Functional Analysis – Lecture script

Prof. D. Salamon

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Coordinating: Lukas Lewark
Writing: Urs Fuchs, Saša Parađ, Andrin Schmidt
Correcting: Philipp Arbenz, David Umbricht, Dominik Staub, Thomas Rast

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Contents

0 Introduction

Remark: Functional Analysis can be viewed as a combination of linear algebra and topology:

Linear Algebra	Topology	**Functional Analysis**
vector spaces	metric spaces	**normed vector spaces**
linear maps	continuous maps	**continuous linear maps**
subspaces	closed subsets	**closed subspaces**

The vector spaces concerned in Functional Analysis generally have infinite dimension.

1 Basic Notions

1.1 Finite dimensional vector spaces

Definition: A *normed vector space* is a pair $(X, \|\cdot\|)$ where X is a vector space (we consider only real spaces) and $X \to [0,\infty), x \mapsto \|x\|$ is a norm, i.e.

1. $\|x\| = 0 \iff x = 0$
2. $\|\lambda x\| = |\lambda| \cdot \|x\| \; \forall x \in X, \lambda \in \mathbb{R}$
3. $\|x + y\| \leq \|x\| + \|y\| \; \forall x, y \in X$

Remark: A norm induces a *metric* on the vector space, by $d(x, y) := \|x - y\|$.

Definition: A *Banach space* is a complete normed vector space $(X, \|.\|)$, i.e. every Cauchy sequence in (X, d) converges.

Definition: Two norms $\|\cdot\|_1, \|\cdot\|_2$ on a real vector space X are called *equivalent*, if

$$\exists c > 0 \, \forall x \in X : \frac{1}{c}\|x\|_1 \leq \|x\|_2 \leq c\|x\|_1.$$

Example:

1. $X = \mathbb{R}^n, x = (x_1, \ldots, x_n) \in \mathbb{R}^n$

$$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}, \quad 1 < p < \infty$$

$$\|x\|_\infty := \max\{|x_i| \mid 1 \leq i \leq n\}$$

2. $(M, \mathcal{A}, \mu)$ measure space

$$L^p(\mu) = \{f : M \to \mathbb{R} \mid f \text{ measurable }, \int_M |f|^p d\mu < \infty\}/\sim$$

where $\sim$ means equal almost everywhere.

$$\|f\|_p := \left(\int_M |f|^p d\mu\right)^{\frac{1}{p}}, 1 \leq p < \infty$$

$(L^p(\mu), \|\cdot\|_p)$ is a Banach space, and if $M = \{1, \ldots, n\}$ we get Example 1.

3. Let M be a locally compact and hausdorff topologic space.

$$C_c(M) := \{f : M \to \mathbb{R} \mid f \text{ is continuous and has compact support}\}$$

$$\|f\|_\infty := \sup_{m \in M} \{|f(m)|\}$$

Combine 2. and 3.:

Let $\mathcal{B} \subset 2^M$ be the Borel σ-Algebra and $\mu : \mathcal{B} \to [0, \infty]$ a Radon measure. Then one can define $\|f\|_p, \|f\|_\infty$ for all $f \in C_c(M)$. These two norms are not equivalent, because there are Cauchy sequences converging in $\|\cdot\|_\infty$ which are not convergent in $\|\cdot\|_p$, e.g.

$$f_n : \mathbb{R} \to \mathbb{R}, \quad f_n = \begin{cases} \left(\frac{1}{n}\right)^{\frac{1}{p}} & x \in [0, n] \\ 0 & \text{otherwise} \end{cases}$$

4.
$$X = C_b^k(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R} \mid f \in C^k \text{ and}$$
$$\sup_{x\in\mathbb{R}^n} |\partial^\alpha f(x)| < \infty \forall \alpha = (\alpha_1, \ldots \alpha_n) \in \mathbb{N}^n\}$$
$$\|f\|_{c^k} := \sup_{|\alpha|\leq k} \sup_{x\in\mathbb{R}^n} |\partial^\alpha f(x)| = \sup_{|\alpha|\leq k} \|\partial^\alpha f\|_\infty$$

with $|\alpha| := \alpha_1 + \ldots + \alpha_n$.
The normed vector space $(C_b^k(\mathbb{R}^n), \|.\|_{c^k})$ is called *Sobolev space*.

Lemma 1: Let X be a finite dimensional vector space.
$\Rightarrow$ Any two norms on X are equivalent.

Proof: w.l.o.g. $X = \mathbb{R}^n$
Let $e_1, \ldots, e_n \in X$ be the standard basis of X.
Let $\mathbb{R}^n \to \mathbb{R} : x \mapsto \|x\|$ be any norm and

$$c := \sqrt{\sum_{i=1}^n \|e_i\|^2}$$

$$\begin{aligned}
\Rightarrow \|x\| &= \|\sum_{i=1}^n x_i e_i\| && (1)\\
&\leq \sum_{i=1}^n \|x_i e_i\| && (2)\\
&= \sum_{i=1}^n |x_i|\|e_i\| \quad \text{by Cauchy-Schwarz} && (3)\\
&\leq \sqrt{\sum_{i=1}^n x_i^2}\sqrt{\sum_{i=1}^n \|e_i\|^2} && (4)\\
&= c\|x\|_2 && (5)
\end{aligned}$$

That proves one half of the inequality.
It follows that the function $\mathbb{R}^n \to \mathbb{R} : x \mapsto \|x\|$ is continuous with respect to the Euclidian norm on $\mathbb{R}^n$:

$$|\|x\| - \|y\|| \leq \|x - y\| \leq c\|x - y\|_2$$

The set $S^n := \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\}$ is compact with respect to the Euclidian norm.

$$\Rightarrow \exists x_0 \in S \forall x \in S^n : \|x\| \geq \|x_0\| =: \delta > 0$$
$$\Rightarrow \forall x \in \mathbb{R}^n : \frac{x}{\|x\|_2} \in S^n$$

and so

$$\left\| \frac{x}{\|x\|_2} \right\| \geq \delta$$

and therefore

$$\|x\| \geq \delta\|x\|_2$$

Which is the other half of the inequality. □

Lemma 2: Every finite dimensional vector space $(X, \|\cdot\|)$ is complete.

Proof: True for $(\mathbb{R}^n, \|\cdot\|_2)$.
$\Rightarrow$ true for $\mathbb{R}^n$ with any norm.
$\Rightarrow$ true for any finite dimensional vector space. □

Lemma 3: Let $(X, \|\cdot\|)$ be any normed vector space and $Y \subset X$ a finite dimensional linear subspace.
$\Rightarrow Y$ is a closed subset of X.

Proof: Y is a finite dimensional normed vector space.
$\stackrel{\text{Lemma 2}}{\Rightarrow} Y$ is complete.

$$(y_n)_{n\in\mathbb{N}} \subset Y, \quad \lim_{n\to\infty} y_n = y \in X$$

$\stackrel{Y \text{ complete}}{\Rightarrow} y \in Y$
$\Rightarrow Y$ is closed. □

Theorem 1: Let $(X, \|.\|)$ be a normed vector space and $B := \{x \in X \mid \|x\| \le 1\}$ be the unit ball. Then

$$\dim(X) < \infty \iff B \text{ is compact.}$$

Proof of Theorem 1, "⇒": Let $e_1, \ldots, e_n$ be a basis of X and define $T : \mathbb{R}^n \to X$ by $T\xi := \sum_{i=1}^n \xi_i e_i$
$\Rightarrow$ The function $\mathbb{R}^n \to \mathbb{R} : \xi \mapsto \|T\xi\|$ is a norm on $\mathbb{R}^n$

$$\stackrel{\text{Lemma 1}}{\Rightarrow} \exists c > 0\, \forall \xi \in \mathbb{R}^n : \max_{i=1,\ldots,n} |\xi_i| \le c\|T\xi\|$$

Let $(x^\nu)_{\nu\in\mathbb{N}} \in B$ be any sequence and denote $\xi^\nu = (\xi_1^\nu, \ldots, \xi_n^\nu) := T^{-1}x^\nu$

$$\Rightarrow |\xi_i^\nu| \le c\|T\xi^\nu\| = c\|x^\nu\| \le c$$

$\stackrel{\text{Heine-Borel}}{\Rightarrow} (\xi^\nu)_{\nu\in\mathbb{N}}$ has a convergent subsequence $(\xi^{\nu_k})_{k\in\mathbb{R},\nu_1<\nu_2<\ldots}$
$\Rightarrow (\xi_i^{\nu_k})_{k\in\mathbb{N}}$ converges in $\mathbb{R}$ for $i = 1, \ldots, n$
$\Rightarrow x^{\nu_k} = \xi_1^{v_k} e_1 + \ldots + \xi_n^{\nu_k} e_n$ converges; so B is sequentially compact.
We use that on metric spaces sequential compactness and compactness defined by existence of finite subcoverings are equivalent; that will be proven in Theorem 2. □

Lemma 4: $0 < \delta < 1$, $(X, \|\cdot\|)$ a normed vector space, $Y \subsetneq X$ a closed subspace.

$$\Rightarrow \exists x \in X \text{ so that } \|x\| = 1, \inf_{y\in Y} \|x - y\| > 1 - \delta$$

Proof: Let $x_0 \in X \setminus Y$. Denote

$$d := \inf_{y\in Y} \|x_0 - y\| > 0$$

($d > 0$ because Y is closed.) $\exists y_0 \in Y$ so that $\|x_0 - y_0\| < \frac{d}{1-\delta}$
Let $x := \frac{x_0-y_0}{\|x_0-y_0\|} \quad \Rightarrow \quad \|x\| = 1$

$$\|x - y\| = \left\| \frac{x_0 - y_0}{\|x_0 - y_0\|} - y \right\| =$$

$$\frac{1}{\|x_0 - y_0\|} \underbrace{\|x_0 - \underbrace{y_0 - \|x_0 - y_0\|y}_{\in Y}\|}_{\ge d} \ge \frac{d}{\|x_0 - y_0\|} > 1 - \delta$$

□

Proof of Theorem 1, "⇐": Suppose $\dim(X) = \infty$
We construct a sequence $x_1, x_2 \dots$ in B so that

$$\|x_i - x_j\| \geq \frac{1}{2} \forall i \neq j$$

Then $(x_i)_{i\in\mathbb{N}}$ has no convergent subsequence.
We construct by induction sets $\{x_1, \dots, x_n\} \subset B$ so that $\|x_i - x_j\| \geq \frac{1}{2} \forall i \neq j$

$n = 1$: pick any vector $x \in B$.

$n \geq 1$: Suppose $x_1, \dots, x_n$ have been constructed.

Define

$$Y := \operatorname{span}\{x_1, \dots, x_n\} = \left\{ \sum_{i_1}^{n} \lambda_i x_i \;\middle|\; \lambda_i \in \mathbb{R} \right\} \subsetneq X$$

$\Rightarrow Y$ is closed.
So by Lemma 4 $\exists x_{n+1} \in X$ so that

$$\|x_{n+1}\| = 1, \|x_{n+1} - y\| \geq \frac{1}{2} \;\forall y \in Y$$

$$\Rightarrow \|x_{n+1} - x_i\| \geq \frac{1}{2} \;\forall i = 1, \dots, n$$

This completes the inductive construction of the sequence. □

1.2 Linear Operators

$(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ normed vector spaces.

Definition: A linear operator $T : X \to Y$ is called *bounded* if $\exists c > 0 \forall x \in X :$ $\|Tx\|_Y \leq c\|x\|_X$
The number $\|T\| := \sup_{x\in X, x\neq 0} \frac{\|Tx\|_Y}{\|x\|_X}$ is called the *norm* of T.

Notation: $\mathcal{L}(X, Y) := \{T : X \to Y \mid T \text{ is a bounded linear operator}\}$ is a normed vector space, and complete whenever Y is complete. (Analysis II)

Lemma 5: $T : X \to Y$ linear operator. Equivalent are

i. T is bounded

ii. T is continuous

iii. T is continuous at 0.

Proof: i. ⇒ ii. $\|Tx - Ty\|_Y \leq \|T\|\|x - y\|_X \Rightarrow$ Lipschitz continuous
ii. ⇒ iii. trivial
iii. ⇒ i. $\varepsilon = 1 \Rightarrow \exists \delta > 0 \,\forall x \in X :$

$$\|x\|_X \leq \delta \Rightarrow \|Tx\|_Y \leq 1$$

$$0 \neq x \in X \Rightarrow \left\| \frac{\delta x}{\|x\|_X} \right\|_X = \delta$$

$$\Rightarrow \left\| \frac{\delta Tx}{\|x\|_X} \right\|_Y \leq 1$$

$$\Rightarrow \|Tx\|_Y \leq \underbrace{\frac{1}{\delta}}_{c} \|x\|_X$$

□

Lemma 6: Let X, Y be normed vector spaces of finite dimension $\Rightarrow$ every linear operator $T : X \to Y$ is bounded.

Proof: Choose a basis $e_1, \dots e_n \in X$.

a)
$$c_1 := \sum_{i=1}^{n} \|Te_1\|_Y$$

b) The map
$$\mathbb{R}^n \to \mathbb{R} : (x_1, \dots x_n) \to \left\| \sum_{i=1}^{n} x_i e_i \right\|_X$$
is a norm on $\mathbb{R}^n$. By Lemma 1, $\exists c_2 > 0$ so that
$$\max_{i=1,\dots n} |x_i| \leq c_2 \left\| \sum_{i=1}^{n} x_i e_i \right\|_X \qquad \forall (x_1, \dots x_n) \in \mathbb{R}^n$$

a)&b) $\Rightarrow \forall x = \sum_{i=1}^{n} x_i e_i \in X$ we have
$$\|Tx\|_Y = \left\| \sum_{i=1}^{n} x_i Te_i \right\|_Y \leq \sum_{i=1}^{n} |x_i| \cdot \|Te_i\|_Y$$
$$\leq (\max_i |x_i|) \cdot \sum_{i=1}^{n} \|Te_i\|_Y = c_i \max |x_i| < c_1 c_2 \|x\|_X$$

□

What for infinite dimensions?

1.3 Infinite dimensional vector spaces

Example 1: Let $X = C^1([0,1]; X)$, $\|x\|_X := \sup_{0 \leq t \leq 1} |f(t)|$, $Y = \mathbb{R}$ and $Tx := \dot{x}(0)$. T is linear and not bounded. This is not a Banach space.

Example 2: X infinite dimensional.
$\exists \{e_i\}_{i \in I}$ basis of X with $\|e_i\| = 1 \, \forall i \in I$ (the axiom of choice is needed to prove this for any vector space).
Choose sequence $i_1, i_2, \dots$
Define $c_i := \begin{cases} k, i = i_k \\ 0, i \notin \{i_1, i_2, \dots\} \end{cases}$
Define $T : X \to \mathbb{R}$ by
$$T\Big(\underbrace{\sum_{i \in I} \lambda_i e_i}_{\text{finite sum}}\Big) = \sum_{i \in I} \lambda_i c_i$$
$$Te_{i_k} = \lambda_{i_k} = k$$

We found three incidences where finite and infinite dimensional space differ:

- Compactness of the unit ball (see Theorem 1)
- Completeness (see Lemma 2)
- Boundedness of Linear Functionals (see Lemma 6 and Example 1)

Definition: A metric space (M, d) is called *totally bounded* if
$$\forall \varepsilon > 0 \, \exists x_1, \dots, x_m \in M : M = \bigcup_{i=1}^{m} B_\varepsilon(x_i)$$
where
$$B_\varepsilon(x) := \{x' \in M \mid d(x, x') < \varepsilon\}$$

Theorem 2: Let (M, d) be a metric space. Equivalent are:

i. Every sequence has a convergent subsequence (sequential compactness).

ii. Every open cover has a finite subcover (compactness).

iii. (M, d) is totally bounded and complete.

Proof: i. $\Rightarrow$ ii.:
$\mathcal{T}(M, d) \subset 2^M$ set of open subsets of M.
Let $\mathcal{U} \subset \mathcal{T}(M, d)$ be an open cover of M.

Step 1

$$\exists \varepsilon > 0 \, \forall x \in M \, \exists U \in \mathcal{U} \text{ so that } B_\varepsilon(x) \subset U$$

Suppose $\forall \varepsilon > 0 \, \exists x \in M \, \forall U \in \mathcal{U}$ so that $B_\varepsilon(x) \not\subset U$.
Pick $\varepsilon = \frac{1}{n}$

$$\Rightarrow \exists x_n \in M \;\; \forall U \in \mathcal{U} : B_{\frac{1}{n}}(x_n) \not\subset U$$

By i. $\exists$ convergent subsequence $x_{n_k} \to x \in M$.
Choose $U \in \mathcal{U}$ so that $x \in U$; choose $\varepsilon > 0$ so that $B_\varepsilon \subset U$.
Choose k so that $d(x, x_{n_k}) < \frac{\varepsilon}{2}$ and $\frac{1}{n} < \frac{\varepsilon}{2}$.

$$\Rightarrow B_{\frac{1}{n_k}}(x_{n_k}) \subset B_{\frac{\varepsilon}{2}}(x_{n_k}) \subset B_\varepsilon(x) \subset U$$

$\Rightarrow$ contradiction.

Step 2 $\mathcal{U}$ has a finite subcover.
Suppose not.
Let $\varepsilon > 0$ be as in Step 1.
Construct sequences $x_1, x_2, \ldots \in M$ and $U_1, U_2, \ldots \in \mathcal{U}$ so that

$$B_\varepsilon(x_n) \subset U_n \text{ and } x_n \notin U_1, \ldots U_{n-1}$$

x_n can be chosen like that because otherwise the $U_1, \ldots U_{n-1}$ would form a finite subcover.
Pick any $x_i \in M$.
By Step 1 $\exists U_i \in \mathcal{U}$ so that $B_\varepsilon(x_i) \subset U_i$.
Suppose $x_1, \ldots, x_n$ and $U_1, \ldots, U_n$ have been found.
$\Rightarrow U_1 \cup U_2 \cup \ldots \cup U_n \neq M$
$\Rightarrow x_{n+1} \in M \setminus (U_1 \cup U_2 \cup \ldots \cup U_n)$
By Step 1 $\exists U_{n+1} \in \mathcal{U}$ so that $B_\varepsilon(x_{n+1}) \subset U_{n+1}$
Given the sequences $(x_k)_{k \in \mathbb{N}}, (U_k)_{k \in \mathbb{N}}$ we observe:
For $k < n : B_\varepsilon(x_k) \subset U_k, x_n \notin U_k$
So $d(x_n, x_k) \geq \varepsilon$
$\Rightarrow d(x_k, x_n) \geq \varepsilon \; \forall k \neq n$
$\Rightarrow$ There is no convergent subsequence.

ii. $\Rightarrow$ iii.:
Assume every open cover has a finite subcover.

a. Take

$$\mathcal{U} := \{B_\varepsilon(x) \mid x \in M\}$$

Then $\exists x_1, \ldots x_m \in M$ so that

$$\bigcup_{i=1}^{m} B_\varepsilon(x_i) = M$$

So M is totally bounded.

b. (M,d) is complete:
Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence.
Assume $(x_n)_{n\in\mathbb{N}}$ does not converge.
$\Rightarrow (x_n)$ has no convergent subsequence.
$\Rightarrow (x_n)$ has no limit point.
$\Rightarrow \forall \xi \in M \ \exists \varepsilon(\xi) > 0$ so that the set $\{n \in \mathbb{N} \mid x_n \in B_{\varepsilon(\xi)}(\xi)\}$ is finite.
Take $\mathcal{U} := \{B_{\varepsilon(\xi)}(\xi) \mid \xi \in M\}$.
Then $\mathcal{U}$ has no finite subcover.

iii. $\Rightarrow$ i.:
Assume (M,d) is totally bounded and complete.
Let $(x_n)_{n\in\mathbb{N}}$ be any sequence in M.
Claim: There is a sequence of infinite subsets

$$\mathbb{N} \supset T_0 \supset T_1 \supset \dots$$

such that $d(x_n, x_m) \le 2^{-k} \ \forall x_n, x_m \in T_k$.
Cover M by finitely many balls

$$\bigcup_{i=1}^{m} B_{\frac{1}{2}}(\xi_i) = M$$

$\Rightarrow \exists i$ so that the set $\{n \in \mathbb{N} \mid x_n \in B_{\frac{1}{2}}(\xi_i)\} =: T_0$ is infinite.
Then $\forall n, m \in T_0$ we have

$$d(x_n, x_m) \le d(x_n, \xi_i) + d(\xi_i, x_m) < 1$$

Suppose T_{k-1} has been constructed.
Cover M by finitely many balls

$$M = \bigcup_{i=1}^{m} B_{\frac{1}{2^{k+1}}}(\xi_i)$$

Then $\exists i$ so that the set

$$T_k := \{n \in T_{k-1} \mid x_n \in B_{\frac{1}{2^{k+1}}}(\xi_i)\}$$

is infinite.
$\Rightarrow \forall n, m \in T_k$:

$$d(x_n, x_m) < d(x_n, \xi_i) + d(\xi_i, x_m) < \frac{1}{2^k}$$

Claim $\Rightarrow \exists$ convergent subsequence.
Pick $n_1 < n_2 < \dots$ so that $n_k \in T_k$.
$\Rightarrow n_l, n_k \in T_k \ \forall l \ge k$
$\Rightarrow d(x_{n_k}, x_{n_l}) \le \frac{1}{2^k} \ \forall l \ge k$
$\Rightarrow$ The sequence $(x_k)_{k\in\mathbb{N}}$ is Cauchy.
$\overset{M \text{ complete}}{\Rightarrow}$ The sequence converges. □

1.4 The Theorem of Arzela-Ascoli

Definition: (M,d) metric space. A subset $D \subset M$ is called *dense* if

$$\forall x \in M \ \forall \varepsilon > 0 : B_\varepsilon(x) \cap D \neq \varnothing$$

Definition: A metric space (M,d) is called *separable* if it contains a countable dense subset.

Corollary: Every compact metric space (M,d) is separable.

Proof: Given $n \in \mathbb{N}$.

$$\exists \xi_1, \dots \xi_n \in M : M = \bigcup_{i=1}^{n} B_{\frac{1}{n}}(\xi_i)$$

Define

$$D_n := \{\xi_1, \dots \xi_n\}$$

Define

$$D = \bigcup_{i=1}^{\infty} D_n \subset M$$

D is countable.
Given $x \in M, \varepsilon > 0$, pick $n \in \mathbb{N}$ so that $\frac{1}{n} < \varepsilon$.
Then $\exists \xi \in D_n$ so that $x \in B_{\frac{1}{n}}(\xi)$.

$$\Rightarrow x \in B_\varepsilon(\xi), \xi \in D \quad \Rightarrow \quad B_\varepsilon(x) \cap D \neq \varnothing$$

So D is dense. □

Exercise: (X, d_X) compact metric space, (Y, d_Y) complete metric space and

$$C(X,Y) := \{f : X \to Y \mid f \text{ continuous}\}$$

$$d(f,g) := \sup_{x \in X} d_Y(f(x), g(x)) < \infty \ \forall f, g \in C(X,Y)$$

Show that $(C(X,Y), d)$ is a complete metric space.
This is exercis 1a) on Series 2.

Definition: A subset $\mathcal{F} \subset C(X,Y)$ is called *equicontinuous* if

$$\forall \varepsilon > 0 \, \exists \delta > 0 : \forall x, y \in X \, \forall f \in \mathcal{F} : d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \varepsilon$$

Theorem 3 (Arzela-Ascoli): (X, d_X) compact metric space and (Y, d_Y) complete metric space, $\mathcal{F} \subset C(X,Y)$.
Equivalent are:

i. $\mathcal{F}$ is compact.

ii. $\mathcal{F}$ is closed, equicontinuous and $\mathcal{F}(x) := \{f(x) \mid f \in \mathcal{F}\} \subset Y$ is compact for every $x \in X$.

Proof: i. $\Rightarrow$ ii.:

- $\mathcal{F}$ is closed (every compact set in a metric space is closed).
- Fix $x \in X$. Then the evaluation map $\mathrm{ev}_x : \mathcal{F} \to Y, \mathrm{ev}_x(f) := f(x)$ is continous. So $\mathrm{ev}_x(\mathcal{F}) = \mathcal{F}(x)$ is compact.
- Pick $\varepsilon > 0$. $\exists f_1, \dots, f_m \in \mathcal{F}$ so that $\mathcal{F} \subset \bigcup_{i=1}^m B_\varepsilon(f_i)$
  Choose $\delta > 0$ so that $\forall i \forall x, x' \in X$:

  $$d_X(x, x') < \delta \Rightarrow d_Y(f_i(x), f_i(x')) < \varepsilon$$

  Given $f \in \mathcal{F}$ choose i so that $d(f, f_i) < \varepsilon$. Now for x, x' with $d_X(x, x') < \delta$:

  $$d_Y(f(x), f(x')) \leq \underbrace{d(f(x), f_i(x))}_{<\varepsilon} + \underbrace{d(f_i(x), f_i(x')}_{<\varepsilon} + \underbrace{d(f_i(x'), f(x'))}_{<\varepsilon} < 3\varepsilon$$

ii. $\Rightarrow$ i.:

To Show: $\mathcal{F}$ is compact. Let $f_n \in \mathcal{F}$ be any sequence.
We know that X is separable, i.e. there is countable dense subset $D \subset X$ with D in the form $D = \{x_1, x_2, \dots\}$.

Claim 1 There is a subsequence $g_i := f_{n_i}$ so that the sequence $g_i(x_k) \in Y$ converges as $i \to \infty$ for every $k \in \mathbb{N}$.

Proof of Claim 1 $\mathcal{F}(x_i)$ is compact and $f_n(x_1) \in \mathcal{F}(x_i)$. Thus there is a subsequence $(f_{n_{1,i}})_i$ so that $(f_{n_{1,i}}(x_1))_i$ converges. By the same argument there is a subsequence $(f_{n_{2,i}})_i$ of $(f_{n_{1,i}})_i$ so that $(f_{n_{2,i}}(x_2))_i$ converges.
Induction: There is a sequence of subsequences $(f_{n_{k,i}})_{i=1}^{\infty}$ so that

- $(f_{n_{k,i}}(x_k))_{i=1}^{\infty}$ converges as $i \to \infty$.
- $(f_{n_{k+1,i}})_{i=1}^{\infty}$ is a subsequence of $(f_{n_{k,i}})_i$ for every $k \in \mathbb{N}$.

Define $g_i := f_{n_{i,i}}$, that is the Diagonal sequence construction. This satisfies $g_i(x_k)$ converges for all $k \in \mathbb{N}$ as $i \to \infty$. But we want more: Namely, convergence in the whole of X, not only D, and uniform convergence.

Claim 2: $(g_i)_i$ is a Cauchy sequence in $C(X, Y)$.
With Claim 2:
Since $C(X, Y)$ is complete the sequence g_i converges. Since $\mathcal{F}$ is closed, its limit belongs to $\mathcal{F}$.

Proof of Claim 2:

- Choose $\varepsilon > 0$ and $\delta > 0$ as in the definition of equicontinuity, i.e.

Condition 1
$$\forall x, x' \in X \forall f \in \mathcal{F} : \quad d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \varepsilon$$

- Since D is dense in X we have
$$X = \bigcup_{k=1}^{\infty} B_\delta(x_k)$$
By Theorem 2

Condition 2
$$\exists m \in \mathbb{N} : \quad X = \bigcup_{k=1}^{m} B_\delta(x_k)$$

- Since $(g_i(x_k))_{i=1}^{\infty}$ is Cauchy for every $k \in \{1, \dots n\}$:

Condition 3
$$\exists N \in \mathbb{N} \forall i, j > N \forall k \in \{1, \dots m\} : d_Y(g_i(x_k), g_j(x_k)) < \varepsilon$$

We prove: $i, j \geq N \Rightarrow d(g_i, g_j) < 3\varepsilon$. Remember that
$$d(g_i, g_j) := \sup_{x \in X} d_Y(g_i(x), g_j(x))$$
Fix an element $x \in X$.
By Condition 2 $\exists k \in \{1, \dots n\}$ so that $d_X(x, x_k) < \delta$.
By Condition 1

Condition 4
$$\forall i \in \mathbb{N} : d_Y(g_i(x), g_i(x_k)) < \varepsilon$$
$$i, j \geq N \Rightarrow d_Y(g_i(x), g_j(x)) \leq$$
$$d_Y(g_i(x), g_i(x_k)) + d_Y(g_i(x_k), g_j(x_k)) + d_Y(g_i(x_k), g_j(x))$$
And this is, by Condition 3 and 4, smaller than $\varepsilon + \varepsilon + \varepsilon = 3\varepsilon$.

□

Looking closely at the proof, one can weaken the three condition of the theorem of Arzéla-Ascoli.

Theorem 3' (Arzéla-Ascoli revisited): Let (X, d_X) be compact, (Y, d_Y) complete metric spaces and $\mathcal{F} \subset C(X, Y)$. Equivalent are

(i) $\mathcal{F}$ has a compact closure.

(ii) $\mathcal{F}$ is equicontinuous and $\mathcal{F}(x) \subset Y$ has a compact closure $\forall x \in X$.

Proof: $\overline{\mathcal{F}}$ is the closure of $\mathcal{F}$ in $C(X, Y)$. From (i) it follows that $\overline{\mathcal{F}}(x) = \overline{\mathcal{F}(x)} \forall x \in X$.

- $\overline{\mathcal{F}}(x) \subset \overline{\mathcal{F}(x)}$ is always true and an exercise.
- $\overline{\mathcal{F}}(x) \supset \overline{\mathcal{F}(x)}$. Proof:
  Let $y \in \overline{\mathcal{F}(x)} \Rightarrow \exists$ sequence $y_k \in \mathcal{F}(x), y_k \to y$.
  $\Rightarrow \exists f_k \in \mathcal{F}$ so that $f_k(x) = y_k$.
  $\Rightarrow f_k$ has a convergent subsequence $f_{k_i} \to f \in C(X, Y)$ where $f_{k_i} \in \mathcal{F}$ and $f \in \overline{\mathcal{F}}$.
  So $y = f(x) \in \overline{\mathcal{F}}(x)$.

"(i)⇒(ii)" $\mathcal{F}$ is equicontinuous by Theorem 4 for $\overline{\mathcal{F}}$.
$\overline{\mathcal{F}(x)} = \overline{\mathcal{F}}(x)$ which is compact by Theorem 3.
"(ii)⇒(i)" Claim 1 and Claim 2 in Theorem 3 only use (ii) in Theorem 3. So every sequence in $\mathcal{F}$ has a Cauchy subsequence. □

Lemma 7: Let (M, d) be a complete metric space, $A \subset M$ any subset. Equivalent are

(i) A has a compact closure.

(ii) Every sequence in A has a Cauchy subsequence.

Proof: "(i)⇒(ii)" follows directly from the definitions.
"(ii)⇒(i)". Let $x_n \in \overline{A}$ be any sequence $\Rightarrow \exists a_n \in A$ so that $d(x_n, a_n) < \frac{1}{n}$. $\Rightarrow \exists$ Cauchy subsequence $(a_{n_i})_{i=1}^{\infty}$. $\Rightarrow (x_{n_i})_{i=1}^{\infty}$ is Cauchy. Because (M, d) is complete $\Rightarrow (x_{n_i})$ converges (to another element of $\overline{A}$). □

Special case: $Y = \mathbb{R}^n$, (X, d_X) compact metric space. $\mathfrak{X} = C(X, \mathbb{R}^n)$ is a normed vector space.

$$\|f\| := \sup_{x \in X} |f(x)|_{\mathbb{R}^n}$$

Theorem 4': Let $\mathcal{F} \subset C(X, \mathbb{R}^n)$. Equivalent are

(i) $\mathcal{F}$ has a compact closure.

(ii) $\mathcal{F}$ is equicontinuous and bounded.

Proof: Theorem 3' (A subset of $\mathbb{R}^n$ has a compact clousre if and only if it is bounded). So condition (ii) in Theorem 4' implies Condition (ii) in Theorem 3' with $Y = \mathbb{R}^n$. Moreover an unbounded subset of $C(X, \mathbb{R}^n)$ cannot be compact. □

Theorem 4: Let $\mathcal{F} \subset C(X, \mathbb{R}^n)$. Equivalent are

(i) $\mathcal{F}$ is compact.

(ii) $\mathcal{F}$ is closed, bounded and equicontinuous.

Proof: Corollary of Theorem 4'. □

This again highlights the difference between finite and infinite dimensional vector spaces, as far as compactness is concerned.

1.5 The Baire Category Theorem

Example for an open and dense set: $\mathbb{Q} \subset \mathbb{R}$. Let $\mathbb{Q} = \{x_1, x_2, \ldots\}$. Let

$$U := \bigcup_{k=1}^{\infty} \left(x_k - \frac{\varepsilon}{2^k}, x_k + \frac{\varepsilon}{2^k} \right)$$

$$\mu^{\text{Leb}}(U) \leq \sum_{k=1}^{\infty} \frac{2\varepsilon}{2^k} = 2\varepsilon$$

U is open and dense.

Theorem 5: Let (M, d) be a complete metric space.

(i) If $U_1, U_2, U_3, \ldots \subset M$ is a sequence of open and dense subset then

$$D := \bigcap_{i=1}^{\infty} U_i$$

is dense in M.

(ii) $M \neq \varnothing$ and $A_1, A_2, A_3, \ldots \subset M$ is a sequence of closed subsets so that

$$M = \bigcup_{i=1}^{\infty} A_i$$

Then $\exists i$ so that A_i contains an open ball.

Example:

1. $M = \mathbb{R} = \bigcup_{x \in \mathbb{R}} \{x\}$ and $\mathbb{R}$ complete; so $\mathbb{R}$ uncountable.
2. $M = \mathbb{Q} = \bigcup_{x \in \mathbb{Q}} \{x\}$ is not complete.

The proof is not so hard. It depends on one ingenious observation which has many important consequences.

Proof:

(i) Let $x \in X$ and $\varepsilon > 0$. To show: $B_\varepsilon(x) \cap D \neq \varnothing$.
Let $B := B_\varepsilon(x) = \{y \in M \mid d(x, y) < \varepsilon\}$. Since U_i is dense $B \cap U_i \neq \varnothing$. Choose $x_1 \in B \cap U_1$. $B \cap U_1$ open $\Rightarrow \exists \varepsilon_1 > 0, \varepsilon_1 \leq \frac{1}{2}$ so that

$$\overline{B_{\varepsilon_1}(x_1)} \subset B \cap U_1$$

Since U_2 is dense, $B_{\varepsilon_1}(x_1) \cap U_2 \neq \varnothing$. Choose $x_2 \in B_{\varepsilon_1}(x_1) \cap U_2$. Because $B_{\varepsilon_1}(x) \cap U_2$ is open, $\exists x_2 > 0$ so that

$$\overline{B_{\varepsilon_2}(x_2)} \subset B_{\varepsilon_1}(x_1) \cap U_2$$

and $0 < \varepsilon_2 \leq \frac{1}{4}$.
By Induction one gets a sequence

$$x_k \in M, 0 < \varepsilon_k \leq \frac{1}{2^k}$$

so that

$$\overline{B_{\varepsilon_k}(x_k)} \subset B_{\varepsilon_{k-1}}(x_{k-1}) \cap U_k$$

In particular $x_k \in B_{\varepsilon_{k-1}}(x_{k-1})$, i.e.

$$d(x_k, x_{k-1}) < \varepsilon_{k-1} \leq \frac{1}{2^{k-1}}$$

So x_k is a Cauchy sequence in M, which converges, because M is complete. Let $x^* := \lim_{k\to\infty} x_k$. Note:

$$\overline{B_{\varepsilon_1}(x_1)} \supset \overline{B_{\varepsilon_2}(x_2)} \supset \overline{B_{\varepsilon_3}(x_3)} \supset \dots$$

So $x_\ell \in \overline{B_{\varepsilon_k}(x_k)} \forall \ell \geq k$ and thus $x^* \in \overline{B_{\varepsilon_k}(x_k)} \subset U_k \forall k$.
So $x^* \in D = \bigcap_{k=1}^\infty U_k$. Also $x^* \in \overline{B_{\varepsilon_i}(x_i)} \subset B$ so $B \cap D \neq \varnothing$.

(ii) Let $U_i := M \setminus A_i$, open. Suppose (by contradiction) that A_i does not contain any open ball for every i. So U_i is open and dense. By (i) $\bigcap_{i=1}^\infty U_i$ is dense; thus $M \setminus \bigcup_{i=1}^\infty A_i \neq \varnothing \Rightarrow M \neq \bigcup_{i=1}^\infty A_i$. □

Reminder Let $A \subset M$, then $A^\circ = \text{int}(A) = \{x \in M \mid \exists \varepsilon > 0 \text{ such that } B_\varepsilon(x) \subset A\}$ is the *interior* of A.

Definition:

- Let (M, d) be a metric space. $A \subset M$ is called *nowhere dense* if $\overline{A}$ has empty interior.
- $A \subset M$ is said to be of *1st category* in the sense of Baire if $A = \bigcup_{i=1}^\infty A_i$, where $A_i \subset M$ is nowhere dense.
- $A \subset M$ is said to be of *2nd category* if it is not of the 1st category.
- $A \subset M$ is called *residual* if $M \setminus A$ is of the 1st category.

Notation: $\text{cat}(A) = 1$ or 2.

Example:

- $\mathbb{Z} \subset \mathbb{R}$ is nowhere dense.
- $\mathbb{Q} \subset \mathbb{R}$ is of the 1st category.

Rules:

1. If $A \subset B$: $\text{cat}(B) = 1 \Rightarrow \text{cat}(A) = 1$
2. $\text{cat}(A) = 2 \Rightarrow \text{cat}(B) = 2$
3. $A = \bigcup_{i=1}^\infty A_i$, $\text{cat}(A_i) = 1 \Rightarrow \text{cat}(A) = 1$

Lemma 8: (M, d) complete metric space, $R \subset M$. Equivalent are:

(i) R is residual

(ii) $R \supset \bigcap_{i=1}^\infty U_i$ with U_i open, dense.

Proof:

(i) $\Rightarrow$ (ii) R residual, $A := M \setminus R \Rightarrow \text{cat}(A) = 1$

$$\Rightarrow A = \bigcup_{i=1}^\infty A_i \subset \bigcup_{i=1}^\infty \overline{A_i}$$

where A_i is nowhere dense. Then $U_i := M \setminus \overline{A_i}$ is open and dense.

$$\Rightarrow R = M \setminus A \supset M \setminus \left(\bigcup_{i=1}^\infty \overline{A_i}\right) = \bigcap \left(M \setminus \overline{A_i}\right) = \bigcap U_i$$

(ii) $\Rightarrow$ (i) Assume $U_i \subset M$ open, dense and $\bigcap_{i=1}^{\infty} U_i \subset R$

$$A := M \setminus R \Rightarrow R = M \setminus \bigcap U_i = \bigcup (M \setminus U_i)$$

with $M \setminus U_i$ nowhere dense $\forall i \in \mathbb{N} \Rightarrow \mathrm{cat}(A) = 1$.

□

Theorem 6 (Baire Category Theorem): Let (M, d) be a complete metric space and $M \neq \varnothing$. Then

(i) $\mathrm{cat}(M) = 2$

(ii) If $A \subset M$, then $\mathrm{cat}(A) = 1 \Rightarrow \mathrm{cat}(M \setminus A) = 2$

(iii) If $A \subset M$, then $\mathrm{cat}(A) = 1 \Rightarrow M \setminus A$ is dense

(iv) $\varnothing \neq U \subset M$ open $\Rightarrow Cat(U) = 2$

(v) $A = \bigcup_{i=1}^{\infty} A_i$, A_i closed with $A_i^\circ = \varnothing \Rightarrow A^\circ = \varnothing$

(vi) $U = \bigcap_{i=1}^{\infty} U_i$ with $U_i \subset M$ open and dense $\Rightarrow U$ is dense in M

Proof:

(i) Suppose $\mathrm{cat}(M) = 1 \Rightarrow M = \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \overline{A_i}$
A_i is nowhere dense $\Rightarrow$(by Thm 5) one of the $\overline{A_i}$ contains an open ball. Contradiction.

(ii) $\mathrm{cat}(M \setminus A \cup A) = \mathrm{cat}(M) = 2 \Rightarrow \mathrm{cat}(M \setminus A) = 2$ using the above rules.

