ERRATA FOR *J-HOLOMORPHIC CURVES AND SYMPLECTIC TOPOLOGY*

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Abstract. The most substantive change here is to the proof of Theorem 6.2.6 (ii): the previous argument did not handle transversality quite correctly. We have listed some other smaller corrections. We intend to update this file from time to time and so welcome further comments.

p 20, line 12–14: Replace the sentence beginning: A smooth map $u : \Sigma \to M$ is conformal ... by: "Every J-holomorphic curve $u : \Sigma \to M$ is conformal with respect to g_J, i.e. its differential preserves angles, or, equivalently, it preserves inner products up to a common positive factor. The converse holds when M has dimension two."

p 48, line 2: The second minus sign should be plus.
pp 102/103: Simplify the proof of Step 3 (the old proof was correct but this one is shorter and more elegant):

Step 3. *We prove the identity*

$$\lim_{R\to\infty}\lim_{\nu\to\infty} E(u^\nu; B_{R\delta^\nu}) = m_0. \tag{1}$$

By definition of m_0 there is a sequence ε^ν such that

$$\lim_{\nu\to\infty} E(u^\nu; B_{\varepsilon^\nu}) = m_0, \qquad \lim_{\nu\to\infty} \varepsilon^\nu = 0. \tag{2}$$

More precisely, for every $\ell \in \mathbb{N}$, there exists $\varepsilon_\ell \in (0, 1/\ell)$ and $\nu_\ell \in \mathbb{N}$ such that

$$|E(u^\nu; B_{\varepsilon_\ell}) - m_0| \le 1/\ell$$

for $\nu \ge \nu_\ell$. Suppose, without loss of generality, that $\varepsilon_{\ell+1} < \varepsilon_\ell$ and $\nu_{\ell+1} > \nu_\ell$ for every ℓ. Then the sequence ε^ν, defined by $\varepsilon^\nu := \varepsilon_\ell$ for $\nu_\ell \le \nu < \nu_{\ell+1}$, satisfies (2). (Note that we may be unable to choose ε^ν so that $E(u^\nu; B_{\varepsilon^\nu})$ is precisely m_0 because the sequence $u^\nu : B_r \to M$ may consist of maps with energy less than m_0 converging away from 0 to a constant.) Since $R\varepsilon^\nu \to 0$ for every $R \ge 1$, we also have $\lim_{\nu\to\infty} E(u^\nu; B_{R\varepsilon^\nu}) = m_0$ and hence

$$\lim_{\nu\to\infty} E\big(u^\nu, A(\delta^\nu, R\varepsilon^\nu)\big) = \delta/2 \tag{3}$$

for every $R \ge 1$. We consider two cases.

If $\delta^\nu/\varepsilon^\nu$ is bounded away from zero then there is a constant $R > 0$ such that $R\delta^\nu \ge \varepsilon^\nu$ for ν sufficiently large. Hence $E\big(u^\nu; B_{R\delta^\nu}\big) \ge E\big(u^\nu; B_{\varepsilon^\nu}\big)$ for large ν and so (1) follows from (2).

Date: 16 August 2007.
2000 *Mathematics Subject Classification.* 53C15.
Key words and phrases. symplectic geometry, J-holomorphic curves.
The first author is partly supported by the NSF grant DMS 0604769.

If $\delta^\nu/\varepsilon^\nu \to 0$ then, by (3) and Lemma 4.7.3 with $e^T = R \geq 2$, we have

$$\lim_{\nu\to\infty} E\big(u^\nu; A(R\delta^\nu, \varepsilon^\nu)\big) \leq \frac{c}{R^{2\mu}} \lim_{\nu\to\infty} E\big(u^\nu; A(\delta^\nu, R\varepsilon^\nu)\big) \leq \frac{c\delta}{2R^{2\mu}}.$$

Hence, by (2),

$$\lim_{\nu\to\infty} E(u^\nu; B_{R\delta^\nu}) \geq m_0 - \frac{c\delta}{2R^{2\mu}}$$

and (1) follows by taking the limit $R \to \infty$. This proves Step 3.

p 107, line 3: Every sequence ... *has a subsequence which* converges ...

p 119, line 15: Replace $du^\nu(z) : \mathbb{C} \to T_{u(z)}M$ by $du^\nu(z) : \mathbb{C} \to T_{u^\nu(z)}M$.

p 128: The last sentence before Step 4 should read: "Now (5.4.5) follows by taking the limit $\varepsilon \to 0$. This proves Step 3."

p 151/152: Change the proof of Theorem 6.2.6 (ii), starting with line -6 on page 151, as follows.

