# Notes on complex Lie groups

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## 1 Complex Lie groups

**Definition 1.1.** A **complex Lie group** is a Lie group G equipped with the structure of a complex manifold such that the structure maps

$$\mathrm{G} \times \mathrm{G} \to \mathrm{G} : (g,h) \mapsto gh, \qquad \mathrm{G} \to \mathrm{G} : g \mapsto g^{-1}$$

are holomorphic.

**Lemma 1.2.** *Let* G *be a connected Lie group. Assume that the Lie algebra* $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$ *is equipped with a complex structure*

$$\mathfrak{g} \to \mathfrak{g} : \xi \mapsto i\xi$$

*and define the complex structure* $J$ *on* G *by*

$$J_g v := (ivg^{-1})g$$

*for* $v \in T_g\mathrm{G}$. *Then the following are equivalent.*
**(i)** $(\mathrm{G}, J)$ *is a complex Lie group.*
**(ii)** *The Lie bracket* $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} : (\xi, \eta) \mapsto [\xi, \eta]$ *is complex bilinear, i.e.*

$$[i\xi, \eta] = [\xi, i\eta] = i[\xi, \eta]$$

*for all* $\xi, \eta \in \mathfrak{g}$.

*Proof.* For $\xi \in \mathfrak{g}$ define the vector fields $X_\xi, Y_\xi \in \mathrm{Vect}(\mathrm{G})$ by

$$X_\xi(g) := \xi g, \qquad Y_\xi(g) := g\xi.$$

We prove that

$$JX_\xi = X_{i\xi}, \qquad (\mathcal{L}_{X_\eta} J)X_\xi = X_{[i\xi,\eta]-i[\xi,\eta]}, \qquad \mathcal{L}_{Y_\xi} J = 0. \tag{1}$$

Here the first equation is obvious from the definitions. The second equation follows from the first and the identities $[X_\xi, X_\eta] = X_{[\xi,\eta]}$ and

$$(\mathcal{L}_{X_\eta} J)X_\xi = \mathcal{L}_{X_\eta}(JX_\xi) - J\mathcal{L}_{X_\eta} X_\xi = [JX_\xi, X_\eta] - J[X_\xi, X_\eta]$$

for $\xi, \eta \in \mathfrak{g}$. To prove the last equation in (1) note that

$$J_{gh}(vh) = (J_g v)h$$

for all $g, h \in \mathrm{G}$ and $v \in T_g\mathrm{G}$. Hence the diffeomorphism $\mathrm{G} \to \mathrm{G} : g \mapsto gh$ is holomorphic for every $h \in \mathrm{G}$. Differentiating with respect to $h$ gives $\mathcal{L}_{Y_\xi} J = 0$ for every $\xi \in \mathfrak{g}$. Thus we have proved (1).

That (i) implies (ii) follows immediately from (1). Conversely assume (ii) and denote by $N_J$ the Nijenhuis tensor of $J$. Then, for all $\xi, \eta \in \mathfrak{g}$, we have

$$N_J(X_\xi, X_\eta) = [X_\xi, X_\eta] + J[JX_\xi, X_\eta] + J[X_\xi, JX_\eta] - [JX_\xi, JX_\eta] = X_\zeta$$

where

$$\zeta := [\xi, \eta] + i[i\xi, \eta] + i[\xi, i\eta] - [i\xi, i\eta] = 0.$$

Here we have used (1) and (ii). Since the vector fields $X_\xi$ span the tangent bundle this shows that $N_J = 0$ and so $J$ is integrable. It follows also from (ii) that

$$g^{-1}(i\xi)g = i(g^{-1}\xi g)$$

for all $\xi \in \mathfrak{g}$ and $g \in \mathrm{G}$ (see Lemma 1.3 below) and hence

$$J_g v := (ivg^{-1})g = g(ig^{-1}v) \tag{2}$$

for $g \in \mathrm{G}$ and $v \in T_g\mathrm{G}$. This implies that the multiplication map is holomorphic. Since $\mathbb{1}$ is a regular value of the multiplication map its preimage is a complex submanifold of $\mathrm{G} \times \mathrm{G}$ and it is the graph of the map $g \mapsto g^{-1}$. Hence this map is holomorphic as well. This proves the lemma. $\square$

**Lemma 1.3.** *Let* $\mathrm{G}$ *be a connected Lie group and* $A : \mathfrak{g} \to \mathfrak{g}$ *be a linear map on its Lie algebra* $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$*. Then the following are equivalent.*

**(i)** *For all* $\xi \in \mathfrak{g}$ *and* $g \in \mathrm{G}$ *we have* $A(g\xi g^{-1}) = g(A\xi)g^{-1}$*.*

**(ii)** *For all* $\xi, \eta \in \mathfrak{g}$ *we have* $A[\xi, \eta] = [A\xi, \eta] = [\xi, A\eta]$*.*

*Proof.* To prove that (i) implies (ii) differentiate the itentity

$$A(\exp(t\xi)\eta\exp(-t\xi)) = \exp(t\xi)(A\eta)\exp(-t\xi)$$

with respect to $t$ at $t = 0$. To prove the converse choose a path $g : [0, 1] \to \mathrm{G}$ such that $g(0) = \mathbb{1}$ and an element $\xi \in \mathrm{G}$. Define the maps $\eta, \zeta : [0, 1] \to \mathfrak{g}$ by

$$\eta := g^{-1}\xi g, \qquad \zeta := g^{-1}(A\xi)g.$$

Since $\partial_t(g^{-1}\xi g) = [g^{-1}\xi g, g^{-1}\dot{g}]$ we obtain

$$\partial_t(A\eta) + [g^{-1}\dot{g}, A\eta] = 0, \qquad \partial_t\zeta + [g^{-1}\dot{g}, \zeta], \qquad A\eta(0) = \zeta(0) = A\xi.$$

Here the first equation follows from (ii). It follows that $A\eta(t) = \zeta(t)$ for all $t$. This proves the lemma. □

**Theorem 1.4.** *Let* $\mathrm{G}$ *be a compact connected Lie group and* $\mathrm{G}^c$ *be a complex connected Lie group with Lie algebras* $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$ *and* $\mathfrak{g}^c = \mathrm{Lie}(\mathrm{G}^c)$*. Let* $\iota : \mathrm{G} \to \mathrm{G}^c$ *be a Lie group homomorphism. Then the following are equivalent.*

**(i)** *For every complex Lie group* $\mathrm{H}$ *and every Lie group homomorphism* $\rho : \mathrm{G} \to \mathrm{H}$ *there is a unique holomorphic homomorphism* $\rho^c : \mathrm{G}^c \to \mathrm{H}$ *such that* $\rho = \rho^c \circ \iota$*:*

$$\begin{array}{ccc} \mathrm{G} & \xrightarrow{\ \iota\ } & \mathrm{G}^c \\ & \searrow^{\rho} & \downarrow{\scriptstyle \rho^c} \\ & & \mathrm{H} \end{array} .$$

**(ii)** $\iota$ *is injective, its image* $\iota(\mathrm{G})$ *is a maximal compact subgroup of* $\mathrm{G}^c$*, and the differential* $d\iota(\mathbb{1}) : \mathfrak{g} \to \mathfrak{g}^c$ *maps* $\mathfrak{g}$ *onto a totally real subspace of* $\mathfrak{g}^c$*.*

**Definition 1.5.** A Lie group homomorphism $\iota : \mathrm{G} \to \mathrm{G}^c$ that satisfies (i) and (ii) in Theorem 1.4 is called a **complexification** of G.

