Notes on the universal determinant bundle

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1 Determinant lines

1.1 (**Fredholm operators).** Let X, Y be real Banach spaces and denote their dual spaces by X^*, Y^*. A bounded linear operator $D : X \to Y$ is called **Fredholm** if it has a closed image and if its kernel and cokernel (the quotient space $Y/\mathrm{im}\, D$) are finite dimensional. Equivalently, there exists a bounded linear operator $T : Y \to X$ such that the operators $TD - \mathrm{id}_X$ and $DT - \mathrm{id}_Y$ are compact. The **Fredholm index** of a Fredholm operator $D : X \to Y$ is the integer

$$\mathrm{index}(D) := \dim(\ker\, D) - \dim(\mathrm{coker}\, D).$$

Denote by $\mathcal{L}(X, Y)$ the space of bounded linear operators from X to Y, by $\mathcal{F}(X, Y) \subset \mathcal{L}(X, Y)$ the space of Fredholm operators, and, for $k \in \mathbb{Z}$, by $\mathcal{F}_k(X, Y) \subset \mathcal{F}(X, Y)$ the space of Fredholm operators of index k. Thus $\mathcal{F}_k(X, Y)$ is an open subset of $\mathcal{L}(X, Y)$ with respect to the norm topology and is invariant under addition of compact operators. If $D \in \mathcal{F}(X, Y)$ and $T \in \mathcal{F}(Y, Z)$ are Fredholm operators then

$$\mathrm{index}(TD) = \mathrm{index}(T) + \mathrm{index}(D).$$

If $D : X \to Y$ is a bounded linear operator then D has a closed image if and only if its dual operator $D^* : Y^* \to X^*$ has a closed image and, in this case,

$$(\ker\, D)^* \cong X^*/\mathrm{im} D^*, \qquad (Y/\mathrm{im} D)^* \cong \ker\, D^*.$$

Hence $D \in \mathcal{F}_k(X, Y)$ if and only if $D^* \in \mathcal{F}_{-k}(Y^*, X^*)$.

1.2 (**Determinant lines).** The **determinant line** of a Fredholm operator $D \in \mathcal{F}(X, Y)$ is the one dimensional real vector space defined by

$$\det(D) := \Lambda^{\max}(\ker D^*) \otimes \Lambda^{\max}(\ker D).$$

If $\dim(\ker D) = k > 0$ and $\dim(\operatorname{coker} D) = \ell > 0$ then an element of $\det(D)$ can be written in the form

$$\theta := (y_\ell^* \wedge \cdots \wedge y_1^*) \otimes (x_1 \wedge \cdots \wedge x_k)$$

where $x_1, \ldots, x_k \in \ker D$ and $y_1^*, \ldots, y_\ell^* \in \ker D^*$. This element is nonzero if and only if the vectors $x_1, \ldots, x_k \in X$ form a basis of $\ker D$ and the covectors $y_1^*, \ldots, y_\ell^* \in Y^*$ form a basis of $\ker D^*$. If $\dim(\operatorname{coker} D) = 0$ then $\det(D) = \Lambda^{\max}(\ker D)$, if $\dim(\ker D) = 0$ then $\det(D) = \Lambda^{\max}(\ker D^*)$, and if $\dim(\ker D) = \dim(\operatorname{coker} D) = 0$ then $\det(D) = \mathbb{R}$.

1.3 (**Product).** The product of two bounded linear operators $D_1 : X_1 \to Y_1$ and $D_2 : X_2 \to Y_2$ is the bounded linear operator

$$D_1 \times D_2 : X_1 \times X_2 \to Y_1 \times Y_2,$$

defined by $(D_1 \times D_2)(x_1, x_2) := (D_1 x_1, D_2 x_2)$ for $x_1 \in X_1$ and $x_2 \in X_2$. If D_1 and D_2 are Fredholm operators, then so is $D_1 \times D_2$ and the Fredholm index of $D_1 \times D_2$ is the sum of the Fredholm indices of D_1 and D_2. Define the isomorphism

$$\rho_{D_1,D_2} : \det(D_1) \otimes \det(D_2) \to \det(D_1 \times D_2)$$

as follows. For $i = 1, 2$ let

$$\begin{aligned} k_i &:= \dim(\ker D_i), \qquad & x_{i,1}, \ldots, x_{i,k_i} &\in \ker D_i, \\ \ell_i &:= \dim(\ker D_i^*), \qquad & y_{i,1}^*, \ldots, y_{i,\ell_i}^* &\in \ker D_i^*. \end{aligned} \tag{1}$$

Abbreviate

$$\theta_i := (y_{i,\ell_i}^* \wedge \cdots \wedge y_{i,1}^*) \otimes (x_{i,1} \wedge \cdots \wedge x_{i,k_i}) \in \det(D_i) \tag{2}$$

and define

$$\begin{aligned} \rho_{D_1,D_2}(\theta_1 \otimes \theta_2) &:= (-1)^{\operatorname{index}(D_1)\cdot\dim(\operatorname{coker} D_2)} \\ &\quad \cdot \Big((0, y_{2,\ell_2}^*) \wedge \cdots \wedge (0, y_{2,1}^*) \wedge (y_{1,\ell_1}^*, 0) \wedge \cdots \wedge (y_{1,1}^*, 0)\Big) \\ &\quad \otimes \Big((x_{1,1}, 0) \wedge \cdots \wedge (x_{1,k_1}, 0) \wedge (0, x_{2,1}) \wedge \cdots \wedge (0, x_{2,k_2})\Big). \end{aligned} \tag{3}$$

It is obvious from the definition that ρ_{D_1,D_2} is well defined and is a vector space isomorphism.

Lemma 1.4. *For $i = 1, 2, 3$ let X_i, Y_i be Banach spaces, $D_i \in \mathcal{F}(X_i, Y_i)$ be a Fredholm operator, and $\xi_i \in \det(D_i)$. Then*

$$\rho_{D_1\times D_2, D_3}(\rho_{D_1,D_2}(\theta_1 \otimes \theta_2) \otimes \theta_3) = \rho_{D_1, D_2\times D_3}(\theta_1 \otimes \rho_{D_2\times D_3}(\theta_2 \otimes \theta_3)) \tag{4}$$

and

$$\rho_{D_2,D_1}(\theta_2 \otimes \theta_1) = (-1)^{\mathrm{index}(D_1)\cdot\mathrm{index}(D_2)} R\left((\rho_{D_1,D_2}(\theta_1 \otimes \theta_2))\right). \tag{5}$$

Here the isomorphism $R : \det(D_1 \times D_2) \to \det(D_2 \times D_1)$ is induced by the Banach space isomorphisms $X_1 \times X_2 \to X_2 \times X_1 : (x_1, x_2) \mapsto (x_2, x_1)$ and $Y_1 \times Y_2 \to Y_2 \times Y_1 : (y_1, y_2) \mapsto (y_2, y_1)$.

