ERRATA FOR *INTRODUCTION TO SYMPLECTIC TOPOLOGY*

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Abstract. This note corrects some typos and some errors in *Introduction to Symplectic Topology* (2nd edition, OUP 1998). In particular, in the latter book the statements of Theorem 6.36 (about Hamiltonian bundles) and Exercise 10.28 (about the structure of the group of symplectomorphisms of an open Riemann surface) need some modification.

We thank everyone who pointed out these errors, and in particular Kotschick and Morita who noticed the problem with Exercise 10.28.

p 32/3: In our discussion of the symplectic camel problem we are tacitly assuming that $n \geq 2$.

p 60: The formula for Vol$E$ should read

$$\mathrm{Vol}\, E = \int_E \frac{\omega_0^n}{n!} = \frac{\pi^n}{n!} \prod_{j=1}^{n} r_j^2.$$

p 64: In line 3 of the proof we should have $J \in \mathbb{R}^{2n\times 2n}$.

p 72: The formula for the unitary basis $\{s_k\}$ is incorrect. One should define s_k for $k \leq n$ as $s_k'/|s_k'|$ where:

$$s_k' = \tilde{s}_k - \sum_{j=1}^{k-1} \langle s_j, \tilde{s}_k\rangle\, s_j - \sum_{j=1}^{k-1} \omega(s_j, \tilde{s}_k)\, J s_j.$$

p 74: The additivity axiom in Theorem 2.69 is stated correctly only when E is a line bundle. For general vector bundles $E_i \to \Sigma$ of rank n_j, where $j = 1, 2$, it should read:

$$c_1(E_1 \oplus E_2) = c_1(E_1) + c_1(E_2), \quad c_1(E_1 \otimes E_2) = n_2 c_1(E_1) + n_1 c_1(E_2).$$

p 77, middle: This should read: Additivity follows from the identities:

$$\det(U_1 \oplus U_2) = \det(U_1)\det(U_2), \quad \det(U_1 \otimes U_2) = \big(\det(U_1)\big)^{n_2}\big(\det(U_2)\big)^{n_1}$$

for unitary matrices $U_j \in \mathrm{U}(n_j)$.

p 82, line -4: The reference should be to Proposition 3.6 not Proposition 3.2.

p 87, line -7: It is clearer to write $L_q = (T_qS)^\omega$ instead of $L_q = T_qS^\omega$.

p 109, line -4: Besides the stated condition a contact isotopy ψ_t should satisfy $\psi_0 = \mathrm{id}$.

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p 132/3: It is easy to deduce from de Rham's theorem and Hodge theory that the Poincaré duality homomorphism PD : $H_{n-k}(M;\mathbb{R}) \to H^k(M;\mathbb{R})$ is well defined and **surjective**. If it is not surjective then there is a nonzero harmonic k-form $\alpha \in \Omega^k(M)$ that is L^2-orthogonal to its image; hence $\int_N *\alpha = 0$ for every closed oriented $(n-k)$-submanifold $N \subset M$ and so, by deRham's theorem, $*\alpha$ is exact; since α is harmonic this implies $\alpha = 0$. Injectivity now follows for dimensional reasons.

p 167, line 1–4: H_ζ is an element of $C^\infty(\mathbb{C}^n)$ not $C^\infty(\mathbb{R}^n)$. The moment map $\mu : \mathbb{C}^n \to \mathfrak{u}(n)$ is related to H_ζ by the formula $H_\zeta(z) = \mathrm{tr}(\mu(z)^*\zeta)$ and so is given by:

$$\mu(z) = -\frac{i}{2} zz^*$$

p 169, Ex 5.26: In line 5 the formula should be

$$\mathfrak{g} \times \mathfrak{g} \to \mathbb{R} : (\xi, \xi') \mapsto \langle \eta, [\xi, \xi'] \rangle.$$

p 170, line -3 and -11: Replace Lie(G) by Lie($\mathcal{G}$).

p 184: Delete the phrase "regular value" in the assertion of Lemma 5.51.

p 186, line 6: Replace p by x.

p 188, line -10: The sentence should start as follows: To see this let $\eta := \mu(x) \in \mu(M)$ and approximate x by a sequence ...

