Notes on compact Lie groups

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1 Lie groups

A **Lie Group** is a smooth manifold G with a group structure such that the multiplication and the inverse map are smooth (C^∞). The tangent space at the identity element $\mathbb{1} \in \mathrm{G}$ is called the **Lie algebra** of G and is denoted by

$$\mathfrak{g} = \mathrm{Lie}(\mathrm{G}) = T_{\mathbb{1}}\mathrm{G}.$$

For every $g \in \mathrm{G}$ the right and left multiplication maps $R_g, L_g : \mathrm{G} \to \mathrm{G}$ are defined by

$$R_g(h) := hg, \qquad L_g(h) := gh$$

for $h \in \mathrm{G}$. We shall denote the derivatives of these maps by

$$vg := dR_g(h)v \in T_{hg}\mathrm{G}, \qquad gv := dL_g(h)v \in T_{gh}\mathrm{G}$$

for $v \in T_h\mathrm{G}$. In particular, for $h = \mathbb{1}$ and $\xi \in T_{\mathbb{1}}\mathrm{G} = \mathfrak{g}$, we have $\xi g, g\xi \in T_g\mathrm{G}$ and hence ξ determines two vector fields $g \mapsto g\xi$ and $g \mapsto \xi g$ on G. These are called the **left-invariant** respectively **right-invariant** vector fields generated by ξ. We shall prove in Lemma 1.2 below that the integral curves of both vector fields through $g_0 = \mathbb{1}$ agree.

Exercise 1.1. (i) Prove that

$$(v_0 g_1)g_2 = v_0(g_1 g_2)$$

for $v_0 \in T_{g_0}\mathrm{G}$ and $g_1, g_2 \in \mathrm{G}$. Similarly

$$(g_0 v_1)g_2 = g_0(v_1 g_2), \qquad (g_0 g_1)v_2 = g_0(g_1 v_2).$$

(ii) Prove that with the above notation the usual multiplication rule holds: If $\alpha, \beta : \mathbb{R} \to \mathrm{G}$ are smooth curves then

$$\frac{d}{dt}\alpha(t)\beta(t) = \dot\alpha(t)\beta(t) + \alpha(t)\dot\beta(t).$$

(**Hint:** Consider the partial derivatives of the map $\mathbb{R}^2 \to \mathrm{G} : (s,t) \mapsto \alpha(s)\beta(t)$ and use the chain rule.)
(iii) Deduce that

$$\frac{d}{dt}\gamma(t)^{-1} = -\gamma(t)^{-1}\dot\gamma(t)\gamma(t)^{-1}$$

for every curve $\gamma : \mathbb{R} \to \mathrm{G}$.
(iv) Prove that the vector fields $g \mapsto g\xi$ and $g \mapsto \xi g$ are complete for every $\xi \in \mathfrak{g}$. (**Hint:** Prove that the length of the existence interval is independent of the initial condition.)

Lemma 1.2. *Let $\xi \in \mathfrak{g}$ and $\gamma : \mathbb{R} \to \mathrm{G}$ be a smooth function. Then the following conditions are equivalent.*

(i)

$$\gamma(t+s) = \gamma(s)\gamma(t), \qquad \gamma(0) = \mathbb{1}, \qquad \dot{\gamma}(0) = \xi. \tag{1}$$

(ii)

$$\dot{\gamma}(t) = \xi\gamma(t), \qquad \gamma(0) = \mathbb{1}. \tag{2}$$

(iii)

$$\dot{\gamma}(t) = \gamma(t)\xi, \qquad \gamma(0) = \mathbb{1}. \tag{3}$$

Moreover, for every $\xi \in \mathfrak{g}$ there exists a unique smooth function $\gamma : \mathbb{R} \to \mathrm{G}$ that satisfies either of these conditions.

Proof. That (i) implies (ii) follows by differentiating the identity (1) with respect to s at $s = 0$. To prove that (ii) implies (i) note that, by Exercise 1.1 (i), the curves $\alpha(t) = \gamma(t+s)$ and $\beta(t) = \gamma(t)\gamma(s)$ are both integral curves of the vector field $g \mapsto \xi g$ such that $\alpha(0) = \beta(0) = \gamma(s)$. Hence they are equal. This shows that (i) is equivalent to (ii). That (i) is equivalent to (iii) follows by analogous arguments, interchanging s and t. The last assertion about the existence of γ follows from Exercise 1.1 (iv). □

The **exponential map** $\exp : \mathfrak{g} \to \mathrm{G}$ is defined by

$$\exp(\xi) := \gamma_\xi(1),$$

where $\gamma_\xi : \mathbb{R} \to \mathrm{G}$ is the unique solution of (1). With this definition the path γ_ξ is given by

$$\gamma_\xi(t) = \exp(t\xi).$$

To see this note that the curve $\alpha(s) = \gamma_\xi(ts)$ satisfies $\dot{\alpha}(s) = t\xi\alpha(s)$. Hence the exponential map satisfies

$$\frac{d}{dt}\exp(t\xi) = \xi\exp(t\xi) = \exp(t\xi)\xi.$$

The **adjoint representation** of G on its Lie algebra $\mathfrak{g}$ is defined by

$$\mathrm{Ad}(g)\eta := g\eta g^{-1} := \left.\frac{d}{dt}\right|_{t=0} g\exp(t\eta)g^{-1}.$$

In other words the linear map $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ is the differential of the map $\mathrm{G} \to \mathrm{G} : h \mapsto ghg^{-1}$ at $h = \mathbb{1}$. The map $\mathrm{G} \to \mathrm{Aut}(\mathfrak{g}) : g \mapsto \mathrm{Ad}(g)$ is a group homomorphism

$$\mathrm{Ad}(gh) = \mathrm{Ad}(g)\mathrm{Ad}(h), \qquad \mathrm{Ad}(\mathbb{1}) = \mathrm{id},$$

and is called the **adjoint action** of G on its Lie algebra. The differential of this map at $g = \mathbb{1}$ in the direction $\xi \in \mathfrak{g}$ is denoted by $\mathrm{ad}(\xi)$. The **Lie bracket** of two elements $\xi, \eta \in \mathfrak{g}$ is defined by

$$[\xi, \eta] := \mathrm{ad}(\xi)\eta = \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi)\eta\exp(-t\xi).$$

Lemma 1.3. *For all $\xi, \eta, \zeta \in \mathfrak{g}$ we have*

$$[\xi, \eta] = -[\eta, \xi],$$

$$[[\xi, \eta], \zeta] + [[\eta, \zeta], \xi] + [[\zeta, \xi], \eta] = 0.$$

Proof. We prove that the map $\mathfrak{g} \to \mathrm{Vect}(\mathrm{G}) : \xi \mapsto X_\xi$ defined by $X_\xi(g) = \xi g$ is a Lie algebra homomorphism. To see this denote by $\psi_t \in \mathrm{Diff}(\mathrm{G})$ the flow generated by X_ξ, that is

$$\psi_t(g) = \exp(t\xi)g.$$

Then, by definition of the Lie bracket of vector fields,

$$\begin{aligned} [X_\xi, X_\eta](g) &= \left.\frac{d}{dt}\right|_{t=0} d\psi_t(\psi_{-t}(g))X_\eta(\psi_{-t}(g)) \\ &= \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi)\eta\exp(-t\xi)g \\ &= [\xi, \eta]g \end{aligned}$$

Here we have used Exercise 1.1 (i). Now the statement follows from the properties of the Lie bracket for vector fields. □

Lemma 1.4. *Let $\xi, \eta \in \mathfrak{g}$ and define $\gamma : \mathbb{R} \to \mathrm{G}$ by*

$$\gamma(t) = \exp(t\xi)\exp(t\eta)\exp(-t\xi)\exp(-t\eta).$$

Then $\dot\gamma(0) = 0$ and

$$\left.\frac{d}{dt}\right|_{t=0} \gamma(\sqrt{t}) = [\xi, \eta].$$

Proof. As in the proof of Lemma 1.3, the flow of the vector field $X_\xi(g) = \xi g$ on G is given by $t \mapsto L_{\exp(t\xi)}$ and $[X_\xi, X_\eta] = X_{[\xi,\eta]}$ for $\xi, \eta \in \mathfrak{g}$. Hence the result follows from the corresponding formula for general vector fields. □

Lemma 1.5. *Let* $\mathbb{R}^2 \to \mathrm{G} : (s,t) \mapsto g(s,t)$ *be a smooth function. Then*

$$\partial_s \left(g^{-1}\partial_t g\right) - \partial_t \left(g^{-1}\partial_s g\right) + \left[g^{-1}\partial_s g, g^{-1}\partial_t g\right] = 0.$$

Proof. If M is a smooth manifold, $\gamma : \mathbb{R}^2 \to M$ is a smooth function, and $X(s,t), Y(s,t) \in \mathrm{Vect}(M)$ are smooth families of vector fields such that

$$\partial_s \gamma = X \circ \gamma, \qquad \partial_t \gamma = Y \circ \gamma,$$

then

$$(\partial_s Y - \partial_t X - [X, Y]) \circ \gamma = 0.$$

To obtain the required formula, apply this to the manifold $M = \mathrm{G}$, the function $g : \mathbb{R}^2 \to \mathrm{G}$, and the vector fields $X(s,t) = X_{\xi(s,t)}$, $Y(s,t) = X_{\eta(s,t)}$, where $\xi = (\partial_s g)g^{-1}$ and $\eta = (\partial_t g)g^{-1}$. □

Exercise 1.6. Prove that for every $g \in \mathrm{G}$ and every $\xi \in \mathfrak{g}$

$$g \exp(\xi) g^{-1} = \exp(\mathrm{Ad}(g)\xi)$$

(**Hint:** Consider the curve $\gamma(t) = g\exp(t\xi)g^{-1}$ and use Exercise 1.1.)

Exercise 1.7. Prove that any two elements $\xi, \eta \in \mathfrak{g}$ satisfy $[\xi, \eta] = 0$ if and only if $\exp(s\xi)$ and $\exp(t\eta)$ commute for all $s, t \in \mathbb{R}$.

2 Lie group homomorphisms

Let G and H be Lie groups with corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. A **Lie group homomorphism** is a smooth map $\phi : \mathrm{G} \to \mathrm{H}$ which is a group homomorphism. A linear map $\Phi : \mathfrak{g} \to \mathfrak{h}$ is called a **Lie algebra homomorphism** if

$$[\Phi(\xi), \Phi(\eta)] = \Phi([\xi, \eta]).$$

for all $\xi, \eta \in \mathfrak{g}$. The next lemma asserts that the differential of a Lie group homomorphism at the identity is a Lie algebra homomorphism. An example is the map $\mathfrak{g} \to \mathrm{Vect}(\mathrm{G}) : \xi \mapsto X_\xi$ in the proof of Lemma 1.3 above. In this case the corresponding Lie group homomorphism is the map $\mathrm{G} \mapsto \mathrm{Diff}(\mathrm{G}) : g \mapsto L_g$. (See Example 10.14 below.)