**Theorem 1.6.** *Every compact connected Lie group admits a complexification, unique up to canonical isomorphism.*

**Example 1.7.** The inclusion of $\mathrm{U}(n)$ into $\mathrm{GL}(n, \mathbb{C})$ is a complexification. The polar decomposition gives rise to a diffeomorphism

$$\phi : \mathrm{U}(n) \times \mathfrak{u}(n) \to \mathrm{GL}(n, \mathbb{C}), \qquad \phi(g, \eta) := \exp(i\eta)g$$

## 2 First existence proof

**Theorem 2.1.** *Let* $\mathrm{G} \subset \mathrm{U}(n)$ *be a connected Lie subgroup with Lie algebra* $\mathfrak{g} \subset \mathfrak{u}(n)$*. Then the set*

$$\mathrm{G}^c := \{\exp(i\eta)g \,|\, g \in \mathrm{G},\, \eta \in \mathfrak{g}\} \subset \mathrm{GL}(n, \mathbb{C})$$

*is a complex Lie subgroup of* $\mathrm{GL}(n.\mathbb{C})$ *and the inclusion of* G *into* $\mathrm{G}^c$ *satisfies (ii) in Theorem 1.4.*

*Proof.* The proof has nine steps.

**Step 1.** $\mathrm{G}^c$ *is a closed submanifold of* $\mathrm{GL}(n, \mathbb{C})$.

This follows immediately from Example 1.7.

**Step 2.** $\mathbb{1} \in \mathrm{G}^c$ *and* $T_{\mathbb{1}}\mathrm{G} = \mathfrak{g} \oplus i\mathfrak{g} =: \mathfrak{g}^c$.

For $\xi, \eta \in \mathfrak{g}$ consider the curve $\gamma(t) := \exp(it\eta)\exp(t\xi) \in \mathrm{G}^c$. It satisfies $\dot\gamma(0) = \xi + i\eta$. Hence $\mathfrak{g}^c \subset T_{\mathbb{1}}\mathrm{G}^c$ and both spaces have the same dimension.

**Step 3.** $T_k\mathrm{G}^c = k\mathfrak{g}^c$ *for every* $k \in \mathrm{G}^c$.

Both spaces have the same dimension, so it suffices to prove that $T_k\mathrm{G}^c \subset k\mathfrak{g}^c$. Let $\phi$ be the diffeomorphism of Exercise 1.7. Fix an element $(g, \eta) \in \mathrm{G} \times \mathfrak{g}$ and let

$$k := \phi(g, \eta) = \exp(i\eta)g \in \mathrm{G}^c.$$

Then, for every $\hat\xi \in \mathfrak{g}$, we obviously have $d\phi(g, \eta)(g\hat\xi, 0) = \exp(i\eta)g\hat\xi \in k\mathfrak{g}^c$. Now let $\hat\eta \in \mathfrak{g}$. We must prove that

$$d\phi(g, \eta)(0, \hat\eta) \in k\mathfrak{g}^c.$$

To see this consider the map $\gamma : \mathbb{R}^2 \to \mathrm{G}^c$ defined by

$$\gamma(s, t) := \phi(g, t(\eta + s\hat\eta)) = \exp(it(\eta + s\hat\eta))g$$

and denote

$$\xi := \gamma^{-1}\partial_s\gamma, \qquad \eta := \gamma^{-1}\partial_t\gamma.$$

Then $\eta(s, t) = g^{-1}i(\eta + s\hat\eta)g \in \mathfrak{g}^c$ for all $s, t$ and

$$\partial_t\xi = \partial_s\eta + [\xi, \eta], \qquad \xi(s, 0) = 0.$$

Since $\eta(s, t) \in \mathfrak{g}^c$ this implies $\xi(s, t) \in \mathfrak{g}^c$ for all $s, t$ and, in particular, $d\phi(g, \eta)(0, \hat\eta) = \gamma(0, 1)\xi(0, 1) \in k\mathfrak{g}^c$. This proves Step 3.

**Step 4.** *Let $a \in \mathrm{GL}(n, \mathbb{C})$. Then $a \in \mathrm{G}^c$ if and only if there exists a smooth path $\alpha : [0,1] \to \mathrm{GL}(n, \mathbb{C})$ satisfying $\alpha(0) = 1$, $\alpha(1) = a$, and $\alpha(t)^{-1}\dot{\alpha}(t) \in \mathfrak{g}^c$ for every $t$.*

To prove that the condition is necessary let $a = \exp(i\eta)h \in \mathrm{G}^c$ be given, choose a smooth path $g : [0,1] \to \mathrm{G}$ with $g(0) = \mathbb{1}$ and $g(1) = h$, and define $\alpha(t) := \exp(it\eta)g(t) = \phi(g(t), t\eta)$. This curve satisfies the required conditions by Step 3. To prove the converse suppose that $\alpha : [0,1] \to \mathrm{GL}(n, \mathbb{C})$ is a smooth curve satisfying $\alpha(0) = \mathbb{1}$, $\alpha(1) = a$, and $\alpha(t)^{-1}\dot{\alpha}(t) \in \mathrm{G}^c$. Consider the set

$$I := \{t \in [0,1] \,|\, \alpha(t) \in \mathrm{G}^c\}$$

this set is nonempty, because $0 \in I$, and it is closed because $\mathrm{G}^c$ is a closed subset of $\mathrm{GL}(n, \mathbb{C})$, by Step 1. To prove that it is open, denote $\eta(t) := \alpha(t)^{-1}\dot{\alpha}(t) \in \mathfrak{g}^c$ and consider the vector fields $X_t$ on $\mathbb{C}^{n\times n}$ given by $X_t(A) := A\xi(t)$. By Step 3, these vector fields are all tangent to $\mathrm{G}^c$. Hence every solution of the differential equation $\dot{A}(t) = A(t)\eta(t)$ that starts in $\mathrm{G}^c$ remains in $\mathrm{G}^c$ on a sufficiently small time interval. In particular this holds for the curve $t \mapsto \alpha(t)$ and so $I$ is open. Thus $I = [0,1]$ and hence $a = \alpha(1) \in \mathrm{G}^c$.

**Step 5.** *If $a \in \mathrm{G}^c$ and $\xi \in \mathfrak{g}^c$ then $a^{-1}\xi a \in \mathfrak{g}^c$.*

Choose $\alpha : [0,1] \to \mathrm{G}^c$ as in Step 4. and denote

$$\zeta(t) := \alpha(t)^{-1}\xi\alpha(t), \qquad \eta(t) := \alpha(t)^{-1}\dot{\alpha}(t).$$

Then

$$\dot{\zeta} + [\eta, \zeta] = 0, \qquad \zeta(0) = \xi \in \mathfrak{g}^c.$$

Since $\eta(t) \in \mathfrak{g}^c$ for all $t$ this implies that $\zeta(t) \in \mathfrak{g}^c$ for all $t$ and, in particular, $a^{-1}\xi a = \zeta(1) \in \mathfrak{g}^c$.