Proof. For $i = 1, 2, 3$ let

$$k_i, \qquad \ell_i, \qquad x_{i,1}, \dots, x_{i,k_i}, \qquad y^*_{i,1}, \dots, y^*_{i,\ell_i}$$

be as in (1) and let

$$\theta_i := (y^*_{i,\ell_i} \wedge \cdots \wedge y^*_{i,1}) \otimes (x_{i,1} \wedge \cdots \wedge x_{i,k_i}) \in \det(D_i)$$

as in (2). Then

$$\begin{aligned}
&(-1)^{\mathrm{index}(D_1)\dim(\mathrm{coker}\, D_2) + (\mathrm{index}(D_1)+\mathrm{index}(D_2))\dim(\mathrm{coker}\, D_3)} \\
&\cdot \rho_{D_1\times D_2, D_3}\Big(\rho_{D_1,D_2}(\theta_1 \otimes \theta_2) \otimes \theta_3\Big) \\
&= \Big((0,0,y^*_{3,\ell_3}) \wedge \cdots \wedge (0,0,y^*_{3,1}) \\
&\qquad \wedge (0, y^*_{2,\ell_2}, 0) \wedge \cdots \wedge (0, y^*_{2,1}, 0) \wedge (y^*_{1,\ell_1}, 0, 0) \wedge \cdots \wedge (y^*_{1,1}, 0, 0)\Big) \\
&\quad \otimes \Big((x_{1,1}, 0, 0) \wedge \cdots \wedge (x_{1,k_1}, 0, 0) \wedge (0, x_{2,1}, 0) \wedge \cdots \wedge (0, x_{2,k_2}, 0) \\
&\qquad \wedge (0, 0, x_{3,1}) \wedge \cdots \wedge (0, 0, x_{3,k_3})\Big) \\
&= (-1)^{\mathrm{index}(D_2)\dim(\mathrm{coker}\, D_3) + \mathrm{index}(D_1)(\dim(\mathrm{coker}\, D_2) + \dim(\mathrm{coker}\, D_3))} \\
&\quad \cdot \rho_{D_1\times D_2, D_3}\Big(\theta_1 \otimes \rho_{D_2,D_3}(\theta_2 \otimes \theta_3)\Big).
\end{aligned}$$

In each step we have used equation (3) twice. This proves (4). Equation (5) follows from the fact that

$$\begin{aligned}
&(-1)^{\mathrm{index}(D_2)\dim(\mathrm{coker}\, D_1)}(-1)^{\mathrm{index}(D_1)\dim(\mathrm{coker}\, D_2)}(-1)^{\mathrm{index}(D_1)\cdot\mathrm{index}(D_2)} \\
&\quad = (-1)^{\dim(\ker\, D_1)\dim(\ker\, D_2) + \dim(\mathrm{coker}\, D_1)\dim(\mathrm{coker}\, D_2)}.
\end{aligned}$$

This proves Lemma 1.4. ∎

2 The isomorphisms

Let X and Y be Banach spaces and $D : X \to Y$ be a Fredholm operator. Let N be a positive integer and $\Phi : \mathbb{R}^N \to Y$ be a linear map. Define the operator $D \oplus \Phi : X \times \mathbb{R}^N \to Y$, by

$$(D \oplus \Phi)(x, \zeta) := Dx + \Phi\zeta.$$

Then $D \oplus \Phi$ is a Fredholm operator and $\mathrm{index}(D \oplus \Phi) = \mathrm{index}(D) + N$.

Theorem 2.1. *Let Y be a Banach space. There exists a unique family of vector space isomorphisms*

$$\iota_{D,\Phi} : \det(D) \to \det(D \oplus \Phi),$$

one for each pair (D, Φ) (consisting of a Fredholm operator $D \in \mathcal{F}(X, Y)$, defined on a Banach space X, and a linear map $\Phi \in \mathcal{L}(\mathbb{R}^N, Y)$, defined on $\mathbb{R}^N$ for some integer $N > 0$), satisfying the following axioms.

(Determinant) *Let $D \in \mathcal{F}(X, Y)$, $\Phi \in \mathcal{L}(\mathbb{R}^N, Y)$, and $g \in \mathrm{GL}(\mathbb{R}^N)$. Then the following diagram commutes*

$$\begin{array}{ccc} \det(D) & \xrightarrow{\det(g)} & \det(D) \\ \downarrow{\scriptstyle \iota_{D,\Phi g}} & & \downarrow{\scriptstyle \iota_{D,\Phi}} \\ \det(D \oplus \Phi g) & \xrightarrow{T_g} & \det(D \oplus \Phi) \end{array} \quad . \tag{6}$$

Here T_g is induced by $\mathrm{id}_X \times g : X \times \mathbb{R}^N \to X \times \mathbb{R}^N$ and $\mathrm{id}_Y : Y \to Y$.

(Normalization) *Let $D \in \mathcal{F}(X, Y)$ and $\Phi \in \mathcal{L}(\mathbb{R}, Y)$ and $y := \Phi 1 \in Y$. Let $\theta = (y_\ell^* \wedge \cdots \wedge y_1^*) \otimes (x_1 \wedge \cdots \wedge x_k) \in \det(D)$, where $x_1, \ldots, x_k \in \ker D$ and $y_1^*, \ldots, y_\ell^* \in \ker D^*$. Then the following holds.*

(a) *If $\xi \in X$ and $D\xi + y = 0$ then*

$$\iota_{D,\Phi}(\theta) = \big(y_\ell^* \wedge \cdots \wedge y_1^*\big) \otimes \big((x_1, 0) \wedge \cdots \wedge (x_k, 0) \wedge (\xi, 1)\big). \tag{7}$$

(b) *If $\langle y_1^*, y\rangle = 1$ and $\langle y_i^*, y\rangle = 0$ for $i \geq 2$ then*

$$\iota_{D,\Phi}(\theta) = (-1)^k \big(y_\ell^* \wedge \cdots \wedge y_2^*\big) \otimes \big((x_1, 0) \wedge \cdots \wedge (x_k, 0)\big). \tag{8}$$

(Stabilization) *Let $D \in \mathcal{F}(X, Y)$ and $\Phi_i \in \mathcal{L}(\mathbb{R}^{N_i}, Y)$ for $i = 1, 2$. Define $\Phi_1 \oplus \Phi_2 : \mathbb{R}^{N_1+N_2} = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \to Y$ by $(\Phi_1 \oplus \Phi_2)(\zeta_1, \zeta_2) := \Phi_1\zeta_1 + \Phi_2\zeta_2$ for $\zeta_1 \in \mathbb{R}^{N_1}$ and $\zeta_2 \in \mathbb{R}^{N_2}$. Then $D \oplus (\Phi_1 \oplus \Phi_2) = (D \oplus \Phi_1) \oplus \Phi_2$ and*

$$\iota_{D,\Phi_1 \oplus \Phi_2} = \iota_{D \oplus \Phi_1, \Phi_2} \circ \iota_{D,\Phi_1}. \tag{9}$$

Proof. That the isomorphisms $\iota_{D,\Phi}$, if they exist, are uniquely determined by the (Normalization) and (Stabilization) axioms is obvious. We prove existence in five steps. The first two steps explain the construction of the isomorphisms $\iota_{D,\Phi}$. The number m in equation (10) below is the decrease in the dimension of the cokernel, and $N-m$ is the increase in the dimension of the kernel, one encounters in replacing D by $D \oplus \Phi$.

