Longrightarrow \quad \langle u \times v \times w, u \times v \times x \rangle = |u \wedge v|^2 \langle w, x \rangle. \tag{59}$$

Using (57) we obtain (58) for every vector $w \in u^\perp \cap v^\perp$. Replacing a general vector w by its projection onto the orthogonal complement of the subspace spanned by u and v we deduce that (58) holds in general. This proves the lemma. □

Let (54) be a triple cross product. If $e \in W$ is a unit vector then the subspace $V_e := e^\perp$ carries a cross product $(u, v) \mapsto u \times_e v$ defined by $u \times_e v := u \times e \times v$. Hence, by Theorem 2.2, the dimension of V_e is 0, 1, 3, or 7. It follows that the dimension of W is 0, 1, 2, 4, or 8.

Lemma 6.4. *Assume* $\dim W = 8$ *and let* (54) *be a triple cross product. Then there is a number* $\varepsilon \in \{\pm 1\}$ *such that*

$$e \times u \times (e \times v \times w) = \varepsilon |e|^2 u \times v \times w \tag{60}$$

whenever $e, u, v \in W$ *are pairwise orthogonal and* $w \in W$ *is orthogonal to* e, u, v, *and* $e \times u \times v$.

Proof. It suffices to assume that the vectors $e, u, v \in W$ are orthonormal. Then the subspace

$$H := \mathrm{span}(e, u, v, e \times u \times v)^\perp$$

has dimension four. It follows from (57) and (59) that the formulas

$$Iw := e \times u \times w, \qquad Jw := e \times v \times w, \qquad Kw := u \times v \times w,$$

define endomorphisms I, J, K of H. Moreover, by (57), these operators are skew adjoint and, by (59), they are complex structures on H. It follows also from (59) that $e \times x \times (e \times x \times w) = -|x|^2 w$ whenever e, x, w are pairwise orthogonal and $|e| = 1$. Assuming $w \in H$ and using this identity with $x = u + v$ we obtain $IJ + JI = 0$. This implies that the automorphisms of H of the form $aI + bJ + cIJ$ with $a^2 + b^2 + c^2 = 1$ belong to the space $\mathcal{J}$ of orthogonal complex structures on H. They form one of the two components of $\mathcal{J}$ and K belongs to this component because it anticommutes with I and J. Hence $K = \varepsilon IJ$ with $\varepsilon = \pm 1$. Since the space of orthonormal triples in W is connected, and the constant ε depends continuously on the triple e, u, v, we have proved (60) under the assumption that e, u, v are orthonormal and w is orthogonal to the vectors $e, u, v, e \times u \times v$. Hence the assertion follows by scaling. $\square$

Definition 6.5. *Assume* $\dim W = 8$. *A triple cross product* (54) *is called* **positive** *if it satisfies* (60) *with* $\varepsilon = 1$ *and is called* **negative** *if it satisfies* (60) *with* $\varepsilon = -1$.

Definition 6.6. *Assume* $\dim W = 8$ *and let* (54) *be a triple cross product. Then, by Lemma 6.2, the map* $\Phi : W \times W \times W \times W \to \mathbb{R}$ *defined by*

$$\Phi(x, u, v, w) := \langle x, u \times v \times w \rangle \tag{61}$$

is an alternating 4*-form. It is called the* **Cayley calibration** *of* W.

Theorem 6.7. *Assume* $\dim W = 8$ *and let* (54) *be a triple cross product with Cayley calibration* $\Phi \in \Lambda^4 W^*$ *given by* (61). *Let* $e \in W$ *be a unit vector.*
(i) *Define the map* $\psi_e : W^4 \to \mathbb{R}$ *by*

$$\begin{aligned} \psi_e(u, v, w, x) := \langle e \times u \times v, e \times w \times x \rangle \\ - \bigl(\langle u, w\rangle - \langle u, e\rangle\langle e, w\rangle\bigr)\bigl(\langle v, x\rangle - \langle v, e\rangle\langle e, x\rangle\bigr) \\ + \bigl(\langle u, x\rangle - \langle u, e\rangle\langle e, x\rangle\bigr)\bigl(\langle v, w\rangle - \langle v, e\rangle\langle e, w\rangle\bigr). \end{aligned} \tag{62}$$

Then $\psi_e \in \Lambda^4 W^*$ *and*

$$\Phi = e^* \wedge \phi_e + \varepsilon \psi_e, \qquad \phi_e := \iota(e)\Phi \in \Lambda^3 W^*, \tag{63}$$

where $\varepsilon \in \{\pm 1\}$ *is as in Lemma 6.4.*
(ii) *The subspace* $V_e := e^\perp$ *carries a cross product*

$$V_e \times V_e \to V_e : (u, v) \mapsto u \times_e v := u \times e \times v, \tag{64}$$

the restriction of ϕ_e *to* V_e *is the associative calibration of* (64), *and the restriction of* ψ_e *to* V_e *is the coassociative calibration of* (64).
(iii) *The space* W *is a normed algebra with unit* e *and multiplication and conjugation given by*

$$uv := u \times e \times v + \langle u, e\rangle v + \langle v, e\rangle u - \langle u, v\rangle e, \qquad \bar{u} := 2\langle u, e\rangle e - u. \tag{65}$$

If the triple cross product is positive then $(u\bar{v})w - (w\bar{v})u = 2u \times v \times w$.

Proof. We prove (i). If the vectors e, u, v, w, x are pairwise orthogonal then

$$\langle e \times u \times x, e \times v \times w \rangle = -\varepsilon \, |e|^2 \, \langle x, u \times v \times w \rangle. \tag{66}$$

To see this, take the inner product of (60) with x. Then it follows from (57) that (66) holds under the additional assumption that w is perpendicular to $e \times u \times v$. Since x is orthogonal to e this additional condition can be dropped, as both sides of the equation remain unchanged if we add to w a multiple of $e \times u \times v$. Thus we have proved (66).

Now fix a unit vector $e \in W$. By definition, ψ_e is alternating in the first two and last two arguments, and satisfies $\psi_e(u, v, w, x) = \psi_e(w, x, u, v)$ for all $u, v, w, x \in W$. By (57) we also have $\psi_e(u, v, u, v) = 0$. Expanding the identity $\psi_e(u, v + x, u, v + x) = 0$ we obtain $\psi_e(u, v, u, x) = 0$ for all $u, v, x \in W$. Using this identity with u replaced by $u + w$ gives

$$\psi_e(u, v, w, x) + \psi_e(w, v, u, x) = 0.$$

Hence ψ_e is also skew-symmetric in the first and third argument and so is an alternating 4-form. To see that it satisfies (63) it suffices to show that $\varepsilon\Phi$ and ψ_e agree on $e^\perp$. Since they are both 4-forms it suffices to show that they agree on every quadrupel of pairwise orthogonal vectors $u, v, w, x \in e^\perp$. But in this case we have $\psi_e(u, x, v, w) = -\varepsilon\Phi(x, u, v, w) = \varepsilon\Phi(u, x, v, w)$, by equation (66). This proves (i).

We prove (ii). That (64) is a cross product on $V_e = e^\perp$ follows immediately from the definitions. By (61) we have

$$\langle u \times_e v, w \rangle = \Phi(w, u, e, v) = \Phi(e, u, v, w) = \phi_e(u, v, w)$$

for $u, v, w \in V_e$, and hence the restriction of ϕ_e to V_e is the associative calibration. Moreover, the associator bracket (16) on V_e is given by

$$[u, v, w]_e = (u \times e \times v) \times e \times w + \langle v, w \rangle u - \langle u, w \rangle v.$$

Hence, for all $u, v, w, x \in V_e$, we have

$$\begin{aligned} \langle [u, v, w]_e, x \rangle &= \langle e \times w \times (u \times e \times v), x \rangle + \langle v, w \rangle \langle u, x \rangle - \langle u, w \rangle \langle v, x \rangle \\ &= \langle e \times u \times v, e \times w \times x \rangle - \langle u, w \rangle \langle v, x \rangle + \langle u, x \rangle \langle v, w \rangle \\ &= \psi_e(u, v, w, x), \end{aligned}$$

where the last equation follows from (62). Hence the restriction of ψ_e to V_e is the coassociative calibration and this proves (ii).

We prove (iii). That e is a unit follows directly from the definitions. To prove that the norm of the product is equal to the product of the norms we observe that $u \times e \times v$ is orthogonal to e, u, and v, by equation (57). Hence

$$\begin{aligned} |uv|^2 &= |u \times e \times v + \langle u,e\rangle v + \langle v,e\rangle u - \langle u,v\rangle e|^2 \\ &= |u \times e \times v|^2 - 2\langle v,e\rangle\langle u,v\rangle\langle v,e\rangle \\ &\quad + \langle u,e\rangle^2 |v|^2 + \langle v,e\rangle^2 |u|^2 + \langle u,v\rangle^2 \\ &= |u|^2 |v|^2 . \end{aligned}$$

Here the last equality uses the fact that $|u \times e \times v|^2 = |u \wedge e \wedge v|^2$. Thus we have proved that W is a normed algebra with unit e.

If the triple cross product (54) is positive then $\varepsilon = 1$ and hence equation (63) asserts that $\Phi = e^* \wedge \phi_e + \psi_e$. Hence it follows from equation (46) in Theorem 5.7 that the Cayley calibration Φ_e associated to the above normed algebra structure is equal to Φ. This implies that the given triple cross product (54) agrees with the triple cross product defined by (40). This proves (iii) and the theorem. ☐

Remark 6.8. Assume $\dim W = 8$ and let (54) be a triple cross product with Cayley calibration $\Phi \in \Lambda^4 W^*$ given by (61). Then, for every unit vector $e \in W$, the subspace $V_e = e^\perp$ is oriented by Lemma 3.4 and Theorem 6.7. We orient W as the direct sum $W = \mathbb{R}e \oplus V_e$. This orientation is independent of the choice of the unit vector e. With this orientation we have $e^* \wedge \phi_e = *\psi_e$, by Theorem 6.7 (ii) and Lemma 4.5. Hence it follows from equation (63) in Theorem 6.7 (i) that $\Phi \wedge \Phi \neq 0$. In fact the triple cross product is positive if and only if $\Phi \wedge \Phi > 0$ with respect to our orientation and negative if and only if $\Phi \wedge \Phi < 0$. In the positive case Φ is self-dual and in the negative case Φ is anti-self-dual.

Corollary 6.9. *Assume* $\dim W = 8$ *and let* (54) *be a triple cross product and let* ε *be as in Lemma 6.4. Then, for all* $e, u, v, w \in W$, *we have*

$$\begin{aligned} e \times u \times (e \times v \times w) =& \varepsilon |e|^2 u \times v \times w - \varepsilon\langle e, u \times v \times w\rangle e \\ & - \varepsilon\langle e,u\rangle e \times v \times w \\ & - \varepsilon\langle e,v\rangle e \times w \times u \\ & - \varepsilon\langle e,w\rangle e \times u \times v \\ & - \big(|e|^2 \langle u,v\rangle - \langle e,u\rangle\langle e,v\rangle\big) w \\ & + \big(|e|^2 \langle u,w\rangle - \langle e,u\rangle\langle e,w\rangle\big) v \\ & + \big(\langle u,v\rangle\langle e,w\rangle - \langle u,w\rangle\langle e,v\rangle\big) e. \end{aligned} \tag{67}$$

Proof. Both sides of the equation remain unchanged if we add to u, v, or w a multiple of e. Hence it suffices to prove (67) under the assumption that u, v, w are all orthogonal to e. Moreover, both sides of the equation are always orthogonal to e. Hence it suffices to prove that the inner products of both sides of (67) with every vector $x \in e^\perp$ agree. It also suffices to assume $|e| = 1$. Thus we must prove that, if $e \in W$ is a unit vector and $u, v, w, x \in W$ are orthogonal to e, then we have

$$\langle e \times u \times (e \times v \times w), x\rangle = \varepsilon\langle u \times v \times w, x\rangle - \langle u, v\rangle\langle w, x\rangle + \langle u, w\rangle\langle v, x\rangle$$

or equivalently

$$-\langle e \times u \times x, e \times v \times w\rangle + \langle u, v\rangle\langle x, w\rangle - \langle u, w\rangle\langle x, v\rangle = \varepsilon\langle x, u \times v \times w\rangle. \quad (68)$$

The right hand side of (68) is $\varepsilon\Phi(x, u, v, w)$ and, by (62), the left hand side of (68) is $-\psi_e(u, x, v, w)$. Hence equation (68) is equivalent to the assertion that the restriction of ψ_e to $e^\perp$ agrees with Φ. But this follows from equation (63) in Theorem 6.7. This proves the corollary. □

Lemma 6.10. *Assume* $\dim W = 8$ *and let* (54) *be a triple cross product with Cayley calibration* $\Phi \in \Lambda^4 W^*$ *given by* (61). *Let* $\Lambda \subset W$ *be a* 4*-dimensional linear subspace. Then the following are equivalent.*

(i) *If* $u, v, w \in \Lambda$ *then* $u \times v \times w \in \Lambda$.

(ii) *If* $u, v \in \Lambda$ *and* $w \in \Lambda^\perp$ *then* $u \times v \times w \in \Lambda^\perp$.

(iii) *If* $u \in \Lambda$ *and* $v, w \in \Lambda^\perp$ *then* $u \times v \times w \in \Lambda$.

(iv) *If* $u, v, w \in \Lambda^\perp$ *then* $u \times v \times w \in \Lambda^\perp$.

(v) *If* $u, v, w \in \Lambda$ *and* $x \in \Lambda^\perp$ *then* $\Phi(x, u, v, w) = 0$.

(vi) *If* x, u, v, w *is an orthonormal basis of* Λ *then* $\Phi(x, u, v, w) = \pm 1$.

(vii) *If* $e \in \Lambda^\perp$ *has norm one then* Λ *is a coassociative subspace of* $V_e := e^\perp$.

(viii) *If* $e \in \Lambda$ *has norm one then* $\Lambda \cap V_e$ *is an associative subspace of* V_e.

A 4*-dimensional subspace that satisfies these equivalent conditions is called a* **Cayley subspace** *of* W. *If* $u, v, w \in W$ *are linearly independent then* $\Lambda := \mathrm{span}\{u, v, w, u \times v \times w\}$ *is a Cayley subspace of* W.

Proof. We prove that (i) is equivalent to (v). If (i) holds and $u, v, w \in \Lambda$, $x \in \Lambda^\perp$ then $u \times v \times w \in \Lambda$ and hence $\Phi(x, u, v, w) = \langle x, u \times v \times w\rangle = 0$. Conversely, if (v) holds and $u, v, w \in \Lambda$ then $\langle x, u \times v \times w\rangle = \Phi(x, u, v, w) = 0$ for every $x \in \Lambda^\perp$ and hence $u \times v \times w \in \Lambda$.

We prove that (i) is equivalent to (vi). If (i) holds and x, u, v, w is an orthonormal basis of Λ then $u \times v \times w$ is orthogonal to u, v, w and has norm one. Since $u \times v \times w \in \Lambda$ we must have $x = \pm u \times v \times w$. Hence $\Phi(x, u, v, w) = \pm |x|^2 = \pm 1$. Conversely, assume (vi), let $u, v, w \in \Lambda$ be orthonormal, and choose x such that x, u, v, w form an orthonormal basis of Λ. Then

$$\langle x, u \times v \times w \rangle^2 = \Phi(x, u, v, w)^2 = 1 = |x|^2 \, |u \times v \times w|^2 .$$

Hence $u \times v \times w$ is a real multiple of x and so $u \times v \times w \in \Lambda$. Since the triple cross product is alternating the general case can be reduced to the orthonormal case by scaling and Gram-Schmidt.

That (vi) is equivalent to (vii) follows from Lemma 4.12 and the fact that $\Phi|_{V_e}$ is the coassociative calibration on V_e. Likewise, that (vi) is equivalent to (viii) follows from Lemma 4.4 and the fact that $\iota(e)\Phi|_{V_e}$ is the associative calibration on V_e.

Thus we have proved that (i), (v), (vi), (vii), (viii) are equivalent. The equivalence of (i), (ii), (iii) for a unit vector $u = e \in \Lambda$ follows from Lemma 4.12 for $V := V_e$ and $H := \Lambda^\perp$, using the fact that $v \times_e w = -e \times v \times w$ is the cross product on V_e. The equivalence of (iii) and (iv) follows from the equivalence of (i) and (ii) by interchanging the roles of Λ and $\Lambda^\perp$. Thus we have proved the equivalence of conditions (i-viii). The last assertion of the lemma follows from (i) and equation (58) in Lemma 6.3. □

7 Cayley calibrations

We assume throughout that W is an 8-dimensional real vector space.

Definition 7.1. *A 4-form $\Phi \in \Lambda^4 W^*$ is called* **nondegenerate** *if, for every triple u, v, w of linearly independent vectors in W, there is a vector $x \in W$ such that $\Phi(u, v, w, x) \neq 0$. An inner product on W is called* **compatible** *with a 4-form Φ if the map $W^3 \to W : (u, v, w) \mapsto u \times v \times w$ defined by*

$$\langle x, u \times v \times w \rangle := \Phi(x, u, v, w) \tag{69}$$

is a triple cross product. A 4-form $\Phi \in \Lambda^4 W^*$ is called a **Cayley-form** *if it admits a compatible inner product.*

Example 7.2. The standard Cayley-form on $W = \mathbb{R}^8$ with coordinates $x_0, x_1, \ldots, x_7$ is given by

$$\begin{aligned} \Phi_0 = & e^{0123} - e^{0145} - e^{0167} - e^{0246} + e^{0257} - e^{0347} - e^{0356} \\ & + e^{4567} - e^{2367} - e^{2345} - e^{1357} + e^{1346} - e^{1256} - e^{1247}. \end{aligned}$$

It is compatible with the standard inner product and induces the standard triple cross product on $\mathbb{R}^8$ (see Example 5.9). Note that $\Phi_0 \wedge \Phi_0 = 14 \, \mathrm{dvol}$.

As in Section 3 we shall see that a compatible inner product, if it exists, is uniquely determined by Φ. However, in contrast to Section 3, nondegeneracy is, in the present setting, not equivalent to the existence of a compatible inner product, but is only a necessary condition. The goal in this section is to give an intrinsic characterization of Cayley-forms. In particular, we shall see that every Cayley-form satisfies the condition

$$\Phi \wedge \Phi \neq 0.$$

It seems to be an open question whether or not every nondegenerate 4-form on W has this property; we could not find a counterexample but also did not see how to prove it. We begin by characterizing compatible inner products.

