MORSE THEORY, THE CONLEY INDEX AND FLOER HOMOLOGY

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1. *Introduction*

In 1965 Arnold [1] conjectured that the number of fixed points of an exact symplectic diffeomorphism on a symplectic manifold M can be estimated below by the sum of the Betti numbers provided that the fixed points are nondegenerate. This estimate is, of course, much sharper than the Lefschetz fixed point theorem which would only give the alternating sum of the Betti numbers as a lower bound. Its proof is based on a Morse type index theory. If the symplectomorphism in question is C^1 close to the identity then the problem can indeed be reduced to classical Morse theory using generating functions [2, 34]. The general case, however, represents a much deeper problem which has recently been addressed by many authors. (We do not attempt here to give a complete overview of the literature and instead refer to [3, 13] for a more extensive discussion of related works.) For the torus $M = T^{2n}$ a remarkable solution was given by Conley and Zehnder [7]. They used a variational principle on the loop space, unbounded on either side, and overcame the problem of an infinite Morse index by means of a finite dimensional reduction. An entirely different approach by Gromov [16] was based on the analysis of holomorphic maps and led to an existence proof for at least one fixed point. Recently, Floer [9–13] combined the ideas of Conley and Zehnder with those by Gromov and gave a beautiful proof of the Arnold conjecture for general symplectic manifolds, only assuming that every holomorphic sphere is constant. (In [13] Floer's assumption is somewhat more general but we will restrict ourselves to this case in order to avoid further complications.) He defined a relative index for a pair of critical points and generalized the Morse complex of critical points and connecting orbits (as described by Witten [35]) to the infinite dimensional situation of the loop space which led to the concept of Floer homology.

In this paper we shall give an exposition of Floer homology including the necessary background on the Conley index and the Morse complex. Moreover, we propose an alternative approach to the connection index which plays an essential role in the case of a coefficient group other than $\mathbb{Z}/2$. In a preliminary section we shall briefly describe the classical Morse inequalities and give a proof which is based on the Conley index (Section 2). In the case of a Morse–Smale gradient flow on a finite dimensional manifold the critical points and connecting orbits determine a chain complex [20, 30, 35] which represents a special case of Conley's connection matrix [14, 15, 22] and plays an essential role in Floer's work [9]. We shall describe this chain complex in Section 3 and for the sake of completeness we include a proof of the fact that it recovers the homology of the manifold M with coefficients in any abelian group

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G. This proof will again be based on the Conley index. We shall also give an interpretation of the connecting orbits as absolute minima of a variational problem and relate the difference of the Morse indices of two critical points to the Fredholm index of a certain first order differential operator acting on vectorfields along a connecting orbit. This operator plays a crucial role in the infinite dimensional situation of the loop space but the finite dimensional analogue has apparently not been studied before. In Section 4 we outline Floer's proof of the Arnold conjecture and at some places suggest slight modifications. These include the definition of the sign associated to a connecting orbit and some parts of the compactness proof in Section 5. Furthermore, we announce an existence result for infinitely many periodic solutions which will be proved in a forthcoming paper.

2. *Morse inequalities and the Conley index*

On an n-dimensional Riemannian manifold M we consider the gradient flow

$$\dot{x} = -\nabla f(x) \tag{2.1}$$

of a smooth function $f: M \to \mathbb{R}$. The flow of (2.1) will be denoted by $\phi^s \in \mathrm{Diff}(M)$ and is defined by

$$\frac{d}{ds}\phi^s = -\nabla f \circ \phi^s, \quad \phi^0 = \mathrm{id}.$$

Of course, f decreases along the orbits of ϕ^s and the rest points of ϕ^s are the critical points of f. They are hyperbolic if f is a Morse function meaning that the Hessian of f is nonsingular at every critical point. In this case the unstable set

$$W^u(x) = \{y \in M; \lim_{s \to -\infty} \phi^s(y) = x\}$$

is a submanifold of M and its dimension is the *index* of the critical point

$$\mathrm{ind}\,(x) = \dim W^u(x).$$

Equivalently, the index of x can be defined as the number of negative eigenvalues of the Hessian $\nabla^2 f(x)$. The *Morse inequalities* relate the number c_k of critical points of index k to the Betti numbers $\beta_k = \mathrm{rank}\, H_k(M; R)$ of the manifold M for any principal ideal domain R. Throughout the paper H_* will denote singular homology.

THEOREM 2.1 (M. Morse).

$$c_k - c_{k-1} + \ldots \pm c_0 \geqslant \beta_k - \beta_{k-1} + \ldots \pm \beta_0$$

for $k = 0, \ldots, n$ and equality holds for $k = n$.

In particular $c_k \geqslant \beta_k$ so that for every Morse function f the minimal number of critical points is the sum of the Betti numbers of M. Moreover, note that Theorem 2.1 provides a simple proof of the fact that the Euler characteristic

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \,\mathrm{rank}\, H_k(M; R) = \sum_{k=0}^{n} (-1)^k c_k$$

is independent of the coefficient ring R.

We shall give a proof of Theorem 2.1 which is based on the Conley index and avoids the usual construction of cells and handlebodies [19, 29, 30]. The fundamental idea of the Conley index is to characterize an isolated invariant set by the behaviour of the flow on the boundary of a neighbourhood. More precisely, we assume that $\phi^t: M \to M$ is a flow on a compact manifold M meaning

$$\phi^{t+s} = \phi^t \circ \phi^s, \quad \phi^0 = \mathrm{id}$$

for $t, s \in \mathbb{R}$. A set $S \subset M$ is called *invariant* if $\phi^t(S) = S$ for every $t \in \mathbb{R}$ and it is called *isolated* if there exists a neighbourhood N of S such that

$$S = I(N) := \bigcap_{t \in \mathbb{R}} \phi^t(N).$$

An *index pair* for an isolated invariant set $S \subset M$ is a pair of compact sets $L \subset N$ such that $S = I(\mathrm{cl}\,(N \backslash L)) \subset \mathrm{int}\,(N \backslash L)$ and

(i) $$x \in L, \phi^{[0,t]}(x) \subset N \Rightarrow \phi^t(x) \in L,$$

(ii) $$x \in N \backslash L \Rightarrow \exists t > 0 \text{ with } \phi^{[0,t]}(x) \subset N.$$

Condition (i) says that L is positively invariant in N and (ii) means that every orbit which leaves N goes through L first.

In **[6, 25]** it is shown that every isolated invariant set S admits an index pair such that the topological quotient N/L has the homotopy type of a finite polyhedron. Moreover, the homotopy type of N/L is independent of the choice of the index pair.

LEMMA 2.2 (C. Conley). *If (N_α, L_α) and (N_β, L_β) are two index pairs for S then the index spaces N_α/L_α and N_β/L_β are homotopy equivalent.*

Proof. Consider the map $\phi^t_{\beta\alpha}: N_\alpha/L_\alpha \to N_\beta/L_\beta$ defined by $\phi^t_{\beta\alpha}(x) = \phi^t(x)$ if

$$\phi^{[0,2t/3]}(x) \subset N_\alpha \backslash L_\alpha \text{ and } \phi^{[t/3,t]}(x) \subset N_\beta \backslash L_\beta$$

and by $\phi^t_{\beta\alpha}(x) = *$ otherwise. This map is continuous for $t > t_{\alpha\beta}$ where $t_{\alpha\beta} \geqslant 0$ is the minimal time such that for $t > t_{\alpha\beta}/3$

$$\phi^{[-t,t]}(x) \subset N_\alpha \backslash L_\alpha \Rightarrow x \in N_\beta \backslash L_\beta, \quad \phi^{[-t,t]}(x) \subset N_\beta \backslash L_\beta \Rightarrow x \in N_\alpha \backslash L_\alpha.$$

(See **[27]**.) In particular, the induced semidynamical system $\phi^t_{\alpha\alpha}$ on N_α/L_α is continuous for $t \geqslant t_{\alpha\alpha} = 0$ if and only if the pair N_α, L_α satisfies the properties (i) and (ii) above. (See **[25]**.) Moreover, $t_{\alpha\gamma} \leqslant t_{\alpha\beta} + t_{\beta\gamma}$ and

$$\phi^t_{\gamma\beta} \circ \phi^s_{\beta\alpha} = \phi^{t+s}_{\gamma\alpha}, \quad \phi^0_{\alpha\alpha} = \mathrm{id}$$

for $t > t_{\beta\gamma}$ and $s > t_{\alpha\beta}$ so that $\phi^t_{\alpha\beta}$ is the homotopy inverse of $\phi^t_{\beta\alpha}$.

The *Conley index* of S is the homotopy type of the pointed space N/L. If L is a neighbourhood deformation retract in N then the homology of the index space N/L agrees with the homology of the pair N, L and is characterized by the index polynomial

$$P_S(s) = \sum_k \mathrm{rank}\, H_k(N, L; R)\, s^k.$$

The Conley index is additive in the sense that $P_S(s) = P_{S_1}(s) + P_{S_2}(s)$ whenever S is the disjoint union of the isolated invariant sets S_1 and S_2.

As an example consider a hyperbolic fixed point $x = 0$ of a differential equation

$\dot{x} = v(x)$ in $\mathbb{R}^n$. Denote by E^s and E^u the stable and unstable subspaces of the linearized equation $d\xi/dt = dv(0)\,\xi$. Then an index pair for the isolated invariant set $S = \{0\}$ in the nonlinear flow is given by

$$N = \{x_s + x_u;\, x_s \in E^s, x_u \in E^u, |x_s| \leqslant \varepsilon, |x_u| \leqslant \varepsilon\},$$

$$L = \{x_s + x_u \in N;\, |x_u| = \varepsilon\}$$

for $\varepsilon > 0$ sufficiently small. It follows that N/L has the homotopy type of a pointed k-sphere where $k = \dim E^u$ is the index of the hyperbolic fixed point and the index polynomial is given by $P_S(s) = s^k$.

In particular, a critical point $x \in M$ of a Morse function $f\colon M \to \mathbb{R}$ is an isolated invariant set with index polynomial $P_x(s) = s^{\operatorname{ind}(x)}$. If c is a critical level of f then an index pair for the isolated invariant set

$$S = \{x \in M;\, \nabla f(x) = 0, f(x) = c\}$$

is given by $N = M^b$, $L = M^a$ where

$$M^a = \{x \in M;\, f(x) \leqslant a\}$$

and $a < b$ are regular values of f such that c is the only critical value of f in the interval (a, b). It follows from Lemma 2.2 and the additivity of the Conley index that

$$\sum_k \operatorname{rank} H_k(M^b, M^a; R)\, s^k = \sum_{x \in S} s^{\operatorname{ind}(x)}. \tag{2.2}$$

Proof of Theorem 2.1. Define $\beta_k^a = \operatorname{rank} H_k(M^a; R)$ and let c_k^a be the number of critical points $x \in M$ of f with $\operatorname{ind}(x) = k$ and $f(x) \leqslant a$. If $a < b$ are chosen as above then it follows from (2.2) that $\operatorname{rank} H_k(M^b, M^a; R) = c_k^b - c_k^a$ and hence the homology exact sequence

$$H_{k+1}(M^b, M^a; R) \xrightarrow{\partial_k} H_k(M^a; R) \longrightarrow H_k(M^b; R) \longrightarrow H_k(M^b, M^a; R) \xrightarrow{\partial_{k-1}} \dots$$

shows that

$$\operatorname{rank} \partial_{k-1} + \operatorname{rank} \partial_k = c_k^b - c_k^a - \beta_k^b + \beta_k^a.$$

Equivalently

$$P_f^b(s) - P_M^b(s) = P_f^a(s) - P_M^a(s) + (1+s)\, Q^{ab}(s)$$

where

$$P_M^a(s) = \sum_{k=0}^n \beta_k^a s^k, \quad P_f^a(s) = \sum_{k=0}^n c_k^a s^k, \quad Q^{ab}(s) = \sum_{k=0}^n \operatorname{rank} \partial_k s^k.$$

In particular, $Q^{ab}(s)$ is a polynomial with nonnegative coefficients. It follows inductively that

$$P_f^a(s) - P_M^a(s) = (1+s)\, Q^a(s)$$

where $Q^a(s)$ is a polynomial whose nonnegative coefficients are given by

$$\rho_k^a = \sum_{j=0}^k (-1)^j (c_{k-j}^a - \beta_{k-j}^a) \geqslant 0.$$

For $a > \sup f$ these are the Morse inequalities.