(iii) Lemma 8 and Theorem 5 (ii)

(iv) $U \subset M$ open, nonempty $\Rightarrow U$ contains an open ball $\Rightarrow M \setminus U$ is not dense.
$\Rightarrow \mathrm{cat}(U) = 2$

(v) $A = \bigcup_{i=1}^{\infty} A_i$ closed $A_i^\circ = \varnothing \Rightarrow \mathrm{cat}(A) = 1 \Rightarrow$ (by (iv)) A does not contain an open ball $\Rightarrow A^\circ = \varnothing$

(vi) Theorem 5 (i)

□

Exercise: Even if (M, d) is not complete, we have (iii)$\Leftrightarrow$ (iv)$\Leftrightarrow$(v)$\Leftrightarrow$ (vi) An Application of Baire's theorem is the following

Theorem 7 (Banach 1931):

$$\mathcal{R} := \{f : [0,1] \to \mathbb{R} \mid f \text{ is continuous and nowhere differentiable}\}$$

is residual in $C([0,1])$.

Proof: Denote

$$U_n := \left\{ f \in C([0,1]) \,\middle|\, \sup_{\substack{0<|h|\leq 1 \\ t+h\in[0,1]}} \left| \frac{f(t+h) - f(t)}{h} \right| > n \quad \forall t \in [0,1] \right\}$$

Claim 1 $\mathcal{R} \supset \bigcap_{i=1}^{\infty} U_n$

Claim 2 U_n is open $\to$. Exercise.

Claim 3 U_n is dense.

Proof of Claim 3 Fix $n \in \mathbb{N}$. Let $g \in C([0,1])$ and $\varepsilon > 0$.
To show: $B_\varepsilon(g) \cap U_n \neq \varnothing$
By Weierstrass there is a polynomial $p : \mathbb{R} \to \mathbb{R}$ such that

$$\|g - p\| = \sup_{t \in [0,1]} |g(t) - p(t)| < \varepsilon/2$$

We must find an $f \in U_n$ such that $\|f - p\| < \varepsilon/2$.
Trick: Define $z : \mathbb{R} \to \mathbb{R}$ by $z_\lambda(t) := \lambda z(\frac{t}{\lambda^2})$
$|\dot{z}(t)| = 1$ and $|z_\lambda(t)| = \frac{1}{\lambda}$
Idea: Choose $f(t) = p(t) + z_\lambda(t)$ then $\|f - p\| = \|z_\lambda| = \lambda$

$$\left|\frac{f(t+h) - f(t)}{h}\right| \geq \underbrace{\left|\frac{z(t+h) - z(t)}{h}\right|}_{\geq \frac{1}{\lambda} \text{ if h is small}} - \underbrace{\left|\frac{p(t+h) - p(t)}{h}\right|}_{\leq \sup_{t\in[0,1]}\left|\frac{d}{dt}p(t)\right| =: c}$$

$$\geq \frac{1}{\lambda} - c > n \text{ if } \lambda \text{ is small} \Rightarrow f \in U_n$$

Proof of Claim 2 See Zehnder's notes. □

1.6 Dual spaces

Let $(X, \|\cdot\|)$ be a Banach space. Three examples for dual spaces:

Example 1: If $X = H$ is a *Hilbert space*, i.e. there is an inner product $H \times H \to \mathbb{R} : (x,y) \mapsto \langle x, y\rangle$ so that

$$\|x\| = \sqrt{\langle x, x\rangle}$$

Each $x \in H$ determines a bounded linear functional $\Lambda_x : H \to \mathbb{R}$ via

(1) $$\Lambda_x(y) := \langle x, y\rangle$$

The map $H \to H^* : x \mapsto \Lambda_x$ is a Banach space isometry, i.e. a bilinear map preserving the norms, so $H \cong H^*$. The difficult part of the proof is that *(1) is onto* (Proof in Measure and Integration).

Example 2: Let $(M, \mathcal{A}, \mu)$ be a σ-finite measure space and

$$X = L^p(\mu) = \left\{ f : M \to \mathbb{R} \,\Big|\, \int_M |f|^p \,\mathrm{d}\mu < \infty \right\} \Big/ \sim$$

and

$$\|f\|_p = \left(\int_M |f|^p \,\mathrm{d}\mu \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty$$

In the measure and integration course it was shown that

$$X^* \cong L^q(\mu), \quad 1 < q \leq \infty, \quad \frac{1}{p} + \frac{1}{q} = 1$$

More precisely the map

$$L^q(\mu) \to L^p(\mu)^* : g \mapsto \Lambda_g$$

$$\Lambda_g(f) := \int_M fg \,\mathrm{d}\mu$$

is a Banach space isometry.
Again it's easy to prove that

$$\|g\|_q = \|\Lambda_g\| = \sup_{0 \neq f \in L^p} \frac{\left|\int_M fg \, \mathrm{d}\mu\right|}{\|f\|_p}$$

and the hard part is that

$$\forall \Lambda \in L^p(\mu)^* : \exists g \in L^q(\mu) \text{ so that } \Lambda = \Lambda_g.$$

Proof in Measure and Integration.

Example 3: Let (M, d) be a compact metric space. Consider $X = C(M) = \{f : M \to \mathbb{R} \mid f \text{ is continuous}\}$. Let

$$\|f\| := \sup_{p \in M} |f(p)|$$

That X is a Banach space is already known from Analysis I & II.

$$X^* = \{\text{All real Borelmeasures on } M\} =: \mathcal{M}$$

Let $\mathcal{B} \subset 2^M$ be the Borel σ-Algebra and a σ-additive $\lambda : \mathcal{B} \to \mathbb{R}$ a real (Borel) measure.
Define $\varphi_\lambda : C(M) \to \mathbb{R}$ by

$$\varphi_\lambda(f) := \int_M f \, \mathrm{d}\lambda$$

Easy: φ_λ is bounded and $\|\varphi_\lambda\| = \|\lambda\| = |\lambda|(M)$.
The map $\mathcal{M} \to C(M)^* : \lambda \to \varphi_\lambda$ is linear. But why is this map surjective?

Exercise with Hints:

1. $\mathcal{U} \subset M$ open $\Rightarrow \mathcal{U}$ is σ-compact.

$$U = \bigcup_{n=1}^{\infty} K_n, \quad K_n := \{x \in M \mid B_{\frac{1}{n}}(x) \subset U\}$$

2. Every finite Borel measure $\mu : \mathcal{B} \to [0, \infty)$ is a Radon measure because of 1.

3. Riesz Representation Theorem

$$\varphi : C(M) \to \mathbb{R}$$

positive linear functional, i.e. if $f \in C(M)$ and $f > 0 \Rightarrow \varphi(f) > 0$, e.g.

$$\varphi(f) = \int_M f \, \mathrm{d}\mu$$

4. For every bounded linear functional $\varphi : C(M) \to \mathbb{R}$ there are two positive linear functionals $\varphi^\pm : C(M) \to \mathbb{R}$, s.t. $\varphi = \varphi^+ - \varphi^-$.
Hint: For $f > 0$ define

$$\psi(f) := \sup\{\varphi(f_1) - \varphi(f_2) \mid f_1 + f_2 = f, f_1, f_2 \in C(M), f_1 \geq 0, f_2 \geq 0\}$$

$\psi(f) \in \mathbb{R}_0^+$. Claim: $\psi(f + g) = \psi(f) + \psi(g)$.
For $f : M \to \mathbb{R}$ continuous define $f^\pm(x) := \max\{\pm f(x), 0\} \Rightarrow f = f^+ - f^-$, $f^\pm$ continuous and nonnegative.
Define $\psi(f) := \psi(f^+) - \psi(f^-)$.
To show: $\psi : C(M) \to \mathbb{R}$ is bounded and linear.

$$f \geq 0 \Rightarrow \psi(f) \geq |\varphi(f)|, \quad \varphi^\pm := \frac{1}{2}(\psi \pm \varphi)$$

$\varphi^\pm$ are positive.

Definition: Let (M, d) be a metric space. A *completion* of (M, d) is triple (M^*, d^*, ι) where

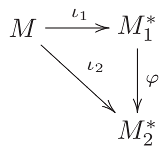
1. (M^*, d^*) is a complete metric space
2. $\iota : M \to M^*$ is an isometric embedding i.e.
   $d^*(\iota(x), \iota(y)) = d(x, y) \quad \forall x, y \in M$
3. The image $\iota(M)$ is dense subset of M^*.

Definition: $(M_1, d_1), (M_2, d_2)$ metric spaces
A map $\phi : M_1 \to M_2$ is called an *isometry*, if it is bijective
and $d_2(\phi(x), \phi(y)) = d_1(x, y) \quad \forall x, y \in M$.

Theorem 8:

(i) Every metric space (M, d) admits a completion.

(ii) If (M_1, d_1, ι_1) and (M_2, d_2, ι_2) are completions of (M, d), then there exists a unique isometry $\phi : M_1 \to M_2$ such that $\phi \circ \iota_1 = \iota_2$



Proof:

(i) Uniqueness $\to$ Exercise, the standard uniqueness proof for objects with universal property.

(ii) Existence

Construction 1 $M^* := \{\text{Cauchy sequence in M}\}/_\sim$
$(x_n) \sim (y_n) \Leftrightarrow \lim_{n\to\infty} d(x_n, y_n) = 0$
$\iota(x) := \{[(x_n)] \text{ where } x_i = x \quad \forall i \in \mathbb{N}\}$
$d^*([(x_n)], [(y_n)]) := \lim_{n\to\infty} d(x_n, y_n)$.
See Topology lecture.

Construction 2 Let $BC(M, \mathbb{R}) := \{f : M \to \mathbb{R} \mid \ f \text{ is continuous and bounded}\}$ and $\|f\| = \sup_{x\in M} |f(x)|$.
Fact: $BC(M, \mathbb{R})$ is a banach space.
Fix a point $x^* \in M$. For every $x \in M$ define the function $f_x : M \to \mathbb{R}$ by $f_x(y) := d(x, y) - d(x^*, y)$

(a) f_x is continuous.
(b) f_x is bounded because $|d(x, y) - d(x^*, y)| \leq d(x, x^*)$.
(c) $i : M \to BC(M, \mathbb{R}) : x \to f_x$ is an isometric embedding.
For all $x, x' \in M$ we have:

$$\begin{aligned} d(f_x, f_{x'}) &= \|f_x - f_{x'}\| && (6)\\ &= \sup_{y\in M} |f_x(y) - f_{x'}(y)| && (7)\\ &= \sup_{y\in M} |d(x, y) - d(x', y)| && (8)\\ &= d(x, x') \text{ (set } y = x') && (9) \end{aligned}$$

Now define $M^* := \text{closure}(\{f_x \mid x \in M\})$ in $B(M, \mathbb{R})$, $d(f, g) := \|f - g\|$ and $\iota(x) := f_x$.

□

Exercise: The completion of a normed vector space is a Banachspace. Hints: Let $(X, \|\cdot\|)$ be a normed vector space and define the metric on X by $d(x,y) := \|x-y\| \quad \forall x,y \in X$

1. Let (X', d', ι) be a completion of (X, d). Then there is a unique pair of vector space structure and norm on X' such that
   (i) $X \to X'$ is linear
   (ii) $\|\iota(x)\|' = \|x\| \quad \forall x \in X$
   (iii) $d'(x', y') = \|x' - y'\|' \quad \forall x', y' \in X'$
2. If $(X_1, \|\cdot\|_1, \iota_1)$ and $(X_2, \|\cdot\|_2, \iota_2)$ are two completions of $(X, \|\cdot\|)$ then the isometry $\phi : X_1 \to X_2$ in Theorem 9(ii) is linear.

Example: $X = C([0,1]) \ni f$

$$\|f\| := \left(\int_0^1 |f(t)|^p \, \mathrm{d}t \right)^{\frac{1}{p}} \qquad 1 \le p < \infty$$

The completion of $(X, \|\cdot\|_p)$ is $L^p([0,1])$ with respect to the Lebesgue measure on $[0,1]$.
More general: Replace $[0,1]$ by a locally compact Hausdorff space M and Lebesgue by a Radonmeasure.

Exercise: $(X, \|\cdot\|)$ normed vector space.
The functions $X \to \mathbb{R} : x \to \|x\|$, $X \times X \to X : (x,y) \to x+y$ and $\mathbb{R} \times X \to X : (\lambda, x) \mapsto \lambda x$ are continuous.
Let M be any set. Then $B(M,X) = \{f : M \to X \mid \text{ f is bounded}\}$
with $\|f\| := \sup_{x \in M} |f(x)| < \infty$ is a normed vector space.
X complete $\Rightarrow B(M,X)$ is complete.
(M,d) metric space $\Rightarrow BC(M,X) := \{f : M \to X \mid \text{ f is continuous and bounded}\}$ is a closed subspace of $B(M,X)$. Recapitulation:

1. Any two norms on a finite dim. vector space are equivalent (Lemma 1).
2. Every finite dimensional normed vector space is complete (Lemma 2).
3. Every finite dimensional subspace of a normed vector space is closed (Lemma 3).
4. A normed vector space $(X, \|\cdot\|)$ is finite dimensional if and only if the unit ball $B := \{x \in X \mid \|x\| \le 1\}$ (resp. the unit sphere $S := \{x \in X \mid \|x\| = 1\}$) is compact (Theorem 1).
5. $(X, \|\cdot\|), (Y, \|\cdot\|)$ normed vector space $A : X \to Y$ linear operator.
   X is finite dimensional $\Leftrightarrow$ Every linear operator $A : X \to Y$ is bounded (Lemma 6).

Definition: $\mathcal{L}(X,Y) := \{A : X \to Y \mid A \text{ is linear and bounded}\}$

$$\|A\| := \sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Ax\|_Y}{\|x\|_X}$$

Theorem 9: Let X, Y, Z be normed vector spaces.

(i) $\mathcal{L}(X,Y)$ is a normed vector space.

(ii) Y complete $\Rightarrow \mathcal{L}(X,Y)$ is complete.

(iii) $A \in \mathcal{L}(X,Y)$, $B \in \mathcal{L}(Y,Z) \Rightarrow BA \in \mathcal{L}(X,Z)$ and
$\|BA\| \le \|B\| \, \|A\| \; (*)$
Moreover the map $\mathcal{L}(X,Y) \times \mathcal{L}(Y,Z) \to \mathcal{L}(X,Z), \quad (A,B) \mapsto BA$ is continuous.

Proof:

(i) Verify axioms.

(iii) $\|BAx\|_Z \le \|B\|\,\|Ax\|_Y \le \|B\|\,\|A\|\,\|x\|_X$
This implies $(*)$. Moreover:
$\|B_2A_2 - B_1A_1\| \overset{(*)}{\le} \|B_2 - B_1\|\,\|A_2\| + \|B_1\|\,\|A_2 - A_1\|$
Now do $\varepsilon, \delta \dots$.

(ii) Assume Y is complete. Let $(A_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{L}(X,Y)$.

$$\|A_n x - A_m x\|_Y \le \|A_n - A_m\|\|x\|_X$$

This shows: For each $x \in X$ the sequence $(A_n x)_{n\in\mathbb{N}}$ is a Cauchy sequence in Y.

Because Y is complete the sequence $(A_n x)_{n\in\mathbb{N}}$ converges for every $x \in X$.

Define $A : X \to Y$ by $Ax := \lim_{n\to\infty} A_n x \quad \Rightarrow \quad A$ is linear.

Claim : A is bounded and A_n converges to A in $\mathcal{L}(X,Y)$.

Proof : Let $\varepsilon > 0$. There $\exists n_0 \in \mathbb{N}$ such that

$$\forall m, n \in \mathbb{N} : n, m \ge n_0 \Rightarrow \|A_n - A_m\| < \varepsilon$$

Hence for $n, m \ge n_0$:

$$\begin{aligned} \|A_n x - A_m x\|_Y &= \|A_n x - \lim_{n\to\infty} A_m x\|_Y \\ &= \lim_{n\to\infty} \|A_n x - A_m x\|_Y \\ &\le \limsup_{n\to\infty} \|A_n - A_m\|\|x\|_X \\ &\le \varepsilon\|x\|_X \end{aligned}$$

So

$$\begin{aligned} \|Ax\|_Y &\le \|Ax - A_n x\|_Y + \|A_n x\|_Y \\ &\le \varepsilon\|x\|_X + \|A_n\|\|x\|_X \\ &= (\varepsilon + \|A_n\|)\|x\|_X \end{aligned}$$

So A is bounded and $\|A\| \le \|A_n\| + \varepsilon$, moreover

$$\|A_n - A\| := \sup_{x\neq 0} \frac{\|A_n x - Ax\|_Y}{\|x\|_X} \le \varepsilon$$

□

Example: $Y = \mathbb{R}$
$X^* := \mathcal{L}(X,\mathbb{R})$ is a Banach space with the norm $\|A\| := \sup_{x\neq 0} \frac{|Ax|}{\|x\|_X}$
X^* is called the *dual space* of X.

Example: $(L^p)^* = L^q$ where $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. See the Measure and Integration lecture for the proof.

Theorem 10: Let X be a normed vector space, Y Banach space.
Let $(A_i)_{i\in\mathbb{N}}$ be a sequence of bounded linear operators such that $\sum_{i=1}^{\infty} \|A_i\| < \infty$
Then the sequence $S_n := \sum_{i=1}^{n} A_i$ converges in $\mathcal{L}(X,Y)$. The limit is denoted by $\sum_{i=1}^{\infty} A_i = \lim_{n\to\infty} \sum_{i=1}^{n} A_i$

Proof: $s_n := \sum_{i=1}^{\infty} \|A_i\| < \infty$ converges in $\mathbb{R}$.

$$\|S_n - S_m\| = \| \sum_{i=m+1}^{n} A_i\| \leq \sum_{i=m+1}^{n} \|A_i\| = s_n - s_m$$

$\Rightarrow S_n$ is a Cauchy Sequence in $\mathcal{L}(X, Y)$.
$\overset{\text{Thm 9}}{\Rightarrow} S_n$ converges. □

Example: $X = Y$ Banach space, $\mathcal{L}(X) := \mathcal{L}(X, X)$
Suppose $f(z) = \sum_{i=0}^{\infty} a_i z^i$ is a convergent power series with convergence radius

$$R := \frac{1}{\limsup\limits_{n\to\infty} |a_n|^{\frac{1}{n}}} > 0$$

Let $A \in \mathcal{L}(X)$ be a bounded linear operator with $\|A\| < R$. Then $\sum_{i=0}^{\infty} |a_i| \|A^i\| \leq \sum_{i=0}^{\infty} |a_i| \|A\|^i < \infty \overset{\text{Thm 10}}{\Rightarrow}$ the limit $f(A) := \sum_{i=0}^{\infty} a_i A^i = \lim_{n\to\infty} \sum_{i=0}^{n} a_i A^i$ exists.

Remark: Works also with $a_i \in \mathbb{C}$ if X is a complex Banach space.

Example: $f(z) = \sum_{i=0}^{\infty} z^i = \frac{1}{1-z}$

Corollary: $\|A\| < 1 \Rightarrow 1 - A$ is bijective with inverse

$$(1 - A)^{-1} = \sum_{i=0}^{\infty} A^i \in \mathcal{L}(X)$$

Proof: $S_n := 1 + A + A^2 + ... + A^n$
$\|A\| < 1 \overset{\text{Thm 10}}{\Rightarrow}$ The sequence S_n converges.
$S_\infty := \lim_{n\to\infty} S_n = \sum_{i=0}^{\infty} A^i$

$$\begin{aligned} S_n(1 - A) &= (1 - A)S_n \\ &= 1 + A + A^2 + \ldots + A^n - A - \ldots - A^{n+1} \\ &= 1 - A^{n+1} \longrightarrow 1 \end{aligned}$$

$\Rightarrow S_\infty(1 - A) = (1 - A)S_\infty = 1$ □

Theorem 11: X Banach space, $A \in \mathcal{L}(X) \Rightarrow$

(i) The limit $r_A := \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} = \inf_{n>0} \|A^n\|^{\frac{1}{n}} \leq \|A\|$ exists.
(It's called the *Spectral radius* of A.)

(ii) $r_A < 1 \Rightarrow \sum_{i=0}^{\infty} \|A^i\| < \infty$ and $\sum_{i=0}^{\infty} A^i = (1 - A)^{-1}$

Proof:

(i) Let $\alpha := \inf_{n>0} \|A\|^{\frac{1}{n}}$. Let $\varepsilon > 0$.
$\exists m \in \mathbb{N} \quad \|A^m\|^{\frac{1}{m}} < \alpha + \varepsilon$
$c := \max\{1, \|A\|, \ldots, \|A^{m-1}\|\}$
Write an integer $n > 0$ in the form $n = km + l \quad k \in \mathbb{N}_0,\ l \in \{0, 1, \ldots m-1\}$

$$\begin{aligned} \|A^n\|^{\frac{1}{n}} &= \|(A^m)^k A^l\|^{\frac{1}{n}} && (10) \\ &\leq \|A^m\|^{\frac{k}{n}} \|A^l\|^{\frac{1}{n}} && (11) \\ &\leq c^{\frac{1}{n}} (\alpha + \varepsilon)^{\frac{km}{n}} && (12) \\ &= c^{\frac{1}{n}} (\alpha + \varepsilon)^{1-\frac{l}{n}} && (13) \\ &\xrightarrow{n\to\infty} \alpha + \varepsilon && (14) \end{aligned}$$

$\Rightarrow \exists n_0 \in \mathbb{N}\ \forall n \geq n_0 : \|A^n\|^{\frac{1}{n}} \leq \alpha + 2\varepsilon$

$\Rightarrow \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} = \alpha$

(ii) $r_A < 1$. Choose $\alpha \in \mathbb{R} : r_A < \alpha < 1 \Rightarrow \exists n_0 \in \mathbb{N}\ \forall n \geq n_0 : \|A^n\|^{\frac{1}{n}} < \alpha \Rightarrow \|A^n\| < \alpha^n \Rightarrow \sum_{i=0}^{\infty} \|A^i\| < \infty \overset{\text{Thm 10}}{\Rightarrow}$ (ii)

□

1.7 Quotient spaces

Definition: Let X normed vector space, $Y \subset X$ closed subspace.
$x + Y := \{x + y \mid y \in Y\} \subset X$
$x + Y = x' + Y \Leftrightarrow x' - x \in Y \Leftrightarrow: x \sim x'$
$X/Y := X/_\sim = \{x + Y \mid x \in X\}$

Notation: $[x] := x + Y$ for the equivalence class of $x \in X$

Remark: X/Y is a normed vector space with

$$\|[x]\|_{X/Y} := \inf_{y\in Y} \|x + y\|_X$$

Exercise:

1. $\|\cdot\|_{X/Y}$ is a norm
2. X Banach space, Y closed subspace $\Rightarrow X/Y$ is a Banach space.

2 Functional Analysis

2.1 Basics

Theorem 1 (Uniform Boundedness): Let X be a Banach space, Y a normed vector space and I an arbitrary set. Let $A_i \in \mathcal{L}(X,Y)$ for $i \in I$ and assume $\forall x \in X : \sup_{i\in I} \|A_i(x)\| < \infty$.
The conclusion says that $\exists c > 0$ such that $\sup_{i\in I} \|A_i x\| \leq c \;\forall x \in X$ with $\|x\| \leq 1$.

Lemma 1: (M,d) complete metric space $M \neq \varnothing$, I any set. $f_i : M \to \mathbb{R}$ continuous for $i \in I$. Assume $\sup_{i\in I} |f_i(x)| < \infty \;\forall x \in M \Rightarrow \exists$ open ball $B \subset M$ such that

$$\sup_{x\in B} \sup_{i\in I} |f_i(x)| < \infty$$

Proof of Lemma 1: Denote

$$A_{n,i} := \{x \in M \mid |f_i(x)| \leq n\} \text{ for } n \in \mathbb{N} \text{ and } i \in I.$$

$$A_n := \bigcap_{i\in I} A_{n,i} = \{x \in M \mid \sup_{i\in I} |f_i(x)| \leq n\}$$

$\Rightarrow \forall x \in M \;\exists n \in \mathbb{N}$ such that $x \in A_n$, i.e. $M = \bigcup_{n=1}^{\infty} A_n$.
Now $A_{n,i} = f_i^{-1}([-n,n])$ is closed. So A_n is closed. So $\exists n \in \mathbb{N}$ such that $\operatorname{int}(A_n) \neq \varnothing$

$$\Rightarrow \exists x_0 \in \operatorname{int}(A_n)$$

$\exists \varepsilon > 0$ such that

$$B_\varepsilon(x_0) = \{x \in M \mid d(x,x_0) < \varepsilon \subset A_n\}$$

□

Proof of Theorem 1: Set $M := X, f_i(x) := \|A_i x\|$, so $f_i : X \xrightarrow{A_i} Y \xrightarrow{\|\cdot\|} \mathbb{R}$.
So f_i is continuous for every $i \in I$.
$\sup_{i\in I} |f_i(x)| < \infty \;\forall x \in X \overset{\text{Lemma 1}}{\Longrightarrow} \exists$ ball $B = B_\varepsilon(x_0) \subset X$ with $x_0 \in X, \varepsilon > 0$ such that

$$c := \sup_{i\in I} \sup_{x\in B} \|A_i x\| < \infty$$

$\Rightarrow \forall i \in I \;\forall x \in X$ we have $\|x - x_0\| \leq \varepsilon \Rightarrow \|A_i x\| \leq c$.
Let $x \in X$ with $\|x\| = 1$.
Then $\|(x_0 + \varepsilon \cdot x) - x_0\| = \varepsilon$
Hence $\|A_i(x_0 + \varepsilon x)\| \leq c$ so

$$\|A_i x\| = \frac{1}{\varepsilon}\|A_i(x_0 + \varepsilon x) - A_i x_0\| \leq \frac{1}{\varepsilon}\|A_i(x_0+\varepsilon x)\| + \frac{1}{\varepsilon}\|A_i x_0\| \leq \frac{c+c}{\varepsilon}$$

□

Theorem 2 (Banach-Steinhaus): X Banach space, Y normed vector space $A_i \in \mathcal{L}(X,Y), i = 1,2,3,\ldots$

(i) Assume the sequence $(A_i x)_{i=1}^{\infty}$ converges in Y for every $x \in X$. Then:

- $\sup_{i\in\mathbb{N}} \|A_i\| < \infty$
- $\exists A \in \mathcal{L}(X,Y)$ such that $Ax = \lim_{i\to\infty} A_i x$, $\|A\| \leq \liminf_i \|A_i\|$

(ii) Assume Y is complete and

- $\sup_{i\in\mathbb{N}} \|A_i\| < \infty$
- $\exists$ dense subset $D \subset X$ such that $(A_i x)_{i=1}^{\infty}$ converges for every $x \in D$

Then $(A_i x)_i$ converges for all $x \in X$

Proof:

1. Since $(A_i x)_i$ converges we have
$$\sup_{i\in\mathbb{N}} \|A_i x\| < \infty \,\forall x \in X \quad \Rightarrow \quad \sup_{i\in\mathbb{N}} \|A_i\| < \infty$$
Define $A : X \to Y$ by $Ax := \lim_{i\to\infty} A_i x$. This operator is linear. Why is A bounded?
$$\|Ax\| = \lim_{i\to\infty} \|A_i x\| = \liminf_{i\to\infty} \|A_i x\| \le \underbrace{\liminf_{i\to\infty} \|A_i\|}_{<\infty} \|x\|$$

2. Let $x \in X$. Need to show that $(A_i x)_{i=1}^{\infty}$ is Cauchy. Let $\varepsilon > 0$.
Denote $c := \sup_{i\in\mathbb{N}} \|A_i\| < \infty$.
Choose $y \in D$ such that $\|x - y\| < \frac{\varepsilon}{4c}$.
Choose $n_0 \in \mathbb{N}$ so that $\forall i, j \ge n_0 : \|A_i - A_j\| < \frac{\varepsilon}{2}$
$$\begin{aligned} \Rightarrow \forall i, j \ge n_0 : \quad \|A_i x - A_j x\| &\le \|A_i x - A_i y\| + \|A_i y - A_j y\| + \|A_j y - A_j x\| \\ &< \|A_i\|\|x - y\| + \|A_i - A_j\|\|y\| + \|A_j\|\|x - y\| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} \\ &< \varepsilon \end{aligned}$$

□

Example 1: $l^\infty = \{\text{bounded sequences} \in \mathbb{R}\}, x \in l^\infty$ with $x = (x_1, x_2, x_3, \dots) = (x_i)_{i=i}^{\infty}$.
$$X := \{x = (x_i)_i \in l^\infty \mid \exists n \in \mathbb{N} \text{ such that } x_i = 0 \;\forall i \ge n\}$$
Define
$$A_n : X \to X \text{ by } A_n x = (x_1, 2x_2, 3x_3, \dots, nx_n, 0, \dots)$$
$\Rightarrow \lim_{n\to\infty} A_n x = Ax$ where $Ax = (x_1, 2x_2, 3x_3, \dots)$.
But $\|A_n\| = n \to \infty$. Completeness of the domain is missing here.
So the assumption that X is complete cannot be removed in Theorem 1 or Theorem 2.

Example 2: X Banach space, Y, Z normed vector spaces and $B : X \times Y \to Z$ bilinear. Equivalent are:

(i) B is continuous

(ii) The functions $X \to Z : x \mapsto B(x, y)$ is continuous $\forall y \in Y$ and
the function $Y \to Z : y \mapsto B(x, y)$ is continuous $\forall x \in X$.

(iii) $\exists c > 0 \;\forall x \in X \;\forall y \in Y : \|B(x, y)\| \le c\|x\| \cdot \|y\|$

This is exercise 2 on Sheet 4.

Theorem 3 (Open Mapping Theorem): X, Y Banach spaces. $A \in \mathcal{L}(X, Y)$ surjective $\Rightarrow A$ is open, i.e. if $U \in X$ is an open set then $AU \subset Y$ is open.

Corollary (Inverse Operator Theorem): X, Y Banach spaces, $A \in \mathcal{L}(X, Y)$ bijective $\Rightarrow A^{-1}$ is bounded, i.e. $A^{-1} \in \mathcal{L}(X, Y)$.

Proof: A open $\Leftrightarrow A^{-1}$ continuous $\Leftrightarrow A^{-1}$ bounded. □

Example 3: X as in example 1; X is not complete. Define $B : X \to X$ by $Bx := (x, \frac{1}{2}x_2, \frac{1}{3}x_3, \ldots)$. Then $\|Bx\| \leq \|x\|$, so B bounded. $B^{-1} = A$, as in example 1, is not bounded.

Lemma 2: X, Y Banach spaces, $A \in \mathcal{L}(X, Y)$ surjective
$\Rightarrow \exists \delta > 0$ such that $\{y \in Y \mid \|y\| < \delta\} \subset \{Ax \mid x \in X, \|x\| < 1\}$ (*)

Remark: $(*)$ means $\forall y \in Y \exists x \in X$ such that $Ax = y$ and $\|x\| \leq \frac{1}{\delta}\|y\|$ $(**)$

Exercise: $(*) \Leftrightarrow (**)$
Use $(*)$ to prove the Corollary.

Proof (Lemma 2 $\Rightarrow$ Theorem 3): Let $U \subset X$ be open, and $y_0 \in AU \Rightarrow \exists x_0 \in U$ such that $y_0 = Ax_0 \overset{(U \text{ open})}{\Longrightarrow} \exists \varepsilon > 0$ such that $B_\varepsilon(x_0) \subset U$.
Claim: $B_{\delta\epsilon}(y_0) \subset AU$. Let

$$y \in B_{\delta\epsilon}(y_0) \quad \Rightarrow \quad \left\| \frac{y - y_0}{\varepsilon} \right\| < \delta$$

$\exists x \in X$ such that $\|x\| < 1$ and $Ax = \frac{y-y_0}{\varepsilon} \quad \Rightarrow \quad x_0 + \varepsilon x \in B_\varepsilon(x_0) \subset U$
$A(x_0 + \varepsilon x) = y_0 + \varepsilon Ax = y \Rightarrow y \in AU$. □

Proof of Lemma 2:

Step 1 $\exists r > 0$ so that

$$\{y \in Y \mid \|y\| < r\} \subset \overline{\{Ax \mid x \in X, \|x\| < 1\}}$$

Proof of Step 1:
Let

$$B := \{x \in X \mid \|x\| < \frac{1}{2}\}$$

$$C := AB = \{Ax \mid x \in X, \|x\| < \frac{1}{2}\}$$

Note that

1. $nC = \{ny \mid y \in C\} = \{Ax \mid x \in X, \|x\| < \frac{1}{2}\}$
2. $y, y' \in C \Rightarrow y - y' \in 2C$
3. $y, y' \in \overline{C} \Rightarrow y - y' \in \overline{2C}$
4. $\overline{nC} = n\overline{C}$

$$\begin{aligned} X &= \bigcup_{n=1}^{\infty} nB \\ \overset{A \text{ onto}}{\Rightarrow} Y &= AX \\ &= \bigcup_{n=1}^{\infty} nAB \\ &= \bigcup_{n=1}^{\infty} nC \\ &= \bigcup_{n=1}^{\infty} \overline{nC} \end{aligned}$$

By Baire: $\exists n \in \mathbb{N} : (\overline{nC})^\circ \neq \varnothing \Rightarrow (\overline{C})^\circ \neq \varnothing$

$$\exists y_0 \in Y \; \exists r > 0 : B_r(y_0) \subset \overline{C}$$

So if $y \in Y$ and $\|y\| < r$ then $y_0 + y \in \overline{C}$.
Thus $\forall y \in Y$ with $\|y\| < r$ we have $y = \underbrace{y_0 + y}_{\in \overline{C}} - \underbrace{y_0}_{\in \overline{C}} \in 2\overline{C}$

$$\Rightarrow \{y \in Y \mid \|y\| < r\} \subset \overline{2C}$$

Step 2 $(*)$ holds with $\delta = \frac{r}{2}$. Proof:
Let $y \in Y$ with $\|y\| < \frac{r}{2}$.
To Show: $\exists x \in X$ so that $Ax = y$ and $\|x\| < 1$.
Denote

$$B_k := \{x \in X \mid \|x\| < \frac{1}{2^k}\} \quad k = 1, 2, 3, \dots$$

$(**)$ Then, by Step 1,

$$\left\{y \in Y \mid \|y\| < \frac{r}{2^k}\right\} \subset \overline{AB_k} \quad k = 1, 2, 3, \dots$$

Since $y \in Y$ and $\|y\| < \frac{r}{2}$, by $(**)$ with $k = 1$:

$$\exists x_1 \in X : \|x_1\| < \frac{1}{2}, \|y - Ax_1\| < \frac{r}{4}$$

and by $(**)$ with $k = 2$

$$\exists x_2 \in X : \|x_2\| < \frac{1}{4}, \|y - Ax_1 - Ax_2\| < \frac{r}{8}$$

So, by induction, using $(**)$, there is a sequence $(x_k)_k \in X$ so that

$$\|x_k\| < \frac{1}{2^k} \quad \|y - Ax_1 - \dots - Ax_k\| < \frac{r}{2^{k+1}}$$

We have

$$\sum_{k=1}^{\infty} \|x_k\| < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$$

By Chapter 1, the limit

$$x := \lim_{n \to \infty} \sum_{i=1}^{n} x_i = \sum_{i=1}^{\infty} x_i$$

exists and $\|x\| < 1$. Since

$$\left\| y - A \sum_{i=1}^{k} x_i \right\| < \frac{r}{2^k} \to \infty$$

we have proved Lemma 2. □

2.2 Product spaces

Example 4: Let X be a Banach space and X_1, X_2 both closed linear subspaces. Assume $X = X_1 + X_2$ and $X_1 \cap X_2 = \{0\}$; these subspaces are called *transverse subspaces*. We say X is the *direct sum* of X_1 and X_2 and write

$$X = X_1 \oplus X_2$$

Every vector in X can be written as sum of a vector in X_1 and one in X_2 in a unique way (Linear Algebra).
Define $A : X_1 \times X_2 \to X$ by $A(x_1, x_2) := x_1 + x_2$. If X, Y are normed vector spaces, then $X \times Y := \{(x, y) \mid x \in X, y \in Y\}$ is again a normed vector space with

$$\|(x, y)\| := \|x\| + \|y\|$$

for $(x, y) \in X \times Y$. Other possibilities are

$$\|(x, y)\|_\infty := \max\{\|x\|, \|y\|\}$$

$$\|(x, y)\|_p := (\|x\|^p + \|y\|^p)^{\frac{1}{p}}, \quad 1 \le p \le \infty$$

- All these norms are equivalent.
- X, Y Banach spaces $\Rightarrow X \times Y$ is a Banach space for any of these norms.

These are exercises.
Return to Example 4: X_1, X_2 are closed subsets of a complete space, hence complete. So $X_1 \times X_2$ is complete by the exercise above and A is a operator between Banach spaces.

- A is bounded and linear, because

$$\|A(x_1, x_2)\| = \|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$$

- A is surjective because $X = X_1 + X_2$
- A is injective because $X_1 \cap X_2 = \varnothing$.

By the open mapping theorem (Theorem 3) A^{-1} is bounded

$$\Rightarrow \exists c > 0 : \ \forall x_1 \in X_1 \ \forall x_2 \in X_2 : \|x_1\| + \|x_2\| \leq c\|x_1 + x_2\|$$

So the projections $\pi_1 : X \to X_1, \ \pi_2 : X \to X_2$ are bounded.

Example 5: $X = Y = C([0,1])$ with supnorm.
$Ax = \dot{x}$ A is only defined on a subset of X namely on

$$D := \{x \in X \mid x \text{ is continuously differentiable}\} =: C^1([0,1])$$

$D \subset X$, $A : D \to Y$.

Definition: Let X, Y be Banach spaces.
$D \subset X$ linear subspace, a linear operator $A : D \to Y$ is called *closed* if its graph $\Gamma = \text{graph}(A) := \{(x, Ax) \mid x \in D\}$ is a closed subspace of $X \times Y$, i.e. for any sequence $(x_n)_{n\in\mathbb{N}}$ in D and $(x, y) \in X \times Y$ we have:

$$\left.\begin{array}{r} x_n \to x \\ Ax_n \to y \end{array}\right\} \quad \Rightarrow \quad x \in D \text{ and } y = Ax$$

Example 5:
$x_n \in C^1([0,1])$ $\quad \lim_{n\to\infty} \sup_{0\leq t\leq 1} |x_n(t) - x(t)| = 0$ $\quad$ and
$x, y \in C([0,1])$ $\quad \lim_{n\to\infty} \sup_{0\leq t\leq 1} |\dot{x}_n(t) - y(t)| = 0$
$\Rightarrow x \in C^1$ and $\dot{x} = y$, so the operator in Example 5 is closed.

Exercise: The *graph norm* on D is defined by $\|x\|_A := \|x\|_X + \|Ax\|_Y$
Prove that $(D, \|\cdot\|_A)$ is complete if and only if A is closed.

Example 5: The graph norm on $C^1([0,1])$ is

$$\|x\|_A = \sup_{0\leq t\leq 1} |x(t)| + \sup_{0\leq t\leq 1} |\dot{x}(t)|$$

The standard norm in C^1.

What if $D = X$?

Theorem (Closed Graph Theorem): X, Y Banach spaces
$A : X \to Y$ linear operator. Equivalent are:

(i) A is bounded

(ii) A has a closed graph

Proof:

(i) ⇒ (ii) $X \ni x_n \to x$ and $Y \ni Ax_n \to y$
$\overset{A \text{ continuous}}{\Longrightarrow} Ax_n \to Ax$
$\overset{\text{Uniqueness of the limit}}{\Longrightarrow} Ax = y$

(ii) ⇒ (i) $\Gamma := \text{graph}(A) \subset X \times Y$, Γ is a Banach space.
Define $\pi : \Gamma \to X$ by $\pi(x, y) = x$
$\Rightarrow \pi$ is a bounded linear operator with norm $= 1$.
π is injective.
π is surjective (because $D \subset X$).
$\overset{\text{Thm 3}}{\Longrightarrow} \pi^{-1} : X \to \Gamma$ is bounded $\Rightarrow \exists c > 0$ such that $\|\pi^{-1}(x)\|_A \leq c\|x\|_X$
But $\|\pi^{-1}(x)\|_A = \|(x, Ax)\|_A = \|x\|_X + \|Ax\|_Y \Rightarrow \|Ax\|_Y \leq c\|x\|_X \ \forall x \in X$. □

Example 6 (Hellinger-Toeplitz-Theorem): H Hilbert space
$A : H \to H$ linear operator which is symmetric,
i.e. $\langle x, Ay\rangle = \langle Ax, y\rangle \ \forall x, y \in H \Rightarrow A$ is bounded.

