Corrigendum: A construction of the Deligne-Mumford orbifold

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Abstract

We correct an error in [3, Lemma 8.2]. As stated the lemma only holds for surfaces of genus greater than 1 or in the case $\alpha = 0$. When the genus is 0 or 1 and in addition $\alpha \neq 0$, equation (8) in [3] (in the present corrigendum this is equation (2)) is only a necessary condition for the integrability of J but is not sufficient. In [3] Lemma 8.2 is only used twice. On page 637 it is used in the trivial case $\alpha = 0$. On page 642 only the "only if" direction is used and the proof of that direction is correct in [3]. In this note we prove a corrected version of [3, Lemma 8.2].

Let $A \subset \mathbb{C}^m$ be an open set and Σ be a compact oriented 2-manifold without boundary. We denote the complex structure on A by i (instead of $\sqrt{-1}$ as in [3].) Let $\mathcal{J}(\Sigma)$ denote the space of (almost) complex structures on Σ that are compatible with the given orientation. An almost complex structure on $A \times \Sigma$ with respect to which the projection $A \times \Sigma \to A$ is holomorphic has the form

$$J = \begin{pmatrix} \mathrm{i} & 0 \\ \alpha & j \end{pmatrix},$$

where $j : A \to \mathcal{J}(\Sigma)$ is a smooth map and $\alpha \in \Omega^1(A, \mathrm{Vect}(\Sigma))$ is a smooth 1-form on A with values in the space of vector fields on Σ that satisfies

$$\alpha(a, \mathrm{i}\hat{a}) + j(a)\alpha(a, \hat{a}) = 0 \tag{1}$$

for $a \in A$ and $\hat{a} \in T_aA$. For $v, w \in \mathrm{Vect}(\Sigma)$ we denote by $\mathcal{L}_v$ the Lie derivative. We use the sign convention $\mathcal{L}_{[v,w]} = \mathcal{L}_w\mathcal{L}_v - \mathcal{L}_v\mathcal{L}_w$ for the Lie bracket.

Lemma A (i) *J is integrable if and only if j and α satisfy*

$$dj(a)\hat{a} + j(a)dj(a)\mathrm{i}\hat{a} + j(a)\mathcal{L}_{\alpha(a,\hat{a})}j(a) = 0, \tag{2}$$

$$d\xi(a)\mathrm{i}\hat{b} - j(a)d\xi(a)\hat{b} - d\eta(a)\mathrm{i}\hat{a} + j(a)d\eta(a)\hat{a} + [\xi(a), \eta(a)] = 0 \tag{3}$$

for all $\hat{a}, \hat{b} \in \mathbb{C}^m$ where $\xi, \eta : A \to \mathrm{Vect}(\Sigma)$ are defined by $\xi(a) := \alpha(a, \hat{a})$ and $\eta(a) := \alpha(a, \hat{b})$.

(ii) *If j and α satisfy (2) and Σ has genus greater than 1 then J is integrable.*

(iii) *If $j : A \to \mathcal{J}(\Sigma)$ is holomorphic and $\alpha = 0$ then J is integrable.*

Lemma B. *Assume* j *and* α *satisfy equation (2). Let* $\hat{a}, \hat{b} \in \mathbb{C}^m$ *and define* $\xi, \eta, \zeta : A \to \mathrm{Vect}(\Sigma)$ *by* $\xi(a) := \alpha(a, \hat{a})$, $\eta(a) := \alpha(a, \hat{b})$, *and*

$$\zeta(a) := d\xi(a)\mathrm{i}\hat{b} - j(a)d\xi(a)\hat{b} - d\eta(a)\mathrm{i}\hat{a} + j(a)d\eta(a)\hat{a} + [\xi(a), \eta(a)]. \tag{4}$$

Then

$$\mathcal{L}_{\zeta(a)} j(a) = 0. \tag{5}$$

Proof. Equation (2) reads

$$\begin{aligned} \mathcal{L}_{\xi(a)} j(a) &= j(a)dj(a)\hat{a} - dj(a)\mathrm{i}\hat{a}, \\ \mathcal{L}_{\eta(a)} j(a) &= j(a)dj(a)\hat{b} - dj(a)\mathrm{i}\hat{b}. \end{aligned} \tag{6}$$