Step 1. *Let $D \in \mathcal{F}(X,Y)$ and $\Phi \in \mathcal{L}(\mathbb{R}^N, Y)$. Then*

$$m := N - \dim\left(\Phi^{-1}\mathrm{im}\, D\right) = \dim(\ker\, D^*) - \dim(\ker\, D^* \cap \ker\, \Phi^*) \qquad (10)$$

and there is a basis $\zeta_1, \dots, \zeta_N \in \mathbb{R}^N$ and vectors $\xi_{m+1}, \dots, \xi_N \in X$ such that

$$D\xi_j + \Phi\zeta_j = 0, \qquad j = m+1, \dots, N. \qquad (11)$$

First, $\dim(\Phi^{-1}\mathrm{im}\, D) = \dim(\ker \Phi) + \dim(\mathrm{im}\, D \cap \mathrm{im}\, \Phi)$. Subtracting this from N gives $m = \dim(\mathrm{im}\,\Phi) - \dim(\mathrm{im}\, D \cap \mathrm{im}\,\Phi) = \dim\left(\frac{\mathrm{im}\,\Phi}{\mathrm{im}\, D \cap \mathrm{im}\,\Phi}\right)$ and so

$$m = \dim\left(\frac{\mathrm{im}\, D + \mathrm{im}\,\Phi}{\mathrm{im}\, D}\right) = \dim\left(\frac{Y}{\mathrm{im}\, D}\right) - \dim\left(\frac{Y}{\mathrm{im}\, D + \mathrm{im}\,\Phi}\right). \qquad (12)$$

This implies (10). Now choose a basis $\zeta_1, \dots, \zeta_N$ of $\mathbb{R}^N$ such that the vectors $\zeta_{m+1}, \dots, \zeta_N$ form a basis of the subspace $\Phi^{-1}(\mathrm{im}\, D)$. Then, for $j > m$, $\Phi\zeta_j \in \mathrm{im}\, D$ and so there is a $\xi_j \in X$ such that $D\xi_j + \Phi\zeta_j = 0$.

Step 2. *Let $D \in \mathcal{F}(X,Y)$ and $\Phi \in \mathcal{L}(\mathbb{R}^N, Y)$ and denote*

$$k := \dim(\ker\, D), \quad \ell := \dim(\mathrm{coker}\, D), \quad m := N - \dim\left(\Phi^{-1}\mathrm{im}\, D\right) \le \ell.$$

Then there is a unique isomorphism $\iota_{D,\Phi} : \det(D) \to \det(D \oplus \Phi)$ satisfying the following condition. Let ζ_j, ξ_j be as in Step 1 and let $\theta \in \det(D)$. Choose $x_1, \dots, x_k \in \ker\, D$ and $y_1^*, \dots, y_\ell^* \in \ker\, D^*$ such that

$$\theta = \left(y_\ell^* \wedge \cdots \wedge y_1^*\right) \otimes \left(x_1 \wedge \cdots \wedge x_k\right), \qquad \Phi^* y_j^* = 0 \text{ for } j > m. \qquad (13)$$

Then

$$\begin{aligned} \iota_{D,\Phi}(\theta) = (-1)^{km} \frac{\det\left(\langle y_j^*, \Phi\zeta_{j'}\rangle_{j,j'=1,\dots,m}\right)}{\det(\zeta_1, \dots, \zeta_N)} \cdot \left(y_\ell^* \wedge \cdots \wedge y_{m+1}^*\right) \\ \otimes \left((x_1, 0) \wedge \cdots \wedge (x_k, 0) \wedge (\xi_{m+1}, \zeta_{m+1}) \wedge \cdots \wedge (\xi_N, \zeta_N)\right). \end{aligned} \qquad (14)$$

If $k = 0$, $\ell = m$, or $m = N$ then the relevant empty wedge products in (13) and (14) are understood as the real number one.

Choose vectors ξ_i and ζ_i as in Step 1 and fix an element $\theta \in \det(D)$; if D is bijective choose $\theta = 1 \in \mathbb{R} = \det(D)$. Then there exist elements $x_1, \dots, x_k \in \ker D$ and $y_1, \dots, y_\ell \in \ker D^*$ such that

$$\theta = \left(y_\ell^* \wedge \cdots \wedge y_1^*\right) \otimes \left(x_1 \wedge \cdots \wedge x_k\right).$$

Here the first, respectively the second, wedge product is understood as the real number one whenever $k = 0$, respectively $\ell = 0$. If $\ell > m$ then, by (10),

$$\dim\left(\ker D^* \cap \ker \Phi^*\right) = \ell - m > 0$$

and hence the y_j^* can be chosen such that $\Phi^* y_j^* = 0$ for $j = m+1, \dots, \ell$. With this understood, we must prove that the right hand side of equation (14) is independent of the choice of the x_j and y_j^*.

We prove first that $\theta = 0$ if and only if the right hand side of (14) vanishes. Indeed, $\theta \neq 0$ if and only if the x_j and the y_j^* are linearly independent. Moreover, the right hand side of (14) is nonzero if and only if

(a) the vectors $x_1, \dots, x_k$ are linearly independent,

(b) the vectors $y_{m+1}^*, \dots, y_\ell^*$ are linearly independent, and

(c) the vectors $\Phi^* y_1^*, \dots, \Phi^* y_m^*$ are linearly independent.

Since $\Phi^* y_j^* = 0$ for $j > m$, conditions (b) and (c) hold if and only if the vectors $y_1^*, \dots, y_\ell^*$ are linearly independent. Hence (a), (b), and (c) hold if and only if $\theta \neq 0$. (Here the empty set of vectors is linearly independent by definition, the empty wedge product is equal to one, and the determinant of the empty matrix is equal to one.) This shows that $\theta = 0$ if and only if the right hand side of (14) vanishes.

Next we prove that the right hand side of (14) is independent of the choice of the x_j and y_j^*. By what we have just shown, we may assume $\theta \neq 0$. Then the vectors $x_1, \dots, x_k$ form a basis of $\ker D$, the vectors $y_{m+1}^*, \dots, y_\ell^*$ form a basis of $\ker D^* \cap \ker \Phi^*$, and the vectors $y_1^*, \dots, y_\ell^*$ form a basis of $\ker D^*$. (One or more of these bases may be empty.) Replacing x_j by $\lambda_j x_j$ and y_j^* by $\mu_j y_j^*$, where $\lambda_1 \cdots \lambda_k \mu_1 \cdots \mu_\ell = 1$, leaves the right hand side of (14) unchanged. Moreover, replacing x_j by $x_j + x_{j'}$ for $j' \neq j$, or y_j by $y_j + y_j'$ for $j' \neq j \leq m$, or for $j' \neq j$ with $j, j' > m$, also leaves the right hand side of (14) unchanged. So does any odd permutation of the x_j, or the $y_j, j > m$, or the $y_j, j \leq m$, followed by a sign change in one of the basis vectors. Now any two bases x_j of $\ker D$ and y_j^* of $\ker D^*$ with $\Phi_j^* y_j^* = 0$ for $j > m$ are related by a finite sequence of such elementary operations. Hence the right hand side of (14) is independent of the choice of the x_j and y_j^*, as claimed.

This shows that the formula (14) defines a vector space isomorphism $\iota_{D,\Phi} : \det(D) \to \det(D \oplus \Phi)$ for every tuple $\xi_{m+1}, \dots, \xi_N, \zeta_1, \dots, \zeta_N$ as in Step 1. It remains to prove that this isomorphism is independent of the choice of the ζ_j and ξ_j. If $k = 0$ then D is injective and hence ξ_j is uniquely determined by ζ_j for $j > m$. If $k > 0$ then ξ_j is uniquely determined by ζ_j up to addition by an element of $\ker D$. Since the vectors $x_1, \dots, x_k$ form a basis of $\ker D$ (when $\theta \neq 0$), this shows that the right hand side of (14) is independent of the choice of the ξ_j. Now the ζ_j must be chosen such that $\Phi\zeta_j \in \operatorname{im} D$ for $j > m$. Any two such bases of $\mathbb{R}^N$ are related by finitely many of the following elementary operations.