Proof: To show: A is closed.
$H \ni x_n$ sequence. Assume $x_n \to x \in H$,
$Ax_n \to y \in H$.
To show: $Ax = y$.
$\langle y, z\rangle = \lim_{n\to\infty}\langle Ax_n, z\rangle = \lim_{n\to\infty}\langle x_n, Az\rangle = \langle x, Az\rangle \quad \forall z \in H$
$\Rightarrow \langle y - Ax, z\rangle = 0 \quad \forall z \in H \overset{z=y-Ax}{\Rightarrow} \|y - Ax\| = 0 \Rightarrow y = Ax$ □

Example 5: $A : D \to Y$ is closed but not bounded:
$x_n(t) = t^n \quad \|x_n\| = \sup_{t\in[0,1]} |x_n(t)| = 1$
$\|Ax_n\| = \|\dot{x}_n\| = \sup_{t\in[0,1]} |\dot{x}_n(t)| = n \to \infty$

Definition: $A : D \subset X \to Y$ is called *closable*, if there is an operator $A' : D' \to Y \quad D \subset D'$ such that A' is closed and $A'|_D = A$.

Remark: Let $\Gamma := \{(x, Ax) \mid x \in D\} := \text{graph}(A)$
A is closable
$\Leftrightarrow \overline{\Gamma}$ is the graph of a closed operator
$\Leftrightarrow \pi : \overline{\Gamma} \to X$ is injective
$\Leftrightarrow D \ni x_n \to 0$, $Ax_n \to y$ implies $y = 0$

Example 7: Let $X = L^2([0, 1])$, $D = C([0, 1])$, $Y = \mathbb{R}$. Let $A : D \to Y, x \mapsto x(0)$ is not closable.

Example 8: $X = L^2(\mathbb{R}) \quad D = \{x \in L^2(\mathbb{R}) \mid \exists c > 0 \quad \forall |t| > c : x(t) = 0\}$
$Y = \mathbb{R} \quad Ax = \int_{-\infty}^{\infty} x(t)dt$

$$x_n(t) := \begin{cases} \frac{1}{n} & |t| \leq n \\ 0 & |t| > n \end{cases}$$

$\|x_n\|_{L^2}^2 = \frac{2}{n} \quad Ax_n = 2$

Example 9: "Every differential operator is closable."
Let $\Omega \subset \mathbb{R}^n$ be an open subset.

$$C_0^\infty = \{f : \Omega \to \mathbb{R} \mid f \text{ is smooth, } \text{supp}(f) \text{ is compact}\}$$

with

$$\begin{aligned}\operatorname{supp}(f) &:= \{x \in \Omega \mid \exists x_n \in \Omega \text{ such that } f(x_n) \neq 0, x_n \to x\} \\ &= \operatorname{cl}_\Omega(\{x \in \Omega \mid f(x) \neq 0\})\end{aligned}$$

$C_0^\infty(\Omega) \subset L^p(\Omega)$. We know:

1. $C_c(\Omega) = \{f : \Omega \to \mathbb{R} \mid f \text{ continuous}, f \text{ has cpct support}\}$ is dense in $L^p(\Omega)$.

2. $C_0^\infty(\Omega)$ is dense in $C_c(\Omega)$, i.e.

   $$\forall f \in C_c(\Omega)\ \exists K \in \Omega, K \text{ compact } \exists f_n \in C_0^\infty(\Omega)$$

   such that

   $$\operatorname{supp}(f_n) \subset K, \quad \lim_{n\to\infty} \sup_{x \in \Omega} |f_n(x) - f(x)| = 0$$

Let $X = Y = L^p(\Omega)$.
Let $D := C_0^\infty(\Omega) \subset X$.
Define $A : D \to Y$ by $(Af)(x) = \sum_{|\alpha| \le m} a_\alpha(x) \partial^\alpha f(x)$
where $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, $\alpha_i \le 0$ and $|\alpha| := \alpha_1 + \ldots + \alpha_n$.

$$\partial^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

Let $a_\alpha : \Omega \to \mathbb{R}$ be smooth.

Claim A is closable.

Proof: Integration by parts: $f, g, \in C_0^\infty(\Omega)$

$$\begin{aligned}\Rightarrow \int_\Omega g(Af)\,\mathrm{d}\mu &= \sum_\alpha \int_\Omega g \cdot a_\alpha \delta^\alpha f \,\mathrm{d}\mu \\ &= \sum_\alpha (-1)^{|\alpha|} \int_\Omega \delta^\alpha(a_\alpha g) \cdot f \,\mathrm{d}\mu \\ &= \int_\Omega \Big(\sum (-1)^{|\alpha|} \delta^\alpha(a_\alpha g)\Big) \cdot f \,\mathrm{d}\mu \\ &= \int_\Omega (Bg) \cdot f \,\mathrm{d}\mu\end{aligned}$$

Let $(0, g) \in \overline{\operatorname{Graph}(A)} \subset L^p(\Omega) \times L^p(\Omega)$
To show: $g = 0$
$\exists f_k \in C_0^\infty(\Omega)$ sequence such that $(f_k, Af_k) \xrightarrow{L^p \times L^p} (0, g)$, i.e.

$$\lim_{k\to\infty} \|f_k\|_{L^p} = 0 = \lim_{k\to\infty} \|Af_k - g\|_{L^p}$$

$\Rightarrow \forall \phi \in C_0^\infty(\Omega)$:

$$\begin{aligned}\int_\Omega \phi g \,\mathrm{d}\mu &= \lim_{k\to\infty} \int_\Omega \phi(Af_k)\,\mathrm{d}\mu \\ &= \lim_{k\to\infty} \int_\Omega (B\phi) \cdot f_k \,\mathrm{d}\mu \\ &= 0\end{aligned}$$

because

$$\left|\int_\Omega (B\phi) f_k \,\mathrm{d}\mu\right| \le \underbrace{\|B\phi\|_{L^q}}_{<\infty}\underbrace{\|f_k\|_{L^p}}_{\to 0}$$

where $\frac{1}{p}+\frac{1}{q}=1$ for $1<p<\infty$.
So

$$\int_\Omega \phi g \,\mathrm{d}\mu = 0 \quad \forall \phi \in \underbrace{C_0^\infty(\Omega)}_{\text{dense in } L^q(\Omega)}$$

$$\Rightarrow \quad \int_\Omega \phi g \,\mathrm{d}\mu = 0 \quad \forall \phi \in L^q(\Omega) \quad \Rightarrow \quad g = 0 \text{ almost everywhere}$$

$$\Rightarrow \quad \int_\Omega \phi g \,\mathrm{d}\mu = \int_\Omega |g|^p \,\mathrm{d}\mu$$

$\phi := (\mathrm{sign}(g))|g|^{p-1} \in L^q(\Omega)$ □

2.3 Extension of bounded linear functionals

Theorem 5 (Hahn-Banach): X normed vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. $Y \subset X$ linear subspace $\phi : Y \to \mathbb{F}$ linear and $c > 0$ such that $|\phi(y)| \le c\|y\|\ \forall y \in Y$ $\Rightarrow \exists \Phi : X \to \mathbb{F}$ linear such that

1. $\Phi|_Y = \phi$
2. $|\Phi(x)| \le c\|x\|\ \forall x \in X$.

Question: If we replace the target $\mathbb{F}$ by another Banach space Z over $\mathbb{F}$, i.e. $\phi : Y \to Z$ bounded linear operator. $\exists ?\Phi : X \to Z$ bounded linear operator, $\Phi|_Y = \phi$.
Answer: No!

Example: $X = l^\infty$, $Y := c_0 =: Z$, $\mathbb{F} = \mathbb{R}$, $\phi = \mathrm{id} : Y \to Z$ does not extend!

Lemma 3: Let X, Y, ϕ, c as in Theorem 5 with $\mathbb{F} = \mathbb{R}$.
Let $x_0 \in X \setminus Y$ and denote

$$Z := Y \oplus \mathbb{R}x_0 = \{y + \lambda x_0 \mid y \in Y, \lambda \in \mathbb{R}\}$$

Then $\exists \psi : Z \to \mathbb{R}$ linear so that

a. $\psi|_Y = \phi$

b. $|\psi(x)| \le c\|x\| \forall x \in Z$

Proof: Need to find a number $a \in \mathbb{R}$ so as to define

$$\psi(x_0) := a \tag{1}$$

Then

$$\psi(y + \lambda x_0) = \phi(y) + \lambda a \ \forall y \in Y \text{ and } \lambda \in \mathbb{R} \tag{2}$$

ψ is well-defined by (2), because $x_0 \notin Y$. Moreover $\psi|_Y = \phi$. To show: a can be chosen such that

$$|\phi(y) + \lambda a| \le c\|y + \lambda x_0\| \ \forall y \in Y\ \forall \lambda \in \mathbb{R} \tag{3}$$

(3) is equivalent to

$$|\phi(y) + a| \le c\|y + x_0\| \ \forall y \in Y \tag{4}$$

(3) $\Rightarrow$ (4) is obvious.
(4) $\Rightarrow$ (3) ok for $\lambda = 0$.

$$\lambda \neq 0 : |\phi(y) + \lambda a| = |\lambda| \cdot |\phi\left(\frac{y}{\lambda}\right) + a| \overset{(4)}{\leq} c|\lambda| \|\frac{y}{\lambda} + x_0\| = c\|y + \lambda x_0\|$$

(4) is equivalent to

$$-c\|y + x_0\| \leq \phi(y) + a \leq c\|y + x_0\| \forall y \in Y$$

$$\Leftrightarrow -\phi(y) - c\|y + x_0\| \leq a \forall y \in Y a \leq c\|y + x_0\| - \phi(y) \forall y \in Y \Leftrightarrow$$
$$\phi(y') - c\|y' - x_0\| \leq y' \in Y a \leq c\|y + x_0\| - \phi(y) \forall y \in Y \tag{5}$$

Is there a real number $a \in \mathbb{R}$ such that (5) holds, i.e.

$$\sup_{y' \in Y} (\phi(y') - c\|y' - x_0\| \leq a \leq \inf_{y \in Y} (c\|y + x_0\| - \phi(y))$$

This is true iff

$$\phi(y') - c\|y' - x_0\| \overset{(6)}{\leq} c\|y + x_0\| - \phi(y) \forall y, y' \in Y \tag{6}$$

□

Proof: of (6)

$$\phi(y) + \phi(y') = \phi(y + y') \leq c\|y + y'\| = c\|y + x_0 + y' - x_0\| \leq c\|y + x_0\| + c\|y' - x_0\|$$

□

Definition: Let $\mathcal{P}$ be a set. A *partial order* on $\mathcal{P}$ is a relation $\leq$ (i.e. a subset of $\mathcal{P} \times \mathcal{P}$, we write $a \leq b$ instead of $(a, b) \in \leq$.)
That satisfies:

- $\leq$ is reflective, i.e. $a \leq a \forall a \in \mathcal{P}$
- $\leq$ is transitive, i.e. $\forall a, b, c \in \mathcal{P}$ we have $a \leq b, b \leq c \Rightarrow a \leq c$
- $\leq$ is anti-symmetric, i.e. $\forall a, b \in \mathcal{P}$: $a \leq b, b \leq a \Rightarrow a = b$

Definition: $(\mathcal{P}, \leq)$ partially ordered set (POS). A subset $\mathcal{C} \subset \mathcal{P}$ is called a *chain* if it is totally ordered, i.e.

$$a, b \in \mathcal{C} \Rightarrow a \leq b \text{ or } b \leq a$$

Definition: $(\mathcal{P}, \leq)$ POS, $\mathcal{C} \subset \mathcal{P}, a \in \mathcal{P}$ a is called the *supremum* of $\mathcal{C}$ if

1. $\forall c \in \mathcal{C} : c \leq a$
2. $\forall b \in \mathcal{P} : (c \leq b \ \forall c \in \mathcal{C} \Rightarrow a \leq b)$

Definition: $(\mathcal{P}, \leq)$ POS, $a \in \mathcal{P}$. a is called a *maximal element* of $\mathcal{P}$ if $\forall b \in \mathcal{P}$ we have $a \leq b \Rightarrow b = a$

Lemma Zorn's Lemma: Let $(\mathcal{P}, \leq)$ be a POS such that every chain $\mathcal{C} \subset \mathcal{P}$ has a supremum. Let $a \in \mathcal{P} \Rightarrow$ There exists a maximal element $b \in \mathcal{P}$ such that $a \leq b$.

Remark: Zorn's Lemma is equivalent to the axiom of choice.

Proof of Theorem 5: Let X, Y, ϕ, c be given as in Theorem 5.
Define

$$\begin{aligned}\mathcal{P} \quad := \{(Z,\psi) \mid \quad & Z \subset X \text{ linear subspace}, \\ & Y \subset Z, \\ & \psi : Z \to \mathbb{R} \text{ linear}, \\ & \psi|_Y = \phi, \\ & |\psi(x)| \leq c\|x\| \forall x \in Z \qquad \}\end{aligned}$$

$$(Z,\psi) \leq (Z',\psi') :\Leftrightarrow Z \subset Z', \quad \psi'|_Z := \psi$$

Note that this is a partial order and every chain $\mathcal{C} \subset \mathcal{P}$ has a supremum $Z_0 := \bigcup_{(Z,\psi)\in\mathcal{C}} Z \ni x$.

- $(Z,\psi) \subset \mathcal{P}$ is maximal $\Leftrightarrow Z = X$ by Lemma 3
- $(Y,\psi) \in \mathcal{P} \Rightarrow \exists$ maximal element
- $(X,\Phi) \in \mathcal{P}$ such that $(Y,\phi) \leq (X,\Phi)$

□

Proof of Theorem 5 for $\mathbb{F} = \mathbb{C}$: X complex normed vector space, $Y \subset X$ complex linear subspace
$\phi : Y \to \mathbb{C}$ such that $|\phi(y)| \leq c\|y\| \quad \forall y \in Y$

Fact: $\phi : Y \to \mathbb{C}$ is complex linear
$\Leftrightarrow \phi$ is real lin. and $\phi(iy) = i\phi(y) \quad \forall y \in Y$ (1)
Write $\phi(y) = \phi_1(y) + i\phi_2(y)$ where $\phi_1(y), \phi_2(y) \in \mathbb{R}$
Then $\phi_1, \phi_2 : Y \to \mathbb{R}$ real linear and
$i\phi(y) = i\phi_1(y) - \phi_2(y) \quad \phi(iy) = \phi_1(iy) + i\phi_2(iy)$
$\Rightarrow \phi$ satisfies (1) $\Leftrightarrow \phi_2(y) = -\phi_1(iy) \quad \phi_1(y) = \phi_2(iy) \, \forall y \in Y$ (2)
We have $|\phi_1(y)| \leq |\phi(y)| \leq c\|y\| \overset{\text{Thm } 5, \mathbb{F}=\mathbb{R}}{\Rightarrow} \exists \Phi_1 : X \to \mathbb{R}$ $\mathbb{R}$-linear
$\Phi_1|_Y = \phi_1 \quad |\Phi_1(x)| \leq c\|x\| \quad \forall x \in X$
Define $\Phi : X \to \mathbb{C}$ by $\Phi(x) := \Phi_1(x) - i\Phi_1(ix) \Rightarrow$

1. Φ is complex linear
2. If $y \in Y$ then $\Phi(y) = \phi_1(y) - i\phi_1(iy) = \phi(y)$
3. Let $x \in X$. Suppose $\Phi(x) \neq 0$
   Then $\frac{\Phi(x)}{|\Phi(x)|} \in S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$
   so $\exists \theta \in \mathbb{R}$ such that $e^{i\theta} = \frac{\Phi(x)}{|\Phi(x)|}$
   $\Rightarrow \Phi(e^{-i\theta}x) = e^{-i\theta}\Phi(x) = |\Phi(x)| \in \mathbb{R}$
   $\Rightarrow |\Phi(x)| = |e^{-i\theta}\Phi(x)| = |\Phi(e^{-i\theta}x)| = |\Phi_1(e^{-i\theta}x)| \leq c\|e^{-i\theta}x\| = c\|x\|$

□

Definition: X real vector space. A function $p : X \to \mathbb{R}$ is called *seminorm* if

1. $p(x+y) \leq p(x) + p(y) \quad \forall x, y \in X$
2. $p(\lambda x) = \lambda p(x) \quad \forall \lambda > 0, \forall x \in X$

Theorem 5': X real vector space, $p : X \to \mathbb{R}$ seminorm
$Y \subset X$ linear subspace, $\phi : Y :\to \mathbb{R}$ linear
Then $\exists$ linear map $\Phi : X \to \mathbb{R}$ such that $\Phi|_Y = \phi$ and $\Phi(x) \leq p(x) \quad \forall x \in X$

Sketch of Proof Theorem 5': As in Lemma 3: $Z := Y \oplus \mathbb{R}x_0, \quad x_0 \notin Y$
$\Rightarrow \phi(y') + \phi(y) = \phi(y + y') \leq p(y + y') \leq p(y + x_0) + p(y' - x_0)$
$\Rightarrow \phi(y') - p(y' - x_0) \leq p(y + x_0) - \phi(y) \quad \forall y, y' \in Y$
$\Rightarrow \exists a \in \mathbb{R}$ such that $\phi(y) - p(y - x_0) \leq a \leq p(y + x_0) - \phi(y) \quad \forall y \in Y$
$\Rightarrow \phi(y) - p(y - \lambda x_0) \leq \lambda a \leq p(y + \lambda x_0) - \phi(y) \quad \forall y \in Y, \forall \lambda > 0$
$\Rightarrow \phi(y) - \lambda a \leq p(y - \lambda x_0)$ and $\phi(y) + \lambda a \leq p(y + \lambda x_0) \quad \forall y \in Y, \forall \lambda > 0$
$\Rightarrow \underbrace{\phi(y) + ta}_{:=\psi(y+tx_0)} \leq p(y + tx_0) \quad \forall t \in \mathbb{R}, \forall y \in Y$
i.e. $\psi(x_0) = a$ so $\exists \psi : Z \to \mathbb{R}\, \psi|_Y = \phi \quad \psi(x) \leq p(x) \quad \forall x \in Z$
For the remainder of the proof, argue as in Theorem 5 using Zorns lemma. □

Remark: Theorem 5' implies Theorem 5 with $\mathbb{F} = \mathbb{R}$, $p(x) = c\|x\|$

Notation: X normed vector space over $\mathbb{F} = \mathbb{R}, \mathbb{C} \quad X^* := \mathcal{L}(X, \mathbb{F})$
Each element of X^* is a bounded $\mathbb{F}$-linear functional $\phi : X \to \mathbb{F}$. We write:

- $x^* \in X^*$ instead of $\phi : X \to \mathbb{F}$
- $\langle x^*, x \rangle \in \mathbb{F}$ instead of $\phi(x)$

Remark: Theorem 5 says, if $Y \subset X$ is a linear subspace and $y^* \in Y^*$ then $\exists x^* \in X^*$ such that $x^*|_Y = y^*$ and $\|x^*\|_{X^*} = \|y^*\|_{Y^*}$

Definition: $Y \subset X$ linear subspace of a normed vector space X.
The *annihilator of* Y is the (closed) subspace $Y^\perp \subset X^*$ defined by
$Y^\perp := \{x^* \in X^* \mid \langle x^*, y \rangle = 0\, \forall y \in Y\}$

Exercise:

1. $Y^* \cong X^*/Y^\perp$
2. $(X/Y)^* \cong Y^\perp$ if Y is closed
3. For $Z \subset X^*$, define $\;^\perp Z := \{x \in X \mid \langle x^*, x \rangle = 0\, \forall x^* \in Z\}$
   Prove that $\;^\perp(Y^\perp) \cong Y$ whenever Y is a closed subspace of X.

Theorem 6: X normed vector space, $A, B \subset X$ convex, $\text{int}(A) \neq \varnothing$, $B \neq \varnothing$, $A \cap B = \varnothing$
$\Rightarrow \exists x^* \in X^*, \exists c \in \mathbb{R}$ such that $\langle x^*, x \rangle < c\; \forall x \in A$ and $\langle x^*, x \rangle \geq c\; \forall x \in B$

Proof:

Case 1: $B = \{0\}$ Let $x_0 \in \text{int}(A)$ and define $p : X \to \mathbb{R}$ by

$$p(x) := \inf\{t > 0 \mid x_0 + \frac{x}{t} \in A\}$$

So $x_0 + \frac{x}{t} \in A$ for $t > p(x)$ and $x_0 + \frac{x}{t} \notin A$ for $t < p(x)$

1. $p(\lambda x) = \lambda p(x)\, \forall \lambda > 0$
2. $p(x + y) \leq p(x) + p(y)$
   Given $\varepsilon > 0\; \exists s, t > 0 : s \leq p(x) + \varepsilon,\; t \leq p(x) + \varepsilon$
   $\Rightarrow x_0 + \frac{x}{s} \in A \quad x_0 + \frac{y}{t} \in A$
   $\Rightarrow x_0 + \frac{x+y}{s+t} = \frac{s}{s+t}\left(x_0 + \frac{x}{s}\right) + \frac{t}{t+s}\left(x_0 + \frac{y}{t}\right) \in A$ (A convex)
   $p(x + y) \leq s + t \leq p(x) + p(y) + 2\varepsilon\, \forall \epsilon > 0$
3. Choose $\delta > 0$ such that $B_\delta(x_0) \subset A$ so $p(x) \leq \frac{\|x\|}{\delta}\, \forall x \in X$

4. $x_0 + x \in A \Rightarrow p(x) \leq 1$ $x_0 + x \notin A \Rightarrow p(x) \geq 1$
   Choose $Y := \mathbb{R}x_0$ $\phi(\lambda x_0) := -\lambda$
   Check: $-1 = \phi(x_0) \leq 0 \leq p(x_0)$ $\quad 1 = \phi(-x_0) \leq p(-x_0)$ (because $0 \notin A$)
   $\overset{\text{Thm 5}}{\Rightarrow} \exists \Phi : X \to \mathbb{R}$ such that $\Phi(x) \leq p(x) \, \forall x \in X$, $\Phi(x_0) = -1$
   (i.e. $\Phi|_Y = \phi$) and $x \in A \Rightarrow \Phi(x) \leq 0$:
   Namely $x \in A \Rightarrow p(x - x_0) \leq 1$
   So if $x \in A$ then $\Phi(x - x_0) \leq p(x - x_0) \leq 1 = \Phi(-x_0) \Rightarrow \Phi(x) \leq 0$
   $\Rightarrow$ Assertion with $x^* = \Phi, c = 0$
   $\left(\Phi \text{ is bounded by 3.: } \pm \Phi(x) \leq p(\pm x) \leq \frac{\|x\|}{\delta}\right)$

Case 2: A arbitrary
$K := \{a - b \mid a \in A \, b \in B\} \Rightarrow K$ convex, int$(K) \neq \varnothing$, $0 \notin K$
$\overset{\text{Case 1}}{\Rightarrow} \exists x^* \in X^*$ such that $\langle x^*, x \rangle \leq 0 \, \forall x \in K$
$\Rightarrow \langle x^*, a \rangle \leq \langle x^*, b \rangle \quad \forall a \in A, b \in B$
$c := \sup_{a \in A} \langle x^*, a \rangle < \infty$ (because $B \neq \varnothing$)
$\Rightarrow \langle x^*, a \rangle \leq c \leq \langle x^*, b \rangle \quad \forall a \in A, b \in B$ □

Theorem 7: X normed vector space over $\mathbb{F} = \mathbb{R}, \mathbb{C}$
$Y \subset X$ linear subspace, $x_0 \in X \setminus \overline{Y}$
Let $\delta := d(x_0, Y) = \inf_{y \in Y} \|x_0 - y\| > 0$
$\Rightarrow \exists x^* \in X^*$ such that $\|x^*\| = 1$, $\langle x^*, x_0 \rangle = \delta$, $\langle x^*, y \rangle = 0 \quad \forall y \in Y$

Note: Hypotheses of Theorem 6 are satisfied with $A = B_\delta(x_0)$, $B = Y$

Proof: Denote $Z := \{y + \lambda x_0 \mid y \in Y, \, \lambda \in \mathbb{F}\} = Y \oplus \mathbb{F}x_0$
Define $\Psi : Z \to \mathbb{F}$ by $\Psi(y + \lambda x_0) := \lambda\delta \quad \forall y \in Y, \, \lambda \in \mathbb{F} \Rightarrow$

1. Ψ is well-defined and linear because $x_0 \notin Y$
2. $\Psi(y) = 0 \, \forall y \in Y$
3. $\Psi(x_0) = \delta$
4. $\sup_{x \in Z z \neq 0} \frac{|\Psi(x)|}{\|x\|} = 1$
   Because:

$$\sup_{(y,\lambda) \neq (0,0)} \frac{|\Psi(y + \lambda x_0)|}{\|y + \lambda x_0\|} = \sup_{(y,\lambda) \neq (0,0)} \frac{|\lambda|\delta}{\|y + \lambda x_0\|} = \sup_{0 \neq \lambda, y} \frac{\delta}{\|\frac{y}{\lambda} + x_0\|}$$
$$= \sup_{y \in Y} \frac{\delta}{\|x_0 + y\|} = \frac{\delta}{\inf_{y \in Y} \|x_0 - y\|} = 1$$

$\overset{\text{Thm 5}}{\Rightarrow} \exists x^* \in X^*$ such that $\|x^*\| = 1$ and $\langle x^*, x \rangle = \Psi(x) \, \forall x \in Z$
$\langle x^*, x_0 \rangle = \Psi(x_0) = \delta \quad \langle x^*, y \rangle = \Psi(y) = 0 \, \forall y \in Y$ □

Corollary 1: X normed vector space, $Y \subset X$ linear subspace, $x \in X$.
Equivalent are:

(i) $x \in \overline{Y}$

(ii) For every $x^* \in X^*$ we have: $\langle x^*, y \rangle = 0 \, \forall y \in Y$ implies $\langle x^*, x \rangle = 0$

Proof:

$(i) \Rightarrow (ii)$ $\quad x = \lim_{n \to \infty} y_n \quad y_n \in Y$
$\langle x^*, y \rangle = 0 \, \forall y \in Y$
$\Rightarrow \langle x^*, x \rangle = \lim_{n \to \infty} \langle x^*, y_n \rangle = 0$
$(ii) \Rightarrow (i)$:
$x \notin \overline{Y} \overset{\text{Thm 7}}{\Rightarrow} \exists x^* \in X^* : \langle x^*, y \rangle = 0 \, \forall y \in Y \quad \langle x^*, x \rangle \neq 0$ □

Corollary 2: X normed vector space, Y linear subspace
Y is dense $\Leftrightarrow Y^\perp = \{0\}$

Proof: Corollary 1. $\square$

Corollary 3: X normed vector space, $0 \neq x_0 \in X \Rightarrow \exists x^* \in X^*$ such that $\|x^*\| = 1, \quad \langle x^*, x_0 \rangle = \|x_0\|$

Proof: Theorem 7 with $Y = \{0\}, \delta = \|x_0\|$ $\square$

2.4 Reflexive Banach Spaces

X real Banach space
$X^* := \mathcal{L}(X, \mathbb{R})$
$X^{**} := \mathcal{L}(X^*, \mathbb{R})$

Example: Every element $x \in X$ determines a bounded linear functional $\phi_x : X^* \to \mathbb{R}$ by $\phi_x(x^*) := \langle x^*, x \rangle$.
Bounded because $|\phi_x(x^*)| \leq \|x^*\| \cdot \|x\|$ hence

$$\|\phi_x\| := \sup_{x^* \neq 0} \frac{\phi_x(x^*)}{\|x^*\|} \leq \|x\|$$

In fact: $\|\phi_x\| = \|x\|$, because $\forall x \neq 0 \; \exists x^* \in X^*$ such that $\|x^*\| = 1$ and $\langle x^*, x \rangle = \|x\|$ (Cor 3). We have proved:

Lemma 4: The map $\iota : X \to X^{**}$ defined by

$$\iota(x)(x^*) := \langle x^*, x \rangle$$

is an isometric embedding.

Definition: A Banach space X is called *reflexive* if the canonical embedding $\iota : X \to X^{**}$ (defined in Lemma 4) is bijective.

Example 1: $X = H$ Hilbert space $\Rightarrow H \cong H^* \cong H^{**}$ so H is reflexive.

Example 2: $(M, \mathcal{A}, \mu)$ measure space, $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow L^p(\mu)^* \cong L^q(\mu), L^q(\mu)^* \cong L^p(\mu)$.
So $L^p(\mu)$ is reflexive for $p > 1$.

$$p = 1 : L^1([0,1])^* = L^\infty([0,1])$$
$$L^\infty([0,1])^* \supsetneqq L^1([0,1])$$

so $L^1([0,1])$ is not reflexive.

Example 3:

$$c_0 := \{(x_n)_{n \in \mathbb{N}} \in \mathbb{R}^\mathbb{N} \mid \lim_{n \to \infty} x_n = 0\}$$
$$\|x\| = \sup_{n \in \mathbb{N}} |x_n|$$
$$(c_0)^* \cong \ell^1, (\ell^1)^* = \ell^\infty \supsetneqq c_0$$

so c_0, ℓ^1, ℓ^∞ are not reflexive

Theorem 8: Let X be a Banach space.

- X is reflexive $\Leftrightarrow X^*$ is reflexive
- suppose X is reflexive and $Y \subset X$ be a closed linear subspace $\Rightarrow Y, X/Y$ are reflexive

Proof:

1. "X reflexive $\Rightarrow X^*$ reflexive":

   Let $\psi : X^{**} \to \mathbb{R}$ be a bounded linear functional. Need to show: $\exists x^* \in X^*$ such that

   $$\psi(x^{**}) = \langle x^{**}, x^* \rangle \, \forall x^{**} \in X^{**}$$

   Consider the diagram

   $$X \xrightarrow{\iota} X^{**} \xrightarrow{\psi} \mathbb{R}$$

   Denote $x^* := \psi \circ \iota : X \to \mathbb{R}$.

   Let $x^{**} \in X$ and denote $x := \iota^{-1}(x^{**}) \in X$. Then

   $$\psi(x^{**}) = \psi(\iota(x)) = \langle x^*, x \rangle = \langle \iota(x), x^* \rangle = \langle x^{**}, x^* \rangle.$$

2. "X^* reflexive $\Rightarrow X$ reflexive":

   Assume X^* is reflexive, but X is not reflexive. Then $\iota(X) \subsetneqq X^{**}$. Pick an element $x_0^{**} \in X^{**} \setminus \iota(X)$.

   Fact: $\iota(X)$ is a closed subspace of X^{**}, because $\iota(X)$ is complete by Lemma 4. $\Rightarrow \exists$ bounded linear functional $\psi : X^{**} \to \mathbb{R}$ such that:

   (i) $\psi(x^{**}) = 0 \, \forall x^{**} \in \iota(X)$, and
   (ii) $\psi(x_0^{**}) = 1$

   X reflexive $\Rightarrow \exists x^* \in X^*$ such that

   $$\psi(x^{**}) = \langle x^{**}, x^* \rangle \forall x^{**} \in X^{**}$$
   $$\Rightarrow \langle x^*, x \rangle = \langle \iota(x), x^* \rangle = 0 \forall x \in X$$
   $$x^* = 0$$
   $$1 = \psi(x_0^{**}) \overset{(3)}{=} \langle x_0^{**}, x^* \rangle = 0$$

   Contradiction.

3. "Y is reflexive":

   Let $\pi : X^* \to Y^*$ be the bounded linear map

   $$\pi(x^*) := x^*|_Y$$

   By Hahn-Banach π is surjective. Let $y^{**} \in Y^{**}$. Consider the diagram

   $$X^* \xrightarrow{\pi} Y^* \xrightarrow{y^{**}} \mathbb{R}$$

   Let $x^{**} := y^{**} \circ \pi : X^* \to \mathbb{R}$. Then, because X is reflexive, $\exists y \in X$ such that $\iota(y) = x^{**}$.

   $\Rightarrow \forall x^* \in Y^\perp$. We have $\pi(x^*) = 0$ and so

   $$\langle x^*, y \rangle = \langle \iota(y), x^* \rangle = \langle x^{**}, x^* \rangle = y^{**}, \pi(x^*) = 0$$

   To show: $\langle y^{**}, y^* \rangle = \langle y^*, y \rangle \forall y^* \in Y^*$

   Given $y^* \in Y^*$ choose $x^* \in X^*$ such that $\pi(x^*) = y^*$

   $$\begin{aligned} \langle y^*, y \rangle &= \langle \pi(x^*), y \rangle \\ \langle y^{**}, y^* \rangle &= \langle y^{**}, \pi(x^*) \rangle \\ &= \langle x^{**}, x^* \rangle \\ &= \langle \iota(y), x^* \rangle \\ &= \langle x^*, y \rangle \\ &= \langle y^*, y \rangle \end{aligned}$$

4. "$Z = X/Y$ reflexive":

   Denote by $\pi : X \to X/Y$ the canonical projection, ie.

$$\pi(x) = [x] = x + Y \forall x \in X$$

   Define the bounded linear operator $T : Z^* \to Y^\perp$ by $Tz^* := z^* \circ \pi : X \to \mathbb{R}$

   Note:

   (a) $\operatorname{im} T \subset Y^\perp$ because $\ker \pi = Y$

   (b) In fact $\operatorname{im} T = Y^\perp$ and T is an isometric isomorphism (Exercise 1,b)).

   Let $z^{**} \in Z^{**}$. Consider the composition

$$Y^\perp \xrightarrow{T^{-1}} Z^* \xrightarrow{z^{**}} \mathbb{R}$$

   This is a bounded linear functional on $Y^\perp \subset X^*$, so by Hahn-Banach: $\exists x^{**} \in X^{**}$ such that

$$\langle x^{**}, x^* \rangle = \langle z^{**}, T^{-1}x^* \rangle \forall x^* \in Y^\perp \tag{7}$$

$$\langle x^{**}, z^* \circ \pi \rangle = \langle z^{**}, z^* \rangle \forall z^* \in Z^* \tag{8}$$

$$\overset{\text{(X reflexive)}}{\Rightarrow} \exists x \in X \text{ such that } \iota(x) = x^{**}$$

   Denote $z := \pi(x) = [x] \in Z$

$$\begin{aligned} \Rightarrow \forall z^* \in Z^* : \quad \langle z^{**}, z^* \rangle &\overset{(7)}{=} \langle x^{**}, z^* \circ \pi \rangle \\ &= \langle \iota(x), z^* \circ \pi \rangle \\ &= \langle z^* \circ \pi, x \rangle \\ &= \langle z^*, \pi(x) \rangle \\ &= \langle z^*, z \rangle \end{aligned}$$

□

Remark: $Y^* \cong X^*/Y^\perp$

$$(X^*/Y^\perp)^* \cong (Y^\perp)^\perp \cong^\perp (Y^\perp)$$

because X is reflexive.

Recall A Banach space X is called *separable* if $\exists$ countable dense subset $D \subset X$.

Remark: Suppose there is a sequence $e_1, e_2, e_3, \ldots$ such that the subspace

$$Y := \operatorname{span}\{e_i \mid i \in \mathbb{N}\} = \left\{ \sum_{i=1}^n \lambda_i e_i \mid n \in \mathbb{N}, \lambda_i \in \mathbb{R} \right\}$$

is dense in X. $\Rightarrow X$ is separable. Indeed the set $D := \{\sum_{i=1}^n \lambda_i e_i \mid n \in \mathbb{N}, \lambda_i \in \mathbb{Q}\}$ is countable and dense.

Theorem: X Banach space

(i) X^* separable $\Rightarrow X$ separable.

(ii) X separable and reflexive $\Rightarrow X^*$ separable.

Proof:

(i) Let $D = \{x_1^*, x_2^*, x_3^*, \dots\}$ be a dense, countable subset of X^*. Assume w.l.o.g that $x_n^* \neq 0\ \forall n \in \mathbb{N}$ such that

$$\|x_n\| = 1, |\langle x_n^*, x_n\rangle| \geq \frac{1}{2}\|x_n^*\|$$

Denote $Y = \operatorname{span}\{x_n \mid n \in \mathbb{N}\}$
Claim: Y is dense in X
By Hahn-Banach

$$Y \text{ is dense} \Leftrightarrow Y^\perp = \{0\}$$

Let $x^* \in Y^\perp \subset X^*$. Because D dense in X^*:

$$\Rightarrow \exists n_1, n_2, n_3, \dots \to \infty \text{ such that } \lim_{i\to\infty} \|x_{n_i}^* - x^*\| = 0$$

Now:

$$\|x_{n_i}^*\| \leq 2|\langle x_{n_i}^*, x_{n_i}\rangle| = 2|\langle x_{n_i}^* - x^*, x_{n_i}\rangle| \leq \|x_{n_i}^* - x^*\| = 0$$

so $x^* = \lim_{i\to\infty} x_{n_i}^* = 0$ so $Y^\perp = 0$ so Y is dense in X so X is separable.

(ii) X reflexive and separable $\Rightarrow X^{**} = \iota(X)$ separable $\Rightarrow X^*$ is separable □

Example:

(i) c_0 separable.
$c_0^* = \ell^1$ separable.
$(\ell^1)^* = \ell^\infty$ not separable.

(ii) (M, d) compact metric space $X = C(M)$ separable

$$X^* = \mathcal{M} = \{\text{finite Borel measures on } M\}$$

not separable, except when M is a finite set.

(iii) $X = L^p(\Omega)$ with Lebesgue-measure $1 \leq p < \infty$. $\varnothing \neq \Omega \subset \mathbb{R}^n$ open $\Rightarrow X$ is separable. $L^\infty(\Omega)$ is not separable.

3 The weak and weak* topologies

3.1 The weak topology

Definition: A *topological vector space* is a pair $(X,\mathcal{U})$, where X is a (real) vector space and $\mathcal{U}$ is a topology such that the maps

$$X \times X \to X \quad (x,y) \mapsto x+y$$

and

$$\mathbb{R} \times X \to X \quad (\lambda, x) \mapsto \lambda x$$

are continuous.

Definition: A topological vector space $(X,\mathcal{U})$ is called *locally convex*, if $\forall x \in X\, \forall U \in \mathcal{U}, x \in U\, \exists V \in \mathcal{U}$ such that $x \in V \subset U$ V convex.

Lemma 1: X topological vector space, $K \subset X$ convex
$\Rightarrow \overline{K}, \mathrm{int}(K)$ are convex

Proof:

(i) $\mathrm{int}(K)$ is convex
$x_0, x_1 \in \mathrm{int}(K), \quad 0 < \lambda < 1$
To show: $x_\lambda = (1-\lambda)x_0 + \lambda x_1 \in \mathrm{int}(K)$
$\exists$ open set $U \subset X$ such that $0 \in U$ and $x_0 + U \subset K$, $x_1 + U \subset K$
$\Rightarrow x_\lambda + U \subset K \Rightarrow x_\lambda \in \mathrm{int}(K)$

(ii) $\overline{K}$ is convex
$x_0, x_1 \in \overline{K}$, $0 < \lambda < 1$
To show: $x_\lambda \in \overline{K}$. Let $U \subset X$ be an open set with $x_\lambda \in U$
$W := \{(y_0, y_1) \in X \times X \mid (1-\lambda)y_0 + \lambda y_1 \in U\}$
$\Rightarrow W \subset X \times X$ is open, $(x_0, x_1) \in W$.
$\exists$ open sets $U_0, U_1 \subset X$ such that: $x_0 \in U_0, x_1 \in U_1 \quad U_0 \times U_1 \subset W$
$\overset{x_0,x_1 \in \overline{K}}{\Rightarrow} \exists y_0 \in U_0 \cap K \quad \exists y_1 \in U_1 \cap K$
$\Rightarrow y_\lambda := (1-\lambda)y_0 + \lambda y_1 \in K \cap U$, so $K \cap U \neq \varnothing$
Hence $x_\lambda \in \overline{K}$.