**Step 6.** *If $a \in \mathrm{G}^c$ and $\xi \in \mathfrak{g}^c$ then $a\xi a^{-1} \in \mathfrak{g}^c$.*

The linear map $\xi \mapsto a^{-1}\xi a$ maps $\mathfrak{g}^c$ to itself, by Step 5, and it is injective. Hence the map $\mathfrak{g}^c \to \mathfrak{g}^c : \xi \mapsto a^{-1}\xi a$ is bijective and this proves Step 6.

**Step 7.** *If $a, b \in \mathrm{G}^c$ then $ab \in \mathrm{G}^c$.*

Choose two curves $\alpha, \beta : [0,1] \to \mathrm{G}^c$ as in Step 4 with $\alpha(0) = \beta(0) = \mathbb{1}$ and $\alpha(1) = a$, $\beta(1) = b$. Then the curve $\gamma := \alpha\beta : [0,1] \to \mathrm{GL}(n, \mathbb{C})$ satisfies

$$\gamma^{-1}\dot{\gamma} = \beta^{-1}\dot{\beta} + \beta^{-1}(\alpha^{-1}\dot{\alpha})\beta.$$

By Step 5, we have $\gamma(t)^{-1}\dot{\gamma}(t) \in \mathfrak{g}^c$ for all $t$ and hence $ab = \gamma(1) \in \mathrm{G}^c$.

**Step 8.** *If $a \in \mathrm{G}^c$ then $a^{-1} \in \mathrm{G}^c$.*

Let $\alpha$ as in Step 4 and denote $\gamma(t) := \alpha(t)^{-1}$. Then

$$\gamma^{-1}\dot{\gamma} = \alpha \frac{d}{dt}\alpha^{-1} = -\dot{\alpha}\alpha^{-1} = \alpha(-\alpha^{-1}\dot{\alpha})\alpha^{-1}.$$

By Step 6, we have $\gamma(t)^{-1}\dot{\gamma}(t) \in \mathfrak{g}^c$ for all $t$ and hence $a^{-1} = \gamma(1) \in \mathrm{G}^c$.

**Step 9.** $\mathrm{G}$ *is a maximal compact subgroup of* $\mathrm{G}^c$.

Let $\mathrm{H} \subset \mathrm{G}^c$ be a subgroup such that $\mathrm{G} \subsetneq \mathrm{H}$. Choose an element $h \in \mathrm{H} \setminus \mathrm{G}$. Since $\mathrm{H} \subset \mathrm{G}^c$, there is a pair $(g, \eta) \in \mathrm{G} \times \mathfrak{g}$ such that $h = \exp(i\eta)g$. since $\mathrm{G} \subset \mathrm{H}$ and $\mathrm{H}$ is a subgroup of $\mathrm{G}^c$ we have

$$P := \exp(i\eta) \in \mathrm{H}.$$

The matrix $P$ is Hermitian and positive definite. Since $h \notin \mathrm{G}$ we also have $P \notin \mathrm{G}$. But this implies $\eta \neq 0$ and so at least one eigenvalue of $P$ is not equal to 1. Hence the sequence

$$P^k = \exp(ik\eta) \in \mathrm{H}, \qquad k = 1, 2, 3, \ldots$$

has no subsequence that converges to an element of $\mathrm{GL}(n, \mathbb{C})$. Thus $\mathrm{H}$ is not compact and this proves the theorem. □

The tangent space of the submanifold $\mathrm{G}^c \subset \mathrm{GL}(n, \mathbb{C})$ in Theorem 2.1 at the identity element is obviously equal to

$$T_{\mathbb{1}}\mathrm{G}^c = \mathfrak{g} \oplus i\mathfrak{g} = \mathfrak{g}^c.$$

Since $\mathrm{G}^c$ is a subgroup we have that, for every pair $\eta, \hat{\eta} \in \mathfrak{g}$, the curve $t \mapsto \exp(-i\eta)\exp(i\eta + ti\hat{\eta})$ lies in $\mathrm{G}^c$ and hence

$$A(\eta)\hat{\eta} := \left.\frac{d}{dt}\right|_{t=0} \exp(-i\eta)\exp(i\eta + ti\hat{\eta}) \in \mathfrak{g}^c.$$

It turns out that $A \in \Omega^1(\mathfrak{g}, \mathfrak{g}^c)$ is a flat connection 1-form that satisfies $A(\eta)\hat{\eta} = i\hat{\eta}$ whenever $\eta$ and $\hat{\eta}$ commute. Conversely, Theorem 3.5 below shows that, for any Lie algebra $\mathfrak{g}$, the connection $A$ is uniquely determined by these conditions and that the group multiplication on $\mathrm{G} \times \mathfrak{g}$ can be reconstructed from $A$. This gives rise to a construction of a complexified Lie group for any compact connected Lie group $\mathrm{G}$ that does not rely on an embedding into the unitary group.

# 3 Second existence proof

**Definition 3.1.** Let $X$ be a connected smooth manifold and $\mathfrak{g}$ be a Lie algebra. A flat connection $A \in \Omega^1(X, \mathfrak{g})$ is called an **infinitesimal group law** if it satisfies the following conditions.

**(Monodromy)** The monodromy representation of $A$ is trivial, i.e. for any two smooth paths $\gamma : [0,1] \to X$ and $\zeta : [0,1] \to \mathfrak{g}$ we have

$$\dot{\zeta} + [A(\gamma)\dot{\gamma}, \zeta] = 0, \quad \gamma(0) = \gamma(1) \qquad \Longrightarrow \qquad \zeta(0) = \zeta(1).$$

**(Parallel)** $A(x) : T_xX \to \mathfrak{g}$ is a vector space isomorphism for every $x \in X$.

**(Complete)** The vector fields $Y_\xi \in \mathrm{Vect}(X)$ defined by

$$A(x)Y_\xi(x) = \xi$$

are complete, i.e. for every smooth path $\mathbb{R} \to \mathfrak{g} : t \mapsto \xi(t)$ the solutions of the differential equation $\dot{\gamma}(t) = Y_{\xi(t)}(\gamma(t))$ exist for all time.

**Example 3.2.** Let G be a Lie group with Lie algebra $\mathfrak{g} := T_1\mathrm{G} = \mathrm{Lie}(\mathrm{G})$. Then the 1-form $A \in \Omega^1(\mathrm{G}, \mathfrak{g})$ defined by

$$A(g)v := g^{-1}v$$

is an infinitesimal group law. The vector fields $Y_\xi$ are given by $Y_\xi(g) = g\xi$ and the curvature $F_A \in \Omega^2(X, \mathfrak{g})$ is

$$\begin{aligned} F_A(Y_\xi, Y_\eta) &= dA(Y_\xi, Y_\eta) + [AY_\xi, AY_\eta] \\ &= \mathcal{L}_{Y_\xi}(AY_\eta) - \mathcal{L}_{Y_\eta}(AY_\xi) + A[Y_\xi, Y_\eta] + [\xi, \eta] \\ &= A[Y_\xi, Y_\eta] + [\xi, \eta] \\ &= 0 \end{aligned}$$