3. *Connecting orbits and Fredholm operators*

The gradient flow ϕ^s of a Morse function $f: M \to \mathbb{R}$ is said to be of *Morse–Smale* type if for any two critical points x and y the stable and unstable manifolds $W^s(x)$ and $W^u(y)$ intersect transversally. This requirement can be achieved by means of an arbitrarily small alteration of the Riemannian metric [31]. If f is of Morse–Smale type then the connecting orbits determine the following chain complex.

We first choose an orientation of the vectorspace $E^u(x) = T_x W^u(x)$ for every critical point of f and denote by $\langle x \rangle$ the pair consisting of a critical point x and this orientation. For every $k = 0, 1, \ldots, n$ we then denote by C_k the free group

$$C_k = \bigoplus_x \mathbb{Z}\langle x \rangle$$

where x runs over all critical points of index k. The function f being of Morse–Smale type implies that $W^s(x) \cap W^u(y)$ consists of finitely many orbits if $\operatorname{ind}(y) - \operatorname{ind}(x) = 1$. In this case one can define an integer $n(y, x)$ by assigning a number $+1$ or -1 to every connecting orbit and taking the sum. Let $\gamma(s)$ be such a connecting orbit meaning a solution of (2.1) with $\lim_{s \to -\infty} \gamma(s) = y$ and $\lim_{s \to \infty} \gamma(s) = x$. Then $\langle y \rangle$ induces an orientation on the orthogonal complement $E^u_\gamma(y)$ of $v = \lim_{s \to -\infty} |\dot\gamma(s)|^{-1} \dot\gamma(s)$ in $E^u(y)$. In the case $\operatorname{ind}(x) = \operatorname{ind}(y) - 1 = k$ the tangent flow induces an isomorphism from $E^u_\gamma(y)$ onto $E^u(x)$ and we define n_γ to be $+1$ or -1 according to whether this map is orientation preserving or orientation reversing. Define

$$n(y, x) = \sum_\gamma n_\gamma$$

where the sum runs over all orbits of (2.1) connecting y to x. Then the boundary operator $\partial^c_k : C_{k+1} \to C_k$ of the chain complex is defined by

$$\partial^c \langle y \rangle = \sum_x n(y, x) \langle x \rangle$$

where the sum runs over all critical points of index k.

One can extend this chain complex to coefficients in any abelian group G by defining $C_k(G) = G \otimes C_k$ and

$$\partial^c_k(G) = \mathbb{1}_G \otimes \partial^c_k : C_{k+1}(G) \longrightarrow C_k(G).$$

The significance of the above construction rests on the following result.

Theorem 3.1 (R. Thom, S. Smale, J. Milnor, C. Conley, E. Witten).

(i) $$\partial^c_{k-1}(G) \circ \partial^c_k(G) = 0,$$

(ii) $$H_k(M; G) = \frac{\ker \partial^c_{k-1}(G)}{\operatorname{im} \partial^c_k(G)}.$$

Note that the second part of this theorem implies the Morse inequalities if $G = R$ is a principal ideal domain.

The above formulation of the chain complex is due to Witten [35]. He actually considers the dual coboundary operator δ and recovers the de Rham cohomology of M. In [20] Milnor proved the above theorem for a selfindexing Morse function on a manifold with boundary and used it in order to establish Poincaré duality. In a somewhat more implicit way Theorem 3.1 was already contained in Smale's handlebody approach to Morse theory [29, 30] and in the earlier work by Thom [32]. More recently Floer [9] has proved Theorem 3.1 for Alexander cohomology with an arbitrary coefficient ring. In [20] the boundary operator ∂^c was characterized in terms of intersection numbers and we shall now describe this construction.

If M is oriented then the level set

$$M_a = \{x \in M \mid f(x) = a\}$$

is an oriented submanifold of M for every regular value a. More precisely, a basis $\xi_2, \ldots, \xi_n$ of $T_x M_a$ will be called positively oriented if $-\nabla f(x), \xi_2, \ldots, \xi_n$ defines a positively oriented basis of $T_x M$. Moreover, the orientation $\langle x \rangle$ of $E^u(x) = T_x W^u(x)$ induces an orientation of $E^s(x) = T_x W^s(x)$ since $T_x M = E^u(x) \oplus E^s(x)$. It follows that the descending sphere $W^u_a(y) = W^u(y) \cap M_a$ inherits an orientation from $W^u(y)$ and the ascending sphere $W^s_a(x) = W^s(x) \cap M_a$ inherits an orientation from $W^s(x)$. The integer $n(y, x)$ in Witten's boundary operator agrees with the intersection number of $W^u_a(y)$ and $W^s_a(x)$ in M_a. The nonorientable case can be treated by considering the $\mathbb{Z}/2$-invariant lift of f to the oriented double cover of M.

We also point out that the above boundary operator represents a special case of Conley's connection matrix and Theorem 3.1 follows directly from Franzosa's work [14, 15, 22]. As a matter of fact, a connection matrix with the properties of Theorem 3.1 can still be defined if ϕ^s is not a Morse–Smale flow, even if the critical points of f are degenerate or if ϕ^s is not even a gradient flow. However, in these cases the connection matrix need no longer be unique [24].

In order to prove Theorem 3.1 we shall now describe Conley's connection matrix for the special case of a Morse–Smale gradient flow. For every critical point x of f let N_x, L_x denote the index pair described in Section 2 and observe that an orientation of $E^u(x) = T_x W^u(x)$ determines a generator of $H_k(N_x, L_x; \mathbb{Z}) \approx \mathbb{Z}$ where $k = \operatorname{ind}(x)$. This shows that the group C_k can be identified with

$$C_k = \bigoplus_x H_k(N_x, L_x; \mathbb{Z})$$

where the sum runs over all critical points of index k. Since $H_k(N_x, L_x; \mathbb{Z})$ is a free group it follows from the universal coefficient theorem that the natural homomorphism $G \otimes H_k(N_x, L_x; \mathbb{Z}) \to H_k(N_x, L_x; G)$ is an isomorphism and hence

$$G \otimes C_k = \bigoplus_x H_k(N_x, L_x; G) = C_k(G).$$

Following Floer we define

$$M(y, x) = W^u(y) \cap W^s(x),$$

the union of the orbits of (2.1) connecting y to x. This set is a submanifold of M of dimension $\operatorname{ind}(y) - \operatorname{ind}(x)$ provided that ϕ^s is a Morse–Smale flow. If moreover $\operatorname{ind}(y) - \operatorname{ind}(x) = 1$ then

$$S(y, x) = M(y, x) \cup \{x, y\}$$

is an isolated invariant set. Let N_2, N_0 be an index pair for $S(y, x)$ and define $N_1 = N_0 \cup (N_2 \cap M^a)$ where $f(x) < a < f(y)$. Then N_2, N_1 is an index pair for y and N_1, N_0 is an index pair for x. Define the homomorphism

$$\Delta_k(x, y; G): H_{k+1}(N_y, L_y; G) \longrightarrow H_k(N_x, L_x; G)$$

to be the composition

$$H_{k+1}(N_y, L_y; G) \longrightarrow H_{k+1}(N_2, N_1; G) \xrightarrow{\partial} H_k(N_1, N_0; G) \longrightarrow H_k(N_x, L_x; G)$$

where the first and third isomorphism is induced by the flow defined homotopy equivalence of Lemma 2.2. This determines a homomorphism

$$\Delta_k(G): C_{k+1}(G) \longrightarrow C_k(G)$$

which is a special case of Conley's connection matrix and agrees with the boundary operator of Milnor and Witten.

LEMMA 3.2. $\partial^c(G) = \Delta(G).$

In the case of $\mathbb{Z}/2$ coefficients this result is due to McCord [18] and it says that in a Morse–Smale flow Conley's connection map $\Delta_k(y, x; \mathbb{Z}/2)$ is given by the number of connecting orbits modulo 2. With integer coefficients Lemma 3.2 was proved by Floer [9] using Milnor's characterization of $n(y, x)$ as an intersection number. For the sake of completeness we give an alternative proof.

Proof of Lemma 3.2. Allowing for an alteration of the function f outside an isolating neighbourhood of $S(y, x)$ we may assume that x and y are the only critical points of f in $f^{-1}([a, b])$ where $a = f(x)$ and $b = f(y)$. More precisely, let S denote the set of all critical points $z \neq y$ with $\operatorname{ind}(z) \geqslant \operatorname{ind}(y)$ together with their connecting orbits. Then S is a repeller and hence there exists a smooth function $g: M \to \mathbb{R}$ such that $N = g^{-1}([0, \infty))$ is an isolating neighbourhood for S and $dg(z)\nabla f(z) > 0$ for $z \in \partial N = g^{-1}(0)$ [25]. With $\varepsilon > 0$ sufficiently small and a smooth, monotonically increasing cutoff function $\rho: \mathbb{R} \to [0, 1]$, satisfying $\rho(r) = 0$ for $r \leqslant 0$ and $\rho(r) = 1$ for $r \geqslant \varepsilon$, define $f_C: M \to \mathbb{R}$ by

$$f_C(z) = f(z) + C\rho(g(z)).$$

Then f_C has the same critical points as f for any positive value of C. With $C > b - \inf(f)$ it follows that $S \subset f_C^{-1}((b, \infty))$. A similar argument can be used for the critical points $z \neq x$ with $\operatorname{ind}(z) \leqslant \operatorname{ind}(x)$. This alteration does not affect either of the homomorphisms ∂^c and Δ.

Given $a < c < b$, a sufficiently small number $\varepsilon > 0$ and a sufficiently large $T > 0$ we define the index pairs

$$N_y = \{z \in M; f(\phi^{-T}(z)) \leqslant b + \varepsilon, f(z) \geqslant c\}, \quad L_y = \{z \in N_y; f(z) = c\},$$

$$N_x = \{z \in M; f(\phi^{T}(z)) \geqslant a - \varepsilon, f(z) \leqslant c\}, \quad L_x = \{z \in N_x; f(\phi^{T}(z)) = a - \varepsilon\},$$

(see Figure 1 for the case of a single connecting orbit). Then an index triple for the attractor-repeller pair x, y in the isolated invariant set $S(y, x)$ is given by

$$N_2 = N_x \cup N_y, \quad N_1 = N_x \cup L_y, \quad N_0 = L_x \cup \operatorname{cl}(L_y \backslash N_x),$$

and we shall use this triple in order to prove that $\partial^c = \Delta$ in the case of integer coefficients.

To this end note that N_y is contractible onto $W^u(y) \cap \{f \geqslant c\}$ by taking the limit $T \to \infty$. Likewise, N_x defines a tubular neighbourhood of $W^s(x) \cap \{f \leqslant c\}$ whose width converges to zero as $T \to \infty$. Since $W^u(y)$ and $W^s(x)$ intersect transversally, it follows that $N_x \cap W^u(y) \cap \{f = c\}$ consists of finitely many components $V_1, \ldots, V_m$

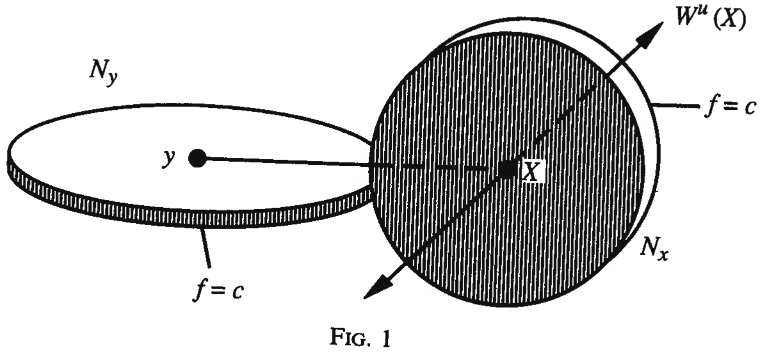

FIG. 1

each containing a unique point $z_j \in M(y,x) \cap V_j$. More precisely, with D^k denoting the closed unit ball in $\mathbb{R}^k$, there exists a diffeomorphism

$$\psi_x : N_x \longrightarrow D^k \times D^{n-k}$$

with $\psi_x(L_x) = \partial D^k \times D^{n-k}$, $\psi_x(W^s(x) \cap N_x) = \{0\} \times D^{n-k}$ and $\psi_x(V_j) = D^k \times \{\theta_j\}$ where $\theta_j \in \partial D^{n-k}$. In particular, V_j is a k-manifold with boundary $W_j = V_j \cap L_x$, diffeomorphic to D^k via $\psi_j = \pi_1 \circ \psi_x | V_j : V_j \to D^k$, and the map $\pi_1 \circ \psi_x : N_x \to D^k$ induces an isomorphism on homology

$$H_k(N_x, L_x) \approx H_k(D^k, \partial D^k) \approx H_k(V_j, W_j).$$

The given orientation of $E^u(x)$ determines a generator of the homology $\alpha \in H_k(N_x, L_x) \approx \mathbb{Z}$ which under the above isomorphism is mapped to a generator $\alpha_j \in H_k(V_j, W_j)$. The homology class α_j is determined by the orientation of $T_{z_j} V_j$ inherited from the orientation of $E^u(x)$ via the flow defined isomorphism $T_{z_j} V_j \to E^u(x)$. This orientation may or may not agree with the one inherited from $W^u(y)$ via the injection

$$T_{z_j} V_j = T_{z_j} W^u(y) \cap \nabla f(z_j)^{\perp} \subset T_{z_j} W^u(y)$$

(by taking $-\nabla f(z_j)$ as the first basis vector). Indeed, both orientations agree if and only if $n_j = 1$ where $n_j \in \{-1, 1\}$ is the sign associated to the connecting orbit $\gamma_j(s) = \phi^s(z_j)$.