□

Let X be a real vector space
Let $\mathcal{F}$ be a set of linear functions $f : X \to \mathbb{R}$
Let $\mathcal{U}_\mathcal{F} \subset 2^X$ be the weakest topology such that $f \in \mathcal{F}$ is continuous w.r.t. $\mathcal{U}_\mathcal{F}$
If $\mathcal{F} \ni f : X \to \mathbb{R}$ we have for $a < b$ $\{x \in X \mid a < f(x) < b\} \in \mathcal{U}_\mathcal{F}$
Let $\mathcal{V}_\mathcal{F} \subset 2^X$ be the set of all subsets of the form
$V := \{x \in X \mid a_i < f_i(x) < b_i \quad i = 1, \ldots, m\} \quad f_i \in \mathcal{F}, a_i, b_i \in \mathbb{R}$
for $i = 1, \ldots, m$

Lemma 2:

(i) Let $U \subset X$. Then $U \in \mathcal{U}_\mathcal{F}$ if and only if
$\forall x \in U\, \exists V \in \mathcal{V}_\mathcal{F}$ such that $x \in V \in U$ $(*)$

(ii) $(X, \mathcal{U}_\mathcal{F})$ is a locally convex topological vector space

(iii) A sequence $x_n \in X$ converges to $x_0 \in X$ if and only if
$f(x_0) = \lim_{n \to \infty} f(x_n)\, \forall f \in \mathcal{F}$

(iv) $(X, \mathcal{U}_\mathcal{F})$ is Hausdorff if and only if $\forall x \in X$, $x \neq 0\, \exists f \in \mathcal{F}$ such that $f(x) \neq 0$.

Proof:

(i) Exercise with hint:
Define $\mathcal{U}'_{\mathcal{F}} := \{U \subset X \mid (*)\}$
Prove:

(a) $\mathcal{U}'_{\mathcal{F}}$ is a topology
(b) Each $f \in \mathcal{F}$ is continuous w.r.t. $\mathcal{U}'_{\mathcal{F}}$
(c) If $\mathcal{U} \subset 2^X$ is another topology such that each $f \in \mathcal{F}$ is continuous w.r.t. $\mathcal{U}$ then $\mathcal{U}'_{\mathcal{F}} \subset \mathcal{U}$

(ii)
- Each $V \in \mathcal{V}_{\mathcal{F}}$ is convex
- scalar multiplication is continuous:
  $\lambda_0 \in \mathbb{R}$, $x_0 \in X$
  Choose $V \in \mathcal{V}_{\mathcal{F}}$ such that $\lambda_0 x_0 \in V$ $\exists \delta > 0$ such that
  $(\lambda_0 - \delta)x_0$, $(\lambda_0 + \delta)x_0 \in V$ and $\delta \neq \pm\lambda_0$
  $\Rightarrow U := \frac{1}{\lambda_0 - \delta} V \cap \frac{1}{\lambda_0 + \delta} V \in \mathcal{V}_{\mathcal{F}}$
  If $x \in U$ and $|\lambda - \lambda_0| < \delta$ then $\lambda x \in V$ (because V is convex)
- addition is continuous:
  $x_0, y_0 \in X,\ x_0 + y_0 \in W,\ W \in \mathcal{V}_{\mathcal{F}}$
  Define $U := \frac{1}{2}W + \frac{x_0 - y_0}{2}$ $\quad V := \frac{1}{2}W + \frac{y_0 - x_0}{2}$
  $\Rightarrow U, V \in \mathcal{V}_{\mathcal{F}},\ x_0 \in U,\ y_0 \in V$
  $x \in U, y \in V \Rightarrow x + y \in W$

(iii) Assume $x_n \overset{\mathcal{U}_{\mathcal{F}}}{\to} x_0$
Let $f \in \mathcal{F}$, $\varepsilon > 0$. Denote $U := \{x \in X \mid |f(x) - f(x_0)| < \varepsilon\} \in \mathcal{V}_{\mathcal{F}}$
$x_0 \in U \Rightarrow \exists n_0 \in \mathbb{N}\, n \geq n_0 : x_n \in U$
$\Rightarrow \forall n \geq n_0 |f(x) - f(x_0)| < \varepsilon$

Assume $f(x_n) \to f(x_0) \forall f \in \mathcal{F}$
Let $U \in \mathcal{U}_{\mathcal{F}}$ with $x \in U \overset{(i)}{\Rightarrow} \exists V \in \mathcal{V}_{\mathcal{F}}$ with $x_0 \in V \in U$
$V = \{x \in X \mid a_i < f_i(x_i) < b_i \quad i = 1, \ldots, m\}$
$\Rightarrow a_i < f_i(x_0) < b_i \quad i = 1, \ldots, m$
$\Rightarrow \exists n_0 \in \mathbb{N} \quad \forall i \in \{1, \ldots, m\} \forall n \geq n_0 : a_i < f_i(x_n) < b_i$
$\Rightarrow \forall n \geq n_0\, x \in V \subset U$

(iv) Exercise without hints

□

Example 1: I any set, $X = \mathbb{R}^I := \{x : I \to \mathbb{R}\} \ni \{x_i\}_{i \in I}$ is a vector space. "product space". $\pi_i : X \to \mathbb{R}$ projection $\pi_i(x) = x(i)$ linear map
$\mathcal{U} \subset 2^X$ weakest topology such that each π_i is continuous

Example 2: X Banach space
$\mathcal{F} := \{\varphi : X \to \mathbb{R} \mid \varphi \text{ is bounded and linear}\} = X^*$
Let $\mathcal{U}^w$ be the weakest topology such that each bounded linear functional is continuous w.r.t. $\mathcal{U}^w$
Facts:

a) $\mathcal{U}^s \subset 2^X$ strong topology; induced by the norm; $\mathcal{U}^w \subset 2^X$ weak topology: $\mathcal{U}^w \subset \mathcal{U}^s$

b) $(X, \mathcal{U}^w)$ is a locally convex vector space

c) A sequence $x_n \in X$ converges to $x_0 \in X$ if and only if
$\langle x^*, x_0 \rangle = \lim_{n \to \infty} \langle x^*, x_n \rangle \quad \forall x^* \in X^*$

Notation: $x_n \rightharpoonup x_0$ or $x_0 = \text{w-}\lim_{n \to \infty} x_n$

Example 3: Let X be a Banach space, $\mathcal{L}(X,\mathbb{R})$ dualspace
Let $\mathcal{U}^{w^*} \subset 2^{X^*}$ be the weakest topology on X^* such that each linear functional of the form $X^* \to \mathbb{R}\, x^* \to \langle x^*, x\rangle$ is continuous (in this case $\mathcal{F} = i(X) \subset X^{**}$)
Facts:

a) $\mathcal{U}^s \subset 2^{X^*}$ strong topology
$\mathcal{U}^w \subset 2^{X^*}$ weak topology $\Rightarrow \mathcal{U}^{w^*} \subset \mathcal{U}^w \subset \mathcal{U}^s$

b) $(X^*, \mathcal{U}^{w^*})$ is a locally convex topological vector space

c) A sequence $x_n^* \in X^*$ converges to x_0^* in the weak*-topology if and only if $\langle x_0^*, x\rangle = \lim\limits_{n\to\infty} \langle x_n^*, x\rangle \quad \forall x \in X$

Notation: $x_n^* \overset{w^*}{\rightharpoonup} x_0^*$ or $x_0^* = w^* - \lim_{n\to\infty} x_n^*$

Remark: Suppose the sequence $\langle x^*, x_n\rangle$ converges $\forall x^* \in X^*$
Does this imply that x_n converges weakly?
No, denote $\varphi(x^*) = \lim_{n\to\infty}\langle x^*, x_n\rangle$
Then $\varphi : X \to \mathbb{R}$ is linear and continuous
x_n converges weakly$\Leftrightarrow \varphi \in i(X) \subset X^{**}$

Exercise: Find an example

Lemma 3: X Banach space, $K \subset X$ convex
Assume: K is closed w.r.t. the strong topology $\Rightarrow$ K is weakly closed

Proof: Let $x_0 \in X \setminus K$, $K \neq \varnothing$
$\exists \varepsilon > 0$ such that $B_\varepsilon(x_0) \cap K = \varnothing$

Chap. II Thm 6 $\Rightarrow$ $\exists x^* \in X^*, \exists c \in \mathbb{R}$ such that $\langle x^*, x\rangle < c \quad \forall x \in B_\varepsilon(x_0)$ and $\langle x^*, x\rangle \geq c \quad \forall x \in K$
$\Rightarrow U := \{x \in X \mid \langle x^*, x\rangle < c\}$ weakly open and $x_0 \in U$, $U \cap K = \varnothing$ □

Lemma 4 (Mazur): x_i sequence, $x_i \rightharpoonup x_0 \Rightarrow \forall \varepsilon > 0\, \exists \lambda_1, \dots, \lambda_n \in \mathbb{R}$
$\lambda_i \geq 0 \quad \sum_{i=1}^n \lambda_i = 1 \quad \|x_0 - \sum_{i=1}^n \lambda_i x_i\| < \varepsilon$

Proof: $K := \{\sum_{i=1}^n \lambda_i x_i \mid n \in \mathbb{N},\, \lambda_i > 0,\, \sum_{i=1}^n \lambda_i = 1\}$ convex
Lemma 1 $\Rightarrow$ the strong closure $\overline{K}$ of K is convex
Lemma 3 $\Rightarrow$ $\overline{K}$ is weakly closed $\Rightarrow x_0 \in \overline{K}$ □

Lemma 5: X Banach space, ∞-dimensional
$S := \{x \in X \mid \|x\| = 1\} \quad B := \{x \in X \mid \|x\| \leq 1\}$
$\Rightarrow B$ is the weak closure of S.

Proof: $x_0 \in B$, $U \in \mathcal{U}^w$ and $x_0 \in U$
$\Rightarrow U \cap S \neq \varnothing$. Choose ε_i, x_i^* such that
$V := \{x \in X \mid |\langle x_i^*, x_0 - x\rangle| < \varepsilon_i\} \subset U$
$V \supset E := \{x \in X \mid \langle x_i^*, x_0 - x\rangle = 0\}$ nontrivial affine subspace, $E \cap S \neq \varnothing$ □

Example: $X = l^1 \ni x_n$ sequence, $l^1 \ni x_0$
Then $x_n \rightharpoonup x_0 \Leftrightarrow x_n \to x_0 \quad \|x_n - x_0\|_1 \to 0$

3.2 The weak* topology

Theorem 1 (Banach Alaoglu, sequentially): X separable Banach space
$\Rightarrow$ every bounded sequence $x_n^* \in X^*$ has a weak*-convergent subsequence.

Proof: $D = \{x_1, x_2, x_3, \ldots\} \subset X$ dense, countable.

$$c := \sup_{n\in\mathbb{N}} \|x_n^*\| < \infty$$

$$\Rightarrow |\langle x_n^*, x_1\rangle| \leq \|x_n^*\| \cdot \|x_1\| \leq c\|x_1\|$$
$$\Rightarrow \exists \text{ subsequence}(x_{n_{i,1}})_{i=1}^{\infty}$$

such that $\langle x_{n_{i,1}}^*, x_1\rangle$ converges.
The sequence $\langle x_{n_{i,1}}^*, x_2\rangle \in \mathbb{R}$ is bounded
$\exists$ further subsequence $(x_{n_{i,2}}^*)_{i=1}^{\infty}$ such that $\langle x_{n_{i,2}}^*, x_2\rangle$ converges.
Induction
$\Rightarrow \exists$ sequence of subsequences $(x_{n_{i,k}})_{i=1}^{\infty}$ such that

- $(x_{n_{i,k+1}}^*)_i$ is a subsequence of $(x_{n_{i,k}}^*)_i$
- the limit $\lim_{i\to\infty}\langle x_{n_{i,k}}^*, x_k\rangle$ exists for every $k \in \mathbb{N}$

Diagonal Subsequence
$x_{n_i}^* := x_{n_{i,i}}^*$
$\Rightarrow (x_{n_i}^*)_{i=1}^{\infty}$ is a subsequence of $(x_n^*)_{n=1}^{\infty}$ and the limit $\lim_{i\to\infty}\langle x_{n_i}^*, x_k\rangle$ exists for every $k \in \mathbb{N}$.
$\Rightarrow$ By Chapter II, Thm 2 (Banach-Steinhaus) with $Y = \mathbb{R}$, $\exists x^* \in X^*$ such that

$$\langle x^*, x\rangle = \lim_{i\to\infty}\langle x_{n_i}^*, x\rangle \forall x \in X$$

So $x_{n_i}^* \rightharpoonup x^*$

□

Example 1: (M, d) compact metric space with $M \neq \varnothing$, $\mathcal{B} \subset 2^M$ Borel σ-algebra. $X := C(M)$ separable
$X^* :\cong \{\text{real Borel measures} \mu : \mathcal{B} \to \mathbb{R}\}$
$f : M \to M$ *homeomorphism.*
A Borel measure $\mu : \mathcal{B} \to [0, \infty)$ is called an *f-invariant Borel probability measure* if

- $\mu(M) = 1$
- $B \in \mathcal{B} \Rightarrow \mu(f(B)) = \mu(B)$

$$\mathcal{M}(f) := \{f\text{-invariant Borel prob. meas. on } M\} \subset X^*$$

Fact 1: $\mu \in \mathcal{M}(f)$
$\Rightarrow \|\mu\| = |\mu|(M) = \mu(M) = 1$
Fact 2: $\mathcal{M}(f)$ is convex $\Rightarrow$ Exercise.
Fact 3: $\mathcal{M}(f) \neq \varnothing$

Proof: Fix an element $x \in M$. Define the Borel-measure $\mu_n : \mathcal{B} \to \mathbb{R}$

$$\int_M u \, d\mu_n := \frac{1}{n}\sum_{k=0}^{n-1} u(f^k(x)) \forall\ u \in C(M)$$

where $f^k := \underbrace{f \circ f \circ \ldots \circ f}_{k} \Rightarrow \|\mu_n\| \leq 1, \quad \mu_n \geq 0 \Rightarrow (Thm1)\exists$ weak* convergent subsequence $\mu_{n_i} \rightharpoonup \mu$
Claim: $\mu \in \mathcal{M}(f)$

- $$\mu \geq 0 \qquad \int_M u \, d\mu = \lim_{n\to\infty}\int_M u \, d\mu_{n_i} \geq 0 \ \forall u \geq 0$$

- $$\mu(M) = \int\limits_M 1 \,\mathrm{d}\mu = \lim_{i\to\infty} \int\limits_M 1 \,\mathrm{d}\mu_{n_i} = 1$$

- $$\mu(f(B)) = \mu(B) \forall B \in \mathcal{B}$$

$$\int\limits_M u \circ f \,\mathrm{d}\mu = \lim_{i\to\infty} \frac{1}{n_i} \sum_{k=1}^{n_i} u(f^k(x)) = \lim_{i\to\infty} \frac{1}{n_i} \sum_{k=1}^{n_i-1} u(f^k(x))$$

$$\int\limits_M u \circ f \,\mathrm{d}\mu = \int\limits_M u \,\mathrm{d}f_*\mu \;\forall u \in C(M)$$

$$\Rightarrow f_*\mu(B) = \mu(B) \forall B \in \mathcal{B}$$

and $f_*\mu(B) = \mu(f^{-1}(B))$ □

Example 2: $X = \ell^\infty$, elements of X are bounded sequences $x = (x_i)_{i=1}^\infty \in \mathbb{R}$. $\|x\|_\infty = \sup_{i\in\mathbb{N}} |x_i|$

Definition: $\phi_n : X \to \mathbb{R}$ by $\phi_n(x) := x_n$ and $\|\phi_n\| = 1$

Exercise: Show that $\phi_n \in X^*$ has no weak*-convergent subsequence, ie. for all subsequences $n_1 < n_2 < n_3 < \ldots : \exists x \in X = \ell^\infty$ such that the sequence $(\phi_{n_i}(x))_{i=1}^\infty \in \mathbb{R}$ does not converge.

Theorem 2 (Banach-Alaoglu, general form): X Banach space $\Rightarrow$ the unit ball $B^* := \{x^* \in X^* \mid \|x^*\| \le 1\}$ in the dual space is weak* compact.

Remark: X^* with the weak* topology is Hausdorff.
$\Rightarrow B^*$ is weak*-closed. Prove it directly without using Thm 2.

Theorem 3 (Tychonoff): Let I be any index set and, for each $i \in I$, let K_i be a compact topological space $\Rightarrow K := \prod_{i\in I} K_i$ is compact wrt the product topology.

Remark: $K = \{x = (x_i)_{i\in I} \mid x_i \in K_i\}$ $\pi_i : K \to K_i$ canonical projection
product topology := weakest topology on K wrt which each π_i is continuous.

Proof Thm 3 ⇒ Thm 2: $I = X$. $K_x := [-\|x\|, \|x\|] \subset \mathbb{R}$

$$K := \prod_{x\in X} K_x = \{f : X \to \mathbb{R} \mid |f(x)| \le \|x\| \forall x \in X\} \subset \mathbb{R}^X$$

$$L := \{f : X \to \mathbb{R} \mid f \text{ is linear}\} \subset \mathbb{R}^X$$

- By Thm 3: K is compact
- L is closed with respect to the product topology. For $x, y \in X, \lambda \in \mathbb{R}$, the functions
$$\phi_{x,y} : \mathbb{R}^X \to \mathbb{R}, \psi_{x,\lambda} : \mathbb{R}^X \to \mathbb{R}$$
given by $\phi_{x,y}(f) := f(x+y) - f(x) - f(y)$ and $\psi_{x,\lambda} := f(\lambda x) - \lambda f(x)$ are continuous wrt product topology. So
$$L = \bigcap_{x,y} \phi_{x,y}^{-1}(0) \cap \bigcap_{x,\lambda} \psi_{x,\lambda}^{-1}(0)$$
is closed $\Rightarrow K \cap L = B^*$ is compact. Product topology on $K \cap L =$ weak*-topology on X^*

□

Definition: K any set. A set $\mathcal{A} \subset 2^K$ is called *FiP* if $A_1, \ldots, A_n \in \mathcal{A} \Rightarrow A_1 \cap \ldots \cap A_n \neq \varnothing$. A set $\mathcal{B} \subset 2^K$ is called *maximal FiP* if $\mathcal{B}$ is FiP and $\forall \mathcal{A} \subset 2^K$ we have
$\mathcal{B} \subset \mathcal{A}$ and $\mathcal{A}$ FiP $\Rightarrow \mathcal{A} = \mathcal{B}$
Fact 1: If $\mathcal{A} \subset 2^K$ is FiP, then $\exists \mathcal{B} \subset 2^K$ max FiP such that $\mathcal{A} \subset \mathcal{B}$ (Zorn's Lemma)
Fact 2: Let $\mathcal{B} \subset 2^K$ is max FiP, then:

(i) $B_1, \ldots, B_n \in \mathcal{B} \Rightarrow B_1 \cap \ldots \cap B_n \in \mathcal{B}$

(ii) $C \in K, C \cap B \neq \varnothing \forall B \in \mathcal{B} \Rightarrow C \in \mathcal{B}$

Fact 3: Let K be a topological space. Then K is compact if and only if every FiP collection $\mathcal{A} \subset 2^K$ of closed subsets satisfies $\bigcap_{A \in \mathcal{A}} A \neq \varnothing$

Proof of Tychonoff's theorem: $K = \prod_i K_i$. Let $\mathcal{A} \subset 2^K$ be a FiP collection of closed sets. By Fact 1 $\exists$ max FiP collection $\mathcal{B} \subset 2^K$ with $\mathcal{A} \subset \mathcal{B}$ (not each $B \in \mathcal{B}$ needs to be closed).
To show: $\bigcap_{B \in \mathcal{B}} \overline{B} \neq \varnothing$

Step 1: Construction of an element $x \in K$.
Fix $i \in I$. $\pi_i : K \to K_i$ projection. Denote $\mathcal{B}_i := \{\pi_i(B) \mid B \in \mathcal{B}\} \subset 2^{K_i} \Rightarrow \mathcal{B}_i$ is FiP (if $B_1, \ldots, B_n \in \mathcal{B}$ then $\pi_i(B_1) \cap \ldots \cap \pi(B_n) \supset \pi_i(B_1 \cap \ldots \cap B_n) \neq \varnothing$)

$$\overset{(K_i \text{ compact,Fact 3})\Rightarrow}{\bigcap} \quad {}_{B \in \mathcal{B}} \overline{\pi_i(B)} \neq \varnothing$$

Pick $x_i \in \bigcap_{B \in \mathcal{B}} \overline{\pi_i(B)}$ Choose $x = (x_i)_{i \in I} \in K$ (Axiom of Choice).

Step 2: $x \in \overline{B} \quad \forall B \in \mathcal{B}$
Let $U \in K$ be open with $x \in U$.
To show: $U \cap B \neq \varnothing \quad \forall B \in \mathcal{B}$ U open, $x \in U \Rightarrow$ (!)$\exists$ finite set $J \subset I$
$\exists$ open sets $U_i \subset K_i, i \in J$ such that $x \in \bigcap_{i \in J} \pi_i^{-1}(U_i) \subset U$ (like Lemma 2 (i)).

$$x_i = \pi_i(x) \in U_i \cap \overline{\pi_i(B)} \forall i \in J, \forall B \in \mathcal{B}$$

$$\overset{(U_i \text{ open})}{\Rightarrow} U_i \cap \pi_i(B) \neq \varnothing \forall i \in J \forall B \in \mathcal{B}$$

$$\Rightarrow \pi_i^{-1}(U_i) \cap B \neq \varnothing \forall i \in J \forall B \in \mathcal{B}$$

$$\overset{(\text{Fact 2})}{\Rightarrow} \pi_i^{-1}(U_i) \in \mathcal{B} \forall i \in J$$

$$\overset{(\text{Fact 2})}{\Rightarrow} \bigcap_{i \in J} \pi_i^{-1}(U_i) \in \mathcal{B}$$

$$\bigcap_{i \in J} \pi_i^{-1}(U_i) \cap B \neq \varnothing \forall B \in \mathcal{B}$$

$$\Rightarrow U \cap B \neq \varnothing \forall B \in \mathcal{B}$$

□

Theorem 4: X separable Banach space, $K \subset X^*$. Equivalent are:

(i) K is weak* compact

(ii) K is bounded and weak* closed

(iii) K is sequentially weak* compact

(iv) K is bounded and sequentially weak* closed

Exercise: $\mathcal{M}(f)$ as in example 1 $\Rightarrow \mathcal{M}(f)$ is weak* compact.

Proof of Thm 4: Exercise.
(i) $\Leftrightarrow$ (Thm 2) (ii): use uniform boundedness (Chapter II, Thm 1)
(ii) $\Rightarrow$ (Thm 1) (iii) $\Rightarrow$ (definitions) (iv) $\Rightarrow$ (ii)
Given $x^* \in$ weak* closure(K). Need to prove $\exists$ sequence $x_n^* \in K$ with $x_n^* \rightharpoonup x^*$.
Then, by (iv), $x^* \in K$ $\square$

Theorem 5: X Banach space, $E \subset X^*$ linear subspace
Assume $E \cap B^*$ is weak*-closed, where $B^* := \{x^* \in X^* \mid \|x^*\| \leq 1\}$
Let $x_0^* \in X^* \setminus E$. Let δ be such that $0 < \delta < \inf\limits_{x^* \in E} \|x_0^* - x^*\|$
$\Rightarrow \exists x_0 \in X$ such that $\langle x_0^*, x_0\rangle = 1$, $\langle x^*, x_0\rangle = 0 \quad \forall x^* \in E$, $\|x_0\| \leq \frac{1}{\delta}$

Remark 1: E is closed
Let $x_n^* \in E$ and $x_n^* \to x^* \in X^*$
$\exists c > 0$ such that $\|x_n^*\| \leq c \quad \forall n \in \mathbb{N}$
$\Rightarrow \frac{x_n^*}{c} \in E \cap B^*$
$\Rightarrow \frac{x^*}{c} \in \overline{E \cap B^*} = E \cap B^* \Rightarrow x^* \in E$
E closed$\Rightarrow \exists \delta > 0$ as in the hypothesis of Theorem 5

Remark 2: B^* is closed in the weak*-topology.
Hence each closed ball $\{x^* \in X^* \mid \|x^* - x_0^*\| \leq r\}$ is weak*-closed.

Proof:

Step 1 There is a sequence of finite sets $S_n \subset B = \{x \in X \mid \|x\| \leq 1\}$ satisfying the following condition for every $x^* \in X^*$:

$$\left.\begin{array}{l} \|x^* - x_0^*\| \leq n\delta \\ \max\limits_{x \in S_k} |\langle x^* - x_0^*, x\rangle| \leq \delta k \\ \text{for } k = 0, \dots, n-1 \end{array}\right\} \Rightarrow x^* \notin E \qquad (*)$$

Proof of Step 1: $n = 1$ Choose $S_0 = \varnothing$
Then $(*)$ holds for $n = 1$
$n \geq 1$: Assume $S_0, \dots, S_{n-1}$ have been constructed such that $(*)$ holds.
To show: There is a finite set $S_n \subset B$ such that $(*)$ holds with n replaced by $n+1$
For any finite set $S \subset B$ denote

$$E(S) := \left\{ x^* \in E \;\middle|\; \begin{array}{c} \|x^* - x_0^*\| \leq \delta(n+1) \\ \max\limits_{x \in S_k} |\langle x^* - x_0^*, x\rangle| \leq \delta k \forall k = 0, \dots n-1 \\ \max\limits_{x \in S} |\langle x^* - x_0^*, x\rangle| \leq \delta n \end{array} \right\}$$

To show: $\exists$ finite set $S \subset B$ such that $E(S) = \varnothing$
Suppose, by contradiction, that $E(S) \neq \varnothing$, for every finite set $S \subset B$

a) The set $K := \{x^* \in E \mid \|x^* - x_0^*\| \leq \delta(n+1)\}$ is weak*-compact.
   Let $R := \|x_0\| + \delta(n+1)$ Then $\|x^*\| \leq R \quad \forall x^* \in K$
   so $K \subset E \cap RB^* = R(E \cap B^*) =: E_R$
   E_R is weak*-closed. So $K = \underbrace{E_R}_{\text{weak}^*\text{cl. by ass.}} \cap \underbrace{\{x^* \in X^* \mid \|x^* - x_0^*\| \leq \delta(n+1)\}}_{\text{weak}^*\text{-closed by Rem. 2}}$
   K is weak*-closed and bounded $\overset{\text{Thm 2}}{\Rightarrow}$ K is weak*-compact.

b) E(S) is weak*-closed for every S
   $E(S)$ is the intersection of K with the weak*-closed subsets
   $\{x^* \in X^* \mid \max\limits_{x \in S_k} |\langle x^* - x_0^*, x\rangle| \leq \delta k\} \quad k = 0, \dots, n-1$
   $\{x^* \in X^* \mid \max\limits_{x \in S} |\langle x^* - x_0^*, x\rangle| \leq \delta n\}$

c) $S^i \subset B$ finite set for $i \in I$, I finite
$\bigcap_{i\in I} E(S^i) = E\left(\bigcup_{i\in I} S^i\right) \neq \varnothing$ by assumption.
Let $\mathcal{S} := \{S \subset B \mid S \text{ finite }\}$ Then by c), the collection $\{E(S) \mid S \in \mathcal{S}\}$ is FIP and by b) it consists of weak*-closed subsets of K.
By a) K is weak*-compact$\Rightarrow \bigcap_{S\in\mathcal{S}} E(S) \neq \varnothing$
Let $x^* \in \bigcap_{S\in\mathcal{S}} E(S) \Rightarrow x^* \in E$, $\|x^* - x_0^*\| \leq \delta(n+1)$,
$\max_{x\in S_k} |\langle x^* - x_0^*, x\rangle| \leq \delta k$ $k = 0 \dots n-1$ and $|\langle x^* - x_0^*, x\rangle| \leq \delta n \quad \forall x \in B$
i.e. $\|x^* - x_0^*\| \leq \delta n$ This contradicts $(*)$

Step 2 Construction of x_0
Choose a sequence $x_i \in B$ which runs successively through all points of the set $S = \bigcup_{i=1}^{\infty} \frac{1}{n} S_n$
Then $\lim_{n\to\infty} \|x_i\| = 0$
Define a linear operator $T : X^* \to c_0$ by $Tx^* := (\langle x^*, x_i\rangle)_{i\in\mathbb{N}}$
Claim: For every $x^* \in E$ there is an $i \in \mathbb{N}$ such that $|\langle x^* - x_0^*, x_i\rangle| \geq \delta$
Let $x^* \in E$ and choose $n \geq \frac{\|x^*-x_0^*\|}{\delta} \Rightarrow \|x^* - x_0^*\| \leq \delta n$
$\overset{\text{Step 1}}{\Rightarrow} \exists k \leq n-1\, \exists x \in S_k$ such that $|\langle x^* - x_0^*, x\rangle| > \delta k$
$\Rightarrow |\langle x^* - x_0^*, \frac{x}{k}\rangle| > \delta \Rightarrow \exists i$ such that $x_i = \frac{x}{k}$
The claim shows: $\|Tx^* - Tx_0^*\| > \delta \quad \forall x^* \in E$
$\Rightarrow Tx_0^* \notin \overline{TE}$
$\overset{\text{Chap II, Thm 7}}{\Rightarrow} \exists \alpha \in c_0^* = l^1$ such that $\langle \alpha, Tx_0^*\rangle = 1$ $\langle \alpha, Tx^*\rangle = 0 \quad \forall x^* \in E$
$\|\alpha\|_1 \leq \frac{1}{\delta}$
Define $x_0 := \sum_{i=1}^{\infty} \alpha_i x_i \in X$.

Note 1 $\sum_{i=1}^{\infty} \|\alpha_i x_i\| \leq \sum_{i=1}^{\infty} |\alpha_i| < \infty$
So by Chapter I Theorem 10, the sequence $\sum_{i=1}^{\infty} \alpha_i x_i$ converges as $n \to \infty$

Note 2

a) $\langle x_0^*, x_0\rangle = \sum_{i=1}^{\infty} \alpha_i \langle x_0^*, x_i\rangle = \langle \alpha, Tx_0^*\rangle = 1$

b) $\langle x^*, x_0\rangle = \langle \alpha, Tx^*\rangle = 0 \quad \forall x^* \in E$

c) $\|x_0\| \leq \sum_{i=1}^{\infty} |\alpha_i| = \|\alpha\|_1 \leq \frac{1}{\delta}$

□

Corollary 1: Let X be a Banach space and $E \subset X$ a linear subspace.
Let $B^* := \{x^* \in X^* \mid \|x^*\| = 1\}$
Equivalent are:

(i) E is weak*-closed

(ii) $E \cap B^*$ is weak*-closed

(iii) $({}^{\perp}E)^{\perp} = E$

Exercise: X Banach space, $\iota : X \to X^{**}$ canonical embedding
$\Rightarrow \iota(X)$ is weak*-dense in X^{**}

Proof of Corollary 1: (i)⇒(ii) obvious
(ii)⇒(iii) $E \subset ({}^\perp E)^\perp$ by definition (${}^\perp E = \{x \in X \mid \langle x^*, x\rangle = 0 \, \forall x^* \in E\}$)
$({}^\perp E)^\perp \supset E$: Let $x_0^* \notin E \overset{\text{Thm 5}}{\Rightarrow} \exists x_0 \in X$ such that $\langle x_0^*, x_0\rangle = 1$
$\langle x^*, x_0\rangle = 0 \, \forall x^* \in E$
$\Rightarrow x_0 \in {}^\perp E$, $\langle x_0^*, x_0\rangle \neq 0 \Rightarrow x_0^* \notin ({}^\perp E)^\perp$
$(iii) \Rightarrow (i)$ $E = ({}^\perp E)^\perp = \bigcap_{x \in {}^\perp E} \underbrace{\{x^* \in X^* \mid \langle x^*, x\rangle = 0\}}_{\text{weak}^*\text{-closed}}$ □

Corollary 2: X Banach space, $\varphi : X^* \to \mathbb{R}$ linear
Equivalent are:

(i) φ is continuous w.r.t. weak*-topology

(ii) $\varphi^{-1}(0) \in X^*$ is weak*-closed

(iii) $\exists x \in X \, \forall x^* \in X^* \quad \varphi(x^*) = \langle x^*, x\rangle$

Proof: $(iii) \Rightarrow (i)$ definition of weak*-topology
$(i) \Rightarrow (ii)$ definition of continuity
$(ii) \Rightarrow (iii)$ w.l.o.g $\varphi \neq 0$
$E := \varphi^{-1}(0) \subset X^*$ is weak*-closed
Choose $x_0^* \in X^*$ such that $\varphi(x_0^*) = 1$, so $x_0^* \notin E$
$\overset{\text{Thm 5}}{\Rightarrow} \exists x_0 \in X$ such that $\langle x_0^*, x_0\rangle = 1$ $\langle x^*, x_0\rangle = 0 \quad \forall x^* \in E$
Let $x^* \in X^*$
$\Rightarrow x^* - \varphi(x^*)x_0^* \in E$
$\Rightarrow \langle x^* - \varphi(x^*)x_0^*, x_0\rangle = 0$
$\Rightarrow \langle x^*, x_0\rangle = \varphi(x^*)\langle x^*, x_0\rangle = \varphi(x^*)$ □

Corollary 3: X Banach space, $\iota : X \to X^{**}$ canonical embedding
$S := \{x \in X \mid \|x\| = 1\}$
The weak*-closure of $\iota(S) \in X^{**}$ is $B^{**} = \{x^{**} \in X^{**} \mid \|x\| \leq 1\}$

Proof: $K :=$ weak* closure of $\iota(S)$

1. $K \subset B^{**}$ because B^{**} is closed

2. K is convex:
   Key fact: $\iota : \underbrace{X}_{\text{weak top}} \to \underbrace{X^{**}}_{\text{weak}^*\text{-top}}$ is continuous
   Hence $\iota(B) \subset K : x \in B$ $U \subset X^{**}$ weak* open, $\iota(x) \in U$
   $\Rightarrow \iota^{-1}(U) \subset X$ weakly open and $x \in \iota^{-1}(U)$
   $\Rightarrow \iota^{-1}(U) \cap S \neq \varnothing$
   $\Rightarrow U \cap \iota(S) \neq \varnothing \overset{\forall U}{\Rightarrow} \iota(x) \in K$
   So K is the weak*-closure of the convex set $\iota(B) \overset{\text{Lemma 1}}{\Rightarrow} K$ is convex

3. $B^{**} \subset K$
   $x_0^{**} \notin K \Rightarrow \exists$ weak*-continuous linear functional $\varphi : X^{**} \to \mathbb{R}$ such that $\sup_{x^{**} \in K} \varphi(x^{**}) < \varphi(x_0^{**})$
   $\overset{\text{Cor 2}}{\Rightarrow} \exists x_0^* \in X^*$ such that $\phi(x^{**}) = \langle x^{**}, x_0^*\rangle$
   $\Rightarrow \langle x_0^{**}, x_0^*\rangle > \sup_{x^{**} \in K} \langle x^{**}, x_0^*\rangle \geq \sup_{x \in S} \langle x_0^*, x\rangle = \|x_0^*\| \Rightarrow \|x_0^{**}\| > 1 \Rightarrow x_0^{**} \notin B^{**}$

□

Lemma 6: X normed vector space

(i) If $x_1^*, \dots, x_n^* \in X^*$ are linearly independent, then $\exists x_1, \dots, x_n \in X$ such that
$$\langle x_i^*, x_j \rangle = \delta_{ij}$$

(ii) If $x_1^*, \dots, x_n^* \in X$ are lin. indep., then
$$X_0 := \{x \in X \mid \langle x_i^*, x \rangle = 0, i = 1, \dots, n\}$$
is a closed subspace of codimension n and $X_0^\perp = \operatorname{span}\{x_1^*, \dots, x_n^*\}$

Proof:

(i) for n $\Rightarrow$(ii) for n: $\forall x \in X$ we have: $x - \sum_{i=1}^n \langle x_i^*, x \rangle x_i \in X_0$
This shows: $X = X_0 \oplus \operatorname{span}\{x_1, \dots, x_n\}$ and, moreover $x^* \in X_0^\perp$

$$\begin{aligned} \Rightarrow 0 &= \langle x^*, x - \sum_i \langle x_i^*, x \rangle x_i \rangle \\ &= \langle x^*, x \rangle - \sum_i \langle x_i^*, x \rangle \cdot \langle x^*, x_i \rangle \\ &= \langle x^* - \sum_{i=1}^n \langle x^*, x_i \rangle x_i^*, x \rangle \forall x \in X \\ \Rightarrow x^* &= \sum_{i=1}^n \langle x^*, x_i \rangle x_i^* \in \operatorname{span}\{x_1^*, \dots, x_n^*\} \end{aligned}$$

(ii) for n $\Rightarrow$(i) for $n+1$: Let $x_1^*, \dots, x_{n+1}^* \in X^*$ be lin. indep. for $i = 1, \dots, n+1$ denote $X_i := \{x \in X \mid \langle x_j^*, x \rangle = 0, j \neq i\}$

$$\overset{\text{(i) for } n}{\Rightarrow} \quad X_i^\perp = \operatorname{span}\{x_j^* \mid j \neq i\}$$
$$\Rightarrow x_i^* \notin X_i^\perp \Rightarrow \exists x_i \in X_i \text{ such that } \langle x_i^*, x_i \rangle = 1$$
$$\Rightarrow \langle x_j^*, x_i \rangle = \delta_{ij}$$

□

Remark 1: Converse if $x_i, \dots, x_n \in X$ are lin. indep., then $\exists x_1^*, \dots, x_n^* \in X^*$ such that $\langle x_i^*, x_j \rangle = \delta_{ij}$

Remark 2: $x_1^*, \dots, x_n^* \in X^*$ lin indep, $c_1, \dots, c_n \in \mathbb{R}$
$\Rightarrow \exists x \in X$ such that $\langle x_i^*, x \rangle = c_i$ (namely: $x = \sum_{i=1}^n c_i x_i$ with x_i as in Lemma 6).

Lemma 7: X normed vector space, $x_1^*, \dots, x_n^* \in X^*, c_1, \dots, c_n \in \mathbb{R}, M \geq 0$. Equivalent are:

(i) $\forall \varepsilon > 0 \exists x \in X$ such that
$$\langle x_i^*, x \rangle = c_i, i = 1, \dots, n \quad \|x\| \leq M + \varepsilon$$

(ii) $\forall \lambda_1, \dots, \lambda_n \in \mathbb{R}$ we have
$$|\sum_{i=1}^n \lambda_i c_i| \leq M \|\sum_{i=1}^n \lambda_i x_i^*\|$$

Proof: (i)⇒(ii):
$x = x_\varepsilon \in X$ as in (i). Then

$$\begin{aligned} |\sum_i \lambda_i c_i| &= |\sum_i \lambda_i \langle x_i^*, x_\varepsilon\rangle| \\ &= |\langle \sum_i \lambda_i x_i^*, x_\varepsilon\rangle| \\ &= \|\sum_i \lambda_i x_i^*\| \cdot \|x_\varepsilon\| \\ &= (M+\varepsilon)\|\sum_i \lambda_i x_i^*\| \ \forall \varepsilon > 0 \end{aligned}$$

(ii)⇒(i):
Assume $x_1^*, \dots, x_n^*$ lin indep (w.l.o.g)
Choose $x \in X$ such that $\langle x_i^*, x\rangle = c_i \forall i$ (Remark 2). X as in Lemma 6.

$$\begin{aligned} \inf_{\xi \in X_0} \|x+\xi\| &= \sup_{0 \neq x^* \in X_0^\perp} \frac{|\langle x^*, x\rangle|}{\|x^*\|} \\ \text{(by Lemma 6)} &= \sup_{\lambda_i \in \mathbb{R}} \frac{|\langle \sum \lambda_i x_i^*, x\rangle|}{\|\sum_i \lambda_i x_i^*\|} \\ &= \sup_{\lambda_i} \frac{|\sum_i \lambda_i c_i}{|\sum_i \lambda_i x_i^*|} \\ &\leq M \end{aligned}$$

□

Theorem 6: X Banach space. Equivalent are:

(i) X is reflexive

(ii) The unit ball $B := \{x \in X \mid \|x\| \leq 1\}$ is weakly compact

(iii) Every bounded sequence in X has a weakly convergent subsequence

Proof: $\iota : X \to X^{**}$ is an isomorphism from X with the weak topology to X^{**} with the weak*topology.
$U \subset X$ weakly open ⇔ $\iota(U) \subset X^{**}$ is weak*-open.