p 204, line 5: $B \times S^{2k}$ should be $B \times S^2$.

p 216, line 3: Integration over the fiber gives a map $H^j(M) \to H^{j-2k}(B)$.

p 216, line -7: $\psi_t(x)$ not $\psi_t(s)$.

p 220, line -10: The reference should be to (6.2) not (6.1).

p 221, line 9: Replace ω by σ.

p 222: The identity in the assertion of Lemma 6.26 should read

$$\iota([v_1^\sharp, v_2^\sharp]^{\mathrm{Ver}})\tau \overset{\mathrm{fibre}}{=} \iota(v_2^\sharp)\iota(v_1^\sharp)d\tau - d\iota(v_2^\sharp)\iota(v_1^\sharp)\tau;$$

the first identity in the proof of Lemma 6.26 should read:

$$\begin{aligned} d\tau(v_1^\sharp, v_2^\sharp, Y) &= \tau([v_1^\sharp, v_2^\sharp], Y) + \tau([v_2^\sharp, Y], v_1^\sharp) + \tau([Y, v_1^\sharp], v_2^\sharp) \\ &\quad + \mathcal{L}_Y(\tau(v_1^\sharp, v_2^\sharp)) + \mathcal{L}_{v_1^\sharp}(\tau(v_2^\sharp, Y)) + \mathcal{L}_{v_2^\sharp}(\tau(Y, v_1^\sharp)); \end{aligned}$$

the last displayed inequality in the proof should read

$$\tau([v_1^\sharp, v_2^\sharp], Y) + \mathcal{L}_Y(\tau(v_1^\sharp, v_2^\sharp)) = \tau([v_1^\sharp, v_2^\sharp]^{\mathrm{Ver}}, Y) + \iota(Y)d\,\iota(v_2^\sharp)\iota(v_1^\sharp)\tau.$$

p 228-229: Theorem 6.36 is wrong as it stands. A counterexample is the quotient

$$M := \frac{S^2 \times \mathbb{T}^2}{\mathbb{Z}_2}$$

where we think of $S^2 \subset \mathbb{R}^3$ as the unit sphere and of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ as the standard torus; the nontrivial element of $\mathbb{Z}_2$ acts by the involution

$$(x, y) \mapsto (-x, y + (1/2, 0)), \qquad x \in S^2,\ y \in \mathbb{T}^2.$$

The closed 2-form

$$\tau = dy_1 \wedge dy_2 \in \Omega^2(S^2 \times \mathbb{T}^2)$$

descends to a closed connection 2-form on M; its holonomy around each contractible loop in $\mathbb{R}P^2$ is the identity and around each noncontractible loop is the symplectomorphism $(y_1, y_2) \mapsto (y_1 + 1/2, y_2)$. Thus M satisfies condition (iii) in Theorem 6.36 and the restriction of M to every loop in $\mathbb{R}P^2$ admits a symplectic trivialization. However, this fibration does not satisfy (i): one can show that the holonomy of every closed connection 2-form on M over every noncontractible loop in $\mathbb{R}P^2$ differs from the half rotation $(y_1, y_2) \mapsto (y_1 + 1/2, y_2)$ by a Hamiltonian symplectomorphism and so cannot itself be Hamiltonian. Another way to see that (i) does not hold is to observe that the classifying map $\mathbb{R}P^2 \to B\mathrm{Symp}_0(\mathbb{T}^2)$ is not null homotopic, while $\mathrm{Ham}(\mathbb{T}^2)$ is contractible.

The mistake in the proof of Theorem 6.36 is the assertion on page 229 that the homomorphism $f_\tau : \pi_1(B) \to H^1(F; \mathbb{R})/\Gamma_\sigma$ given by the flux of the holonomy of τ around the loops in B lifts to a homomorphism

$$\widetilde{f}_\tau : \pi_1(B) \to H^1(F; \mathbb{R}).$$

If this lift exists then the rest of the proof that $M \to B$ has a Hamiltonian structure goes through. But there is a lift if and only if the flux of the holonomy of every torsion element in $\pi_1(B)$ is zero (in $H^1(F; \mathbb{R})/\Gamma_\sigma$). In the above example $\pi_1(B)$ is torsion and f_τ is nonzero. Hence the lift $\widetilde{f}_\tau$ does not exist.