Now choose a triangulation of the k-manifolds V_j and extend it to a triangulation of the $k+1$-manifold $W^u(y) \cap \{f \geqslant c\}$ with boundary $W^u(y) \cap \{f = c\}$. Together with the given orientation of $W^u(y)$ this determines a generator

$$\beta \in H_{k+1}(W^u(y) \cap N_y, W^u(y) \cap L_y) \approx H_{k+1}(N_y, L_y).$$

The homology class $\partial_j \beta \in H_k(W^u(y) \cap L_y, \mathrm{cl}(W^u(y) \cap L_y \setminus V_j)) \approx H_k(V_j, W_j)$ is represented by the original triangulation of V_j together with the orientation inherited from $W^u(y)$ and therefore agrees with $n_j \alpha_j$. Using the above isomorphism $H_k(N_x, L_x) \approx H_k(V_j, W_j)$ we obtain

$$\Delta\beta = \sum_{j=1}^{m} n_j \alpha = n(y,x)\, \alpha \in H_k(N_x, L_x)$$

and this proves the statement in the case $G = \mathbb{Z}$.

The general case then follows from the identity $\Delta(G) = \mathbb{1}_G \otimes \Delta(\mathbb{Z})$ which is a consequence of the fact that the homomorphism $G \otimes H_k(.;\mathbb{Z}) \to H_k(.;G)$ commutes with the boundary operators.

Proof of Theorem 3.1. For $j \leqslant k$ let S_{kj} denote the union of the sets $M(y,x)$ over all pairs (x,y) of critical points of f with $j \leqslant \operatorname{ind}(x) \leqslant \operatorname{ind}(y) \leqslant k$. These sets are compact provided that the gradient flow ϕ^s of f is of Morse–Smale type. Indeed, if $\gamma_\nu(s)$ is a sequence of orbits connecting y to x then a subsequence converges in an appropriate sense to a finite collection of orbits $\gamma^j(s)$ connecting x^j to x^{j-1} for $j = 1, \ldots, m$ where $x^0 = x$, $x^m = y$ and $\operatorname{ind}(x^{j-1}) < \operatorname{ind}(x^j)$.

It follows that S_{kj} is an isolated invariant set for $j \leqslant k$. In particular, $S_{n0} = M$ and S_{kk} consists of all critical points of index k. In [6] Conley proved that there exists an index filtration $N_0 \subset N_1 \subset \ldots \subset N_n$ such that $N_n = M$ and (N_k, N_{j-1}) is an index pair for S_{kj} where $j \leqslant k$ and $N_{-1} = \varnothing$.

By Lemma 3.2 that there is a commuting diagram

$$
\begin{array}{ccccc}
C_{k+1}(G) & \xrightarrow{\partial_k^c} & C_k(G) & \xrightarrow{\partial_{k-1}^c} & C_{k-1}(G) \\
\downarrow & & \downarrow & & \downarrow \\
H_{k+1}(N_{k+1}, N_k; G) & \xrightarrow{\partial_k} & H_k(N_k, N_{k-1}; G) & \xrightarrow{\partial_{k-1}} & H_{k-1}(N_{k-1}, N_{k-2}; G)
\end{array}
$$

in which the vertical isomorphisms are given by Lemma 2.2 and it follows that

$$\partial_{k-1}^c(G) \circ \partial_k^c(G) = 0.$$

In addition, Lemma 2.2 shows that $H_j(N_k, N_{k-1}; G) = \{0\}$ for $j \neq k$ and it follows from the homology exact sequence that the inclusion induced map

$$H_j(N_k; G) \to H_j(N_{k+1}; G)$$

is an isomorphism for $j \neq k, k+1$. This shows that $H_j(N_k; G) \to H_j(M; G)$ is an isomorphism for $j < k$ and $H_j(N_k) = \{0\}$ for $j > k$. The latter identity shows that in the commuting diagram

$$
\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \downarrow \\
0 & \longrightarrow & H_k(N_k; G) & \longrightarrow & H_k(N_k, N_{k-1}; G) & \xrightarrow{\partial} & H_{k-1}(N_{k-1}; G) \\
 & & & & & \searrow^{\partial} & \downarrow \\
 & & & & & & H_{k-1}(N_{k-1}, N_{k-2}; G)
\end{array}
$$

the horizontal and vertical sequences are exact. In particular the homomorphism $H_{k-1}(N_{k-1}; G) \to H_{k-1}(N_{k-1}, N_{k-2}; G)$ is injective so that the kernels of the two boundary homomorphisms agree. They are isomorphic to both $H_k(N_k; G)$ and $\ker \partial_{k-1}^c(G) \subset C_k(G)$. We conclude that the homology exact sequence

$$H_{k+1}(N_{k+1}, N_k; G) \xrightarrow{\partial_k} H_k(N_k; G) \longrightarrow H_k(N_{k+1}; G) \longrightarrow 0$$

is isomorphic to an exact sequence

$$C_{k+1}(G) \xrightarrow{\partial^c_k} \ker \partial^c_{k-1}(G) \longrightarrow H_k(M;G) \longrightarrow 0$$

and hence $H_k(M;G) \approx \ker \partial^c_{k-1}(G)/\mathrm{im}\,\partial^c_k(G)$.

The same arguments as above are used in **[21]** in order to characterize the homology of a CW-complex. Moreover, note that Theorem 3.1 remains valid if $\mathbb{Z}$ is replaced by a principal ideal domain R and G by any module over R.

We illustrate Theorem 3.1 with the example of a Morse function on $M = \mathbb{R}P^2$ having three critical points (Figure 2). In this example the connection matrix is given by

$$C_2 \xrightarrow{2} C_1 \xrightarrow{0} C_0$$

with $C_k = \mathbb{Z}$ and determines the integral homology $H_* = (\mathbb{Z}, \mathbb{Z}/2, 0)$ of $\mathbb{R}P^2$.

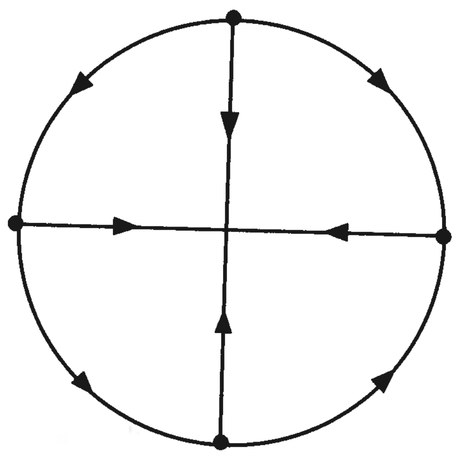

FIG. 2

As a side remark we point out that the connecting orbits of (2.1) can be interpreted as the solutions of the variational problem

$$\Phi_f(\gamma) = \frac{1}{2}\int_{\mathbb{R}} \left(\left| \frac{d\gamma}{ds} \right|^2 + |\nabla f(\gamma)|^2 \right) ds \tag{3.1}$$

for smooth curves $\gamma: \mathbb{R} \to M$ subject to the boundary conditions

$$\lim_{s\to-\infty} \gamma(s) = y, \quad \lim_{s\to\infty} \gamma(s) = x \tag{3.2}$$

where x and y are critical points of f. Indeed, if the space $\mathcal{M}(y, x)$ of all solutions $\gamma(t)$ of (2.1) satisfying (3.2) is nonempty then it consists of the absolute minima of the energy functional Φ_f. This follows immediately from the identity

$$\Phi_f(\gamma) = \frac{1}{2}\int_{\mathbb{R}} \left| \frac{d\gamma}{ds} + \nabla f(\gamma) \right|^2 ds + f(y) - f(x)$$

for every smooth curve $\gamma: \mathbb{R} \to M$ which satisfies (3.2). Note that the solutions of $\dot{x} - \nabla f(x) = 0$ also define absolute minima of $\Phi_f = \Phi_{-f}$ and that at most one of the spaces $\mathcal{M}(y,x)$ and $\mathcal{M}(x,y)$ can be nonempty. These observations indicate that in the rather elementary context of finite dimensional Morse theory there is some similarity to the theory of Yang–Mills equations on 4-manifolds with the spaces $\mathcal{M}(y,x)$ of connecting orbits playing the role of the moduli spaces [5].

If $\mathcal{M}(y,x)$ is nonempty then the relative index $\operatorname{ind}(y)-\operatorname{ind}(x)$ can be expressed as the Fredholm index of a certain linear first order differential operator. The latter is defined on vectorfields $\xi(s)$ along a connecting orbit $\gamma(s)$ by linearizing equation (2.1). More precisely, given a vectorfield $\xi(s)\in T_{\gamma(s)}M$ we define

$$F_\gamma \xi = \nabla_s \xi + \nabla_\xi \nabla f(\gamma) \tag{3.3}$$

where ∇ denotes the covariant derivative. We define the Hilbert space

$$L^2(\gamma) = \left\{\xi:\mathbb{R}\longrightarrow TM;\, \xi(s)\in T_{\gamma(s)}M,\, \int_{-\infty}^{\infty} |\xi(s)|^2\,ds < \infty\right\}$$

and consider F_γ as a linear operator from $W^{1,2}(\gamma) = \{\xi\in L^2(\gamma);\, \nabla_s\xi\in L^2(\gamma)\}$ into $L^2(\gamma)$.

THEOREM 3.3. *If $f: M\to\mathbb{R}$ is a Morse function with critical points $x, y\in M$ and $\gamma:\mathbb{R}\to M$ is a smooth curve satisfying* (3.2) *then F_γ is a Fredholm operator and*

$$\operatorname{ind}(F_\gamma) = \operatorname{ind}(y)-\operatorname{ind}(x).$$

If, moreover, the gradient flow of f is of Morse–Smale type and γ satisfies (2.1) *then F_γ is onto.*

As a matter of fact, it turns out that the kernel of F_γ consists of vectorfields tangent to $W^u(y)\cap W^s(x)$ whenever γ is a solution of (2.1). The proof of Theorem 3.3 makes use of the following observation.

LEMMA 3.4. *Let X, Y, Z be Banach spaces and suppose that the bounded linear operator $F\in L(X, Y)$ and the compact operator $K\in L(X, Z)$ satisfy an estimate*

$$\|x\|_X \leqslant c(\|Fx\|_Y + \|Kx\|_Z)$$

for every $x\in X$. Then F has a closed range and $\dim\ker F<\infty$.

Proof of Theorem 3.3. In our proof of the operator F_γ being Fredholm we closely follow the line of argument used by Floer [11] in an analogous situation. For every C^∞ vectorfield ξ along γ and sufficiently large constants $c>0$ and $T>0$ we shall prove the estimate

$$\int_{-\infty}^{\infty} (|\xi|^2+|\nabla\xi|^2)\,ds \leqslant c\left(\int_{-\infty}^{\infty} |F_\gamma\xi|^2\,ds + \int_{-T}^{T} |\xi|^2\,ds\right). \tag{3.4}$$

Then Lemma 3.4 shows that F_γ has a closed range and a finite dimensional kernel. Subsequently we will characterize the kernel and the cokernel of F_γ in order to establish the index formula.