Lemma 2.1. *If* $\phi : \mathrm{G} \to \mathrm{H}$ *is a Lie group homomorphism then* $\Phi = d\phi(\mathbb{1}) : \mathfrak{g} \to \mathfrak{h}$ *is a Lie algebra homomorphism*

Proof. We show first that Φ and ϕ intertwine the the exponential maps, i.e.

$$\exp(\Phi(\xi)) = \phi(\exp(\xi)) \tag{4}$$

for all $\xi \in \mathfrak{g}$. To see consider the curve $\gamma(t) := \phi(\exp(t\xi)) \in \mathrm{H}$. This curve satisfies $\gamma(s+t) = \gamma(s)\gamma(t)$ for all $s, t \in \mathbb{R}$ and $\dot{\gamma}(0) = \Phi(\xi)$. Hence, by Lemma 1.2, $\gamma(t) = \exp(t\Phi(\xi))$. With $t = 1$ this proves (4).

Next we prove that

$$\Phi(g^{-1}\xi g) = \phi(g)^{-1}\Phi(\xi)\phi(g) \tag{5}$$

for $\xi \in \mathfrak{g}$ and $g \in \mathrm{G}$. To see this, consider the curve $\gamma(t) := g\exp(t\xi)g^{-1}$. By (4),

$$\phi(\gamma(t)) = \phi(g)\exp(t\Phi(\xi))\phi(g)^{-1}.$$

Differentiating this curve at $t = 0$ we obtain (5). It follows from (4) and (5) that

$$\begin{aligned} \Phi([\xi,\eta]) &= \left.\frac{d}{dt}\right|_{t=0} \Phi(\exp(t\xi)\eta\exp(-t\xi)) \\ &= \left.\frac{d}{dt}\right|_{t=0} \exp(t\Phi(\xi))\Phi(\eta)\exp(-t\Phi(\xi)) \\ &= [\Phi(\xi), \Phi(\eta)] \end{aligned}$$

for all $\xi, \eta \in \mathfrak{g}$. This proves the lemma. □

A **representation** of G is a Lie group homomorphism $\rho : \mathrm{G} \to \mathrm{Aut}(V)$ where V is a real or complex vector space. Differentiating such a map at $g = \mathbb{1}$ gives a Lie algebra homomorphism $\dot{\rho} : \mathfrak{g} \to \mathrm{End}(V)$ defined by

$$\dot{\rho}(\xi) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \rho(\exp(t\xi))$$

for $\xi \in \mathfrak{g}$. For example the map $\mathrm{G} \to \mathrm{Aut}(\mathfrak{g}) : g \mapsto \mathrm{Ad}(g)$ is the adjoint representation of G on its Lie algebra with corresponding Lie algebra homomorphism $\mathfrak{g} \mapsto \mathrm{End}(\mathfrak{g}) : \xi \mapsto \mathrm{ad}(\xi)$. Also there is the obvious action of $\mathrm{U}(n)$ on $\mathbb{C}^n$ and the induced actions on spaces of symmetric polynomials or exterior forms.

3 The Haar measure

Let G be a compact Lie group and denote by $\mathcal{C}(\mathrm{G})$ the space of continuous functions $f : \mathrm{G} \to \mathbb{R}$ with the norm

$$\|f\| := \sup_{g\in\mathrm{G}} |f(g)| .$$

The next theorem asserts the existence of a translation invariant measure on every compact Lie group. The result and its proof extend to every compact topological group that satisfies the second axiom of countability (i.e. it has a finite or countable basis).

Theorem 3.1. *Let* G *be a compact Lie group. Then there exists a bounded linear functional* $M : \mathcal{C}(\mathrm{G}) \to \mathbb{R}$ *that satisfies the following conditions.*

(i) $M(1) = 1$.

(ii) *M is left invariant, i.e.* $M(f \circ L_g) = M(f)$ *for* $f \in \mathcal{C}(G)$ *and* $g \in \mathrm{G}$.

(iii) *M is right invariant, i.e.* $M(f \circ R_g) = M(f)$ *for* $f \in \mathcal{C}(G)$ *and* $g \in \mathrm{G}$.

(iv) *If* $f \geq 0$ *and* $f \neq 0$ *then* $M(f) > 0$.

(v) *Let* $\phi : \mathrm{G} \to \mathrm{G}$ *denote the diffeomorphism defined by* $\phi(g) = g^{-1}$. *Then* $M(f \circ \phi) = M(f)$ *for every* $f \in \mathcal{C}(\mathrm{G})$.

M is uniquely determined by (i) and either (ii) or (iii). It is called the **Haar measure** *on* G.

Proof. We follow notes by Moser which in turn are based on a proof by Pontryagin. Let $\mathcal{A}$ denote the set of all measures on G of the form

$$A = \sum_{i=1}^{k} \alpha_i \delta_{a_i}$$

where $\alpha_i \in \mathbb{Q}$ and $\sum_i \alpha_i = 1$. If $B = \sum_{j=1}^{\ell} \beta_j \delta_{b_j}$ is another such measure, denote

$$A \cdot B := \sum_{i=1}^{k} \sum_{j=1}^{\ell} \alpha_i \beta_j \delta_{a_i b_j}.$$

This defines a group structure on $\mathcal{A}$. For $A \in \mathcal{A}$ we define two linear operators $L_A, R_A : \mathcal{C}(\mathrm{G}) \to \mathcal{C}(\mathrm{G})$ by

$$(L_A f)(g) := \sum_{i=1}^{m} \alpha_i f(a_i g), \qquad (R_A f)(g) := \sum_{i=1}^{m} \alpha_i f(g a_i)$$

for $f \in \mathcal{C}(\mathrm{G})$ and $g \in \mathrm{G}$. Then

$$L_A(f \circ R_h) = (L_A f) \circ R_h, \qquad R_A(f \circ L_h) = (R_A f) \circ L_h, \tag{6}$$

$$L_{A \cdot B} = L_B \circ L_A, \qquad R_{A \cdot B} = R_A \circ R_B, \qquad L_A \circ R_B = R_B \circ L_A, \tag{7}$$

$$\min f \leq L_A f \leq \max f, \qquad \min f \leq R_A f \leq \max f. \tag{8}$$

We make use of the following three observations.

Observation 1: *Denote* $\mathrm{Osc}(f) := \max f - \min f$*. If* $f \in \mathcal{C}(\mathrm{G})$ *is non-constant then there exists an* $A \in \mathcal{A}$ *such that* $\min f < \min L_A f$ *and hence* $\mathrm{Osc}(L_A f) < \mathrm{Osc}(f)$*.*

Suppose f assumes its maximum at a point $g_0 \in \mathrm{G}$. Choose a neighbourhood $U \subset \mathrm{G}$ of $\mathbb{1}$ such that

$$g {g_0}^{-1} \in U \qquad \Longrightarrow \qquad f(g) > \frac{1}{2}(\max f + \min f).$$

Now $\mathrm{G} = \bigcup_{a \in \mathrm{G}} a^{-1} U$. Since G is compact, there exist finitely many points $a_1, \ldots, a_m \in \mathrm{G}$ such that

$$\mathrm{G} = \bigcup_{i=1}^{m} {a_i}^{-1} U.$$

This means that for every $h \in \mathrm{G}$ there exists an i such that $a_i h \in U$. Consider the measure $A := m^{-1} \sum_i \delta_{a_i}$. Since, for every $g \in \mathrm{G}$, at least one of the points $a_i g {g_0}^{-1}$ lies in U we obtain

$$\begin{aligned} (L_A f)(g) &= \frac{1}{m} \sum_{i=1}^{m} f(a_i g) \\ &\geq \frac{m-1}{m} \min f + \frac{1}{2m}(\max f + \min f) \\ &> \min f. \end{aligned}$$

Hence $\min L_A f > \min f$ and, by (8), $\mathrm{Osc}(L_A f) < \mathrm{Osc}(f)$.

Observation 2: *For every $f \in \mathcal{C}(\mathrm{G})$ the set $\mathcal{L}(f) := \{L_A f \,|\, A \in \mathcal{A}\}$ is bounded and equicontinuous.*

Boundedness follows from (8). To prove equicontinuity, note that, since G is compact and second countable, it is a metrizable topological space. Let $d : \mathrm{G} \times \mathrm{G} \to \mathbb{R}$ be a distance function which induces the given topology. Fix a function $f \in \mathcal{C}(\mathrm{G})$ and an $\varepsilon > 0$. Since G is compact, f is uniformly continuous. Hence there is a $\delta > 0$ such that, for all $g, h \in \mathrm{G}$,

$$d(g,h) < \delta \qquad \Longrightarrow \qquad |f(g) - f(h)| < \varepsilon. \tag{9}$$

We prove that there is an open neighbourhood $U \subset \mathrm{G}$ of $\mathbb{1}$ such that

$$g^{-1}h \in U \qquad \Longrightarrow \qquad d(g,h) < \delta. \tag{10}$$

We argue by contradiction. Suppose that there exist sequences $g_\nu, h_\nu \in \mathrm{G}$ such that ${g_\nu}^{-1}h_\nu \to \mathbb{1}$ and $d(g_\nu, h_\nu) \geq \delta$. Passing to a subsequence we may assume that g_ν converges to g. Then $h_\nu = g_\nu({g_\nu}^{-1}h_\nu)$ converges also to g. Hence, for ν sufficiently large, we have $d(g_\nu, g) < \delta/2$ and $d(h_\nu, g) < \delta/2$, contradicting the assumption that $d(g_\nu, h_\nu) \geq \delta$. This proves (10). A similar argument shows that there is a constant $\delta' > 0$ such that

$$d(g,h) < \delta' \qquad \Longrightarrow \qquad g^{-1}h \in U. \tag{11}$$

Now let $g, h \in \mathrm{G}$ such that $d(g,h) < \delta'$. Then, by (11), $g^{-1}h \in U$, hence $(ag)^{-1}(ah) = g^{-1}h \in U$ for every $a \in \mathrm{G}$, hence, by (10), $d(ag, ah) < \delta$, and hence it follows from (9) that $|f(ag) - f(ah)| < \varepsilon$ for every $a \in \mathrm{G}$. This implies $|(L_A f)(g) - (L_A f)(h)| < \varepsilon$ for every $A \in \mathcal{A}$. Thus we have proved equicontinuity.