Lemma 7.3. *Fix an inner product on W and a 4-form $\Phi \in \Lambda^4 W^*$. Then the following are equivalent.*

(i) *The inner product is compatible with Φ.*

(ii) *There is an orientation on W such that*

$$\iota(v)\iota(u)\Phi \wedge \iota(v)\iota(u)\Phi \wedge \Phi = 6\,|u \wedge v|^2\,\mathrm{dvol} \tag{70}$$

for all $u, v \in W$, where dvol $\in \Lambda^8 W^*$ *denotes the volume form associated to the inner product and the orientation.*

(iii) *There is an orientation on W such that*

$$\iota(v)\iota(u)\Phi \wedge \iota(w)\iota(u)\Phi \wedge \Phi = 6\left(|u|^2\,\langle v, w\rangle - \langle v, u\rangle\langle u, w\rangle\right)\mathrm{dvol} \tag{71}$$

for all $u, v, w \in W$, where dvol *is as in (ii).*

Each of these conditions implies that Φ is nondegenerate and $\Phi \wedge \Phi \neq 0$.

Proof. We prove that (i) implies (ii). Assume the inner product is compatible with Φ and let $W^3 \to W : (u, v, w) \mapsto u \times v \times w$ be the triple cross product on W defined by (69). Assume $u, v \in W$ are linearly independent. Then the subspace

$$W_{u,v} := \{w \in W \mid \langle u, w\rangle = \langle v, w\rangle = 0\}$$

carries a symplectic form $\omega_{u,v} : W_{u,v} \times W_{u,v} \to \mathbb{R}$ and a compatible complex structure $J_{u,v} : W_{u,v} \to W_{u,v}$ given by

$$\omega_{u,v}(x, w) := \frac{\Phi(x, u, v, w)}{|u \wedge v|}, \qquad J_{u,v} w := -\frac{u \times v \times w}{|u \wedge v|}.$$

Equation (56) asserts that $J_{u,v}$ is an isometry on $W_{u,v}$ and equation (57) asserts that $J_{u,v}$ is skew adjoint. Hence $J_{u,v}$ is a complex structure on $W_{u,v}$ and equation (69) shows that, for all $x, w \in W_{u,v}$, we have

$$\omega_{u,v}(x, w) = \frac{\langle x, u \times v \times w\rangle}{|u \wedge v|} = -\langle x, J_{u,v}w\rangle.$$

Thus the inner product $\omega_{u,v}(\cdot, J_{u,v}\cdot)$ on $W_{u,v}$ is the one inherited from W. It follows that

$$\omega_{u,v} \wedge \omega_{u,v} \wedge \omega_{u,v} = 6\mathrm{dvol}_{u,v}, \tag{72}$$

where $\mathrm{dvol}_{u,v} \in \Lambda^6 W^*_{u,v}$ denotes the volume form on $W_{u,v}$ with the symplectic orientation. Since the space of linearly independent pairs $u, v \in W$ is connected, there is a unique orientation on W such that, for every pair u, v of linearly independent vectors in W and every symplectic basis $e_1, \dots, e_6$ of $W_{u,v}$, the basis $u, v, e_1, \dots, e_6$ of W is positively oriented. Let $\mathrm{dvol} \in \Lambda^8 W^*$ be the volume form of W^* for this orientation. Then

$$\mathrm{dvol}_{u,v} = \frac{1}{|u \wedge v|}\iota(v)\iota(u)\mathrm{dvol}|_{W_{u,v}}$$

and hence equation (70) follows from (72). This shows that (i) implies (ii). That (ii) implies (iii) follows by using (70) with v replaced by $v + w$.

We prove that (iii) implies (i). Assume there is an orientation on W such that (71) holds, and define the map $W^3 \to W : (u, v, w) \mapsto u \times v \times w$ by (69). That this map is alternating and satisfies (55) is obvious. We prove that it satisfies (56). Fix a unit vector $e \in W$ and denote

$$V_e := \{v \in W \,|\, \langle e, v\rangle = 0\}, \quad \phi_e := \iota(e)\Phi|_{V_e}, \quad \mathrm{dvol}_e := \iota(e)\mathrm{dvol}|_{V_e}.$$

Then equation (71) asserts that

$$\iota(u)\phi_e \wedge \iota(v)\phi_e \wedge \phi_e = 6\langle u, v\rangle\mathrm{dvol}_e$$

for every $u \in V_e$. Hence ϕ_e satisfies condition (ii) in Lemma 3.4 and therefore is compatible with the inner product. This means that the bilinear map $V_e \times V_e \to V_e : (u, v) \mapsto u \times_e v$ defined by $\langle u \times_e v, w\rangle := \phi_e(u, v, w)$ is a cross product on V_e. Since $\phi_e(u, v, w) = \Phi(w, u, e, v) = \langle u \times e \times v, w\rangle$, we have $u \times_e v = u \times e \times v$. This implies $|u \times e \times v| = |u \wedge v|$ whenever u and v are orthogonal to e and e has norm one. Using Gram–Schmidt and scaling, we deduce that our map $(u, v, w) \mapsto u \times v \times w$ satisfies (56) and hence is a triple cross product. Thus we have proved that (i), (ii), and (iii) are equivalent. Moreover, condition (ii) implies that Φ is nondegenerate and (i) implies that $\Phi \wedge \Phi \neq 0$, by Remark 6.8. This proves the lemma. □

We are now in a position to characterize Cayley-forms intrinsically. First observe that a 4-form Φ on W is nondegenerate if and only if the 2-form $\iota(v)\iota(u)\Phi \in \Lambda^2 W^*$ descends to a symplectic form on the quotient space $W/\mathrm{span}\{u,v\}$ for every pair of linearly independent vectors u, v. This means that the 8-form $\iota(v)\iota(u)\Phi \wedge \iota(v)\iota(u)\Phi \wedge \Phi$ is nonzero whenever $u, v \in W$ are linearly independent. The question to be adressed is under which additional condition we can find an inner product on W that satisfies (70).

Theorem 7.4. *A 4-form $\Phi \in \Lambda^4 W^*$ admits a compatible inner product if and only if it satisfies the following condition.*

(C) *Φ is nondegenerate and, if $u, v, w \in W$ are linearly independent and*

$$\iota(v)\iota(u)\Phi \wedge \iota(w)\iota(u)\Phi \wedge \Phi = \iota(u)\iota(v)\Phi \wedge \iota(w)\iota(v)\Phi \wedge \Phi = 0, \tag{73}$$

then, for all $x \in W$, we have

$$\iota(w)\iota(u)\Phi \wedge \iota(x)\iota(u)\Phi \wedge \Phi = 0 \iff \iota(w)\iota(v)\Phi \wedge \iota(x)\iota(v)\Phi \wedge \Phi = 0. \tag{74}$$

If this holds then the compatible inner product is uniquely determined by Φ.

To understand condition (C) geometrically, assume Φ satisfies (71) for some inner product on W. Then $\iota(v)\iota(u)\Phi \wedge \iota(w)\iota(u)\Phi \wedge \Phi = 0$ if and only if $|u|^2 \langle v, w\rangle - \langle v, u\rangle\langle u, w\rangle = 0$. Hence, if u, v, w are linearly independent, equation (73) asserts that w is orthogonal to u and v. Under this assumption both conditions in (74) assert that w and x are orthogonal.

Next we observe that every Cayley-form Φ induces two orientations on W. First, since the 8-form $\iota(v)\iota(u)\Phi \wedge \iota(v)\iota(u)\Phi \wedge \Phi$ is nonzero for every linearly independent pair $u, v \in W$ and the space of linearly independent pairs in W is connected, there is a unique orientation on W such that $\iota(v)\iota(u)\Phi \wedge \iota(v)\iota(u)\Phi \wedge \Phi > 0$ whenever $u, v \in W$ are linearly independent. The second orientation of W is induced by the 8-form $\Phi \wedge \Phi$. This leads to the following definition.

Definition 7.5. *A Cayley-form $\Phi \in \Lambda^4 W^*$ is called* **positive** *if the 8-forms $\Phi \wedge \Phi$ and $\iota(v)\iota(u)\Phi \wedge \iota(v)\iota(u)\Phi \wedge \Phi$ induce the same orientation whenever $u, v \in W$ are linearly independent. It is called* **negative** *if it is not positive.*

Thus Φ is negative if and only if $-\Phi$ is positive. Moreover, it follows from Remark 6.8 that a Cayley-form $\Phi \in \Lambda^4 W^*$ is positive if and only if the associated triple cross product is positive.

Theorem 7.6. *If $\Phi, \Psi \in \Lambda^4 W^*$ are two positive Cayley-forms then there is an automorphism $g \in \mathrm{Aut}(W)$ such that $g^*\Phi = \Psi$.*

Lemma 7.7. *Let W be a real vector space and $g : W^4 \to \mathbb{R}$ be a multi-linear map satisfying*

$$g(u, v; w, x) = g(w, x; u, v) = -g(v, u; w, x) \tag{75}$$

for all $u, v, w, x \in W$ and

$$g(u, v; u, v) > 0 \tag{76}$$

whenever $u, v \in W$ are linearly independent. Then the matrices

$$\Lambda_u(v, w) := \begin{pmatrix} g(u, v; u, v) & g(u, v; u, w) \\ g(u, w; u, v) & g(u, w; u, w) \end{pmatrix} \in \mathbb{R}^{2\times 2},$$

$$A(u, v, w) := \begin{pmatrix} g(v, w; v, w) & g(v, w; w, u) & g(v, w; u, v) \\ g(w, u; v, w) & g(w, u; w, u) & g(w, u; u, v) \\ g(u, v; v, w) & g(u, v; w, u) & g(u, v; u, v) \end{pmatrix} \in \mathbb{R}^{3\times 3}$$

are positive definite whenever $u, v, w \in W$ are linearly independent. Moreover, the following are equivalent.

(i) *If u, v, w are linearly independent and $g(u, v; w, u) = g(v, w; u, v) = 0$ then, for all $x \in W$, we have*

$$g(u, w; u, x) = 0 \qquad \Longleftrightarrow \qquad g(v, w; v, x) = 0. \tag{77}$$

(ii) *If u, v, w and u, v, w' are linearly independent then*

$$\frac{\det(\Lambda_u(v, w))}{\det(\Lambda_v(u, w))} = \frac{\det(\Lambda_u(v, w'))}{\det(\Lambda_v(u, w'))}. \tag{78}$$

(iii) *If u, v, w and u, v', w' are linearly independent then*

$$\frac{\det(\Lambda_u(v, w))}{\sqrt{\det(A(u, v, w))}} = \frac{\det(\Lambda_u(v', w'))}{\sqrt{\det(A(u, v', w'))}}. \tag{79}$$

(iv) *There is an inner product on W such that*

$$g(u, v; u, v) = |u|^2 \, |v|^2 - \langle u, v\rangle^2 \tag{80}$$

for all $u, v \in W$.

If these equivalent conditions are satisfied then the inner product in (iv) is uniquely determined by g and it satisfies

$$\det(\Lambda_u(v, w)) = |u|^2 \, |u \wedge v \wedge w|^2 , \qquad \det(A(u, v, w)) = |u \wedge v \wedge w|^4 . \tag{81}$$

Proof. Let $u, v, w \in W$ be linearly independent. We prove that the matrices $\Lambda_u(v, w)$ and $A(u, v, w)$ are positive definite. By (76) they have positive diagonal entries. Since the determinant of $\Lambda_u(v, w)$ agrees with the determinant of the lower right 2×2 block of $A(u, v, w)$ it suffices to prove that both matrices have positive determiants. To see this, we observe that the determinants of $\Lambda_u(v, w)$ and $A(u, v, w)$ remain unchanged if we add to v a multiple of u and to w a linear combination of u and v. With the appropriate choices both matrices become diagonal and thus have positive determinants. Hence $\Lambda_u(v, w)$ and $A(u, v, w)$ are positive definite, as claimed.

We prove that (iv) implies (81). The matrix $\Lambda_u(v, w)$ and the volume $|u \wedge v \wedge w|^2$ remain unchanged if we add to v and w multiples of u. Hence we may assume that v and w are orthogonal to u. In this case

$$\Lambda_u(v, w) = |u|^2 \begin{pmatrix} |v|^2 & \langle v, w \rangle \\ \langle w, v \rangle & |w|^2 \end{pmatrix}$$

and this implies the first equation in (81). Since the determinant of the matrix $A(u, v, w)$ remains unchanged if we add to v a multiple of u and to w a linear combination of u and v we may assume that u, v, w are pairwise orthogonal. In this case the second equation in (81) is obvious. Thus we have proved that (iv) implies (81). It follows from (81) that the inner product is uniquely determined by g.

We prove that (i) implies (ii). If $u, v \in W$ are linearly independent then the subspace

$$W_{u,v} := \{w \in W \mid g(u, v; w, u) = g(v, w; u, v) = 0\}$$

has codimension two and $W = W_{u,v} \oplus \operatorname{span}\{u, v\}$. Now fix an element $w \in W_{u,v}$. Then (77) asserts that the linear functionals $x \mapsto g(u, w; u, x)$ and $x \mapsto g(v, w; v, x)$ on W have the same kernel. Hence there exists a constant $\lambda \in \mathbb{R}$ such that $g(v, w; v, x) = \lambda g(u, w; u, x)$ for all $x \in W$. With $x = w$ we obtain $\lambda = g(v, w; v, w)/g(u, w; u, w)$ and hence

$$g(u, w; u, x)g(v, w; v, w) = g(u, w; u, w)g(v, w; v, x)$$

for all $x \in W$. This equation asserts that the differential of the map

$$W_{u,v} \setminus \{0\} \to \mathbb{R} : w \mapsto \frac{g(u, w; u, w)}{g(v, w; v, w)}$$

vanishes everywhere and so the map is constant. This proves (78) for all $w, w' \in W_{u,v} \setminus \{0\}$. Since adding to w a linear combination of u and v does not change the determinants of $\Lambda_u(v,w)$ and $\Lambda_v(u,w)$, equation (78) continues to hold for all $w,w' \in W$ that are linearly independent of u and v. Thus we have proved that (i) implies (ii).

We prove that (ii) implies (iii). It follows from (78) that

$$w,w' \in W_{u,v} \setminus \{0\} \qquad \Longrightarrow \qquad \frac{g(u,w;u,w)}{g(v,w;v,w)} = \frac{g(u,w';u,w')}{g(v,w';v,w')}.$$

Using this identity with u replaced by $u+v$ we obtain

$$w,w' \in W_{u,v} \setminus \{0\} \qquad \Longrightarrow \qquad \frac{g(u,w;v,w)}{g(v,w;v,w)} = \frac{g(u,w';v,w')}{g(v,w';v,w')}.$$

Now let $w,w' \in W_{u,v}$ and assume $g(u,w;v,w) = 0$. Then we also have $g(u,w';v,w') = 0$ and all the off-diagonal terms in the matrices $\Lambda_u(v,w)$, $\Lambda_u(v,w')$, $A(u,v,w)$, and $A(u,v,w')$ vanish. Hence

$$\begin{aligned}
\frac{\det(\Lambda_u(v,w))^2}{\det(A(u,v,w))} &= \frac{g(u,v;u,v)g(u,w;u,w)}{g(v,w;v,w)} \\
&= \frac{g(u,v;u,v)g(u,w';u,w')}{g(v,w';v,w')} = \frac{\det(\Lambda_u(v,w'))^2}{\det(A(u,v,w'))}.
\end{aligned}$$

Thus we have proved (79) under the assumption $w,w' \in W_{u,v} \setminus \{0\}$ and $g(w,u;w,v) = 0$. Since the determinants of $\Lambda_u(v,w)$ and $A(u,v,w)$ remain unchanged if we add to w a linear combination of u and v and if we add to v a multiple of u equation (79) continues to hold when $v = v'$. If u,v,w and u,v,w' and u,v',w' are all linearly independent triples we obtain

$$\frac{\det(\Lambda_u(v,w))^2}{\det(A(u,v,w))} = \frac{\det(\Lambda_u(v,w'))^2}{\det(A(u,v,w'))} = \frac{\det(\Lambda_u(v',w'))^2}{\det(A(u,v',w'))}.$$

Here the last equation follows from the first by symmetry in v and w. This proves (79) under the additional assumption that u,v,w' is a linearly independent triple. This assumption can be dropped by continuity. Thus we have proved that (ii) implies (iii).

We prove that (iii) implies (iv). Define a function $W \to [0,\infty) : u \mapsto |u|$ by $|u| := 0$ for $u = 0$ and by

$$|u|^2 := \frac{g(u,w;u,w)g(u,v;u,v) - g(u,v;u,w)^2}{\sqrt{\det(A(u,v,w))}} \tag{82}$$

for $u \neq 0$, where $v,w \in W$ are chosen such that u,v,w are linearly independent. By (79) the right hand side of (82) is independent of v and w. It follows from (82) with u replaced by $u+v$ that

$$|u+v|^2 - |u|^2 - |v|^2 = 2\frac{g(u,w;v,w)g(u,v;u,v) - g(u,v;u,w)g(u,v;v,w)}{\sqrt{\det(A(u,v,w))}}.$$

Replacing v by $-v$ gives $|u+v|^2 + |u-v|^2 = 2|u|^2 + 2|v|^2$. Thus the map $W \to [0,\infty) : u \mapsto |u|$ is continuous, satisfies the parallelogram identity, and vanishes only for $u=0$. Hence it is a norm on W and the associated inner product of two linearly independent vectors $u, v \in W$ is given by

$$\langle u, v \rangle := \frac{g(u,w;v,w)g(u,v;u,v) - g(u,v;u,w)g(u,v;v,w)}{\sqrt{\det(A(u,v,w))}} \tag{83}$$

whenever $w \in W$ is chosen such that u, v, w are linearly independent. That this inner product satisfies (80) for every pair of linearly independent vectors follows from (82) and (83) with $w \in W_{u,v}$. This proves that (iii) implies (iv).