Let $X_1(s),\dots,X_n(s)$ be an orthonormal basis of $T_{\gamma(s)}M$ for $s\in\mathbb{R}$ such that $\nabla X_\nu(s)\equiv 0$ and let $(\xi_1,\dots,\xi_n)\in\mathbb{R}^n$ denote the coordinates of ξ with respect to this basis. Then

$$F_\gamma\xi = F_\gamma\sum_{\nu=1}^{n}\xi_\nu X_\nu = \sum_{\nu=1}^{n}\left(\frac{d\xi_\nu}{ds}+\sum_{\mu=1}^{n} a_{\nu\mu}\xi_\mu\right)X_\nu$$

where

$$a_{\nu\mu}(s) = \langle X_\nu(s), \nabla_{X_\mu(s)} \nabla f(\gamma(s)) \rangle = a_{\mu\nu}(s).$$

We shall from now on denote by ξ the n-vector with components $\xi_1, \ldots, \xi_n$ and by F the operator F_y in the new coordinates so that

$$F\xi(s) = \frac{d\xi(s)}{ds} + A(s)\,\xi(s)$$

where $A(s) \in \mathbb{R}^{n\times n}$ denotes the symmetric matrix with entries $a_{\nu\mu}(s)$. Observe that the matrices $A_x = \lim_{s\to\infty} A(s)$ and $A_y = \lim_{s\to-\infty} A(s)$ represent the Hessian of f at x and y, respectively. Since f is a Morse function, it follows that A_x and A_y are nonsingular.

We claim that the operator

$$F_x\xi = \frac{d\xi}{ds} + A_x\xi$$

from $W^{1,2}(\mathbb{R}, \mathbb{R}^n)$ to $L^2(\mathbb{R}, \mathbb{R}^n)$ is boundedly invertible. Using the Fourier transform

$$\hat{\xi}(i\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega s}\xi(s)\,ds$$

we observe that $F_x\xi = \eta$ if and only if

$$(i\omega I + A_x)\,\hat{\xi}(i\omega) = \hat{\eta}(i\omega), \quad \omega \in \mathbb{R}.$$

Since A_x is nonsingular, it follows that $(1+|\omega|^2)\,|\hat{\xi}(i\omega)|^2 \leqslant c|\hat{\eta}(i\omega)|^2$ and hence

$$\|\xi\|^2_{W^{1,2}} = \int_{-\infty}^{\infty} (1+|\omega|^2)\,|\hat{\xi}|^2\,d\omega \leqslant c\int_{-\infty}^{\infty} |\hat{\eta}|^2\,d\omega = c\|\eta\|^2_{L^2}.$$

Thus we have proved that the operator F_x, or F_y for that matter, is boundedly invertible. This property is preserved under sufficiently small perturbations and hence we obtain the following estimate for large T

$$\int_{-\infty}^{\infty} \left(|\xi|^2 + \left|\frac{d\xi}{ds}\right|^2\right) ds \leqslant c\int_{-\infty}^{\infty} \left|\frac{d\xi}{ds} + A\xi\right|^2 ds, \quad \text{if } \xi(s) = 0 \text{ for } -T \leqslant s \leqslant T.$$

On bounded time intervals

$$\begin{aligned}
\int_{-T}^{T} \left|\frac{d\xi}{ds} + A\xi\right|^2 ds &= \int_{-T}^{T} \left(\left|\frac{d\xi}{ds}\right|^2 + 2\left\langle \frac{d\xi}{ds}, A\xi \right\rangle + |A\xi|^2\right) ds \\
&\geqslant \int_{-T}^{T} \left(\frac{1}{2}\left|\frac{d\xi}{ds}\right|^2 - |A\xi|^2\right) ds \\
&\geqslant \frac{1}{2}\int_{-T}^{T} \left|\frac{d\xi}{ds}\right|^2 ds - c\int_{-T}^{T} |\xi|^2\,ds
\end{aligned}$$

and hence

$$\int_{-T}^{T} \left(|\xi|^2 + \left|\frac{d\xi}{ds}\right|^2\right) ds \leqslant c\int_{-T}^{T} \left(|\xi|^2 + \left|\frac{d\xi}{ds} + A\xi\right|^2\right) ds. \tag{3.6}$$

Now choose a cutoff function $\beta: \mathbb{R} \to [0, 1]$ such that $\beta(t) = 1$ for $|t| \leqslant T$ and

$\beta(t) = 0$ for $|t| \geqslant T+1$. Then, with a sufficiently large constant $c > 0$, we obtain from (3.5) and (3.6) that

$$\begin{aligned}\|\xi\|_{W^{1,2}} &\leqslant \|\beta\xi\|_{W^{1,2}} + \|(1-\beta)\,\xi\|_{W^{1,2}}\\ &\leqslant c(\|\beta\xi\|_{L^2} + \|F\beta\xi\|_{L^2} + \|F(1-\beta)\,\xi\|_{L^2})\\ &\leqslant c\left(\|\beta\xi\|_{L^2} + 2\left\|\frac{d\beta}{ds}\xi\right\|_{L^2} + \|\beta F\xi\|_{L^2} + \|(1-\beta)\,F\xi\|_{L^2}\right)\\ &\leqslant 3c(\|\xi\|_{L^2[-T-1,\,T+1]} + \|F\xi\|_{L^2[\mathbb{R}]})\end{aligned}$$

and this proves (3.4).

It remains to be shown that range F_γ is of finite codimension and $\operatorname{ind} F_\gamma = \operatorname{ind}(y) - \operatorname{ind}(x)$. For this, we denote by

$$\Phi(s,t)\colon T_{\gamma(t)}M \longrightarrow T_{\gamma(s)}M$$

the solution operator of the linear differential equation $\nabla\xi + \nabla_\xi \nabla f(\gamma) = 0$. (Note that $\Phi(s,t) = d\phi^{s-t}(\gamma(t))$ whenever γ satisfies (2.1).) Then $\xi \in \ker F_\gamma$ if and only if $\xi(s) = \Phi(s,t)\,\xi(t)$ for $s, t \in \mathbb{R}$ and $\xi \in L^2(\gamma)$. Define

$$E^u(s) = \{\xi \in T_{\gamma(s)}M;\ \lim_{t\to-\infty} \Phi(t,s)\,\xi(s) = 0\}$$

and

$$E^s(s) = \{\xi \in T_{\gamma(s)}M;\ \lim_{t\to\infty} \Phi(t,s)\,\xi(s) = 0\}.$$

Using local coordinates near x and y one sees that $\xi(s)$ converges to zero exponentially as $|s| \to \infty$ if $\xi(s) \in E^s(s) \cap E^u(s)$. We conclude that $\xi \in \ker F_\gamma$ if and only if $\xi(s) = \Phi(s,t)\,\xi(t)$ and $\xi(s) \in E^s(s) \cap E^u(s)$ so that

$$\dim\ker F_\gamma = \dim(E^s \cap E^u).$$

In the same way as above the kernel of the adjoint operator

$$F_\gamma^*\colon W_\gamma^{1,2} \longrightarrow L_\gamma^2, \quad F_\gamma^*\eta = -\nabla\eta + \nabla_\eta \nabla f(\gamma),$$

is related to the map

$$\Psi(s,t) = \Phi(t,s)^*\colon T_{\gamma(t)}M \longrightarrow T_{\gamma(s)}M.$$

In particular, it follows from the identity $\langle \Psi(s,t)\,\eta(t), \Phi(s,t)\,\xi(t)\rangle = \langle \eta(t), \xi(t)\rangle$ that $-\nabla\eta + \nabla_\eta \nabla f(\gamma) = 0$ is equivalent to $\eta(s) = \Psi(s,t)\,\eta(t)$ for $s, t \in \mathbb{R}$. If this is satisfied then $\lim_{s\to\infty} \eta(s) = 0$ if and only if $\eta(s) \perp E^s(s)$ and likewise $\lim_{s\to-\infty} \eta(s) = 0$ if and only if $\eta(s) \perp E^u(s)$. Moreover, in both cases $\eta(s)$ converges to zero exponentially. Hence F_γ^* has a finite dimensional kernel consisting of those vectorfields η along γ which satisfy $\eta(s) = \Psi(s,t)\,\eta(t)$ and $\eta(s) \perp E^s(s) + E^u(s)$ so that

$$\dim\ker F_\gamma^* = \dim(E^s + E^u)^\perp.$$

Since $\operatorname{ind} F_\gamma$ depends only on the homotopy class of γ we may assume without loss of generality that $\gamma(s)$ satisfies (2.1) outside an interval $[-T, T]$. In this case

$$E^u(s) = T_{\gamma(s)}W^u(y), \ s \leqslant -T, \quad E^s(s) = T_{\gamma(s)}W^s(x), \ s \geqslant T,$$

and since $\Phi(s,t)\,E^{u,s}(t) = E^{u,s}(s)$ it follows that

$$\dim E^u(s) = \operatorname{ind}(y), \quad \dim E^s(s) = n - \operatorname{ind}(x)$$

for all $s \in \mathbb{R}$. We conclude that

$$\begin{aligned} \operatorname{ind} F_\gamma &= \dim\ker F_\gamma - \dim\ker F_\gamma^* \\ &= \dim(E^s \cap E^u) + \dim(E^s + E^u) - n \\ &= \dim E^u + \dim E^s - n \\ &= \operatorname{ind}(y) - \operatorname{ind}(x). \end{aligned}$$

If the gradient flow of f is of Morse–Smale type and $\gamma(s)$ satisfies (2.1), then $E^s(s) + E^u(s) = T_{\gamma(s)}M$ for every $s \in \mathbb{R}$ and it follows from the above that in this case F_γ is onto.

4. *Floer homology and the Arnold conjecture*

The Arnold conjecture states that the minimal number of fixed points of an exact symplectomorphism on a symplectic manifold is the sum of the Betti numbers provided that the fixed points are nondegenerate [**1**, **2**]. Recently this was proved by Floer [**9–13**] under the assumption that over $\pi_2(M)$ the cohomology class of ω agrees up to a constant with the first Chern class $c_1 \in H^2(M)$ of TM (regarded as a complex vectorbundle via an almost complex structure). His proof is based on a Morse type index theory for an indefinite functional on the loop space which was already employed by Conley and Zehnder [**7**, **8**] for the case of the $2n$-torus. Floer used an operator analogous to F_γ in order to define a relative index for two critical points with infinite Morse index. Moreover, he generalized the chain complex described in Section 3 in order to derive Morse type inequalities in the infinite dimensional situation. In this section we shall outline the main ideas of Floer's proof with some minor modifications. In order to avoid additional difficulties we shall content ourselves with the weaker assumption that the integral of ω vanishes over every sphere (see (4.2) below).

Let (M, ω) be a compact $2n$-dimensional symplectic manifold meaning that $\omega \in \Omega^2(M)$ is a nondegenerate closed 2-form. A *symplectomorphism* of M is a diffeomorphism $\psi \in \mathrm{Diff}(M)$ satisfying $\psi^*\omega = \omega$. It is called *exact* (or *homologous to the identity*) if it can be interpolated by a time dependent Hamiltonian differential equation

$$\dot{x}(t) = X_H(x(t), t). \tag{4.1}$$

Here $H: M \times \mathbb{R} \to \mathbb{R}$ is a smooth function satisfying $H(x, t+1) = H(x, t)$ and the associated Hamiltonian vectorfield $X_H: M \times \mathbb{R} \to TM$ is defined by

$$\omega(X_H(x, t), \xi) = -d_x H(x, t)\,\xi, \quad \xi \in T_x M.$$

The solutions $x(t)$ of (4.1) determine a 1-parameter family of symplectomorphisms $\psi_t \in \mathrm{Diff}(M)$ satisfying $\psi_t(x(0)) = x(t)$ and any symplectomorphism $\psi = \psi_1$ which can be generated this way is called *exact*. We denote by

$$\mathscr{P}_0 = \{x: \mathbb{R} \longrightarrow M;\, x \text{ satisfies (4.1)}, x(t+1) = x(t), x \text{ is null-homotopic}\}$$

the space of contractible 1-periodic solutions of (4.1). A periodic solution $x \in \mathscr{P}_0$ is called *nondegenerate* if $\det(I - d\psi_1(x(0))) \neq 0$.