(i)⇒(ii) $\iota(B)$ is the unit ball in X^{**}, hence is weak*compact (Thm 2), so B is weakly compact.

(i)⇒(iii) X separable and reflexive ⇒ X^* separable (Chapter II, Thm 9). $(x_n) \in X$ bounded sequence ⇒ $\iota(x_n) \in X^{**}$ is a bounded sequence. So, by Thm 1, $\iota(x_n)$ has a weak*-convergent subsequence $\iota(x_{n_i})$
⇒ x_{n_i} converges weakly.

(i)⇒(iii): The nonseparable case Let $x_n \in X$ be a bounded sequence. Denote $Y := \overline{\left\{\sum_{n=1}^N \lambda_n x_n \mid N \in \mathbb{N}, \lambda_n \in \mathbb{R}\right\}}$
⇒ Y is separable and reflexive (Chapter II, Thm 8)
⇒ x_n by separable case has a subsequence x_{n_i} converging weakly in Y,ie.

$$\exists x \in Y \forall y^* \in Y^* : \langle y^*, x\rangle = \lim_{i\to\infty} \langle y^*, x_{n_i}\rangle$$

$$\overset{\text{by Hahn-Banach}}{\Rightarrow} \quad \forall x^* \in X^* : \langle x^*, x\rangle = \lim_{i\to\infty} \langle x^*, x_{n_i}\rangle$$

(ii)⇒(i) Let $x^{**} \in X^{**}, x^{**} \neq 0$.
Claim: For every finite set $S \subset X^*$ there exists an $x \in X$ such that

$$\langle x^*, x\rangle = \langle x^{**}, x^*\rangle \ \forall x^* \in S \quad \|x\| \leq 2\|x^{**}\|$$

$\mathcal{S} := \{S \subset X^* \mid \text{S finite subset}\}$ and $K(S) := \{x \in X \mid \|x\| \leq 2\|x^{**}\|, \langle x^{**}, x^*\rangle = \langle x^*, x\rangle \forall x^* \in S\}$
Note:

- $K(S)$ is a weakly closed subset of $cB = \{x \in X \mid \|x\| \leq c\}$ where $c = 2\|x^{**}\|$.
- cB is weakly compact, by (ii).
- The collection $\{K(S) \mid S \in \mathcal{S}\}$ is FiP, because

$$K(S_1) \cap \ldots \cap K(S_n) = K(S_1 \cup \ldots \cup S_n) \neq \varnothing \Rightarrow \bigcap_{S \in \mathcal{S}} K(S) \neq \varnothing$$

$$\Rightarrow \exists x \in X \text{ such that} \langle x^*, x\rangle = \langle x^{**}, x^*\rangle \forall x^* \in X^*$$

Proof of claim:
Write $S = \{x_1^*, \ldots, x_n^*\}, c_i := \langle x^{**}, x_i^*\rangle$.

$$|\sum_i \lambda_i c_i| = |\sum_i \lambda_i \langle x^{**}, x_i^*\rangle| = |\langle x^{**}, \sum_i \lambda_i x_i^*\rangle| \leq \|x^{**}\| \cdot \|\sum_i \lambda_i x_i^*\|$$

Assumption of Lemma 7 holds with $M = \|x^{**}\| > 0$ Choose $\varepsilon = \|x^{**}\| > 0$.

(iii)⇒(i) Let $x_0^{**} \in X^{**}, \|x_0^{**}\| \leq 1$. Denote $E := \{x^* \in X^* \mid \langle x_0^{**}, x^*\rangle = 0\}$ and $B^* := \{x^* \in X^* \mid \|x^*\| \leq 1\}$
Claim 1: $E \bigcap B^*$ is weak*closed.
Claim 1 $\overset{\text{by Cor 1}}{\Rightarrow}$ E is weak*-closed $\overset{\text{by Cor 2}}{\Rightarrow}$

$$\exists x_0 \in X \forall x^* \in X^* : \langle x_0^{**}, x^*\rangle = \langle x^*, x_0\rangle \Rightarrow x_0^{**} = \iota(x_0)$$

So $\iota : X \to X^{**}$ is surjective.
Claim 2: $\forall x_1^*, \ldots, x_n^* \in X^*, \exists x \in X$ such that

$$\langle x_i^*, x\rangle = \langle x_0^{**}, x_i^*\rangle, i = 1, \ldots, n \quad \|x\| \leq 1$$

Proof of Claim 2 Denote $U_m := \{x^{**} \in X^{**} \mid |\langle x^{**} - x_0^{**}, x_i^*\rangle| < \frac{1}{m}; i = 1, \ldots n\}$
$\Rightarrow x_0^{**} \in U_m$, U_m is weak*open. Moreover $\|x_0^{**}\| \leq 1$.
Recall the weak*closure of $\iota(S), S := \{x \in X \mid \|x\| = 1\}$ is the closed unit ball in X^{**} (Cor 3). $\Rightarrow U_m \bigcap \iota(S) \neq \varnothing \quad \exists x_m \in X$ such that

$$\|x_m\| = 1 \quad |\langle x_i^*, x_m\rangle - \langle x_0^{**}, x_i^*\rangle| < \frac{1}{m} \quad i = 1, \ldots, n$$

$\Rightarrow$ by (iii) $\exists$ weakly convergent subsequence $x_{m_k} \rightharpoonup x$.
$\Rightarrow \|x\| \leq 1$ and

$$\langle x_i^*, x\rangle = \lim_{k \to \infty} \langle x_i^*, x_{m_k}\rangle = \langle x_0^{**}, x_i^*\rangle \quad i = 1, \ldots n$$

Proof of Claim 1 Let $x_0^* \in$ weak*closure of $E \bigcap B^*$. We must prove that $x_0^* \in E \bigcap B^*$. Clearly $\|x_0^*\| \leq 1$. So it remains to prove $\langle x_0^{**}, x_0^*\rangle = 0$

Step 1 Let $\varepsilon > 0$. Then $\exists$ sequences $x_n \in X, n \geq 1, x_n^* \in X^*$ such that

(1) $\|x_n\| \leq 1, \|x_n^*\| \leq 1, \langle x_0^{**}, x_n^*\rangle = 0$

(2) $\langle x_i^*, x_n\rangle = \langle x_0^{**}, x_i^*\rangle \quad i = 0, \ldots, n-1$

(3) $|\langle x_n^* - x_0^*, x_i\rangle| < \varepsilon \quad i = 1, \ldots, n$

Proof of Step 1 Induction $n = 1$: a) By Claim 2, $\exists x_1 \in X$ such that $\|x_1\| \leq 1, \langle x_0^*, x_1\rangle = \langle x_0^{**}, x_0^*\rangle$ b) Because $x_0^* \in$ weak*closure $(E \bigcap B^*)$ $\exists x_1^* \in E \bigcap B^*$ such that $|\langle x_1^* - x_0^*, x_1\rangle| \leq \varepsilon$
$\Rightarrow$ (1),(2),(3) hold for $n = 1$.
$n \geq 1$: Suppose x_i, x_i^* have been constructed for $i = 1, \dots, n$.

a) By Claim 2,

$$\exists x_{n+1} \in X : \langle x_i^*, x_{n+1}\rangle = \langle x_0^{**}, x_i^*\rangle \quad i = 0, \dots, n \quad \|x_{n+1}\| \leq 1$$

b) $\exists x_{n+1}^* \in E \bigcap B^*$ such that $|\langle x_{n+1}^* - x_0^*, x_i\rangle| \leq \varepsilon, i = 1, \dots, n$

Step 2 $\langle x_0^{**}, x_0^*\rangle = 0$
By (iii) $\exists$ weakly convergent subsequence

$$x_{n_i} \rightharpoonup x_0 \in X \quad \|x_0\| \leq 1$$

$\Rightarrow$ by Lemma 4 $\exists m \in \mathbb{N}\ \exists \lambda_1, \dots, \lambda_m \geq 0$
(4) $\sum_{i=1}^m \lambda_i = 1 \quad \|x_0 - \sum_{i=1}^m \lambda_i x_i\| < \varepsilon$

a) $\langle x_m^*, x_0\rangle = \lim\limits_{i\to\infty} \langle x_m^*, x_{n_i}\rangle = \lim\limits_{i\to\infty} \langle x_0^{**}, x_m^*\rangle = 0$

b)

$$\begin{aligned}
|\langle x_0^{**}, x_0^*\rangle| &\overset{\text{(by a)}}{\leq} \left|\langle x_0^{**}, x_0^*\rangle - \left\langle x_M^*, \sum_i^m \lambda_i x_i \right\rangle\right| + \left|\left\langle x_m^*, \sum_{i=1}^m \lambda_i x_i - x_o\right\rangle\right| \\
&\leq \sum_i \lambda_i \underbrace{|\langle x_0^{**}, x_0^*\rangle - \langle x_m^*, x_i\rangle|}_{\leq \varepsilon \text{by (3)}} + \underbrace{\left\|\sum_i^m \lambda_i x_i - x_0\right\|}_{\leq (3)\ \text{by}(4)} \\
&\leq 2\varepsilon
\end{aligned}$$

□

3.3 Ergodic measures

(M, d) compact metric space, $f : M \to M$ homeomorphism
$\mathcal{M}(f) := \{f\text{-invariant Borel probability measure } \mu : \mathcal{B} \to [0, \infty)\}$
$\mathcal{B} \subset 2^M$ Borel σ-algebra, $\mu(M) = 1$, $\mu(f(E)) = \mu(E) \quad \forall E \in \mathcal{B}$
We know: $\mathcal{M}(f)$ nonempty, convex, weak*-compact.

Definition: An f-invariant Borel-measure $\mu \in \mathcal{M}(f)$ is called *ergodic*, if $\forall \Lambda \in \mathcal{B} \quad \Lambda = f(\Lambda) \Rightarrow \mu(\Lambda) \in \{0, 1\}$

Example: $M = S^2 \quad \delta_N(\Lambda) = \begin{cases} 1 & N \in \Lambda \\ 0 & N \notin \Lambda \end{cases} \quad \delta_S(\Lambda) = \begin{cases} 1 & S \in \Lambda \\ 0 & S \notin \Lambda \end{cases}$ where N stands for north pole and S for south pole.

Definition: X vectorspace, $K \subset X$ convex
$x \in K$ is called an *extremal point* of K, if the following holds:

$$\left.\begin{array}{l} x_0, x_1 \in K \\ x = (1-\lambda)x_0 + \lambda x_1 \\ 0 < \lambda < 1 \end{array}\right\} \Rightarrow x_0 = x_1 = x$$

Lemma 8: $\mu \in \mathcal{M}(f)$ extremal point$\Rightarrow \mu$ is ergodic

Proof: Suppose not.
$\Rightarrow \exists \Lambda \in \mathcal{B}$ such that $\Lambda = f(\Lambda)$, $0 < \mu(\Lambda) < 1$
Define $\mu_0, \mu_1 : \mathcal{B} \to [0, \infty)$ by $\mu_0(E) := \frac{\mu(E \setminus \Lambda)}{1-\mu(\Lambda)}$ $\quad \mu_1(E) = \frac{\mu(E \cap \Lambda)}{\mu(\Lambda)}$
$\mu_0, \mu_1 \in \mathcal{M}(\{)$, $\quad \mu_0 \neq \mu$, $\mu_1 \neq \mu$
$\mu = (1-\lambda)\mu_0 + \lambda\mu_1 \quad \lambda = \mu(\Lambda)$
$\Rightarrow \mu$ is not extreme □

Theorem 7 (Krein-Milman): X locally convex topological T2 vectorspace
$K \subset X$ nonempty, compact, convex
$E :=$ set of extremal points of K
$C :=$convex hull of E $:=\{\sum_{i=1}^m \lambda_i e_i \mid e_i \in E \, \lambda_i \geq 0 \, \sum_{i=1}^m \lambda_i = 1\}$
$\Rightarrow \overline{C} = K$ (in particular $C \neq \varnothing$)

Corollary: Every homeomorphism of a compact metric space has an ergodic measure

Proof: Apply Theorem 7 to the case $X = C(M)^*$ with weak*-topology and $K = \mathcal{M}(f)$ □

Proof of Theorem 7:

Step 1 $A, B \subset X$ nonempty, disjoint, convex sets, A open $\Rightarrow \exists$ continuous linear functional $\varphi : X \to \mathbb{R}$ such that $\varphi(a) < \inf_{b \in B} \varphi(b) \quad \forall a \in A$
Proof of Step 1: Hahn-Banach as in Chapter II Theorem 6

Step 2 $\forall x \in X$, $x \neq 0 \quad \exists$ linear functional $\varphi : X \to \mathbb{R}$ with $\varphi(x) \neq 0$
Proof: Choose an open convex neighborhood $A \subset X$ of 0 such that $x \notin A$. Denote $B = \{x\}$. Now apply Step 1.

Step 3 $E \neq \varnothing$
Proof:
A nonempty compact convex subset $K' \subset X$ is called a *face* of K if

$$K' \subset K \text{ and } \left.\begin{array}{l} x \in K',\, x_0, x_1 \in K \\ x = (1-\lambda)x_0 + \lambda x_1 \\ 0 < \lambda < 1 \end{array}\right\} \Rightarrow x_0, x_1 \in K'$$

Denote $\mathcal{K} := \{K \subset X \mid K \text{ is nonempty, compact, convex}\}$
$\mathcal{K}$ is partially ordered by $K' \subset K \overset{\text{def}}{\Leftrightarrow} K'$ is a face of K

- $K \preccurlyeq K$
- $K \preccurlyeq K'$, $K' \preccurlyeq K \Rightarrow K' = K$
- $K'' \preccurlyeq K'$, $K' \preccurlyeq K \Rightarrow K'' \preccurlyeq K$

If $\mathcal{C} \subset \mathcal{K}$ is a chain, then $K_0 := \bigcap_{C \in \mathcal{C}} C \in \mathcal{K}$!
Because of the FIP characterisation of compactness we have $K_0 \neq \varnothing$
Zorns Lemma implies: For every $K \in \mathcal{K}$, there is a minimal element $K_0 \in \mathcal{K}$ such that $K_0 \preccurlyeq K$
Claim: $K_0 = \{\text{pt}\}$
Suppose $K_0 \ni x_0, x_1 \quad x_0 \neq x_1$
$\overset{\text{Step 2}}{\Rightarrow} \exists \varphi : X \to \mathbb{R}$ continuous linear such that $\varphi(x_1 - x_0) > 0$ so $\varphi(x_0) < \varphi(x_1)$
$K_1 := K_0 \cap \varphi^{-1}(\sup_{x \in K_0} \varphi(x)) \in \mathcal{K} \Rightarrow K_1 \preccurlyeq K_0$ and $K_0 \neq K_1$ since $x_0 \in K_0 \setminus K_1$
So K_0 is not minimal. Contradiction.
Claim$\Rightarrow K_0 = \{x_0\} \Rightarrow x_0 \in E$ (by definition of face)

Step 4 $K = \overline{C}$
Clearly $\overline{C} \subset K$
Suppose $\overline{C} \subsetneqq K$, let $x_0 \in K \setminus \overline{C}$
$\overset{\text{Step 1}}{\Rightarrow}$ $\exists \varphi : X \to \mathbb{R}$ continuous linear such that $\varphi(x_0) > \sup_{\overline{C}} \varphi$
$K_0 := K \cap \varphi^{-1}(\sup_K \varphi)$ is a face of K and $K_0 \cap \overline{C} = \varnothing$
$\overset{\text{Step 3}}{\Rightarrow}$ K_0 has an extremal point e
$\Rightarrow e$ is extremal point of K
Contradiction, because $e \notin \overline{C}$ □

(M, d) compact metric space
$f : M \to M$ homeomorphismus, $\mu \in \mathcal{M}(f)$ ergodic
$u : M \to \mathbb{R}$ continuous, $x \in M$
Question: $\frac{1}{n}\sum_{k=0}^{n-1} u(f^k(x)) \overset{?}{\to} \int_M u\, d\mu$

Theorem (Birkhoff): $\forall u \in C(M)\, \exists \Lambda \in \mathcal{B}$ such that $f(\Lambda) = \Lambda$ $\mu(\Lambda) = 1$
$\int_M u\, d\mu = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} u(f^k(x)) \quad \forall x \in \Lambda$
Without proof

Theorem 8 (von Neumann): (M, d) compact metric space, $f : M \to M$ homeomorphismus, $\mu \in \mathcal{M}(f)$ ergodic, $1 < p < \infty$
$\Rightarrow \lim_{n\to\infty} \left\| \frac{1}{n}\sum_{k=0}^{n-1} u(f^k(x)) - \int_M u\, d\mu \right\|_{L^p} = 0$

Theorem 9 (Abstract Ergodic Theorem): X Banach space, $T \in \mathcal{L}(X)$, $c \geq 1$
Assume $\|T^n\| \leq c \quad \forall n \in \mathbb{N}$
Denote $S_n := \frac{1}{n}\sum_{k=0}^{n-1} T^k \in \mathcal{L}(X)$
Then the following holds:

(i) For $x \in X$ we have: $(S_n x)_{n\in\mathbb{N}}$ converges $\Leftrightarrow$ $(S_n x)_{n\in\mathbb{N}}$ has a weakly convergent subsequence.

(ii) The set $Z := \{x \in X \mid S_n x \text{ converges}\}$ is a closed linear subspace of X and
$Z = \operatorname{Ker}(1-T) \oplus \overline{\operatorname{Im}(1-T)}$
If X is reflexive $Z = X$

(iii) Define $S : Z \to Z$ by $S(x+y) := x \quad x \in \ker(1-T)$, $y \in \overline{\operatorname{Im}(1-T)}$
Then $Sz := \lim_{n\to\infty} S_n z \quad \forall z \in Z$ and $ST = TS = S^2 = S$, $\|S\| \leq c$

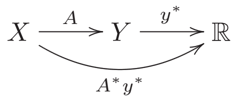
Proof of Theorem 9$\Rightarrow$ Theorem 8 $X = L^p(\mu)$ $Tu := u \circ f$
$\int_M |u \circ f|^p\, d\mu = \int_M |u|^p\, d\mu$, so $\|Tu\|_p = \|u\|_p$
$\|T\| = 1 \quad \forall k \in \mathbb{N}$
$(S_n u)(x) = \frac{1}{n}\sum_{i=1}^{n-1} u(f^k(x))$
To show: $\lim_{n\to\infty} \left\|S_n u - \int_M u\, d\mu\right\|_{L^p(\mu)} = 0$
Equivalently Claim 1: $(Su)(x) = \int_M u\, d\mu \quad \forall x \in M$
(By Theorem 9 we have $S_n u \to Su$ in $L^p(\mu)$)

Claim 2: $Tv = v \Rightarrow v \equiv \text{const.}$
Claim 2$\Rightarrow$ Claim 1: $Su \in \operatorname{Ker}(1-T) \Rightarrow Su \equiv c$
$c = \int_M Su\, d\mu = \lim_{n\to\infty} \int_M S_n u\, d\mu = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \int_M u \circ f^k\, d\mu = \int_M u\, d\mu$

Proof of Claim 2:
$v : M \to \mathbb{R}$ measurable, $\int_M |v|^p\, d\mu < \infty$
$[v] \in L^p(\mu) \quad T[v] = [v] \Leftrightarrow v \circ f = v$ almost everywhere
$E_0 := \{x \in M \mid v(x) \neq v(f(x))\}$ measure zero
$\Rightarrow E := \bigcup_{k\in\mathbb{N}} f^k(E_0) \quad \mu(E) = 0$

$M \setminus E = \Lambda_0 \cup \Lambda_+ \cup \Lambda_-$
$\Lambda_0 = \{x \in M \mid v(x) = c\} \quad \Lambda_\pm = \{x \in M \mid \pm v(x) > c\}$ where $c := \int_M v\, d\mu$
$\Rightarrow \Lambda_0, \Lambda_\pm$ $f-$invariant, $\mu(\Lambda_0) + \mu(\Lambda_+) + \mu(\Lambda_-) = 1$
$\overset{\mu \text{ ergodic}}{\Rightarrow} \quad \mu(\Lambda_0) = 1$
because otherwise: $\mu(\Lambda_+) = 1$ so $\int_M v\, d\mu > c$
or $\mu(\Lambda_-) = 1$ so $\int_M v\, d\mu < c$ □
Let $A \in \mathcal{L}(X, Y)$.

Definition: The *dual operator* of A is the bounded linear operator $A^* \in \mathcal{L}(Y^*, X^*)$ defined by $\langle A^*y^*, x\rangle := \langle y^*, Ax\rangle$ for $y^* \in Y^*$ and $x \in X$, ie.



Remark: $\|A^*\| = \|A\|$

Proof:

$$\begin{aligned}
\|A^*\| &= \sup_{y^* \neq 0} \frac{\|A^*y^*\|}{\|y^*\|} \\
&= \sup_{y^* \neq 0} \sup_{x \neq 0} \frac{\langle A^*y^*, x\rangle}{\|x\| \cdot \|y^*\|} \\
&= \sup_{\substack{y^* \neq 0 \\ x \neq 0}} \frac{\langle y^*, Ax\rangle}{\|y^*\| \cdot \|x\|} \\
&\overset{\text{by Hahn-Banach}}{=} \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} \\
&= \|A\|
\end{aligned}$$

□

Proof of Thm 9: X is space. $T \in \mathcal{L}(X), c \geq 1, \|T^k\| \leq c, k = 0, 1, 2, \ldots$. Denote $S_n := \frac{1}{n}\sum_{k=0}^{n-1} T^k$

Step 1

$$\|S_n\| \leq c, \|S_n(\mathbb{1} - T\| \leq \frac{1+c}{n}$$

Proof

$$\|S_n\| \leq \frac{1}{n}\sum_{k=0}^{n-1} \|T^k\| \leq c$$

$$S_n(\mathbb{1} - T) = \frac{1}{n}\sum_{k=0}^{n-1} T^k - \frac{1}{n}\sum_{k=1}^{n} T^k = \frac{1}{n}(\mathbb{1} - T^n)$$

Step 2 $\forall x, \xi \in X$ with $Tx = x$ we have $\|x\| \leq c\|x + \xi - T\xi\|$.

Proof $x = Tx = T^2x = \ldots \Rightarrow S_n x = x \; \forall n \in \mathbb{N}$.
Also, by Step 1, $\lim_{n\to\infty} \|S_n(\xi - T\xi)\| = 0$

$$\Rightarrow \|x\| = \lim_{n\to\infty} \underbrace{\|S_n(x + \xi - T\xi)\|}_{\leq c\|x+\xi-T\xi\|} \leq c\|x + \xi - T\xi\|$$

Step 3

$$x \in \ker(\mathbb{1}-T), y \in \overline{\operatorname{im}(\mathbb{1}-T)} \Rightarrow \|x\| \leq c\|x+y\|$$

Proof

$$\exists \xi_n \in X \text{ such that } \xi_n - T\xi_n \to y$$
$$\Rightarrow \text{ by Step 2 } \|x\| \leq c\|x + \underbrace{\xi_n - T\xi_n}_{\to y}\|$$
$$\overset{n\to\infty}{\Longrightarrow} \|x\| \leq c\|x+y\|$$

Step 4 $\ker(\mathbb{1}-T) \cap \overline{\operatorname{im}(\mathbb{1}-T)} = \{0\}$ and the subspace $X_0 := \ker(\mathbb{1}-T) \oplus \overline{\operatorname{im}(\mathbb{1}-T)}$ is closed

Proof For $x \in \ker(\mathbb{1}-T) \cap \overline{\operatorname{im}(\mathbb{1}-T)}$ choose $y := -x$ in Step 3 $\Rightarrow x = 0$
Let $z_n \in X_0, z = \lim_{n\to\infty} z_n \in X$.
Write $z_n = x_n + y_n, x_n \in \ker(\mathbb{1}-T), y_n \in \overline{\operatorname{im}(\mathbb{1}-T)}$

$$\overset{\text{by Step 3}}{\Rightarrow} \|x_n - x_m\| \leq c\|z_n - z_m\|$$
$$\Rightarrow x_n \text{ is Cauchy}$$

$\Rightarrow$ The limits exist and $x := \lim_{n\to\infty} x_n \in \ker(\mathbb{1}-T), y := \lim_{n\to\infty} y_n \in \overline{\operatorname{im}(\mathbb{1}-t)}, z = x+y \in X_0$

Step 5 Let $z = x + y \in X_0$ where $x \in \ker(\mathbb{1}-T), y \in \overline{\operatorname{im}(\mathbb{1}-T)}$.
$\Rightarrow S_n \in X_0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} S_n z = x$

Proof

a) $x = Tx \Rightarrow S_n x = x \in X_0$

b) Choose $\xi_k \in X$ such that $\xi_k - T\xi_k \to y$. Then

$$\begin{aligned} S_n y &= \lim_{k\to\infty} S_n(\mathbb{1}-T)\xi_k \\ &= \lim_{k\to\infty} \underbrace{(\mathbb{1}-T)S_n\xi_k}_{\in \operatorname{im}(\mathbb{1}-T)} \\ &\in \overline{\operatorname{im}(\mathbb{1}-T)} \end{aligned}$$

c) By Step 1,

$$\|S_n(\xi T\xi)\| \leq \frac{1+c}{n}\|\xi\| \forall \xi \in X$$

So $S_n y \to 0 \ \forall y \in \operatorname{im}(\mathbb{1}-T)$
$\Rightarrow$ (by Ch II, Thm 2 (ii)) $S_n y \to 0 \forall y \in \overline{\operatorname{im}(\mathbb{1}-T)}$
$\Rightarrow S_n z = x + S_n y \to x$.

Step 6 Let $x, z \in X$. Equivalent are

(i) $\lim_{n\to\infty} S_n z = x$

(ii) $\exists$ subsequence $n_1 < n_2 < n_3 < \ldots$ such that

$$w - \lim_{i\to\infty} S_{n_i} z = x$$

(iii) $Tx = x, z - x \in \overline{\operatorname{im}(\mathbb{1}-T)}$

Proof (iii) $\overset{\text{by Step 5}}{\Rightarrow}$ (i) $\overset{\text{obvious}}{\Rightarrow}$ (ii).
(ii) $\Rightarrow$ (iii)

$$\begin{aligned}\langle x^*, Tx - x\rangle &= \langle T^*x^* - x^*, x\rangle \\ &\overset{\text{by (ii)}}{=} \lim_{i\to\infty} \langle T^*x^* - x^*, S_{n_i} z\rangle \\ &= \lim_{i\to\infty} \langle x^*, \underbrace{T - \mathbb{1})S_{n_i}}_{\|\cdot\|\le \frac{1+c}{n}} z\rangle \\ &= 0 \quad \forall x^* \in X^* \text{ by Step 1}\end{aligned}$$

$\Rightarrow$ by Hahn-Banach $Tx - x = 0$.
Suppose $x^* \in (\mathrm{im}(\mathbb{1} - T))^{\perp}$

$$\Rightarrow \langle x^*, \xi - T\xi\rangle = 0 \ \forall \xi \in X$$

$$\Rightarrow \langle x^* - T^*x^*, \xi\rangle = 0 \forall \xi$$

$$\Rightarrow x^* = T^*x^* = (T^*)^2 x^* \cdots, x^* = S_{n_i}^* x^* \forall i$$

$$\langle x^*, z - x\rangle \overset{(ii)}{=} \lim_{i\to\infty} \langle x^*, z - S_{n_i} z\rangle$$

$$= \lim_{i\to\infty} \langle x^* - S_{n_i}^* x^*, z\rangle = 0$$

So $z - x \in^{\perp} ((\mathrm{im}(\mathbb{1} - T))^{\perp}) = \overline{\mathrm{im}(\mathbb{1} - T)}$. $\overset{\text{Step 6}}{\Rightarrow}$ Theorem 9.
$X_0 = X$ in the reflexive case. □

4 Compact operators and Fredholm theory

4.1 Compact operators

Lemma 1: X, Y Banach spaces, $K \in \mathcal{L}(X, Y)$. Equivalent are:

(i) If $x_n \in X$ is a bounded sequence then $Kx_n \in Y$ has a convergent subsequence

(ii) If $S \subset X$ is a bounded subset then $\overline{KS}$ is a compact subset of Y.

(iii) The set $\overline{\{Kx \mid \|x\| \leq 1\}} \subset Y$ is compact.

The operator K is called *compact* if it satisfies these equivalent conditions.

Proof: (i) $\Rightarrow$ (ii):
To Show: Every sequence in KS has a Cauchy subsequence (see Ch I, Lemma 7). Let $y_n \in KS$. Choose $x_n \in S$ such that $y_n = Kx_n$.
$\Rightarrow x_n$ is a bounded subsequence $\overset{(i)}{\Rightarrow} Kx_n$ has a convergent subsequence $Kx_{n_i} = y_{n_i} \Rightarrow y_{n_i}$ is Cauchy.
(ii) $\Rightarrow$ (iii): take $S := \{x \in X \mid \|x\| \leq 1\}$
(iii) $\Rightarrow$ (i): $x_n \in X$ bounded. Choose $c > 0$ such that $\|x_n\| \leq c \forall n$
$\overset{(iii)}{\Rightarrow} K\frac{x_n}{c}$ has a convergent subsequence $\Rightarrow Kx_n$ has a convergent subsequence
□

Example 1: $T : X \to Y$ surjective, $\dim Y = \infty \Rightarrow T$ not compact. (open mapping theorem) $\{Tx \mid \|x\| < 1\} \supset \{y \in Y \mid \|y\| \leq \delta\}$ for some $\delta > 0$ not compact.

Example 2: $K \in \mathcal{L}(X, Y), \mathrm{im} K$ finite dimensional $\Rightarrow K$ is compact.

Example 3: $X = C^1([0, 1]), Y = C^0([0, 1]), K : X \to Y$ obvious inclusion $\Rightarrow K$ is compact. Arzela-Ascoli.

Example 4: $X = Y = \ell^p, 1 \leq p \leq \infty \quad \lambda_1, \lambda_2, \ldots \in \mathbb{R}$ bounded.
$Kx := (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \ldots)$

$$K \text{ compact} \Leftrightarrow \lim_{n\to\infty} \lambda_n = 0$$

(exercise).

Theorem 1: X, Y, Z Banach spaces

(i) $A \in \mathcal{L}(X, Y), B \in \mathcal{L}(Y, Z)$

$$A \text{ compact or } B \text{ compact} \Rightarrow BA \text{ is compact}$$

(ii) $K_\nu \in \mathcal{L}(X, Y)$ compact $\nu = 1, 2, 3, \ldots$ $K \in \mathcal{L}(X, Y)$ such that $\lim_{\nu\to\infty} \|K_\nu - K\| = 0 \Rightarrow K$ is compact.

(iii) K compact $\Leftrightarrow K^*$ is compact.

Proof: (i): Exercise
(ii): $x_n \in X$ bounded, $c := \sup_{n\in\mathbb{N}} \|x_n\| < \infty$
Diagonal Sequence Argument: $\exists$ subsequence x_{n_i} such that $(K_\nu x_{n_i})_{i=1}^\infty$ is Cauchy for every $\nu \in \mathbb{N}$.
Claim: $(Kx_{n_i})_{i=1}^\infty$ is Cauchy.
$\varepsilon > 0$. Choose ν such that $\|K_\nu - K\| < \frac{\varepsilon}{3c}$.
Choose $N \forall i, j \geq N : \|K_\nu x_{n_i} - K_\nu x_{n_j}\| < \frac{\varepsilon}{3}$.

$$\overset{i,j\geq N}{\Rightarrow} \|Kx_{n_i} - Kx_{n_j}\| \leq \underbrace{\|(K-K_\nu)x_{n_i}\|}_{<\|K-K_\nu\|\cdot\|x_{n_i}\|<\frac{\varepsilon}{3}\|} + \underbrace{\|K_\nu x_{n_i} - K_\nu x_{n_j}\|}_{<\frac{\varepsilon}{3}} + \underbrace{\|(K_\nu - K)x_{n_j}\|}_{<\frac{\varepsilon}{3}} < \varepsilon$$

(iii): K compact $\Rightarrow K^*$ compact.
Denote $M := \overline{\{Kx \mid \|x\| \leq 1\}} \subset Y$.
M is a nonempty compact metric space. For $y^* \in Y^*$ denote $f_{y^*} := y^*|_M \in C(M)$. Let $\mathcal{F} := \{f_{y^*} \mid \|y^*\| \leq 1\} \subset C(M)$.
Note:

$$\begin{aligned}
\|f_{y^*}\| &:= \sup_{y\in M} |f_{y^*}(y)| \\
&= \sup_{y\in M} \langle y^*, y\rangle \\
&= \sup_{\|x\|\leq 1} \langle y^*, Kx\rangle \\
&= \sup_{\|x\|\leq 1} \langle K^*y^*, x\rangle \\
&= \|K^*y^*\|
\end{aligned}$$

$\mathcal{F}$ is bounded

$$f = f_{y^*} \in \mathcal{F} \Rightarrow \|f\| = \|K^*y^*\| \leq \|K^*\| \cdot \underbrace{\|y^*\|}_{\leq 1} \leq \|K\|$$

$\mathcal{F}$ is equicontinuous
$f = f_{y^*} \in \mathcal{F}, y^* \in Y^*, \|y^*\| \leq 1$.

$$\begin{aligned}
\Rightarrow |f(y_1) - f(y_2)| &= |\langle y^*, y_1 - y_2\rangle| \\
&\leq \|y^*\|\|y_1 - y_2\| \\
&\leq \|y_1 - y_2\|
\end{aligned}$$

Arzela-Ascoli $\Rightarrow \overline{\mathcal{F}}$ is compact in $C(M) \overset{!}{\Rightarrow} K^*$ is compact. $y_n^* \in Y^*, \|y^*\| \leq 1$
$\Rightarrow f_n := y_n^*|_M \in \mathcal{F} \Rightarrow f_n$ has a convergent subsequence f_{n_i}
$\|K^*y_{n_i}^*\| = \|f_{n_i} - f_{n_j}\| \Rightarrow (K^*y_{n_i}^*)_{i=1}^*$ is a Cauchy sequence $\Rightarrow K^*y_{n_i}^*$ converges.
K^* compact $\Rightarrow K$ compact K^* compact $\Rightarrow K^{**}$ compact.

$$\begin{array}{ccc}
X & \xrightarrow{K} & Y \\
{\scriptstyle \iota_X}\downarrow & & \downarrow{\scriptstyle \iota_Y} \\
X^{**} & \xrightarrow[K^{**}]{} & Y^{**}
\end{array}$$

$\overset{(i)}{\Rightarrow} \iota_Y \circ K = K^{**} \circ \iota_X : X \to Y^{**}$ is compact.
If $x_n \in X$ is bounded $\Rightarrow \iota_Y(Kx_n)$ has a convergent subsequence $\Rightarrow Kx_n$ has a convergent subsequence.