Observation 3: *For every $f \in \mathcal{C}(\mathrm{G})$, $\inf_{A \in \mathcal{A}} \mathrm{Osc}(L_A f) = 0$.*

Choose a sequence $A_\nu \in \mathcal{A}$ such that

$$\lim_{\nu \to \infty} \mathrm{Osc}(L_{A_\nu} f) = \inf_{A \in \mathcal{A}} \mathrm{Osc}(L_A f).$$

By Observation 2 and the Arzéla-Ascoli theorem, the sequence $f_\nu := L_{A_\nu} f$ has a uniformly convergent subsequence (still denoted by f_ν). Let f_0 denote the limit of this subsequence. Then

$$\mathrm{Osc}(f_0) = \inf_{A \in \mathcal{A}} \mathrm{Osc}(L_A f). \tag{12}$$

Now, for every $B \in \mathcal{A}$,

$$\mathrm{Osc}(L_B f_0) = \lim_{\nu\to\infty} \mathrm{Osc}(L_B L_{A_\nu} f) = \lim_{\nu\to\infty} \mathrm{Osc}(L_{A_\nu \cdot B} f) \geq \mathrm{Osc}(f_0).$$

The penultimate equality follows from (7) and the last inequality from (12). By Observation 1, f_0 is constant. Hence $\mathrm{Osc}(f_0) = 0$ and so Observation 3 follows from (12).

Observation 3 shows that there is a sequence $A_\nu \in \mathcal{A}$ such that $L_{A_\nu} f$ converges uniformly to a constant $p \in \mathbb{R}$ (called a **left mean of** f). Similarly, there exists a sequence $B_\nu \in \mathcal{A}$ such that $R_{B_\nu} f$ converges uniformly to a constant $q \in \mathbb{R}$ (called a **right mean of** f). Since

$$\|L_A R_B f - R_B f\| \leq \mathrm{Osc}(R_B f), \qquad \|R_B L_A f - L_A f\| \leq \mathrm{Osc}(L_A f) \tag{13}$$

for $f \in \mathcal{C}(\mathrm{G})$ and $A, B \in \mathcal{A}$ it follows that the right and left means agree and hence are independent of the choices of the sequences A_ν and B_ν. Namely,

$$p = \lim_{\nu\to\infty} R_{B_\nu} L_{A_\nu} f = \lim_{\nu\to\infty} L_{A_\nu} R_{B_\nu} f = q.$$

Let us define the operator $M : \mathcal{C}(\mathrm{G}) \to \mathbb{R}$ by

$$M(f) := \lim_{\nu\to\infty} L_{A_\nu} f = \lim_{\nu\to\infty} R_{B_\nu} f,$$

where $A_\nu, B_\nu \in \mathcal{A}$ are chosen such that $\mathrm{Osc}(L_{A_\nu} f)$ and $\mathrm{Osc}(R_{B_\nu} f)$ converge to zero. Thus $M(f)$ is the left mean and the right mean of f. It is immediate from this definition that $M(1) = 1$, $M(\lambda f) = \lambda M(f)$ for $\lambda \in \mathbb{R}$, that M is left and right invariant, and

$$\min f \leq M(f) \leq \max f.$$

Now let $f, f' \in \mathcal{C}(M)$ and choose sequences $A_\nu, B_\nu \in \mathcal{A}$ such that

$$M(f) = \lim_{\nu\to\infty} L_{A_\nu} f, \qquad M(f') = \lim_{\nu\to\infty} R_{B_\nu} f'.$$

Since M is left and right invariant, we have $M(L_{A_\nu} R_{B_\nu}(f+f')) = M(f+f')$. Hence there is a sequence $C_\nu \in \mathcal{A}$ such that

$$M(f+f') = \lim_{\nu\to\infty} L_{C_\nu} R_{B_\nu} L_{A_\nu}(f+f').$$

By (13), the right hand side also converges to $M(f)+M(f')$ and hence

$$M(f+f') = M(f) + M(f')$$

for $f, f' \in \mathcal{C}(\mathrm{G})$. Thus we have proved that M is a nonnegative bounded linear functional that satisfies the assertions (i), (ii), and (iii) of the theorem.

We prove that M satisfies (iv). Hence let $f \in \mathcal{C}(\mathrm{G})$ be a function such that $f \geq 0$ and $f \not\equiv 0$. Then, by Observation 1, there exists an $A \in \mathcal{A}$ such that

$$\min L_A f > 0.$$

Choose $B_\nu \in \mathcal{A}$ such that $R_{B_\nu} f$ converges to $M(f)$. Then

$$M(f) = \lim_{\nu\to\infty} L_A R_{B_\nu} f = \lim_{\nu\to\infty} R_{B_\nu} L_A f \geq \min L_A f > 0$$

as claimed.

Next we prove that M is uniquely determined by conditions (i) and (ii). To see this, let M' be another bounded linear functional on $\mathcal{C}(\mathrm{G})$ that satisfies (i) and (ii). Then $M'(c) = c$ for every constant c and

$$M'(L_A f) = M'(f)$$

for every $f \in \mathcal{C}(\mathrm{G})$ and every $A \in \mathcal{A}$. Given $f \in \mathcal{C}(\mathrm{G})$ choose a sequence $A_\nu \in \mathcal{A}$ such that $L_{A_\nu} f$ converges uniformly to $M(f)$. Then

$$M(f) = M'(M(f)) = \lim_{\nu\to\infty} M'(L_{A_\nu} f) = M'(f).$$

This proves uniqueness. That M satisfies condition (v) follows from uniqueness and the fact that the map $\mathcal{C}(\mathrm{G}) \to \mathbb{R} : f \mapsto M(f \circ \phi)$ is a bounded linear functional that satisfies (i) and (ii). This proves the theorem. □

4 Invariant inner products

An inner product $\langle .,. \rangle$ on $\mathfrak{g}$ is called **invariant** if it is invariant under the adjoint action of G, i.e.

$$\langle g^{-1}\xi g, g^{-1}\eta g \rangle = \langle \xi, \eta \rangle$$

for $\xi, \eta \in \mathfrak{g}$ and $g \in \mathrm{G}$. A Riemannian metric on G is called **bi-invariant** if the left and right translations L_h and R_h are isometries for every $h \in \mathrm{G}$. Every invariant inner product on $\mathfrak{g}$ determines a bi-invariant metric on G via

$$\langle v, w \rangle := \langle g^{-1}v, g^{-1}w \rangle = \langle vg^{-1}, wg^{-1} \rangle \tag{14}$$

for $v, w \in T_g\mathrm{G}$. In turn, such a metric determines a volume form and hence a bi-invariant measure on G. By Theorem 3.1 this agrees with the Haar measure up to a constant factor. Conversely, if G is compact, one can use the existence of a translation invariant measure to prove the existence of an invariant inner product.

Proposition 4.1. *Let* G *be a compact Lie group. Then* $\mathfrak{g}$ *carries an invariant inner product.*

Proof. Let $M : \mathcal{C}(\mathrm{G}) \to \mathbb{R}$ denote the Haar measure and $Q : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ be any inner product. For $\xi, \eta \in \mathfrak{g}$ define $f_{\xi,\eta} : \mathrm{G} \to \mathbb{R}$ by $f_{\xi,\eta}(g) := Q(g\xi g^{-1}, g\eta g^{-1})$. Then the formula $\langle \xi, \eta\rangle := M(f_{\xi,\eta})$ defines an inner product on $\mathfrak{g}$. That it is invariant follows from the formula $f_{h\xi h^{-1},h\eta h^{-1}} = f_{\xi,\eta} \circ R_h$. □

Remark 4.2. **(i)** *The proof of Proposition 4.1 shows that the existence of a right invariant measure on* G *suffices to establish the existence of an invariant inner product on* $\mathfrak{g}$*, and hence the existence of a bi-invariant measure on* G*.*
(ii) *On any Lie group the existence of a right invariant measure is easy to prove. Choose any inner product on* $\mathfrak{g}$ *and extend it to a Riemannian metric on* G *by left translation. Then the right translations are isometries and hence the volume form defines a right invariant measure on* G*.*
(iii) *Combining (i) and (ii) gives rise to a simpler proof of the existence of a Haar measure for compact Lie groups.*
(iv) *Uniqueness in Theorem 3.1 implies that every left invariant measure is right invariant. Here is a direct proof for compact Lie Groups: If* ω *is a left invariant volume form on* G *then so is* ${R_g}^*\omega$*. Hence there exists a group homomorphism* $\lambda : \mathrm{G} \to \mathbb{R}$ *such that* ${R_g}^*\omega = e^{\lambda(g)}\omega$*. Since* G *is compact, the only group homomorphism from* G *to* $\mathbb{R}$ *is* $\lambda = 0$*.*

Lemma 4.3. *Let* G *be a compact Lie group with a bi-invariant Riemannian metric. Then the geodesics have the form* $\gamma(t) = \exp(t\xi)g$ *for* $g \in \mathrm{G}$ *and* $\xi \in \mathfrak{g}$*.*

Proof. Let $I = [a, b] \subset \mathbb{R}$ be a closed interval and $\gamma_0 : I \to \mathrm{G}$ be a geodesic. Let $\xi : I \to \mathfrak{g}$ be a smooth function such that $\xi(a) = \xi(b) = 0$ and consider the function

$$\gamma(s, t) = \gamma_0(t) \exp(s\xi(t)).$$

Then $\gamma^{-1}\partial_s\gamma = \xi$ and hence

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\int_a^b \langle \gamma^{-1}\partial_t\gamma, \gamma^{-1}\partial_t\gamma\rangle\, dt &= \int_a^b \langle \partial_s(\gamma^{-1}\partial_t\gamma), \gamma^{-1}\partial_t\gamma\rangle\, dt \\ &= \int_a^b \langle \partial_t(\gamma^{-1}\partial_s\gamma), \gamma^{-1}\partial_t\gamma\rangle\, dt \\ &= -\int_a^b \langle \xi, \partial_t(\gamma^{-1}\partial_t\gamma)\rangle\, dt.\end{aligned}$$

Here the penultimate equality follows from Lemma 1.5 and Exercise 4.5. Since the left hand side vanishes for $s = 0$ and every ξ it follows that $\partial_t({\gamma_0}^{-1}\partial_t\gamma_0) \equiv 0$. This proves the lemma. □

Exercise 4.4. **(i)** Prove that the group $\mathrm{GL}^+(n,\mathbb{R})$ of real $n \times n$-matrices with positive determinant is connected.

(ii) Prove that the exponential map $\exp : \mathbb{R}^{n\times n} \to \mathrm{GL}^+(n,\mathbb{R})$ is not surjective. (**Hint:** Every negative eigenvalue of an exponential matrix $\Phi = \exp(A)$ must have even multiplicity.)

(iii) Prove that Φ^2 is an exponential matrix for every $\Phi \in \mathrm{GL}(n,\mathbb{R})$.

(iv) Prove that for every compact connected Lie group G the exponential map $\exp : \mathfrak{g} \to \mathrm{G}$ is onto. (**Hint:** Use Proposition 4.1 (existence of an invariant inner product), Lemma 4.3 (geodesics and exponential map), and the Hopf-Rinow theorem (the existance of minimal geodesics).