We shall assume throughout that the integral of ω vanishes over every sphere

$$\int_{S^2} u^*\omega = 0, \quad u: S^2 \longrightarrow M. \tag{4.2}$$

THEOREM 4.1 (Floer). *Suppose that* (4.2) *holds and the contractible* 1-*periodic solutions of* (4.1) *are nondegenerate. Then their minimal number is the sum of the Betti numbers of* M (*with coefficients in any principal ideal domain* R).

If $H(x,t) = H(x)$ is time independent then this result follows directly from Theorem 2.1 since the critical points of H are periodic solutions of (4.1). In the general case the proof of Theorem 4.1 is based on a version of Morse theory on the loop space. More precisely, we denote by LM the loop space of M and by $L_0 M \subset LM$ the subspace of contractible loops. We shall identify $S^1 = \mathbb{R}/\mathbb{Z}$ and represent a loop in M by its cover $\gamma: \mathbb{R} \to M$ satisfying $\gamma(t+1) = \gamma(t)$. The contractible 1-periodic solutions of (4.1) can then be characterized as the critical points of the functional $f_H: L_0 M \to \mathbb{R}$ defined by

$$f_H(\gamma) = -\int_{D^2} u^*\omega + \int_0^1 H(\gamma(t), t)\, dt$$

where $D^2 \subset \mathbb{C}$ denotes the unit disc and $u: D^2 \to M$ is a smooth function satisfying $u(e^{2\pi i t}) = \gamma(t)$. Such a function u exists whenever γ is a contractible loop and it follows from (4.2) that $\int u^*\omega$ is independent of the choice of u. Now the tangent space $T_\gamma L_0 M$ can be represented as the space of vectorfields $\xi \in C^\infty(\gamma^* TM)$ along γ satisfying $\xi(t+1) = \xi(t)$ and a simple calculation shows that the 1-form $df_H: TL_0 M \to \mathbb{R}$ is given by

$$df_H(\gamma)\,\xi = \int_0^1 (\omega(\dot\gamma, \xi) + dH(\gamma, t)\,\xi)\, dt.$$

It follows that the critical points of f_H are indeed periodic solutions of (4.1).

In order to determine the gradient of f_H we choose an *almost complex structure* on M meaning an endomorphism $J \in C^\infty(\mathrm{End}\,(TM))$ such that $J^2 = -1$ and

$$\langle \xi, \eta \rangle = \omega(\xi, J(x)\eta), \quad \xi, \eta \in T_x M, \tag{4.3}$$

defines a Riemannian metric on M. Such an almost complex structure exists on every symplectic manifold and J is an isometry with respect to the metric (4.3). Moreover, $T_x M$ becomes an n-dimensional complex vectorspace with scalar multiplication defined by $z\xi = s\xi + tJ(x)\xi$ for $z = s + it \in \mathbb{C}$. A *holomorphic curve* is a solution $u: S \to M$ of the nonlinear Cauchy–Riemann equations

$$\bar\partial u = \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = 0$$

defined on a Riemann surface S. Holomorphic curves have been studied extensively by Gromov **[16]** and their analysis plays an essential role in Floer's work. In particular, they satisfy the identity

$$\int_S u^*\omega = \frac{1}{2}\int_S |\nabla u|^2$$

and hence condition (4.2) implies that there are no nonconstant holomorphic spheres. It actually turns out that both conditions are equivalent. We also point out that if $\nabla H: M \times \mathbb{R} \to TM$ denotes the gradient of H with respect to the x-variable then the associated Hamiltonian vectorfield can be written as $X_H(x,t) = J(x)\nabla H(x,t)$.

Now the gradient of f_H with respect to the induced metric on $L_0 M$ is given by

$$\nabla f_H(\gamma) = J(\gamma)\dot\gamma + \nabla H(\gamma, t) \in T_\gamma L_0 M.$$

A gradient flow line of f_H will therefore be a smooth map $u: \mathbb{R}\times S^1 = \mathbb{C}/i\mathbb{Z} \to M$ satisfying

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \nabla H(u,t) = 0. \tag{4.4}$$

This equation with the initial condition $u(0,t) = \gamma(t)$ does not define a wellposed Cauchy problem and, moreover, at any critical point the Morse index for both f_H and $-f_H$ is infinite. Nevertheless one can do Morse theory for f_H by studying only the space of bounded solutions, an idea which goes back to C. Conley.

In order to describe the space of bounded solutions of (4.4) we choose any two periodic solutions $x \in \mathscr{P}_0$ and $y \in \mathscr{P}_0$ and denote by $\mathscr{M}(y,x)$ the space of connecting orbits with respect to the 'gradient flow' of f_H. These are the solutions of (4.4) which satisfy the boundary conditions

$$\lim_{s\to-\infty} u(s,t) = y(t), \quad \lim_{s\to\infty} u(s,t) = x(t) \tag{4.5}$$

and they minimize the energy functional

$$\Phi_H(u) = \frac{1}{2}\int_{-\infty}^{\infty}\int_0^1 \left(\left|\frac{\partial u}{\partial s}\right|^2 + \left|\frac{\partial u}{\partial t} - X_H(u,t)\right|^2\right) dt\, ds.$$

As in the case of the functional (3.1) in finite dimensional Morse theory this follows from the identity

$$\Phi_H(u) = \frac{1}{2}\int_{-\infty}^{\infty}\int_0^1 \left|\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \nabla H(u,t)\right|^2 dt\, ds + f_H(y) - f_H(x).$$

In particular, if the space $\mathscr{M}(y,x)$ is nonempty and $x \neq y$ one gets

$$\inf \Phi_H = f_H(y) - f_H(x) > 0$$

where the infimum is taken over all smooth functions $u: \mathbb{R}\times S^1 \to M$ satisfying (4.5) and is attained in $\mathscr{M}(y,x)$. It also follows that $\mathscr{M}(y,x)$ is contained in the space

$$\mathscr{M} = \{u \in C^\infty(\mathbb{R}\times S^1, M);\ u \text{ satisfies (4.4)}, \Phi_H(u) < \infty \text{ and } u \text{ is null-homotopic}\}$$

of *bounded solutions*. Equivalently, $\mathscr{M}$ can be defined as the space of solutions u of (4.4) along which the decreasing function $f_H(u_s)$ with $u_s(t) = u(s,t)$ remains bounded. We point out that the real numbers act naturally on the space $\mathscr{M}$ by shifting $u(s,t)$ in the s-direction. This action corresponds to the gradient flow of f_H restricted to the space of bounded solutions and the spaces $\mathscr{M}(y,x)$ are invariant under this action.

PROPOSITION 4.2 (Floer). *$\mathscr{M} = \bigcup \mathscr{M}(y,x)$, where the union is taken over all pairs $x, y \in \mathscr{P}_0$. Moreover, if (4.2) is satisfied then $\mathscr{M}$ is compact. More precisely, let u_ν be any sequence in $\mathscr{M}(y,x)$. Then there exists a subsequence (still denoted by u_ν) and sequences of times $s_\nu^j \in \mathbb{R}$, $j = 1, \ldots, m$, such that $u_\nu(s + s_\nu^j, t)$ converges with its derivatives uniformly on compact sets to $u^j \in \mathscr{M}(x^j, x^{j-1})$ where $x^j \in \mathscr{P}_0$ for $j = 0, \ldots, m$ and $x^0 = x$, $x^m = y$.*

For a proof of this result we refer to **[11, 13, 17]** and to the next section where we present a modified version of Floer's compactness proof.

It was Floer's idea to reverse the approach to Morse theory which we have described in Section 3. In particular, he used the spaces $\mathscr{M}(y,x)$ of connecting orbits in order to define a relative Morse index.

PROPOSITION 4.3. (Floer). *Suppose that the contractible* 1*-periodic solutions of* (4.1) *are nondegenerate. Then there exists a dense set $J_{\mathrm{reg}} \subset C^\infty(\mathrm{End}(TM))$ of almost complex structures such that for every $J \in J_{\mathrm{reg}}$ and every pair $x, y \in \mathscr{P}_0$ the space $\mathcal{M}(y,x)$ is a finite-dimensional manifold.*

In finite-dimensional Morse theory the condition on the metric to be in J_{reg} corresponds to the gradient flow being of Morse–Smale type. This analogy suggests the dimension of the spaces $\mathcal{M}(y,x)$ as a candidate for the relative Morse index.

The proof of Proposition 4.3 makes use of a first order differential operator F_u defined on vectorfields $\xi \in C^\infty(u^*TM)$ by linearizing equation (4.4). More precisely, we shall denote by ∇ the Riemannian connection corresponding to the metric (4.3). Then for $u \in \mathcal{M}$ and a vectorfield $\xi(s,t) \in T_{u(s,t)}M$ we define

$$F_u \xi = \nabla_s \xi + J(u)\nabla_t \xi + \nabla_\xi J(u)\frac{\partial u}{\partial t} + \nabla_\xi \nabla H(u,t). \tag{4.6}$$

More abstractly, one can consider the left-hand side of (4.4) as a vectorfield on the infinite dimensional manifold of smooth maps $u: \mathbb{R} \times S^1 \to M$ satisfying (4.5). The differential operator (4.6) can then be interpreted as the covariant derivative of this vectorfield in the direction of a tangent vector $\xi \in C^\infty(u^*TM)$. We also point out that F_u can be written in the form

$$F_u \xi = \nabla_s \xi + J(u)(\nabla_t \xi - \nabla_\xi X_H(u,t)) + \nabla_\xi J(u)\left(\frac{\partial u}{\partial t} - X_H(u,t)\right).$$

Observing that the formal adjoint operator of F_u with respect to the metric (4.3) is given by

$$F_u^* \eta = -\nabla_s \eta + J(u)(\nabla_t \eta - \nabla_\eta X_H(u,t)) + \nabla_\eta J(u)\left(\frac{\partial u}{\partial t} - X_H(u,t)\right)$$

we define a locally square-integrable vectorfield ξ along u to be a *weak solution of* $F_u \xi = \eta$ if

$$\int_{-\infty}^{\infty}\int_0^1 \langle F_u^* \phi, \xi\rangle \, dt\, ds = \int_{-\infty}^{\infty}\int_0^1 \langle \phi, \eta\rangle \, dt\, ds, \quad \phi \in C_0^\infty(u^*TM). \tag{4.7}$$

Then the local regularity theory for elliptic operators yields the following estimate.

LEMMA 4.4. *Let ξ and η be locally square integrable vectorfields along u satisfying* (4.7). *Then $\nabla_s \xi$ and $\nabla_t \xi$ (defined in the distributional sense) are locally square integrable and*

$$\int_{-T}^{T}\int_0^1 (|\xi|^2 + |\nabla_s \xi|^2 + |\nabla_t \xi|^2)\, dt\, ds \leqslant c \int_{-T}^{T}\int_0^1 (|\xi|^2 + |F_u \xi|^2)\, dt\, ds$$

with a constant $c > 0$ depending on T but not on ξ. Moreover, if $\eta = F_u \xi$ is smooth (C^∞) then so is ξ and any $\xi \in C^\infty(u^*TM)$ is uniquely determined by $F_u \xi$ together with the values of ξ on $\{s\} \times S^1$ for any $s \in \mathbb{R}$.

We define the Hilbert space

$$L^2(u) = \left\{\xi : \mathbb{R} \times S^1 \longrightarrow TM; \xi(s,t) \in T_{u(s,t)}M, \int_{-\infty}^{\infty}\int_0^1 |\xi(s,t)|^2\, dt\, ds < \infty\right\}$$

and consider F_u as a linear operator from $W^{1,2}(u) = \{\xi \in L^2(u); \nabla_s \xi, \nabla_t \xi \in L^2(u)\}$ into $L^2(u)$.

PROPOSITION 4.5 (Floer). *If the contractible* 1*-periodic solutions of* (4.1) *are nondegenerate then F_u is a Fredholm operator for every smooth function $u: \mathbb{R} \times S^1 \to M$ which satisfies* (4.5). *Moreover, there exists a dense set of metrics $J_{\text{reg}} \subset C^\infty(\text{End}(TM))$ such that F_u is onto for every $J \in J_{\text{reg}}$ and every $u \in \mathcal{M}$.*

For the proof of this result we refer to **[11]** and **[13]**. That the operator F_u is Fredholm can be shown by arguments similar to those used in the proof of Theorem 3.3 with the estimate (3.6) replaced by Lemma 4.4.