□

Lemma 2: X, Y Banach spaces, $A \in \mathcal{L}(X, Y)$

(i) $(\operatorname{Im} A)^\perp = \operatorname{Ker} A^*$ $\quad {}^\perp(\operatorname{Im} A^*) = \operatorname{Ker} A$

(ii) A^* injective $\Leftrightarrow \operatorname{Im} A$ dense in Y

(iii) A injective $\Leftrightarrow \operatorname{Im} A^*$ is weak* dense in X^*

Proof:

(i) Let $y^* \in Y^*$
$y^* \in (\operatorname{Im} A)^\perp \Leftrightarrow \underbrace{\langle y^*, Ax\rangle}_{\langle A^*y^*, y\rangle} = 0 \quad \forall x \in X \Leftrightarrow A^*y^* = 0$

Let $x \in X$. Then $x \in {}^\perp(\operatorname{Im} A^*) \Leftrightarrow$
$\underbrace{\langle A^*y^*, x\rangle}_{=\langle y^*, Ax\rangle} = 0 \quad \forall y^* \in Y^* \overset{\text{Hahn-Banach}}{\Leftrightarrow} Ax = 0$

(ii) A^* injective $\Leftrightarrow \operatorname{Ker} A^* = 0 \overset{(i)}{\Leftrightarrow} (\operatorname{Im} A)^\perp = 0$
$\Leftrightarrow \operatorname{Im} A$ is dense in Y (Chap. II, Cor. 2 of Theorem 7)

(iii) A injective $\Leftrightarrow \operatorname{Ker} A = 0 \Leftrightarrow \underbrace{({}^\perp \operatorname{Im} A^*)^\perp}_{\text{weak}^* \text{ closure of } \operatorname{Im} A^*} = X^*$

□

Example 1: $X = Y = H = l^2$
$Ax := (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots) \quad X^* = Y^* \cong H$
$A^* = A$ injective, $\operatorname{Im} A \neq H \quad \left(\frac{1}{n}\right)_{n=1}^\infty \in l^2 \setminus \operatorname{Im} A$

Example 2: $X = l^1$, $Y = c_0$, $A : l^1 \to c_0$ inclusion
$A^* : l^1 \to l^\infty$ inclusion, $\operatorname{Im} A^* = l^1 \subset l^\infty$ not dense
$A^{**} : (l^\infty)^* \to l^\infty$ not injective.
When is $\operatorname{Im} A \overset{?}{=} {}^\perp(\operatorname{Im} A^*)$ or $\operatorname{Im} A^* \overset{?}{=} (\operatorname{Ker} A)^\perp$

Theorem 2: X, Y Banach spaces, $A \in \mathcal{L}(X, Y)$
Equivalent are:

(i) $\operatorname{Im} A$ is closed in Y

(ii) $\exists c \geq 0 \, \forall x \in X \ \inf_{A\xi = 0} \|x + \xi\| \leq c\|Ax\|$

(iii) $\operatorname{Im} A^*$ is weak*-closed in X^*

(iv) $\operatorname{Im} A^*$ is closed in X^*

(v) $\exists c \geq 0 \, \forall y^* \in Y^* \ \inf_{A^*\eta^* = 0} \|y^* + \eta^*\| \leq c\|A^*y^*\|$

If these equivalent conditions are satisfied, then:
$\operatorname{Im} A = {}^\perp(\operatorname{Ker} A^*)$, $\operatorname{Im} A^* = (\operatorname{Ker} A)^\perp$

Lemma 3: X, Y Banach spaces, $A \in \mathcal{L}(X, Y)$, $\varepsilon > 0$
Assume $\{y \in Y \mid \|y\| < \varepsilon\} \subset \overline{\{Ax \mid \|x\| < 1\}}$ $(*)$
Then: $\{y \in Y \mid \|y\| < \frac{\varepsilon}{2}\} \subset \{Ax \mid \|x\| < 1\}$

Proof: Chapter II, Lemma 2, Step 2 □

Remark 1: In Chapter II we proved:
A surjective $\overset{\text{Baire}}{\Rightarrow} (*) \overset{\text{Lemma 3}}{\Rightarrow} A$ is open

Remark 2: $(*) \Rightarrow A$ is surjective

Proof of Theorem 2 (i)⇒(ii) Denote $X_0 := X/\operatorname{Ker} A$, $Y_0 := \operatorname{Im} A$
A induces an operator $A_0 : X_0 \to Y_0$ by $A_0[x] := Ax$
Note: $[x_1] = [x_2] \Rightarrow x_1 - x_2 \in \operatorname{Ker} A \Rightarrow Ax_1 = Ax_2$, so A_0 is well defined
A_0 is a bijective, linear operator
$\overset{\text{open mapping thm}}{\Rightarrow} A_0^{-1} : Y_0 \to X_0$ bounded
$\Rightarrow \exists c \geq 0\, \forall x \in X : \inf_{A\xi=0} \|x+\xi\| = \|[x]\|_{X/\operatorname{Ker} A} \leq c\|A_0[x]\| = c\|Ax\|$
(ii)⇒(iii) Claim: $\operatorname{Im} A^* = (\operatorname{Ker} A)^\perp = \bigcap_{x\in\operatorname{Ker} A} \{x^* \in X^* \mid \langle x^*, x\rangle = 0\}$
Let $x^* \in (\operatorname{Ker} A)^\perp$ i.e. $\langle x^*, x\rangle = 0 \quad \forall x \in \operatorname{Ker} A$
Define $\psi : \operatorname{Im} A \to \mathbb{R} \quad \psi(Ax) := \langle x^*, x\rangle$ well defined
ψ is bounded: $\forall \xi \in \operatorname{Ker} A$:
$|\psi(Ax)| = |\langle x^*, x+\xi\rangle| \leq \|x^*\|\|x+\xi\|$
so $|\psi(Ax)| \leq \|x^*\| \inf_{A\xi=0} \|x+\xi\| \overset{(ii)}{\leq} c\|x^*\|\,\|Ax\|$
$\overset{\text{Hahn-Banach}}{\Rightarrow} \exists y^* \in Y^*$ such that $\langle y^*, y\rangle = \psi(y) \quad \forall y \in \operatorname{Im} A$
$\Rightarrow \langle A^*y^*, x\rangle = \langle y^*, Ax\rangle = \psi(Ax) = \langle x^*, x\rangle \quad \forall x \in X$
$\Rightarrow A^*y^* = x^* \in \operatorname{Im} A^*$
(iii)⇒(iv) obvious
(iv)⇒(v) follows from (i)⇒(ii) with A^* instead of A
(v)⇒(i) **Case 1**:A^* is injective
If $\exists c > 0\, \forall\, y^* \in Y^*\, \|y^*\| \leq c\|A^*y^*\|$, then A is surjective
Claim: A satisfies $(*)$ in Lemma 3 with $\varepsilon = \frac{1}{c}$
(Then by Lemma 3, A is surjective)
Proof of the claim: Denote $K := \overline{\{Ax \mid \|x\| < 1\}}$ closed, convex, nonempty
To show: $y_0 \in Y \setminus K \Rightarrow \|y_0\| \geq \varepsilon$
Let $y_0 \in Y \setminus K \overset{\text{Chap. II, Thm 6}}{\Rightarrow} \exists y_0^* \in Y^*$ such that $\langle y_0^*, y_0\rangle > \sup_{y\in K} \langle y_0^*, y\rangle$
$\Rightarrow \|A^*y_0^*\| = \sup_{\|x\|<1} \langle A^*y_0^*, x\rangle = \sup_{\|x\|<1} \langle y_0^*, Ax\rangle = \sup_{y\in K} \langle y_0^*, y\rangle < \langle y_0^*, y_0\rangle \leq \|y_0^*\|\,\|y_0\|$
$\Rightarrow \|y_0\| > \frac{\|A_0^*y_0^*\|}{\|y_0^*\|} \geq \frac{1}{c} = \varepsilon$

Case 2: A^* not injective
Denote $Y_0 := \overline{\operatorname{Im} A^*} \quad Y_0^* \cong Y^*/(\operatorname{Im} A)^\perp = Y^*/\operatorname{Ker} A^*$
$A_0 : X \to Y_0 \quad A_0^* : Y_0^* = Y^*/\operatorname{Ker} A^* \to X^*$
A_0^* is the operator induced by A^*
By (v) we have: $\|[y^*]\|_{Y^*/\operatorname{Ker} A^*} \leq c\|A^*y^*\|$
$\Rightarrow A_0^*$ satisfies the hypotheses of Case 1
$\Rightarrow A_0$ is surjective $\operatorname{Im} A = \operatorname{Im} A_0 = Y_0 = \overline{\operatorname{Im} A} \Rightarrow \operatorname{Im} A$ is closed
$\Rightarrow \operatorname{Im} A = {}^\perp((\operatorname{Im} A)^\perp) = {}^\perp(\operatorname{Ker} A^*)$ □

Corollary: X, Y Banach spaces, $A \in \mathcal{L}(X,Y)$

(i) A is surjective if and only if $\exists c > 0\, \forall y^* \in Y^*\, \|y^*\| \leq c\|A^*y^*\|$

(ii) A^* is surjective if and only if $\exists c > 0\, \forall x \in X\, \|x\| \leq c\|Ax\|$

Proof:

(i) A is surjective ⇔ $\operatorname{Im} A$ closed and $\operatorname{Im} A$ dense
⇔ (v) in Theorem 2 and $\operatorname{Ker} A^* = 0$

(ii) A^* is surjective ⇔ $\operatorname{Im} A^*$ weak* closed and $\operatorname{Im} A^*$ weak* dense
⇔ (ii) in Theorem 2 and $\operatorname{Ker} A = 0$

□

Remark 1: X, Y, Z Banach spaces, $A \in \mathcal{L}(X,Y), B \in \mathcal{L}(Z,Y)$, $\operatorname{Im} B \subset \operatorname{Im} A$
$\Rightarrow \exists T \in \mathcal{L}(Z,X)\; AT = B$ (Douglas Factorization)
Hint: $T := A^{-1}B : Z \to X$ is closed

Remark 2: $A \in \mathcal{L}(X,Y)$, $B \in \mathcal{L}(X,Z)$
Equivalent are:

(i) $\operatorname{Im} B^* \subset \operatorname{Im} A^*$

(ii) $\exists c \geq 0 \, \forall x \in X \|Bx\| \leq c\|Ax\|$

Hint for the proof:$B = id$ see Corollary (ii)
(i)⇒(ii) Douglas Factorization (when A^* is injective)
(ii)⇒(i) Prove that
$x^* \in \operatorname{Im} A^* \Leftrightarrow \exists c \geq 0 \, \forall x \in X \, |\langle x^*, x\rangle| \leq c\|Ax\|$ as in Proof of Theorem 2

Remark 3: X reflexive, $A \in \mathcal{L}(X,Y)$, $B \in \mathcal{L}(Z,Y)$
Equivalent are:

(i) $\operatorname{Im} B \subset \operatorname{Im} A$

(ii) $\exists c \geq 0 \, \forall y^* \in Y^* \, \|B^*y^*\| \leq c\|A^*y^*\|$

Hint for the proof: (ii)$\overset{\text{Rem 2}}{\Leftrightarrow}$ $\operatorname{Im} B^{**} \subset \operatorname{Im} A^{**}$

Example: X reflexive cannot be removed in Remark 3:
$X = c_0$, $Y = l^2$, $Z = \mathbb{R}$
$A : X \to Y \; Ax := \left(\frac{x_n}{n}\right)_{n=1}^{\infty}$ $\quad B : Z \to Y \; Bz := \left(\frac{z}{n}\right)_{n=1}^{\infty} \in l^2$
A, B satisfy (ii) in Remark 3, but not (i)

4.2 Fredholm operators

Definition: X, Y Banach spaces and $A \in \mathcal{L}(X,Y)$. $\ker A := \{x \in X \mid Ax = 0\}, \operatorname{im} A := \{Ax \mid x \in X\}$. Define $\operatorname{coker} A := Y/A$. A is called a *Fredholm operator* if

- $\operatorname{im} A$ is a closed subspace of Y
- $\ker A$ and $\operatorname{coker} A$ are finite dimensional

The *Fredholm index* of A is the integer $\operatorname{index}(A) := \dim\ker A - \dim\operatorname{coker} A$.

Lemma 3: X, Y Banach spaces, $A \in \mathcal{L}(X,Y)$
$\dim\operatorname{coker} A < \infty \Rightarrow \operatorname{im} A$ closed.

Proof: $\dim\operatorname{coker} A < \infty \Rightarrow \exists y_1, \dots, y_m \in Y$ such that $[y_i] \in Y \setminus \operatorname{im} A$ form a basis
$\Rightarrow Y = \operatorname{im} A \oplus \operatorname{span}\{y_1, \dots, y_m\}$
Denote $X \; := X \times \mathbb{R}^m, (x, \lambda) \in X$
$\|(x,\lambda)\|_X \; := \|x\|_X + \|\lambda\|_{\mathbb{R}^m}$.
Define $A \; : X \; \to Y$ by $A \; (x,\lambda) := Ax + \sum_{i=1}^m \lambda_i y_i$
$\Rightarrow A \; \in \mathcal{L}(X,Y)$. A is surjective and $\ker A \; = \ker A \times \{0\}$.

$$\overset{\text{Thm2}}{\Rightarrow} \exists c \geq 0 \forall x \in X : \inf_{A\xi=0} \|x + \xi\|_X \leq c\|Ax\|_Y$$

i.e.

$$\forall x \in X \forall \lambda \in \mathbb{R}^m : \inf_{A\xi=0} \|x+\xi\|_X + \|\lambda\|_{\mathbb{R}^m} \leq c\|Ax + \sum \lambda_i y_i\|_Y$$

$$\Rightarrow \inf_{A\xi=0} \|x+\xi\|_X \leq c\|Ax\|_Y \overset{\text{Thm2}}{\Rightarrow} \operatorname{im} A \text{ closed}$$

□

Remark: Y Banach space, $Y_0 \subset Y$ linear subspace, $\dim Y/Y_0 < \infty$
$\not\Rightarrow Y_0$ is closed.

Lemma 4: X, Y Banach spaces, $A \in \mathcal{L}(X,Y)$
A Fredholm $\Leftrightarrow A^*$ is Fredholm.
In this case: $\operatorname{index}(A^*) = -\operatorname{index}(A)$.

Proof: By Thm 2 $\operatorname{im} A$ closed $\Leftrightarrow \operatorname{im} A^*$ closed. In this case we have $\operatorname{im}(A^*) = (\ker(A))^\perp$ and $\ker(A^*) = (\operatorname{im}(A))^{perp}$. Hence

$$(\ker A)^* \cong X^*/(\ker A)^\perp = X^*/(\operatorname{im} A^*) = \operatorname{coker}(A^*)$$

$$(\operatorname{coker} A)^* = (Y/\operatorname{im} A)^* \cong (\operatorname{im} A)^\perp = \ker(A^*)$$

$\Rightarrow \dim \operatorname{coker}(A^*) = \dim \ker(A)$ $\dim \ker(A^*) = \dim \operatorname{coker}(A)$ □

Example 1: X, Y finite dimensional $\Rightarrow$ Every linear operator $A : X \to Y$ is Fredholm and $\operatorname{index}(A) = \dim X - \dim Y$.

Proof: $\dim X = \dim \ker A + \dim \operatorname{im} A$ (by Linear Algebra) □

Example 2: X, Y arbitrary Banach spaces, $A \in \mathcal{L}(X,Y)$ bijective $\Rightarrow A$ Fredholm and $\operatorname{index}(A) = 0$

Example 3: Hilbert spaces $X = Y = H = \ell^2 \ni x = (x_1, x_2, \dots)$
Define $A_k : H \to H$ by $A_k x := (x_{k+1}, x_{k+2}, \dots)$ shift $\Rightarrow A_k$ is Fredholm and $\operatorname{index}(A_k) = k$
$H \cong H^*$ with $\langle x, y\rangle = \sum_{i=1}^\infty x_i y_i$. So $A^* : H \to H$ is defined by $\langle x, A^* y\rangle := \langle Ax, y\rangle$.
$A_k^* y = (\underbrace{0, \dots, 0} k, x_1, x_2, \dots)$ so $A_k^* y =: A_{-k} y$, where A_{-k} is Fredholm and $\operatorname{index}(A_{-k}) = -k$.

Lemma (Main Lemma) 5: X, Y Banach spaces and $D \in \mathcal{L}(X,Y)$. Equivalent are:

(i) D has a closed image and a finite dimensional kernel.

(ii) $\exists$ Banach space Z and $\exists$ compact operator $K \in \mathcal{L}(X,Z)$ and $\exists c \leq 0$ such that $\forall x \in X$ $\|x\|_X \leq c\,(\|Dx\|_Y + \|Kx\|_Z)$

(*) **Proof:** (i)$\Rightarrow$(ii): $Z := \mathbb{R}^m$ $m := \dim \ker D < \infty$. Choose an isomorphism $\Phi : \ker D \to \mathbb{R}^m$. $\overset{\text{Hahn-Banach}}{\Rightarrow} \exists$ bounded linear operator $K : X \to \mathbb{R}^m$ such that $Kx = \Phi x \forall x \in \ker D$. Define $D : X \to Y \times \mathbb{R}^m$ by $D x := (Dx, Kx)$
$\Rightarrow \operatorname{im} D = \operatorname{im} D \times \mathbb{R}^m$ closed and $\ker D = \{0\} \overset{Thm2}{\Rightarrow} \exists c \leq 0 \forall x \in X$ with $\|x\|_X \leq c\|D x\|_{Y\times\mathbb{R}^m} = c\,(\|Dx\|_Y + \|Kx\|_{\mathbb{R}^m})$
(ii)$\Rightarrow$(i):

Claim 1 Every bounded sequence in $\ker D$ has a convergent subsequence.
($\Rightarrow \dim \ker D < \infty$, by Chapter I, Thm 1)

Proof of Claim 1 Let $x_n \in \ker D$ be a bounded sequence $\overset{K\text{ compact}}{\Rightarrow} \exists$ subsequence $(x_{n_i})_{i=1}^\infty$ such that $(Kx_{n_i})_i$ converges
$\Rightarrow (Dx_{n_i})_i$ and $(Kx_{n_i})_i$ are Cauchy sequences
$\Rightarrow (x_{n_i})$ is Cauchy, because

$$\|x_{n_i} - x_{n_j}\| \leq c\|Dx_{n_i} - Dx_{n_j}\| + c\|Kx_{n_i} - Kx_{n_j}\|$$

$\overset{X\text{ complete}}{\Rightarrow} (x_{n_i})$ converges.

Claim 2 $\exists C > 0 \forall x \in X : \inf_{D\xi=0} \|x + \xi\| \leq C\|Dx\|$ (By Thm 2, this implies that $\operatorname{im} D$ is closed).

Proof of Claim 2 Suppose not. $\Rightarrow \forall n \in \mathbb{N} \exists x_n \in X$ such that $\inf_{D\xi=0} \|x_n + \xi\| > n\|Dx_n\|$.
Without loss of generality we can assume $\inf_{D\xi=0} \|x_n+\xi\| = 1$ and $1 \le \|x_n\| \le 2$ $\Rightarrow \|Dx_n\| < \frac{1}{n}$ so $\Rightarrow$

a) $Dx_n \to 0$

b) $\exists$ subsequence $(x_{n_i})_i$ such that (Kx_{n_i}) converges

$\Rightarrow (Dx_{n_i})_i$ and $(Kx_{n_i})_i$ are Cauchy
$\overset{(*)}{\Rightarrow} (x_{n_i})_i$ is Cauchy
$\Rightarrow (x_{n_i})$ converges. Denote $x := \lim_{i\to\infty} x_{n_i}$
$\Rightarrow Dx = \lim_{i\to\infty} Dx_{n_i} \overset{a)}{=} 0$ and $1 = \inf_{D\xi=0} \|x_{n_i} + \xi\| \le \|x_{n_i} - x\|$ □

Theorem 3: (another characterization of Fredholm operators) X, Y Banach spaces, $A \in \mathcal{L}(X,Y)$ bounded linear operator. Equivalent are:

(i) A is Fredholm

(ii) $\exists F \in \mathcal{L}(Y,X)$ such that $1_X - FA, 1_Y - AF$ are compact.

Proof: (i)$\Rightarrow$(ii):
$X_0 := \ker A \subset X$ is finite dimensional. So $\exists$ a closed subspace $X_1 \subset X$ such that $X = X_0 \oplus X_1$.
$Y_1 := \operatorname{im} A \subset Y$ is a closed subspace of finite codimension. So $\exists$ finite dimensional subspace $Y_0 \subset Y$ such that $Y = Y_0 \oplus Y_1$.
Consider the operator $A_1 := A|_{X_1} : X_1 \to Y_1$. Then $A \in \mathcal{L}(X_1, Y_1)$ and A_1 is bijective. $\overset{\text{open mapping}}{\Rightarrow} A_1^{-1} \in \mathcal{L}(Y_1, X_1)$
Define $F : Y \to X$ by $F(y_0 + y_1) := A_1^{-1} y_1$ for $y_0 \in Y_0, y_1 \in Y_1$.
Then $F \in \mathcal{L}(Y,X)$:

$$FA(x_0 + x_1) = FA_1 x_1 = x_1$$
$$AF(y_0 + y_1) = AA_1^{-1} y_1 = y_1$$

$\Rightarrow 1_X - FA = \Pi_{X_0} : X \cong X_0 \oplus X_1 \to X_0 \subset X$ and $1_Y - AF = \Pi_{Y_0} : Y \cong Y_0 + Y_1 \to Y_0 \subset Y$ compact
(ii)$\Rightarrow$(i):
$K := 1_X - FA \in \mathcal{L}(X,Y)$ compact

$$\begin{aligned}\Rightarrow \|x\|_X &= \|FAx + Kx\|_X \\ &\le \|FAx\|_X + \|Kx\|_X \\ &\le \|F\| \cdot \|Ax\|_Y + \|Kx\|_X \\ &\le c(\|Ax\|_Y + \|Kx\|_X)\end{aligned}$$

where $C := \max\{1, \|F\|\}$
$\overset{Lemma5}{\Rightarrow}$ A has a finite dimensionl kernel and a closed image.
$L := 1_Y - AF \in \mathcal{L}(Y)$ compact $\overset{Thm1}{\Rightarrow}$ L^* is compact and $y^* = L^*y^* + F^*A^*y^* \forall y^* \in Y^*$
$\Rightarrow$ with $c := \max\{1, \|L\|\}$ we have
$\|y^*\| \le c(\|A^*y^*\| \|L^*y^*\|) \overset{lemma5}{\Rightarrow} \dim\ker A^* < \infty$ with $\dim\ker A^* = \dim\operatorname{coker} A$.
□

Theorem 4: X, Y, Z Banach spaces,
$A \in \mathcal{L}(X,Y)$, $B \in \mathcal{L}(Y,Z)$ Fredholm operators
$\Rightarrow BA \in \mathcal{L}(X,Z)$ is a Fredholm operator and index(BA) =index(A)+index(B)

Proof: By Theorem 3, $\exists F \in \mathcal{L}(Y,X)$, $\exists G \in \mathcal{L}(Z,Y)$ such that $1_X - FA$, $1_Y - AF$, $1_Y - GB$, $1_Z - BG$ are compact $\Rightarrow$

a) $1_X - FGBA = \underbrace{1_X - FA}_{\text{compact}} + F(\underbrace{1_Y - GB}_{\text{compact}})A$ compact by Theorem 1

b) $1_Z - BAFG = 1_Z - BG + B(1_Y - GB)A$ is compact

$\Rightarrow BA$ is Fredholm by Theorem 3.

Proof of the index formula:
$A_0 : \ker BA/\ker A \to \ker B \quad [x] \to Ax$
$B_0 : Y/\operatorname{im} A \to \operatorname{im} B/\operatorname{im} BA \quad [y] \to [By]$
$\Rightarrow A_0$ is injective, B_0 is surjective
$\operatorname{im} A_0 = \operatorname{im} A \cap \ker B$
$\operatorname{coker} A_0 = \ker B/(\operatorname{im} A \cap \ker B)$
$\ker B_0 = \{[y] \in Y/\operatorname{im} A \mid By \in \operatorname{im} BA\}$
$= \{[y] \in Y/\operatorname{im} A \mid \exists x \in X \text{ such that } By = BAx\}$
$= \{[y] \in Y/\operatorname{im} A \mid \exists x \in X \text{ such that } y - Ax \in \ker B\} = (\operatorname{im} A + \ker B)/\operatorname{im} A$
$\cong \ker B/(\operatorname{im} A \cap \ker B) = \operatorname{coker} A_0$
$\Rightarrow 0 = \dim\ker B_0 - \dim\operatorname{coker} A_0 - \dim\operatorname{coker} B_0 + \dim\ker A_0$
$= \operatorname{index} A_0 + \operatorname{index} B_0$
$= \dim(\ker BA/\ker A) - \dim\ker B + \dim\operatorname{coker} A - \dim\operatorname{im} B/\ker BA$
$= \dim\ker BA - \dim\ker A - \dim\ker B + \dim\operatorname{coker} A - \dim\ker Y/\operatorname{im} BA + \dim Y/\operatorname{im} B$
$= \operatorname{index} BA - \operatorname{index} A - \operatorname{index} B$ □

Theorem 5 (Stability): X, Y Banach spaces, let $D \in \mathcal{L}(X,Y)$ be a Fredholm operator

(i) $\exists \varepsilon > 0$ such that $\forall P \in \mathcal{L}(X,Y)$ we have
$\|P\| < \varepsilon \Rightarrow D + P$ is Fredholm and $\operatorname{index}(D+P) = \operatorname{index}(D)$

(ii) If $K \in \mathcal{L}(X,Y)$ is a compact operator, then $D + K$ is Fredholm and $\operatorname{index}(K+P) = \operatorname{index}(D)$

Proof: (i) By Lemma 5, $\exists$ Banach space Z, $\exists$ compact operator $L \in \mathcal{L}(X,Z)$, $\exists c > 0$ such that $\|x\| \le c(\|Dx\| + \|Lx\|) \quad \forall x \in X$
(because $\operatorname{im} D$ closed, $\dim\ker D < \infty$)
Suppose $P \in \mathcal{L}(X,Y)$ with $\|P\| < \frac{1}{c}$
Then $\|x\| \le c(\|Dx\| + \|Lx\|) \le c(\|(D+P)x\| + \|Px\| + \|Lx\|)$
$\le c(\|(D+P)x\| + \|Lx\|) + c\|P\|\,\|x\|$
$\Rightarrow (1 - c\|P\|)\|x\| \le c(\|(D+P)x\| + \|Lx\|)$
$\Rightarrow \|x\| \le \frac{c}{1-c\|P\|}(\|(D+P)x\| + \|Lx\|)$
$\overset{\text{Lemma 5}}{\Rightarrow} D + P$ has a closed image and a finite dimensional kernel
(provided $\|P\| < \frac{1}{c}$)
$(\operatorname{coker} D)^* = (Y/\operatorname{im} D)^* = (\operatorname{im} D)^\perp = \ker D^*$
$\dim\ker(D+P)^* < \infty$ for $\|P\| = \|P^*\|$ sufficiently small
(by the same argument as for $D + P$)

Index formula:
$X = X_0 \oplus X_1 \quad X_0 = \ker D$, $Y = Y_0 \oplus Y_1 \quad Y_1 = \operatorname{im} D$
$P_{ji} : X_i \hookrightarrow X \overset{P}{\to} Y \overset{\pi_j}{\to} Y_j$
Then $P(x_0 + x_1) = \underbrace{P_{00}x_0 + P_{01}x_1}_{\in Y_0} + \underbrace{P_{10}x_0 + P_{11}x_1}_{\in Y_1}$

$$P = \begin{pmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 0 \\ 0 & D_{11} \end{pmatrix}$$

$D_{11} : X_1 \to Y_1$ is bijective
$\Rightarrow 1 + D_{11}^{-1}P_{11}$ is bijective as well for $\|P_{11}\|$ small.

$\Rightarrow D_{11} + P_{11}$ is bijective for $\|P\|$ small.
Let $x = x_0 + x_1 \quad x_0 \in X_0,\, x_1 \in X_1$. Then:
$(D + P)x = 0 \Leftrightarrow P_{00}x_0 + P_{01}x_1 = 0,\, P_{10}x_0 + (P_{11} + D_{11})x_1 = 0$
$\Leftrightarrow x_1 = -(D_{11} + P_{11})^{-1}P_{10}x_0,\, (P_{00} - P_{01}(D_{11} + P_{11})^{-1}P_{10})x_0 = 0$
Denote $A_0 := P_{00} - P_{01}(D_{11} + P_{11})^{-1}P_{10} : X_0 \to Y_0$
Then $\ker(D + P) = \{x_0 - (D_{11} + P_{11})^{-1}P_{10}x_0 \mid x_0 \in \ker A_0\}$
so dim $\ker(D + P)$=dim ker A_0
Let $x = x_0 + x_1 \quad x_0 \in X_0,\, x_1 \in X_1$ $y = y_0 + y_1 \quad y_0 \in Y_0,\, y_1 \in Y_1$
Then: $y = (D + P)x \Leftrightarrow y_0 = P_{00}x_0 + P_{01}x_1,\, y_1 = P_{10}x_0 + (D_{11} + P_{11})x_1$
$\Leftrightarrow y_0 = P_{00}x_0 + P_{01}(D_{11} + P_{11})^{-1}(y_1 - P_{10}x_0),\, x_1 = (D_{11} + P_{11})^{-1}(y_1 - P_{10}x_0)$
$\Leftrightarrow y_0 = A_0x_0 + P_{01}(D_{11} + P_{11})^{-1}y_1,\, x_1 = (D_{11} + P_{11})^{-1}(y_1 - P_{10}x_0)$
Hence $y \in \operatorname{im}(D + P) \Leftrightarrow y_0 - P_{01}(D_{11} + P_{11})^{-1}y_1 \in \operatorname{im} A_0$ $(*)$
This implies $\operatorname{coker} A_0 \cong \operatorname{coker}(D + P)$!
Indeed, choose a subspace $Z \subset Y_0 \subset Y$ such that $Y_0 = \operatorname{im} A_0 \oplus Z$
Then by $(*)$ $Y = \operatorname{im}(D + P) \oplus Z$
$\operatorname{im}(D + P) \cap Z = 0$:
$y = y_0 + y_1 \in \operatorname{im}(D + P) \cap Z$, then $y_0 - P_{01}(D_{11} + P_{11})^{-1}y_1 \in \operatorname{im} A_0$, $y_1 = 0$
so $y_0 \in Z \cap \operatorname{im} A$ so $y_0 = 0$, $y_1 = 0$
Given $y = y_0 + y_1 \in Y$, write
$y_0 - P_{01}(D_{11} + P_{11})^{-1}y_1 = A_0x_0 + z \quad z \in Z$, then
$(y_0 - z) + y_1 = y - z \overset{(*)}{\in} \operatorname{im}(D + P)$
Hence $\operatorname{index}(D + P) = \operatorname{index} A_0 = \dim X_0 - \dim Y_0$
$= \dim\ker D - \dim\operatorname{coker} D = \operatorname{index} D$
(ii)By Theorem 3 $\exists T \in \mathcal{L}(X, Y)$ such that $1_X - TD$, $1_Y - DT$ compact
$\Rightarrow 1_X - T(D + K)$, $1_Y - (D + K)T$ compact $\overset{\text{Thm 3}}{\Rightarrow}$ $D + K$ Fredholm
$\mathcal{F}_k := \{A \in \mathcal{L}(X; Y) \mid A \text{ Fredholm, } \operatorname{index} A = k\}$
open subset of $\mathcal{L}(X, Y)$ by (i)
$\bigcup_{k\in\mathbb{Z}} \mathcal{F}_k =: \mathcal{F}(X, Y) = \{\text{Fredholm operators } X \to Y\}$
We have proved $D + tK \in \mathcal{F}(X, Y) \quad \forall t \in \mathbb{R}$
Consider the map $\gamma : \mathbb{R} \to \mathcal{F}(X, Y)\, t \to D + tK$
γ is continuous
So $I_k := \gamma^{-1}(\mathcal{F}_k) = \{t \in \mathbb{R} \mid \operatorname{index}(D + K) = k\}$ is open $\forall k \in \mathbb{Z}$
$\Rightarrow \mathbb{R} = \bigcup_{k\in\mathbb{Z}} I_k$ disjoint union
$\Rightarrow$ each I_k is open and closed
$\exists k \in \mathbb{Z}$ such that $I_k = \mathbb{R} \Rightarrow \operatorname{index}(D) = k = \operatorname{index}(D + K)$ □

Example: X Banach space, $K \in \mathcal{L}(X)$ compact
$\Rightarrow 1 - K$ Fredholm and $\operatorname{index}(1 - K) = 0 \Rightarrow \dim\ker(1 - K) = \dim\operatorname{coker}(1 - K)$

Fredholm alternative
Either the equation $x - Kx = y$ has a unique solution $\forall y \in Y$
or the homogeneous equation $x - Kx = 0$ has a nontrivial solution

Application to integral equations like $x(t) + \int_0^1 k(t, s)x(s)\, ds = y(t) \quad 0 \le t \le 1$

Remark: We well prove in Chapter V: $\exists m \ge 0$ such that:
$\operatorname{index}(1 - K)^m = \operatorname{index}(1 - K)^{m+1}$
By Theorem 4 this implies $m \operatorname{index}(1 - K) = (m + 1) \operatorname{index}(1 - K)$
so it follows also $\operatorname{index}(1 - K) = 0$

5 Spectral Theory

5.1 Eigenvectors

For this whole chapter: X complex Banach space and $\|\lambda x\| = |\lambda|\|x\|\forall\lambda \in \mathbb{C}$. $\mathcal{L}(X,Y) = \{A : X \to Y \mid A \text{ complex linear bounded}\}$, $X^* = \mathcal{L}(X,\mathbb{C})$ and $\mathcal{L}(X) := \mathcal{L}(X,X)$.
Let $A \in \mathcal{L}(X)$:

$$\lambda \in \mathbb{C} \textit{ eigenvalue} \Leftrightarrow \exists x \in X \quad x \neq 0 \quad Ax = \lambda x$$

Definition: Let X complex Banach space and $A \in \mathcal{L}(x)$. The *spectrum* of A is the set

$$\sigma(A) := \{\lambda \in \mathbb{C} \mid \lambda \cdot 1 - A \text{ is not bijective}\} = P\sigma(A) \cup R\sigma(A) \cup C\sigma(A)$$

$P\sigma(A) = \{\lambda \in \mathbb{C} \mid \lambda 1 - A \text{ is not injective}\}$	point spectrum
$R\sigma(A) = \{\lambda \in \mathbb{C} \mid \lambda 1 - A \text{ is injective and } \overline{\text{im}(\lambda 1 - A)} \neq X\}$	residual spectrum
$C\sigma(A) = \{\lambda \in \mathbb{C} \mid \lambda 1 - A \text{ is injective and } \overline{\text{im}(\lambda 1 - A)} = X, \text{im}(\lambda 1 - A) \neq X\}$	continuous spectrum
$\rho(A) := \mathbb{C} \setminus \sigma(A) = \{\lambda \in \mathbb{C} \mid \lambda 1 - A \text{ is bijective}\}$	resolvent set

Remark: X real Banach space, $A \in \mathcal{L}(X)$ bounded real linear operator. *spectrum* of A: $\sigma(A) := \sigma(A_{\mathbb{C}}$
$A_{\mathbb{C}} : X_{\mathbb{C}} \to X_{\mathbb{C}}$ complexified operator
$X_{\mathbb{C}} = X \times X = X \oplus iX \ni x + iy$
$A_{\mathbb{C}}(x + iy) := Ax + iAy$

Example 1: $X = \ell^2_{\mathbb{C}} \ni (x_1, x_2, x_3, \ldots)$ and $\lambda = (\lambda_1, \lambda_2, \ldots)$ bounded sequence in $\mathbb{C}$. Set $Ax := (\lambda_1 x_1, \lambda_2, x_2, \ldots)$, $P\sigma(A) = \{\lambda_1, \lambda_2, \ldots\}$, $\sigma(A) = \overline{P\sigma(A)}$

Example 2: $X = \ell^2_{\mathbb{C}}, \mathbb{D} := \{z \in \mathbb{C} \mid |z| \leq 1\}$

$$\begin{array}{rcl} Ax := (x_2, x_3, \ldots) & Bx := (0, x_1, x_2, \ldots) & \\ P\sigma(A) = \text{int}(\mathbb{D}) & P\sigma(B) = \varnothing & \\ R\sigma(A) = \varnothing & R\sigma(B) = \text{int}(\mathbb{D}) & \\ C\sigma(A) = S^1 & C\sigma(B) = S^1 & \\ \sigma(A) = & \mathbb{D} & = \sigma(B) \end{array}$$

Lemma 1: $A \in \mathcal{L}(X)$. Then $\sigma(A^*) = \sigma(A)$.

Proof:

$$(\lambda 1 - A)^* = \lambda 1 - A^*$$

Claim

$$A \text{ bijective} \Leftrightarrow A^* \text{ bijective} \quad (A^{-1})^* = (A^*)^{-1}$$

$$\begin{array}{rcl} A \text{ bijective} & \Leftrightarrow & \text{im}\, A \text{ closed} \quad \overline{\text{im}\, A} = X \quad \ker A = 0 \\ & \overset{ChIV,Thm2,Lemma2}{\Leftrightarrow} & \text{im}\, A^* \text{ weak*closed} \quad \ker A^* = 0 \quad \text{im}\, A^* \text{ weak*dense} \\ & \Leftrightarrow & A^* \text{ bijective} \end{array}$$

$$\begin{aligned} \langle (A^{-1})^* x^*, y \rangle &= \langle x^*, A^{-1} y \rangle \\ &= \langle A^*(A^*)^{-1} x^*, A^{-1} y \rangle \\ &= \langle (A^*)^{-1} x^*, A A^{-1} y \rangle \\ &= \langle (A^*)^{-1} x^*, y \rangle \forall y \in X, \forall x^* \in X^* \end{aligned}$$

$$\Rightarrow (A^{-1})^* x^* = (A^*)^{-1} x^* \forall x^* \in X^*$$

□

Remark:

a) $P\sigma(A^*) \subset P\sigma(A) \cup R\sigma(A)$
$R\sigma(A^*) \subset P\sigma(A) \cup C\sigma(A)$
$C\sigma(A^*) \subset C\sigma(A)$

b) $P\sigma(A) \subset (A^*) \cup R\sigma(A^*)$
$R\sigma(A) \subset P\sigma(A^*)$
$C\sigma(A) \subset P\sigma(A^*) \cup C\sigma(A^*)$

5.2 Integrals

Lemma 2: X Banach space, $x : [a,b] \to X$ continuous

(*)
$$\Rightarrow \exists! \xi \in X \forall x^* \in X^* : \langle x^*, \xi \rangle = \int_a^b \langle x^*, x(t) \rangle \, \mathrm{d}t$$

Notation: $\int_a^b x(t)\,\mathrm{d}t := \xi$ is called the *Integral of* x.

Proof (exercise with hint): Define $\xi_n := \sum_{k=0}^{2^n-1} \frac{b-a}{2^n} x(a + \frac{k(b-a)}{2^n})$
$\delta_n := \sup_{|s-t| \leq \frac{b-a}{2^n}} \|x(t) - x(s)\| \to 0$

Show that: $\|\xi_{n+m} - \xi_n\| \leq (b-a)\delta_n \forall m > 0 \Rightarrow \xi_n$ is Cauchy.
Denote $\xi := \lim_{n\to\infty} \xi_n$ and check (*). □

Remark:

1. $\int_a^b x(t)\,\mathrm{d}t + \int_b^c x(t)\,\mathrm{d}t = \int_a^c x(t)\,\mathrm{d}t$
2. $\int_a^b x(t) + y(t)\,\mathrm{d}t = \int_a^b x(t)\,\mathrm{d}t + \int_a^b y(t)\,\mathrm{d}t$
3. $\phi : [\alpha, \beta] \to [a,b]$ C^1-map
   $x : [a,b] \to X$ continuous. Then $\int_{\phi(\alpha)}^{\phi(\beta)} x(t)\,\mathrm{d}t = \int_\alpha^\beta x(\phi(s))\dot{\phi}(s)\,\mathrm{d}s$
4. If $x : [a,b] \to X$ is continuously differentiable, then $\int_a^b \dot{x}(t)\,\mathrm{d}t = x(b) - x(a)$
5. $A \in \mathcal{L}(X,Y)$ and $x : [a,b] \to X$ continuous. Then $\int_a^b Ax(t)\,\mathrm{d}t = A\int_a^b x(t)\,\mathrm{d}t$
6. $\|\int_a^b x(t)\,\mathrm{d}t\| \leq \int_a^b \|x(t)\|\,\mathrm{d}t$

Proof: By definition: $\forall x^* \in X^*$.

$$\begin{aligned} \left|\langle x^*, \int_a^b x(t)\,\mathrm{d}t\rangle\right| &= \left|\int_a^b \langle x^*, x(t)\,\mathrm{d}t\right| \\ &\leq \int_a^b |\langle x^*, x(t)\rangle|\,\mathrm{d}t \\ &\leq \int_a^b \|x^*\| \cdot \|x(t)\|\,\mathrm{d}t \\ &= \|x^*\| \cdot \int_a^b \|x(t)\|\,\mathrm{d}t \end{aligned}$$

Now use:

$$\left\|\int_a^b x(t)\,\mathrm{d}t\right\| = \sup_{x^* \neq 0} \frac{|\int_a^b \langle x^*, x(t)\rangle\,\mathrm{d}t}{\|x^*\|}$$

□

Notation: Let X be a Banach space, $\Omega \subset \mathbb{C}$ an open subset, $f : \Omega \to X$ a continuous map, and $\gamma : [a, b] \to \Omega$ be continuously differentiable.

Denote: $\int_\gamma f(z)\,\mathrm{d}z := \int_a^b f(\gamma(t))\dot{\gamma}(t)\,\mathrm{d}t \in X$

Remark:

$$\left\|\int_\gamma f(z)\,\mathrm{d}z\right\| \leq l(\gamma) \cdot \sup a \leq t \leq b\|f(\gamma(t))\|$$

$$l(\gamma) = \int_a^b |\dot{\gamma}|\,\mathrm{d}t$$

Definition: $\Omega \subset \mathbb{C}$ open set. X complex Banach space, $f : \Omega \to X$ is called *holomorphic* if the limit $f'(z) := \lim_{\mathbb{C} \ni h \to 0} \frac{f(z+h)-f(z)}{h}$ exists for every $z \in \Omega$ and the map $f' : \Omega \to X$ is continuous.

Lemma 3: X, Y complex Banach spaces. $\Omega \in \mathbb{C}$ open set. $A \in \mathcal{L}(X, Y)$ continuous. Equivalent are:

(i) A is holomorphic

(ii) For every $x \in X$ and every $y^* \in Y^*$ the function $\Omega \to \mathbb{C} : z \mapsto\mapsto \langle y^*, A(z)x\rangle$ is holomorphic

(iii) Proof $\overline{B}_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| \leq r\} \subset \Omega$ then $A(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{A(z)}{z-z_0}\,\mathrm{d}z$, where $\gamma(t) := z_0 + r \cdot e^{2\pi i t}, 0 \leq t \leq 1$.

Exercise: $A : \Omega \to X$ holomorphic $\Rightarrow A$ is C^∞ and

$$\frac{A^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_\gamma \frac{A(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z \quad \gamma \text{ as in (iii)}$$

Proof: of Lemma 3 (i)⇒(ii): obvious
(ii)⇒(iii): Cauchy integral formula of Complex Analysis.
(iii)⇒(i): usual argument from Complex Analysis.
Denote $B := \frac{1}{2\pi i} \int\limits_\gamma \frac{A(z)}{(z-z_0)^2} \, \mathrm{d}z$ with γ as in (iii).

Claim For $0 < |h| < r$ and $c := \sup_{|z-z_0|<r} \|A(z)\|$ we have

$$\|\frac{1}{h}(A(z_0+h) - A(z)) - B\| \leq \frac{c|h|}{r(r-|h|)}$$

This implies $A'(z_0) = \frac{1}{2\pi i} \int\limits_\gamma \frac{A(z)}{(z-z_0)^2} \, \mathrm{d}z$.
Continuity of A': Exercise.