for $\xi, \eta \in \mathfrak{g}$. Thus the connection is flat. The *(Monodromy)* condition holds because, for any path $g : [0,1] \to \mathrm{G}$, the solutions of the equation

$$\dot{\xi} + [g^{-1}\dot{g}, \xi] = 0$$

have the form

$$\xi(t) = g(t)^{-1}\xi_0 g(t).$$

**Example 3.3.** Let $\mathfrak{g}$ be a Lie algebra. Then there is a unique flat connection $A \in \Omega^1(\mathfrak{g}, \mathfrak{g})$ such that

$$[\xi, \hat{\xi}] = 0 \qquad \Longrightarrow \qquad A(\xi)\hat{\xi} = \hat{\xi} \tag{3}$$

for all $\xi, \hat{\xi} \in \mathfrak{g}$. In general, this connection is not an infinitesimal group law. The idea behind this example is as follows. If we have a Lie group G with Lie algebra $\mathfrak{g}$ we might attempt to reconstruct the group multiplication locally as an operation $m : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that $\exp(\xi)\exp(\eta) = \exp(m(\xi, \eta))$. While this is not possible globally in most cases, the associated connection $A(\xi)\hat{\xi} := \exp(-\xi) d \exp(\xi)\hat{\xi}$ does exist globally and satisfying (3).

To prove uniqueness note that a connection $A \in \Omega^1(\mathfrak{g}, \mathfrak{g})$ is flat if and only if every smooth map $\gamma : \mathbb{R}^2 \to \mathfrak{g}$ satisfies the equation

$$\partial_s(A(\gamma)\partial_t\gamma) - \partial_t(A(\gamma)\partial_s\gamma) + [A(\gamma)\partial_s\gamma, A(\gamma)\partial_t\gamma] = 0. \tag{4}$$

If in addition the connection satisfies (3) then, with $\gamma(s, t) := t(\xi + s\hat{\xi})$, we obtain

$$A(\gamma)\partial_t\gamma = \xi + s\hat{\xi}, \qquad A(\gamma)\partial_s\gamma = A(t(\xi + s\hat{\xi}))t\hat{\xi}.$$

Setting $s = 0$ we find that the function

$$\zeta(t) := A(t\xi)t\hat{\xi}$$

satisfies the differential equation

$$\dot{\zeta} + [\xi, \zeta] = \hat{\xi}, \qquad \zeta(0) = 0. \tag{5}$$

Thus

$$A(\xi)\hat{\xi} = \zeta(1) = \int_0^1 \exp(-t\mathrm{ad}(\xi))\hat{\xi}\, dt = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!}\mathrm{ad}(\xi)^k\hat{\xi}.$$

Conversely, let $A \in \Omega^1(\mathfrak{g}, \mathfrak{g})$ be defined by this formula. If $[\xi, \hat{\xi}] = 0$ then $\zeta(t) = t\hat{\xi}$ is the unique solution of (5) and so $A(\xi)\hat{\xi} = \zeta(1) = \hat{\xi}$. To prove that $A$ is flat we fix three elements $\xi, \hat{\xi}_1, \hat{\xi}_2 \in \mathfrak{g}$, define $\zeta_j : [0, 1] \to \mathfrak{g}$ as the solutions of (5) with $\hat{\xi} = \hat{\xi}_j$, and define $\zeta_{ij} : [0, 1] \to \mathfrak{g}$ as the solution of the linearized equation

$$\dot{\zeta}_{ij} + [\xi, \zeta_{ij}] + [\hat{\xi}_i, \zeta_j] = 0, \qquad \zeta_{ij}(0) = 0.$$

Then $A(\xi)\hat{\xi}_j = \zeta_j(1)$ and $(dA(\xi)\hat{\xi}_i)\hat{\xi}_j = \zeta_{ij}(1)$. Moreover,

$$\dot{\zeta} + [\xi, \zeta] = 0, \qquad \zeta := \zeta_{12} - \zeta_{21} + [\zeta_1, \zeta_2],$$

so $\eta \equiv 0$ and thus $A$ is flat.

**Theorem 3.4.** *Let $X$ be a connected smooth manifold, $\mathfrak{g}$ be a Lie algebra, and $A \in \Omega^1(X, \mathfrak{g})$ be an infinitesimal group law. Fix an element $1 \in X$. Then there is a unique Lie group structure on $X$ with unit 1 such that*

$$A(x)v = A(1)x^{-1}v$$

*for $x \in X$ and $v \in T_xX$. Moreover, the map $A(1) : T_1X = \mathrm{Lie}(X) \to \mathfrak{g}$ is a Lie algebra isomorphism.*

*Proof.* The proof has seven steps.

**Step 1.** *There is a unique function $\Phi : X \to \mathrm{Aut}(\mathfrak{g})$ satisfying*

$$\Phi(0) = \mathrm{id}, \qquad (d\Phi(x)v)\xi + [A(x)v, \Phi(x)\xi] = 0 \tag{6}$$

*for all $x \in X$, $v \in T_xX$, and $\xi \in \mathfrak{g}$.*

Given $x \in X$ and $\xi \in \mathfrak{g}$ choose a smooth path $\gamma : [0, 1] \to X$ with endpoints $\gamma(0) = 1$ and $\gamma(1) = x$, let $\zeta : [0, 1] \to \mathfrak{g}$ be the unique solution of the differential equation

$$\dot{\zeta} + [A(\gamma)\dot{\gamma}, \zeta] = 0, \qquad \zeta(0) = \xi, \tag{7}$$

and define

$$\Phi(x)\xi := \zeta(1).$$

The *(Monodromy)* axiom guarantees that $\zeta(1)$ is independent of the choice of the path $\gamma$. The resulting function $\Phi$ is obviously smooth and satisfies (6).

**Step 2.** *For any two smooth paths $\beta, \gamma : [0, 1] \to X$ we have*

$$\beta(0) = \beta(1), \quad A(\gamma)\dot{\gamma} = A(\beta)\dot{\beta} \qquad \Longrightarrow \qquad \gamma(0) = \gamma(1).$$

Assume without loss of generality that $\beta(0) = \beta(1) = 1$. Choose a smooth path $[0, 1] \to X : \lambda \mapsto x_\lambda$ such that $x_0 = 1$ and $x_1 = \gamma(0)$. For $\lambda \in [0, 1]$ let $\gamma_\lambda : [0, 1] \to X$ be the solution of the differential equation

$$A(\gamma_\lambda(t))\partial_t\gamma_\lambda(t) = A(\beta(t))\partial_t\beta(t), \qquad \gamma_\lambda(0) = x_\lambda.$$

Then $\lambda \mapsto \gamma_\lambda$ is a smooth homotopy from $\beta$ to $\gamma$. We observe that

$$A(\gamma_\lambda(t))\partial_\lambda\gamma_\lambda(t) = \Phi(\gamma_0(t))A(x_\lambda)\partial_\lambda x_\lambda, \tag{8}$$

where $\Phi$ is as in Step 1. Namely, both the left and right hand side of (8), as functions of $t$, satisfy the differential equation $\dot{\zeta} + [A(\gamma_0)\dot{\gamma}_0, \zeta] = 0$ with initial condition $\zeta(0) = A(x_\lambda)\partial_\lambda x_\lambda$. It follows from (8) with $t = 1$ that $A(\gamma_\lambda(1))\partial_\lambda\gamma_\lambda(1) = A(x_\lambda)\partial_\lambda x_\lambda$ for all $\lambda$. Since $\gamma_0(1) = x_0 = 1$ we obtain $\gamma_\lambda(1) = x_\lambda = \gamma_\lambda(0)$ for all $\lambda$. This proves Step 2.