Using an abstract implicit function theorem in connection with quadratic estimates on the higher order terms of the Taylor expansion for the nonlinear partial differential equation (4.4) one can then prove Proposition 4.3. It follows that if F_u is onto then a neighbourhood of u in $\mathcal{M}(y, x)$ is diffeomorphic to a neighbourhood of zero in $\ker F_u$ and hence

$$\dim \mathcal{M}(y, x) = \text{ind}\, F_u$$

locally near $u \in \mathcal{M}(y, x)$. The details of this argument become quite technical **[11]** and we shall not describe them here.

PROPOSITION 4.6 (Floer). *If* (4.2) *holds and the contractible* 1*-periodic solutions of* (4.1) *are nondegenerate then the index $\mu(y, x) = \text{ind}\, F_u$ is independent of $u \in \mathcal{M}(y, x)$ and*

$$\mu(z, y) + \mu(y, x) = \mu(z, x), \quad \mu(x, x) = 0$$

for any triple $x, y, z \in \mathcal{P}_0$.

In his proof of Proposition 4.6 Floer uses analytical methods **[13]**. Alternatively, one can use the Maslov index in order to give a topological characterization of the Fredholm index as was done by Floer **[12]** in the framework of Lagrangian intersections based on a construction due to Viterbo **[33]**. For the above operator F_u this will be carried out in a forthcoming paper.

It follows from Proposition 4.6 that one can assign an integer $\text{ind}\,(x) \in \mathbb{Z}$ to every periodic solution $x \in \mathcal{P}_0$ such that

$$\mu(y, x) = \text{ind}\, F_u = \text{ind}\,(y) - \text{ind}\,(x)$$

for every $u \in \mathcal{M}(y, x)$. (In **[13]** Floer does not assume (4.2) and defines the index modulo a subgroup $\Gamma \subset \mathbb{Z}$.) Observe that this index is only determined up to a common additive constant. Using the Maslov index, however, it is possible to determine whether the index of $x \in \mathcal{P}_0$ is odd or even according to whether or not $d\psi_1(x(0))$ lies in the component of $-I$ in the set of those linear symplectomorphisms of $T_{x(0)} M$ whose spectrum does not contain 1.

In the case of $\mathbb{Z}/2$ coefficients one now proceeds as in finite-dimensional Morse theory and defines the chain group C_k to be the $\mathbb{Z}/2$ vectorspace generated by the 1-periodic solutions of (4.1) with index k. It follows from Proposition 4.3 that $\mathcal{M}(y, x)$ consists of finitely many orbits whenever $\text{ind}\,(y) - \text{ind}\,(x) = 1$ and $J \in J_{\text{reg}}$. The boundary operator $\partial_k: C_{k+1} \to C_k$ is then given by counting the connecting orbits modulo 2. The homology groups of this chain complex are called the *Floer homology groups* with $\mathbb{Z}/2$ coefficients.

In order to define Floer homology with coefficients in any abelian group G we need to assign an integer $+1$ or -1 to a connecting orbit $u \in \mathcal{M}(y, x)$ provided that $\operatorname{ind}(y) - \operatorname{ind}(x) = 1$ and F_u is onto. Here we propose a slightly different approach from the one in [13]. For a periodic solution $x \in \mathcal{P}_0$ we consider the differential operator A_x on vectorfields $\xi(t) = \xi(t+1)$ along x defined by

$$A_x \xi = J(x)(\nabla \xi - \nabla_\xi X_H(x, t)) = J(x) \nabla \xi + \nabla_\xi \nabla H(x, t) + \nabla_\xi J(x) \dot{x}. \tag{4.8}$$

This operator represents the Hessian of f_H at the critical point $x \in L_0 M$ and is selfadjoint on the Hilbert space $L^2(x^*TM)$ with $D(A_x) = W^{1,2}(x^*TM)$. Observe that this operator A_x has a bounded inverse if and only if x is nondegenerate as a periodic solution of (4.1). Moreover, since A_x has a compact resolvent operator it follows that the linear subspace

$$E^u(x; \alpha) = \bigoplus_{-\alpha < \lambda < 0} \ker(\lambda I - A_x) \subset L^2(x^*TM)$$

is finite-dimensional for every $\alpha > 0$. We assume that there exists an integer $N > 0$ and numbers $\alpha_x > 0$ such that $-\alpha_x \notin \sigma(A_x)$ and

$$\dim E^u(x; \alpha_x) = N + \operatorname{ind}(x)$$

for every 1-periodic solution x of (4.1). Such numbers α_x and N exist for example if A_x has simple eigenvalues and this can be achieved by an arbitrarily small perturbation of the almost complex structure J. Returning to $u \in \mathcal{M}(y, x)$ with $\operatorname{ind}(y) - \operatorname{ind}(x) = 1$, we denote by $\mathcal{E}^u(u; \alpha_x)$ the space of all vectorfields $\xi \in C^\infty(u^*TM)$ such that $F_u \xi = 0$ and

$$\int_{-\infty}^{0} \int_0^1 |\xi|^2 \, dt \, ds + \int_0^\infty \int_0^1 e^{-2\alpha_x s} |\xi|^2 \, dt \, ds < \infty.$$

Since F_u is onto this defines a vectorspace of dimension $N + \operatorname{ind}(x) + 1$. Observing that $L^2(u_s^* TM)$ can be identified with $L^2(y^*TM)$ if $-s$ is sufficiently large we obtain a linear transformation

$$P_y : \mathcal{E}^u(u; \alpha_x) \longrightarrow E^u(y; \alpha_y)$$

defined by first restricting ξ to $\{s\} \times S^1$ then using the aforementioned identification and finally projecting orthogonally onto $E^u(y; \alpha_y) \subset L^2(y^*TM)$. If N was chosen sufficiently large then this map is actually an isomorphism. Similarly, identifying $L^2(u_s^* TM)$ with $L^2(x^*TM)$ for large s one can define a surjective linear transformation $P_x : \mathcal{E}^u(u; \alpha_x) \to E^u(x; \alpha_x)$. Both transformations remain surjective even if F_u is not. If, however, F_u is onto then they induce an isomorphism

$$P_x \circ P_y^{-1} : E^u(y; \alpha_y) \cap \xi_u^\perp \longrightarrow E^u(x; \alpha_x) \tag{4.9}$$

where

$$\xi_u = \lim_{s \to -\infty} \frac{\partial u / \partial s}{\| \partial u / \partial s \|_{L^2}} \in L^2(y^*TM).$$

We point out that we must have $\operatorname{ind}(y) - \operatorname{ind}(x) = 1$ in order for the spaces $E^u(x; \alpha_x)$ and $E^u(y; \alpha_y) \cap \xi_u^\perp$ to be of the same dimension. Now we fix an orientation of the spaces $E^u(x; \alpha_x)$ for every $x \in \mathcal{P}_0$. This induces an orientation on $E^u(y; \alpha_y) \cap \xi_u^\perp$ by taking ξ_u as the first basis vector in $E^u(y; \alpha_y)$. For $u \in \mathcal{M}(y, x)$ we then define n_u to be $+1$ or -1 according to whether the isomorphism (4.9) is orientation-preserving or orientation-reversing.

As in the finite-dimensional situation we denote by $\langle x \rangle$ the pair consisting of a periodic orbit $x \in \mathscr{P}_0$ together with the above mentioned orientation of the space $E^u(x;\alpha_x)$. We then define C_k to be the free group

$$C_k = \bigoplus_x \mathbb{Z}\langle x\rangle$$

where x runs over all $x \in \mathscr{P}_0$ with $\operatorname{ind}(x) = k$. For any pair $x, y \in \mathscr{P}_0$ with $\operatorname{ind}(y) - \operatorname{ind}(x) = 1$ we define the integer $n(y,x) \in \mathbb{Z}$ by

$$n(y,x) = \sum_u n_u$$

where the sum runs over all orbits in $\mathscr{M}(y,x)$. The boundary operator $\partial_k^F : C_{k+1} \to C_k$ of Floer's chain complex is then defined by

$$\partial^F\langle y\rangle = \sum_x n(y,x)\langle x\rangle, \quad y \in \mathscr{P}_0,$$

where the sum runs over all $x \in \mathscr{P}_0$ with $\operatorname{ind}(x) = \operatorname{ind}(y) - 1$. This chain complex is independent of N and of the choice of the orientations.

LEMMA 4.7. *For large enough integers N and N' and orientations $\langle x\rangle$ and $\langle x\rangle'$ of the subspaces $E^u(x;\alpha_x)$ and $E^u(x;\alpha_x')$, respectively, the associated chain complexes C_* and C'_* are chain isomorphic.*

We point out that increasing N corresponds to taking the product of the original flow with a repelling fixed point. In the finite-dimensional case this is reflected in increasing the Morse–Conley index by a common additive constant.

The above chain complex can be generalized to coefficients in any abelian group G by defining $C_k(G) = G \otimes C_k$ and $\partial^F(G) = \mathbb{1}_G \otimes \partial^F$.

THEOREM 4.8 (Floer). *The boundary operator satisfies $\partial^F \circ \partial^F = 0$. Moreover, the homology groups*

$$H_k^F(M;G) = \frac{\ker \partial_{k-1}^F(G)}{\operatorname{im} \partial_k^F(G)}$$

are independent of the Hamiltonian H and the almost complex structure J used to define them. They agree with the singular homology groups of M.

The homology groups of the chain complex $\partial^F : C_* \to C_*$ are called the *Floer homology groups*. The fact that they recover the homology of M proves Theorem 4.1 by choosing $G = R$ to be a principal ideal domain.

To prove this last part of Theorem 4.8 one continues H to a time-independent Morse function. Then the minimal period of the nonconstant periodic orbits of (4.1) is bounded away from zero. Multiplying H by a sufficiently small positive constant one can guarantee that the nonconstant periodic orbits are of period greater than 1. Under these conditions f_H agrees with H on the subspace of constant loops and Floer's chain complex contains the Morse complex of H as a subcomplex. More precisely, if $u \in \mathscr{M}$ is independent of t then $\gamma(s) = u(s,t)$ is a connecting orbit in the gradient flow of H and $\operatorname{ind} F_u = \operatorname{ind} F_\gamma$, where F_γ is the Fredholm operator defined by (3.3) with $f = H$. (This statement is not entirely obvious. It follows from a topological characterization of the Fredholm index in terms of the Maslov index for which we refer to a forthcoming paper.) It then follows from Theorem 3.3 that the index of a critical point x of H in the sense of Floer (regarded as a 1-periodic solution of

(4.1)) can be chosen as to agree with the Morse index. In the case $J \in J_{\mathrm{reg}}$ and $\operatorname{ind}(y) - \operatorname{ind}(x) = 1$ it also follows that every $u \in \mathcal{M}(y, x)$ is independent of t since otherwise $\operatorname{ind} F_u = \dim \ker F_u \geqslant 2$. Hence Floer's boundary operator is in this case entirely determined by the connecting orbits in the gradient flow of H. In order to determine the number n_u we observe that for a critical point x of H the tangent space $T_x L_0 M$ consists of 1-periodic vectorfields $\xi : S^1 \to T_x M$. On $T_x M$ we choose a symplectic orthonormal basis so that with respect to these coordinates the operator (4.8) is given by

$$A_x \xi = J \frac{d\xi}{dt} + S\xi$$

where J denotes the standard complex structure

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \in \mathbb{R}^{2n \times 2n}$$

and $S = S^T \in \mathbb{R}^{2n \times 2n}$ represents the Hessian of H at x. In terms of the standard Fourier series expansion of a loop $\xi : S^1 \to \mathbb{R}^{2n}$ with coordinates $a_0, a_k, b_k \in \mathbb{R}^{2n}$, we obtain that $\xi' = A_x \xi$ if and only if

$$a_0' = Sa_0, \quad a_k' = Sa_k + 2\pi k J b_k, \quad b_k' = -2\pi k J a_k + S b_k.$$

Recall that we may assume S to be arbitrarily small (but nondegenerate) and that the dimension of the negative part of S is the index of x. We may therefore choose $N = 0$ and $\alpha_x = \pi$ and it follows that $E^u(x; \alpha_x)$ consists of the constant orbits taking values in the unstable subspace $E^u(x) = T_x W^u(x)$ defined by the gradient flow of H. Moreover, note that the operator F_u is in the case $u(s, t) \equiv \gamma(s)$ given by

$$F_u \xi = \nabla_s \xi + J(u) \nabla_t \xi + \nabla_\xi \nabla H(u)$$

and hence a dimension argument shows that $\mathscr{E}^u(u; \alpha_x)$ consists of all vectorfields $\xi : \mathbb{R} \times S^1 \to TM$ along γ which are independent of t and satisfy $F_u \xi = 0$. We conclude that the isomorphism (4.9) in this case agrees with the flow defined map $E^u_\gamma(y) \to E^u(x)$ of Section 2. This shows that $n_u = n_\gamma$ and therefore the last statement of Theorem 4.8 follows from Theorem 3.1.