Differentiating the first equation with respect to a in the direction $\hat{b}$ gives

$$\mathcal{L}_{d\xi(\hat{b})} j + \mathcal{L}_\xi(dj(\hat{b})) = dj(\hat{b})dj(\hat{a}) + jd^2j(\hat{a}, \hat{b}) - d^2j(\mathrm{i}\hat{a}, \hat{b}).$$

Here we omit the argument a and abbreviate $d\xi(\hat{b}) := d\xi(a)\hat{b}$, $dj(\hat{b}) := dj(a)\hat{b}$, $d^2j(\hat{a}, \hat{b}) := d^2j(a)(\hat{a}, \hat{b})$, etc. Multiplying the last equation by j, respectively replacing $\hat{b}$ by $\mathrm{i}\hat{b}$, we obtain

$$\begin{aligned} \mathcal{L}_{d\xi(\mathrm{i}\hat{b})} j + \mathcal{L}_\xi(dj(\mathrm{i}\hat{b})) - dj(\mathrm{i}\hat{b})dj(\hat{a}) &= jd^2j(\hat{a}, \mathrm{i}\hat{b}) - d^2j(\mathrm{i}\hat{a}, \mathrm{i}\hat{b}), \\ \mathcal{L}_{jd\xi(\hat{b})} j + j\mathcal{L}_\xi(dj(\hat{b})) - jdj(\hat{b})dj(\hat{a}) &= -d^2j(\hat{a}, \hat{b}) - jd^2j(\mathrm{i}\hat{a}, \hat{b}). \end{aligned}$$

Here we have used the identity $j\mathcal{L}_\xi j = \mathcal{L}_{j\xi} j$. Similarly, Replacing ξ by η, and interchanging $\hat{a}$ with $\hat{b}$ we obtain

$$\begin{aligned} \mathcal{L}_{d\eta(\mathrm{i}\hat{a})} j + \mathcal{L}_\eta(dj(\mathrm{i}\hat{a})) - dj(\mathrm{i}\hat{a})dj(\hat{b}) &= jd^2j(\mathrm{i}\hat{a}, \hat{b}) - d^2j(\mathrm{i}\hat{a}, \mathrm{i}\hat{b}), \\ \mathcal{L}_{jd\eta(\hat{a})} j + j\mathcal{L}_\eta(dj(\hat{a})) - jdj(\hat{a})dj(\hat{b}) &= -d^2j(\hat{a}, \hat{b}) - jd^2j(\hat{a}, \mathrm{i}\hat{b}). \end{aligned}$$