We prove that (iv) implies (i). Replacing v in equation (80) by $v+w$ we obtain

$$g(u,v;u,w) = |u|^2 \langle v, w \rangle - \langle u, v \rangle \langle u, w \rangle.$$

for all $u, v, w \in W$. Hence

$$g(u,v;w,u) = g(v,w;u,v) = 0 \qquad \Longleftrightarrow \qquad \langle u, w \rangle = \langle v, w \rangle = 0.$$

If $w \in W$ is orthogonal to u and v then $g(u,w;u,x) = |u|^2 \langle w, x \rangle$ and $g(v,w;v,x) = |v|^2 \langle w, x \rangle$. This implies (77) and proves the lemma. □

Proof of Theorem 7.4. If Φ is nondegenerate and $u \in W$ is nonzero then $\iota(u)\Phi$ descends to a nondegenerate 3-form on the 7-dimensional quotient space $W/\mathbb{R}u$. By Lemma 3.4 this implies that $\iota(v)\iota(u)\Phi \wedge \iota(v)\iota(u)\Phi \wedge \iota(u)\Phi$ descends to a nonzero 7-form on $W/\mathbb{R}u$ for every vector $v \in W \setminus \mathbb{R}u$. Hence the 8-form $\iota(v)\iota(u)\Phi \wedge \iota(v)\iota(u)\Phi \wedge \Phi$ on W is nonzero whenever u, v are linearly independent. The orientation on W induced by this form is independent of the choice of the pair u, v. Choose any volume form $\Omega \in \Lambda^8 W^*$ compatible with this orientation and, for $\lambda > 0$, define a multi-linear function $g_\lambda : W^4 \to \mathbb{R}$ by

$$g_\lambda(u,v;w,x) := \frac{\iota(v)\iota(u)\Phi \wedge \iota(x)\iota(w)\Phi \wedge \Phi}{6\lambda^4\Omega} \tag{84}$$

This function satisfies (75) and (76) and, if Φ satisfies (C), it also satisfies (77). Hence it follows from Lemma 7.7 that there is a unique inner product $\langle \cdot, \cdot \rangle_\lambda$ on W such that, for all $u, v \in W$, we have

$$g_\lambda(u,v;u,v) = |u|_\lambda^2 |v|_\lambda^2 - \langle u, v \rangle_\lambda^2. \tag{85}$$

Let dvol_λ be the volume form associated to the inner product and the orientation. Then there is a constant $\mu(\lambda) > 0$ such that $\mathrm{dvol}_\lambda = \mu(\lambda)^2\Omega$. We have $g_\lambda = \lambda^{-4}g_1$, hence $|u|_\lambda = \lambda^{-1}|u|_1$ for every $u \in W$, and hence $\mathrm{dvol}_\lambda = \lambda^{-8}\mathrm{dvol}_1$. Thus $\mu(\lambda) = \lambda^{-4}\mu(1)$. With $\lambda := \mu(1)^{1/6}$ we obtain $\mu(\lambda) = \lambda^{-4}\mu(1) = \mu(1)^{1/3} = \lambda^2$. With this value of λ we have $\lambda^4\Omega = \mathrm{dvol}_\lambda$. Hence it follows from (84) and (85) that

$$\iota(v)\iota(u)\Phi \wedge \iota(v)\iota(u)\Phi \wedge \Phi = 6\left(|u|_\lambda^2\,|v|_\lambda^2 - \langle u, v\rangle_\lambda^2\right)\mathrm{dvol}_\lambda.$$

Hence, by Lemma 7.3, Φ is compatible with the inner product $\langle\cdot,\cdot\rangle_\lambda$. This shows that every 4-form $\Phi \in \Lambda^4W^*$ that satisfies (C) is compatible with a unique inner product.

Conversely, suppose that Φ is compatible with an inner product. Then, by Lemma 7.3, there is an orientation on W such that the associated volume form $\mathrm{dvol} \in \Lambda^8W^*$ satisfies (70). Define $g : W^4 \to \mathbb{R}$ by

$$g(u, v; w, x) := \frac{\iota(v)\iota(u)\Phi \wedge \iota(x)\iota(w)\Phi \wedge \Phi}{6\mathrm{dvol}}.$$

By (70) this map satisfies condition (iv) in Lemma 7.7 and it obviously satisfies (75) and (76). Hence it satisfies condition (i) in Lemma 7.7 and this implies that Φ satisfies (C). This proves the theorem. $\square$

Proof of Theorem 7.6. Suppose that $\Phi \in \Lambda^4W^*$ is a positive Cayley-form with the associated inner product, orientation, and triple cross product. Let $\phi_0 \in \Lambda^3(\mathbb{R}^7)^*$ and $\psi_0 \in \Lambda^4(\mathbb{R}^7)^*$ be the standard associative and coassociative calibrations defined in Example 2.7 and in the proof of Lemma 4.5. Then $\Phi_0 := 1^* \wedge \phi_0 + \psi_0 \in \Lambda^4(\mathbb{R}^8)^*$ is the standard Cayley-form on $\mathbb{R}^8$.

Choose a unit vector $e \in W$ and denote

$$V_e := e^\perp, \qquad \phi_e := \iota(e)\Phi|_{V_e} \in \Lambda^3V_e^*, \qquad \psi_e := \Phi|_{V_e} \in \Lambda^4V_e^*.$$

Then ϕ_e is a nondegenerate 3-form on V_e and hence, by Theorem 3.2, there is an isomorphism $g : \mathbb{R}^7 \to V_e$ such that $g^*\phi_e = \phi_0$. It follows also from Theorem 3.2 that g identifies the standard inner product on $\mathbb{R}^7$ with the unique inner product on V_e that is compatible with ϕ_e, and the standard orientation on $\mathbb{R}^7$ with the orientation determined by ϕ_e via Lemma 3.4. Hence it follows from Lemma 4.5 that g also identifies the two coassociative calibrations, i.e. $g^*\psi_e = \psi_0$. Since Φ is a positive Cayley-form we have $\Phi = e^*\wedge\phi_e+\psi_e$. Hence, if we extend g to an isomorphism $\mathbb{R}^8 = \mathbb{R}\oplus\mathbb{R}^7 \to W$, which is still denoted by g and sends $e_0 = 1 \in \mathbb{R} \subset \mathbb{R}^8$ to e, we obtain $g^*\Phi = \Phi_0$ and this proves the theorem. $\square$

Remark 7.8. The space $S^2\Lambda^2W^*$ of symmetric bilinear forms on Λ^2W can be identified with the space of multi-linear maps $g : W^4 \to \mathbb{R}$ that satisfy (75). Denote by $S^2_0\Lambda^2W^* \subset S^2\Lambda^2W^*$ the subspace of all $g \in S^2\Lambda^2W^*$ that satisfy the algebraic Bianchi identity

$$g(u,v;w,x) + g(v,w;u,x) + g(w,u;v,x) = 0 \tag{86}$$

for all $u, v, w, x \in W$. Then there is a direct sum decomposition

$$S^2\Lambda^2W^* = \Lambda^4W^* \oplus S^2_0\Lambda^2W^*$$

and the projection $\Pi : S^2\Lambda^2W^* \to \Lambda^4W^*$ is given by

$$(\Pi g)(u,v,w,x) := \frac{1}{3}\big(g(u,v;w,x) + g(v,w;u,x) + g(w,u;v,x)\big).$$

Note that

$$\dim \Lambda^2W = 28, \qquad \dim S^2\Lambda^2W = 406,$$
$$\dim \Lambda^4W = 70, \qquad \dim S^2_0\Lambda^2W = 336.$$

Moreover, there is a natural quadratic map $q^\Lambda : S^2W^* \to S^2_0\Lambda^2W^*$ given by

$$\big(q^\Lambda(\gamma)\big)(u,v;x,y) := \gamma(u,x)\gamma(v,y) - \gamma(u,y)\gamma(v,x)$$

for $\gamma \in S^2W^*$ and $u, v, x, y \in W$. Lemma 7.7 asserts, in particular, that the restriction of this map to the subset of inner products is injective and, for each element $g \in S^2\Lambda^2W^*$, it gives a necessary and sufficient condition for the existence of an inner product γ on W such that

$$g - \Pi g = q^\Lambda(\gamma).$$

We shall see in Corollary 9.6 below that, if $\Phi \in \Lambda^4W^*$ is a positive Cayley-form and $g = g_\Phi \in S^2\Lambda^2W^*$ is given by

$$g_\Phi(u,v;x,y) := \frac{\iota(v)\iota(u)\Phi \wedge \iota(y)\iota(x)\Phi \wedge \Phi}{\mathrm{dvol}}, \qquad \mathrm{dvol} := \frac{\Phi \wedge \Phi}{14},$$

then

$$g_\Phi = 6q^\Lambda(\gamma) + 7\Phi$$

for a unique inner product $\gamma \in S^2W^*$, and the volume form of γ is indeed dvol. Thus, in particular, we have $\Pi g_\Phi = 7\Phi$.

Remark 7.9. The space $S^2S^2W^*$ of symmetric bilinear forms on S^2W can be identified with the space of multi-linear maps $\sigma : W^4 \to \mathbb{R}$ that satisfy

$$\sigma(u,v;x,y) = \sigma(x,y;u,v) = \sigma(v,u;x,y). \tag{87}$$

Denote by $S_0^2S^2W^*$ the subspace of all $\sigma \in S^2S^2W^*$ that satisfy the algebraic Bianchi identity (86). Then

$$S^2S^2W^* = S^4W^* \oplus S_0^2S^2W^*,$$

where

$$\dim S^2W = 36, \qquad \dim S^2S^2W = 666,$$

$$\dim S^4W = 330, \qquad \dim S_0^2S^2W = 336.$$

The projection $\Pi : S^2S^2W^* \to S^4W^*$ is given by the same formula as above. Thus

$$(\sigma - \Pi\sigma)(u,v;x,y) = \frac{2}{3}\sigma(u,v;x,y) - \frac{1}{3}\sigma(v,x;u,y) - \frac{1}{3}\sigma(x,u;v,y).$$

There is a natural quadratic map $q^S : S^2W^* \to S^2S^2W^*$ given by

$$\bigl(q^S(\gamma)\bigr)(u,v;x,y) := \gamma(u,v)\gamma(x,y).$$

Polarizing the quadratic map $q^\Lambda : S^2W^* \to S^2\Lambda^2W^*$ one obtains a linear map $T : S^2S^2W^* \to S^2\Lambda^2W^*$ given by

$$\bigl(T\sigma\bigr)(u,v;x,y) := \sigma(u,x;v,y) - \sigma(u,y;v,x)$$

such that $q^\Lambda = T \circ q^S$. The image of T is the subspace $S_0^2\Lambda^2W^*$ of solutions of the algebraic Bianchi identity (86) and its kernel is the subspace S^4W^*. A pseudo-inverse of T is the map $S : S^2\Lambda^2W^* \to S^2S^2W^*$ given by

$$\bigl(Sg\bigr)(u,v;x,y) := \frac{1}{3}\bigl(g(u,x;v,y) + g(u,y;v,x)\bigr)$$

whose kernel is Λ^4W^* and whose image is $S_0^2S^2W^*$. Thus

$$TSg = g - \Pi g, \qquad ST\sigma = \sigma - \Pi\sigma$$

for $g \in S^2\Lambda^2W^*$ and $\sigma \in S^2S^2W^*$. Given $g \in S^2\Lambda^2W^*$ and $\gamma \in S^2W^*$, we have

$$g - \Pi g = q^\Lambda(\gamma) \qquad \iff \qquad Sg = (\mathbb{1} - \Pi)q^S(\gamma).$$

Namely, if $q^\Lambda(\gamma) = g - \Pi g$ then $Sg = S(g - \Pi g) = Sq^\Lambda(\gamma) = q^S(\gamma) - \Pi q^S(\gamma)$, and if $(\mathbb{1} - \Pi)q^S(\gamma) = Sg$ then $(\mathbb{1} - \Pi)g = TSg = T(\mathbb{1} - \Pi)q^S(\gamma) = q^\Lambda(\gamma)$.

8 The group G_2

Let V be a 7-dimensional real Hilbert space equipped with a cross product and let $\phi \in \Lambda^3 V^*$ be the associative calibration defined by (5). We orient V as in Lemma 3.4 and denote by $* : \Lambda^k V^* \to \Lambda^{7-k} V^*$ the associated Hodge $*$-operator and by $\psi := *\phi \in \Lambda^4 V^*$ the coassociative calibration. Recall that V is equipped with an associator bracket via (16), related to ψ via (19), and with a coassociator bracket (25).

The group of automorphisms of ϕ will be denoted by

$$\mathrm{G}(V,\phi) := \{ g \in \mathrm{GL}(V) \,|\, g^*\phi = \phi \} .$$

By Lemma 2.9, we have $\mathrm{G}(V,\phi) \subset \mathrm{SO}(V)$ and hence, by (5),

$$\mathrm{G}(V,\phi) = \{ g \in \mathrm{SO}(V) \,|\, gu \times gv = g(u \times v) \; \forall \, u, v \in V \} .$$

For the standard structure ϕ_0 on $\mathbb{R}^7$ in Example 2.7 we denote the structure group by $\mathrm{G}_2 := \mathrm{G}(\mathbb{R}^7, \phi_0)$. By Theorem 3.2, the group $\mathrm{G}(V,\phi)$ is isomorphic to G_2 for every nondegenerate 3-form on a 7-dimensional vector space.

Theorem 8.1. *The group* $\mathrm{G}(V,\phi)$ *is a* 14*-dimensional simple, connected, simply connected Lie group. It acts transitively on the unit sphere and, for every unit vector* $u \in V$*, the isotropy subgroup* $\mathrm{G}_u := \{ g \in \mathrm{G}(V,\phi) \,|\, gu = u \}$ *is isomorphic to* $\mathrm{SU}(3)$*. Thus there is a fibration*

$$\mathrm{SU}(3) \hookrightarrow \mathrm{G}_2 \longrightarrow S^6 .$$

Proof. As we have observed in Step 4 in the proof of Lemma 3.4, the group $\mathrm{G} = \mathrm{G}(V,\phi)$ has dimension at least 14, as it is an isotropy subgroup of the action of the 49-dimensional group $\mathrm{GL}(V)$ on the 35-dimensional space $\Lambda^3 V^*$. Since $\mathrm{G} \subset \mathrm{SO}(V)$, by Lemma 2.9, the group acts on the unit sphere

$$S := \{ u \in V \mid |u| = 1 \} .$$

Thus, for every $u \in S$, the isotropy subgroup G_u has dimension at least 8. And G_u preserves the subspace $W_u := u^\perp$, the symplectic form $\omega_u(v,w) = \langle u, v \times w \rangle$, and the complex structure $J_u v = u \times v$ on W_u (see Lemma 2.8). Hence G_u is isomorphic to a subgroup of $\mathrm{U}(W_u, \omega_u, J_u) \cong \mathrm{U}(3)$. Now consider the complex valued 3-form $\theta_u \in \Lambda^{3,0} W_u^*$ given by

$$\theta_u(x,y,z) := \phi(x,y,z) - i\phi(u \times x, y, z) = \phi(x,y,z) - i\psi(u,x,y,z)$$

for $x, y, z \in W_u$. (See (16) and (19) for the last equality.) This form is nonzero and is preserved by G_u. Hence G_u is isomorphic to a subgroup of

$\mathrm{SU}(W_u, \omega_u, J_u)$. Since $\mathrm{SU}(W_u, \omega_u, J_u) \cong \mathrm{SU}(3)$ is a connected Lie group of dimension 8 and G_u has dimension at least 8, it follows that

$$\mathrm{G}_u \cong \mathrm{SU}(W_u, \omega_u, J_u) \cong \mathrm{SU}(3).$$

In particular, $\dim \mathrm{G}_u = 8$ and so $\dim \mathrm{G} \leq \dim \mathrm{G}_u + \dim S = 14$. This implies $\dim \mathrm{G} = 14$ and, since S is connected, G acts transitively on S. Thus we have proved that there is a fibration $\mathrm{SU}(3) \hookrightarrow \mathrm{G} \to S$. It follows from the homotopy exact sequence of this fibration that G is connected and simply connected and that $\pi_3(\mathrm{G}) \cong \mathbb{Z}$. Hence G is simple.

Here is another proof that G is simple. Let $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$ denote its Lie algebra and, for every $u \in S$, let $\mathfrak{g}_u := \mathrm{Lie}(\mathrm{G}_u)$ denote the Lie algebra of the isotropy subgroup. Then, for every $\xi \in \mathfrak{g}$, we have $\xi \in \mathfrak{g}_u$ if and only if $u \in \ker \xi$. Since every $\xi \in \mathfrak{g}$ is skew-adjoint, it has a nontrivial kernel and hence belongs to $\mathfrak{g}_u$ for some $u \in S$.

Now let $I \subset \mathfrak{g}$ be a nonzero ideal. Then, by what we have just observed, there is an element $u \in S$ such that $I \cap \mathfrak{g}_u \neq \{0\}$. Thus $I \cap \mathfrak{g}_u$ is a nonzero ideal in $\mathfrak{g}_u$ and, since $\mathfrak{g}_u$ is simple, this implies $\mathfrak{g}_u \subset I$. Next we claim that, for every $v \in u^\perp$, there is an element $\xi \in I$ such that $\xi u = v$. To see this, choose any element $\eta \in \mathfrak{g}_u \subset I$ such that $\ker \eta = \langle u \rangle$. Then there is a unique element $w \in u^\perp$ such that $\eta w = v$. Since G acts transitively on S there is an element $\zeta \in \mathfrak{g}$ such that $\zeta u = w$. Hence $\xi = [\eta, \zeta] \in I$ and $\xi u = \eta \zeta u = \eta w = v$. This proves that $\dim(I/\mathfrak{g}_u) \geq 6$, hence $\dim I \geq 14$, and hence $I = \mathfrak{g}$. This proves the theorem. □

We examine the action of the group $\mathrm{G}(V, \phi)$ on the space

$$\mathscr{S} := \left\{ (u,v,w) \in V \,\middle|\, \begin{array}{l} |u| = |v| = |w| = 1, \\ \langle u, v\rangle = \langle u, w\rangle = \langle v, w\rangle = \langle u \times v, w\rangle = 0 \end{array} \right\}.$$

Let $S \subset V$ denote the unit sphere. Then each tangent space $T_uS = u^\perp$ carries a natural complex structure $v \mapsto u \times v$. The space $\mathscr{S}$ is a bundle over S whose fiber over u is the space of Hermitian orthonormal pairs in T_uS. Hence $\mathscr{S}$ is a bundle of 3-spheres over a bundle of 5-spheres over a 6-sphere and therefore is a compact connected simply connected 14-dimensional manifold.

Theorem 8.2. *The group* $\mathrm{G}(V, \phi)$ *acts freely and transitively on* $\mathscr{S}$.