**Step 3.** *For any four smooth paths $\beta_0, \beta_1, \gamma_0, \gamma_1 : [0,1] \to X$ satisfying $\beta_0(0) = \beta_1(0)$ and $\beta_0(1) = \beta_1(1)$ and $A(\gamma_j)\dot{\gamma}_j = A(\beta_j)\dot{\beta}_j$ we have*

$$\gamma_0(0) = \gamma_1(0) \qquad \Longleftrightarrow \qquad \gamma_0(1) = \gamma_1(1).$$

Assume without loss of generality that $\gamma_j$ and $\beta_j$ are constant near the endpoints and that $\beta_0(0) = \beta_1(0) = 1$ and $\gamma_0(0) = \gamma_1(0)$. Define $\beta : [0,1] \to X$ by $\beta(t) := \beta_0(2t)$ for $0 \le t \le 1/2$ and $\beta(t) := \beta_1(2-2t)$ for $1/2 \le t \le 1$. Let $\gamma : [0,1] \to X$ be the unique solution of the differential equation $A(\gamma)\dot{\gamma} = A(\beta)\dot{\beta}$ with initial condition $\gamma(0) = \gamma_0(0)$. Then $\gamma_0(t) = \gamma(t/2)$ for $0 \le t \le 1$. Moreover, since $\beta(0) = \beta(1) = 1$, it follows from Step 2 that $\gamma(1) = \gamma(0) = \gamma_0(0) = \gamma_1(0)$. Hence $\gamma_1(t) = \gamma((1-t)/2)$. with $t = 1$ we obtain that $\gamma_1(1)$ and $\gamma_0(1)$ both agree with $\gamma(1/2)$. This proves Step 3.

**Step 4.** *There is a unique smooth map*

$$X \times X \to X : (x, y) \mapsto \phi_x(y) = \psi_y(x)$$

*such that $\phi_x(1) = x$ and $\phi_x^* A = A$ for every $x \in X$. Moreover, $\phi_x$ and $\psi_y$ are diffeomorphisms for all $x, y$ and $\psi_1 = \phi_1 = \mathrm{id}$.*

Fix an element $x \in X$. It follows from Step 3 that, for every smooth path $\beta : [0,1] \to X$ with $\beta(0) = 1$, the endpoint of the path $\gamma : [0,1] \to X$, defined by

$$A(\gamma)\dot{\gamma} = A(\beta)\dot{\beta}, \qquad \gamma(0) = x, \tag{9}$$

depends only on the endpoint of $\beta$. Hence there is a well defined map $\phi_x : X \to X$ satisfying

$$\phi_x(\beta(1)) = \gamma(1)$$

whenever $\beta(0) = 1$ and $\gamma$ is given by (9). Since the solutions of a differential equation depend smoothly on the initial condition and the parameter it follows that the map $(x, y) \mapsto \phi_x(y)$ is smooth. (Namely, choose a local smooth family of paths $\beta_y : [0,1] \to X$ with $\beta_y(0) = 1$ and $\beta_y(1) = y$.) It follows directly from the construction that $\phi_x(1) = x$ and $\phi_x^* A = A$ for every $x$. That $\phi_x$ is a diffeomorphism follows by reversing the roles of the pairs $(1, \beta)$ and $(x, \gamma)$ to construct an inverse. That $\psi_y$ is a diffeomorphism follows by interchanging 1 and $y$ and reversing $\beta$. That $\phi_1$ is the identity is obvious from the definition (we get $\gamma = \beta$ when $x = 1$). That $\psi_1$ is the identity follows by choosing $\beta(t) \equiv 1$. Uniqueness is left as an exercise. This proves Step 4.

**Step 5.** *The map* $(x, y) \mapsto \phi_x(y) = \psi_y(x) =: xy$ *in Step 4 defines a Lie group structure on* $X$ *with unit* $1$.

It suffices to prove associativity, i.e.

$$\phi_x(\phi_y(z)) = \psi_z(\psi_y(x)) \tag{10}$$

for $x, y, z \in X$. That 1 is the unit follows then from the fact that $\phi_1 = \psi_1 = \mathrm{id}$ and that every element has an inverse follows from the fact that $\phi_x$ and $\psi_y$ are diffeomorphisms. The inverse map $x \mapsto \phi_x^{-1}(1)$ is smooth by Step 4.

To prove (10) we fix $x, y, z \in X$ and choose paths $\beta, \gamma : [0,1] \to X$ with endpoints $\beta(0) = \gamma(0) = 1$ and $\beta(1) = y$, $\gamma(1) = z$. Define the paths $\beta', \gamma', \gamma'' : [0,1] \to X$ by

$$A(\beta')\dot{\beta}' = A(\beta)\dot{\beta}, \qquad A(\gamma'')\dot{\gamma}'' = A(\gamma')\dot{\gamma}' = A(\gamma)\dot{\gamma},$$

and

$$\beta'(0) = x, \qquad \gamma'(0) = y, \qquad \gamma''(0) = \beta'(1) = \phi_x(y) = \psi_y(x).$$

We claim that

$$\phi_x(\phi_y(z)) = \gamma''(1) = \psi_z(\psi_y(x)).$$

To prove the first identity note that $\gamma'(1) = \phi_y(z)$ and so the catenation $\beta \# \gamma'$ (first $\beta$ then $\gamma'$) runs from 1 to $\phi_y(z)$. The catenation $\beta' \# \gamma''$ is the lift of this path starting at $x$ and hence ends at $\gamma''(1) = \phi_x(\phi_y(z))$, by definition of $\phi_x$ in the proof of Step 4. On the other hand $\gamma''$ is also the lift of $\gamma$ starting at $\psi_y(x)$ and hence ends at $\gamma''(1) = \psi_z(\psi_y(x))$, by definition of $\psi_z$ in the proof of Step 4. This proves Step 5.

**Step 6.** *The map* $A(1) : T_1X = \mathrm{Lie}(X) \to \mathfrak{g}$ *is a Lie algebra homomorphism and satisfies* $A(x)xv = A(1)v$ *for* $x \in X$ *and* $v \in T_1X$.

The formula $A(x)xv = A(1)v$ with $xv := d\phi_x(1)v$ follows immediately from the fact that $\phi_x^*A = A$ and $\phi_x(1) = x$. This formula shows that the vector fields $Y_\xi \in \mathrm{Vect}(X)$ in Definition 3.1 satisfy $\xi = A(x)Y_\xi(x) = A(1)x^{-1}Y_\xi(x)$. Hence

$$Y_\xi(x) = xv, \qquad v := A(1)^{-1}\xi \in \mathrm{Lie}(X).$$

The map $\mathrm{Lie}(X) \to \mathrm{Vect}(X)$ that assigns to every tangent vector $v \in \mathrm{Lie}(X)$ the left invariant vector field $x \mapsto xv$ is a Lie algebra anti-homomorphism. Since $A$ is flat we have

$$0 = F_A(Y_\xi, Y_\eta) = dA(Y_\xi, Y_\eta) + [AY_\xi, AY_\eta] = A[Y_\xi, Y_\eta] + [\xi, \eta].$$

Hence the map $\mathfrak{g} \to \mathrm{Vect}(X) : \xi \mapsto Y_\xi$ is also a Lie algebra anti-homomorphism and so $A(1)$ is a Lie algebra isomorphism. This completes the proof of the existence statement.