- Multiply one of the ζ_j by a nonzero real number.
- Permute the ζ_j for $j > m$.
- Permute the ζ_j for $j \leq m$.
- Replace ζ_j by $\zeta_j + \zeta_{j'}$, where $j' \neq j$ and $j, j' > m$.
- Replace ζ_j by $\zeta_j + \zeta_{j'}$, where $j' \neq j$ and $j \leq m$.

None of these operations change the right hand side of equation (14) and this proves Step 2.

Step 3. *The isomorphisms $\iota_{D,\Phi}$ in Step 2 satisfy the (Determinant) axiom.*

Let $D \in \mathcal{F}(X, Y)$ and $\Phi \in \mathcal{L}(\mathbb{R}^N, Y)$ and choose ξ_i, ζ_i as in Step 1. Let $\theta \in \det(D)$ and choose $x_1, \dots, x_k \in \ker D$ and $y_1^*, \dots, y_\ell^* \in \ker D^*$ such that (13) holds. Let $g \in \mathrm{GL}(\mathbb{R}^N)$. Then it follows from equation (14), with Φ replaced by Φg and ζ_j replace by $g^{-1}\zeta_j$, that

$$\begin{aligned}
\iota_{D,\Phi g}(\theta) &= (-1)^{km} \frac{\det\big(\langle y_j^*, \Phi\zeta_{j'}\rangle_{j,j'=1,\dots,m}\big)}{\det(g^{-1}\zeta_1, \dots, g^{-1}\zeta_N)} \cdot \big(y_\ell^* \wedge \cdots \wedge y_{m+1}^*\big) \\
&\quad \otimes \big((x_1, 0) \wedge \cdots \wedge (x_k, 0) \wedge (\xi_{m+1}, g^{-1}\zeta_{m+1}) \wedge \cdots \wedge (\xi_N, g^{-1}\zeta_N)\big) \\
&= (-1)^{km} \det(g) \frac{\det\big(\langle y_j^*, \Phi\zeta_{j'}\rangle_{j,j'=1,\dots,m}\big)}{\det(\zeta_1, \dots, \zeta_N)} \cdot \big(y_\ell^* \wedge \cdots \wedge y_{m+1}^*\big) \\
&\quad \otimes \big((x_1, 0) \wedge \cdots \wedge (x_k, 0) \wedge (\xi_{m+1}, g^{-1}\zeta_{m+1}) \wedge \cdots \wedge (\xi_N, g^{-1}\zeta_N)\big) \\
&= \det(g) \cdot T_g^{-1}\big(\iota_{D,\Phi}(\theta)\big).
\end{aligned}$$

This proves Step 3.

Step 4. *The isomorphisms $\iota_{D,\Phi}$ in Step 2 satisfy the (Normalization) axiom.*

Equation (7) follows from (14) with $m = 0$, $N = 1$ by taking $\zeta_1 := 1$ and $\xi_1 := \xi$. Equation (8) follows from (14) with $m = N = 1$ by taking $\zeta_1 := 1$. This proves Step 4.

Step 5. *The isomorphisms $\iota_{D,\Phi}$ in Step 2 satisfy the (Stabilization) axiom.*

Assume first that $N_2 = 1$. Thus let $\Phi : \mathbb{R}^N \to Y$ and $\Psi : \mathbb{R} \to Y$ be linear maps and choose ζ_i and ξ_i as in Step 1. Define

$$y_{N+1} := \Psi 1 \in Y, \qquad \widetilde{\Phi} := \Phi \oplus \Psi : \mathbb{R}^{N+1} \to Y, \qquad \widetilde{\Phi}\widetilde{\zeta} := \Phi\zeta + \zeta_{N+1} y_{N+1}$$

for $\widetilde{\zeta} = (\zeta, \zeta_{N+1}) \in \mathbb{R}^N \times \mathbb{R} = \mathbb{R}^{N+1}$. There are two cases to consider.

Case 1. $y_{N+1} \in \operatorname{im} D + \operatorname{im} \Phi$.

In this case $\ker D^* \cap \ker \widetilde{\Phi}^* = \ker D^* \cap \ker \Phi^*$ and

$$m := N - \dim\bigl(\Phi^{-1}\operatorname{im} D\bigr) = N + 1 - \dim\bigl(\widetilde{\Phi}^{-1}\operatorname{im} D\bigr).$$

Choose $\xi_{N+1} \in X$ and $\zeta \in \mathbb{R}^N$ such that $D\xi_{N+1} + \Phi\zeta + y_{N+1} = 0$ and define

$$\widetilde{\zeta}_j := (\zeta_j, 0) \in \mathbb{R}^{N+1} \quad \text{for } 1 \le j \le N, \qquad \widetilde{\zeta}_{N+1} := (\zeta, 1) \in \mathbb{R}^{N+1}.$$

Then $D\xi_j + \widetilde{\Phi}\widetilde{\zeta}_j = 0$ for $j = m+1, \dots, N+1$. Fix a nonzero element $\theta \in \det(D)$ and choose bases $x_1, \dots, x_k$ of $\ker D$ and $y_1^*, \dots, y_\ell^*$ of $\ker D^*$ such that (13) holds for Φ, and hence also for $\widetilde{\Phi}$. Since

$$\det(\widetilde{\zeta}_1, \dots, \widetilde{\zeta}_{N+1}) = \det(\zeta_1, \dots, \zeta_N)$$

and

$$\widetilde{\Phi}\widetilde{\zeta}_j = \Phi\zeta_j \qquad \text{for } j = 1, \dots, m,$$

it follows from equation (14) that

$$\begin{aligned}
\iota_{D,\widetilde{\Phi}}(\theta) &= (-1)^{km} \frac{\det\bigl(\langle y_j^*, \widetilde{\Phi}\widetilde{\zeta}_{j'}\rangle_{j,j'=1,\dots,m}\bigr)}{\det(\widetilde{\zeta}_1, \dots, \widetilde{\zeta}_{N+1})} \cdot \bigl(y_\ell^* \wedge \cdots \wedge y_{m+1}^*\bigr) \\
&\qquad \otimes \bigl((x_1, \widetilde{0}) \wedge \cdots \wedge (x_k, \widetilde{0}) \wedge (\xi_{m+1}, \widetilde{\zeta}_{m+1}) \wedge \cdots \wedge (\xi_{N+1}, \widetilde{\zeta}_{N+1})\bigr) \\
&= (-1)^{km} \frac{\det\bigl(\langle y_j^*, \Phi\zeta_{j'}\rangle_{j,j'=1,\dots,m}\bigr)}{\det(\zeta_1, \dots, \zeta_N)} \cdot \bigl(y_\ell^* \wedge \cdots \wedge y_{m+1}^*\bigr) \\
&\qquad \otimes \bigl((x_1, 0, 0) \wedge \cdots \wedge (x_k, 0, 0) \wedge (\xi_{m+1}, \zeta_{m+1}, 0) \wedge \cdots \wedge (\xi_N, \zeta_N, 0) \\
&\qquad\qquad \wedge (\xi_{N+1}, \zeta, 1)\bigr) \\
&= \iota_{D\oplus\Phi,\Psi}\bigl(\iota_{D,\Phi}(\theta)\bigr).
\end{aligned}$$

Here $\widetilde{0}$ is the zero vector in $\mathbb{R}^{N+1}$. This proves the assertion in Case 1.

Case 2. $y_{N+1} \notin \operatorname{im} D + \operatorname{im} \Phi$.