Putting things together we obtain

$$\begin{aligned} 0 &= \mathcal{L}_{d\xi(\mathrm{i}\hat{b})} j + \mathcal{L}_\xi(dj(\mathrm{i}\hat{b})) - dj(\mathrm{i}\hat{b})dj(\hat{a}) \\ &\quad -\mathcal{L}_{jd\xi(\hat{b})} j - j\mathcal{L}_\xi(dj(\hat{b})) + jdj(\hat{b})dj(\hat{a}) \\ &\quad -\mathcal{L}_{d\eta(\mathrm{i}\hat{a})} j - \mathcal{L}_\eta(dj(\mathrm{i}\hat{a})) + dj(\mathrm{i}\hat{a})dj(\hat{b}) \\ &\quad +\mathcal{L}_{jd\eta(\hat{a})} j + j\mathcal{L}_\eta(dj(\hat{a})) - jdj(\hat{a})dj(\hat{b}) \\ &= \mathcal{L}_{d\xi(\mathrm{i}\hat{b})} j - \mathcal{L}_{jd\xi(\hat{b})} j - \mathcal{L}_{d\eta(\mathrm{i}\hat{a})} j + \mathcal{L}_{jd\eta(\hat{a})} j \\ &\quad +\mathcal{L}_\xi(dj(\mathrm{i}\hat{b})) - j\mathcal{L}_\xi(dj(\hat{b})) - \mathcal{L}_\eta(dj(\mathrm{i}\hat{a})) + j\mathcal{L}_\eta(dj(\hat{a})) \\ &\quad +(\mathcal{L}_\eta j)dj(\hat{a}) - (\mathcal{L}_\xi j)dj(\hat{b}) \\ &= \mathcal{L}_{d\xi(\mathrm{i}\hat{b})} j - \mathcal{L}_{jd\xi(\hat{b})} j - \mathcal{L}_{d\eta(\mathrm{i}\hat{a})} j + \mathcal{L}_{jd\eta(\hat{a})} j \\ &\quad +\mathcal{L}_\xi(dj(\mathrm{i}\hat{b})) - \mathcal{L}_\xi(jdj(\hat{b})) - \mathcal{L}_\eta(dj(\mathrm{i}\hat{a})) + \mathcal{L}_\eta(jdj(\hat{a})) \\ &= \mathcal{L}_{d\xi(\mathrm{i}\hat{b})} j - \mathcal{L}_{jd\xi(\hat{b})} j - \mathcal{L}_{d\eta(\mathrm{i}\hat{a})} j + \mathcal{L}_{jd\eta(\hat{a})} j - \mathcal{L}_\xi\mathcal{L}_\eta j + \mathcal{L}_\eta\mathcal{L}_\xi j \\ &= \mathcal{L}_{d\xi(\mathrm{i}\hat{b})} j - \mathcal{L}_{jd\xi(\hat{b})} j - \mathcal{L}_{d\eta(\mathrm{i}\hat{a})} j + \mathcal{L}_{jd\eta(\hat{a})} j + \mathcal{L}_{[\xi,\eta]} j \\ &= \mathcal{L}_\zeta j. \end{aligned}$$

Here the second and fourth equations follow from (6). □

Proof of Lemma A. The proof has three steps.

Step 1. *Fix a vector* $\hat{a} \in \mathbb{C}^m$ *and let* $\xi : A \to \mathrm{Vect}(\Sigma)$ *be as in Lemma B. Fix a vector field* $v \in \mathrm{Vect}(\Sigma)$. *Then the Nijenhuis tensor on the pair*

$$X(a, z) := (\hat{a}, 0), \qquad Y(a, z) := (0, v(z))$$

is

$$N_J(X, Y) = \big(0, j\big(dj(\hat{a}) + j dj(\mathrm{i}\hat{a}) + j\mathcal{L}_\xi j\big)v\big)\,.$$

We have

$$JX(a, z) = \big(\mathrm{i}\hat{a}, \xi(a)(z)\big), \qquad JY(a, z) = \big(0, (j(a)v)(z)\big)$$

and hence

$$\begin{aligned} N_J(X, Y) &= [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y] \\ &= (0, -dj(\mathrm{i}\hat{a})v + [\xi, jv] + j dj(\hat{a})v - j[\xi, v]) \\ &= (0, -dj(\mathrm{i}\hat{a})v + j dj(\hat{a})v - (\mathcal{L}_\xi j)v)\,. \end{aligned}$$

Step 2. *Fix two vectors* $\hat{a}, \hat{b} \in \mathbb{C}^m$ *and let* $\zeta : A \to \mathrm{Vect}(\Sigma)$ *be as in Lemma B. Then then Nijenhuis tensor on the pair*

$$X(a, z) := (\hat{a}, 0), \qquad Y(a, z) := (\hat{b}, 0)$$

is

$$N_J(X, Y) = (0, \zeta).$$

Let $\xi, \eta : A \to \mathrm{Vect}(\Sigma)$ be as in Lemma B. Then

$$JX(a, z) = (\mathrm{i}\hat{a}, \xi(a)(z))\,, \qquad JY(a, z) = \Big(\mathrm{i}\hat{b}, \eta(a)(z)\Big)$$

and hence

$$\begin{aligned} N_J(X, Y) &= [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y] \\ &= \Big(0, d\xi(\mathrm{i}\hat{b}) - d\eta(\mathrm{i}\hat{a}) + [\xi, \eta] + j d\eta(\hat{a}) - j d\xi(\hat{b})\Big) \\ &= (0, \zeta). \end{aligned}$$