**Step 7.** *The Lie group structure on $X$ is uniquely determined by $A$ and $1$.*

Let $X \times X \to X : (x, y) \mapsto xy$ be a Lie group structure with unit 1 such that $A(x)v = A(1)x^{-1}v$ for $x \in X$ and $v \in T_xX$. Fix two elements $x, y \in X$, choose a path $\beta : [0,1] \to X$ such that $\beta(0) = 1$ and $\beta(1) = y$, and define $\gamma(t) := x\beta(t)$. Then $A(\gamma)\dot\gamma = A(1)\gamma^{-1}\dot\gamma = A(1)\beta^{-1}\dot\beta = A(\beta)\dot\beta$. Hence the Lie group structure on $X$ agrees with the one constructed in Step 5. This proves the theorem. □

**Theorem 3.5.** *Let $\mathrm{G}$ be a compact Lie group with Lie algebra $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$ and denote*

$$\mathrm{G}^c := \mathrm{G} \times \mathfrak{g}, \qquad \mathfrak{g}^c := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g} \oplus i\mathfrak{g}.$$

*Then the following holds.*

**(i)** *There is a unique flat connection $A_1 \in \Omega^1(\mathfrak{g}, \mathfrak{g}^c)$ such that, for all $\eta, \hat\eta \in \mathfrak{g}$, we have*

$$[\eta, \hat\eta] = 0 \qquad \Longrightarrow \qquad A_1(\eta)\hat\eta = i\hat\eta \tag{11}$$

**(ii)** *If $A_1$ is as in (i) then the 1-form $A \in \Omega^1(\mathrm{G}^c, \mathfrak{g}^c)$ defined by*

$$A(g, \eta)(v, \hat\eta) := g^{-1}v + g^{-1}(A_1(\eta)\hat\eta)g$$

*is an infinitesimal group law.*

**(iii)** *Suppose $\mathrm{G}^c$ is equipped with the Lie group structure associated to $A$ in Theorem 3.4. Then $\mathrm{G}$ is a maximal compact subgroup of $\mathrm{G}^c$.*

*Proof.* First assume that $A_1$ satisfies the requirements of (i). Let $\eta, \hat\eta \in \mathcal{G}$ and define $\zeta : \mathbb{R} \to \mathfrak{g}^c$ by $\zeta(t) := A_1(t\eta)t\hat\eta$. Then

$$\dot\zeta + [i\eta, \zeta] = i\hat\eta, \qquad \zeta(0) = 0. \tag{12}$$

(Apply equation (4) to the function $(s, t) \mapsto t(\eta + s\hat\eta)$ and set $s = 0$.) Thus we must define $A_1(\eta)\hat\eta = \zeta(1)$ where $\zeta : \mathbb{R} \to \mathfrak{g}$ is the unique solution of (12). That this 1-form satisfies (11) follows from the fact that $\zeta(t) := it\hat\eta$ satisfies (12) whenever $\eta$ and $\hat\eta$ commute. That it is flat follows from the same argument that was used in Example 3.3. This proves (i).

We prove (ii). First we observe that $A$ is flat. Namely the $\mathfrak{g}$-connection $A_0$ on G defined by $A_0(g)v := g^{-1}v$ is flat by Example 3.2 and $A_1$ is flat by (i). Hence, for two tangent vectors $w_j = (v_j, \hat{\eta}_j) \in T_{(g,\eta)}(\mathrm{G}\times\mathfrak{g})$, $j = 1, 2$, we obtain

$$\begin{aligned} F_A(w_1, w_2) &= dA(w_1, w_2) + [A(g,\eta)w_1, A(g,\eta)w_2] \\ &= dA_0(v_1, v_2) + g^{-1}dA_1(\hat{\eta}_1, \hat{\eta}_2)g \\ &\quad + [g^{-1}A_1(\hat{\eta}_2)g, g^{-1}v_1] - [g^{-1}A_1(\hat{\eta}_1)g, g^{-1}v_2] \\ &\quad + [g^{-1}v_1 + g^{-1}(A_1(\eta)\hat{\eta}_1)g, g^{-1}v_2 + g^{-1}(A_1(\eta)\hat{\eta}_2)g] \\ &= F_{A_0}(v_1, v_2) + g^{-1}F_{A_1}(\hat{\eta}_1, \hat{\eta}_2)g \\ &= 0. \end{aligned}$$

For the *(Monodromy)* axiom it suffices to consider curves based at 1. It is obviously satisfied for curves in G and hence follows from the fact that the connection $A$ is flat and that every based curve in $\mathrm{G}^c$ is homotopic to one in G. The *(Parallel)* and *(Complete)* axioms follow from the inequality

$$|\hat{\eta}| \leq |\mathrm{Im}(A_1(\eta)\hat{\eta})|\,. \qquad (13)$$

This inequality shows that the linear map $A(g,\eta) : T_{(g,\eta)}\mathrm{G}^c \to \mathfrak{g}^c$ is invertible for every pair $(g,\eta) \in \mathrm{G}\times\mathfrak{g}$. It also shows that, for every curve $\zeta : \mathbb{R} \to \mathfrak{g}^c$, the solutions $[0,T] \to \mathrm{G}^c : t \mapsto (g(t), \eta(t))$ of the differential equation

$$g(t)^{-1}\dot{g}(t) + g(t)^{-1}(A_1(\eta(t))\dot{\eta}(t))g(t) = \zeta(t)$$

satisfy $\sup_{0\leq t\leq T}|\eta(t) - \eta(0)| \leq cT$, where $c := \sup_{0\leq t\leq T}|\mathrm{Im}\zeta(t)|$. Hence the solutions must exist for all time. To prove (13) consider the imaginary part $\xi := \mathrm{Im}(\zeta)$ of a solution $\zeta : [0,1] \to \mathfrak{g}^c$ of equation (12). It satisfies the second order differential equation

$$\ddot{\xi} + [\eta, [\eta, \xi]] = 0, \qquad \xi(0) = 0, \qquad \dot{\xi}(0) = \hat{\eta}.$$

By Lemma 3.6 below every solution of this equation satisfies $|\xi(1)| \geq |\hat{\eta}|$ and this is equivalent to (13). Thus we have proved (ii).