Proof. We give two proofs of this result. The first proof uses the fact that the isotropy subgroup $\mathrm{G}_u \subset \mathrm{G} := \mathrm{G}(V, \phi)$ of a unit vector $u \in V$ is isomorphic to $\mathrm{SU}(3)$ and the isotropy subgroup in $\mathrm{SU}(3)$ of a Hermitian orthonormal pair is the identity. Hence G acts freely on $\mathscr{S}$. Since G and $\mathscr{S}$ are compact connected manifolds of the same dimension, this implies that G acts transitively on $\mathscr{S}$.

For the second proof we assume that $\phi = \phi_0$ is the standard structure on $V = \mathbb{R}^7$. Given $(u, v, w) \in \mathscr{S}$, define $g : \mathbb{R}^7 \to \mathbb{R}^7$ by

$$\begin{gathered} ge_1 = u, \quad ge_2 = v, \quad ge_3 = u \times v, \quad ge_4 = w \\ ge_5 = w \times u, \quad ge_6 = w \times v, \quad ge_7 = w \times (u \times v). \end{gathered}$$

By construction g preserves the cross product and the inner product. Hence $g \in \mathrm{G}_2$. Moreover, g is the unique element of G_2 that maps the triple (e_1, e_2, e_4) to (u, v, w). This proves the theorem. □

Corollary 8.3. *The group* $\mathrm{G}(V, \phi)$ *acts transitively on the space of associative subspaces of* V *and on the space of coassociative subspaces of* V*.*

Proof. This follows from Theorem 8.2, Lemma 4.4, and Lemma 4.12. □

Theorem 8.4. *There are orthogonal splittings*

$$\begin{aligned} \Lambda^2 V^* &= \Lambda^2_7 \oplus \Lambda^2_{14}, \\ \Lambda^3 V^* &= \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}, \end{aligned}$$

where $\dim \Lambda^k_d = d$ *and*

$$\begin{aligned} \Lambda^2_7 &:= \{\iota(u)\phi \,|\, u \in V\} \\ &= \left\{\omega \in \Lambda^2 V^* \,|\, *(\phi \wedge \omega) = 2\omega\right\}, \\ \Lambda^2_{14} &:= \left\{\omega \in \Lambda^2 V^* \,|\, \psi \wedge \omega = 0\right\} \\ &= \left\{\omega \in \Lambda^2 V^* \,|\, *(\phi \wedge \omega) = -\omega\right\}, \\ \Lambda^3_1 &:= \langle \phi \rangle, \\ \Lambda^3_7 &:= \{\iota(u)\psi \,|\, u \in V\}, \\ \Lambda^3_{27} &:= \left\{\omega \in \Lambda^3 V^* \,|\, \phi \wedge \omega = 0,\ \psi \wedge \omega = 0\right\}. \end{aligned}$$

Each of the spaces Λ^k_d *is an irreducible representation of* $\mathrm{G}(V, \phi)$ *and the representations* Λ^2_7 *and* Λ^3_7 *are both isomorphic to* V*. Moreover, the operator*

$$\Lambda^2 V^* \to \Lambda^2 V^* : \omega \mapsto \frac{1}{3} * (\psi \wedge *(\psi \wedge \omega))$$

is the projection onto Λ^2_7 *with respect to the splitting, i.e.*

$$* (\psi \wedge *(\psi \wedge \omega)) = \omega + *(\phi \wedge \omega) = *(\phi \wedge *(\phi \wedge \omega)) - \omega \tag{88}$$

for every $\omega \in \Lambda^2 V^*$*.*

Proof. For $u \in V$ denote by $A_u \in \mathfrak{so}(V)$ the endomorphism $A_u v := u \times v$. Then the Lie algebra $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$ of $\mathrm{G} = \mathrm{G}(V, \phi)$ is given by

$$\mathfrak{g} = \{\xi \in \mathrm{End}(V) \,|\, \xi + \xi^* = 0,\ A_{\xi u} + [A_u, \xi] = 0\ \forall\, u \in V\}\,.$$

Step 1. *There is an orthogonal decomposition*

$$\mathfrak{so}(V) = \mathfrak{g} \oplus \mathfrak{h}, \qquad \mathfrak{h} := \{A_u \,|\, u \in V\}$$

with respect to the inner product $\langle \xi, \eta \rangle := -\frac{1}{2}\mathrm{trace}(\xi\eta)$ *on* $\mathfrak{so}(V)$.

The group G acts on the space $\mathfrak{so}(V)$ of skew-adjoint endomorphisms by conjugation and this action preserves the inner product. Both subspaces $\mathfrak{g}$ and $\mathfrak{h}$ are invariant under this action, because $gA_ug^{-1} = A_{gu}$ for all $u \in V$ and $g \in \mathrm{G}$. If $\xi = A_u \in \mathfrak{g} \cap \mathfrak{h}$ then $0 = \mathcal{L}_{A_u}\phi = 3\iota(u)\psi$ (see equation (23)) and hence $u = 0$. This shows that $\mathfrak{g} \cap \mathfrak{h} = \{0\}$. Since $\dim \mathfrak{g} = 14$, $\dim \mathfrak{h} = 7$, and $\dim \mathfrak{so}(V) = 21$, we have $\mathfrak{so}(V) = \mathfrak{g} \oplus \mathfrak{h}$. Moreover, $\mathfrak{g}^\perp$ is another G-invariant complement of $\mathfrak{g}$. Hence $\mathfrak{h}$ is the graph of a G-equivariant linear map $\mathfrak{g}^\perp \to \mathfrak{g}$. The image of this map is an ideal in $\mathfrak{g}$ and hence must be zero. This shows that $\mathfrak{h} = \mathfrak{g}^\perp$.

Step 2. Λ^2_{14} *is the orthogonal complement of* Λ^2_7.

It follows from equation (24) with $\alpha = \phi$ that $u^* \wedge \psi = *\iota(u)\phi$. Hence $u^* \wedge \omega \wedge \psi = \omega \wedge *\iota(u)\phi$ and this proves Step 2.

Step 3. *The isomorphism* $\mathfrak{so}(V) \to \Lambda^2 V^* : \xi \mapsto \omega_\xi := \langle \cdot, \xi \cdot \rangle$ *is an* $\mathrm{SO}(V)$-*equivariant isometry and maps* $\mathfrak{g}$ *onto* Λ^2_{14}.

That the isomorphism $\xi \mapsto \omega_\xi$ is an $\mathrm{SO}(V)$-equivariant isometry follows directly from the definitions. The image of $\mathfrak{h}$ under this isomorphism is obviously the subspace Λ^2_7. Hence, by Step 1, the orthogonal complement of Λ^2_7 is the image of $\mathfrak{g}$ under this isomorphism. Hence the assertion follows from Step 2.

Step 4. *For every* $u \in V$ *we have*

$$*(\phi \wedge \iota(u)\phi) = 2\iota(u)\phi, \qquad |\iota(u)\phi|^2 = 3\,|u|^2\,, \tag{89}$$

$$*(\psi \wedge \iota(u)\phi) = 3u^*, \qquad *(\psi \wedge *(\psi \wedge \iota(u)\phi)) = 3\iota(u)\phi. \tag{90}$$

It suffices to prove these identities for unit vectors $u \in V$. Moreover, since $\mathrm{G}(V, \phi) \subset \mathrm{SO}(V)$ acts transitively on the unit sphere, it suffices to prove them for a single unit vector. By Theorem 3.2 we may in fact assume that

$V = \mathbb{R}^7$ and $\phi = e^{123} - e^{145} - e^{167} - e^{246} - e^{275} - e^{347} - e^{356}$ as in Example 2.7. Then, with $u = e_1$, we have

$$\iota(u)\phi = e^{23} - e^{45} - e^{67}, \qquad \phi \wedge \iota(u)\phi = -2e^{12345} - 2e^{12367} + 2e^{14567}.$$

This proves (89). To prove (90), we recall from Remark 4.6 that

$$\psi = -e^{1247} - e^{1256} + e^{1346} - e^{1357} - e^{2345} - e^{2367} + e^{4567},$$

and hence $\psi \wedge \iota(e_1)\phi = 3e^{234567}$. This proves the first equation in (90). The second follows from the first and the fact that $*\iota(u)\phi = u^* \wedge \psi$.

Step 5. *If* $\omega \in \Lambda^2 V^*$ *satisfies* $\omega \wedge \psi = 0$ *then* $*(\phi \wedge \omega) = -\omega$.

The operator $\omega \mapsto *(\phi \wedge \omega)$ on $\Lambda^2 V^*$ is obviously self-adjoint and equivariant under the action of G. Moreover, the action of G on Λ^2_{14} is irreducible, by Step 3. Hence Λ^2_{14} is (contained in) an eigenspace of the operator $\omega \mapsto *(\phi \wedge \omega)$. We claim that the operator is traceless. To see this, let $e_1, \dots, e_7$ be an orthonormal basis of V and denote by $e^1, \dots, e^7$ the dual basis of V^*. Then the 2-forms $e^{ij} := e^i \wedge e^j$ with $i < j$ form an orthonormal basis of $\Lambda^2 V^*$ and we have

$$\sum_{i<j} \langle e^{ij}, *(\phi \wedge e^{ij}) \rangle = \sum_{i<j} (e^{ij} \wedge e^{ij} \wedge \phi)(e_1, \dots, e_7) = 0.$$

By Step 4, our operator has eigenvalue 2 on the 7-dimensional subspace Λ^2_7 and $\dim \Lambda^2 V^* = 21$. Hence the operator has eigenvalue -1 on the 14-dimensional subspace Λ^2_{14}. This proves Step 5.

Step 6. *The subspaces* Λ^3_1, Λ^3_7, *and* Λ^3_{27} *form an orthogonal decomposition of* $\Lambda^3 V^*$ *and* $\dim \Lambda^3_d = d$.

That $\dim \Lambda^3_d = d$ for $d = 1, 7$ is obvious. Since $*\iota(u)\psi = -u^* \wedge \phi$ it follows that Λ^3_1 is orthogonal to Λ^3_7. Moreover, for every $\omega \in \Lambda^3 V^*$, we have

$$\phi \wedge \omega = 0 \quad \iff \quad u^* \wedge \phi \wedge \omega = 0 \; \forall u \in V \quad \iff \quad \omega \perp \Lambda^3_7$$

and $\psi \wedge \omega = 0$ if and only if $\omega \perp \Lambda^3_1$. Since $\dim \Lambda^3 V^* = 35$, this proves Step 6. We also point out that Λ^3_7 is the tangent space of the orbit of ϕ under the action of $\mathrm{SO}(V)$. The irreducibility of Λ^3_1 and $\Lambda^2_7 \cong \Lambda^3_7$ is obvious, for Λ^2_{14} it follows from Step 3, and for Λ^3_{27} we refer to [1]. (The space Λ^3_{27} can be identified with the space of traceless symmetric endomorphisms $S : V \to V$ via $S \mapsto S^*\phi$.) Equation (88) follows from the fact that the operator $\omega \mapsto *(\psi \wedge *(\psi \wedge \omega))$ vanishes on Λ^2_{14}, by definition, and has eigenvalue 3 on Λ^2_7, by (90) in Step 4. This proves the theorem. □

9 The group Spin(7)

Let W be an 8-dimensional real Hilbert space equipped with a positive triple cross product and let $\Phi \in \Lambda^4 W^*$ be the Cayley calibration defined by (61). We orient W so that $\Phi \wedge \Phi > 0$ and denote by $* : \Lambda^k W^* \to \Lambda^{8-k} W^*$ the associated Hodge $*$-operator. Then Φ is self-dual, by Remark 6.8. Recall that, for every unit vector $e \in W$, the subspace $V_e := e^\perp$ is equipped with a cross product $u \times_e v := u \times e \times v$ and that

$$\Phi = e^* \times \phi_e + \psi_e, \quad \phi_e := \iota(e)\Phi \in \Lambda^3 W^*, \quad \psi_e := *(e^* \wedge \phi_e) \in \Lambda^4 W^*,$$

(see Theorem 6.7). The orientation of W is compatible with the decomposition $W = \langle e \rangle \oplus V_e$ (see Remark 6.8).

The group of automorphisms of Φ will be denoted by

$$\mathrm{G}(W, \Phi) := \{ g \in \mathrm{GL}(W) \,|\, g^*\Phi = \Phi \}.$$

By Theorem 7.4, we have $\mathrm{G}(W, \Phi) \subset \mathrm{SO}(W)$ and hence

$$\mathrm{G}(W, \Phi) = \{ g \in \mathrm{SO}(W) \,|\, gu \times gv \times gw = g(u \times v \times w) \; \forall \, u, v, w \in W \}.$$

For the standard structure Φ_0 on $\mathbb{R}^8$ in Example 5.9 we denote the structure group by $\mathrm{Spin}(7) := \mathrm{G}(\mathbb{R}^8, \Phi_0)$. By Theorem 7.6, the group $\mathrm{G}(W, \Phi)$ is isomorphic to Spin(7) for every positive Cayley-form on an 8-dimensional vector space.

Theorem 9.1. *The group* $\mathrm{G}(W, \Phi)$ *is a* 21*-dimensional simple, connected, simply connected Lie group. It acts transitively on the unit tangent bundle of the unit sphere and, for every unit vector* $e \in W$*, the isotropy subgroup* $\mathrm{G}_e := \{ g \in \mathrm{G}(W, \Phi) \,|\, ge = e \}$ *is isomorphic to* G_2*. Thus there is a fibration*

$$\mathrm{G}_2 \hookrightarrow \mathrm{Spin}(7) \longrightarrow S^7.$$

Proof. The isotropy subgroup G_e is obviously isomorphic to $\mathrm{G}(V_e, \phi_e)$ and hence to G_2. We prove that $\mathrm{G}(W, \Phi)$ acts transitively on the unit sphere. Let $u, v \in W$ be two unit vectors and choose a unit vector $e \in W$ which is orthogonal to u and v. By Theorem 8.1, the isotropy subgroup G_e acts transitively on the unit sphere in V_e. Hence there is an element $g \in \mathrm{G}_e$ such that $gu = v$. That $\mathrm{G}(W, \Phi)$ acts transitively on the set of pairs of orthonormal vectors now follows immediately from Theorem 8.1. In particular, there is a fibration $\mathrm{G}_2 \hookrightarrow \mathrm{Spin}(7) \longrightarrow S^7$. It follows from the homotopy exact sequence of this fibration and Theorem 8.1 that Spin(7) is connected and simply connected, and that $\pi_3(\mathrm{Spin}(7)) \cong \mathbb{Z}$. Hence Spin(7) is simple. This proves the theorem. $\square$

Lemma 9.2. *Abbreviate*

$$\mathrm{G} := \mathrm{G}(W, \Phi), \qquad \mathfrak{g} := \mathrm{Lie}(\mathrm{G}) \subset \mathfrak{so}(W).$$

The homomorphism $\rho : \mathrm{G}(W, \Phi) \to \mathrm{SO}(\mathfrak{g}^\perp)$ *is a nontrivial double cover. Hence* Spin(7) *is isomorphic to the universal cover of* SO(7)*.*

Proof. Consider the set

$$I := \{\xi \in \mathfrak{g} \,|\, [\xi, \mathfrak{so}(W)] \subset \mathfrak{g}\}.$$

If $\xi \in I$ and $\eta \in \mathfrak{g}$ then

$$[[\xi, \eta], \zeta] = -[[\eta, \zeta], \xi] - [[\zeta, \xi], \eta] \in \mathfrak{g}$$

for every $\zeta \in \mathfrak{so}(W)$, and hence $[\xi, \eta] \in I$. Thus I is an ideal in $\mathfrak{g}$. Since $\mathfrak{so}(W)$ is simple we have $I \subsetneq \mathfrak{g}$. Since $\mathfrak{g}$ is simple we have $I = \{0\}$. This implies $\mathrm{im}\,\mathrm{Ad}(\xi) \not\subset \mathfrak{g}$ for $0 \neq \xi \in \mathfrak{g}$. Since $\mathrm{Ad}(\xi) : \mathfrak{so}(W) \to \mathfrak{so}(W)$ is skew-adjoint this implies $\mathfrak{g}^\perp \not\subset \ker \mathrm{Ad}(\xi)$ for $0 \neq \xi \in \mathfrak{g}$. This means that the infinitesimal adjoint action defines an isomorphism $\mathfrak{g} \to \mathfrak{so}(\mathfrak{g}^\perp)$. Hence the adjoint action gives rise to a covering map $\mathrm{G} \to \mathrm{SO}(\mathfrak{g}^\perp)$. Since G is connected and simply connected, this implies that G is the universal cover of $\mathrm{SO}(\mathfrak{g}^\perp) \cong \mathrm{SO}(7)$ and this proves the lemma. ☐

We examine the action of the group $\mathrm{G}(W, \Phi)$ on the space

$$\mathscr{S} := \big\{(u, v, w, x) \in W \,\big|\, u, v, w, u \times v \times w, x \text{ are orthonormal}\big\}.$$

The space $\mathscr{S}$ is a bundle of 3-spheres over a bundle of 5-spheres over a bundle of 6-spheres over a 7-sphere. Hence it is a compact connected simply connected 21-dimensional manifold.