Proof of Claim

$$\begin{aligned}
\frac{1}{h}(A(z_0+h) - A(z_0)) - B &\overset{(iii)}{=} \frac{1}{2\pi i} \int\limits_\gamma \frac{1}{h}\left(\frac{1}{z-z_0-h} - \frac{1}{z-z_0}\right) - \frac{1}{(z-z_0)^2} A(z) \, \mathrm{d}z \\
&= \frac{1}{2\pi i} \int\limits_\gamma \frac{h}{(z-z_0)^2(z-z_0-h)} A(z) \, \mathrm{d}z \\
&\Rightarrow \|\frac{1}{h}(A(z_0+h) - A(z_0)) - B\| \\
&\leq \frac{1}{2\pi} l(\gamma) \sup_{0\leq t\leq 1} \frac{|h| \|A(\gamma(t))\|}{|\gamma(t) - z_0|^2 |\gamma(t) - z_0 - h|} \\
&\leq \frac{|h|}{r} \sup_{0\leq t\leq 1} \frac{\|A(\gamma(t))\|}{|re^{2\pi i t} - h|} \leq \frac{c|h|}{r(r-|h|)}
\end{aligned}$$

□

Lemma 4: $A \in \mathcal{L}(X) \Rightarrow \rho(A) \subset \mathbb{C}$ is open, the function
$\rho(A) \to \mathcal{L}(X) : \lambda \mapsto (\lambda 1 - A)^{-1} =: R_\lambda(A)$ is holomorphic, and

(*) $$R_\lambda(A) - R_\mu(A) = (\mu - \lambda) R_\lambda(A) R_\mu(A) \forall \lambda, \mu \in \rho(A)$$

Proof: Proof of (*)

$$(\lambda 1 - A)(R_\lambda(A) - R_\mu(A))(\mu 1 - A) = (\mu 1 - A) - (\lambda 1 - A) = (\mu - \lambda) \cdot 1 \Rightarrow (*)$$

$\rho(A)$ is open:
$\lambda \in \rho(A), |\mu - \lambda| \|(\lambda 1 - A)^{-1}\| < 1 \overset{ChI, Thm11}{\Rightarrow} 1 + (\mu - \lambda)(\lambda 1 - A)^{-1}$ is bijective,
where $(\mu - \lambda) = (\mu 1 - A)(\lambda 1 - A)^{-1} \Rightarrow \mu 1 - A$ bijective $\Rightarrow \mu \in \rho(A)$
Also:
$\|(\mu 1 - A)^{-1} - (\lambda 1 - A)^{-1}\| \leq \frac{|\mu - \lambda| \|(\mu 1 - A)^{-1}\|^2}{1 - |\mu - \lambda| \|(\lambda 1 - A)^{-1}\|}$ Continuity
(*)$\Rightarrow \lim_{\mu \to \lambda} \frac{R_\mu(A) - R_\lambda(A)}{\mu - \lambda} = -R_\lambda(A)^2$
$\overset{Continuity}{\Rightarrow} \lambda \mapsto R_\lambda(A)$ is holomorphic. □

Theorem 1: Let $A \in \mathcal{L}(X)$, then

1. $\sigma(A) \neq \varnothing$ and
2. $\sup\limits_{\lambda \in \sigma(A)} |\lambda| = \lim\limits_{n \to \infty} \|A^n\|^{\frac{1}{n}} = \inf\limits_{n \in \mathbb{N}} \|A^n\|^{\frac{1}{n}} =: r_A$

(Spectral radius, Chapter I, Theorem 11)

Proof:

1. $\sup\limits_{\lambda\in\sigma(A)} |\lambda| \le r_A$
   Let $\lambda \in \mathbb{C}$ with $|\lambda| > r_A \Rightarrow r_{\lambda^{-1}A} < 1$
   $\overset{\text{Chap 1 Thm 11}}{\Rightarrow} 1 - \lambda^{-1}A$ bijective $\Rightarrow \lambda \in \rho(A)$

2. $r_A \le \sup\limits_{\lambda\in\sigma(A)} |\lambda|$
   $R(z) := \begin{cases} (z^{-1}1 - A)^{-1} & z \ne 0,\ \frac{1}{z} \in \rho(A) \\ 0 & z = 0 \end{cases}$
   $R : \Omega \to \mathcal{L}(X) \quad \Omega = \{\frac{1}{z} \mid z \in \rho(A)\} \cup \{0\}$
   Fact: Ω open and $R : \Omega \to \mathcal{L}(X)$ is holomorphic
   Proof: Lemma 4 and removal of singularity for holom. $\Omega \setminus \{0\} \to \mathbb{C}$
   Or for small z :
   $R(z) = z(1 - zA)^{-1} = \sum_{i=1}^{\infty} z^{k+1}A^k$ (Chap. I, Thm. 11)
   $\Rightarrow \frac{R^{(n)}(0)}{n!} = A^{n-1}$
   $\overset{\text{Cauchy integral formula}}{\Rightarrow} A^{n-1} = \frac{1}{2\pi i}\int_\gamma \frac{R(z)}{z^{n+1}}\,dz \qquad (1)$
   $\gamma = \frac{1}{r}e^{2\pi i t}\ t \in [0,1]$ provided $\left\{z \in \mathbb{C} \mid |z| \le \frac{1}{r}\right\} \subset \Omega \qquad (2)$
   If $r > \sup\limits_{\lambda\in\sigma(A)} |\lambda|$ then (2) holds, so
   $\|A^{n-1}\| = \left\|\frac{1}{2\pi i}\int_\gamma \frac{R(z)}{z^{n+1}}\,dz\right\| \le l(\gamma) \sup\limits_{|z|=\frac{1}{r}} \frac{\|R(z)\|}{2\pi|z^{n+1}|}$
   $= l(\gamma)\frac{r^n}{2\pi\frac{1}{r}} \sup\limits_{|z|=\frac{1}{r}} \|(z^{-1}1 - A)^{-1}\| = r^n \underbrace{\sup\limits_{|\lambda|=r} (\lambda 1 - A)^{-1}\|}_{=:c}$
   $\Rightarrow \|A^{n-1}\| \le cr^n \Rightarrow \|A^n\| \le cr^{n+1} \Rightarrow \|A^n\|^{\frac{1}{n}} \le r\underbrace{(cr)^{\frac{1}{n}}}_{\to 1}$
   $\Rightarrow \lim\limits_{n\to\infty} \|A^n\|^{\frac{1}{n}} \le r \quad \forall r > \sup\limits_{\lambda\in\sigma(A)} |\lambda|$

3. $\sigma(A) \ne \varnothing$
   Suppose $\sigma(A) = \varnothing$ and so $\rho(A) = \mathbb{C}$.
   Pick $x \in X, x^* \in X^*$ and define
   $f : \mathbb{C} \to \mathbb{C} \quad f(\lambda) := \langle x^*, (\lambda 1 - A)^{-1}x\rangle$
   $\overset{\text{Lemmma 4}}{\Rightarrow} f$ is holomorphic and
   $|f(\lambda)| \le \|x^*\|\,\|x\|\,\|(\lambda 1 - A)^{-1}\| \le \frac{\|x^*\|\,\|x\|}{|\lambda| - \|A\|}$ if $|\lambda| > \|A\|$
   $\overset{\text{Liouville}}{\Rightarrow} f \equiv 0 \Rightarrow (\lambda 1 - A)^{-1} = 0\,\forall \lambda \in \mathbb{C}$ Contradiction

□

H **complex (real inner product) Hilbert space** $\|ix\| = \|x\|\,\forall x \in H \Leftrightarrow \langle ix, iy\rangle = \langle x, y\rangle\,\forall x,\, y \in H$
recover inner product from norm: $\langle x, y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$
$\Leftrightarrow \langle ix, y\rangle + \langle x, iy\rangle = 0 \quad \forall x,\, y \in H$
Define the *Hermitian inner product*:
$H \times H \to \mathbb{C}\,(x, y) \to \langle x, y\rangle_{\mathbb{C}}$ by $\langle x, y\rangle_{\mathbb{C}} := \langle x, y\rangle + i\langle ix, y\rangle$

a) real bilinear

b) $\langle x, y\rangle_{\mathbb{C}} = \overline{\langle x, y\rangle}_{\mathbb{C}}$

c) $\langle x, \lambda y\rangle_{\mathbb{C}} = \lambda\langle x, y\rangle_{\mathbb{C}},\ \langle \lambda x, y\rangle_{\mathbb{C}} = \overline{\lambda}\langle x, y\rangle_{\mathbb{C}}$

d) $|\langle x, y\rangle_{\mathbb{C}}| \le \|x\|\,\|y\|$

e) $\|x\| = \sqrt{\langle x, x\rangle_{\mathbb{C}}}$

Proof: Exercise □

$H^* := \{\varphi : H \to \mathbb{C} \mid \varphi \text{ complex, linear}\}$
Define $\iota : H \to H^*$ $\quad \langle \iota(x), y\rangle_{H^*,H} := \langle x, y\rangle_{\mathbb{C}}$
Riesz Representation Theorem: $\iota : H \to H^*$ isometric isomorphism
Warning: ι is anti-linear: $\iota(\lambda x) = \overline{\lambda}\iota(x) \quad \forall x \in H\, \forall \lambda \in \mathbb{C}$

The Hilbert space adjoint of a linear operator $A : H \to H$ is the operator $A^* : H \to H$ defined by $\langle A^*x, y\rangle_{\mathbb{C}} := \langle x, Ay\rangle_{\mathbb{C}}\, \forall x, y \in H$

Remark:

(i) $A^*_{\text{new}} = \iota^{-1} \circ A^*_{\text{old}} \circ \iota$

(ii) $(\lambda A)^*_{\text{old}} = \lambda A^*_{\text{old}} \quad (\lambda A)^*_{\text{new}} = \overline{\lambda} A^*_{\text{new}}$

(iii) $\sigma(A^*_{\text{old}}) = \sigma(A) \quad \sigma(A^*_{\text{new}}) = \overline{\sigma(A)}$

Proof: $\overline{\lambda}1 - A^*_{\text{new}} \overset{(i)}{=} \iota^{-1}(\lambda 1 - A^*_{\text{old}})\iota$ bijective
$\Leftrightarrow \lambda 1 - A^*_{\text{old}}$ bijective $\overset{\text{Lemma 1}}{\Leftrightarrow} \lambda 1 - A$ bijective □

Definition: H complex Hilbertspace
An operator $A \in \mathcal{L}(H)$ is called *normal*, if $AA^* = A^*A$, *selfadjoint*, if $A = A^*$ and *unitary*, if $AA^* = A^*A = I$.

Example 1: $H = l^2_{\mathbb{C}}$, $Ax = (\lambda_1 x_1, \lambda_2 x_2, \ldots)$
$\lambda_n \in \mathbb{C}$, $\sup_{n\in\mathbb{N}} |\lambda_n| < \infty$
$A = A^* \Leftrightarrow \lambda_n \in \mathbb{R}$
Because $A^*x = (\overline{\lambda_1} x_1, \overline{\lambda_2} x_2, \ldots)$
$\langle x, y\rangle_{\mathbb{C}} = \sum_{n=1}^{\infty} \overline{x_n} y_n$
A is normal

Example 2: $H = l^2_{\mathbb{C}}$
$Ax = (x_2, x_3, \ldots) \quad A^*x = (0, x_1, x_2, \ldots) \Rightarrow AA^* \neq A^*A$

Example 3: selfadjoint$\Rightarrow$ normal
A unitary ($\|Ax\| = \|x\|\, \forall x \in H$) isomorphism $\Rightarrow A$ normal, because
$\langle x, A^*Ax\rangle = \|Ax\|^2 = \|x\|^2 \Leftrightarrow \langle y, A^*Ax\rangle = \langle y, x\rangle \quad \forall x, y \in H$
$\Leftrightarrow A^*A = 1 \overset{A \text{ onto} \Rightarrow}{A} A^* = 1 \quad A^* = A^{-1}$
In Example 2 we have $\|A^*x\| = \|x\|$ but $A^* \neq A^{-1}$ (A^* is not onto)

Lemma 5: H complex Hilbert space, $A \in \mathcal{L}(H)$
Equivalent are:

(i) A is normal

(ii) $\|Ax\| = \|A^*x\| \quad \forall x \in H$

Proof:

(i)⇒(ii) $\|Ax\|^2 = \langle Ax, Ax\rangle_{\mathbb{C}} = \langle x, A^*Ax\rangle_{\mathbb{C}} = \langle x, AA^*x\rangle_{\mathbb{C}}$
$= \langle A^*x, A^*x\rangle_{\mathbb{C}} = \|A^*x\|^2$

(ii)⇒(i) Same argument gives $\langle x, A^*Ax\rangle_{\mathbb{C}} = \langle x, AA^*x\rangle_{\mathbb{C}} \quad \forall x \in H$
$\Rightarrow \langle y, A^*Ax\rangle_{\mathbb{C}} = \langle y, AA^*x\rangle_{\mathbb{C}} \quad \forall x, y \in H$
$\Rightarrow A^*A = AA^*$ □

Theorem 2: H complex Hilbert space, $A \in \mathcal{L}(H)$ normal $\Rightarrow$

(i) $\sup\limits_{\lambda \in \sigma(A)} |\lambda| = \|A\|$

(ii) $A = A^* \Rightarrow \sigma(A) \subset \mathbb{R}$

Proof:

(i) Claim: $\|A^n\| = \|A\|^n \quad \forall n \in \mathbb{N}$ (Then (i) follows from Theorem 1)
$n = 2$: $\|A\|^2 = \sup\limits_{x \neq 0} \frac{\|Ax\|^2}{\|x\|^2} = \sup\limits_{x \neq 0} \frac{\langle x, A^*Ax\rangle}{\|x\|^2}$
$\overset{\text{CS}}{\leq} \sup\limits_{x \neq 0} \frac{\|A^*Ax\|^2}{\|x\|^2} \overset{\text{Lemma 5}}{=} \sup\limits_{x \neq 0} \frac{\|A^2x\|}{\|x\|^2} = \|A^2\|$
$\Rightarrow \|A^2\|^2 \overset{\text{normal}}{\leq} \|A^2\| \overset{\text{always}}{\leq} \|A\|^2$
$n = 2^m$ Induction $\|A^{2^m}\| = \|A\|^{2^m}$
n arbitrary:

$$\|A\|^n = \frac{\|A\|^{2^m}}{\|A\|^{2^m-n}} = \frac{\|A^{2^m}\|}{\|A\|^{2^m-n}} \leq \frac{\|A^n\| \, \|A^{2^m-n}\|}{\|A\|^{2^m-n}} \leq \|A^n\|$$

(ii) Assume $A = A^*$, let $\lambda \in \mathbb{C} \setminus \mathbb{R}$
To show: $\lambda \in \rho(A)$
Claim: $\|\lambda x - Ax\|^2 \geq (\operatorname{Im}\lambda)^2 \|x\|^2$
Claim $\overset{\text{Chap. IV Thm 2}}{\Rightarrow}$ $\lambda 1 - A$ injective, $\operatorname{im}(\lambda 1 - A)$ closed
$\overset{A=A^*}{\Rightarrow}$ $\overline{\lambda}1 - A^*$ injective, so $\lambda 1 - A$ bijective
Proof of Claim: $\|\lambda x - Ax\|^2 = \langle \lambda x - Ax, \lambda x - Ax\rangle_{\mathbb{C}}$
$= \overline{\lambda}\lambda\langle x, x\rangle_{\mathbb{C}} - \lambda\langle Ax, x\rangle_{\mathbb{C}} - \overline{\lambda}\langle x, Ax\rangle_{\mathbb{C}} + \langle Ax, Ax\rangle_{\mathbb{C}}$
$= |\lambda|^2\|x\|^2 - 2(\operatorname{Re}\lambda)\langle x, Ax\rangle_{\mathbb{C}} + \|Ax\|^2 = (\operatorname{Im}\lambda)^2\|x\|^2 + (\operatorname{Re}\lambda)^2\|x\|^2 - 2(\operatorname{Re}\lambda)\langle x, Ax\rangle_{\mathbb{C}} + \|Ax\|^2$
$= (\operatorname{Im}\lambda)^2\|x\|^2 + \|(\operatorname{Re}\lambda)x - Ax\| \geq (\operatorname{Im}\lambda)^2\|x\|^2$

□

Theorem 3: H complex Hilbert space, $A = A^* \in \mathcal{L}(H)$ selfadjoint. Then

a) $R\sigma(A) = \varnothing$ and

b) (1) $\sup \sigma(A) = \sup_{\|x\|=1} \langle x, Ax\rangle$
   (2) $\inf \sigma(A) = \inf_{\|x\|=1} \langle x, Ax\rangle$
   (3) $\|A\| = \sup_{\|x\|=1} |\langle x, Ax\rangle|$

Proof:

a) $R\sigma(A) = \varnothing$

$$\begin{aligned} \lambda \notin P\sigma(A) \quad &\Leftrightarrow \quad \lambda 1 - A \text{ injective} \\ &\Leftrightarrow \quad H = \ker(\lambda 1 - A)^{\perp} \\ &= \quad \{x \in H \mid \langle x, \xi\rangle = 0 \forall \xi \in \ker(1 - A)\} \\ &= \quad \overline{\operatorname{im}(1 - A)} \\ &\Leftrightarrow \quad \lambda \in C\sigma(A) \end{aligned}$$

(1)⇒(2)
replace A by $-A$.

(1)&(2)⇒(3)

$$\begin{aligned}\|A\| &\overset{Thm2}{=} \sup_{\lambda\in\sigma(A)} |\lambda| \\ &= \max\{\sup\sigma(A), -\inf\sigma(A)\} \\ &\overset{(1),(2)}{=} \sup_{\|x\|=1} |\langle x, Ax\rangle|\end{aligned}$$

Proof of (1) Assume wlog

(4) $$\langle x, Ax\rangle \geq 0 \;\forall x \in H$$

(replace A by $A + \|A\|\cdot 1$).
Claim 1: (4) $\Rightarrow \sigma(A) \subset [0,\infty)$
Claim 2: (4) $\Rightarrow \|A\| = \sup_{\|x\|=1}\langle x, Ax\rangle$

Claims 1&2 ⇒ (1)

$$\begin{aligned}\sup_{\|x\|=1}\langle x, Ax\rangle &= \|A\| \text{ by Claim 2} \\ &= \sup_{\lambda\in\sigma(A)}|\lambda| \text{ by Thm 2} \\ &= \sup\sigma(A) \text{ by Claim 1}\end{aligned}$$

Proof of Claim 1 Let $\varepsilon > 0$. Then $\forall x \in H$:

$$\begin{aligned}\varepsilon\|x\|^2 &\overset{(4)}{\leq} \varepsilon\|x\|^2 + \langle x, Ax\rangle \\ &= \langle x, \varepsilon x + Ax\rangle \\ &\leq \|x\|\cdot\|\varepsilon x + Ax\|\end{aligned}$$

$$\Rightarrow \varepsilon\|x\| \leq \|\varepsilon x + Ax\|\forall x \in H$$

$\varepsilon\mathbb{1} + A$ is injective and has a closed image (see Chapter IV).
$\overset{A=A^*}{\Rightarrow} \varepsilon 1 + A$ is bijective
$\Rightarrow -\varepsilon \notin \sigma(A)$

Proof of Claim 2 Let $a := \sup_{\|x\|=1}\langle x, Ax\rangle \leq \|A\|$ (Cauchy-Schwarz). To show: $\|A\| \leq a$.
For $x, y \in H$ we have

$$\langle y, Ax\rangle = \frac{1}{4}(\langle x+y, A(x+y)\rangle - \langle x-y, A(x-y)\rangle)$$

⇒ For $\|x\| = \|y\| = 1$:

$$-a \leq -\frac{1}{4}a\|x-y\|^2 \leq -\frac{1}{4}\langle x-y, A(x-y)\rangle \leq \langle y, Ax\rangle$$

$$\leq \frac{1}{4}\langle x+y, A(x+y)\rangle \leq \frac{1}{4}a\|x+y\| \leq a$$

$$\Rightarrow |\langle y, Ax\rangle| \leq a \forall x, y \in H \text{ and } \|x\| = \|y\| = 1$$

$$\Rightarrow \|A\| = \sup_{\substack{\|x\|=1,\\ \|y\|=1}} |\langle y, Ax\rangle| \leq a$$

□

Remark: Given any operator $T \in \mathcal{L}(H)$ we have

$$\begin{aligned} \|T\|^2 &= \sup_{\|x\|=1} \|Tx\|^2 \\ &= \sup_{\|x\|=1} \langle Tx, Tx\rangle \\ &= \sup_{\|x\|=1} \langle x, T^*Tx\rangle \\ &\leq \|T^*T\| \\ &\leq \|T^*\|\|T\| \\ &\leq \|T\|^2 \end{aligned}$$

$$\Rightarrow \|T\|^2 \sup_{\|x\|=1} \langle x, T^*Tx\rangle \overset{\text{Thm 3 for } A=T^*T}{=} \|T^*T\|$$

$\|T\| = \sqrt{\|T^*T\|}$. We can use this to compute $\|T\|$.

5.3 Compact operators on Banach spaces

X complex Banach space, $A \in \mathcal{L}(X)$ bounded, complex linear.

Facts

a) $\ker(\lambda\mathbb{1} - A) \subset \ker(\lambda\mathbb{1} - A)^2 \subset \ker(\lambda\mathbb{1} - A)^3 \subset \ldots$.

b) $\ker(\lambda\mathbb{1} - A)^m = \ker(\lambda\mathbb{1} - A)^{m+1}$

$$\Rightarrow \ker(\lambda\mathbb{1} - A)^m = \ker(\lambda\mathbb{1} - A)^{m+k} \forall k \geq 0$$

Notation: $E_\lambda := E_\lambda(A) := \bigcup_{m=1}^\infty \ker(\lambda\mathbb{1} - A)^m$

Theorem 4: $A \in \mathcal{L}(X)$ compact

(i) If $\lambda \in \sigma(A), \lambda \neq 0$, then $\lambda \in P\sigma(A)$ and $\dim E_\lambda < \infty$.
Hence $\exists m \in \mathbb{N}$ such that $\ker(\lambda\mathbb{1} - A)^m = \ker(\lambda\mathbb{1} - A)^{m+1}$.

(ii) Eigenvalues of A can only accumulate at 0, ie

$$\forall \lambda \in \sigma(A), \lambda \neq 0, \exists \varepsilon > 0 \forall \mu \in \mathbb{C} :$$

$$0 < |\mu - \lambda| < \varepsilon \Rightarrow \mu \in \rho(A)$$

Proof:

(i) If $\lambda \neq 0 \Rightarrow \lambda\mathbb{1}$ Fredholm, index = 0.
$\overset{ChIV}{\Rightarrow} \lambda\mathbb{1} - A$ Fredholm, index = 0
$\overset{Thm5A \text{ cpct}}{\Rightarrow}$ either $\lambda\mathbb{1} - A$ is bijective ($\lambda \notin \sigma(A)$) or $\lambda\mathbb{1} - A$ is not injective ($\lambda \in P\sigma(A)$). Moreover: $\dim(\ker(\lambda\mathbb{1} - A)^m) < \infty \forall m \in \mathbb{N}$, because $(\lambda\mathbb{1} - A)^m = \sum_{k=0}^m \binom{m}{k} \lambda^k (-A)^{m-k} = \lambda^m \mathbb{1} +$ cpct
Let $K := \lambda^{-1}A$ and $E_n = \ker(\mathbb{1} - K)^n = \ker(\lambda\mathbb{1} - A)^n$.

To show: $\exists m$ such that $E_m = E_{m+1}$.
Suppose not. Then $E_n \subsetneq E_{n+1} \forall n \in \mathbb{N}$

$$\overset{ChIILemma4}{\Rightarrow} \forall n \in \mathbb{N} \exists x_n \in E_n \text{ such that}$$

$$\|x_n\| = 1 \quad \inf_{x \in E_{n-1}} \|x_n - x\| \geq \frac{1}{2}$$

Now: for $m < n$ we have

$$Kx_m \in E_m \subset E_{n-1} \quad x_n - Kx_n \in E_{n-1}$$

$$\Rightarrow \|Kx_n - Kx_m\| = \|x_n - \underbrace{(x_n - Kx_n + Kx_m}_{\in E_{n-1}}\| \geq \frac{1}{2}$$

So $(Kx_n)_n$ has no convergent subsequence: contradiction!

(ii) Let $\lambda \in \sigma(A), \lambda \neq 0$.
$\overset{(i)}{\Rightarrow} \exists m \in \mathbb{N}$ such that $\ker(\lambda 1 - A)^m = \ker(\lambda 1 - A)^{m+1}$
$\Rightarrow X = \ker(\underbrace{\lambda 1 - A}_{=:X_0})^m \oplus \operatorname{im}(\underbrace{\lambda 1 - A}_{=:X_1}$
This is an exercise (use Hahn-Banach).
$AX_0 \subset X_0 \quad AX_1 \subset X_1$.
Note: $(\lambda 1 - A)^m : X_1 \to X_1$ is bijective

$$\overset{\text{open mapping}}{\Rightarrow} \exists c > 0 \forall x_1 \in X_1$$
$$\|x_1\| \leq c\|(\lambda 1 - A)^m x_1\|$$

Choose $\varepsilon > 0$ such that for all $\mu \in \mathbb{C}$:

$$|\lambda - \mu| < \varepsilon \Rightarrow \|(\lambda 1 - A)^m - (\mu 1 - A)^m\| < \frac{1}{c}$$
$$\Rightarrow (\mu 1 - A)^m : X_1 \to X_1 \text{ is bijective for } |\lambda - \mu| < \varepsilon$$
$$(\mu 1 - A)^m : X_0 \to X_0 \text{ is bijective for } \mu \neq \lambda$$

The rest by induction. X_0 is finite dimensional. $A|_{X_0} \cong$
$\Rightarrow (\mu 1 - A)^m : X \to X$ only true if bijective!
$\Rightarrow \mu 1 - A : X \to X$ is bijective.

□

H real or complex Hilbert space

Notation: $\langle .,. \rangle$ real or Hermitian inner product.

Definition: A collection of vectors $\{e_i\}_{i\in I}$ in H is called an *orthonormal basis* if

(1)
$$\langle e_i, e_j \rangle = \delta_{ij}$$

(2)
$$H = \overline{\operatorname{span}\{e_i \mid i \in I\}}$$

Remark 1: (2) holds if and only if

$$\forall x \in H : \langle x, e_i \rangle = 0 \forall i \in I \Rightarrow x = 0$$

Remark 2: H separable $\Leftrightarrow I$ is finite or countable

Remark 3: $x \in H, \{e_i\}_{i\in I}$ ONB

$$\Rightarrow x = \sum_{i\in I} \langle e_i, x \rangle e_i \quad \|x\|^2 = \sum_{i\in I} |\langle e_i, x \rangle|^2$$

Theorem 5: H real or complex Hilbert space. $A = A^* \in \mathcal{L}$ selfadjoint, compact. $\Rightarrow A$ admits an ONB $\{e_i\}_{i\in I}$ of eigenvectors

$$Ae_i = \lambda_i e_i, \lambda_i \in \mathbb{R} \quad Ax = \sum_{i\in I} \lambda_i \langle e_i, x \rangle e_i$$

Proof:

Step 1

$$Ax = \lambda x \quad x \neq 0 \quad Ay = \mu y \quad y \neq 0 \text{ and } \lambda \neq \mu$$
$$\Rightarrow \langle x, y \rangle = 0$$

proof

$$\begin{aligned}(\lambda - \mu)\langle x, y\rangle &= \langle \lambda x, y\rangle - \langle x, \mu y\rangle \\ &= \langle Ax, y\rangle - \langle x, Ay\rangle\end{aligned}$$

Step 2 $\ker(\lambda 1 - A)^m = \ker(\lambda 1 - A) \forall m \geq 1$

proof

$$\lambda \in \mathbb{R} \quad \lambda \in P\sigma(A) \quad (\lambda 1 - A)^2 x = 0$$
$$\Rightarrow 0 = \langle x, (\lambda 1 - A)^2 x\rangle$$
$$= \langle (\lambda 1 - A)x, (\lambda 1 - A)x\rangle = \|\lambda x - Ax\|^2$$

Step 3

(*) $$\langle x, y\rangle = 0 \forall y \in \ker(\lambda 1 - A) \forall \lambda \in \mathbb{R} \Rightarrow x = 0$$
$$H_0 := \{x \in H \mid (*)\} \quad H_0 \neq 0 \quad \Rightarrow A|_{H_0} \neq 0$$
$$\Rightarrow \|A|_{H_0}\| \in \sigma(A|_{H_0}) \text{ or } -\|A|_{H_0}\| \in \sigma(A|_{H_0}$$

$\Rightarrow A|_{H_0}$ has a nonzero eigenvalue, eigenvector. This is also an eigenvector of A: contradiction! □

Definition: A C^*-*algebra* is a complex Banach space $\mathcal{A}$ equipped with

- an assoziative, distributive product $\mathcal{A} \times \mathcal{A} \to \mathcal{A} : (a, b) \to ab$ with a unit $1 \in \mathcal{A}$ such that $\|ab\| \leq \|a\| \, \|b\|$
- a complex anti-linear involution $\mathcal{A} \to \mathcal{A} : a \to a^*$ such that $(ab)^* = b^* a^*$, $1^* = 1$, and $\|a^*\| = \|a\|$

Remark: antilinear: $(\lambda a)^* = \overline{\lambda} a^*$, involution: $a^{**} = a$

Example 1: H complex Hilbert space, then $\mathcal{L}(H)$ is a C^*-algebra

Example 2: $A \in \mathcal{L}(H)$
$\mathcal{A}$:= smallest C^*-algebra containing A
$A = A^*$, $p(\lambda) := a_0 + a_1\lambda + \ldots + a_n\lambda^n \quad a_k \in \mathbb{C}$
$p(A) := a_0 + a_1 A + \ldots + a_n A^n \quad p(A)^* = \overline{a_0} + \overline{a_1} A + \ldots + \overline{a_n} A^n$
$(pq)(A) = p(A)q(A)$
$\mathcal{A} := \text{closure}(\{p(A) \mid p : \mathbb{R} \to \mathbb{C})$

Example 3: Σ compact metric space
$C(\Sigma) := \{f : \Sigma \to \mathbb{C} \mid f \text{ continuous}\}$
C^*-Algebra, sup-norm, involution: $f \to \overline{f}$
Goal: $A = A^*$, $\Sigma = \sigma(A) \Rightarrow \mathcal{A} \cong C(\Sigma) : p(A) \leftarrow p$

Theorem 6: H complex Hilbert space, $A = A^* \in \mathcal{L}(H)$ selfadjoint
$\Sigma := \sigma(A) \subset (\mathbb{R})$
$\Longrightarrow$ There is a unique bounded linear operator
$C(\Sigma) \to \mathcal{L}(H) : f \to f(A)$ $(*)$
such that $(fg)(A) = f(A)g(A)$, $\mathbb{1}_{\mathbb{R}}(A) = 1_H$ (1)
$\overline{f}(A) = f(A)^*$ (2)
$f(\lambda) = \lambda \, \forall \lambda \in \Sigma \Rightarrow f(A) = A$ (3)
Denote $\Phi_A(f) := f(A)$
Then $(*)$ is the operator $\Phi_A : C(\Sigma) \to \mathcal{L}(H)$

(1): $\Phi_A(fg) = \Phi_A(f)\Phi_A(g) \quad \Phi_A(1) = 1$

(2): $\Phi_A(\overline{f}) = \Phi_A(f)^*$

(3): $\Phi_A(id : \Sigma \to \Sigma \subset \mathbb{C}) = A$

Lemma 6: H complex Hilbert space, $A \in \mathcal{L}(H)$
$p(\lambda) = \sum_{k=0}^{n} a_k \lambda^k \; a_k \in \mathbb{C}$ complex polynomial $\Rightarrow$

(i) $p(A)^* = \overline{p}(A^*)$, $\overline{p}(\lambda) = \sum_{k=0}^{n} \overline{a_k} \lambda^k$, $(pq)(A) = p(A)q(A)$

(ii) $\sigma(p(A)) = p(\sigma(A)) = \{p(\lambda) \mid \lambda \in \sigma(A)\}$

(iii) $A = A^* \Rightarrow \|p(A)\|_{L^\infty(\sigma(A))} = \sup_{\lambda \in \sigma(A)} |p(\lambda)|$

Proof:

(i) Exercise

(ii) $\lambda \in \sigma(A)$, to show $p(\lambda) \in \sigma(p(A))$
The polynomial $t \to p(t) - p(\lambda)$ vanishes at $t = \lambda$
$\exists$ polynomial q such that $p(t) - p(\lambda) = (t - \lambda)q(t)$
$\Rightarrow p(A) - p(\lambda)1 = (A - \lambda 1)q(A) = q(A)(A - \lambda 1)$
$\Rightarrow p(A) - p(\lambda)1$ is not bijective $\Rightarrow p(\lambda) \in \sigma(p(A))$

$\mu \in \sigma(p(A)) \Rightarrow \exists \lambda \in \sigma(A) : \mu = p(\lambda)$
$n := \deg(p) \Rightarrow p(t) - \mu = a(t - \lambda_1) \dots (t - \lambda_n) \quad a \neq 0$
$p(A) - \mu 1 = a(A - \lambda_1 1) \dots (A - \lambda_n 1)$ not bijective
$\Rightarrow \exists i$ such that $A - \lambda_i 1$ not bijective
$\Rightarrow \lambda_i \in \sigma(A)$, $p(\lambda_i) - \mu = 0$

(iii) $A = A^* \Rightarrow p(A)^* = \overline{p}(A)$
$\Rightarrow p(A)$ is normal: $q(A)p(A) = (pq)(A)) = p(A)q(A)$
so $p(A)p(A)^* = p(A)^*p(A) \overset{\text{Thm 2}}{\Rightarrow} \|p(A)\| = \sup_{\mu \in \sigma(p(A))} |\mu| \overset{\text{(ii)}}{=} \sup_{\lambda \in \sigma(A)} |p(\lambda)|$

□

Remark 1: If $p(\lambda) = q(\lambda) \, \forall \lambda \in \sigma(A)$, then $p(A) = q(A)$
i.e. the operator $p(A)$ only depends on the restriction $p|_{\sigma(A)}$

Remark 2: Why is $P(\Sigma) := \{p|_\Sigma \mid p : \mathbb{R} \to \mathbb{C} \text{ polynomial}\}$ dense in $C(\Sigma)$?
Stone-Weierstrass:

- $P(\Sigma)$ is a subalgebra of $C(\Sigma)$
- $P(\Sigma)$ seperates points (i.e. $\forall x, y \in \Sigma, x \neq y \, \exists p \in P(\Sigma)$ s.t. $p(x) \neq p(y)$)
- $p \in P(\Sigma) \Rightarrow \overline{p} \in P(\Sigma)$

$\Rightarrow P(\Sigma)$ is dense in $C(\Sigma)$

Proof of Theorem 6:

1. Existence: Given $f \in C(\Sigma)$, construct $f(A) \in \mathcal{L}(H)$
   By Remark 2, $\exists$ sequence $p_n \in P(\Sigma)$ such that $\lim_{n\to\infty} \|f - p_n\|_{L^\infty(\Sigma)} = 0$
   p_n is a Cauchy sequence in $C(\Sigma)$
   $\overset{\text{Lemma 6 (iii)}}{\Rightarrow} p_n(A)$ is a Cauchy sequence in $\mathcal{L}(H)$
   $\|p_n(A) - p_m(A)\| = \|p_n - p_m\|_{L^\infty(\Sigma)}$
   $\Rightarrow p_n(A)$ converges in $\mathcal{L}(H)$
   Define $f(A) := \lim_{n\to\infty} p_n(A)$
   (This is the only way of defining $f(A)$, so we have proved uniqueness)

2. $f(A)$ is well-defined
   If $q_n \in P(\Sigma)$ is another sequence converging uniformly to f, then
   $\|p_n(A) - q_n(A)\| = \|p_n - q_n\|_{L^\infty(\Sigma)} \to 0$
   So $\lim_{n\to\infty} p_n(A) = \lim_{n\to\infty} q_n(A)$

3. f is linear, continuous and satisfies (1), (2), (3)

   - $p_n \to f$, $q_n \to q$
     $\Rightarrow p_n q_n \to fg$, $p_n + q_n \to f + g$
     $\Rightarrow (fg)(A) = \lim_{n\to\infty} p_n q_n(A) = \lim_{n\to\infty} p_n(A) \lim_{n\to\infty} q_n(A) = f(A)g(A)$
     Same for addition and for $f(tA) = tf(A)$
     $1(A) = 1$
   - $f \to f(A)$ bounded, indeed $\|f(A)\| = \|f\|_{L^\infty(\Sigma)} = \sup_{\lambda\in\sigma(A)} |f(\lambda)|$
     True for $f = p \in P(\Sigma)$; $p_n \to f$
     $\Rightarrow \|f(A)\| = \lim_{n\to\infty} \|p_n(A)\| = \lim_{n\to\infty} \|p_n\|_{L^\infty(\Sigma)}$
   - (2) and (3)
     (2) obvious, (3) true for polynomials, take limits □

Theorem 7: Let $A = A^* \in \mathcal{L}(H)$, $\Sigma := \sigma(A) \subset \mathbb{R}$ and $f \in C(\Sigma) \Rightarrow$

(i) $\sigma(f(A)) = f(\sigma(A))$

(ii) $\|f(A)\| = \|f\|_{L^\infty(\Sigma)}$

(iii) $Ax = \lambda x \Rightarrow f(A)x = f(\lambda)x$

(iv) $AB = BA \Rightarrow f(A)B = Bf(A)$

(v) If $f(\Sigma) \subset \mathbb{R}$ then $f(A) = f(A)^*$

(vi) $f \geq 0 \Leftrightarrow \langle x, f(A)x\rangle \geq 0 \, \forall x \in H$

Proof:

(ii) already proved, also follows from (i)

(iii) true for polynomials, hence true in the limit

(iv) true for polynomials, take the limit

(v) Use Theorem 3:
$\inf f(\Sigma) = \inf \sigma(f(A)) = \inf\langle x, f(A)x\rangle$

(vi) $\sigma(f(A) \subset f(\Sigma)$
Let $\mu \notin f(\Sigma)$
Define $g(\lambda) := \frac{1}{f(\lambda)-\mu}$, $\lambda \in \Sigma$
$\Rightarrow g(f - \mu) = (f - \mu)g = 1$
$\Rightarrow g(A)(f(A) - \mu 1) = 1 = (f(A) - \mu 1)g(A)$
$\Rightarrow f(A) - \mu$ bijective, i.e. $\mu \notin \sigma(f(A))$

$f(\Sigma) \subset \sigma(f(A))$
Let $\lambda \in \Sigma = \sigma(A)$. Claim: $f(\lambda) \in \sigma(f(A))$
Suppose not
$\Rightarrow f(\lambda)1 - f(A)$ is bijective
Pick a sequence of polynomials $p_n \in P(\Sigma)$ such that $\|p_n - f\|_{L^\infty(\Sigma)} \to 0$
$\Rightarrow f(\lambda)1 - f(A) = \lim_{n\to\infty}(p_n(\lambda)1 - p_n(A))$ (in norm topology)
$\Rightarrow p_n(\lambda)1 - p_n(A)$ is bijective for n large
$p_n(\lambda) \notin \sigma(p_n(A))$, this contradicts Lemma 6 (ii)

□

5.4 Spectral Measure

Let $A = A^* \in \mathcal{L}, \Sigma = \sigma(A) \subset \mathbb{R}$. We have defined

$$\begin{aligned} C(\Sigma) &= \{\text{continuous functions} f : \Sigma \to \mathbb{C}\} \\ B(\Sigma) &= \{\text{bounded measurable functions} f : \Sigma \to \mathbb{C}\} \\ F(\Sigma) &= \{\text{bounded functions} f : \Sigma \to \mathbb{C}\} \end{aligned}$$

$$\|f\| := \sup_{\lambda\in\Sigma} |f(\lambda)|$$

Remark 1: $C(\Sigma) \subset B(\Sigma) \subset F(\Sigma)$ closed subspace and $F(\Sigma)$ is a Banachspace.

Remark 2: We take a sequence $f_n \in F(\Sigma)$ *bp-converges* to $f \in F(\Sigma)$ iff

$$\sup_{n\in\mathbb{N}} \|f_n\| < \infty, \quad f(\lambda) = \lim_{n\to\infty} f_n(\lambda) \forall \lambda \in \Sigma$$

We write

$$f = \text{bp-lim}_{n\to\infty} f_n$$

1. $B(\Sigma)$ is closed under bp-convergence.
2. $C(\Sigma)$ is bp-dense in $B(\Sigma)$.

Therefore

Remark 3: $B(\Sigma)$ is the smallest C^*-subalgebra in $F(\Sigma)$ so that

1. $C(\Sigma) \subset B(\Sigma)$
2. $B(\Sigma)$ is bp-closed.

Recall from Theorem 6 that $\exists!$ continuous C^*-homomorphism $\Phi_A : C(\Sigma) \to \mathcal{L}(H)$ so that

$$\Phi_A(\text{id}) = A$$

Theorem 8: $\exists!$ C^*-homomorphism $\Psi_A : B(\Sigma) \to \mathcal{L}(H)$ such that

1. If $f(\lambda) = \lambda \forall \lambda \in \Sigma$ then $\Psi_A(f) = A$.
2. $\|\Psi_A(f)\| \le \|f\| \forall f \in B(\Sigma)$
3. $f = \text{bp-lim}_{n\to\infty} f_n \quad \Rightarrow \quad \Psi_A(f)x = \lim_{n\to\infty} \Psi_A(f_n)x \forall x \in H$

Remark: Let $A_n \in \mathcal{L}(H)$ and $A \in \mathcal{L}(H)$. We say A_n converges *strongly* to A if

$$Ax = \lim_{n\to\infty} A_n x \quad \forall x \in H$$

A_n converges *in norm* to A if

$$\lim_{n\to\infty} \|A - A_n\| = 0$$

Fact Norm convergence $\Rightarrow$ Strong convergence.