Step 3. *We prove the lemma.*

If J is integrable then equation (2) follows from Step 1 and equation (3) follows from Step 2. Conversely, suppose j and α satisfy (2) and (3). Then, by Step 2, the Nijenhuis tensor vanishes on every pair of horizontal vector fields. That it vanishes on every pair consisting of a horizontal and a vertical vector field follows from (2) and Step 1. That it vanishes on every pair of vertical vector fields follows from the integrability of every almost complex structure on Σ. Hence J is integrable whenever j and α satisfy (2) and (3). This proves (i).

If Σ has genus greater then 1 then there are no nonzero holomorphic vector fields on Σ for any almost complex structure. Hence it follows from Lemma B and (2) that ζ vanishes for all $\hat{a}, \hat{b} \in \mathbb{C}^m$. This proves (ii). If $\alpha = 0$ then ζ vanishes by definition for all $\hat{a}, \hat{b} \in \mathbb{C}^m$. This proves (iii) and the lemma. □

Remark. Let $\omega \in \Omega^2(\Sigma)$ be a symplectic form and

$$TA \to C^\infty(\Sigma) : (a, \hat{a}) \mapsto H_{a,\hat{a}}$$

be a smooth 1-form. We think of H as a connection on the principal bundle $A \times \mathrm{Diff}(\Sigma, \omega)$ and there is an induced connection on the associated bundle $A \times \mathcal{J}(\Sigma)$. The **covariant derivative** of a smooth map $j : A \to \mathcal{J}(\Sigma)$ is the 1-form $\nabla^H j \in \Omega^1(A, j^*T\mathcal{J}(\Sigma))$ with values in the pullback tangent bundle of $\mathcal{J}(\Sigma)$ given by

$$\nabla^H_{\hat{a}} j(a) := dj(a)\hat{a} - \mathcal{L}_{v_{a,\hat{a}}} j(a), \qquad \iota(v_{a,\hat{a}})\omega := H_{a,\hat{a}}.$$

Thus $v_{a,\hat{a}}$ is the Hamiltonian vector field of $H_{a,\hat{a}}$. The complex structure on $\mathcal{J}(\Sigma)$ induces a nonlinear Cauchy-Riemann operator $j \mapsto \bar{\partial}^H j$ which assigns to every section $j : A \to \mathcal{J}(\Sigma)$ the $(0,1)$-form $\bar{\partial}^H j \in \Omega^{0,1}(A, j^*T\mathcal{J}(\Sigma))$ with values in the pullback tangent bundle of $\mathcal{J}(\Sigma)$ given by

$$\bar{\partial}^H j(a, \hat{a}) := \frac{1}{2}\left(\nabla^H_{\hat{a}} j(a) + j(a)\nabla^H_{\mathrm{i}\hat{a}} j(a)\right)$$

Now suppose

$$\alpha(a, \hat{a}) = j(a)\left(v_{a,\hat{a}} + j(a)v_{a,\mathrm{i}\hat{a}}\right).$$

(In the case $\Sigma = S^2$ every 1-form $\alpha : TA \to \mathrm{Vect}(\Sigma)$ that satisfies (1) can be written in this form.) Then the formula (2) asserts that $\bar{\partial}^H j = 0$ and the function $\zeta : A \to \mathrm{Vect}(\Sigma)$ in (4) corresponds to the $(0,2)$-part of the curvature of the induced connection on $A \times \mathcal{J}(\Sigma)$. This point of view is motivated by the observation, due to Donaldson and Fujiki, that the action of $\mathrm{Diff}(\Sigma, \omega)$ on $\mathcal{J}(\Sigma)$ can be viewed as a Hamiltonian group action with the moment map given by the Gauss curvature [2]. Thus, in the case $\dim^{\mathbb{C}} A = 1$, the integrability equation $\bar{\partial}^H j = 0$ can be viewed as part of the symplectic vortex equations (see [1]) in an infinite dimensional setting, where the second equation combines the Gauss curvature in the fiber with the curvature of the connection form H.

References

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