We claim that the existence of the lift $\widetilde{f}_\tau$ does not depend on the choice of the closed connection form τ. To see this, we fix a symplectic trivialization $\Phi : B_1 \times F \to M$ over the 1-skeleton and a homology class $a \in H^2(M; \mathbb{R})$ such that $\iota_b^* a = [\sigma_b]$ for some (and hence every) $b \in B$. Each such pair (a, Φ) defines a homomorphism

$$\mu_{\Phi^* a} : H_1(B_1; \mathbb{Z}) \to H^1(F; \mathbb{R})$$

via

$$\big\langle \mu_{\Phi^* a}([\beta]), [\gamma] \big\rangle := \int_{S^1 \times S^1} (\beta \times \gamma)^* \Phi^* a$$

for every pair of based loops $\beta : S^1 \to B_1$ and $\gamma : S^1 \to F$. If τ is a closed connection form in class a, then $\Phi^*\tau$-parallel translation around the loop β in $B_1 \times F$ gives a path $\{g_t\}_{0 \le t \le 1}$ in $\mathrm{Symp}(F)$ that starts at the identity and is such that

$$\mu_{\Phi^* a}([\beta]) = \mathrm{Flux}\big(\{g_t\}\big).$$

Hence $\mu_{\Phi^* a}$ does lift the pullback

$$H_1(B_1; \mathbb{Z}) \to H_1(B; \mathbb{Z}) \xrightarrow{f_\tau} H^1(F; \mathbb{R})/\Gamma_\sigma$$

of f_τ to the free group $H_1(B_1; \mathbb{Z})$. But we need a lift $\widetilde{f}_\tau$ that is defined on $H_1(B; \mathbb{Z})$, and this exists if and only if $\mu_{\Phi^* a}([\beta]) \in \Gamma_\sigma$ for every loop $\beta : S^1 \to B_1$ that represents a torsion class in $H_1(B; \mathbb{Z})$.

It remains to prove that the latter condition is independent of a. Let $\beta : S^1 = \mathbb{R}/\mathbb{Z} \to B_1$ represent a torsion class in $H_1(B; \mathbb{Z})$. Then there is a positive integer k such that the loop $s \mapsto \beta(ks)$ is homologous to zero in B. This implies that the k-fold multiple of the cycle $\Phi \circ (\beta \times \gamma) : \mathbb{T}^2 \to M$ is homologous to a cycle $v : \mathbb{T}^2 \to F_{b_0}$ in the fiber. To be explicit, note that there is an oriented Riemann surface Σ with two boundary components $\partial_0 \Sigma$ and $\partial_1 \Sigma$, diffeomorphisms $\iota_0 : S^1 \to \partial_0 \Sigma$ (orientation reversing) and $\iota_1 : S^1 \to \partial_1 \Sigma$ (orientation preserving), and a smooth map $u : \Sigma \to B$ such that $u \circ \iota_1(s) = \beta(ks)$ and $u \circ \iota_0(s) = b_0$ for $s \in S^1$. Since the fibration M admits a (symplectic) trivialization over every loop in B, the given trivialization of M over β extends to a symplectic trivialization $\Psi : \Sigma \times F \to M$ over u such that $\Psi(\iota_1(s), q) = \Phi(ks, q)$ for $s \in S^1$ and $q \in F$. Hence the cycle

$$\mathbb{T}^2 \to M : \quad (s,t) \mapsto \Phi(\beta(ks), \gamma(t)) = \Psi(\iota_1(s), \gamma(t))$$

is homologous to the cycle

$$\mathbb{T}^2 \to F_{b_0} : \quad (s,t) \mapsto v(s,t) := \Psi(\iota_0(s), \gamma(t)).$$

It follows that

$$\langle \mu_{\Phi^* a}([\beta]), [\gamma] \rangle := \frac{1}{k} \int_{\mathbb{T}^2} v^* \sigma_{b_0}.$$

Since the map u and the trivialization Ψ along u are independent of a so is the cycle v and the cohomology class $\mu_{\Phi^* a}([\beta]) \in H^1(F;\mathbb{Z})$.