Theorem 9.3. *The group* $\mathrm{G}(W, \Phi)$ *acts freely and transitively on* $\mathscr{S}$*.*

Proof. Since Spin(7) acts transitively on S^7 with isotropy subgroup G_2, the result follows immediately from Theorem 8.2. ☐

Corollary 9.4. *The group* $\mathrm{G}(W, \Phi)$ *acts transitively on the space of Cayley subspaces of* W

Proof. This follows from Theorem 9.3 and Lemma 6.10. ☐

Theorem 9.5. *There are orthogonal splittings*

$$\begin{aligned}\Lambda^2W^* &= \Lambda^2_7 \oplus \Lambda^2_{21},\\ \Lambda^3W^* &= \Lambda^3_8 \oplus \Lambda^3_{48},\\ \Lambda^4W^* &= \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27} \oplus \Lambda^4_{35},\end{aligned}$$

where $\dim \Lambda^k_d = d$ *and*

$$\begin{aligned}\Lambda^2_7 &:= \big\{\omega \in \Lambda^2W^* \,|\, *(\Phi \wedge \omega) = 3\omega\big\}\\ &= \{u^* \wedge v^* - \iota(u)\iota(v)\Phi \,|\, u, v \in W\},\\ \Lambda^2_{21} &:= \{\omega_\xi \,|\, \xi \in \mathfrak{g}\}\\ &= \big\{\omega \in \Lambda^2W^* \,|\, *(\Phi \wedge \omega) = -\omega\big\}\\ &= \big\{\omega \in \Lambda^2W^* \,|\, \langle \omega, \iota(u)\iota(v)\Phi\rangle = \omega(u,v)\ \forall\, u, v \in W\big\},\\ \Lambda^3_8 &:= \{\iota(u)\Phi \,|\, u \in W\},\\ \Lambda^3_{48} &:= \big\{\omega \in \Lambda^3W^* \,|\, \Phi \wedge \omega = 0\big\},\\ \Lambda^4_1 &:= \langle \Phi \rangle,\\ \Lambda^4_7 &:= \{\mathcal{L}_\xi\Phi \,|\, \xi \in \mathfrak{so}(W)\},\\ \Lambda^4_{27} &:= \big\{\omega \in \Lambda^4W^* \,|\, *\,\omega = \omega,\, \omega \wedge \Phi = 0,\, \omega \wedge \mathcal{L}_\xi\Phi = 0\,\forall \xi \in \mathfrak{so}(W)\big\},\\ \Lambda^4_{35} &:= \big\{\omega \in \Lambda^4W^* \,|\, *\,\omega = -\omega\big\}.\end{aligned}$$

Here we denote by $\mathfrak{so}(W) \to \Lambda^4W^* : \xi \mapsto \mathcal{L}_\xi\Phi$ *the contravariant infinitesimal action, given by*

$$\mathcal{L}_\xi\Phi := \frac{d}{dt}\bigg|_{t=0} \exp(t\xi)^*\Phi,$$

by $\mathfrak{g} := \mathrm{Lie}(\mathrm{G}(W,\Phi)) \subset \mathfrak{so}(W)$ *the Lie algebra of the structure group, and by* $\mathfrak{so}(W) \to \Lambda^2W^* : \xi \mapsto \omega_\xi$ *the isomorphism given by* $\omega_\xi(u,v) := \langle u, \xi v\rangle$ *for* $\xi \in \mathfrak{so}(W)$ *and* $u, v \in W$*. Each of the spaces* Λ^k_d *is an irreducible representation of* $\mathrm{G}(W,\Phi)$*.*

Proof. By Theorem 9.1, $\mathrm{G} := \mathrm{G}(W,\Phi)$ is simple and so the action of G on $\mathfrak{g}$ by conjugation is irreducible. Hence the 21-dimensional subspace Λ^2_{21} must be contained in an eigenspace of the operator $\omega \mapsto *(\Phi \wedge \omega)$ on Λ^2W^*. We prove that the eigenvalue is -1. To see this, we choose a unit vector $e \in W$ and an element $\xi \in \mathfrak{g}$ with $\xi e = 0$. Let

$$V_e := e^\perp$$

and denote by $\iota_e : V_e \to W$ and $\pi_e : W \to V_e$ the inclusion and orthogonal projection and by $*_e : \Lambda^kV_e^* \to \Lambda^{7-k}V_e^*$ the Hodge $*$-operator on the subspace. Then

$$*(e^* \wedge \pi_e^*\alpha_e) = \pi_e^* *_e \alpha_e \qquad \forall\, \alpha_e \in \Lambda^kV_e^*.$$

Moreover, the alternating forms

$$\phi_e := \iota_e^*(\iota(e)\Phi), \qquad \psi_e := \iota_e^*\Phi$$

are the associative and coassociative calibrations of V_e. Since $\xi e = 0$, we have $\omega_\xi = \pi_e^*\iota_e^*\omega_\xi$ and, by Theorem 8.4,

$$\psi_e \wedge \iota_e^*\omega_\xi = 0, \qquad *_e(\phi_e \wedge \iota_e^*\omega_\xi) = -\iota_e^*\omega_\xi.$$

Since $\Phi = e^* \wedge \pi_e^*\phi_e + \pi_e^*\psi_e$, this gives

$$\begin{aligned} *\big(\Phi \wedge \omega_\xi\big) &= *\big((e^* \wedge \pi_e^*\phi_e + \pi_e^*\psi_e) \wedge \pi_e^*\iota_e^*\omega_\xi\big) \\ &= *\big(e^* \wedge \pi_e^*\left(\phi_e \wedge \iota_e^*\omega_\xi\right)\big) + *\pi_e^*\big(\psi_e \wedge \iota_e^*\omega_\xi\big) \\ &= \pi_e^* *_e \left(\phi_e \wedge \iota_e^*\omega_\xi\right) = -\pi_e^*\iota_e^*\omega_\xi = -\omega_\xi. \end{aligned}$$

By Lemma 9.2 the adjoint action of G on $\mathfrak{g}^\perp \subset \mathfrak{so}(W)$ is irreducible, and $\mathfrak{g}^\perp$ is mapped under $\xi \mapsto \omega_\xi$ onto the orthogonal complement of Λ^2_{21}. Hence the 7-dimensional orthogonal complement of Λ^2_{21} is also contained in an eigenspace of the operator

$$\omega \mapsto *(\Phi \wedge \omega).$$

Since this operator is self-adjoint and has trace zero, its eigenvalue on the orthogonal complement of Λ^2_{21} must be 3 and therefore this orthogonal complement is equal to Λ^2_7. It follows that the orthogonal projection of $\omega \in \Lambda^2W^*$ onto Λ^2_7 is given by

$$\pi_7(\omega) = \frac{1}{4}\left(\omega + *(\Phi \wedge \omega)\right).$$

Hence, for $0 \neq e \in W$, we have

$$\begin{aligned} \Lambda^2_7 &= \Big\{e^* \wedge u^* - \iota(e)\iota(u)\Phi \,|\, u \in e^\perp\Big\}, \\ \Lambda^2_{21} &= \Big\{\omega \in \Lambda^2W^* \,|\, \langle\omega, \iota(e)\iota(u)\Phi\rangle = \omega(e,u) \,\forall\, u \in e^\perp\Big\}. \end{aligned}$$

This proves the decomposition result for Λ^2W^*.

We verify the decomposition of Λ^3W^*. For $u \in W$ and $\omega \in \Lambda^3W^*$ we have the equation

$$u^* \wedge \Phi \wedge \omega = -\omega \wedge *\iota(u)\Phi.$$

Hence $\Phi \wedge \omega = 0$ if and only if ω is orthogonal to $\iota(u)\Phi$ for all $u \in W$. This shows that Λ^3_{48} is the orthogonal complement of Λ^3_8. Since Φ is nondegenerate, we have $\dim \Lambda^3_8 = 8$ and, since $\dim \Lambda^3W^* = 56$, it follows that $\dim \Lambda^3_{48} = 48$.

We verify the decomposition of $\Lambda^4 W^*$. The 4-form $g^*\Phi$ is self-dual for every $g \in \mathrm{G} = \mathrm{G}(W, \Phi)$, because Φ is self-dual and $\mathrm{G} \subset \mathrm{SO}(W)$. This implies that $\mathcal{L}_\xi \Phi$ is self-dual for every $\xi \in \mathfrak{g} = \mathrm{Lie}(\mathrm{G})$. Since $\mathrm{SO}(W)$ has dimension 28 and the isotropy subgroup G of Φ has dimension 21, it follows that the tangent space Λ^4_7 to the orbit of Φ under the action of G has dimension 7. As Λ^4_1 has dimension 1 and the space of self-dual 4-forms has dimension 35, the orthogonal complement of $\Lambda^4_1 \oplus \Lambda^4_7$ in the space of self-dual 4-forms has dimension 27. This proves the dimension and decomposition statements.

That the action of G on $\Lambda^2_{21} \cong \mathfrak{g}$ is irreducible follows from the fact that G is simple. Irreducibility of the action on Λ^4_1 is obvious. For $\Lambda^3_8 \cong W$ it follows from the fact that G acts transitively on the unit sphere in W, and for $\Lambda^2_7 \cong \mathfrak{g}^\perp \cong \Lambda^4_7$ it follows from the fact that the istropy subgroup G_e of a unit vector $e \in W$ acts transitively on the unit sphere in $V_e = e^\perp$. For Λ^4_{27}, Λ^4_{35}, and Λ^3_{48} we refer to [1]. This proves the theorem. □

Corollary 9.6. *For $u, v \in W$ denote $\omega_{u,v} := \iota(v)\iota(u)\Phi = \Phi(u, v, \cdot, \cdot)$. Then, for all $u, v, x, y \in W$ we have*

$$*(\Phi \wedge u^* \wedge v^*) = \omega_{u,v}, \qquad *(\Phi \wedge \omega_{u,v}) = 3u^* \wedge v^* + 2\omega_{u,v}, \tag{91}$$

$$\langle \omega_{u,v}, \omega_{x,y} \rangle = 3\big(\langle u, x\rangle\langle v, y\rangle - \langle u, y\rangle\langle v, x\rangle\big) + 2\Phi(u, v, x, y), \tag{92}$$

$$\frac{\omega_{u,v} \wedge \omega_{x,y} \wedge \Phi}{\mathrm{dvol}} = 6\big(\langle u, x\rangle\langle v, y\rangle - \langle u, y\rangle\langle v, x\rangle\big) + 7\Phi(u, v, x, y). \tag{93}$$

Proof. The first equation in (91) is a general statement about the Hodge $*$-operator in any dimension. Moreover, by Theorem 9.5, the 2-form $u^* \wedge v^* + \omega_{u,v}$ is an eigenvector of the operator $\omega \mapsto *(\Phi \wedge \omega)$ with eigenvalue 3. Hence the second equation in (91) follows from the first. To prove (92), take the inner product of the second equation in (91) with $x^* \wedge y^*$ and use the identities

$$\langle \omega_{u,v}, x^* \wedge y^* \rangle = \Phi(u, v, x, y), \tag{94}$$

$$\langle u^* \wedge v^*, x^* \wedge y^* \rangle = \langle u, x\rangle\langle v, y\rangle - \langle u, y\rangle\langle v, x\rangle, \tag{95}$$

and the fact that the operator $\omega \mapsto *(\Phi \wedge \omega)$ is self-adjoint. To prove (93), we observe that

$$\begin{aligned}
\frac{\omega_{u,v} \wedge \omega_{x,y} \wedge \Phi}{\mathrm{dvol}} &= \langle \omega_{u,v}, *(\Phi \wedge \omega_{x,y}) \rangle \\
&= \langle \omega_{u,v}, 3x^* \wedge y^* + 2\omega_{x,y} \rangle \\
&= 6\big(\langle u, x\rangle\langle v, y\rangle - \langle u, y\rangle\langle v, x\rangle\big) + 7\Phi(u, v, x, y),
\end{aligned}$$

where the second equation follows from (91) and the last equation follows from (92) and (95). This proves the lemma. □

10 Spin structures

Let W be an 8-dimensional oriented real Hilbert space, equipped with a positive triple cross product (54), let $\Phi \in \Lambda^4 W^*$ be the Cayley calibration defined by (61), and assume that $\Phi \wedge \Phi > 0$. Recall that, for every unit vector $e \in W$, there is a normed algebra structure on W, defined by (65). This normed algebra structure can be recovered from an intrinsic product map

$$m : W \times W \to \Lambda^0 \oplus \Lambda^2_7$$

(which does not depend on e) and an ismorphism $\gamma(e) : \Lambda^0 \oplus \Lambda^2_7 \to \Lambda^1$ (which does depend on e). The product map is given by

$$m(u,v) = \Bigl(\langle u,v\rangle, \frac{1}{2}\bigl(\omega_{u,v} + u^* \wedge v^*\bigr) \Bigr) \tag{96}$$

for $u, v \in W$ and the isomorphism $\gamma(e)$ by

$$\gamma(e)(\lambda, \omega) := \lambda e^* + 2\iota(e)\omega \tag{97}$$

for $\lambda \in \mathbb{R} = \Lambda^0$ and $\omega \in \Lambda^2_7$. Here it is convenient to identify W with its dual space $W^* = \Lambda^1 W^* = \Lambda^1$. Thus we not do distinguish in notation between W and W^* and also write $\gamma(e) : \Lambda^0 \oplus \Lambda^2_7 \to W$.

Lemma 10.1. *For all $u, v, e \in W$ we have*

$$\gamma(e)m(u,v) = \langle u,v\rangle e + \langle u,e\rangle v - \langle v,e\rangle u + u \times e \times v, \tag{98}$$

$$|m(u,v)| = |u|\,|v|, \qquad |\gamma(e)(\lambda,\omega)|^2 = |e|^2 \Bigl(|\lambda|^2 + |\omega|^2\Bigr). \tag{99}$$

Proof. Equation (98) follows directly from the definitions. Moreover, it follows from (92) that

$$|m(u,v)|^2 = \langle u,v\rangle^2 + \frac{1}{4}|u^* \wedge v^*|^2 + \frac{1}{4}|\omega_{u,v}|^2 = \langle u,v\rangle^2 + |u \wedge v|^2 = |u|^2\,|v|^2.$$

This proves the first equation in (99). To prove the second equation in (99) we observe that $\gamma(e)m(e,v) = v$ and hence $|\gamma(e)m(e,v)| = |v| = |m(e,v)|$ whenever $|e| = 1$. Since the map $W \to \Lambda^0 \oplus \Lambda^2_7 : v \mapsto m(e,v)$ is bijective this proves the lemma. □

Remark 10.2. It turns out that the linear map $\gamma(e) : \Lambda^0 \oplus \Lambda^2_7 \to \Lambda^1 = W^*$ is dual to the map $m(e,\cdot) : W \to \Lambda^0 \oplus \Lambda^2_7$ for every $e \in W$ (see Lemma 10.4). Moreover, if we denote $\bar{v} := 2\langle e,v\rangle e - v$ then the product (65) is given by

$$uv = -\langle u,v\rangle e + \langle u,e\rangle v + \langle v,e\rangle u + u \times e \times v = \overline{\gamma(e)m(u,\bar{v})}.$$

Combining the product map m with the triple cross product we obtain an alternating multi-linear map $\tau : W^4 \to \Lambda^0 \oplus \Lambda^2_7$ defined by

$$\begin{aligned}\tau(x,u,v,w) = \frac{1}{4}\Big(& m(u\times v\times w, x) - m(v\times w\times x, u) \\ & + m(w\times x\times u, v) - m(x\times u\times v, w)\Big).\end{aligned} \tag{100}$$

This map corresponds to the four-fold cross product (see Definition 5.10) and has the following properties (see Theorem 5.11).

Lemma 10.3. *Let $\chi : W^4 \to \Lambda^2_7$ denote the second component of τ. Then, for all $u, v, w, x \in W$, we have*

$$\tau(x,u,v,w) = (\Phi(x,u,v,w), \chi(x,u,v,w)),$$

$$\Phi(x,u,v,w)^2 + |\chi(x,u,v,w)|^2 = |x\wedge u\wedge v\wedge w|^2.$$

Proof. That the first component of τ is equal to Φ follows directly from the definitions. Moreover, for $u, v, w, x \in W$, we have

$$\begin{aligned}2\chi(x,u,v,w) = \,& (u\times v\times w)^* \wedge x^* + \omega_{u\times v\times w, x} \\ & - (v\times w\times x)^* \wedge u^* - \omega_{v\times w\times x, u} \\ & + (w\times x\times u)^* \wedge v^* + \omega_{w\times x\times u, v} \\ & - (x\times u\times v)^* \wedge w^* - \omega_{x\times u\times v, w}.\end{aligned} \tag{101}$$

We claim that the four rows on the right agree whenever u, v, w, x are pairwise orthogonal. Under this assumption the first two rows remain unchanged if we add to x a multiple of $u\times v\times w$. Thus we may assume that x is orthogonal to u, v, w, and $u\times v\times w$. By Theorem 9.3, we may therefore assume that $W = \mathbb{R}^8$ with the standard triple cross product and

$$u = e_0, \qquad v = e_1, \qquad w = e_2, \qquad x = e_4.$$

In this case a direct computation proves that the first two rows agree. Thus we have proved that, if $u, v, w, x \in W$ are pairwise orthogonal, then

$$\tau(x,u,v,w) = m(u\times v\times w, x).$$

In this case it follows from (99) that

$$\begin{aligned}|\tau(x,u,v,w)| &= |m(u\times v\times w, x)| \\ &= |x|\,|u\times v\times w| \\ &= |x|\,|u|\,|v|\,|w| \\ &= |x\wedge u\wedge v\wedge w|.\end{aligned}$$

Since τ is alternating, this proves the lemma. □

Denote

$$S^+ := \mathbb{R} \oplus \Lambda^2_7, \qquad S^- := W^*.$$

Equation (97) defines a vector space homomorphism $\gamma(e) : S^+ \to S^-$ for every $e \in W$. The next lemma asserts that this is a spin structure on W (see Proposition 4.13, Definition 4.32, and Exercise 4.48 in [6]).

Lemma 10.4. *The homomorphism* $\gamma : W \to \mathrm{Hom}(S^+, S^-)$ *satisfies*

$$\gamma(u)^* v^* = m(u,v) = \Big(\langle u, v\rangle, \frac{1}{2}\big(\omega_{u,v} + u^* \wedge v^*\big)\Big) \tag{102}$$

and

$$\gamma(u)^*\gamma(u) = |u|^2\, \mathbb{1} \tag{103}$$

for all $u, v \in W$.

Proof. For $u \in W$, $\lambda \in \mathbb{R}$, $\omega \in \Lambda^2_7$, and $v \in W$ we compute

$$\begin{aligned} \langle \gamma(u)(\lambda, \omega), v^*\rangle &= \langle \lambda u^* + 2\iota(u)\omega, v^*\rangle \\ &= \lambda\langle u, v\rangle + 2\langle \omega, u^* \wedge v^*\rangle \\ &= \lambda\langle u, v\rangle + \frac{1}{2}\left\langle \omega, \omega_{u,v} + u^* \wedge v^*\right\rangle. \end{aligned}$$

The last equation follows from the fact that

$$\pi_7(u^* \wedge v^*) = \frac{1}{4}(\omega_{u,v} + u^* \wedge v^*).$$

This proves (102). Equation (103) follows immediately from (99). This proves the lemma. □

Next we assume that V is a 7-dimensional oriented real Hilbert space equipped with a cross product (1), that $\phi \in \Lambda^3 W^*$ is the associative calibration defined by (5), that $\iota(u)\phi \wedge \iota(u)\phi \wedge \phi > 0$ for $0 \neq u \in V$, and that $\psi = *\phi \in \Lambda^4 V^*$ is the coassociative calibration. Define

$$S := \mathbb{R} \oplus V$$

and, for $u \in V$, define $\gamma(u) : S \to S$ by

$$\gamma(u)(\lambda, v) := (-\langle u, v\rangle, \lambda u + u \times v) \tag{104}$$

for $\lambda \in \mathbb{R}$ and $v \in V$. The next lemma shows that this is a spin structure on V and is compatible with the orientation (see [6, Definition 4.32 and (4.22)]).