Exercise 4.5. *Let* G *be a compact connected Lie group. Prove that an inner product* $\langle\cdot,\cdot\rangle$ *on* $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$ *is invariant if and only if*

$$\langle [\xi,\eta],\zeta\rangle = \langle \xi, [\eta,\zeta]\rangle$$

for $\xi,\eta,\zeta \in \mathfrak{g}$.

5 Maximal toral subgroups

Let G be a compact connected Lie group. A **Lie subgroup** of G is a closed subgroup H which is a submanifold. A linear subspace $\mathfrak{h} \subset \mathfrak{g}$ is called a **Lie subalgebra** if it is invariant under the Lie bracket. If $\mathrm{H} \subset \mathrm{G}$ is a Lie subgroup then, by definition of the Lie bracket, $\mathfrak{h} = T_{\mathbb{1}}\mathrm{H}$ is a Lie subalgebra of $\mathfrak{g}$. A **maximal torus** in G is a connected abelian subgroup $T \subset \mathrm{G}$ which is not properly contained in any other connected abelian subgroup. The fundamental example is the subgroup of diagonal matrices in $\mathrm{U}(n)$ or $\mathrm{SU}(n)$.

Exercise 5.1. Let $T \subset \mathrm{G}$ be a maximal torus with Lie algebra $\mathfrak{t} = \mathrm{Lie}(T)$. Let $\eta \in \mathfrak{g}$ such that $[\eta, \tau] = 0$ for every $\tau \in \mathfrak{t}$. Prove that $\eta \in \mathfrak{t}$.

Lemma 5.2. *Let* G *be a compact connected Lie group and* $T \subset \mathrm{G}$ *be a maximal torus. Then every element in* G *is conjugate to an element in* T.

Proof. Given $h \in \mathrm{G}$ choose $\xi \in \mathfrak{g}$ with $\exp(\xi) = h$. Such an element exists by Exercise 4.4 (iv). Then, by Exercise 1.6, $ghg^{-1} = \exp(g\xi g^{-1})$ for every $g \in \mathrm{G}$. Hence we must find $g \in \mathrm{G}$ such that $g\xi g^{-1} \in \mathrm{Lie}(T) = \mathfrak{t}$. Choose an invariant inner product on $\mathfrak{g}$ and fix a generator $\tau \in \mathfrak{t}$ such that $\{\exp(s\tau) \mid s \in \mathbb{R}\}$ is dense in T. Since the orbit of ξ under the adjoint action of G is compact there is an $\eta \in \mathfrak{g}$, conjugate to ξ, which minimizes the distance to τ in this conjugacy class, i.e.

$$|\eta - \tau|^2 = \inf_{g \in \mathrm{G}} |g\eta g^{-1} - \tau|^2.$$

We must prove that $\eta \in \mathfrak{t}$. To see this differentiate the map

$$\mathrm{G} \to \mathbb{R} : g \mapsto |g\eta g^{-1} - \tau|^2$$

at $g = \mathbb{1}$ to obtain $\langle \eta - \tau, [\zeta, \eta] \rangle = 0$ for all $\zeta \in \mathfrak{g}$. This implies $\langle \zeta, [\eta, \tau] \rangle = 0$ for all $\zeta \in \mathfrak{g}$ and hence $[\eta, \tau] = 0$. By Exercise 1.7, $\exp(t\eta)$ commutes with $\exp(s\tau)$ for all s and t. Since τ generates the torus, it follows that $\exp(t\eta)$ commutes with T for every t and hence $[\eta, \mathfrak{t}] = 0$. By Exercise 5.1, this implies $\eta \in \mathfrak{t}$. □

Lemma 5.3. *Any two maximal tori in* G *are conjugate.*

Proof. Let $T_1, T_2 \subset \mathrm{G}$ be two maximal tori and choose an element $g_2 \in T_2$ such that $T_2 = \mathrm{cl}\left(\{{g_2}^k \mid k \in \mathbb{Z}\}\right)$. By Lemma 5.2, there exists a $g \in \mathrm{G}$ such that $g_2 \in gT_1g^{-1}$. Hence $T_2 \subset gT_1g^{-1}$, and hence $T_2 = gT_1g^{-1}$. □

Lemma 5.3 shows that any two maximal tori in G have the same dimension. This dimension is called the **rank** of G. The rank of G agrees with the dimension of a maximal abelian Lie subalgebra of $\mathfrak{g}$. (Prove this!)

Lemma 5.4. *Let* G *be a compact connected Lie group and* $T \subset \mathrm{G}$ *be a maximal torus. Then* T *is a maximal abelian subgroup of* G.

Proof. We follow the argument of Frank Adams in *Lectures on Lie groups.* Let $h \in \mathrm{G}$ be an element that commutes with T. We shall prove that $h \in T$. To see this let $S \subset \mathrm{G}$ be a maximal torus containing h and denote by

$$H := \mathrm{cl}\left(\{h^k \mid k \in \mathbb{Z}\}\right)$$

the subgroup of G generated by h. Examining closed subgroups of tori we see that H is a Lie subgroup of S. Moreover, the Lie algebra $\mathfrak{h} = \mathrm{Lie}(H)$ commutes with $\mathfrak{t} = \mathrm{Lie}(T)$ and hence must be contained in $\mathfrak{t}$. Hence the identity component of H is equal to $H \cap T$ and the quotient $H/H \cap T$ is a finite group. This finite group is generated by a single element $[h] \in \mathrm{G}/T \cap H$ and hence is isomorphic to $\mathbb{Z}_m$ for some integer m. This implies $h^m \in T$. Hence the set

$$\widehat{T} := \{h^i t \mid t \in T,\, 1 \le i \le m-1\}$$

is a Lie subgroup of G such that

$$\widehat{T}/T \cong \mathbb{Z}_m.$$

Any such group is generated by a single element $\widehat{h}$. To see this, let $\phi : \mathbb{R}^n/\mathbb{Z}^n \to T$ be an isomorphism, choose a vector $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n$ such that the numbers $1, \omega_1, \ldots, \omega_n$ are rationally independent. Choose $\tau = (\tau_1, \ldots, \tau_n) \in \mathbb{R}^n$ such that $\phi(\tau) = h^m$. Then the element

$$\widehat{h} := h\phi((\omega - \tau)/m) \in \mathrm{G}$$

generates $\widehat{T}$. By Lemma 5.2, there exists a maximal torus containing $\widehat{h}$ and hence both h and T. Since T is a maximal torus it follows that $h \in T$. □

Example 5.5. In general, a maximal abelian subgroup need not be a torus. For example the $n \times n$-matrices with diagonal entries ± 1 and determinant 1 form a maximal abelian subgroup of $\mathrm{G} = \mathrm{SO}(n)$.

For every maximal torus $T \subset \mathrm{G}$ denote

$$\mathrm{G}_T := \left\{ g \in \mathrm{G} \mid g^{-1} T g = T \right\}.$$

The quotient $\mathrm{W} := \mathrm{G}_T/T$ is called the **Weyl group** of T. The next lemma shows that every adjoint orbit in $\mathfrak{t}$ intersects $\mathfrak{t}/\mathrm{W}$ in precisely one point.

Lemma 5.6. *Let* G *be a compact connected Lie group and* $T \subset \mathrm{G}$ *be a maximal torus. Let* $\xi, \eta \in \mathfrak{t}$. *Then the following are equivalent.*

(i) *There exists a* $g \in \mathrm{G}$ *such that* $g^{-1}\xi g = \eta$.

(ii) *There exists a* $g \in \mathrm{G}_T$ *such that* $g^{-1}\xi g = \eta$.

Proof. That (ii) implies (i) is obvious. Hence suppose that ${g_0}^{-1}\xi g_0 = \eta$ for some $g_0 \in \mathrm{G}$. Choose sequences $\xi_\nu, \eta_\nu \in \mathfrak{t}$ such that $\xi_\nu \to \xi$, $\eta_\nu \to \eta$, and

$$\mathrm{cl}(\{\exp(s\xi_\nu) \,|\, s \in \mathbb{R}\}) = \mathrm{cl}(\{\exp(t\eta_\nu) \,|\, t \in \mathbb{R}\}) = T$$

for every ν. Choose $g_\nu \in \mathrm{G}$ such that

$$\left|{g_\nu}^{-1}\xi_\nu g_\nu - \eta_\nu\right| = \inf_{g\in\mathrm{G}} \left|g^{-1}\xi_\nu g - \eta_\nu\right|. \tag{15}$$

Since $|{g_0}^{-1}\xi_\nu g_0 - \eta_\nu|$ converges to zero it follows that

$$\lim_{\nu\to\infty} \left|{g_\nu}^{-1}\xi_\nu g_\nu - \eta_\nu\right| = 0. \tag{16}$$

Now differentiate the function $g \mapsto \left|g^{-1}\xi_\nu g - \eta_\nu\right|^2$ at $g = g_\nu$. Then, by (15), we obtain

$$\begin{aligned} 0 &= \frac{1}{2}\frac{d}{dt}\Big|_{t=0} \left|\exp(-t\zeta){g_\nu}^{-1}\xi_\nu g_\nu \exp(t\zeta) - \eta_\nu\right|^2 \\ &= \langle {g_\nu}^{-1}\xi_\nu g_\nu - \eta_\nu, [{g_\nu}^{-1}\xi_\nu g_\nu, \zeta]\rangle \\ &= \langle [{g_\nu}^{-1}\xi_\nu g_\nu, \eta_\nu], \zeta\rangle \end{aligned}$$

for every $\zeta \in \mathfrak{g}$. Hence $[{g_\nu}^{-1}\xi_\nu g_\nu, \eta_\nu] = 0$. Since η_ν generates the torus T this implies ${g_\nu}^{-1}\xi_\nu g_\nu \in \mathfrak{t}$. Since ξ_ν generates the torus, this implies $g_\nu \in \mathrm{G}_T$. Passing to a convergent subsequence, we may assume that g_ν converges to some element $g \in \mathrm{G}_T$. By (16), we have

$$g^{-1}\xi g - \eta = \lim_{\nu\to\infty} \left({g_\nu}^{-1}\xi_\nu g_\nu - \eta_\nu\right) = 0$$

and this proves the lemma. □

6 The center

Let G be a connected Lie group. The subgroup

$$Z(\mathrm{G}) = \{g \in \mathrm{G} \,|\, gh = hg \,\forall\, h \in \mathrm{G}\}$$

is called the **center** of G It is a Lie subgroup with corresponding Lie subalgebra

$$Z(\mathfrak{g}) = \{\xi \in \mathfrak{g} \,|\, [\xi, \eta] = 0 \,\forall\, \eta \in \mathfrak{g}\}.$$

Note that $Z(\mathrm{G})$ is a normal subgroup and the center of the quotient $\mathrm{G}/Z(\mathrm{G})$ is trivial. The following theorem is due to Herman Weyl.

Theorem 6.1. *Let* G *be a compact connected Lie group. Then the first Betti number of* G *is given by* $\dim H^1(\mathrm{G};\mathbb{R}) = \dim Z(\mathfrak{g})$.