We prove (iii). The group operation on $\mathrm{G}\times\mathfrak{g}$ associated to $A$ satisfies

$$[\xi, g\eta g^{-1}] = 0 \qquad \Longrightarrow \qquad (g,\xi)\cdot(h,\eta) = (gh, \xi + g\eta g^{-1}) \qquad (14)$$

for all $g, h \in \mathrm{G}$ and $\xi, \eta \in \mathfrak{g}$. To see this choose a path $\alpha : [0,1] \to \mathrm{G}$ with $\alpha(0) = 1$ and $\alpha(1) = h$ and define $\beta, \gamma : [0,1] \to \mathrm{G}\times\mathfrak{g}$ by

$$\beta(t) := (\alpha(t), t\eta), \qquad \gamma(t) := (g\alpha(t), \xi + tg\eta g^{-1}).$$

Then $\gamma(0) = (g, \xi)$ and, inspecting the definition of $A$, we find.

$$A(\beta)\dot{\beta} = \alpha^{-1}\dot{\alpha} + i\alpha^{-1}\eta\alpha = A(\gamma)\dot{\gamma}.$$

Hence $(g, \xi) \cdot (h, \eta) = \gamma(1) = (gh, \xi + g\eta g^{-1})$ as claimed. Thus we have proved (14). The formula shows that $\mathrm{G} \cong \mathrm{G} \times \{0\}$ is a Lie subgroup of $\mathrm{G}^c$. We prove that it is maximal compact. Suppose that $\mathrm{H} \subset \mathrm{G}^c$ is a subgroup such that $\mathrm{G} \subsetneq \mathrm{H}$. Then H contains an element of the form $(g, \xi)$ with $\xi \neq 0$. Hence, by (14), the pair $(1, \xi) = (g, \xi) \cdot (g^{-1}, 0)$ is also an element of H and hence, so is $(1, k\xi)$ for every integer $k \geq 1$. This sequence has no convergent subsequence and so H is not compact. This proves the theorem. □

**Lemma 3.6.** *Let $\mathfrak{g}$ be a Lie algebra with an inner product $\langle \cdot, \cdot \rangle$ such that*

$$\langle \xi, [\eta, \zeta] \rangle = \langle [\xi, \eta], \zeta \rangle$$

*for all $\xi, \eta, \zeta \in \mathfrak{g}$. Fix an element $\eta \in \mathfrak{g}$ and let $\xi : \mathbb{R} \to \mathfrak{g}$ be a solution of the second order differential equation*

$$\ddot{\xi} + [\eta, [\eta, \xi]] = 0, \qquad \xi(0) = 0. \tag{15}$$

*Then $|\xi(t)| \geq |t|\,|\dot{\xi}(0)|$ for every $t \in \mathbb{R}$.*

*Proof.* We have

$$\frac{d}{dt}\left(|\dot{\xi}|^2 - |[\xi, \eta]|^2\right) = 2\langle \dot{\xi}, \ddot{\xi} \rangle + 2\langle [\dot{\xi}, \eta], [\eta, \xi] \rangle = 0.$$

Since $\xi(0) = 0$ this implies

$$|\dot{\xi}(t)|^2 = |\dot{\xi}(0)|^2 + |[\xi(t), \eta]|^2 \geq |\dot{\xi}(0)|^2$$

for all $t \in \mathbb{R}$. Moreover, it follows from (15), by taking the inner product with $\xi$ and integrating by parts, that

$$\begin{aligned} 0 &= \int_0^t \langle \xi(s), \ddot{\xi}(s) + [\eta, [\eta, \xi(s)]] \rangle \, ds \\ &= \langle \xi(t), \dot{\xi}(t) \rangle - \int_0^t \left( |\dot{\xi}(s)|^2 + |[\xi(s), \eta]|^2 \right) ds \\ &\leq \langle \xi(t), \dot{\xi}(t) \rangle - t|\dot{\xi}(0)|^2. \end{aligned}$$

The last inequality holds for $t \geq 0$. Hence

$$|\xi(t)|^2 = 2\int_0^t \langle \xi(s), \dot{\xi}(s) \rangle \, ds \geq 2\int_0^t s|\dot{\xi}(0)|^2 \, ds = t^2|\dot{\xi}(0)|^2$$

for $t \geq 0$. Since equation (15) is time reversible, this proves the lemma. □

# 4 Hadamard's theorem

**Theorem 4.1** (Hopf-Rinow). *Let $M$ be a connected Riemannian manifold and denote by $d : M \times M \to [0, \infty)$ the distance function associated to the Riemannian metric. Fix a point $p_0 \in M$. Then the following are equivalent.*

**(i)** *The geodesics starting at $p_0$ exist for all time.*

**(ii)** *For every $p_1 \in M$ there exists a geodesic $\gamma : [0, 1] \to M$ such that*

$$\gamma(0) = p_0, \qquad \gamma(1) = p_1, \qquad L(\gamma) := \int_0^1 |\dot\gamma(t)|\, dt = d(p_0, p_1).$$

**(iii)** *Every closed and bounded subset of $(M, d)$ is compact.*

**(iv)** *$(M, d)$ is a complete metric space.*

The implications $(ii) \implies (iii) \implies (iv) \implies (i)$ are quite easy to establish. The hard part is to prove that (i) implies (ii). A connected Rimannian manifold satisfying the conditions of Theorem 4.1 is called **complete**. In such a manifold any two points can be connected by a (minimal) geodesic. If, in addition, $M$ is simply connected and has nonpositive sectional curvature, then any two points can be connected by a unique geodesic.

**Theorem 4.2** (Hadamard). *Let $M$ be a complete, connected, simply connected Riemannian manifold with nonpositive sectional curvature. Then, for every $p \in M$, the exponential map $\exp_p : T_pM \to M$ is a diffeomorphism.*

*Proof.* (Explained to me by Urs Lang.) There are three steps. The first step asserts that there are no conjugate points. We denote by $\nabla$ the Levi-Civita connection and by $R \in \Omega^2(M, \mathrm{End}(TM))$ the Riemann curvature tensor.

**Step 1.** *If $\gamma : [0, 1] \to M$ is a smooth curve and $X : [0, 1] \to TM$ is a vector field along $\gamma$ (i.e. $X(t) \in T_{\gamma(t)}M$ for all $t$) satisfying the* **Jacobi equation**

$$\nabla_t \nabla_t X + R(X, \dot\gamma)\dot\gamma = 0 \tag{16}$$

*and the boundary conditions $X(0) = 0$, $X(1) = 0$ then $X \equiv 0$.*

We have

$$\frac{d}{dt} \langle \nabla_t X, X \rangle = |\nabla_t X|^2 + \langle \nabla_t \nabla_t X, X \rangle = |\nabla_t X|^2 - \langle R(X, \dot\gamma)\dot\gamma, X \rangle$$

and hence

$$\int_0^1 \left( |\nabla_t X|^2 - \langle R(X, \dot\gamma)\dot\gamma, X \rangle \right) dt = 0.$$

Since $\langle R(X, \dot\gamma)\dot\gamma, X \rangle \le 0$ everywhere, we obtain $\nabla_t X \equiv 0$ and hence $X \equiv 0$.

**Step 2.** *The differential* $d\exp_p(v) : T_pM \to T_{\exp_p(v)}M$ *of the exponential map is bijective for every* $v \in T_pM$.