Theorem 6.36 becomes correct if we add to (iii) the existence of the lift $\widetilde{f}_\tau$ and to (iv) the condition that the homomorphism $\mu_{\Phi^* a} : H_1(B_1;\mathbb{Z}) \to H^1(F;\mathbb{R})$ takes the elements of $H_1(B_1;\mathbb{Z})$ that descend to torsion classes in $H_1(B;\mathbb{Z})$ to the subgroup $\Gamma_\sigma \subset H^1(F;\mathbb{R})$, for some (and hence every) symplectic trivialization Φ over the 1-skeleton. For further discussion of these questions see [2].

p 236, line -6: The homology group is $H_2(X;\mathbb{Z})$.

p 240: The first displayed formula should read: $F(z) = f(|z|)z/|z|$. The second displayed formula should read:

$$\begin{aligned}\omega(\lambda) &= \frac{i}{2}\left(\partial\bar{\partial}(|z|^2 + \lambda^2 \log(|z|^2)\right) \\ &= \frac{i}{2}\left(dz \wedge d\bar{z} + \lambda^2 \frac{dz \wedge d\bar{z}}{|z|^2} - \lambda^2 \frac{\bar{z} \cdot dz \wedge z \cdot d\bar{z}}{|z|^4}\right),\end{aligned}$$

p 291: In Figure 9.3 the second point should be x_N not x_1.

p 330, line 10: The calculation should start with $-\int G\omega^n$ not $\int G\omega^n$.

p 332: We state in Exercise 10.28 that if (M,ω) is a connected symplectic manifold such that $\omega = -d\lambda$ then the direct sum Flux $\oplus$ CAL defines a homomorphism from the identity component $\mathrm{Symp}_0^c M$ of the group of compactly supported symplectomorphisms to the sum $H_c^1(M;\mathbb{R}) \oplus \mathbb{R}$. As pointed out by Kotschick–Morita in [1] this is correct when $\dim M = 2n \geq 4$ but may fail when $\dim M = 2$. They observe that CAL extends to a map

$$\mathrm{CAL} : \mathrm{Symp}_0^c M \to \mathbb{R}, \quad \phi \mapsto -\frac{1}{n+1} \int_M \phi^*(\lambda) \wedge \lambda \wedge \omega^{n-1},$$

that satisfies the identity

$$\mathrm{CAL}(\phi\psi) = \mathrm{CAL}(\phi) + \mathrm{CAL}(\psi) + \frac{1}{n+1} \int_M \mathrm{Flux}(\phi) \wedge \mathrm{Flux}(\psi) \wedge \omega^{n-1}.$$

Hence this map is a group homomorphism only if the cup product pairing

$$H_c^1(M;\mathbb{R}) \times H_c^1(M;\mathbb{R}) \to \mathbb{R}, \quad (\alpha, \beta) \mapsto \int_M \alpha \wedge \beta \wedge \omega^{n-1}$$

vanishes. This pairing does vanish when $n > 1$ because ω is exact. However, it need not vanish when $n = 1$. In this case, the correct target group is the central extension of $H_c^1(M;\mathbb{R})$ by $\mathbb{R}$ that is defined by this pairing. For more detail see [1, Prop 11].

p 334, lines -1 and -7: Replace ψ_t by ϕ_t.

p 342: In Example 11.3 the second function should be

$$H(x, y) = \sin 2\pi x + \sin 2\pi y.$$

p 350, line -5 ff: The homology exact sequence of the triple (N^b, N^a, L) shows that

$$d_{k-1}^{ab} + d_k^{ab} = c_k^b - c_k^a - b_k^b + b_k^a.$$

p 374, line 7: Replace "by the monotonicity axiom" by "by the monotonicity and conformality axioms."

p 387/8: The diffeomorphism f whose description begins at the bottom of p 387 takes a neighbourhood of the arc $(-\sqrt{2}r, -r)$ into a neighbourhood of $\partial U_1 \cup L$.

p 399, line -9: The equation $\iota(X)\omega = \omega$ should be replaced by $\mathcal{L}_X\omega = \omega$.

References

[1] D. Kotschick and S. Morita, Signatures of foliated surface bundles and the symplectomorphism groups of surfaces, SG/0305182, to appear in *Topology*.

[2] D. McDuff, Enlarging the Hamiltonian group (2004).

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