Proof. The proof consists of three steps.

Step 1: *Suppose* G *is equipped with a bi-invariant Riemannian metric. Then*

$$\nabla_v X(g) = \left(d\xi(g)v + \frac{1}{2}[\xi(g), \eta] \right) g,$$

where $\xi : \mathrm{G} \to \mathfrak{g}$, $\eta \in \mathfrak{g}$, $v = \eta g \in T_g\mathrm{G}$, *and* $X(g) = \xi(g)g$.

Suppose first that $\xi(g) \equiv \xi$ is constant. Then, by Lemma 4.3, the integral curves of X_ξ are geodesics. Hence $\nabla_{X_\xi} X_\xi = 0$ for every $\xi \in \mathfrak{g}$. Replace ξ by $\xi + \eta$ to obtain

$$\nabla_{X_\eta} X_\xi + \nabla_{X_\xi} X_\eta = 0$$

for all $\xi, \eta \in \mathfrak{g}$. Since $\nabla_{X_\eta} X_\xi - \nabla_{X_\xi} X_\eta = [X_\xi, X_\eta] = X_{[\xi,\eta]}$ it follows that

$$\nabla_{X_\eta} X_\xi = \frac{1}{2} X_{[\xi,\eta]}.$$

This proves Step 1 in the case where $\xi : \mathrm{G} \to \mathfrak{g}$ is constant. The general case is an immediate consequence.

Step 2: *The Riemann curvature tensor of* G *is given by*

$$R(\xi g, \eta g)\zeta g = -\frac{1}{4}[[\xi, \eta], \zeta]g$$

for $g \in \mathrm{G}$ *and* $\xi, \eta, \zeta \in \mathfrak{g}$.

Consider the right invariant vector fields $X_\xi(g) = \xi g$ for $\xi \in \mathfrak{g}$. By Step 1,

$$\nabla_{X_\eta} X_\xi = \frac{1}{2} X_{[\xi,\eta]}.$$

Hence Step 2 follows by straight forward calculation from the identity

$$R(X_\xi, X_\eta)X_\zeta = \nabla_{X_\xi}\nabla_{X_\eta}X_\zeta - \nabla_{X_\eta}\nabla_{X_\xi}X_\zeta + \nabla_{[X_\xi, X_\eta]}X_\zeta.$$

Step 3: *We prove the theorem.*

Let $e_1, \dots, e_k$ be an orthonormal basis of $\mathfrak{g}$. Then, by Step 2, the Ricci tensor of G is given by

$$\mathrm{Ric}(\xi g, \eta g) = \sum_{i=1}^{k} \langle R(e_i g, \xi g)\eta g, e_i g\rangle = \frac{1}{4}\sum_{i=1}^{k} \langle [\xi, e_i], [\eta, e_i]\rangle \tag{17}$$

Hence $\mathrm{Ric}(\xi g, \xi g) \geq 0$ with equality iff $\xi \in Z(\mathfrak{g})$. Now let $\alpha \in \Omega^1(\mathrm{G})$ and choose $\xi : \mathrm{G} \to \mathfrak{g}$ such that

$$\alpha_g(\eta g) = \langle \xi(g), \eta\rangle.$$

The Bochner-Weitzenböck formula asserts that

$$\|d\alpha\|_{L^2}^2 + \|d^*\alpha\|_{L^2}^2 = \|\nabla\alpha\|_{L^2}^2 + \int_{\mathrm{G}} \mathrm{Ric}(\alpha, \alpha)\mathrm{dvol}. \tag{18}$$

Since $\mathrm{Ric}(\alpha, \alpha) \geq 0$, this shows that α is harmonic if and only if $\nabla\alpha \equiv 0$ and $\mathrm{Ric}(\alpha, \alpha) \equiv 0$. By (17) and Step 1 this means that

$$d\xi(g)\eta g = \frac{1}{2}[\eta, \xi(g)] = 0$$

for every $g \in \mathrm{G}$ and every $\eta \in \mathfrak{g}$. Equivalently, $\xi : \mathrm{G} \to \mathfrak{g}$ is constant and takes values in the center of $\mathfrak{g}$. Thus we have proved that the space of harmonic 1-forms can be identified with $Z(\mathfrak{g})$. This proves the theorem. □

Theorem 6.2. *Let* G *be a compact Lie group. Then the following holds.*

(i) *The fundamental group of* G *is abelian.*

(ii) *If* $Z(\mathrm{G})$ *is finite then so is* $\pi_1(\mathrm{G})$.

Proof. Assertion (i) holds for every topological group. To see this let $\alpha, \beta : [0, 1] \to \mathrm{G}$ be loops with $\alpha(0) = \alpha(1) = \beta(0) = \beta(1) = \mathbb{1}$. Denote

$$\alpha\#\beta(t) := \begin{cases} \alpha(2t), & \text{if } 0 \leq t \leq 1/2, \\ \alpha(1)\beta(2t-1), & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

(Here the term $\alpha(1)$ can be dropped, but the more general form will be needed below.) Moreover, define

$$\alpha_s(t) := \begin{cases} \alpha(2t - s), & \text{if } s/2 \leq t \leq (s+1)/2, \\ \mathbb{1}, & \text{otherwise,} \end{cases}$$

and

$$\beta_s(t) := \begin{cases} \beta(2t+s-1), & \text{if } (1-s)/2 \le t \le 1-s/2, \\ \mathbb{1}, & \text{otherwise}, \end{cases}$$

for $0 \le s,t \le 1$. Then $\gamma_s(t) = \alpha_s(t)\beta_s(t)$ is a homotopy from $\gamma_0 = \alpha\#\beta$ to $\gamma_1 = \beta\#\alpha$. This proves (i). To prove (ii) note that, by (i), the fundamental group

$$\Gamma := \pi_1(\mathrm{G}),$$

is abelian and, Theorem 6.1,

$$\mathrm{Hom}(\Gamma,\mathbb{R}) \cong H^1(\mathrm{G};\mathbb{R}) = 0.$$

This implies that Γ is finite. To see this, note first that Γ is finitely generated. Let $\gamma_1,\dots,\gamma_n \in \Gamma$ be generators. Since Γ is abelian, the set $R \subset \mathbb{Z}^n$ of all integer vectors $m = (m_1,\dots,m_n)$ that satisfy

$${\gamma_1}^{m_1}\cdots{\gamma_n}^{m_n} = 1$$

form a subgroup of $\mathbb{Z}^n$ and there is a natural isomorphism

$$\Gamma \cong \mathbb{Z}^n/R.$$

Since $\mathrm{Hom}(\Gamma,\mathbb{R}) = R^\perp = \{0\}$ it follows that R spans $\mathbb{R}^n$. Hence the quotient $\mathbb{R}^n/R$ is compact, and hence $\Gamma \cong \mathbb{Z}^n/R$ is a finite set. □

Now let us denote by $\widetilde{\mathrm{G}}$ the universal cover of G. In explicit terms,

$$\widetilde{\mathrm{G}} = \{\gamma : [0,1] \to \mathrm{G} \,|\, \gamma(0) = \mathbb{1}\} \,/\sim$$

where $\sim$ denotes homotopy with fixed endpoints. The projection

$$\pi : \widetilde{\mathrm{G}} \to \mathrm{G}$$

is given by $\pi([\gamma]) = \gamma(1)$.

Proposition 6.3. *Let* G *be a connected Lie group. Then*

$$Z(\widetilde{\mathrm{G}}) = \pi^{-1}(Z(\mathrm{G})).$$

Proof. Let $\alpha, \beta : [0,1] \to \mathrm{G}$ such that $\alpha(0) = \beta(0) = 1\!\!1$ and $\alpha(1) \in Z(\mathrm{G})$. Define

$$\alpha_s(t) := \begin{cases} \alpha((1+s)t), & \text{if } 0 \le t \le 1/(s+1), \\ \alpha(1), & \text{otherwise,} \end{cases}$$

and

$$\beta_s(t) := \begin{cases} \beta((2t-s)/(2-s)), & \text{if } s/2 \le t \le 1, \\ 1\!\!1, & \text{otherwise,} \end{cases}$$

for $0 \le s, t \le 1$. Since $\alpha(1) \in Z(\mathrm{G})$, we have

$$\alpha_1\beta_1 = \beta_1\alpha_1 = \alpha\#\beta.$$

Moreover $\alpha_0 = \alpha$ and $\beta_0 = \beta$. Hence both $\alpha\beta$ and $\beta\alpha$ are homotopic to $\alpha\#\beta$. This proves that $\pi^{-1}(Z(\mathrm{G})) \subset Z(\widetilde{\mathrm{G}})$. The converse inclusion is obvious. □

The **commutator subgroup** $[\mathrm{G}, \mathrm{G}] \subset \mathrm{G}$ is defined as the smallest subgroup of G that contains all commutators $[a, b] := aba^{-1}b^{-1}$ for $a, b \in \mathrm{G}$. Thus $[\mathrm{G}, \mathrm{G}]$ is the subset of all products of finitely many such commutators. It is a normal subgroup of G.

Proposition 6.4. *Let* G *be a compact connected Lie group. Then* $Z(\mathrm{G})$ *is finite if and only if* $[\mathrm{G}, \mathrm{G}] = \mathrm{G}$.