Proof of Theorem 8, only sketch:

Existence

1. For every $x \in H$ there is a unique real Borel measure μ_x on Σ such that

$$\langle x, \Phi_A(f)x\rangle_{\mathbb{C}} = \int_\Sigma f \,\mathrm{d}\mu_x \quad \forall f \in C(\Sigma)$$

Namely

$$\{\text{real Borel measures on } \Sigma\} = C(\Sigma, \mathbb{R})^*$$
$$= \{\phi : C(\Sigma) \to \mathbb{C} \mid \phi \text{ complex linear}, \phi \text{ bounded}, \phi(\overline{f}) = \overline{\phi(f)} \forall f\}$$

That was discussed in detail when we introduced the dual space.

Therefore for every bounded complex linear function $\phi : C(\Sigma) \to \mathbb{C}$ with $\phi(\overline{f}) = \overline{\phi(f)}$ $\exists!$ real Borel-measure μ on Σ so that

$$\phi(f) = \int_\Sigma f \,\mathrm{d}\mu \quad \forall f \in C(\Sigma)$$

Example $\phi_x(f) := \langle x, \Phi_A(f)x\rangle$.

(a)

$$\begin{aligned} \phi_x(\overline{f}) &= \langle x, \Phi_A(\overline{f})x\rangle \\ &= \langle x, \Phi(f)^* x\rangle \\ &= \langle \Phi_A(f)x, x\rangle \\ &= \overline{\langle x, \Phi_A(f)x\rangle} \\ &= \overline{\phi_x(f)} \end{aligned}$$

(b)

$$\begin{aligned} |\phi_x(f)| &= |\langle x, \Phi_A(f)x\rangle| \\ &\le \|x\| \cdot \|\Phi_A(f)x\| \\ &\le \|\Phi_A(f)\| \cdot \|x\|^2 \\ \text{by Theorem 7} \quad &= \|f\| \cdot \|x\|^2 \end{aligned}$$

This concludes the proof of the first statement: Choose μ_x so that

$$\phi_x(f) = \int_\Sigma f \,\mathrm{d}\mu_x \quad \forall f \in C(\Sigma)$$

2. $\exists!$ complex linear operator $\Psi_A : B(\Sigma) \to \mathcal{L}(H)$ so that $\|\Psi_A f\| \le \|f\| \forall f \in B(\Sigma)$ and
$$\langle x, \Psi_A(f)x\rangle_{\mathbb{C}} = \int\limits_{\Sigma} f \, \mathrm{d}\mu_x \quad \forall f \in B\Sigma$$
Namely: Let $f : \Sigma \to \mathbb{R}$ be bounded and measurable. The map $H \times H \to \mathbb{R}$,
$$(y, x) \mapsto \frac{1}{4}\left(\int\limits_{\Sigma} f \, \mathrm{d}\mu_{x+y} - \int\limits_{\Sigma} f \, \mathrm{d}\mu_{x-y}\right)$$
is bilinear and symmetric and bounded (which we will not verify).
Define $\Psi_A(f)$ by
$$\langle y, \Psi_A(f)x\rangle := \frac{1}{4}\left(\int\limits_{\Sigma} f \, \mathrm{d}\mu_{x+y} - \int\limits_{\Sigma} f \, \mathrm{d}\mu_{x-y}\right)$$
$$\Rightarrow \langle x, \Psi_A(f)x\rangle = \int\limits_{\Sigma} f \, \mathrm{d}\mu_x \le \|f\| \cdot \|x\|^2$$
$$\Rightarrow \|\Phi_A(f)\| = \sup_{\|x\|=1} |\langle x, \Psi_A(f)x\rangle \le \|f\|$$
This is the end of the proof of step 2.

□

Proof: For $f = f_1 + if_2 : \Sigma \to \mathbb{C}$ bounded measurable define
$\Psi_A(f) := \Psi_A(f_1) + i\Psi_A(f_2)$

Exercise: $\Psi_A : B(\Sigma) \to \mathcal{L}(H)$ is a C*-algebra homomorphism

1. $f_n \stackrel{\text{bp}}{\to} f \Rightarrow \Psi_A(f_n) \to \Psi_A(f)$
By Lebesgue dominated convergence: $\lim\limits_{n\to\infty} \langle x, \Psi_A(f_n)x\rangle = \lim \int_\Sigma f_n \, d\mu_x = \int_S igmaf \, d\mu_x = \langle x, \Psi_A(f)x\rangle \forall x \in H$
$\Rightarrow \langle y, \Psi_A(f)x\rangle = \lim\limits_{n\to\infty} \langle y, \Psi_A(f_n)x\rangle \quad \forall x, y \in H$

Also: $\|\Psi_A(f)x\|^2 = \langle x, \Psi_A(f)^*\Psi_A(f)x\rangle = \langle x, \Psi_A(\overline{f}f)x\rangle = \lim\limits_{n\to\infty} \langle x, \Psi_A(\overline{f_n}f_n)x\rangle = \lim\limits_{n\to\infty} \|\Psi_A(f_n)x\|^2 \stackrel{\text{Chap. III}}{\Rightarrow} \Psi_A(f)x = \lim\limits_{n\to\infty} \Psi_A(f_n)x \quad \forall x \in H$
Here we use: If $\xi_n \in H$ and $\xi \in H$, $\lim\limits_{n\to\infty} \langle x, \xi_n\rangle = \langle y, \xi\rangle \quad \forall y \in H$,
$\lim\limits_{n\to\infty} \|\xi_n\| = \|\xi\|$ then $\lim\limits_{n\to\infty} \|\xi_n - \xi\| = 0$. Apply this to $\xi_n := \Psi_A(f_n)x$

Uniqueness: $C(\Sigma)$ is bp-dense in $B(\Sigma)$ □

Exercise:

(i) $f \ge 0 \Rightarrow \Psi_A(f) \ge 0$, i.e. $\langle x, \Psi_A(f)x\rangle \ge 0$ and $\Psi_A(f)$ selfadjoint

(ii) $Ax = \lambda x \Rightarrow \Psi_A(f)x = f(\lambda)x$

(iii) $AB = BA \Rightarrow \Psi_A(f)B = B\Psi_A(f)$

Notation: $f(A) := \Psi_A(f)$
Theorem 6: "continuous functional calculus"
Theorem 8: "measurable functional calculus"

Remark:

1. Recall $\langle x, \Psi_A(f)x\rangle = \int_\Sigma f\, d\mu_x$
2. f continuous $\Rightarrow \|\Psi_A(f)\| = \|f\|$
3. f bounded measurable $\not\Rightarrow \|\Psi_A(f)\| = \|f\|$

Warning: $B(\Sigma) \neq L^\infty(\Sigma, \mu_x)$
Literature: Reed-Simon

Spectral projections
Let $\Omega \subset \Sigma$ be a Borel set. Define

$$\chi_\Omega(\lambda) := \begin{cases} 1 & \lambda \in \Omega \\ 0 & \lambda \in \Sigma \setminus \Omega \end{cases}$$

Then $\chi_\Omega \in B(\Sigma)$ and $\chi_\Omega^2 = \chi_\Omega = \overline{\chi_\Omega} \Rightarrow$ The operator $P_\Omega := \Psi_A(\chi_\Omega)$ is an orthogonal projection: $P_\Omega^2 = P_\Omega = P_\Omega^*$

Corollary: The orthogonal projections $P_\Omega \in \mathcal{L}(H)$ satisfy the following conditions:

(i) $P_\varnothing = 0 \quad P_\Sigma = 1$

(ii) $P_{\Omega_1 \cap \Omega_2} = P_{\Omega_1} P_{\Omega_2}$

(iii) $\Omega = \bigcup_{i=1}^\infty \Omega_i \quad \Omega_l \cap \Omega_k = \varnothing\; k \neq l$
$\Rightarrow P_\Omega x = \lim\limits_{n\to\infty} \sum_{k=1}^n P_{\Omega_k} x \quad \forall x \in H$

Proof: Theorem 8 □

$\Sigma \subset \mathbb{R}$ compact set
$\mathcal{B}(\Sigma) \subset 2^\Sigma$ Borel σ-algebra
The map $B(\Sigma) \to \mathcal{L}(H)$ satisfying the axioms of the corollary above, is called a *projection valued measured on* Σ. The projection valued measure of the corollary is called the *spectral measure of* A

Remark: From the spectral measure we can recover the operator A via

$$\mu_x(\Omega) = \langle x, P_\Omega x\rangle \text{ and } \langle x, f(A)x\rangle = \int_\Sigma f\, d\mu_x$$

Example: A compact and self-adjoint
$\Sigma = \{\lambda_0, \lambda_1, \ldots\}$, $\lambda_n \to \lambda_0 = 0$
$E_n := \ker(\lambda_n 1 - A)$, $P_n \in (H)$ orthogonal projection onto E_n
$f(A)x = \sum_{n=0}^\infty f(\lambda_n) P_n x$

$$"P_\Omega = \sum_{\lambda_n \in \Omega} P_n \delta_{\lambda_n}" \text{ (not convergent in the norm)}$$

Definition: $x \in H$ is called *cyclic for* A, if $\overline{\mathrm{span}\{x, Ax, \ldots\}} = H$

Theorem 9: $A = A \in \mathcal{L}(H)$, $x \in H$ cyclic
$\Rightarrow \exists$ Hilbert space isometry (unitary operator) $U : H \to L^2(\Sigma, \mu_x)$ such that

$$(UAU^{-1}f)(\lambda) = \lambda f(\lambda)$$

Proof: μ_x Borel measure on Σ defined by $\int_\Sigma f\, d\mu_x = \langle x, f(A)x\rangle_{\mathbb{C}} \quad \forall f \in C(\Sigma)$
Claim: $\exists$ isometric isomorphism $U : H \to L^2(\Sigma, \mu_x)$ such that

$$\begin{array}{ccc} H & \xrightarrow[A]{} & H \\ {\scriptstyle U}\big\downarrow & & \big\downarrow{\scriptstyle U} \\ L^2 & \xrightarrow[\Lambda]{} & L^2 \end{array} \qquad (\Lambda f)(\lambda) = \lambda f(\lambda)$$

Define $T : C(\Sigma) \to H$ by $Tf := \Phi_A(f)x$

$$\|Tf\|^2 = \|\Phi_A(f)x\|^2 = \langle \Phi_A(f)x, \Phi_A(f)x\rangle$$

$$= \langle x, \Phi_A(f)^*\Phi_A(f)x\rangle = \langle x, \Phi_A(\overline{f})\Phi_A(f)x\rangle \qquad (*)$$

$$= \langle x, \Phi_A(\overline{f}f)x\rangle = \int_\Sigma \overline{f}f\, d\mu_x = \int_\Sigma |f|^2\, d\mu_x = \|f\|^2_{L^2}$$

Recall: $C(\Sigma)$ dense in $L^2(\Sigma, \mu_x)$
$\overset{(*)}{\Rightarrow} T$ extends uniquely to an isometric embedding $T : L^2(\Sigma, \mu_x) \to H$
Claim: T is surjective: $f(\lambda) = \lambda^n \Rightarrow \Phi_A(f) = A^n \Rightarrow Tf = A^n x$
Hence $A^n x \in \operatorname{im} T\, \forall n \in \mathbb{N}$
$\overset{\text{x cyclic}}{\Rightarrow} \operatorname{im} T \supset \operatorname{span}\{A^n x\}$ is dense in H
Moreover $\|Tf\|_H = \|f\|_{L^2}\, \forall f$, so T is injective and has a closed image
$\Rightarrow T$ is bijective $U := T^{-1}$
To show: $UAU^{-1} = \Lambda$ or equivalently $AT = T\Lambda$
$ATf = A\Phi_A(f)x = \Phi_A(id)\Phi_A(f)x = \Phi_A(id\, f)x = T(id\, f) = T\Lambda f$ □

Remark: In general, if $A = A^* \in \mathcal{L}(H)$ and H is separable $\exists$ orthogonal decomposition $H = \bigoplus_k H_k$ such that $AH_k = H_k$ and $A|_{H_k}$ admits a cyclic vector.

Exercise: A compact and selfadjoint $\Rightarrow$
$\exists$ cyclic $x \in H \Leftrightarrow$ every eigenspace of A is 1-dimensional
Similar to the following example:

Example 1: $A = A^* \in \mathbb{C}^{n\times n}$ $A^* = \overline{A}^T$, $\langle x, y\rangle = \sum\limits_{j=1}^{n} \overline{x_j} y_j$
$\exists$ ONB $e_1, \dots e_n$ of eigenvectors of A; $Ae_j = \lambda_j e_j$ $\lambda_j \in \mathbb{R}$
Assume $\lambda_j \neq \lambda_k$ for $k \neq j$, then (LA) $x = \sum\limits_{i=1}^{n} e_i$ is cyclic.
$A^k x = \sum_{i=1}^n \lambda_i^k e_i$
$\Sigma = \{\lambda_1, \dots \lambda_n\}$ $\quad f(A)\xi = \sum\limits_{i=1}^{n} f(\lambda_i)\langle e_i, \xi\rangle e_i$
$C(\Sigma) = L^2(\Sigma) \cong \mathbb{C}^n$
μ_x is defined by
$\int_\Sigma f\, d\mu_x = \langle x, f(A)x\rangle \sum\limits_{i=1}^{n} f(\lambda_i)\langle e_i, x\rangle\langle x, e_i\rangle = \sum\limits_{i=1}^{n} f(\lambda_i)$
$\Rightarrow \mu_x = \sum_{i=1}^n \delta_{\lambda_i}$
$U : H = \mathbb{C}^n \to \mathbb{C}^n = L^2$ $\quad U\xi = (\langle e_i, \xi\rangle)_{i=1}^n$
$\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$

Example 2: $H = l^2(\mathbb{Z}) = \{x = (x_n)_{n\in\mathbb{Z}} \mid \sum\limits_{n=-\infty}^{\infty} |x_n|^2 < \infty\}$
$(Lx)_n = x_{n+1}$ $\quad (L^*x)_n = x_{n-1}$
$A = L + L^*$ selfadjoint $\sigma(A) = [-2, 2]$ and $H = H^{\text{ev}} \oplus H^{\text{odd}}$
where $H^{\text{ev}} := \{x \mid x_{-n} = x_n\}$ and $H^{\text{odd}} := \{x \mid x_{-n} = -x_n\}$ are invariant under A

$(Ax)_n = x_{n-1} + x_{n+1}$
$x^{\mathrm{ev}} := (\dots 0, \underbrace{1}_{=x_0^{\mathrm{ev}}}, 0 \dots)$ and $x^{\mathrm{odd}} := (\dots 0, 0, -1, \underbrace{0}_{=x_0^{\mathrm{odd}}}, 1, 0, 0 \dots)$ cyclic vectors for $A|_{H^{\mathrm{ev}}}, A|_{H^{\mathrm{odd}}}$
Define $U : H \to L^2([0,1])$ by $(Ux)(t) := \sum\limits_{n=-\infty}^{\infty} x_n e^{2\pi int}$
$(ULx)(t) = e^{-2\pi it}(Ux)(t) \quad (UL^*x)(t) = e^{2\pi it}(Ux)(t)$
$(UAU^{-1}f)(t) = 2\cos(2\pi t) f(t)$
A multiplication operators on $L^2([-2,2],\mu_1) \times L^2([-2,2],\mu_2)$

6 Unbounded operators

X, Y Banach spaces, $D \subset X$ dense, $A : D \to Y$ closed graph
For $x \in D$, $\|x\|_A := \|x\|_X + \|Ax\|_Y$ graph norm
Then $(D, \|\cdot\|_A)$ is a Banach space and $A : D \to Y$ is bounded

In Spectral theory one studies $\lambda 1 - A : D \to X$ where X Banach space, $D \subset X$ dense subspace and $A : D \to X$ linear

Recapitulation

1. $\text{graph}(A) := \{(x, Ax) \mid x \in D\} \subset X \times X$
   A closed $\overset{\text{def}}{\Leftrightarrow}$ graph(A) is a closed subspace of $X \times X$
2. $B : \text{dom}(B) \to X$ is an *extension of* A, if $\text{dom}(A) \subset \text{dom}(B)$ and $B|_{\text{dom}}(A) = A$
3. A is *closable*, if A admits a closed extension
4. A is closable $\Leftrightarrow \overline{\text{graph}(A)}$ is a graph i.e. $(0, y) \in \overline{\text{graph}(A)} \Rightarrow y = 0$
   Denote by $\overline{A}$ the smallest closed extension of A, $\text{graph}(\overline{A}) = \overline{\text{graph}(A)}$
5. $\Omega \subset \mathbb{R}^n$ open, $X = L^p(\Omega)$ $1 < p < \infty$ $D := C_0^\infty(\Omega)$,
   $A : D \to X$ differential operator
   $\Rightarrow A$ is closable

Adjoint Operator
$A : \text{dom}(A) \Rightarrow X$ densly defined linear operator on a Banach space. The adjoint operator $A^* : \text{dom}(A^*) \to X^*$ is defined as follows:
$\text{dom}(A^*) := \{y^* \mid \exists c > 0\, \forall x \in \text{dom}(A)\, |\langle y^*, Ax\rangle| \le c\|x\|\}$
For $y^* \in \text{dom}(A^*)$ the linear functional $\text{dom}(A) \to \mathbb{R}\, x \to \langle y^*, Ax\rangle$ is bounded
$\Rightarrow \exists x^* \in X^*$ such that $\langle x^*, x\rangle = \langle y^*, Ax\rangle\, \forall x \in \text{dom}(A)$
Define $A^* y^* := x^*$
Note that $\langle A^* y^*, x\rangle = \langle y^*, Ax\rangle \quad x \in \text{dom}(A)$, $y^* \in \text{dom}(A^*)$

Remark 1: Let $y^* \in X^*$, then
$\exists c \ge 0\, \forall x \in \text{dom(A)}\, |\langle y^*, Ax\rangle| \le c\|x\|$
$\Leftrightarrow \exists x^* \in X^*$ sucht that $\langle x^*, x\rangle = \langle y^*, Ax\rangle$
In this case we have $y^* \in \text{dom}(A^*), x \in \text{dom}(A)$

Remark 2: $(y^*, x^*) \in \text{graph}(A^*) \Leftrightarrow \langle x^*, x\rangle = \langle y^*, Ax\rangle\, \forall x \in \text{dom}(A)$
$\Leftrightarrow \langle(-x^*, y^*), (x, Ax)\rangle = 0\, \forall x \in \text{dom}(A) \Leftrightarrow (-x^*, y^*) \in (\text{graph}(A))^\perp$
Hence $\text{graph}(A^*) \subset X^* \times X^* \cong (X \times X)^*$ is always weak*-closed

Remark 3: A closable $\Rightarrow \text{graph}(A)^\perp = \overline{\text{graph}(A)}^\perp = \text{graph}(\overline{A})^\perp$
$\overset{\text{Rem. 2}}{\Rightarrow} \overline{A}^* = A^*$

Lemma 1: X reflexive, $A : \text{dom}(A) \to X$ densely defined linear operator. Then

(i) A^* is closed

(ii) A closable $\Leftrightarrow \text{dom}(A^*)$ is dense in X^*

(iii) A closable $\Rightarrow \overline{A}^* = A^*$ and

$$\begin{array}{ccc} X \supset \text{dom}(\overline{A}) & \xrightarrow[\overline{A}]{} & X \\ \downarrow \iota & & \downarrow \iota \\ X^{**} \supset \text{dom}(A^{**}) & \xrightarrow[A^{**}]{} & X^{**} \end{array}$$

Proof: Define $J : X \times X \to X \times X$ by $J(x,y) = (-y,x)$
Then by Remark 2, $\mathrm{graph}(A^*) = (J\,\mathrm{graph}(A))^\perp = J^*\,\mathrm{graph}(A)^\perp$
$J^*(x^*,y^*) = (y^*,-x^*)$

(iii) $(J * \mathrm{graph}(A^*))^\perp = \mathrm{graph}(A^{**})$ and
$(J^*\,\mathrm{graph}(A^*))^\perp = (\mathrm{graph}(A)^\perp)^\perp = \iota_{X\times X}({}^\perp(\mathrm{graph}(A)^\perp)) = \iota_{X\times X}(\overline{\mathrm{graph}(A)})$

(ii) Assume $\mathrm{dom}(A^*)$ is dense
Let $(0,y) \in \overline{\mathrm{graph}(A)} \Rightarrow \exists x_n \in \mathrm{dom}(A)$ such that $x_n \to 0$, $Ax_n \to y$
$\Rightarrow \forall y^* \in \mathrm{dom}(A^*)$ we have $\langle y^*, y\rangle = \lim\limits_{n\to\infty} \langle y^*, Ax_n\rangle = \lim\limits_{n\to\infty} \langle A^*y^*, x_n\rangle = 0$
$\overset{\mathrm{dom}(A^*)\text{ dense}}{\Rightarrow} y = 0$

□

Proof: of Lemma 1 (continued) X Banach space, $A : \mathrm{dom}(A) \to X$ linear, $\mathrm{dom}(A) \subset X$ dense supspace, X reflexive.
A closable $\Rightarrow \mathrm{dom}(A^*)$ is dense in X^*.

1. $\mathrm{graph}(A)^\perp = \{(x^*,y^*) \in X^* \times X^* \mid \langle x^*, x\rangle + \langle y^*, Ax\rangle = 0\}\forall x \in \mathrm{dom}(A) = \{(-A^*y^*, y^*) \mid y^* \in \mathrm{dom}(A^*)\}$
2. $\overline{\mathrm{graph}(A)} ={}^\perp (\mathrm{graph}(A)^\perp)$
3. $(x,y) \in \overline{\mathrm{graph}(A)}$
$\overset{2.}{\Leftrightarrow} \langle x^*, x\rangle + \langle y^*, y\rangle = 0 \forall (x^*,y^*) \in \mathrm{graph}(A)^\perp$
$\overset{1.}{\Leftrightarrow} \langle -A^*y^*, x\rangle + \langle y^*, y\rangle = 0 \forall y^* \in \mathrm{dom}(A^*)$
4. Because X is reflexive we have: $\overline{\mathrm{dom}(A^*)} = X^* \Leftrightarrow^\perp \mathrm{dom}(A^*) = 0$.
5. $y \in^\perp \mathrm{dom}(A^*)$
$\Rightarrow \langle y^*, y\rangle = 0 \forall y^* \in \mathrm{dom}(A^*)$
$\overset{3.}{\Rightarrow} (0,y) \in \overline{\mathrm{dom}(A)}$
$\overset{A\text{ closable}}{\Rightarrow} y = 0$

□

Remark 1: The spectrum of an unbounded operator $A : \mathrm{dom}(A) \subset X \to X$ is defined exactly as in the bounded case:
$\rho(A) := \{\lambda \in \mathbb{C} \mid \lambda 1 - A : \mathrm{dom}(A) \to X \text{ is bijective}\}$
$\sigma(A) = \mathbb{C} \setminus \rho(A)$
$P\sigma(A) = \{\lambda \mid \lambda 1 - A \text{ not injective}\}$
$R\sigma(A) = \{\lambda \mid \lambda 1 - A \text{ injective}, \overline{\mathrm{im}(\lambda 1 - A)} \neq X\}$
$C\sigma(A) = \{\lambda \mid \lambda 1 - a \text{ injective}, \overline{\mathrm{im}(\lambda 1 - A)} = X, \mathrm{im}(\lambda 1 - A) \neq X\}$

Remark 2: X reflexive, $A : \mathrm{dom}(A) \to X$ closed, densely defined.

$$\mathrm{im}\, A^\perp = \ker A^* \quad {}^\perp(\mathrm{im}\, A^*) = \ker A$$
$$\overline{\mathrm{im}\, A} ={}^\perp (\ker A^*) \quad \overline{\mathrm{im}\, A^*} = (\ker A)^\perp$$

Remark 3:

$$\rho(A) = \rho(A^*) \quad C\sigma(A) = C\sigma(A^*)$$
$$R\sigma(A) \subset P\sigma(A^*) \quad R\sigma(A^*) \subset P\sigma(A)$$
$$P\sigma(A) \cup R\sigma(A) = P\sigma(A^*) \cup R\sigma(A^*)$$

Example 1: $X = \ell^2_{\mathbb{C}} \ni x = (x_1, x_2, x_3, \ldots)$ $D := \{x \in \ell^2 \mid \sum_{n=1}^{\infty} n^2|x_n|^2 < \infty\}$
$Ax := (2x_2, 3x_3, \ldots)$ $A : D \to \ell^2$ closed.
$P\sigma(A) = \mathbb{C} \ni \lambda$ $x_\lambda = (\lambda, \frac{\lambda^2}{2!}, \frac{\lambda^3}{3!}, \ldots) \in D$ $Ax_\lambda = \lambda x_\lambda$.

Example 2: $Ax := (x_1, 2x_2, 3x_3, \ldots)$. Eigenvectors: $e_n := (0, \ldots, 0, \overbrace{1}^{n}, 0, \ldots)$
$Ae_n = ne_n$ $P\sigma(A) = \sigma(A) = \mathbb{N}$

Lemma 2: X complex Banach space. $D \subset X$ dense subset. $A : D \to X$ closed operator. Assume $\lambda_0 \in \rho(A)$ (then $R_{\lambda_0}(A) := (\lambda_0 1 - A)^{-1} : X \to D \subset X$ is bounded, cf. Closed Graph Theorem). Then the following holds:

i) If $\lambda \in \mathbb{C} \setminus \{\lambda_0\}$ then
$\ker(\lambda 1 - A) = \ker(\frac{1}{\lambda_0 - \lambda} 1 - R_{\lambda_0}(A))$
$\operatorname{im}(\lambda 1 - A) = \operatorname{im}(\frac{1}{\lambda_0 - \lambda} 1 - R_{\lambda_0}(A))$

ii) $\sigma(A) = \{\lambda \in \mathbb{C} \setminus \lambda_0 \mid \frac{1}{\lambda_0 - \lambda} \in \sigma(R_{\lambda_0}(A))\}$. Same for $P\sigma, R\sigma, C\sigma$.

iii) $\rho(A)$ is open, the map $\rho(A) \to \mathcal{L}(X, D) : \lambda \mapsto R_\lambda(A)$ is holomorphic, and $R_\mu(A) - R_\lambda(A) = (\lambda - \mu) R_\lambda(A) R_\mu(A) \forall \lambda, \mu \in \rho(A)$. Here D is equipped with the graph norm of A: $\|x\|_D := \|x\|_X + \|Ax\|_X$ for $x \in D$.

Proof: $\lambda \neq \lambda_0$.

$$\begin{aligned}
\lambda 1 - A &= \lambda_0 1 - A + (\lambda - \lambda_0) 1 \\
&= (\lambda_0 1 - A)(1 + (\lambda - \lambda_0) R_{\lambda_0}(A)) \\
&\overset{(*)}{=} (\lambda - \lambda_0)(\lambda_0 1 - A)(\frac{1}{\lambda - \lambda_0} 1 + R_{\lambda_0}(A)) \\
&= (\lambda_0 - \lambda)(\lambda_0 1 - A)(\frac{1}{\lambda_0 - \lambda} 1 - R_{\lambda_0 - \lambda} 1 - R_{\lambda_0}(A)) \\
\lambda 1 - A &\Rightarrow (i), (ii)
\end{aligned}$$

Proof of (iii) $|\lambda - \lambda_0| \cdot \|R_{\lambda_0}(A)\| < 1 \Rightarrow 1 \cdot (\lambda - \lambda_0) R_{\lambda_0}(A)$ bijective
$\overset{(*)}{\Rightarrow} \lambda 1 - A = (1 - (\lambda - \lambda_0) R_{\lambda_0}(A))(\lambda_0 1 - A) : D \to X$ is bijective
Hence $\rho(A)$ is open.

$$R_\lambda(A) \overset{(*)}{=} \frac{1}{\lambda_0 - \lambda} R_{\lambda_0}(\frac{1}{\lambda_0 - \lambda} 1 - R_{\lambda_0}(A))^{-1}$$

So $\rho(A) \to \mathcal{L}(X, D) : \lambda \mapsto R_\lambda(A)$ is holomorphic. □

Definition: A closed, densely defined unbounded operator $A : \operatorname{dom}(A) \subset X \to X$ is said to have a *compact resolvent* if $\rho(A) \neq \varnothing$ and $R_\lambda(A) : X \to X$ is compact $\forall \lambda \in \rho(A)$.

Remark 4: $\lambda_0 \in \rho(A)$ $R_{\lambda_0}(A)$ compact.
$\Rightarrow R_\lambda(A)$ compact $\forall \lambda \in \rho(A)$, because

$$R_\lambda(A) = \underbrace{R_{\lambda_0}}_{\text{compact}} \underbrace{(1 + (\lambda_0 - \lambda) R_\lambda(A))}_{\textit{bounded}}$$

Remark 5: Suppose $\rho(A) \neq \varnothing$ and let $D := \operatorname{dom}(A)$ be equipped with a graph norm. Then:

$$A \text{ has a compact resolvent} \Leftrightarrow \text{the inclusion } D \to X \text{ is compact}$$

Remark 6: $A : D = \mathrm{dom}(A) \to X$ closed, densely defined.
$X^0 := X \quad \|x\|_0 = \|x\|_X$
$X^1 := D = \mathrm{dom}(A) \quad \|x\|_1 := \|x\|_X + \|Ax\|_X$
$X^2 := \{x \in D \mid Ax \in X_1\} \quad \|x\|_2 := \|x\| + \|Ax\| + \|A^2x\|$
$X^3 := \{x \in D \mid Ax \in X_2\}$ and so on with
$\ldots \subset X^3 \subset X^2 \subset X^1 \subset X^0$.

Assume $\rho(A) \neq \varnothing$, let $\lambda_0 \in \rho(A)$ and denote $T := \lambda_0 1 - A$. Then $T : X^{k+1} \to X^k$ is an isomorphism for all K and
Moreover: $(\lambda 1 - A)^m : X^m \to X^0$ and $\ker(\lambda 1 - A)^m \subset X^\infty = \bigcup_{m=1}^\infty X^m$. And: if the inclusion $X^1 \to X^0$ is compact, then $X^{k+1} \to X^k$ is compact $\forall k$.

Lemma 3: $A : \mathrm{dom}(A) \to X$ closed, densely defined, compact resolvent $\Rightarrow$

(i) $\sigma(A) = P\sigma(A)$

(ii) The space $E_\lambda(A) := \bigcup_{m=1}^\infty \ker(\lambda 1 - A)^m$ is finite dimensional $\forall \lambda \in \sigma(A)$.

(iii) $\sigma(A)$ is discrete.

Proof: Let $\lambda_0 \in \rho(A)$ and denote $K := R_{\lambda_0}(A) \in \mathcal{L}(X)$. Let $\lambda \in \sigma(A) \Rightarrow \lambda \neq \lambda_0$ and $\mu := \frac{1}{\lambda_0 - \lambda} \in \sigma(K)$
Moreover: $\mu \neq 0$ and $E_\lambda(A) = E_\mu(K)$ finite dimensional. $\Rightarrow (i), (ii)$ see Ch IV.
Let $\lambda_n \in \sigma(A) \quad \lambda_n \neq \lambda_m \forall n \neq m$
$\Rightarrow \mu_n := \frac{1}{\lambda_0 - \lambda} \in \sigma(K)$
$\overset{\text{Ch IV}}{\Rightarrow} \mu_n \to 0 \quad |\lambda_n| \to \infty$. □

$X = H$ Hilbert space and $\mathrm{dom}(A) \subset H$ dense subset
$A : \mathrm{dom}(A) \to H$ closed linear operator.

Definition: The *Hilbert space adjoint* of A is the (closed, densely defined) operator $A^* : \mathrm{dom}(A^*) \to H$ given by $\mathrm{dom}(A^*) := \{y \in H \mid \exists c \geq 0 \forall x \in \mathrm{dom}(A), |\langle y, Ax\rangle| \leq c\|x\|\}$ with $A^*y := z$, where $z \in H$ is the unique vector with $\langle z, x\rangle = \langle y, Ax\rangle$.

Remark 7: A closed $\Rightarrow A^{**} = A$

Definition:

a) A is called *self-adjoint* if $A^* = A$, ie. $\mathrm{dom}(A^*) = \mathrm{dom}(A)$ and $A^x = Ax \forall x \in \mathrm{dom}(A)$

b) A is called *symmetric* if $\langle x, Ay\rangle = \langle Ax, y\rangle \forall x, y \in \mathrm{dom}(A)$.

Remark 8: A symmetric $\Rightarrow \mathrm{dom}(A) \subset \mathrm{dom}(A^*)$ and $A^*|_{\mathrm{dom}(A)} = A$.

Lemma 4: $A : \mathrm{dom}(A) \to H$ densely defined, self-adjoint $\Rightarrow$

i) $\sigma(A) \subset \mathbb{R}$

ii) If in addition, A has a compact resolvent, then $\sigma(A) = P\sigma(A)$ is discrete subset of $\mathbb{R}$ and H has a ONB of eigenvectors of A.

Proof: Easy exercise. □

Exercise 1: $A \in \mathcal{L}(X), U \subset \mathbb{C}$ open, $\sigma(A) \subset U$
$f : U \to \mathbb{C}$ holomorphic, γ a path in U around $\sigma(A)$

$$f(A) := \frac{1}{2\pi i} \int_\gamma f(\lambda)(\lambda 1 - A)^{-1} \, d\lambda$$

Prove:

(i) $fg(A) = f(A)g(A)$, $1(A) = 1$ $id(A) = A$

(ii) $\sigma(A) = \sigma_0 \cup \sigma_1$ σ_0, σ_1 compact and disjoint
$f(\lambda) = \begin{cases} 0 & \lambda \in U_0 \\ 1 & \lambda \in U_1 \end{cases}$
where U_0, U_1 are disjoint open sets, U_i containing σ_i
$\Rightarrow P := f(A)$ satisfies $P^2 = P \quad PA = AP$
$X_0 := \ker P \quad X_1 := \operatorname{im} P$, so $X = X_0 \oplus X_1$ and $\sigma(A|_{X_i}) = \sigma_i$

What is $W^{1,p}([0,1])$?
$W^{1,p}([0,1]) := \{f : [0,1] \to \mathbb{R} | f \text{ cont. } \exists g \in L^p([0,1]) : f(x) = f(0) + \int_0^x g(t)\, dt \, \forall x\}$
$\|f\|_{W^{1,p}} := \left(\int_0^1 |f(x)|^p \, dx + \int_0^1 |g(x)|^p \, dx \right)^{\frac{1}{p}}$
Fact: $f \in W^{1,p} \Rightarrow f$ is differentiable almost everywhere and $\dot{f}(x) = g(x)$ for almost all $x \in [0,1]$
Warning: f almost everywhere differentiable
$\dot{f} \in L^p \not\Rightarrow f \in W^{1,p}$ (Cantor-function)

Remark: $g \in L^1$, $\int_0^x g(t)\, dt = 0 \, \forall x \in [0,1] \Rightarrow g \equiv 0$ a.e.
(measure and integration)

Definition: A function $f : [0,1] \to \mathbb{R}$ is said to have *bounded variation* if $\operatorname{Var}_{[0,1]} f < \infty$ where

$$\operatorname{Var}_{[0,x]} f := \sup_{0=t_0<t_1<\ldots t_n = x} \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|$$

$\sin(\frac{1}{x})$ is not of bounded variation

$\varphi_f(u) = \lim_{\delta \to 0} \sum_{i=0}^{m-1} u(t_i)(f(t_{i+1}) - f(t_i)) = \int_0^1 u \, df = \int u \, d\mu_f$
where $0 = t_0 < t_1 < \ldots < t_m = 1$ and $\delta := \max_i |t_{i+1} - t_i|$

Exercise 2: $BV := \{f : [0,1] \to \mathbb{R} \mid f$ is of bounded variation and right continuous$\}$
$\|f\|_{BV} := |f(0)| + \operatorname{Var}_{[0,1]} f$
Prove BV is a Banach space

Exercise 3: Every (right continuous) function of bounded variation is the difference of two monotone (right continuous) functions.
Hint: Denote $F(x) := |f(0)| + \operatorname{Var}_{[0,x]} f$
Show that $F \pm f$ are monotone, right continuous.
$f^\pm := \frac{F \pm f}{2} \quad f = f^+ - f^-$

Exercise 4:

a) $f : [0,1] \to \mathbb{R}$ monotone, right continuous, $f(0) \geq 0$
$\exists!$ Borel measure μ_f on $[0,1]$ such that $\mu_f([0,x]) = f(x)$

b) $f \in BV \Rightarrow \exists!$ Borel measure μ_f such that $\mu_f([0,x]) = f(x) \quad \forall x \in [0,1]$

c) $f(x) = \int_0^x g(t)\, dt \, \forall x \in [0,1]$, $g \in L^1$
$\Rightarrow \mu_f(E) = \int_E g \, d\lambda \leftarrow$ Lebesgue measure
hint for a): construct outer measure ν_f,
$\nu_f((a,b)) = \lim_{t \nearrow b} f(t) - f(a) \rightsquigarrow \nu_f(\text{open sets})$

Exercise 5*: $f \in BV, F(x) := |f(0)| + \mathrm{Var}_{[0,x]} f \Rightarrow$

(i) $|\mu_f| = \mu_F$

(ii) $f(x) = \int_0^x g(t)\,dt \Rightarrow F(x) = \int_0^x |g(t)|\,dt$

Hint: (i)$\Rightarrow$ (ii) $P := \{g > 0\} \quad N := \{g < 0\}$
$\Rightarrow |\mu_f|(E) = \mu_f(E \cap P) - \mu_f(E \cap N) = \int_{E\cap P} g - \int_{E \cap N} g = \int_E |g| \qquad (*)$
$\Rightarrow F(x) = \mu_F([0,x]) \overset{(i)}{=} |\mu_f|([0,x]) \overset{(*)}{=} \int_0^x |g|$
Proof of (i): Easy $\mu_F([0,x]) \leq |\mu_F|([0,x])$
Hard " $\geq$ "

Exercise 6: $f \in BV$. Equivalent are:

(i) $\mu_f \ll \lambda$

(ii) $\exists g \in L^1([0,1])$ such that $f(x) = \int_0^x g(t)\,dt$

(iii) $f(0) = 0$ and f is *absolutely continuous* i.e.
$\forall \varepsilon > 0\, \exists \delta > 0\, \forall n \in \mathbb{N}\, 0 \leq s_1 < t_1 < s_2 < t_2 < \ldots < s_n < t_n \leq 1$

$$\sum_{i=1} |t_i - s_i| < \delta \Rightarrow \sum_{i=1}^{n} |f(t_i) - f(s_i)| < \varepsilon$$

Hint: First assume f is monotone

Remark:

a) f abs. continuous $\Leftrightarrow$ f is diff. a.e., $f \in L^1$ and $f(x) = f(0) + \int_0^x \dot{f}(t)\,dt\,\forall x$

b) $W^{1,p}([0,1]) = \{f : [0,1] \to \mathbb{R} \mid f \text{ abs. continuous}, f \in L^p\}$
$\|f\|_{W^{1,p}} = \left(\|f\|_p^p + \|\dot{f}\|_p^p\right)^{\frac{1}{p}}$

Lemma: $p > 1$ The inclusion $W^{1,p}([0,1]) \to C([0,1])$ is a compact operator

Proof: $|f(t) - f(s)| = |\int_s^t \dot{f}(v)\,dv| \leq \int_s^t |\dot{f}(v)|\,dv \leq \left(\int_s^t |\dot{f}(v)|^p\,dv\right)^{\frac{1}{p}} |t-s|^{\frac{1}{q}}$
$\frac{1}{p} + \frac{1}{q} = 1$
$|f(t) - f(s)| \leq \|\dot{f}\|_{L^p} |t-s|^{\frac{1}{q}}$
$\Rightarrow$ The set $\{f \in W^{1,p} \mid \|f\|_{W^{1,p}} \leq 1\}$ is bounded and equicontinuous, so the result follows from Arzéla-Ascoli $\square$

Example: $H = L^2([0,1],\mathbb{C}) \quad \langle f, g\rangle = \int_0^1 \overline{f} g\,dt$
$D := \{f \in W^{1,2}([0,1]) \mid f(0) \in \mathbb{R}, f(1) \in \mathbb{R}\}$
$Af := i\dot{f}$

Exercise 7: A is selfadjoint, $\sigma(A) = 2\pi\mathbb{Z}$

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