Now consider the projections

$$p^\ell : \mathcal{M}^*(\{A_\alpha\}; \mathcal{J}^\ell) \to \mathcal{J}^\ell, \qquad \pi^\ell : \widetilde{\mathcal{M}}^*_{0,T}(\{A_\alpha\}; \mathcal{J}^\ell) \to \mathcal{J}^\ell.$$

These are Fredholm maps of indices

$$\operatorname{index}(p^\ell) = 2n(1 + e(T)) + 2c_1(A), \qquad \operatorname{index}(\pi^\ell) = \mu(A, T) + \dim \mathrm{G}_T.$$

Hence, by the Sard–Smale theorem A.5.1, the set $\mathcal{J}^\ell_{\mathrm{reg}}(T, \{A_\alpha\})$ of common regular values of p^ℓ and π^ℓ is of the second category in $\mathcal{J}^\ell$ for ℓ sufficiently large. Moreover, an almost complex structure $J \in \mathcal{J}^\ell$ is a common regular value of p^ℓ and π^ℓ if and only if it satisfies the conditions of Definition 6.2.1

Now, for every $K > 0$, consider the subset $\mathcal{M}^*_K(\{A_\alpha\}; \mathcal{J}^\ell) \subset \mathcal{M}^*(\{A_\alpha\}; \mathcal{J}^\ell)$ of all tuples $(\mathbf{u}, J) \in \mathcal{M}^*(\{A_\alpha\}; \mathcal{J}^\ell)$ that satisfy

$$\|du_\alpha\|_{L^\infty} \leq K$$

and

$$\inf_{\zeta \in S^2 \setminus \{z_\alpha\}} \frac{d(u_\alpha(z_\alpha), u_\alpha(\zeta))}{d(z_\alpha, \zeta)} \geq \frac{1}{K}, \qquad \inf_{\zeta \in S^2} d(u_\alpha(z_\alpha), u_\beta(\zeta)) \geq \frac{1}{K}$$

for every $\alpha \in T$, $\beta \in T \setminus \{\alpha\}$, and some collection of points $\{z_\alpha\}_{\alpha \in T}$ in S^2. Likewise, let $Z_K(T) \subset Z(T)$ be the set of all tuples $\mathbf{z} \in Z(T)$ that satisfy

$$d(z_{\alpha\beta}, z_{\alpha\gamma}) \geq \frac{1}{K}, \qquad d(z_i, z_j) \geq \frac{1}{K}, \qquad d(z_{\alpha\beta}, z_i) \geq \frac{1}{K}$$

for all α, $\beta \neq \gamma$, $i \neq j$ with $\alpha E \beta$, $\alpha E \gamma$, and $\alpha_i = \alpha_j = \alpha$, and denote

$$\widetilde{\mathcal{M}}^*_{0,T;K}(\{A_\alpha\}; J) := \widetilde{\mathcal{M}}^*_{0,T}(\{A_\alpha\}; J) \cap \Big(\mathcal{M}^*_K(\{A_\alpha\}; \mathcal{J}^\ell) \times Z_K(T)\Big).$$

Then the projections

$$p^\ell_K : \mathcal{M}^*_K(\{A_\alpha\}; \mathcal{J}^\ell) \to \mathcal{J}^\ell, \qquad \pi^\ell_K : \widetilde{\mathcal{M}}^*_{0,T;K}(\{A_\alpha\}; \mathcal{J}^\ell) \to \mathcal{J}^\ell$$

are proper Fredholm maps and so the set $\mathcal{J}^\ell_{\mathrm{reg,K}}(T, \{A_\alpha\})$ of common regular values of p^ℓ_K and π^ℓ_K is open and dense in $\mathcal{J}^\ell$ for ℓ sufficiently large. By the same reasoning the set

$$\mathcal{J}_{\mathrm{reg,K}}(T, \{A_\alpha\}) := \mathcal{J}^\ell_{\mathrm{reg,K}}(T, \{A_\alpha\}) \cap \mathcal{J}_\tau(M, \omega)$$