Let $m := N - \dim\bigl(\Phi^{-1}\operatorname{im} D\bigr)$ as before. Then

$$\widetilde{\Phi}^{-1}\operatorname{im} D = \Phi^{-1}\operatorname{im} D, \qquad m+1 = N+1-\dim\bigl(\widetilde{\Phi}^{-1}\operatorname{im} D\bigr).$$

Define $\widetilde{\zeta}_j \in \mathbb{R}^{N+1}$ by

$$\widetilde{\zeta}_j := \begin{cases} (\zeta_j, 0), & \text{for } j = 1, \ldots, m, \\ (0, \ldots, 0, 1), & \text{for } j = m+1, \\ (\zeta_{j-1}, 0), & \text{for } j = m+2, \ldots, N+1. \end{cases}$$

Fix a nonzero element $\theta \in \det(D)$ and choose bases $x_1, \ldots, x_k$ of $\ker D$ and $y_1^*, \ldots, y_\ell^*$ of $\ker D^*$ such that (13) holds for Φ and

$$\langle y_j^*, y_{N+1} \rangle = \begin{cases} 1, & \text{for } j = m+1, \\ 0, & \text{for } j = m+2, \ldots, \ell. \end{cases}$$

Then $\widetilde{\Phi}^* y_j^* = 0$ for $j = m+2, \ldots, \ell$ and

$$\det\bigl(\langle y_j^*, \widetilde{\Phi}\widetilde{\zeta}_{j'} \rangle_{j,j'=1,\ldots,m+1}\bigr) = \det\bigl(\langle y_j^*, \Phi\zeta_{j'} \rangle_{j,j'=1,\ldots,m}\bigr).$$

Hence it follows from equation (14) that

$$\begin{aligned}
\iota_{D,\widetilde{\Phi}}(\theta) &= (-1)^{k(m+1)} \frac{\det\bigl(\langle y_j^*, \widetilde{\Phi}\widetilde{\zeta}_{j'} \rangle_{j,j'=1,\ldots,m+1}\bigr)}{\det(\widetilde{\zeta}_1, \ldots, \widetilde{\zeta}_{N+1})} \cdot \bigl(y_\ell^* \wedge \cdots \wedge y_{m+2}^*\bigr) \\
&\qquad \otimes \bigl((x_1, \widetilde{0}) \wedge \cdots \wedge (x_k, \widetilde{0}) \wedge (\xi_{m+1}, \widetilde{\zeta}_{m+2}) \wedge \cdots \wedge (\xi_N, \widetilde{\zeta}_{N+1})\bigr) \\
&= (-1)^{k+N-m}(-1)^{km} \frac{\det\bigl(\langle y_j^*, \Phi\zeta_{j'} \rangle_{j,j'=1,\ldots,m}\bigr)}{\det(\zeta_1, \ldots, \zeta_N)} \cdot \bigl(y_\ell^* \wedge \cdots \wedge y_{m+2}^*\bigr) \\
&\qquad \otimes \bigl((x_1, 0, 0) \wedge \cdots \wedge (x_k, 0, 0) \wedge (\xi_{m+1}, \zeta_{m+1}, 0) \wedge \cdots \wedge (\xi_N, \zeta_N, 0)\bigr) \\
&= \iota_{D\oplus\Phi,\Psi}\bigl(\iota_{D,\Phi}(\theta)\bigr).
\end{aligned}$$

Here the second step follows from the fact that

$$\det(\widetilde{\zeta}_1, \ldots, \widetilde{\zeta}_{N+1}) = (-1)^{N-m} \det(\zeta_1, \ldots, \zeta_N)$$

and the last step follows from the fact that

$$k + N - m = \dim(\ker(D \oplus \Phi)).$$

This proves the assertion in Case 2.

Thus we have proved the (Stabilization) axiom for $N_2 = 1$. The general case follows from the case $N_2 = 1$ by induction. This proves Theorem 2.1. □

3 Compatibility with products

Theorem 3.1. *Let $D_1 : X_1 \to Y_1$ and $D_2 : X_2 \to Y_2$ be Fredholm operators and $\Phi_1 : \mathbb{R}^{N_1} \to Y_1$ and $\Phi_2 : \mathbb{R}^{N_2} \to Y_2$ be linear maps. Then*

$$\begin{aligned}&\rho_{D_1\oplus\Phi_1,D_2\oplus\Phi_2}\big(\iota_{D_1,\Phi_1}(\theta_1)\otimes\iota_{D_2,\Phi_2}(\theta_2)\big)\\ &= (-1)^{N_1\mathrm{index}(D_2)}\cdot R\big(\iota_{D_1\times D_2,\Phi_1\times\Phi_2}\big(\rho_{D_1,D_2}(\theta_1\otimes\theta_2)\big)\big)\end{aligned} \tag{15}$$

for $\theta_1 \in \det(D_1)$ and $\theta_2 \in \det(D_2)$. Here ρ_{D_1,D_2} and $\rho_{D_1\oplus\Phi_1,D_2\oplus\Phi_2}$ are as in 1.3, the isomorphisms ι_{D_i,Φ_i} and $\iota_{D_1\times D_2,\Phi_1\times\Phi_2}$ are those of Theorem 2.1, and the isomorphism

$$R : \det\big((D_1\times D_2)\oplus(\Phi_1\times\Phi_2)\big) \to \det\big((D_1\oplus\Phi_1)\times(D_2\oplus\Phi_2)\big) \tag{16}$$

is induced by the identity on $Y_1 \times Y_2$ and the isomorphism

$$\left\{\begin{array}{ccc}(X_1\times X_2)\times(\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}) & \to & (X_1\times\mathbb{R}^{N_1})\times(X_2\times\mathbb{R}^{N_2})\\ ((x_1,x_2),(\zeta_1,\zeta_2)) & \mapsto & (x_1,\zeta_1;x_2,\zeta_2).\end{array}\right.$$

Proof. For $i = 1, 2$ define

$$k_i := \dim(\ker\, D_i),\quad \ell_i := \dim(\mathrm{coker}\, D_i),\quad m_i := N_i - \dim(\Phi_i^{-1}\mathrm{im}\, D_i),$$

choose $x_{i,1}, \dots, x_{i,k_i} \in \ker D_i$, $y^*_{i,1}, \dots, y^*_{i,\ell_i} \in \ker D_i^*$, $\xi_{i,m_i+1}, \dots, \xi_{i,N_i} \in X_i$, and a basis $\zeta_{i,1}, \dots, \zeta_{i,N_i} \in \mathbb{R}^{N_i}$, such that

$$D_i\xi_{i,j} + \Phi_i\zeta_{i,j} = 0, \qquad \Phi_i^* y^*_{i,j} = 0 \qquad \text{for } j = m_i+1, \dots, N_i,$$

and denote

$$\theta_i := \big(y^*_{i,\ell_i}\wedge\cdots\wedge y^*_{i,1}\big)\otimes\big(x_{i,1}\wedge\cdots\wedge x_{i,k_i}\big) \in \det(D_i).$$