Proof. Consider the subbundle

$$E := \{(g, \xi g) \,|\, \xi \perp Z(\mathfrak{g})\} \subset T\mathrm{G}.$$

By Exercise 4.5, the Lie bracket of any two right invariant vector fields $X_\xi(g) = \xi g$ and $X_\eta(g) = \eta g$ is contained in E. Hence, by Frobenius' theorem, E is integrable. Let H be the leaf of E through $1\!\!1$, i.e.

$$H := \left\{\gamma(1) \,|\, \gamma : [0,1] \to \mathrm{G},\ \gamma(0) = 1\!\!1,\ \gamma(t)^{-1}\dot\gamma(t) \perp Z(\mathfrak{g})\right\}.$$

If $\alpha, \beta : [0,1] \to \mathrm{G}$ are paths that are tangent to E then so are $\alpha\beta$ and α^{-1}. Hence H is a subgroup of G. Next we prove that

$$[\mathrm{G}, \mathrm{G}] \subset H.$$

To see this note that, for every pair $\xi, \eta \in \mathfrak{g}$ the curve

$$\gamma(t) := \exp(t\xi)\exp(t\eta)\exp(-t\xi)\exp(-t\eta)$$

is tangent to E. Since the exponential map is surjective it follows that every commutator $[a, b] = aba^{-1}b^{-1}$ of two elements in G lies in H. Hence $[\mathrm{G}, \mathrm{G}] \subset H$. Next we prove that

$$H \subset [\mathrm{G}, \mathrm{G}]$$

To see this note that, by Exercise 4.5, the orthogonal complement of $Z(\mathfrak{g})$ is spanned by vectors of the form $[\xi, \eta]$ for $\xi, \eta \in \mathfrak{g}$. Choose $\xi_1, \ldots, \xi_k, \eta_1, \ldots, \eta_k$ such that the vectors $[\xi_i, \eta_i]$ form a basis of $Z(\mathfrak{g})^\perp$. For $i = 1, \ldots, k$ define $\gamma_i : \mathbb{R} \to [\mathrm{G}, \mathrm{G}]$ by

$$\gamma_i(t) := \exp(\sqrt{t}\xi_i)\exp(\sqrt{t}\eta_i)\exp(-\sqrt{t}\xi_i)\exp(-\sqrt{t}\eta_i)$$

for $t \geq 0$ and $\gamma_i(t) := \gamma_i(-t)$ for $t < 0$. Then γ_i is continuously differentiable and $\dot{\gamma}_i(0) = [\xi_i, \eta_i]$. Hence the function $\phi : \mathbb{R}^k \to [\mathrm{G}, \mathrm{G}]$, defined by

$$\phi(t_1, \ldots, t_k) := \gamma_1(t_1) \cdots \gamma_k(t_k),$$

is a continuously differentiable embedding near $t = 0$ and it is everywhere tangent to E. Hence the image U_0 of a sufficiently small neighbourhood of $0 \in \mathbb{R}^k$ under ϕ is a neighbourhood of $\mathbb{1}$ in H with respect to the intrinsic topology of H and it is contained in $[\mathrm{G}, \mathrm{G}]$. More generally, for every $h \in H$ the set $U = U_0 h \subset H$ is a neighbourhood of h with respect to the intrinsic topology and $Uh^{-1} \subset [\mathrm{G}, \mathrm{G}]$. Hence the sets $H \cap [\mathrm{G}, \mathrm{G}]$ and $H \setminus [\mathrm{G}, \mathrm{G}]$ are both open with respect to the intrinsic topology of H. Since $H \cap [\mathrm{G}, \mathrm{G}] \neq \emptyset$ it follows that $H \subset [\mathrm{G}, \mathrm{G}]$, as claimed. Thus we have proved that $[\mathrm{G}, \mathrm{G}] = H$ is a leaf of the foliation determined by E. Hence $[\mathrm{G}, \mathrm{G}] = \mathrm{G}$ if and only if $E = T\mathrm{G}$ if and only if $Z(\mathrm{G})$ is finite. This proves the proposition. □

Corollary 6.5. *Let* G *be a compact connected Lie group with finite center. Then every principal* G*-bundle* $P \to \Sigma$ *over a compact oriented Riemann surface of sufficiently large genus carries a flat connection.*

Proof. By Theorem 6.2, $\pi_1(\mathrm{G})$ is finite, and hence $\widetilde{\mathrm{G}}$ is compact. By Proposition 6.3, we have

$$\pi_1(\mathrm{G}) = \pi^{-1}(\mathbb{1}) \subset Z(\widetilde{\mathrm{G}}).$$

There is a one-to-one correspondence between isomorphism classes of principal G-bundles over a Riemann surface and elements $\gamma \in \pi_1(\mathrm{G})$. Suppose that Σ is a Riemann surface of genus g and let $P_\gamma \to \Sigma$ be the principal bundle corresponding to $\gamma \in \pi_1(\mathrm{G})$. Then a gauge equivalence class of flat connection on P_γ (with respect to the identity component of the gauge group) can be represented by elements

$$\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g \in \widetilde{\mathrm{G}}$$

that satisfy

$$\prod_{j=1}^{g} [\alpha_j, \beta_j] = \gamma.$$

By Proposition 6.4, every element $\gamma \in \widetilde{\mathrm{G}}$ can be expressed in this form whenever $Z(\widetilde{\mathrm{G}})$ is finite. This proves the corollary. □

7 Isotropy subgroups

Let G be a compact connected Lie group and M be a compact smooth manifold equipped with a left action of G. The action will be denoted by

$$\mathrm{G} \times M \to M : (g, x) \mapsto gx.$$

The **isotropy subgroup** of an element $x \in M$ is defined by

$$\mathrm{G}_x := \{h \in \mathrm{G} \,|\, hx = x\}.$$

Since $\mathrm{G}_{gx} = g\mathrm{G}_x g^{-1}$ the set of isotropy subgroups is invariant under conjugation. The next theorem asserts that the set of conjugacy classes of isotropy subgroups is finite.

Theorem 7.1. *There exist finitely many Lie subgroups $H_1, \ldots, H_N$ of* G *such that for every $x \in M$ there exists a j such that G_x is conjugate to H_j.*

Proof. The proof is by induction on the dimension of M. If M is zero dimensional then the result is obvious. Now assume that dim $M = n > 0$ and that the result has been proved for all manifolds of dimensions less than n. We prove that every point $x_0 \in M$ has a neighbourhood U in which only finitely many isotropy subgroups occur up to conjugacy. To see this, let $\mathrm{G}_0 := \mathrm{G}_{x_0}$ choose a G-invariant metric on M, denote by $L_x : \mathfrak{g} \to T_xM$ the infinitesimal action, and consider the horizontal space $H_0 := \ker L_{x_0}^* \subset T_{x_0}M$. Then the exponential map

$$\mathrm{G} \times H_0 \to M : (g, v_0) \mapsto g \exp_{x_0}(v_0)$$

descends to a map

$$\phi_0 : \mathrm{G} \times_{\mathrm{G}_0} H_0 \to M,$$

where $(g, v_0) \sim (gg_0, g_0^{-1}v_0)$ for $g \in \mathrm{G}$, $v_0 \in H_0$, and $g_0 \in \mathrm{G}_0$. The restriction of ϕ_0 to a sufficiently small neighbourhood of the zero section in the vector bundle $\mathrm{G} \times_{\mathrm{G}_0} H_0 \to \mathrm{G}/\mathrm{G}_0$ is a G-equivariant diffeomorphism onto a neighbourhood of the G-orbit of x_0. It follows that the isotropy groups of points $x \in M$ belonging to this neighbourhood are all conjugate to subgroups of G_0 that appear as isotropy subgroups of the action of G_0 on H_0. By considering the action of G_0 on the unit sphere in H_0 we obtain from the induction hypothesis that there are only finitely many such isotropy subgroups. This proves the local statement. Cover M by finitely many such neighbourhoods to prove the global assertion. □

8 Centralizers

Let G be a compact connected Lie group. For any subset $H \subset \mathrm{G}$ the **centralizer** of H is defined by

$$Z(H) := Z(H; \mathrm{G}) := \{g \in \mathrm{G} \,|\, gh = hg \,\forall\, h \in H\}$$

This set is a Lie subgroup of G with Lie algebra

$$\mathrm{Lie}(Z(H)) = \left\{\xi \in \mathfrak{g} \,|\, h\xi h^{-1} = \xi \,\forall\, h \in H\right\} = \bigcap_{h \in H} \ker(\mathbb{1} - \mathrm{Ad}(h)).$$

Moreover, $Z(\mathrm{G}) = Z(\mathrm{G}; \mathrm{G})$ is the center of G and $Z(Z(\mathrm{G}); \mathrm{G}) = \mathrm{G}$. A subgroup $H \subset \mathrm{G}$ is abelian iff $H \subset Z(H; \mathrm{G})$ and is maximal abelian iff $H = Z(H; \mathrm{G})$. For a singleton $H = \{h\}$ we denote $Z(h) := Z(\{h\})$.

Lemma 8.1. *Let* G *be a group. Then, for every subset* $H \subset \mathrm{G}$*,*

$$H \subset Z(Z(H)), \qquad Z(Z(Z(H)) = Z(H).$$

Proof. The first assertion follows directly from the definition and the second follows from the first. Namely, since $H \subset Z(Z(H))$ we have $Z(Z(Z(H)) \subset Z(H)$, and the converse inclusion follows by applying the first assertion to $Z(H)$ instead of H. This proves the lemma. □

A subgroup $H \subset \mathrm{G}$ is called a **centralizer subgroup** if there exists a subset of G whose centralizer is equal to H. By Lemma 8.1 this condition is equivalent to

$$H = Z(Z(H)). \tag{19}$$

A Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called a **centralizer subalgebra** if there exists a centralizer subgroup $H \subset \mathrm{G}$ such that $\mathfrak{h} = \mathrm{Lie}(H)$. Let us denote by $\mathscr{Z} \subset 2^{\mathrm{G}}$ the set of all centralizer subgroups. By (19), the map

$$\mathscr{Z} \to \mathscr{Z} : H \mapsto Z(H)$$

is an involution. Moreover the group G acts on $\mathscr{Z}$ by conjugation and the involution $H \mapsto Z(H)$ is equivariant under this action, i.e.

$$Z(gHg^{-1}) = gZ(H)g^{-1}.$$

The fixed points of the involution are the maximal abelian subgroups of G and hence are also fixed points of the conjugate action. Consider the equivalence relation $H \sim H'$ iff $H' = gHg^{-1}$ for some $g \in \mathrm{G}$. The following theorem asserts that the quotient $\mathscr{Z}/\sim$ is a finite set.

Theorem 8.2. *Let* G *be a compact connected Lie group. Then there exist finitely many centralizer subgroups $H_1, \dots, H_m$ of* G *such that every centralizer subgroup $H \subset \mathrm{G}$ is conjugate to one of the H_i.*

Proof. Since G is compact it admits a faithful representation $\rho : \mathrm{G} \to \mathrm{U}(n)$. Now let $H \subset \mathrm{G}$ be a subgroup and $g \in Z(H)$. Then $\rho(g)$ commutes with all matrices in the span of $\rho(H)$. Thus it suffices to pick n^2 elements $h_1, \dots, h_{n^2} \in H$ such that $\rho(H)$ is contained in the span of the matrices $\rho(h_i)$. Then $Z(H)$ can be characterized as the set of all $g \in \mathrm{G}$ such that $\rho(g)$ commutes with $\rho(h_i)$ for $i = 1, \dots, n^2$. In other words, if the group G acts on the vector space $V := (C^{n\times n})^{n^2}$ by $g \cdot A_i := \rho(g)A_i\rho(g)^{-1}$ for $i = 1, \dots, n^2$, then every centralizer subgroup of G is the isotropy subgroup of some element of V. By Theorem 7.1, the set of conjugacy classes of such isotropy subgroups is finite. □

9 Simple groups

A Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called an **ideal** if $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$. The Lie algebra of a normal Lie subgroup of G is necessarily an ideal. A Lie algebra $\mathfrak{g}$ is called simple if it has no nontrivial ideals (that is $\{0\}$ and $\mathfrak{g}$ are the only ideals in $\mathfrak{g}$). It is called **semi-simple** if it is a direct sum of simple Lie algebras. A Lie group is called **simple** (respectively **semi-simple**) if its Lie algebra is simple (respectively semi-simple).