Lemma 10.5. *The homomorphism* $\gamma : V \to \mathrm{End}(S)$ *satisfies*

$$\gamma(u)^* + \gamma(u) = 0, \qquad \gamma(u)^*\gamma(u) = |u|^2\, \mathbb{1}, \tag{105}$$

and

$$\gamma(u)\gamma(v)(\mu, w) + \langle u, v\rangle(u, w) = \gamma(u \times v)(\mu, w) - (0, 2[u, v, w]) \tag{106}$$

for all $u, v, w \in V$ *and* $\mu \in \mathbb{R}$. *If* $e_1, \ldots, e_7$ *is a positive orthonormal basis of* V *then*

$$\gamma(e_7)\gamma(e_6)\cdots\gamma(e_1) = -\mathbb{1}. \tag{107}$$

Proof. For $u, v, w \in V$ and $\lambda, \mu \in \mathbb{R}$ we have

$$\langle(\lambda, v), \gamma(u)(\mu, w)\rangle = \mu\langle u, v\rangle - \lambda\langle u, w\rangle + \phi(v, u, w).$$

This expression is skew-symmetric in (λ, v) and (μ, w) and so $\gamma(u)$ is skew-adjoint. Moreover, for $u, v, w \in V$ and $\mu \in \mathbb{R}$, we have

$$\begin{aligned}
&\gamma(u)\gamma(v)(\mu, w) + \langle u, v\rangle(\mu, w) \\
&= (-\langle u, \mu v + v \times w\rangle, -\langle v, w\rangle u + u \times (\mu v + v \times w)) + \langle u, v\rangle(\mu, w) \\
&= (-\langle u \times v, w\rangle, \mu(u \times v) + u \times (v \times w) - \langle v, w\rangle u + \langle u, v\rangle w) \\
&= \gamma(u \times v)(\mu, w) + (0, -(u \times v) \times w - \langle v, w\rangle u + \langle u, w\rangle v) \\
&\quad + (0, -(v \times w) \times u - \langle u, w\rangle v + \langle u, v\rangle w) \\
&= \gamma(u \times v)(\mu, w) - 2(0, [u, v, w]).
\end{aligned}$$

Here the last equation follows from (16). This proves (105) and (106). For the proof of (107) it is convenient to use the standard basis for the standard cross product on $V = \mathbb{R}^7$ in Example 2.7. The left hand side of (107) is independent of the choice of the positive orthonormal basis and we know from general principles that the composition $\gamma(e_7)\cdots\gamma(e_1)$ must equal $\pm\mathbb{1}$ (see [6, Proposition 4.34]). The sign can thus be determined by evaluating the composition of the $\gamma(e_j)$ on a single nonzero vector. We leave the verification to the reader. This proves the lemma. □

Remark 10.6. In the setting of Lemma 10.4 we introduce the 16-dimensional real Hilbert space

$$S := S^+ \oplus S^- = \mathbb{R} \oplus \Lambda^2_7 \oplus W^* = \Lambda^0 \oplus \Lambda^2_7 \oplus \Lambda^1$$

and define the homomorphism $\Gamma : W \to \mathrm{End}(S)$ by

$$\Gamma(u) := \begin{pmatrix} 0 & \gamma(u) \\ -\gamma(u)^* & 0 \end{pmatrix}$$

for $u \in W$. Equation (103) guarantees that Γ extends uniquely to an algebra isomorphism from the Clifford algebra $C(W)$ to $\mathrm{End}(S)$, still denoted by Γ (see [6, Proposition 4.33]). Moreover, complexifying S we obtain an algebra isomorphism $\Gamma : C^c(W) \to \mathrm{End}(S^c)$ from the complexified Clifford algebra $C^c(W) := C(W) \otimes_{\mathbb{R}} \mathbb{C}$ to the complex endomorphisms of $S^c := S \otimes_{\mathbb{R}} \mathbb{C}$.

Remark 10.7. In the setting of Lemma 10.5 equation (105) guarantees that the homomorphism $\gamma : V \to \mathrm{End}(S)$ extends uniquely to an algebra homomorphism $\gamma : C(V) \to \mathrm{End}(S)$ (see [6, Proposition 4.33]). It follows from (107) that the kernel of this isomorphism is given by

$$\gamma(x) = 0 \qquad \Longleftrightarrow \qquad \varepsilon x = x,$$

where

$$\varepsilon := e_7 \cdots e_1 \in C_7(V)$$

for a positive orthonormal basis $e_1, \ldots, e_7$ of V (see [6, Proposition 3.34]). Since ε is an odd element of $C(V)$, this implies that the restrictions of γ to both $C^{\mathrm{ev}}(V)$ and $C^{\mathrm{odd}}(V)$ are injective. Since

$$\dim\, C^{\mathrm{ev}}(V) = \dim\, C^{\mathrm{odd}}(V) = \dim \mathrm{End}(S) = 64,$$

it follows that γ restricts to an algebra isomorphism from $C^{\mathrm{ev}}(V)$ to $\mathrm{End}(S)$ and to a vector space isomorphism from $C^{\mathrm{odd}}(V)$ to $\mathrm{End}(S)$.

11 Octonians and complex linear algebra

Let W be a $2n$-dimensional real vector space. An **$\mathrm{SU}(n)$-structure on W** is a triple (ω, J, θ) consisting of a nondegenerate 2-form $\omega \in \Lambda^2 W^*$, an ω-compatible complex structure $J : W \to W$ (so that $\langle \cdot, \cdot \rangle := \omega(\cdot, J\cdot)$ is an inner product), and a complex multi-linear map $\theta : W^n \to \mathbb{C}$ which has norm $2^{n/2}$ with respect to the metric determined by ω and J. The archetypal example is $W = \mathbb{C}^n$ with the standard symplectic form

$$\omega := \sum_j dx_j \wedge dy_j,$$

the standard complex structure $J := i$, and the standard $(n, 0)$-form

$$\theta := dz_1 \wedge \cdots \wedge dz_n.$$

In this section we examine the relation between SU(3)-structures and cross products and between SU(4)-structures and triple cross products. We also explain the decompositions of Theorems 8.4 and 9.5 in this setting.

Theorem 11.1. *Let W be 6-dimensional real vector space equipped with an* $\mathrm{SU}(3)$*-structure* (ω, J, θ)*. Then the space $V := \mathbb{R}\oplus W$ carries a natural cross product defined by*

$$v \times w := (\omega(v_1, w_1), v_0 J w_1 - w_0 J v_1 + v_1 \times_\theta w_1) \tag{108}$$

for $u = (u_0, u_1), v = (v_0, v_1) \in \mathbb{R} \oplus W$, where $v_1 \times_\theta w_1 \in V$ is defined by $\langle u_1, v_1 \times_\theta w_1\rangle := \mathrm{Re}\,\theta(u_1, v_1, w_1)$ for all $u_1 \in W$. The associative calibration of this cross product is

$$\phi := e^0 \wedge \omega + \mathrm{Re}\,\theta \in \Lambda^3 V^* \tag{109}$$

and the coassociative calibration is

$$\psi := *\phi = \frac{1}{2}\omega \wedge \omega - e^0 \wedge \mathrm{Im}\,\theta \in \Lambda^4 V^*. \tag{110}$$

Moreover, the subspaces $\Lambda^k_d \subset \Lambda^k V^*$ in Theorem 8.4 are given by

$$\begin{aligned}
\Lambda^2_7 &= \mathbb{R}\omega \oplus \{e^0 \wedge u^* - \iota(u)\mathrm{Im}\,\theta \,|\, u \in W\},\\
\Lambda^2_{14} &= \left\{\tau - e^0 \wedge *_W(\tau \wedge \mathrm{Re}\,\theta) \,|\, \tau \in \Lambda^2 W^*,\ \tau \wedge \omega \wedge \omega = 0\right\},\\
\Lambda^3_7 &= \mathbb{R} \cdot \mathrm{Im}\,\theta \oplus \left\{u^* \wedge \omega - e^0 \wedge \iota(u)\mathrm{Re}\,\theta \,|\, u \in W\right\},\\
\Lambda^3_{27} &= \mathbb{R} \cdot \left(3\mathrm{Re}\,\theta - 4e^0 \wedge \omega\right)\\
&\quad \oplus \left\{e^0 \wedge \tau \,|\, \tau \in \Lambda^{1,1}W^*,\ \tau \wedge \omega \wedge \omega = 0\right\}\\
&\quad \oplus \left\{\beta \in \Lambda^{2,1}W^* + \Lambda^{1,2}W^* \,|\, \beta \wedge \omega = 0\right\}\\
&\quad \oplus \left\{u^* \wedge \omega + e^0 \wedge \iota(u)\mathrm{Re}\,\theta \,|\, u \in W\right\}.
\end{aligned}$$

Proof. For $v, w \in W$ we define $\alpha_{v,w} \in \Lambda^1 W^*$ by $\alpha_{v,w} := \mathrm{Re}\,\theta(\cdot, v, w)$. Then $|\alpha_{v,w}| = |\theta(u, v, w)| = |v|\,|w|$ whenever u, Ju, v, Jv, w, Jw are pairwise orthogonal and $|u| = 1$. This implies

$$|\alpha_{v,w}|^2 + \omega(v, w)^2 + \langle v, w\rangle^2 = |v|^2\,|w|^2 \tag{111}$$

for all $v, w \in W$. (Add to w a suitable linear combination of v and Jv.) It follows from (111) by direct computation that the formula (108) defines a cross product on $\mathbb{R}\times W$. By (108) and (109), we have $\phi(u, v, w) = \langle u, v \times w\rangle$ so that ϕ is the associative calibration of (108) as claimed. That ϕ is compatible with the orientation of $\mathbb{R}\oplus W$ follows from the fact that $\iota(e_0)\phi = \omega$ and $\omega \wedge \mathrm{Re}\,\theta = 0$ so that $\iota(e_0)\phi \wedge \iota(e_0)\phi \wedge \phi = e^0 \wedge \omega^3 = 6\mathrm{dvol}$. The formula (110) for $\psi := *\phi$ follows from the fact that $\omega \wedge \theta = 0$ and $\mathrm{Im}\,\theta = *\mathrm{Re}\,\theta$ so that $\mathrm{Re}\,\theta \wedge \mathrm{Im}\,\theta = 4\mathrm{dvol}_W$. It remains to examine the subspaces $\Lambda^k_d \subset \Lambda^k V^*$ introduced in Theorem 8.4.

The formula for Λ^2_7 follows directly from the formula for ϕ in (109) and the fact that Λ^2_7 consists of all 2-forms $\iota(v)\phi$ for $v \in \mathbb{R} \oplus W$. With $v = (1,0)$ we obtain $\iota(v)\phi = \omega$ and with $v = (0, Ju)$ we obtain

$$\iota(v)\phi = -e^0 \wedge \iota(Ju)\omega + \iota(Ju)\mathrm{Re}\,\theta = e^0 \wedge u^* - \iota(u)\mathrm{Im}\,\theta.$$

Similarly, the formula for Λ^3_7 follows directly from the formula for ψ in (109) and the fact that Λ^3_7 consists of all 3-forms $\iota(v)\psi$ for $v \in \mathbb{R} \oplus W$. With $v = (-1, 0)$ we obtain $\iota(v)\psi = \mathrm{Im}\,\theta$ and with $v = (0, -Ju)$ we obtain

$$\iota(v)\psi = -(\iota(Ju)\omega) \wedge \omega - e^0 \wedge \iota(Ju)\mathrm{Im}\,\theta = u^* \wedge \omega - e^0 \wedge \iota(u)\mathrm{Re}\,\theta.$$

To prove the formula for Λ^2_{14} we choose $\alpha \in \Lambda^1 W^*$ and $\tau \in \Lambda^2 W^*$. Then $\tau + e^0 \wedge \alpha \in \Lambda^2_{14}$ if and only if $(\tau + e^0 \wedge \alpha) \wedge \psi = 0$. By (110), we have

$$\begin{aligned}(e^0 \wedge \alpha + \tau) \wedge \psi &= \big(e^0 \wedge \alpha + \tau\big) \wedge \left(\frac{1}{2}\omega \wedge \omega - e^0 \wedge \mathrm{Im}\,\theta\right) \\ &= e^0 \wedge \left(\frac{1}{2}\omega \wedge \omega \wedge \alpha - \tau \wedge \mathrm{Im}\,\theta\right) + \frac{1}{2}\tau \wedge \omega \wedge \omega.\end{aligned}$$

The expression on the right vanishes if and only if $\tau \wedge \omega \wedge \omega = 0$ and $\omega \wedge \omega \wedge \alpha = 2\mathrm{Im}\,\theta \wedge \tau$. Since $\alpha \circ J = \frac{1}{2} *_W (\omega \wedge \omega \wedge \alpha)$, the last equation is equivalent to $\alpha = -\left(*_W(\mathrm{Im}\,\theta \wedge \tau)\right) \circ J = - *_W (\mathrm{Re}\,\theta \wedge \tau)$.

To prove the formula for Λ^3_{27} we choose $\tau \in \Lambda^2 W^*$ and $\beta \in \Lambda^3 W^*$. Then

$$\begin{aligned}\big(\beta + e^0 \wedge \tau\big) \wedge \phi &= e^0 \wedge (\tau \wedge \mathrm{Re}\,\theta - \beta \wedge \omega) + \beta \wedge \mathrm{Re}\,\theta, \\ \big(\beta + e^0 \wedge \tau\big) \wedge \psi &= e^0 \wedge \left(\frac{1}{2}\tau \wedge \omega \wedge \omega + \beta \wedge \mathrm{Im}\,\theta\right).\end{aligned}$$

Both terms vanish simultaneously if and only if

$$\tau \wedge \mathrm{Re}\,\theta = \beta \wedge \omega, \quad \beta \wedge \mathrm{Re}\,\theta = 0, \quad \beta \wedge \mathrm{Im}\,\theta = -\frac{1}{2}\tau \wedge \omega \wedge \omega.$$

These equations hold in the following four cases.

(a) $\beta = 3\lambda\mathrm{Re}\,\theta$ and $\tau = -4\lambda\omega$ with $\lambda \in \mathbb{R}$.

(b) $\beta = 0$ and $\tau \in \Lambda^{1,1}W^*$ with $\tau \wedge \omega \wedge \omega = 0$.

(c) $\beta \in \Lambda^{1,2}W^* + \Lambda^{2,1}W^*$ with $\beta \wedge \omega = 0$ and $\tau = 0$.

(d) $\beta = u^* \wedge \omega$ and $\tau = \iota(u)\mathrm{Re}\,\theta$ with $u \in W$.

In case (d) this follows from $(\iota(u)\mathrm{Re}\ \theta) \wedge \mathrm{Re}\,\theta = 2 * (Ju)^* = u^* \wedge \omega \wedge \omega$. The subspaces determined by these conditions are pairwise orthogonal and have dimensions 1 in case (a), 8 in case (b), 12 in case (c), and 6 in case (d). Thus, for dimensional reasons, their direct sum is the space Λ^3_{27}. This proves the theorem. □

Theorem 11.2. *Let W be an 8-dimensional real vector space equipped with an* $\mathrm{SU}(4)$*-structure* (Ω, J, Θ)*. Then the alternating multi-linear map*

$$\Phi := \frac{1}{2}\Omega \wedge \Omega + \mathrm{Re}\,\Theta \in \Lambda^4 W^*$$

is a positive Cayley calibration, compatible with the complex orientation and the inner product. Moreover, in the notation of Theorem 9.5, we have

$$\begin{aligned}
&\Lambda^2_7 = \mathbb{R}\Omega \oplus \left\{\tau \in \Lambda^{2,0} + \Lambda^{0,2} \,|\, *\,(\mathrm{Re}\,\Theta \wedge \tau) = 2\tau\right\}, \\
&\Lambda^2_{21} = \left\{\tau \in \Lambda^{1,1} \,|\, \tau \wedge \Omega^3 = 0\right\} \oplus \left\{\tau \in \Lambda^{2,0} + \Lambda^{0,2} \,|\, *\,(\mathrm{Re}\,\Theta \wedge \tau) = -2\tau\right\}.
\end{aligned}$$

Proof. We prove that Φ is compatible with the inner product $\langle\cdot,\cdot\rangle := \Omega(\cdot, J\cdot)$ and the complex orientation on W. The associated volume form is $\frac{1}{24}\Omega^4$. Hence, by Lemma 7.3, we must show that

$$\omega_{u,v} \wedge \omega_{u,v} \wedge \Phi = \frac{1}{4}\left|u \wedge v\right|^2 \Omega^4 \tag{112}$$

for all $u, v \in W$, where

$$\omega_{u,v} := \iota(v)\iota(u)\Phi = \Omega(u,v)\Omega - \iota(u)\Omega \wedge \iota(v)\Omega + \iota(v)\iota(u)\mathrm{Re}\,\Theta.$$

To see this, we observe that

$$\iota(v)\iota(u)\mathrm{Re}\,\Theta \wedge \iota(u)\Omega \wedge \iota(v)\Omega \wedge \Omega^2 = \left(\iota(v)\iota(u)\mathrm{Re}\,\Theta\right)^2 \wedge \mathrm{Re}\,\Theta = 0. \tag{113}$$

If $v = Ju$ then (113) follows from the fact that $\iota(u)\Omega \wedge \iota(Ju)\Omega$ is a $(1,1)$-form and $\iota(Ju)\iota(u)\mathrm{Re}\,\Theta = 0$. If v is orthogonal to u and Ju then (113) follows from the explicit formulas in Remark 11.3 below. The general case follows from the special cases by adding to v a linear combination of u and Ju. Using (113) and the identity $\iota(u)\Omega \wedge \iota(v)\Omega \wedge \Omega^3 = \frac{1}{4}\Omega(u,v)\Omega^4$ we obtain

$$\begin{aligned}
\omega_{u,v} \wedge \omega_{u,v} \wedge \Phi &= \frac{1}{2}\Omega(u,v)^2\Omega^4 + \frac{1}{2}\iota(v)\iota(u)\mathrm{Re}\,\Theta \wedge \iota(v)\iota(u)\mathrm{Re}\,\Theta \wedge \Omega^2 \\
&\quad -\Omega(u,v)\iota(u)\Omega \wedge \iota(v)\Omega \wedge \Omega^3 \\
&\quad -2\iota(v)\iota(u)\mathrm{Re}\,\Theta \wedge \iota(u)\Omega \wedge \iota(v)\Omega \wedge \mathrm{Re}\,\Theta \\
&= \frac{1}{4}\Omega(u,v)^2\Omega^4 + \frac{1}{2}\iota(v)\iota(u)\mathrm{Re}\,\Theta \wedge \iota(v)\iota(u)\mathrm{Re}\,\Theta \wedge \Omega^2 \\
&\quad -2\iota(v)\iota(u)\mathrm{Re}\,\Theta \wedge \iota(u)\Omega \wedge \iota(v)\Omega \wedge \mathrm{Re}\,\Theta.
\end{aligned}$$

One can now verify equation (112) by first considering the case $v = Ju$ and using $\iota(Ju)\iota(u)\mathrm{Re}\,\Theta = 0$ (here the last two terms on the right vanish). Next one can verify (112) in the case where v is orthogonal to u and Ju by using the $\mathrm{SU}(4)$-symmetry and the explicit formulas in Remark 11.3 below (here the first term on the right vanishes). Finally, one can reduce the general case to the special cases by adding to v a linear combination of u and Ju.