Then, by equation (3) in 1.3,

$$\begin{aligned}&\rho_{D_1,D_2}(\theta_1\otimes\theta_2)\\ &= (-1)^{(k_1-\ell_1)\ell_2}\Big((0,y^*_{2,\ell_2})\wedge\cdots\wedge(0,y^*_{2,1})\wedge(y^*_{1,\ell_1},0)\wedge\cdots\wedge(y^*_{1,1},0)\Big)\\ &\quad\otimes\Big((x_{1,1},0)\wedge\cdots\wedge(x_{1,k_1},0)\wedge(0,x_{2,1})\wedge\cdots\wedge(0,x_{2,k_2})\Big)\end{aligned} \tag{17}$$

and for $i = 1, 2$, by equation (14) in Step 2 in the proof of Theorem 2.1,

$$\begin{aligned}\iota_{D_i,\Phi_i}(\theta_i) &= (-1)^{k_im_i}\frac{\det\big(\langle y^*_{i,j},\Phi_i\zeta_{i,j'}\rangle_{j,j'=1,\dots,m_i}\big)}{\det(\zeta_{i,1},\dots,\zeta_{i,N_i})}\cdot\big(y^*_{i,\ell_i}\wedge\cdots\wedge y^*_{i,m_i+1}\big)\\ &\otimes\big((x_{i,1},0)\wedge\cdots\wedge(x_{i,k_i},0)\wedge(\xi_{i,m_i+1},\zeta_{i,m_i+1})\wedge\cdots\wedge(\xi_{i,N_i},\zeta_{i,N_i})\big).\end{aligned} \tag{18}$$

Abbreviate

$$\widehat{X}_i := X_i \times \mathbb{R}^{N_i}, \qquad \widehat{D}_i := D_i \oplus \Phi_i : \widehat{X}_i \to Y_i,$$

$$\widehat{k}_i := \dim\big(\ker \widehat{D}_i\big) = k_i + N_i - m_i, \qquad \widehat{\ell}_i := \dim\big(\operatorname{coker} \widehat{D}_i\big) = \ell_i - m_i.$$

Then, by equation (18),

$$\begin{aligned}\widehat{\theta}_i &:= \iota_{D_i,\Phi_i}(\theta_i) \in \det(\widehat{D}_i) \\ &= (-1)^{k_i m_i} \frac{\det\big(\langle y^*_{i,j}, \Phi_i \zeta_{i,j'}\rangle_{j,j'=1,\dots,m_i}\big)}{\det(\zeta_{i,1},\dots,\zeta_{i,N_i})} \\ &\quad \cdot \big(\widehat{y}^*_{i,\widehat{\ell}_i} \wedge \cdots \wedge \widehat{y}^*_{i,1}\big) \otimes \big(\widehat{x}_{i,1} \wedge \cdots \wedge \widehat{x}_{i,\widehat{k}_i}\big),\end{aligned}$$

where $\widehat{x}_{i,j} = (x_{1,j}, 0)$ for $j = 1,\dots,k_i$ and $\widehat{x}_{i,k_i+j} = (\xi_{i,m_i+j}, \zeta_{i,m_i+j})$ for $j = 1,\dots,N_i - m_i$, and $\widetilde{y}^*_{i,j} = y^*_{i,m_i+j}$ for $j = 1,\dots,\ell_i - m_i$. Hence it follows from (17) that

$$\begin{aligned}&\rho_{\widehat{D}_1,\widehat{D}_2}\left(\widehat{\theta}_1 \otimes \widehat{\theta}_2\right) \\ &= (-1)^{(\widehat{k}_1-\widehat{\ell}_1)\widehat{\ell}_2}(-1)^{k_1 m_1}(-1)^{k_2 m_2} \\ &\quad \cdot \frac{\det\big(\langle y^*_{1,j}, \Phi_1 \zeta_{1,j'}\rangle_{j,j'=1,\dots,m_1}\big)}{\det(\zeta_{1,1},\dots,\zeta_{1,N_1})} \cdot \frac{\det\big(\langle y^*_{2,j}, \Phi_2 \zeta_{2,j'}\rangle_{j,j'=1,\dots,m_2}\big)}{\det(\zeta_{2,1},\dots,\zeta_{2,N_2})} \\ &\quad \cdot \Big((0, \widehat{y}^*_{2,\widehat{\ell}_2}) \wedge \cdots \wedge (0, \widehat{y}^*_{2,1}) \wedge (\widehat{y}^*_{1,\widehat{\ell}_1}, 0) \wedge \cdots \wedge (\widehat{y}^*_{1,1}, 0)\Big) \\ &\quad \otimes \Big((\widehat{x}_{1,1}; 0) \wedge \cdots \wedge (\widehat{x}_{1,\widehat{k}_1}; 0) \wedge (0; \widehat{x}_{2,1}) \wedge \cdots \wedge (0; \widehat{x}_{2,\widehat{k}_2})\Big) \\ &= (-1)^{(k_1-\ell_1+N_1)(\ell_2-m_2)}(-1)^{k_1 m_1}(-1)^{k_2 m_2} \\ &\quad \cdot \frac{\det\big(\langle y^*_{1,j}, \Phi_1 \zeta_{1,j'}\rangle_{j,j'=1,\dots,m_1}\big)}{\det(\zeta_{1,1},\dots,\zeta_{1,N_1})} \cdot \frac{\det\big(\langle y^*_{2,j}, \Phi_2 \zeta_{2,j'}\rangle_{j,j'=1,\dots,m_2}\big)}{\det(\zeta_{2,1},\dots,\zeta_{2,N_2})} \\ &\quad \cdot \big((0, y^*_{2,\ell_2}) \wedge \cdots \wedge (0, y^*_{2,m_2+1}) \wedge (y^*_{1,\ell_1}, 0) \wedge \cdots \wedge (y^*_{1,m_1+1}, 0)\big) \\ &\quad \otimes \big((x_{1,1}, 0; 0, 0) \wedge \cdots \wedge (x_{1,k_1}, 0; 0, 0) \\ &\qquad \wedge (\xi_{1,m_1+1}, \zeta_{1,m_1+1}; 0, 0) \wedge \cdots \wedge (\xi_{1,N_1}, \zeta_{1,N_1}; 0, 0) \\ &\qquad \wedge (0, 0; x_{2,1}, 0) \wedge \cdots \wedge (0, 0; x_{2,k_2}, 0) \\ &\qquad \wedge (0, 0; \xi_{2,m_2+1}, \zeta_{2,m_2+1}) \wedge \cdots \wedge (0, 0; \xi_{2,N_2}, \zeta_{2,N_2})\big).\end{aligned} \tag{19}$$

This is an explicit expression for the left hand side of equation (15).

To compute the right hand side of equation (15), abbreviate

$$\widetilde{k} := k_1 + k_2, \quad \widetilde{\ell} := \ell_1 + \ell_2, \quad \widetilde{m}_1 := m_1 + m_2, \quad \widetilde{N} := N_1 + N_2,$$
$$\widetilde{X} := X_1 \times X_2, \quad \widetilde{Y} := Y_1 \times Y_2, \quad \widetilde{D} := D_1 \times D_2, \quad \widetilde{\Phi} := \Phi_1 \times \Phi_2.$$