Theorem 9.1. *Every compact connected simply connected simple Lie group is isomorphic to one in the following list*

$$\begin{aligned} A_n &= \mathrm{SU}(n+1), && n \geq 1, \\ B_n &= \mathrm{Spin}(2n+1), && n \geq 2, \\ C_n &= \mathrm{Sp}(n), && n \geq 3, \\ D_n &= \mathrm{Spin}(2n), && n \geq 4, \end{aligned}$$

or to one of the exceptional groups G_2, F_4, E_6, E_7, E_8.

There are relations such as $\mathrm{Spin}(3) = \mathrm{SU}(2) = \mathrm{Sp}(1)$, $\mathrm{Spin}(5) = \mathrm{Sp}(2)$, and $\mathrm{Spin}(6) = \mathrm{SU}(4)$. The groups $D_1 = \mathrm{Spin}(2) = \mathrm{U}(1) = S^1$ and $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$ are not simple. (See Exercises 10.10 and 10.11 below for Spin(3) and Spin(4).)

The Killing form

Every Lie algebra carries a natural pairing

$$\kappa(\xi, \eta) = \mathrm{trace}(\mathrm{ad}(\xi)\mathrm{ad}(\eta))$$

called the **Killing form**. On $\mathfrak{su}(n)$ this form is negative definite. In general the Killing form may have a kernel and/or be indefinite.

Theorem 9.2 (Cartan). *The Killing form is nondegenerate (and negative definite) if and only if* G *is semisimple.*

Exercise 9.3. Prove that the Killing forms on $\mathfrak{su}(n)$ and $\mathfrak{so}(2n)$ are given by

$$\kappa(\xi, \eta) = -(2n-1)\,\mathrm{trace}(\xi^*\eta), \qquad \xi, \eta \in \mathfrak{su}(n),$$

$$\kappa(\xi, \eta) = -(n-2)\,\mathrm{trace}(\xi^T\eta), \qquad \xi, \eta \in \mathfrak{so}(2n).$$

Exercise 9.4. Prove that the Killing form on $\mathrm{SL}(2, \mathbb{R})$ is indefinite.

Root systems

Let G be a compact Lie group with maximal torus T. The exponential map is onto by Exercise 4.4 (iv). It determines an isomorphism

$$T \cong \mathfrak{t}/\Lambda$$

where $\mathfrak{t} = \mathrm{Lie}(T)$ and

$$\Lambda := \{\tau \in \mathfrak{t} \,|\, \exp(\tau) = 1\!\!1\}$$

is a lattice which spans $\mathfrak{t}$. A one-dimensional complex representation is a homomorphism $T \to S^1$. Under the identification $T \cong \mathfrak{t}/\Lambda$ any such homomorphism is of the form $\tau \mapsto e^{2\pi i \mathrm{w}(\tau)}$ where $\mathrm{w} : \mathfrak{t} \to \mathbb{R}$ is a linear map with

$$\mathrm{w}(\Lambda) \subset \mathbb{Z}.$$

Any such map $\mathrm{w} \in \mathfrak{t}^*$ is called a **weight**. Now consider the adjoint representation of T on $\mathfrak{g}$. Since the action preserves any invariant inner product the commuting endomorphisms $\mathrm{Ad}(\tau)$ for $\tau \in \mathfrak{t}$ are simultaneously diagonalizable (over $\mathbb{C}$). It follows that there exists a decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha} V_\alpha$$

where $V_\alpha \subset \mathfrak{g}$ are two dimensional representations of T. In other words there exists a complex structure J_α on V_α and weights $\mathrm{w}_\alpha \in \mathfrak{t}^*$ such that

$$[\tau, \xi] = 2\pi J_\alpha \mathrm{w}_\alpha(\tau)\xi, \qquad \tau \in \mathfrak{t}, \qquad \xi \in V_\alpha.$$

The weights w_α are called the **roots** of the Lie algebra $\mathfrak{g}$. For each α define $\tau_\alpha \in \mathfrak{t}$ to be the dual element with respect to the Killing form:

$$\kappa(\tau_\alpha, \sigma) = \mathrm{w}_\alpha(\sigma), \qquad \sigma \in \mathfrak{t}.$$

The **length** of the root w_α is defined by

$$\ell(\alpha) = \sqrt{-\kappa(\tau_\alpha, \tau_\alpha)}.$$

The length of the longest root is an important invariant of the Lie group G. We denote the square of its inverse by

$$a(\mathrm{G}) = \frac{1}{\sup_\alpha \ell(\alpha)^2}.$$

Here is a list of these invariants for the simple groups.

G	dim(G)	a(G)
SU(n)	n^2-1	n
Spin(n)	$\frac{1}{2}n(n-1)$	$n-2$
Sp(n)	$n(2n+1)$	$n+1$
G_2	14	4
F_4	52	9
E_6	78	12
E_7	133	18
E_8	248	30

10 Examples

Example 10.1 (General linear group). The group $\mathrm{GL}(n,\mathbb{R})$ of invertible real $n\times n$-matrices is a Lie group. This space is an open set in $\mathbb{R}^{n\times n}$ and hence is obviously a manifold. Its Lie algebra is the vector space $\mathbb{R}^{n\times n}$ of all real $n\times n$ matrices with Lie bracket operation

$$[A,B]=AB-BA.$$

In this case the exponential map

$$\exp:\mathbb{R}^{n\times n}\to\mathrm{GL}(n,\mathbb{R})$$

is the usual exponential map for matrices and the expressions gv and vg for $v\in T_hG$ are given by matrix multiplication. The example $\mathrm{GL}(n,\mathbb{C})$ of invertible complex $n\times n$-matrices is similar. However, the group $\mathrm{GL}(n,\mathbb{C})$ is connected while the group $\mathrm{GL}(n,\mathbb{R})$ has two components distinguished by the sign of the determinant.

Example 10.2 (Special linear group). The determinant map

$$\det:\mathrm{GL}(n,\mathbb{C})\to\mathbb{C}^*$$

is a Lie group homomorphism and its kernel is a Lie group denoted by

$$\mathrm{SL}(n,\mathbb{C})=\left\{\Phi\in\mathbb{C}^{n\times n}\mid\det\,\Phi=1\right\}.$$

The formula $\det(\exp(A)) = \exp(\mathrm{trace}(A))$ shows that the Lie algebra of $\mathrm{SL}(n,\mathbb{C})$ is given by

$$\mathfrak{sl}(n,\mathbb{C}) = \left\{ A \in \mathbb{C}^{n\times n} \,|\, \mathrm{trace}\, A = 0 \right\}.$$

The Lie group $\mathrm{SL}(n,\mathbb{R})$ with Lie algebra $\mathfrak{sl}(n,\mathbb{R})$ is defined analogously.

Example 10.3 (Circle). The unit circle $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$ in the complex plane is a Lie group (under multiplication of complex numbers). Its Lie algebra is the space $i\mathbb{R}$ of imaginary numbers with zero Lie bracket. (See Exercise 1.7.) There is a Lie group isomorphism

$$\mathbb{R}/\mathbb{Z} \to S^1 : t \mapsto e^{2\pi i t}.$$

Example 10.4 (Torus). Let V be an n-dimensional real vector space and $\Lambda \subset V$ be a lattice (a discrete additive subgroup) which spans V. Then

$$T = V/\Lambda$$

is a compact abelian Lie group (the group operation is the addition in V) with Lie algebra V. The exponential map is the projection $V \to V/\Lambda$. Any such Lie group is called a **torus**. Tori can be characterized as compact connected finite dimensional abelian Lie groups. The basic example is the standard torus

$$\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$$

and every n-dimensional torus is isomorphic to $\mathbb{T}^n$.

Example 10.5 (Orthogonal group). The orthogonal $n \times n$-matrices form a Lie group

$$\mathrm{O}(n) = \left\{ \Phi \in \mathbb{R}^{n\times n} \,|\, \Phi^T\Phi = \mathbb{1} \right\}.$$

This group has two components distinguished by the determinant $\det \Phi = \pm 1$ and the component of the identity is denoted by

$$\mathrm{SO}(n) = \{\Phi \in \mathrm{O}(n) \,|\, \det\, \Phi = 1\}.$$

Its Lie algebra is the space of antisymmetric matrices

$$\mathfrak{so}(n) = \left\{ A \in \mathbb{R}^{n\times n} \,|\, A^T + A = 0 \right\}.$$

The group $\mathrm{SO}(n)$ is compact and connected and the exponential map is surjective (see Exercise 4.4).

Example 10.6 (Unitary group). The unitary $n \times n$-matrices form a Lie group

$$\mathrm{U}(n) = \left\{ U \in \mathbb{C}^{n\times n} \,|\, U^* U = \mathbb{1} \right\}$$

where U^* denotes the conjugate transpose of U. This group is connected and its Lie algebra is given by

$$\mathfrak{u}(n) = \left\{ A \in \mathbb{C}^{n\times n} \,|\, A^* + A = 0 \right\}.$$

The case $n = 1$ corresponds to the circle $S^1 = \mathrm{U}(1)$. The subgroup of unitary matrices of determinant 1 is denoted by

$$\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}(n, \mathbb{C})$$

and its Lie algebra by

$$\mathfrak{su}(n) = \{ A \in \mathfrak{u}(n) \,|\, \mathrm{trace}(A) = 0 \}.$$

Both groups $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are compact and connected.

Example 10.7 (Unit quaternions). Denote by $\mathbb{H} = \mathbb{R}^4$ the space of quaternions

$$x = x_0 + ix_1 + jx_2 + kx_3$$

with (noncommutative) multiplicative structure

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i \quad ki = -ik = j.$$

The norm of $x \in \mathbb{H}$ is defined by

$$|x|^2 = x\bar{x} = {x_0}^2 + {x_1}^2 + {x_2}^2 + {x_3}^2, \qquad \bar{x} = x_0 - ix_1 - jx_2 - kx_3,$$

and satisfies the rule $|xy| = |x| \cdot |y|$. Hence the unit quaternions form a group

$$\mathrm{Sp}(1) = \{ x \in \mathbb{H} \,|\, |x| = 1 \}$$

with unit 1 and inverse map $x \mapsto \bar{x}$. Its Lie algebra consists of the imaginary quaternions

$$\mathfrak{sp}(1) = \{ x \in \mathbb{H} \,|\, x_0 = 0 \}.$$

The exponential map is given by the usual formula $\exp(x) = \sum_{k=0}^{\infty} x^k / k!$. The quaternion multiplication defines a group structure on $S^3 = \mathrm{Sp}(1)$ and a Lie algebra structure on $\mathbb{R}^3 \simeq \mathfrak{sp}(1)$. This Lie algebra structure corresponds to the vector product.