is open in $\mathcal{J}_\tau(M,\omega)$. Moreover, since $\mathcal{J}^\ell_{\mathrm{reg,K}}(T,\{A_\alpha\})$ is dense in $\mathcal{J}^\ell$ for ℓ sufficiently large it follows as in the proof of Theorem 3.1.5 that $\mathcal{J}_{\mathrm{reg,K}}(T,\{A_\alpha\})$ is dense in $\mathcal{J}_\tau(M,\omega)$. Hence the set

$$\mathcal{J}_{\mathrm{reg}}(T,\{A_\alpha\}) = \bigcap_{K>0} \mathcal{J}_{\mathrm{reg,K}}(T,\{A_\alpha\})$$

is a countable intersection of open and dense sets in $\mathcal{J}_\tau(M,\omega)$. This proves (ii).

p 160: The condition "$\overline{F(W)}$ is compact" is needed in the definition of bordant pseudocycles.

p 161, before Lemma 6.5.5: Replace "dim M" by "dim X".

p 251: The condition $\int_M H_t\omega^n = 0$ should be mentioned in the construction of Remark 8.2.11 (i).

p 272/273: The condition $\int_M H_t\omega^n = 0$ is required in Corollary 8.6.10. It is used in the last displayed equation of the proof which refers to the construction of Remark 8.2.11.

p 285: Lemma 9.1.9 requires the assumption $\int_M H_t^\lambda\omega^n = 0$ for all t and λ.

p 340: The last line in the proof of Proposition 9.7.2 should read:

Hence the loop $t \mapsto \phi^{-1}\circ\psi_t\circ\phi$ is smoothly isotopic to $t\mapsto\psi_t$ and preserves the symplectic form ω_λ. This proves the proposition.

p 356: The constant c_0 in Proposition 10.5.1 depends not only on p but also on c.

p 373: The assertion of Step 2 should read: *"For every $\varepsilon > 0$ there are positive constants δ_2 and ε_2 such that, for every $(\delta, R) \in \mathcal{A}(\delta_2)$, the following holds..."*

p 378/9: The proof of Step 4 should read: "First choose $\varepsilon_1 > 0$ such that the assertion of Step 1 holds. Then choose $\varepsilon_2 > 0$ and $\delta_2 < \delta_0(c)$ such that the assertion of Step 2 holds with this constant ε_1. Finally, choose $\varepsilon_3 > 0$ and $\delta_3 < \delta_2$ such that the assertion of Step 3 holds with this constant ε_2. Now..."

p 379/381: Replace S_0 by S^0 in Theorem 10.8.1 (twice), Remark 10.8.2 (once), and in equation (10.8.1) (once).

p 392: Add the following remark after Example 11.1.4 and before Remark 11.1.5.

The notation $e^A := \phi(\delta_A)\in\Lambda$ is meaningful only for $A\in K^{\mathrm{eff}}(M,\omega)$. However, in Example 11.1.4 (i-vii) the homomorphism $\phi:\Gamma(M,\omega)\to\Lambda$ extends naturally to the group ring of $H_2(M)$ and then the notation $e^A := \phi(\delta_A)\in\Lambda$ is meaningful for every $A\in H_2(M)$. (In some cases the restriction $c_1(A)\ge 0$ is required.) In Example 11.1.4 (ii-iv) we have

$$\iota(e^A) = \begin{cases} 1, & \text{if } A = 0,\\ 0, & \text{otherwise,}\end{cases}$$

and Example 11.1.4 (i), (v), (vi) we have

$$\iota(e^A) = \begin{cases} 1, & \text{if } c_1(A) = \omega(A) = 0,\\ 0, & \text{otherwise.}\end{cases}$$

These two formulas agree if we restrict attention to classes $A\in K^{\mathrm{eff}}(M,\omega)$.

p 393: At the end of Remark 11.1.6 the sign # is missing in the finiteness condition.

p 394: Add the following remark before Remark 11.1.7.