and

$$\widetilde{x}_j := \begin{cases} (x_{1,j}, 0), & \text{for } j = 1, \dots, k_1, \\ (0, x_{2,j-k_1}), & \text{for } j = k_1 + 1, \dots, k_1 + k_2, \end{cases}$$
$$\widetilde{y}_j^* := \begin{cases} (y_{1,j}^*, 0), & \text{for } j = 1, \dots, m_1, \\ (0, y_{2,j-m_1}^*), & \text{for } j = m_1 + 1, \dots, m_1 + m_2, \\ (y_{1,j-m_2}^*, 0), & \text{for } j = m_1 + m_2 + 1, \dots, \ell_1 + m_2, \\ (0, y_{2,j-\ell_1}^*), & \text{for } j = \ell_1 + m_2 + 1, \dots, \ell_1 + \ell_2, \end{cases}$$
$$\widetilde{\zeta}_j := \begin{cases} (\zeta_{1,j}, 0), & \text{for } j = 1, \dots, m_1, \\ (0, \zeta_{2,j-m_1}), & \text{for } j = m_1 + 1, \dots, m_1 + m_2, \\ (\zeta_{1,j-m_2}, 0), & \text{for } j = m_1 + m_2 + 1, \dots, N_1 + m_2, \\ (0, \zeta_{2,j-N_1}), & \text{for } j = N_1 + m_2 + 1, \dots, N_1 + N_2, \end{cases}$$
$$\widetilde{\xi}_j := \begin{cases} (\widetilde{\xi}_{1,j-m_2}, 0), & \text{for } j = m_1 + m_2 + 1, \dots, N_1 + m_2, \\ (0, \widetilde{\xi}_{2,j-N_1}), & \text{for } j = N_1 + m_2 + 1, \dots, N_1 + N_2, \end{cases}$$
$$\begin{aligned} \widetilde{\theta} &:= \rho_{D_1,D_2}(\theta_1 \otimes \theta_2) \\ &= (-1)^{(k_1-\ell_1)\ell_2}(-1)^{(\ell_1-m_1)m_2}\big(\widetilde{y}_{\widetilde{\ell}}^* \wedge \cdots \wedge \widetilde{y}_1^*\big) \otimes \big(\widetilde{x}_1 \wedge \cdots \wedge \widetilde{x}_{\widetilde{k}}\big). \end{aligned}$$

Here the last equation follows from (17). Since $\widetilde{D}\widetilde{\zeta}_j + \widetilde{\Phi}\widetilde{\xi}_j = 0$ and $\widetilde{\Phi}^*\widetilde{y}_i^* = 0$ for $j > \widetilde{m}$, it follows from (18) that

$$\begin{aligned} \iota_{\widetilde{D},\widetilde{\Phi}}(\widetilde{\theta}) &= (-1)^{(k_1-\ell_1)\ell_2}(-1)^{(\ell_1-m_1)m_2}(-1)^{\widetilde{k}\widetilde{m}} \cdot \frac{\det\big(\langle \widetilde{y}_j^*, \widetilde{\Phi}\widetilde{\zeta}_{j'}\rangle_{j,j'=1,\dots,\widetilde{m}}\big)}{\det(\widetilde{\zeta}_1, \dots, \widetilde{\zeta}_{\widetilde{N}})} \\ &\cdot \big(\widetilde{y}_{\widetilde{\ell}}^* \wedge \cdots \wedge \widetilde{y}_{\widetilde{m}+1}^*\big) \otimes \big((\widetilde{x}_1; 0) \wedge \cdots \wedge (\widetilde{x}_{\widetilde{k}}; 0) \wedge (\widetilde{\xi}_{\widetilde{m}+1}; \widetilde{\zeta}_{\widetilde{m}+1}) \wedge \cdots \wedge (\widetilde{\xi}_{\widetilde{N}}, \widetilde{\zeta}_{\widetilde{N}})\big) \\ &= (-1)^{(k_1-\ell_1)\ell_2}(-1)^{(\ell_1-m_1)m_2}(-1)^{(k_1+k_2)(m_1+m_2)}(-1)^{(N_1-m_1)m_2} \\ &\quad \cdot \frac{\det\big(\langle y_{1,j}^*, \Phi_1\zeta_{1,j'}\rangle_{j,j'=1,\dots,m_1}\big)}{\det(\zeta_{1,1}, \dots, \zeta_{1,N_1})} \cdot \frac{\det\big(\langle y_{2,j}^*, \Phi_2\zeta_{2,j'}\rangle_{j,j'=1,\dots,m_2}\big)}{\det(\zeta_{2,1}, \dots, \zeta_{2,N_2})} \\ &\quad \cdot \big((0, y_{2,\ell_2}^*) \wedge \cdots \wedge (0, y_{2,m_2+1}^*) \wedge (y_{1,\ell_1}^*, 0) \wedge \cdots \wedge (y_{1,m_1+1}^*, 0)\big) \\ &\quad \otimes \big(((x_{1,1}, 0), (0, 0)) \wedge \cdots \wedge ((x_{1,k_1}, 0), (0, 0)) \\ &\qquad \wedge ((0, x_{2,1}), (0, 0)) \wedge \cdots \wedge ((0, x_{2,k_2}), (0, 0)) \\ &\qquad \wedge ((\xi_{1,m_1+1}, \zeta_{1,m_1+1}), (0, 0)) \wedge \cdots \wedge ((\xi_{1,N_1}, \zeta_{1,N_1}), (0, 0)) \\ &\qquad \wedge ((0, 0), (\xi_{2,m_2+1}, \zeta_{2,m_2+1})) \wedge \cdots \wedge ((0, 0), (\xi_{2,N_2}, \zeta_{2,N_2}))\big). \end{aligned}$$

This implies

$$
\begin{aligned}
&R\left(\iota_{\widetilde{D},\widetilde{\Phi}}(\widetilde{\theta})\right) \\
&= (-1)^{(k_1-\ell_1)\ell_2}(-1)^{(\ell_1-m_1)m_2}(-1)^{(k_1+k_2)(m_1+m_2)}(-1)^{(N_1-m_1)m_2} \\
&\quad \cdot \frac{\det\big(\langle y^*_{1,j}, \Phi_1\zeta_{1,j'}\rangle_{j,j'=1,\ldots,m_1}\big)}{\det(\zeta_{1,1},\ldots,\zeta_{1,N_1})} \cdot \frac{\det\big(\langle y^*_{2,j}, \Phi_2\zeta_{2,j'}\rangle_{j,j'=1,\ldots,m_2}\big)}{\det(\zeta_{2,1},\ldots,\zeta_{2,N_2})} \\
&\quad \cdot \big((0,y^*_{2,\ell_2})\wedge\cdots\wedge(0,y^*_{2,m_2+1})\wedge(y^*_{1,\ell_1},0)\wedge\cdots\wedge(y^*_{1,m_1+1},0)\big) \\
&\quad \otimes \big((x_{1,1},0;0,0)\wedge\cdots\wedge(x_{1,k_1},0;0,0) \\
&\qquad\quad \wedge(0,0;x_{2,1},0)\wedge\cdots\wedge(0,0;x_{2,k_2},0) \\
&\qquad\quad \wedge(\xi_{1,m_1+1},\zeta_{1,m_1+1};0,0)\wedge\cdots\wedge(\xi_{1,N_1},\zeta_{1,N_1};0,0) \\
&\qquad\quad \wedge(0,0;\xi_{2,m_2+1},\zeta_{2,m_2+1})\wedge\cdots\wedge(0,0;\xi_{2,N_2},\zeta_{2,N_2})\big) \qquad (20)\\
&= (-1)^{(k_1-\ell_1)\ell_2}(-1)^{(\ell_1-m_1)m_2}(-1)^{(k_1+k_2)(m_1+m_2)}(-1)^{(N_1-m_1)(m_2+k_2)} \\
&\quad \cdot \frac{\det\big(\langle y^*_{1,j}, \Phi_1\zeta_{1,j'}\rangle_{j,j'=1,\ldots,m_1}\big)}{\det(\zeta_{1,1},\ldots,\zeta_{1,N_1})} \cdot \frac{\det\big(\langle y^*_{2,j}, \Phi_2\zeta_{2,j'}\rangle_{j,j'=1,\ldots,m_2}\big)}{\det(\zeta_{2,1},\ldots,\zeta_{2,N_2})} \\
&\quad \cdot \big((0,y^*_{2,\ell_2})\wedge\cdots\wedge(0,y^*_{2,m_2+1})\wedge(y^*_{1,\ell_1},0)\wedge\cdots\wedge(y^*_{1,m_1+1},0)\big) \\
&\quad \otimes \big((x_{1,1},0;0,0)\wedge\cdots\wedge(x_{1,k_1},0;0,0) \\
&\qquad\quad \wedge(\xi_{1,m_1+1},\zeta_{1,m_1+1};0,0)\wedge\cdots\wedge(\xi_{1,N_1},\zeta_{1,N_1};0,0) \\
&\qquad\quad \wedge(0,0;x_{2,1},0)\wedge\cdots\wedge(0,0;x_{2,k_2},0) \\
&\qquad\quad \wedge(0,0;\xi_{2,m_2+1},\zeta_{2,m_2+1})\wedge\cdots\wedge(0,0;\xi_{2,N_2},\zeta_{2,N_2})\big).
\end{aligned}
$$