Now recall from Theorem 9.5 that, for every $\tau \in \Lambda^2 W^*$, we have

$$\begin{aligned} \tau \in \Lambda^2_7 \qquad &\Longleftrightarrow \qquad *(\Phi \wedge \tau) = 3\tau, \\ \tau \in \Lambda^2_{21} \qquad &\Longleftrightarrow \qquad *(\Phi \wedge \tau) = -\tau. \end{aligned}$$

Since $\operatorname{Re}\Theta \wedge \Omega = 0$ we have

$$*\,(\Phi \wedge \Omega) = \frac{1}{2} *(\Omega \wedge \Omega \wedge \Omega) = 3\Omega$$

and hence $\mathbb{R}\Omega \subset \Lambda^2_7$. Moreover, Λ^2_{21} is the image of the Lie algebra $\mathfrak{g}$ of $\mathrm{G}(W, \Phi)$ under the isomorphism $\mathfrak{so}(W) \to \Lambda^2 W^* : \xi \mapsto \omega_\xi$ given by $\omega_\xi(u, v) := \langle u, \xi v\rangle$. The image of $\mathfrak{su}(W)$ under this inclusion is the subspace $\left\{\tau \in \Lambda^{1,1} W^* \,|\, \tau \wedge \Omega^3 = 0\right\}$ and, since $\mathrm{SU}(W) \subset \mathrm{G}(W, \Phi)$, this space is contained in Λ^2_{21}. By considering the standard structure on $\mathbb{C}^4$ we obtain $*(\Omega \wedge \Omega \wedge \tau) = 2\tau$ for $\tau \in \Lambda^{2,0} + \Lambda^{0,2}$. Hence

$$*(\Phi \wedge \tau) = \frac{1}{2} *(\Omega \wedge \Omega \wedge \tau) + *(\operatorname{Re}\Theta \wedge \tau) = \tau + *(\operatorname{Re}\Theta \wedge \tau).$$

for $\tau \in \Lambda^{2,0} + \Lambda^{0,2}$. Since the operator $\tau \mapsto *(\operatorname{Re}\Theta \wedge \tau)$ has eigenvalues ± 2 on the subspace $\Lambda^{2,0} + \Lambda^{0,2}$ the result follows. □

Remark 11.3. If (Ω, J, Θ) is the standard SU(4)-structure on $W = \mathbb{C}^4$ with coordinates $(x_1 + iy_1, \ldots, x_4 + iy_4)$ then

$$\begin{aligned} \operatorname{Re}\Theta \;=\; & dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + dy_1 \wedge dy_2 \wedge dy_3 \wedge dy_4 \\ & - dx_1 \wedge dx_2 \wedge dy_3 \wedge dy_4 - dy_1 \wedge dy_2 \wedge dx_3 \wedge dx_4 \\ & - dx_1 \wedge dy_2 \wedge dx_3 \wedge dy_4 - dy_1 \wedge dx_2 \wedge dy_3 \wedge dx_4 \\ & - dx_1 \wedge dy_2 \wedge dy_3 \wedge dx_4 - dy_1 \wedge dx_2 \wedge dx_3 \wedge dy_4 \end{aligned}$$

$$\begin{aligned} \frac{1}{2}\Omega \wedge \Omega \;=\; & dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 + dx_3 \wedge dy_3 \wedge dx_4 \wedge dy_4 \\ & + dx_1 \wedge dy_1 \wedge dx_3 \wedge dy_3 + dx_2 \wedge dy_2 \wedge dx_4 \wedge dy_4 \\ & + dx_1 \wedge dy_1 \wedge dx_4 \wedge dy_4 + dx_2 \wedge dy_2 \wedge dx_3 \wedge dy_3. \end{aligned}$$

These forms are self-dual. The first assertion in Theorem 11.2 also follows from the fact that the isomorphism $\mathbb{R}^8 \to \mathbb{C}^4$ which sends $e_0, \ldots, e_7$ to $\partial/\partial x_1, \partial/\partial y_1, \partial/\partial x_2, \partial/\partial y_2, \partial/\partial x_3, -\partial/\partial y_3, -\partial/\partial x_4, \partial/\partial y_4$ pulls back Φ to the standard form Φ_0 in Example 5.9.

Theorem 11.4. *Let V be a 7-dimensional real Hilbert space equipped with a cross product and its induced orientation. Let $\phi \in \Lambda^3 V^*$ be the associative calibration and $\psi := *_V\phi \in \Lambda^4 V^*$ the coassociative calibration. Denote $W := \mathbb{R} \oplus V$ and define $\Phi \in \Lambda^4 W^*$ by*

$$\Phi := e^0 \wedge \phi + \psi.$$

Then Φ is a positive Cayley-form on W and, in the notation of Theorems 8.4 and 9.5, we have

$$\begin{aligned}
\Lambda^2_7 W^* &= \left\{ e^0 \wedge *_V(\psi \wedge \tau) + 3\tau \,|\, \tau \in \Lambda^2_7 V^* \right\} \\
\Lambda^2_{21} W^* &= \left\{ e^0 \wedge *_V(\psi \wedge \tau) - \tau \,|\, \tau \in \Lambda^2 V^* \right\} \\
\Lambda^3_8 W^* &= \mathbb{R}\phi \oplus \left\{ \iota(u)\psi - e^0 \wedge \iota(u)\phi \,|\, u \in V \right\} \\
\Lambda^3_{48} W^* &= \Lambda^3_{27} V^* \oplus \left\{ e^0 \wedge \tau \,|\, \tau \in \Lambda^2_{14} V^* \right\} \oplus \left\{ 3\iota(u)\psi + 4e^0 \wedge \iota(u)\phi \,|\, u \in V \right\} \\
\Lambda^4_7 W^* &= \left\{ e^0 \wedge \iota(u)\psi - u^* \wedge \phi \,|\, u \in V \right\} \\
\Lambda^4_{27} W^* &= \left\{ e^0 \wedge \beta + *_V\beta \,|\, \beta \in \Lambda^3_{27} V^* \right\} \\
\Lambda^4_{35} W^* &= \left\{ e^0 \wedge \beta - *_V\beta \,|\, \beta \in \Lambda^3 V^* \right\}.
\end{aligned}$$

Proof. By Theorem 5.2, W is a normed algebra with product (30). Hence, by Theorem 5.7, W carries a triple cross product (45) and Φ is the associated Cayley calibration. By Theorem 7.4, Φ is a Cayley form. By (43) the triple cross product on W satisfies (60) with $\varepsilon = +1$ and so is positive (Definition 6.5). Thus, by Theorem 7.6, Φ is positive.

Recall that, by Theorem 9.5, $\Lambda^2_7 W^*$ and $\Lambda^2_{21} W^*$ are the eigenspaces of the operator $*_W(\Phi \wedge \cdot)$ with eigenvalues 3 and -1 and, by Theorem 8.4, $\Lambda^2_7 V^*$ and $\Lambda^2_{14} V^*$ are the eigenspaces of the operator $*_V(\phi \wedge \cdot)$ with eigenvalues 2 and -1. With $\alpha \in \Lambda^1 V^*$ and $\tau \in \Lambda^2 V^*$ we have

$$\begin{aligned}
*_W\left(\Phi \wedge \left(e^0 \wedge \alpha + \tau\right)\right) &= *_W\left(e^0 \wedge \left(\psi \wedge \alpha + \phi \wedge \tau\right) + \psi \wedge \tau\right) \\
&= e^0 \wedge *_V(\psi \wedge \tau) + *_V(\phi \wedge \tau) + *_V(\psi \wedge \alpha)
\end{aligned}$$

and hence

$$e^0 \wedge \alpha + \tau \in \Lambda^2_7 W^* \qquad \iff \qquad \left\{ \begin{array}{l} *_V(\psi \wedge \tau) = 3\alpha, \\ *_V(\phi \wedge \tau) + *_V(\psi \wedge \alpha) = 3\tau. \end{array} \right.$$

Since $*_V(\psi \wedge *_V(\psi \wedge \tau)) = \tau + *_V(\phi \wedge \tau)$, by equation (88) in Theorem 8.4, we deduce that $e^0 \wedge \alpha + \tau \in \Lambda^2_7 W^*$ if and only if $*_V(\phi \wedge \tau) = 2\tau$ and $3\alpha = *_V(\psi \wedge \tau)$. This proves the formula for $\Lambda^2_7 W^*$. Likewise, we have $e^0 \wedge \alpha + \tau \in \Lambda^2_{21} W^*$ if and only if $\alpha = - *_V(\psi \wedge \tau)$. In this case the second equation $*_V(\phi \wedge \tau) + *_V(\psi \wedge \alpha) = -\tau$ is automatically satisfied.

The formula for the subspace $\Lambda^3_8 W^*$ follows from the fact that it consists of all 3-forms of the form $\iota(u)\Phi$ for $u \in W$ (see Theorem 9.5). Now let $\tau \in \Lambda^2 V^*$ and $\beta \in \Lambda^3 V^*$. Then $e^0 \wedge \tau + \beta \in \Lambda^3_{48} W^*$ if and only if

$$0 = \Phi \wedge \left(e^0 \wedge \tau + \beta\right) = e^0 \wedge (\phi \wedge \beta + \psi \wedge \tau) + \psi \wedge \beta$$

(see again Theorem 9.5). Hence

$$e^0 \wedge \tau + \beta \in \Lambda^3_{48} W^* \qquad \iff \qquad \left\{ \begin{array}{l} \phi \wedge \beta + \psi \wedge \tau = 0, \\ \psi \wedge \beta = 0. \end{array} \right.$$

These conditions are satisfied in the following three cases.
(a) $\beta = 0$ and $\psi \wedge \tau = 0$ (or equivalently $\tau \in \Lambda^2_{14} V^*$).
(b) $\tau = 0$ and $\phi \wedge \beta = 0$ and $\psi \wedge \beta = 0$ (or equivalently $\beta \in \Lambda^3_{27} V^*$).
(c) $\beta = 3\iota(u)\psi$ and $\tau = 4\iota(u)\phi$ with $u \in V$.
In the case (c) this follows from the equations $\psi \wedge \iota(u)\psi = 0$ and

$$3\phi \wedge \iota(u)\psi + 4\psi \wedge \iota(u)\phi = 0 \tag{114}$$

for $u \in V$. This last identity can be verified by direct computation using the standard structure on $V = \mathbb{R}^7$ with

$$\phi_0 = e^{123} - e^{145} - e^{167} - e^{246} + e^{257} - e^{347} - e^{356}$$

$$\psi_0 = -e^{1247} - e^{1256} + e^{1346} - e^{1357} - e^{2345} - e^{2367} + e^{4567},$$

and $u := e_1$ (see the proof of Lemma 4.5). In this case

$$\iota(u)\phi_0 = e^{23} - e^{45} - e^{67}, \qquad \iota(u)\psi_0 = -e^{247} - e^{256} + e^{346} - e^{357}$$

and so

$$\psi_0 \wedge \iota(u)\phi_0 = 3e^{234567}, \qquad \phi_0 \wedge \iota(u)\psi_0 = -4e^{234567}.$$

This proves (114). The subspaces determined by the above conditions are pairwise orthogonal and have dimensions 14 in case (a), 27 in case (b), and 7 in case (c). Thus, for dimensional reasons, their direct sum is $\Lambda^3_{48} W^*$.

Now $\Lambda^4_7 W^*$ is the tangent space of the $\mathrm{SO}(W)$-orbit of Φ. For $u \in V$ define the endomorphism $A_u \in \mathfrak{so}(V)$ by $A_u v := u \times v$. Then, by Remark 4.7, we have $\mathcal{L}_{A_u}\phi = 3\iota(u)\psi$ and $\mathcal{L}_{A_u}\psi = -3u^* \wedge \phi$. Hence $e^0 \wedge \iota(u)\psi - u^* \wedge \phi \in \Lambda^4_7 W^*$ for every $u \in V$. Since $\Lambda^4_7 W^*$ has dimension 7, each element of $\Lambda^4_7 W^*$ has this form.

Next we recall that $\Lambda^4_{27}W^*$ is contained in the subspace of self-dual 4-forms, and every self-dual 4-form can be written as $e^0 \wedge \beta + *_V\beta$ with $\beta \in \Lambda^3V^*$. By Theorem 9.5 we have

$$\begin{aligned}
e^0 \wedge \beta + *_V\beta \in \Lambda^4_{27}W^* \quad &\iff \quad \left\{ \begin{array}{c} \beta \wedge *_V\phi + *_V\beta \wedge *_V\psi = 0, \\ \beta \wedge *_V(\iota(u)\psi) = *_V\beta \wedge *_V(u^* \wedge \phi) \;\forall u, \end{array} \right. \\
&\iff \quad \psi \wedge \beta = 0, \;\; \phi \wedge \beta = 0 \\
&\iff \quad \beta \in \Lambda^3_{27}V^*.
\end{aligned}$$

The last equivalence follows from Theorem 8.4. This proves the formula for $\Lambda^4_{27}W^*$. The formula for $\Lambda^4_{35}W^*$ follows from the fact that this subspace consists of the anti-self-dual 4-forms. This proves the theorem. □

12 Donaldson–Thomas theory

The motivation for the discussion in these notes comes from our attempt to understand Riemannian manifolds with special holonomy in dimensions six, seven, and eight [1, 4, 5] and the basic setting of Donaldson–Thomas theory on such manifolds [2, 3].

12.1 Manifolds with special holonomy

Definition 12.1. *Let Y be a smooth 7-manifold and X a smooth 8-manifold. An* **almost** G_2**-structure** *on Y is a nondegenerate 3-form $\phi \in \Omega^3(Y)$; in this case the pair (Y, ϕ) is called an* **almost** G_2**-manifold**. *An* **almost** $\mathrm{Spin}(7)$**-structure** *on X is a 4-form $\Phi \in \Omega^4(X)$ which restricts to a positive Cayley-form on each tangent space; in this case the pair (X, Φ) is called an* **almost** $\mathrm{Spin}(7)$**-manifold**.

Remark 12.2. An almost G_2-manifold (Y, ϕ) admits a unique Riemannian metric and a unique orientation that, on each tangent space, are compatible with the nondegenerate 3-form ϕ as in Definition 3.1 (see Theorem 3.2). Thus each tangent space of Y carries a cross product

$$T_yY \times T_yY \to T_yY : (u, v) \mapsto u \times v$$

such that

$$\phi(u, v, w) = \langle u \times v, w \rangle$$

for all $u, v, w \in T_yY$. Moreover, Theorem 8.4 gives rise to a natural splitting of the space $\Omega^k(Y)$ of k-forms on Y for each k.

Remark 12.3. An almost Spin(7)-manifold (X, Φ) admits a unique Riemannian metric that, on each tangent space, is compatible with the Cayley-form Φ as in Definition 7.1 (see Theorem 7.4). Moreover, the positivity hypothesis asserts that the 8-forms

$$\Phi \wedge \Phi, \qquad \iota(v)\iota(u)\Phi \wedge \iota(v)\iota(u)\Phi \wedge \Phi$$

induce the same orientation whenever $u, v \in T_xX$ are linearly independent (see Definition 7.5). Thus each tangent space of X carries a positive triple cross product

$$T_xX \times T_xX \times T_xX \to T_xX : (u, v, w) \mapsto u \times v \times w$$

such that

$$\Phi(\xi, u, v, w) = \langle \xi, u \times v \times w \rangle$$

for all $\xi, u, v, w \in T_xX$. Moreover, Theorem 9.5 gives rise to a natural splitting of the space $\Omega^k(X)$ of k-forms on X for each k.

Examples of almost G_2-manifolds are S^7 (considered as unit sphere in the octonions), $S^1 \times Z$ where Z is a Calabi-Yau 3-fold and various resolutions of $\mathbb{T}^7/\Gamma$ where Γ is an appropriate finite group (see [5].) Almost Spin(7)-manifolds can be obtained from almost G_2-manifolds, Calabi-Yau 4-folds and various resolutions of $\mathbb{T}^8/\Gamma$.

Definition 12.4. *An almost* G_2*-manifold* (Y, ϕ) *is called a* G_2**-manifold** *if* ϕ *is harmonic with respect to the Riemannian metric in Remark 12.2. An almost* Spin(7)*-manifold* (X, Φ) *is called a* Spin(7)**-manifold** *if* Φ *is closed (and hence harmonic with respect to the Riemannian metric in Remark 12.3).*

Remark 12.5. Let (Y, ϕ) be an almost G_2-manifold equipped with the metric of Remark 12.2. Then ϕ is harmonic if and only if ϕ is parallel with respect to the Levi–Civita connection and hence is preserved by parallel transport. It follows that the holonomy of a G_2-manifold is contained in the group G_2. It also follows that the splitting of Theorem 8.4 is preserved by the Hodge Laplace operator and hence passes on to the DeRham cohomology. Exactly the same holds for an almost Spin(7)-manifold (X, Φ) equipped with the metric of Remark 12.3. The 4-form Φ is closed (and hence harmonic) if and only if it is parallel with respect to the Levi–Civita connection. Thus the holonomy of a Spin(7) manifold is contained in Spin(7) and the splitting of its spaces of differential forms in Theorem 9.5 descends to the DeRham cohomology.