In the notation of Remark 11.1.6 the quantum product of two quantum cohomology classes $a = \sum_A a_A \otimes e^A$ and $b = \sum_B b_B \otimes e^B$ is

$$a * b = \sum_{A,B,C,\nu,\mu} \mathrm{GW}^M_{C,3}(a_A, b_B, e_\nu) g^{\nu\mu} e_\mu \otimes e^{A+B+C}$$

and the pairing (11.1.4) is

$$\langle a, b\rangle = \sum_{A,B} \iota(e^{A+B}) \int_M a_A \smile b_B = \alpha(a * b), \qquad \alpha(a) := \sum_A \iota(e^A) \int_M a_A.$$

Here the classes a_A are not required to have pure degree and the integral over M is understood as the integral of the component in degree $2n$. (For the notation $\iota(e^A)$ see the Remark after Example 11.1.4.)

p 404, line 7: Replace $\varepsilon(\nu) = r(r+1)/2$ by $\varepsilon(\nu) = (-1)^{r(r+1)/2}$.

p 421: In line 4 from below it should read $\deg(u) = \langle x_1, L\rangle = 1$.

p 443, line -6: The displayed formula and subsequent text should read

$$\alpha(a_t) := \int_M a_t = t_N \in \mathbb{C}, \qquad a_t := \sum_i t_i e_i.$$

The corresponding pairing can be written in the form (11.1.4) with ι equal to the identity map from $\Lambda = \mathbb{C}$ to $R = \mathbb{C}$. Hence Proposition 11.1.9 shows that $\mathcal{H}$ is a Frobenius algebra over $\mathbb{C}$. If we must use ...

p 447, line 7: Replace (11.5.3) by (11.5.6). and on line -15 replace g_j by e_j.

p 454, Rmk 12.1.1: Replace "Cohen–James–Segal" by "Cohen–Jones–Segal".

p 457: The discussion uses nonexistence of holomorphic spheres with negative Chern numbers for generic 2-parameter families of almost complex structures (in the proof that $\Phi^{\alpha\beta}$ is independent of the homotopy from J^α to J^β used to define it). This holds only under the strong semipositivity assumption (8.5.1). If one wants to prove the Arnold conjecture in the general semipositive case with the methods described in the book, then one has to fix a generic almost complex structure J once and for all, and then construct Floer homology groups that are independent of H but, apriori, might depend on J. The best way around this subtlety would be to assume (8.5.1) and allow J to depend on t.

p 509: Refer to "Abraham–Robbin, *Transversal Mappings and Flows*, Benjamin, 1970" for the proof of Sard's theorem with sharp differentiability hypotheses. This doesn't follow from the proof in Milnor's book.

p 512: The proof of Exercise B.1.2 (ii) is surprisingly nontrivial. The hard part is to prove that, if $u \in W^{1,p}(\Omega)$ and $v \in W^{1,\infty}(\Omega)$, then the weak derivatives of uv are given by the Leibnitz rule $\partial_i(uv) = (\partial_i u)v + u(\partial_i v)$. These functions are obviously in L^p and the result then follows by induction. The proof of the Leibnitz rule requires Proposition B.1.4. Prove the result first when u is smooth and then approximate u on a compact subset of Ω by a sequence of smooth functions.

p 513, line 1/2: Replace "a unit vector ..." by "a nonzero vector $\xi \in \mathbb{R}^n$, a constant $\delta > 0$, and a Lipschitz continuous function $f : \xi^\perp \to \mathbb{R}$ such that $f(0) = 0$, $|f(\eta)| < \delta$ for $|\eta| \le \delta$, and".

p 520, lines 1-3: Replace the first three lines by the text. "n times with $m = n-1$. In the kth step we integrate over x_k and obtain

$$\int |u|^{n/(n-1)}\, dx_1 \cdots dx_k$$
$$\leq \prod_{i=1}^{k} \left(\int |\partial_i u|\, dx_1 \cdots dx_k \right)^{1/(n-1)} \prod_{i=k+1}^{n} \left(\int |\partial_i u|\, dx_1 \cdots dx_k dx_i \right)^{1/(n-1)}.$$

(where the kth factor doesn't depend on x_k). With $k = n$ this gives ..."

p 522: In Propositions B.1.21 and B.1.22 assume that $\Omega \subset \mathbb{R}^n$ is bounded.