Comparing the sign

$$(-1)^{(k_1-\ell_1+N_1)(\ell_2-m_2)}(-1)^{k_1m_1}(-1)^{k_2m_2}$$

in equation (19) with the sign

$$(-1)^{(k_1-\ell_1)\ell_2}(-1)^{(\ell_1-m_1)m_2}(-1)^{(k_1+k_2)(m_1+m_2)}(-1)^{(N_1-m_1)m_2}(-1)^{(N_1-m_1)k_2}$$

in equation (20) we find

$$\rho_{\widehat{D}_1,\widehat{D}_2}\left(\widehat{\theta}_1\otimes\widehat{\theta}_2\right) = (-1)^{N_1(k_2-\ell_2)}\cdot R\left(\iota_{\widetilde{D},\widetilde{\Phi}}(\widetilde{\theta})\right).$$

This proves equation (15) and Theorem 3.1. □

4 Remarks

1. The sign convention in the definition of the product isomorphism ρ_{D_1,D_2} in equation (3) in 1.3 follows the book of Seidel [3, page 150].

2. For two Banach spaces X, Y define the set

$$\det(X,Y) := \{(D,\theta) \,|\, D \in \mathcal{F}(X,Y),\ \theta \in \det(D)\}.$$

Denote by

$$\mathcal{F}^*(X,Y) \subset \mathcal{F}(X,Y)$$

the set of surjective Fredholm operators and define

$$\det{}^*(X,Y) := \{(D,\theta) \,|\, D \in \mathcal{F}^*(X,Y),\ \theta \in \det(D)\}.$$

Then $\det^*(X,Y)$ has a canonical topology and a canonical structure of a real line bundle over $\mathcal{F}^*(X,Y)$. Local trivializations are induced by local trivializations of the kernel bundle over $\mathcal{F}^*(X,Y)$.

3. For two Banach spaces X, Y and a linear map $\Phi : \mathbb{R}^N \to Y$, the map

$$\det{}^*(X,Y) \to \det{}^*(X \times \mathbb{R}^N, Y) : (D,\theta) \mapsto (D \oplus \Phi, \iota_{D,\Phi}(\theta)),$$

determined by the isomorphisms of Theorem 2.1, is a vector bundle homomorphism, bijective on each fiber, and is a homeomorphism onto its image.

4. For two Banach spaces X, Y and a linear map $\Phi : \mathbb{R}^N \to Y$, define

$$\mathcal{U}_\Phi := \left\{D \in \mathcal{F}(X,Y) \,|\, D \oplus \Phi \in \mathcal{F}^*(X \times \mathbb{R}^N, Y)\right\}.$$

Then the map

$$\det(X,Y)|_{\mathcal{U}_\Phi} \to \det{}^*(X \times \mathbb{R}^N, Y) : (D,\theta) \mapsto (D \oplus \Phi, \iota_{D,\Phi}(\theta))$$

determines a topology and vector bundle structure on the restriction of $\det(X,Y)$ to $\mathcal{U}_\Phi$. That this vector bundle structure is independent of the choice of the linear map Φ (with the same domain and the same image) follows from the (Determinant) axiom in Theorem 2.1. That the vector bundle structures on $\mathcal{U}_\Phi$ and $\mathcal{U}_\Psi$ agree on the intersection $\mathcal{U}_\Phi \cap \mathcal{U}_\Psi$ follows from the (Stabilization) axiom in Theorem 2.1.

5. Let X and Y be infinite-dimensional Banach spaces. Then $\mathcal{F}(X,Y)$ is a classifying space for K-theory. Let M be a topological space and $\xi \in K(M)$ be represented by a continuous map $M \to \mathcal{F}(X,Y) : p \mapsto D_p$. Then the first Stiefel–Whitney class $\mathrm{w}_1(\xi)$ is determined by the pullback of the determinant line bundle $\det(X,Y) \to \mathcal{F}(X,Y)$ under the classifying map $M \to \mathcal{F}(X,Y)$. Thus $\mathrm{w}_1(\xi)$ is nonzero over a loop $\gamma : S^1 \to M$ whenever the determinant lines $\det(D_{\gamma(t)})$ form a Möbius strip. When $\mathrm{index}(D_{\gamma(t)}) = 0$, this is equivalent to the condition that the mod-2 crossing index of the loop $t \mapsto D_{\gamma(t)}$ is nonzero.

6. The isomorphism $\det(D) \to \det(D \oplus \Phi)$ in [1, Exercise A.2.3] has only been defined when $D \oplus \Phi$ is surjective or, equivalently, when $m = \ell$ in the notation of Step 2 in the proof of Theorem 2.1. It differs from the isomorphism of Theorem 2.1 by the sign

$$\varepsilon(D, \Phi) = (-1)^{k\ell + \binom{\ell}{2} + \ell(N-\ell)},$$

where $\binom{0}{2} = \binom{1}{2} = 0$ and

$$k := \dim(\ker D), \qquad \ell := \dim(\mathrm{coker}\, D) = N - \dim(\Phi^{-1}\mathrm{im}\, D).$$

If $\Phi_1 : \mathbb{R}^{N_1} \to Y$ and $\Phi_2 : \mathbb{R}^{N_2} \to Y$ are two linear maps such that $D \oplus \Phi_1$ is surjective, then $\varepsilon(D \oplus \Phi_1, \Phi_2) = 1$ and hence

$$\varepsilon(D, \Phi_1 \oplus \Phi_2) \cdot \varepsilon(D \oplus \Phi_1, \Phi_2) \cdot \varepsilon(D, \Phi_1) = (-1)^{\ell(N_2 - \ell)}.$$

Thus the stabilization formula (9) in Theorem 2.1 holds for the isomorphism of [1, Remark A.2.3] whenever N_1 and $N_1 + N_2$ have the same parity. In this case the isomorphisms of [1, Remark A.2.3] can be used to define a topology and vector bundle structure on $\det(X,Y)$. However, the resulting topology on $\det(X,Y)$ will depend on the parity of N and be different from the topology determined by the isomorphisms of Theorem 2.1.

7. The isomorphisms of Theorem 2.1 agree with the isomorphisms constructed by McDuff–Wehrheim [2] in the proof of Proposition 7.4.8 (under the assumption that Φ is injective and $D \oplus \Phi$ is surjective).

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