We close this section with an existence result for periodic orbits of period greater than 1 which in the case of the $2n$-torus was proved by Conley and Zehnder [7]. For general symplectic manifolds the proof is based on Floer homology and will be carried out in a forthcoming joint paper with E. Zehnder.

THEOREM 4.9. *Suppose that* (4.2) *holds and the contractible periodic solutions of* (4.1) *with integer period are nondegenerate. Then there are infinitely many of them.*

The assumptions of this result imply in particular that no root of 1 occurs as a Floquet multiplier of a periodic solution with integer period. It seems to be an open question whether the non-degeneracy condition in Theorem 4.9 can be removed or replaced by a weaker assumption.

5. *Compactness*

The proof of Proposition 4.2 rests on the following two lemmas. In the first we denote by B_r the open disc of radius $r > 0$ centred at $0 \in \mathbb{C}$.

LEMMA 5.1. *There exists a constant $\varepsilon > 0$ such that if $u: B_r \to M$ is a smooth solution of* (4.4) *satisfying*

$$\int_{B_r} \left|\frac{\partial u}{\partial s}\right|^2 dt\, ds \leqslant \varepsilon$$

then

$$\left|\frac{\partial u}{\partial s}(0)\right|^2 \leqslant 1 + \frac{8}{\pi r^2}\int_{B_r}\left|\frac{\partial u}{\partial s}\right|^2 dt\, ds. \tag{5.1}$$

The number $\varepsilon > 0$ in the estimate (5.1) depends on the manifold M, the symplectic form ω, the almost complex structure J and on the Hamiltonian function H. In the case of holomorphic curves ($H = 0$) a similar estimate was proved by Gromov [**16**] and Wolfson [**36**]. The proof in [**36**] carries through to the case $H \neq 0$ with only minor modifications. We point out that in the case $H = 0$ the additive constant 1 on the right-hand side of (5.1) can be dropped.

It follows from Lemma 5.1 that for every $u \in \mathscr{M}$

$$\|\nabla u\|_{L^\infty} = \sup\left\{\max\left\{\left|\frac{\partial u}{\partial s}(s,t)\right|, \left|\frac{\partial u}{\partial t}(s,t)\right|\right\}; s \in \mathbb{R}, 0 \leqslant t \leqslant 1\right\} < \infty.$$

LEMMA 5.2. *Let $G \subset \mathbb{C}$ be an open domain. Then every sequence of smooth solutions $u_\nu: G \to M$ of* (4.4) *satisfying*

$$\sup_{\nu \in \mathbb{N}} \|\nabla u_\nu\|_{L^\infty(G)} < \infty$$

has a subsequence converging (*with its derivatives*) *uniformly on every compact subset of G.*

The proof of Lemma 5.2 is based on a well-known elliptic bootstrapping argument. For the sake of completeness we shall carry out the proof of both lemmas below. On the basis of these results we are now in the position to carry out the proof of Floer's compactness result [**11, 17**].

Proof of Proposition 4.2. Recall from Lemma 5.1 that $\|\nabla u\|_{L^\infty} < \infty$ for every $u \in \mathscr{M}$. It then follows from Lemma 5.2 that $\mathscr{M}$ is the union of the sets $\mathscr{M}(y, x)$ over all pairs $x, y \in \mathscr{P}_0$. Otherwise there would exist a number $\varepsilon > 0$, a bounded solution $u \in \mathscr{M}$ of (4.4) and a sequence $(s_\nu, t_\nu) \in \mathbb{R} \times [0,1]$ such that $|s_\nu|$ converges to ∞ and $d(u(s_\nu, t_\nu), x(t_\nu)) \geqslant \varepsilon$ for every $\nu \in \mathbb{N}$ and every $x \in \mathscr{P}_0$. By Lemma 5.2 the sequence $u_\nu(s,t) = u(s + s_\nu, t)$ has a subsequence (still denoted by u_ν) converging with its derivatives, uniformly on compact sets, to a function $u^* \in \mathscr{M}$. Assuming without loss of generality that t_ν converges to $t^* \in [0,1]$ we then obtain that $d(u^*(0, t^*), x(t^*)) \geqslant \varepsilon$ for every $x \in \mathscr{P}_0$. But since $|s_\nu|$ converges to ∞ it follows that

$$\int_{-T}^{T}\int_0^1 \left|\frac{\partial u^*}{\partial s}\right|^2 dt\, ds = \lim_{\nu\to\infty}\int_{-T}^{T}\int_0^1\left|\frac{\partial u_\nu}{\partial s}\right|^2 dt\, ds = 0$$

for every $T > 0$ and hence $u^*(s,t) = x(t)$ for some $x \in \mathscr{P}_0$, a contradiction.

We shall now assume that condition (4.2) is satisfied and prove that $\mathscr{M}$ is compact. In view of Lemma 5.2 it is enough to prove that

$$\sup_{u \in \mathscr{M}} \|\nabla u\|_{L^\infty} < \infty.$$

We proceed indirectly and assume that there exists a sequence $u_\nu \in \mathcal{M}$ such that $c_\nu := \|\nabla u_\nu\|_{L^\infty} \to \infty$. We choose $z_\nu = s_\nu + it_\nu \in \mathbb{C}$ with

$$\max\left\{\left|\frac{\partial u_\nu}{\partial s}(z_\nu)\right|, \left|\frac{\partial u_\nu}{\partial t}(z_\nu)\right|\right\} \geqslant \tfrac{1}{2}c_\nu$$

and define $v_\nu(z) := u_\nu(z_\nu + c_\nu^{-1}z)$. Then

(i) $$|\nabla v_\nu(0)| \geqslant \tfrac{1}{2}, \quad \|\nabla v_\nu\|_{L^\infty(\mathbb{C})} \leqslant 1,$$

(ii) $$\frac{\partial v_\nu}{\partial s} + J(v_\nu)\frac{\partial v_\nu}{\partial t} + \frac{1}{c_\nu}\nabla H(v_\nu, t_\nu + c_\nu^{-1}t) = 0,$$

(iii) $$\int_{B_{c_\nu}(0)} \left|\frac{\partial v_\nu}{\partial s}\right|^2 = \int_{B_1(z_\nu)} \left|\frac{\partial u_\nu}{\partial s}\right|^2 \leqslant 2\Phi_H(u_\nu) \leqslant 2 \max_{x,y \in \mathcal{P}_0} f_H(y) - f_H(x).$$

It follows from (i) and Lemma 5.2 that v_ν has a subsequence converging together with its derivatives uniformly on every compact subset of $\mathbb{C}$. The limit function $v: \mathbb{C} \to M$ is smooth and satisfies

(i) $$\nabla v(0) \neq 0,$$

(ii) $$\frac{\partial v}{\partial s} + J(v)\frac{\partial v}{\partial t} = 0,$$

(iii) $$\int_{\mathbb{C}} \left|\frac{\partial v}{\partial s}\right|^2 < \infty.$$

Now define $\gamma_r: S^1 \to M$ by $\gamma_r(\theta) = v(re^{2\pi i\theta})$ and observe that

$$|\gamma_r'(\theta)| = 2\pi r\left|\frac{\partial v}{\partial s}(re^{2\pi i\theta})\right|$$

and hence

$$\int_{\mathbb{C}} \left|\frac{\partial v}{\partial s}\right|^2 = \int_0^\infty \frac{1}{2\pi r}\int_0^1 |\gamma_r'(\theta)|^2 \, d\theta \, dr < \infty.$$

Therefore the length $\ell(\gamma_R) \leqslant \|\gamma_R'\|_{L^2}$ can be made arbitrarily small by choosing R sufficiently large. Let $\alpha: U_\alpha \to \mathbb{R}^{2n}$ be a chart of M such that $\alpha(U_\alpha)$ is convex and suppose that $\gamma_R(S^1) \subset U_\alpha$ and $\alpha \circ \gamma_R(1) = 0$. Defining $w: S^2 = \mathbb{C} \cup \{\infty\} \to M$ by

$$w(re^{2\pi i\theta}) = \begin{cases} v(re^{2\pi i\theta}), & r \leqslant R, \\ \alpha^{-1}\left(\dfrac{R}{r}\alpha \circ v(Re^{2\pi i\theta}\right), & r \geqslant R, \end{cases}$$

one easily checks that for R sufficiently large

$$\int_{S^2} w^*\omega \geqslant \int_{D_R} v^*\omega - \varepsilon(R) = \int_{D_R} \left|\frac{\partial v}{\partial s}\right|^2 - \varepsilon(R) > 0$$

and this contradicts condition (4.2). Alternatively one can use a removable singularity theorem **[16, 23]** in order to establish that v extends to a (nonconstant) holomorphic sphere $v: S^2 \to M$ again contradicting (4.2).

In order to establish the more detailed convergence result we choose a number

$\varepsilon > 0$ such that $d(x(t), y(t)) > 2\varepsilon$ for every pair $x, y \in \mathscr{P}_0$ and every $t \in \mathbb{R}$. Given a sequence $u_\nu \in \mathscr{M}(y, x)$ we then define

$$s_\nu^1 = \sup\{s \in \mathbb{R}; d(u_\nu(s,t), x(t)) > \varepsilon \text{ for some } t \in \mathbb{R}\}.$$

By Lemma 5.2 we can choose a subsequence such that $u_\nu(s+s_\nu^1, t)$ converges to $u^1 \in \mathscr{M}$. It follows that $d(u^1(s,t), x(t)) \leqslant \varepsilon$ for all $s \geqslant 0, t \in \mathbb{R}$, and $d(u^1(0,t), x(t)) = \varepsilon$ for some $t \in \mathbb{R}$ so that $u^1 \in \mathscr{M}(x^1, x)$ for some $x^1 \in \mathscr{P}_0$, $x^1 \neq x$. We are done if $x^1 = y$ and otherwise we proceed by induction. Having established the existence of sequences s_ν^j such that $u_\nu(s+s_\nu^j, t)$ converges to $u^j \in \mathscr{M}(x^j, x^{j-1})$ for $j = 1, \ldots, k$ with $x^k \neq y$ we choose $s^* > 0$ such that $d(u^k(s,t), x^k(t)) < \varepsilon$ for some $s \leqslant -s^*$. For ν sufficiently large we then have $d(u_\nu(s_\nu^k - s^*, t), x^k(t)) < \varepsilon$ and define

$$s_\nu^{k+1} = \inf\{s \in \mathbb{R}; s \leqslant s_\nu^k - s^* \text{ and } d(u_\nu(\sigma, t), x^k(t)) < \varepsilon \text{ for } s \leqslant \sigma \leqslant s_\nu^k - s^*\}.$$

Then $s_\nu^k - s^* - s_\nu^{k+1}$ converges to ∞ and choosing a further subsequence we obtain that $u_\nu(s+s_\nu^{k+1}, t)$ converges to $u^{k+1} \in \mathscr{M}(x^{k+1}, x^k)$ with $x^{k+1} \neq x^k$. This finishes the induction step and the proof of Proposition 4.2.

We point out that Proposition 4.2 becomes false if condition (4.2) is not satisfied. This is due to the phenomenon of 'bubbling off of holomorphic spheres' which was first observed by Sacks and Uhlenbeck [**26**] and played an essential role in Gromov's work on pseudoholomorphic curves [**16**, **23**, **36**].

The proof of Lemma 5.1 is based on the next result which is a reformulation of an estimate in [**36**]. The main idea of the proof is a well-known trick in the theory of nonlinear partial differential equations which is due to Heinz and was also used by R. Schoen [**28**] in a similar context.