Example 10.8. The quaternion matrices $\Phi \in \mathbb{H}^{n\times n}$ with $\Phi^*\Phi = \mathbb{1}$ form a compact connected group denoted by $\mathrm{Sp}(n)$. Its Lie algebra $\mathfrak{sp}(n)$ consists of the quaternion matrices $A \in \mathbb{H}^{n\times n}$ with $A^* + A = 0$. Here A^* denotes the conjugate transpose as in the complex case.

Exercise 10.9. **(i)** *Prove that the map* $\mathrm{Sp}(1) \to \mathrm{SU}(2) : x \mapsto U(x)$ *defined by*

$$U(x) = \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix}$$

is a Lie group isomorphism.

(ii) *Prove that the corresponding Lie algebra homomorphism* $\mathfrak{sp}(1) \to \mathfrak{su}(2) : \xi \mapsto u(\xi)$ *is given by*

$$u(\xi) = \begin{pmatrix} i\xi_1 & \xi_2 + i\xi_3 \\ -\xi_2 + i\xi_3 & -i\xi_1 \end{pmatrix}.$$

Show that the matrices

$$I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

satisfy the quaternion relations. In other words, the Lie algebra $\mathfrak{su}(2)$ *is isomorphic to the imaginary quaternions and the isomorphism is given by* $i \mapsto I$, $j \mapsto J$, $k \mapsto K$. *The natural orientation of* $\mathrm{SU}(2)$ *is determined by the basis* $\{I, J, K\}$ *of* $\mathfrak{su}(2)$.

(iii) *Prove that*

$$[u(\xi), u(\eta)] = 2u(\xi \times \eta), \qquad \mathrm{trace}(u(\xi)^* u(\eta)) = 2\langle \xi, \eta\rangle$$

for $\xi, \eta \in \mathbb{R}^3 \cong \mathrm{Im}(\mathbb{H})$.

Exercise 10.10 (Spin(3)). The unit quaternions act on the imaginary quaternions by conjugation. This determines a homomorphism $\mathrm{Sp}(1) \to \mathrm{SO}(3) : x \mapsto \Phi(x)$ defined by

$$\Phi(x)\xi = x\xi\bar{x}$$

for $x \in \mathrm{Sp}(1)$ and $\xi \in \mathrm{Im}(\mathbb{H}) \cong \mathbb{R}^3$. On the left the multiplication is understood as a product of matrix and vector and on the right as a product of quaternions.

(i) Prove that

$$\Phi(x) = \begin{pmatrix} x_0^2 + x_1^2 - x_2^2 - x_3^2 & 2(x_1x_2 - x_0x_3) & 2(x_0x_2 - x_1x_3) \\ 2(x_0x_3 - x_1x_2) & x_0^2 - x_1^2 + x_2^2 - x_3^2 & 2(x_2x_3 - x_0x_1) \\ 2(x_1x_3 - x_0x_2) & 2(x_0x_1 - x_2x_3) & x_0^2 - x_1^2 - x_2^2 + x_3^2 \end{pmatrix}.$$

(ii) Verify that the map $\mathrm{SU}(2) \to \mathrm{SO}(3) : U(x) \mapsto \Phi(x)$ is a Lie group homomorphism and a double cover. Deduce that $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2$.

(iii) Let $\mathfrak{su}(2) \to \mathfrak{so}(3) : u(\xi) \mapsto A(\xi)$ denote the corresponding Lie algebra homomorphism. Prove that

$$A(\xi) = 2\begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}.$$

Prove that $[A(\xi), A(\eta)] = 2A(\xi \times \eta)$ and $\mathrm{trace}(A(\xi)^T A(\eta)) = 8\langle \xi, \eta \rangle$.

Exercise 10.11 (Spin**(4)).** The group $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ acts on $\mathbb{H}$ by the orthogonal transformations $x \mapsto ux\bar{v}$ for $(u, v) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1)$. Prove that this axtion determines a double cover $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{SO}(4)$ and find an explicit formula for the matrix $\Psi(u, v) \in \mathbb{R}^{4\times 4}$ defined by $\Psi(u, v)x = ux\bar{v}$.

Lemma 10.12. **(i)** $\mathrm{SO}(n)$ *is connected and in the case* $n \geq 3$ *its fundamental group is isomorphic to* $\mathbb{Z}_2$. *Hence for* $n \geq 3$ *the universal cover of* $\mathrm{SO}(n)$ *is a compact group (with the same Lie algebra). It is denoted by* $\mathrm{Spin}(n)$.

(ii) $\mathrm{SU}(n)$ *is connected and simply connected and* $\pi_2(\mathrm{SU}(n)) = 0$.

(iii) *The fundamental group of* $\mathrm{U}(n)$ *is isomorphic to the integers. The determinant homomorphism* $\det : \mathrm{U}(n) \to S^1$ *induces an isomorphism of fundamental groups.*

Proof. The subgroup of all matrices $\Phi \in \mathrm{SO}(n)$ whose first column is the first unit vector $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$ is isomorphic to $\mathrm{SO}(n-1)$. Hence there is a fibration $\mathrm{SO}(n-1) \hookrightarrow \mathrm{SO}(n) \to S^{n-1}$ where the second map sends a matrix in $\mathrm{SO}(n)$ to its first column. The homotopy exact sequence of this fibration has the form

$$\pi_{k+1}(S^{n-1}) \to \pi_k(\mathrm{SO}(n-1)) \to \pi_k(\mathrm{SO}(n)) \to \pi_k(S^{n-1})$$

By Exercise 10.10, $\pi_1(\mathrm{SO}(3)) \simeq \mathbb{Z}_2$. For $n \geq 4$ this follows from the exact sequence with $k = 1$. The connectedness of $\mathrm{SO}(n)$ is obvious for $n = 1, 2$. For $n \geq 3$ it follows from the exact sequence with $k = 0$. This proves (i).

To prove (ii) consider the fibration $\mathrm{SU}(n-1) \hookrightarrow \mathrm{SU}(n) \to S^{2n-1}$ where the last map sends $U \in \mathrm{SU}(n)$ to the first column of U. The homotopy exact sequence of this fibration has the form

$$\pi_{k+1}(S^{2n-1}) \to \pi_k(\mathrm{SU}(n-1)) \to \pi_k(\mathrm{SU}(n)) \to \pi_k(S^{2n-1}).$$

For $n = 1$ the group $\mathrm{SU}(1) = \{1\}$ is obviously connected and simply connected. For $n \geq 2$ use the exact sequence inductively (over n) with $k = 0, 1$. The statement about π_2 is proved similarly with $k = 2$.

To prove (iii) consider the fibration $\mathrm{SU}(n) \hookrightarrow \mathrm{U}(n) \to S^1$. The homotopy exact sequence of this fibration has the form

$$1 = \pi_1(\mathrm{SU}(n)) \to \pi_1(\mathrm{U}(n)) \to \pi_1(S^1) \to \pi_0(\mathrm{SU}(n)) = 1.$$

In view of statement (ii) this shows that $\pi_1(\mathrm{U}(n)) \simeq \pi_1(S^1) \simeq \mathbb{Z}$. □

Let Y be a compact oriented smooth 3-manifold, and recall from Exercise 10.9 that $\mathrm{SU}(2)$ is diffeomorphic to S^3 and carries a natural orientation. Hence every smooth map $g : Y \to \mathrm{SU}(2)$ has a well defined degree. The next proposition shows that this degree can be expressed as as the integral of natural 3-form over Y.

Lemma 10.13. *For every compact oriented smooth 3-manifold Y and every smooth map $g : Y \to \mathrm{SU}(2)$ we have*

$$\int_Y \mathrm{trace}\left(g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g\right) = -24\pi^2 \deg(g).$$

Proof. Denote

$$\omega_g := \mathrm{trace}\left(g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g \wedge g^{-1}\mathrm{d}g\right) \in \Omega^3(Y).$$

If $f : Y' \to Y$ is a smooth map then $\omega_{g\circ f} = f^*\omega_g$. In particular, with $\omega_0 := \omega_{\mathrm{id}} \in \Omega^3(\mathrm{SU}(2))$, we have $\omega_g = g^*\omega_0$ and hence

$$\int_Y \omega_g = \deg(g) \int_{\mathrm{SU}(2)} \omega_0. \tag{20}$$

To compute the integral of ω_0 consider the diffeomorphism $U : S^3 \to \mathrm{SU}(2)$ defined in Exercise 10.9. With the standard orientations of S^3 and $\mathrm{SU}(2)$ this map has degree 1. Moreover, it follows from the symmetry of this map that ω_U is a constant multiple of the volume form on S^3. To find out the factor we compute the form on the tangent space T_xS^3 for $x = (1, 0, 0, 0)$. On this space

$$U^{-1}dU = Idx_1 + Jdx_2 + Kdx_3,$$

hence

$$U^{-1}dU \wedge U^{-1}dU = 2Idx_2 \wedge dx_3 + 2Jdx_3 \wedge dx_1 + 2Kdx_1 \wedge dx_2,$$

and hence

$$U^{-1}dU \wedge U^{-1}dU \wedge U^{-1}dU = 2(I^2 + J^2 + K^2)dx_1 \wedge dx_2 \wedge dx_3$$

This implies $\omega_U = -12\,\mathrm{dvol}_{S^3}$ and hence, by (20) with $g = U$,

$$\int_{\mathrm{SU}(2)} \omega_0 = \int_{S^3} \omega_U = -12\,\mathrm{Vol}(S^3) = -24\pi^2.$$

This proves the proposition. □

Example 10.14 (Diffeomorphisms). Let M be a compact manifold. Then the diffeomorphisms of M form an *infinite dimensional Lie group* $\mathrm{Diff}(M)$ with group multiplication given by composition $(f, g) \mapsto f \circ g$. Its Lie algebra is the space $\mathrm{Vect}(M)$ of vector fields on M. The Lie algebra structure on $\mathrm{Vect}(M)$ is the usual one if the sign in the definition of the Lie bracket of two vector fields is chosen appropriately. The one-parameter subgroup generated by a vector field $X \in \mathrm{Vect}(M)$ is its flow $R \to \mathrm{Diff}(M) : t \mapsto \phi_t$ defined by

$$\frac{d}{dt}\phi_t = X \circ \phi_t, \qquad \phi_0 = \mathrm{id}.$$

Note also that the inverse of the adjoint action of $\mathrm{Diff}(M)$ on $\mathrm{Vect}(M)$ is given by pullback, i.e.

$$\mathrm{Ad}(\phi^{-1})X = \phi^*X = \mathrm{d}\phi \circ X \circ \phi^{-1}.$$

Interesting subgroups are given by the volume preserving diffeomorphisms or the isometries on a Riemannian manifold, or by the symplectomorphisms on a symplectic manifold.

Example 10.15. For invertible operators on an infinite dimensional Hilbert space H the relation between Lie-group and Lie-algebra is somewhat subtle. Not every one parameter group $t \mapsto S(t)$ of invertible linear operators is differentiable. Such groups can be generated by unbounded operators and this leads to the theory of semigroups of linear operators.