12.2 Donaldson–Thomas theory in dimensions 7 and 8: the gauge theory picture

We close these notes with a brief review of certain partial differential equations arising in Donaldson–Thomas theory [2]. We first discuss the gauge theoretic setting. Let (Y, ϕ) be a G_2-manifold with coassociative calibration $\psi := *\phi$ and $E \to Y$ a G-bundle with structure group $\mathrm{G} = \mathrm{SU}(2)$. In [2] Donaldson and Thomas introduce a **G_2-Chern–Simons functional**

$$\mathcal{CS}^{\psi} : \mathscr{A}(E) \to \mathbb{R}$$

on the space of G-connections on E. The functional depends on the choice of a reference connection $A_0 \in \mathscr{A}(E)$ satisfying $F_{A_0} \wedge \psi = 0$ and is given by

$$\mathcal{CS}^{\psi}(A_0 + a) := \frac{1}{2} \int_Y \left(\langle d_{A_0} a \wedge a \rangle + \frac{1}{3} \langle a \wedge [a \wedge a] \rangle \right) \wedge \psi \tag{115}$$

for $a \in \Omega^1(Y, \mathrm{End}(E))$. The differential of $\mathcal{CS}$ has the form

$$\delta \mathcal{CS}^{\psi}(A)\hat{A} = \int_N \left\langle F_A \wedge \hat{A} \right\rangle \wedge \psi$$

for $A \in \mathscr{A}(E)$ and $\alpha \in T_A\mathscr{A}(E) = \Omega^1(Y, \mathrm{End}(E))$. Thus a connection A is a critical point of $\mathcal{CS}^{\psi}$ if and only if

$$F_A \wedge \psi = 0. \tag{116}$$

By Theorem 8.4 this is equivalent to the equation $*(F_A \wedge \phi) = -F_A$ and hence to $\pi_7(F_A) = 0$. A connection A that satisfies equation (116) is called a **G_2-instanton**. As in the case of flat connections on 3-manifolds equation (116) becomes elliptic with index zero after augmenting by a suitable gauge fixing condition (which we do not elaborate on here). The negative gradient flow lines of the G_2-Chern–Simons functional are the 1-parameter families of connections $\mathbb{R} \to \mathscr{A}(E) : t \mapsto A(t)$ satisfying the partial differential equation

$$\partial_t A = - * (F_A \wedge \psi), \tag{117}$$

where $F_A = F_{A(t)}$ is understood as the curvature of the connection $A(t) \in \mathscr{A}(E)$ for a fixed value of t. For the study of the solutions of (117) it is interesting to observe that, by equation (88) in Theorem 8.4, every connection A on Y satisfies the energy identity

$$\int_Y |F_A|^2 \, \mathrm{dvol}_Y = \int_Y |F_A \wedge \psi|^2 \, \mathrm{dvol}_Y - \int_Y \langle F_A \wedge F_A \rangle \wedge \phi.$$

A smooth solution of (117) can also be thought of as connection $\mathbb{A}$ on the pullback bundle $\mathbb{E}$ of E over $\mathbb{R} \times Y$. The curvature of this connection is given by

$$F_{\mathbb{A}} = F_A + dt \wedge \partial_t A = F_A - dt \wedge *(F_A \wedge \psi).$$

Hence it follows from Theorems 9.5 and 11.4 that $F_{\mathbb{A}}$ satisfies

$$*(F_{\mathbb{A}} \wedge \Phi) = -F_{\mathbb{A}} \tag{118}$$

or, equivalently, $\pi_7(F_{\mathbb{A}}) = 0$. Conversely, a connection on $\mathbb{E}$ satisfying equation (118) can be transformed into temporal gauge and hence corresponds to a solution of (117). It is interesting to observe that equation (118) makes sense over any Spin(7)-manifold. Solutions of (118) are called **Spin(7)-instantons**. This discussion is completely analogous to Floer–Donaldson theory in $3 + 1$ dimensions. The hope is that one can construct an analogous quantum field theory in dimension $7 + 1$. Moreover, as is apparent from Theorems 11.1 and 11.2, this theory will interact with theories in complex dimensions 3 and 4. The ideas for the real and complex versions of this theory are outlined in [2, 3].

12.3 Donaldson–Thomas theory in dimensions 7 and 8: the submanifold picture

There is an analogue of the G_2-Chern–Simons functional on the space of 3-dimensional submanifolds of Y, whose critical points are the associative submanifolds of Y and whose gradient flow lines are Cayley submanifolds of $\mathbb{R} \times Y$ [2]. This is the submanifold version of the conjectural Donaldson–Thomas field theory.

More precisely, let (Y, ϕ) be a G_2-manifold with coassociative calibration $\psi = *\phi$ and let S be a compact oriented 3-manifold without boundary. Denote by $\mathscr{F}$ the space of smooth embeddings $f : S \to Y$ such that $f^*\phi$ vanishes nowhere. Then the group $\mathscr{G} := \mathrm{Diff}^+(S)$ of orientation preserving diffeomorphism of S acts on $\mathscr{F}$ by composition. The quotient space

$$\mathscr{S} := \mathscr{F}/\mathscr{G}$$

can be identified with the space of oriented 3-dimensional submanifolds of Y that are diffeomorphic to S and have the property that the restriction of ϕ to each tangent space is nonzero; the identification sends the equivalence class $[f]$ of an element $f \in \mathscr{F}$ to its image $f(S)$.

Given $f \in \mathscr{F}$ the tangent space of $\mathscr{S}$ at $[f]$ can be identified with the quotient

$$T_{[f]}\mathscr{S} = \frac{\Omega^0(S, f^*TY)}{\{df \circ \xi \,|\, \xi \in \mathrm{Vect}(S)\}}.$$

If $g \in \mathscr{G}$ is an orientation preserving diffeomorphism of S then $g^*f := f \circ g$ is another representative of the equivalence class $[f]$ and the two quotient spaces can be naturally identified via $[\hat{f}] \mapsto [\hat{f} \circ g]$.

Let us fix an element $f_0 \in \mathscr{F}$ and denote by $\widetilde{\mathscr{F}}$ the universal cover of $\mathscr{F}$ based at f_0. Thus the elements of $\widetilde{\mathscr{F}}$ are equivalence classes of smooth maps $\widetilde{f} : [0,1] \times S \to Y$ such that $\widetilde{f}(0,\cdot) = f_0$ and $\widetilde{f}(t,\cdot) =: f_t \in \mathscr{F}$ for all t. Thus we can think of $\widetilde{f} = \{f_t\}_{0 \le t \le 1}$ as a smooth path in $\mathscr{F}$ starting at f_0, and two such paths are *equivalent* iff they are smoothly homotopic with fixed endpoints. The projection $\widetilde{\mathscr{F}} \to \mathscr{F}$ sends $\widetilde{f}$ to $f := \widetilde{f}(1,\cdot)$. The universal cover of $\mathscr{S}$ is the quotient

$$\widetilde{\mathscr{S}} := \widetilde{\mathscr{F}}/\widetilde{\mathscr{G}}$$

where $\widetilde{\mathscr{G}}$ denotes the group of smooth isotopies $[0,1] \to \mathrm{Diff}(S) : t \mapsto g_t$ starting at the identity. Now the space $\widetilde{\mathscr{F}}$ carries a natural $\widetilde{\mathscr{G}}$-invariant action functional $\mathscr{A} : \widetilde{\mathscr{F}} \to \mathbb{R}$ defined by

$$\mathscr{A}(\widetilde{f}) := -\int_{[0,1]\times S} \widetilde{f}^*\psi = -\int_0^1 \int_S f_t^* \left(\iota(\partial_t f_t)\psi\right) dt.$$

This functional is well defined because ψ is closed and it evidently descends to $\widetilde{\mathscr{S}}$. Its differential is the 1-form $\delta\mathscr{A}$ on $\mathscr{F}$ given by

$$\delta\mathscr{A}(f)\hat{f} = -\int_S f^* \left(\iota(\hat{f})\psi\right)$$

This 1-form is $\mathscr{G}$-invariant in that $\delta\mathscr{A}(g^*f)g^*\hat{f} = \delta\mathscr{A}(f)\hat{f}$ and horizontal in that $\delta\mathscr{A}(f)df\xi = 0$ for $\xi \in \mathrm{Vect}(S)$. Hence $\delta\mathscr{A}$ descends to a 1-form on $\mathscr{S}$.

Lemma 12.6. *An element $[\widetilde{f}] = [\{f_t\}] \in \widetilde{\mathscr{F}}$ is a critical point of $\mathscr{A}$ if and only if the image of $f := f_1 : S \to Y$ is an associative submanifold of Y.*

Proof. We have $\delta\mathscr{A}(f) = 0$ iff $\psi(\hat{f}(x), df(x)\xi, df(x)\eta, df(x)\zeta) = 0$ for all $\hat{f} \in \Omega^0(S, f^*TY)$, $x \in S$, and $\xi, \eta, \zeta \in T_xS$. This means that $\psi(u, v, w, \cdot) = 0$ for all $q \in f(S)$ and all $u, v, w \in T_qf(S)$. By definition of the coassociative calibration ψ in Lemma 4.5 this means that $[u, v, w] = 0$ for all $u, v, w \in T_qf(S)$ where $T_qY \times T_qY \times T_qY \to T_qY : (u, v, w) \mapsto [u, v, w]$ denotes the associator bracket defined by (16). By Definition 4.3 this means that $T_qf(S)$ is an associative subspace of T_qY for every $q \in f(S)$. This proves the lemma. □

The tangent space of $\mathscr{F}$ at f carries a natural L^2 inner product given by

$$\left\langle \hat{f}_1, \hat{f}_2 \right\rangle_{L^2} := \int_S \langle \hat{f}_1, \hat{f}_2 \rangle \, f^*\phi \tag{119}$$

for $\hat{f}_1, \hat{f}_2 \in \Omega^0(S, f^*TY)$. This can be viewed as a $\mathscr{G}$-invariant metric on $\mathscr{F}$.

Lemma 12.7. *The gradient of $\mathscr{A}$ at an element $f \in \mathscr{F}$ with respect to the inner product* (119) *is given by*

$$\operatorname{grad} \mathscr{A}(f) = \frac{[df \wedge df \wedge df]}{f^*\phi} \in \Omega^0(S, f^*TY),$$

where $[df \wedge df \wedge df] \in \Omega^3(S, f^*TY)$ denotes the 3-form

$$T_xS \times T_xS \times T_xS \to T_{f(x)}Y : (\xi, \eta, \zeta) \mapsto [df(x)\xi, df(x)\eta, df(x)\zeta].$$

Proof. The gradient of $\mathscr{A}$ at an element $f \in \mathscr{F}$ is the vector field $\operatorname{grad} \mathscr{A}(f)$ along f defined by

$$\int_S \langle \operatorname{grad} \mathscr{A}(f), \hat{f} \rangle f^*\phi = -\int_S f^*\left(\iota(\hat{f})\psi\right) = \int_S \langle [df \wedge df \wedge df], \hat{f} \rangle.$$

Here the last equation follows from the identity

$$-\psi(\hat{f}, u, v, w) = \psi(u, v, w, \hat{f}) = \langle [u, v, w], \hat{f} \rangle$$

(see equation (19) in Lemma 4.5). This proves the lemma. □

We emphasize that the gradient of $\mathscr{A}$ at f is pointwise orthogonal to the image of df. This is of course a consequence of the fact that the 1-form $\delta\mathscr{A}$ on $\mathscr{F}$ and the inner product on $T\mathscr{F}$ are $\mathscr{G}$-invariant. Now a negative gradient flow line of $\mathscr{A}$ is a smooth map

$$\mathbb{R} \times S \to Y : (t, x) \mapsto u_t(x)$$

that satisfies the partial differential equation

$$\partial_t u_t(x) + \frac{[du_t(x)e_1, du_t(x)e_1, du_t(x)e_3]}{\phi(du_t(x)e_1, du_t(x)e_2, du_t(x)e_3)} = 0 \tag{120}$$

for all $(t, x) \in \mathbb{R} \times S$ and every frame e_1, e_2, e_3 of T_xS. Moreover, we require of course that u_t is an embedding for every t and that $u_t^*\phi$ vanishes nowhere.

Lemma 12.8. *Let $\mathbb{R} \times S \to Y : (t, x) \mapsto u_t(x)$ be a smooth map such that $u_t \in \mathscr{F}$ for every t. Let $\xi_t \in \mathrm{Vect}(S)$ be chosen such that*

$$\partial_t u_t(x) - du_t(x)\xi_t(x) \perp \operatorname{im} du_t(x) \qquad \forall (t, x) \in \mathbb{R} \times S. \tag{121}$$

Then the set

$$\Sigma := \{(t, u_t(x)) \,|\, t \in \mathbb{R},\, x \in S\} \tag{122}$$

is a Cayley submanifold of $\mathbb{R} \times Y$ with respect to the Cayley calibration $\Phi := dt \wedge \phi + \psi$ if and only if

$$\partial_t u_t(x) - du_t(x)\xi_t(x) + \frac{[du_t(x)e_1, du_t(x)e_3), du_t(x)e_3]}{\phi(du_t(x)e_1, du_t(x)e_3), du_t(x)e_3)} = 0 \tag{123}$$

for every pair $(t, x) \in \mathbb{R} \times S$ and every frame e_1, e_2, e_3 of T_xS.

Proof. Fix a pair $(t, x) \in \mathbb{R} \times S$ and choose a basis e_1, e_2, e_3 of T_xS. By Theorem 5.7 (iii) the triple cross product of the three tangent vectors

$$(0, du_t(x)e_1), \qquad (0, du_t(x)e_2), \qquad (0, du_t(x)e_3)$$

of Σ is the pair

$$\bigl(\phi(du_t(x)e_1, du_t(x)e_3), du_t(x)e_3), -[du_t(x)e_1, du_t(x)e_3), du_t(x)e_3]\bigr).$$

Since this pair is orthogonal to the three vectors $(0, du_t(x)e_i)$ and its first component is nonzero, it follows that our pair is tangent to Σ if and only if it is a scalar multiple of the pair $(1, \partial_t u_t(x) - du_t(x)\xi_t(x))$. This is the case if and only if (123) holds. Hence it follows from Lemma 6.10 that Σ is a Cayley submanifold of $\mathbb{R} \times Y$ if and only if u satisfies equation (123). This proves the lemma. $\square$

Lemma 12.8 shows that every negative gradient flow line of $\mathscr{A}$ determines a Cayley submanifold $\Sigma \subset \mathbb{R} \times Y$ via (122) and, conversely, every Cayley submanifold $\Sigma \subset \mathbb{R} \times Y$, with the property that the projection $\Sigma \to \mathbb{R}$ is a proper submersion, can be parametrized as a negative gradient flow line of $\mathscr{A}$ (for some S). Thus the negative gradient trajectories of $\mathscr{A}$ are solutions of an elliptic equation, after taking account of the action of the infinite dimensional reparametrization group $\mathscr{G}$. They minimize the energy

$$\begin{aligned}
E(u, \xi) &:= \frac{1}{2} \int_{-\infty}^{\infty} \int_S \left(|\partial_t u_t - du_t\xi_t|^2 + \left| \frac{[du_t \wedge du_t \wedge du_t]}{u_t^*\phi} \right|^2 \right) u_t^*\phi \, dt \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \int_S \left| \partial_t u_t - du_t\xi_t + \frac{[du_t \wedge du_t \wedge du_t]}{u_t^*\phi} \right|^2 u_t^*\phi \, dt + \int_{\mathbb{R} \times S} u^*\psi.
\end{aligned}$$

For studying the solutions of (123) it will be interesting to introduce the energy density $e_f : S \to \mathbb{R}$ of an embedding $f \in \mathscr{F}$ via

$$e_f(x) := \frac{\det\big(\langle df(x)e_i, df(x)e_j\rangle_{i,j=1,2,3}\big)}{\phi(df(x)e_1, df(x)e_2, df(x)e_3)^2}$$

for every $x \in S$ and every frame e_1, e_2, e_3 of T_xS. Then $e_{f\circ g} = e_f \circ g$ for every (orientation preserving) diffeomorphism g of S and so the energy

$$\mathscr{E}(f) := \int_S e_f f^*\phi \tag{124}$$

is a $\mathscr{G}$-invariant function on $\mathscr{F}$. Moreover, it follows from Lemma 4.2 that

$$\mathscr{E}(f) = \int_S \left|\frac{[df \wedge df \wedge df]}{f^*\phi}\right|^2 f^*\phi + \int_S f^*\phi.$$

If ϕ is closed then the last term on the right is a topological invariant. Moreover, the first term vanishes if and only if f is a critical point of the action functional $\mathscr{A}$. Thus the critical points of $\mathscr{A}$ are also the absolute minima of the energy $\mathscr{E}$ (in a given homology class).

These observations are the starting point of a conjectural Floer–Donaldson type theory in dimensions seven and eight, as outlined in the paper by Donaldson and Thomas [2]. The analytical difficulties one encounters when making this precise are formidable and, so far, have not been overcome. The recent work of Donaldson and Segal [3] also shows that new geometric phenomena are to be expected and that the theories outlined in Sections 12.2 and 12.3 may have to be combined to obtain new invariants.

References

[1] R.L. Bryant, Metrics with exceptional holonomy, *Annals of Math.* **126** (1987), 525–576.

[2] S.K. Donaldson, R.P. Thomas, Gauge theory in higher dimensions, in *The Geometric Universe (Oxford, 1996)*, 31–47, OUP, 1998.

[3] S.K. Donaldson, Ed Segal, Gauge theory in higher dimensions II, Preprint, Imperial College, 2009.

[4] R. Harvey and H.B. Lawson Jr., Calibrated Geometries. *Acta Mathematica* **148** (1982), 47–157.

[5] D. Joyce, *Compact Manifolds with Special Holonomy.* Oxford Mathematical Monographs, OUP 2000.

[6] D.A. Salamon, *Spin Geometry and Seiberg–Witten Invariants.* In preparation.

[7] T. Walpuski, Donaldson–Thomas theory. MSc thesis, ETH Zürich, April 2009.