The hint in the proof of Proposition B.1.22 only works for functions in $C^1(\bar{\Omega})$. To deal with general functions $u \in W^{1,p}(\Omega)$ one can argue as follows. Assume that u vanishes on the boundary and extend u to all of $\mathbb{R}^n$ by $u(x) := 0$ for $x \in \mathbb{R}^n \setminus \Omega$. Then the extended function is in $W^{1,p}(\mathbb{R}^n)$. To see this one can approximate u on $\bar{\Omega}$ by a sequence of smooth functions $u_j : \bar{\Omega} \to \mathbb{R}$, using Proposition B.1.4. Then it follows from Proposition B.1.21 that $u_j|_{\partial\Omega}$ converges to zero in $L^p(\partial\Omega)$. Hence it follows from the divergence theorem that

$$\int_\Omega \big(u(\partial_i \phi) + (\partial_i u)\phi\big) = \lim_{j\to\infty} \int_\Omega \big(u_j(\partial_i \phi) + (\partial_i u_j)\phi\big) = \lim_{j\to\infty} \int_{\partial\Omega} \nu_i u_j \phi = 0$$

for every test function $\phi \in C^\infty(\bar{\Omega})$ (and not just for $\phi \in C_0^\infty(\Omega)$). This proves that the extended function u belongs to $W^{1,p}(\mathbb{R}^n)$. Since u vanishes outside of Ω we can now approximate u by a sequence in $W^{1,p}(\Omega)$ which vanishes near $\partial\Omega$ and hence belongs to $W_0^{1,p}(\Omega)$.

p 529: In the assertion of Step 2 (ii) (and in the proof of Step 3) replace Q by $B := \bigcup_i Q_i$. Replace the proof of Step 2 by the following argument.

For $k \in \mathbb{Z}^n$ and $\ell \in \mathbb{Z}$ denote

$$Q(k,\ell) := \left\{ x \in \mathbb{R}^n \,|\, 2^{-\ell} k_i \leq x_i \leq 2^{-\ell}(k_i + 1),\, i = 1, \ldots, n \right\}.$$

Let $\mathcal{Q} := \{Q(k,\ell) \,|\, k \in \mathbb{Z}^n,\, \ell \in \mathbb{Z}\}$ and $\mathcal{Q}_0 \subset \mathcal{Q}$ be the set of all $Q \in \mathcal{Q}$ satisfying

$$t\mathrm{Vol}(Q) < \|f\|_{L^1(Q)}$$

and

$$Q \subsetneq Q' \in \mathcal{Q} \qquad \Longrightarrow \qquad \|f\|_{L^1(Q')} \leq t\mathrm{Vol}(Q').$$

Then every decreasing sequence of cubes in $\mathcal{Q}$ contains at most one element of $\mathcal{Q}_0$. Hence every $Q \in \mathcal{Q}_0$ satisfies assertion (i) and any two cubes in $\mathcal{Q}_0$ have disjoint interiors. Now let

$$B := \bigcup_{Q \in \mathcal{Q}_0} Q.$$

Then

$$x \in \mathbb{R}^n \setminus B, \quad x \in Q \in \mathcal{Q} \qquad \Longrightarrow \qquad \frac{1}{\mathrm{Vol}(Q)} \|f\|_{L^1(Q)} \leq t.$$

(Otherwise take a maximal cube $Q \in \mathcal{Q}$ that satisfies $t\mathrm{Vol}(Q) < \|f\|_{L^1(Q)}$ and contains x. This cube would belong to $\mathcal{Q}_0$ and so $x \in B$.) Thus we have proved that, for every $x \in \mathbb{R}^n \setminus B$, there is a sequence of decreasing cubes $Q_\ell \in \mathcal{Q}$ containing x such that $\mathrm{Vol}(Q_\ell)^{-1} \|f\|_{L^1(Q_\ell)} \leq t$. Hence it follows from Lebesgue's differentiation theorem that $|f(x)| \leq t$ for almost every $x \in \mathbb{R}^n \setminus B$. This proves Step 2.

p 560: The proof of equation (C.4.2) is wrong. To correct it, choose a family of (nonlocal) Lagrangian boundary conditions for the operator $D_{01} \oplus D_{12}$ connecting $F_1 \oplus F_1$ to the diagonal in $\bar{E}|_{\Gamma_1} \oplus E|_{\Gamma_1}$.

p 587, line 12: Replace "$\alpha_i = i$" by "$\alpha_i = \alpha$".

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