LEMMA 5.3. *If $\phi: \mathbb{R}^2 \supset B_r \to \mathbb{R}$ is a function of class C^2 satisfying*

$$\Delta\phi \geqslant -A(1+\phi^2), \quad \phi \geqslant 0, \quad \int_{B_r} \phi < \frac{\pi}{12A}, \tag{5.2}$$

for some constant $A > 0$ then

$$\phi(0) \leqslant 1 + \frac{8}{\pi r^2}\int_{B_r}\phi. \tag{5.3}$$

Proof. First observe that if $\phi \geqslant 0$ and $\Delta\phi \geqslant -C$ then

$$\begin{aligned} 2\pi\rho\phi(0) &= -\rho\int_{B_\rho}(\log\rho - \log|x|)\,\Delta\phi + \int_{\partial B_\rho}\phi \\ &\leqslant C\rho\int_{B_\rho}(\log\rho - \log|x|) + \int_{\partial B_\rho}\phi = \frac{C\pi\rho^3}{2} + \int_{\partial B_\rho}\phi \end{aligned}$$

and integrating over $0 \leqslant \rho \leqslant r$ we obtain

$$\phi(0) \leqslant \frac{Cr^2}{8} + \frac{1}{\pi r^2}\int_{B_r}\phi. \tag{5.4}$$

Secondly note that it is enough to prove Lemma 5.3 for $r = 1$. The general case then follows by rescaling.

Given a C^2 function $\phi:\bar{B}_1 \to \mathbb{R}$ which satisfies (5.2) we define

$$f(\rho) = (1-\rho)^2 \sup_{B_\rho} \phi, \quad 0 \leqslant \rho \leqslant 1,$$

and observe that $f(0) = \phi(0)$ and $f(1) = 0$. Choose $0 \leqslant \rho^* < 1$ and $\xi^* \in \bar{B}_{\rho^*}$ such that

$$f(\rho^*) = \sup_{[0,1]} f, \quad c^* = \phi(\xi^*) = \sup_{B_{\rho^*}} \phi,$$

and define $\varepsilon = \frac{1}{2}(1-\rho^*)$ so that

$$\sup_{B_\varepsilon(\xi^*)} \phi \leqslant \sup_{B_{\rho^*+\varepsilon}(0)} \phi = 4\frac{f(\rho^*+\varepsilon)}{(1-\rho^*)^2} \leqslant 4\frac{f(\rho^*)}{(1-\rho^*)^2} = 4c^*.$$

From now on we shall assume that $c^* \geqslant 1$ since otherwise $\phi(0) \leqslant c^* \leqslant 1$ and the Lemma is proved. Then in $B_\varepsilon(\xi^*)$ the function ϕ satisfies the estimate

$$\Delta\phi \geqslant -A(1+\phi^2) \geqslant -A(1+16c^{*2}) \geqslant -24Ac^{*2}$$

and this in connection with (5.4) shows that

$$c^* = \phi(\xi^*) \leqslant 3Ac^{*2}\rho^2 + \frac{1}{\pi\rho^2}\int_{B_\rho(\xi^*)} \phi\, dx, \quad 0 \leqslant \rho \leqslant \varepsilon. \tag{5.5}$$

Now suppose that $3Ac^*\varepsilon^2 \geqslant \frac{1}{2}$. Then we may choose $\rho = (6Ac^*)^{-\frac{1}{2}} \leqslant \varepsilon$ in (5.5) and obtain the inequality

$$c^* \leqslant \frac{c^*}{2} + \frac{6Ac^*}{\pi}\int_{B_1(0)} \phi\, dx$$

which contradicts (5.2). Hence $3Ac^*\varepsilon^2 \leqslant \frac{1}{2}$ and it follows from (5.5) with $\rho = \varepsilon$ that

$$c^* \leqslant 3Ac^{*2}\varepsilon^2 + \frac{1}{\pi\varepsilon^2}\int_{B_\varepsilon(\xi^*)} \phi\, dx \leqslant \frac{c^*}{2} + \frac{1}{\pi\varepsilon^2}\int_{B_1(0)} \phi\, dx$$

and this implies

$$c^*\varepsilon^2 \leqslant \frac{2}{\pi}\int_{B_1(0)} \phi\, dx.$$

We conclude that

$$\phi(0) = f(0) \leqslant f(\rho^*) = (1-\rho^*)^2 c^* = 4\varepsilon^2 c^* \leqslant \frac{8}{\pi}\int_{B_1(0)} \phi\, dx$$

and this finishes the proof of Lemma 5.3.

Proof of Lemma 5.1. In view of Lemma 5.3 it is enough to prove that for every solution $u: G \to M$ of (4.4) the function $\phi: G \to \mathbb{R}$ defined by

$$\phi(s,t) = \frac{1}{2}\left|\frac{\partial u}{\partial s}(s,t)\right|^2$$

satisfies an estimate of the form $\Delta\phi \geqslant -A(1+\phi^2)$ with a suitable constant $A > 0$ which only depends on M, ω, J and H.

For this we first observe that every solution of (4.4) satisfies

$$\nabla_s\left(\frac{\partial u}{\partial s} + \nabla H\right) + \nabla_t\left(\frac{\partial u}{\partial t} - X\right) = Y$$

where $X = X_H$ is the Hamiltonian vector field associated to H and the vector field Y along u is defined by

$$Y = \nabla_t J \frac{\partial u}{\partial s} - \nabla_s J \frac{\partial u}{\partial t} = (\nabla_{\nabla H} J) J \frac{\partial u}{\partial s} - (\nabla_{J\partial u/\partial s} J) \nabla H.$$

Now a somewhat technical but straightforward calculation shows that

$$\Delta\phi = \left|\nabla_s \frac{\partial u}{\partial s}\right|^2 + \left|\nabla_t \frac{\partial u}{\partial s}\right|^2 + |\nabla_s X|^2 - \left\langle \nabla_t \frac{\partial u}{\partial s}, \nabla_s X \right\rangle - \left\langle \nabla_s \frac{\partial u}{\partial s}, Y - \nabla_s \nabla H \right\rangle$$
$$- \left\langle \frac{\partial u}{\partial s}, R\left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}\right)\left(\frac{\partial u}{\partial t} - X\right)\right\rangle + \left\langle \frac{\partial u}{\partial s}, \nabla_t \nabla_s X \right\rangle - \left\langle \frac{\partial u}{\partial t} - X, \nabla_s \nabla_s X \right\rangle.$$

Hence the inequalities

$$|\nabla_t \nabla_s X| \leqslant c\left(\left|\frac{\partial u}{\partial s}\right| \cdot \left|\frac{\partial u}{\partial t}\right| + \left|\nabla_t \frac{\partial u}{\partial s}\right|\right), \quad |\nabla_s \nabla_s X| \leqslant c\left(\left|\frac{\partial u}{\partial s}\right|^2 + \left|\nabla_s \frac{\partial u}{\partial s}\right|^2\right), \quad |Y| \leqslant c\left|\frac{\partial u}{\partial s}\right|,$$

and

$$\left|\frac{\partial u}{\partial s}\right|^2 + \left|\frac{\partial u}{\partial t}\right|^2 \leqslant c\sqrt{1+\phi^2}$$

yield the required estimate for $\Delta\phi$.

Proof of Lemma 5.2. In view of Rellich's theorem it suffices to prove that for every compact domain $K \subset G$, every constant $c_1 > 0$ and numbers $k \in \mathbb{N}, p \geqslant 1$ with $kp > 2$ there exists a constant $C > 0$ such that

$$\|u\|_{W^{k+1,p}(K)} \leqslant C(\|\bar\partial u + \nabla H\|_{W^{k,p}(G)} + \|u\|_{W^{k,p}(G)} + 1) \tag{5.6}$$

for every smooth function $u: G \to M$ satisfying

$$\left|\frac{\partial u}{\partial s}(s,t)\right| + \left|\frac{\partial u}{\partial t}(s,t)\right| \leqslant c_1, \quad z = s + it \in G.$$

In order to establish (5.6) we shall reproduce the argument given by Floer [11] in the case $H = 0$. This again is a standard trick in the regularity theory for nonlinear elliptic problems. We shall need the inequality

$$\|w\|_{W^{k+1,p}(B_1)} \leqslant C_{k,p} \|\bar\partial_0 w\|_{W^{k,p}(B_1)}, \quad k = 0, 1, \ldots, 1 \leqslant p < \infty \tag{5.7}$$

for every smooth function $w: B_1 \to \mathbb{R}^{2n}$ with compact support, where

$$\bar\partial_0 w = \frac{\partial w}{\partial s} + J_0 \frac{\partial w}{\partial t}, \quad J_0 = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

The estimate (5.7) is indeed a straightforward consequence of the interior regularity for Laplace's equation.

We shall prove (5.6) locally in the neighbourhood of a point $z_0 = s + it \in K$. We choose local coordinates such that $J(u(z_0))$ is represented by the matrix J_0. Moreover, we choose $\varepsilon > 0$ sufficiently small and determine $\delta > 0$ such that $\|J(u(z)) - J_0\| < \varepsilon$ for $|z - z_0| < \delta$. Furthermore, we choose a cutoff function $\beta: \mathbb{R}^2 \to [0,1]$ such that $\beta(z) = 1$ for $|z| \leqslant \frac{1}{2}$ and $\beta(z) = 0$ for $|z| \geqslant \frac{3}{4}$ and define

$$\beta_\delta(z) = \beta(\delta^{-1}(z - z_0)).$$

Then

$$\|u\|_{W^{k+1,p}(B_{\delta/2}(z_0))} \leqslant \|\beta_\delta u\|_{W^{k+1,p}(B_\delta(z_0))}$$

and hence it suffices to estimate $\|\beta_\delta u\|_{k+1,p}$. For this we shall need the following fundamental inequalities for smooth functions $f, g: \mathbb{R}^n \to \mathbb{R}^m$ and numbers $k \in \mathbb{N}$, $1 \leqslant p < \infty$ with $kp > n$:

$$\|fg\|_{k,p} \leqslant C_{k,p}(\|f\|_{k,p}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{k,p}),$$

$$\|F \circ f\|_{k,p} \leqslant C_{k,p}\|F\|_{C^k}(\|f\|_{k,p} + 1).$$

We point out that in the latter inequality the constant $C_{k,p}$ depends on a bound for the function f.

We are now in the position to carry out the estimate for $\|\beta_\delta u\|_{k+1,p}$ denoting by $C > 0$ a generic constant which is independent of u and δ.

$$\begin{aligned}
\|\beta_\delta u\|_{k+1,p} &\leqslant C_{k,p}\|\bar{\partial}_0(\beta_\delta u)\|_{k,p} \\
&\leqslant C_{k,p}(\|\beta_\delta \bar{\partial}_0 u\|_{k,p} + C_\delta\|u\|_{k,p}) \\
&= C_{k,p}\left(\left\|\beta_\delta \bar{\partial} u + \beta_\delta(J_0 - J(u))\frac{\partial u}{\partial t}\right\|_{k,p} + C_\delta\|u\|_{k,p}\right) \\
&= C_{k,p}\left(\left\|\beta_\delta \bar{\partial} u + (J_0 - J(u))\frac{\partial(\beta_\delta u)}{\partial t} - \frac{\partial \beta_\delta}{\partial t}(J_0 - J(u))\,u\right\|_{k,p} + C_\delta\|u\|_{k,p}\right) \\
&\leqslant C\Bigg(\|\beta_\delta\|_{L^\infty}\|\bar{\partial} u\|_{k,p;B_\delta} + \|\beta_\delta\|_{k,p}\|\bar{\partial} u\|_{L^\infty} + C_\delta\|u\|_{k,p} + C_\delta \\
&\qquad + \|J_0 - J(u)\|_{L^\infty(B_\delta)}\left\|\frac{\partial(\beta_\delta u)}{\partial t}\right\|_{k,p} + \|J_0 - J(u)\|_{k,p}\left\|\frac{\partial(\beta_\delta u)}{\partial t}\right\|_{L^\infty}\Bigg) \\
&\leqslant C_\delta(\|\bar{\partial} u\|_{k,p} + \|u\|_{k,p} + 1) + C\|J_0 - J(u)\|_{L^\infty(B_\delta)}\left\|\frac{\partial(\beta_\delta u)}{\partial t}\right\|_{k,p} \\
&\leqslant C_\delta(\|\bar{\partial} u + \nabla H(u,t)\|_{k,p} + \|u\|_{k,p} + 1) + C\varepsilon\|\beta_\delta u\|_{k+1,p}.
\end{aligned}$$

With $\delta > 0$ sufficiently small we obtain $C\varepsilon < 1$ and this proves Lemma 5.2.

We point out that a similar argument as in the proof of Lemma 5.2 can be used in order to establish Lemma 4.4 without referring to the general theory of elliptic operators.

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