Let $\hat{v} \in T_pM$ be a tangent vector such that $d\exp_p(v)\hat{v} = 0$. Define the map $\gamma : \mathbb{R}^2 \to M$ and the vector field $X : \mathbb{R}^2 \to TM$ along $\gamma$ by

$$\gamma(s,t) := \exp_p(t(v + s\hat{v})), \qquad X(s,t) := \partial_s\gamma(s,t) = d\exp_p(t(v + s\hat{v}))t\hat{v}.$$

Then

$$\begin{aligned}
\nabla_t\nabla_t X &= \nabla_t\nabla_t\partial_s\gamma \\
&= \nabla_t\nabla_s\partial_t\gamma \\
&= \nabla_s\nabla_t\partial_t\gamma - R(\partial_s\gamma, \partial_t\gamma)\partial_t\gamma \\
&= -R(X, \partial_t\gamma)\partial_t\gamma.
\end{aligned}$$

Here the second equation follows from the fact that the Levi-Civita connection is torsion free, the third equation follows from the definition of the Riemann curvature tensor, and the last equation from the fact that the curve $t \mapsto \gamma(s,t)$ is a geodesic for every $s$. Since $X(0,0) = 0$ and $X(0,1) = 0$, by assumption, it follows from Step 1 that $X(0,t) = 0$ for all $t$. By choosing $t$ small we find that $\hat{v} = 0$.

**Step 3.** *The exponential map* $\exp_p : T_pM \to M$ *is a covering, i.e. it is surjective and, for every continuous path* $\gamma : [0,1] \to M$ *and every* $v_0 \in T_pM$ *with* $\gamma(0) = \exp_p(v_0)$ *there is a unique continuous path* $v : [0,1] \to T_pM$ *such that* $v(0) = v_0$ *and* $\gamma(t) = \exp_p(v(t))$ *for every* $t$.

That the map $\exp_p : T_pM \to M$ is surjective follows immediately from the Hopf-Rinow theorem. By Step 2 we may consider the space $T_pM$ with the pullback metric under the map $\exp_p$. Thus $\exp_p$ is a local isometry for this metric and so the rays $t \mapsto tv$ are geodesics in $T_pM$ for this metric (because they are mapped to geodesics in $M$ under $\exp_p$). Now we can apply the Hopf-Rinow theorem again to the pullback metric and obtain that it is complete (use the implication $(i) \implies (iv)$ in Theorem 4.1). This implies the covering property by a standard open and closed argument (given $\gamma$, let $I \subset [0,1]$ be the set of all $t$ such that the lift exists on the interval $[0,1]$. Then $I$ is obvious nonempty and open. That $I$ is closed follows from completeness of $T_pM$ with the pullback metric). This proves Step 3.

By Step 3, the map $\exp_p : T_pM \to M$ is a universal covering of $M$. Since $M$ is simply connected, this implies that $\exp_p$ is a diffeomorphism. This proves the theorem. □

# 5 Proof of the main theorem

**Lemma 5.1.** *Let* $\mathrm{G}^c$ *be a complex Lie group with Lie algebra* $\mathfrak{g}^c$ *and* $\mathrm{G} \subset \mathrm{G}^c$ *be a maximal compact Lie subgroup such that* $\mathfrak{g} := \mathrm{Lie}(\mathrm{G})$ *is a totally real subspace of* $\mathfrak{g}^c$*, i.e.* $\mathfrak{g}^c = \mathfrak{g} \oplus i\mathfrak{g}$*. Then the map*

$$\mathrm{G} \times \mathfrak{g} \to \mathrm{G}^c : (g, \eta) \mapsto \exp(i\eta)g$$

*is a diffeomorphism. In particular the quotient space* $\mathrm{G}^c/\mathrm{G}$ *is simply connected.*

The proof of Lemma 5.1 relies on Hadamard's theorem for the quotient space of $\mathrm{G}^c$ by the right action of G. We denote this space by

$$\mathrm{G}^c/\mathrm{G} := \{[k] \mid k \in \mathrm{G}^c\}, \qquad [k] := k\mathrm{G} = \{kg \mid g \in \mathrm{G}\}.$$

The tangent space of $\mathrm{G}^c/\mathrm{G}$ at $[k]$ is the quotient of the tangent spaces

$$T_{[k]}\mathrm{G}^c/\mathrm{G} = \frac{T_k\mathrm{G}^c}{T_k k\mathrm{G}} = \frac{T_k\mathrm{G}^c}{\{k\xi \mid \xi \in \mathfrak{g}\}}.$$

Throughout we use the notation $\mathrm{Re}(\zeta) = \xi$ and $\mathrm{Im}(\zeta) = \eta$ for $\zeta = \xi + i\eta \in \mathfrak{g}^c$ with $\xi, \eta \in \mathfrak{g}$. Thus the equivalence class of a tangent vector $[\zeta] \in T_{[k]}\mathrm{G}^c/\mathrm{G}$ is uniquely determined by $\mathrm{Im}(\zeta)$. Now choose an invariant inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on G and define a Riemannian metric on $\mathrm{G}^c/\mathrm{G}$ by

$$\langle [k\zeta], [k\zeta'] \rangle := \langle \eta, \eta' \rangle_{\mathfrak{g}}, \qquad \zeta, \zeta' \in \mathfrak{g}, \qquad \eta := \mathrm{Im}(\zeta), \qquad \eta' := \mathrm{Im}(\zeta').$$

It is convenient to leave out the square bracket when writing $[k\zeta]$ with $\zeta \in i\mathfrak{g}$. Thus we write $ki\xi \in T_{[k]}\mathrm{G}^c/\mathrm{G}$ insetad of $[ki\xi]$. In particular, we use this notation to avoid any possible confusion with the Lie bracket.

**Lemma 5.2.** *Let* $\mathrm{G}^c$ *and* G *be as in Lemma 5.1 and choose a Riemannian metric on the quotient* $\mathrm{G}^c/\mathrm{G}$ *as above. Then the following holds.*

**(i)** *The geodesics in* $\mathrm{G}^c/\mathrm{G}$ *have the form*

$$\gamma(t) = [k_0 \exp(it\eta)]$$

*for* $k_0 \in \mathrm{G}^c$ *and* $\eta \in \mathfrak{g}$.

**(ii)** *The Riemann curvature tensor on* $\mathrm{G}^c/\mathrm{G}$ *is given by*

$$R(ki\xi, ki\eta)ki\zeta = ki[[\xi, \eta], \zeta]$$

*for* $k \in \mathrm{G}^c$ *and* $\xi, \eta, \zeta \in \mathfrak{g}$*. In particular,* $\mathrm{G}^c/\mathrm{G}$ *has nonpositive sectional curvature.*

*Proof.* The proof has three steps. The first step gives a formula for the Levi–Civita connection on $\mathrm{G}^c/\mathrm{G}$.

**Step 1.** *Let $k:\mathbb{R}\to\mathrm{G}^c$ and $\xi:\mathbb{R}\to\mathfrak{g}^c$ be smooth curves and denote*

$$\gamma(t) := [k(t)] \in \mathrm{G}^c, \qquad X(t) := [k(t)\xi(t)] \in T_{\gamma(t)}\mathrm{G}^c/\mathrm{G}.$$

*Then*

$$\nabla_t X(t) = [k(t)\eta(t)], \qquad \eta(t) := \dot{\xi}(t) + [\mathrm{Re}(k(t)^{-1}\dot{k}(t), \xi(t)].$$

To prove that the formula is well defined we must choose a smooth map $g:\mathbb{R}\to\mathrm{G}$ and replace $k$, $\xi$, $\eta$ by

$$\tilde{k} := kg, \qquad \tilde{\xi} := g^{-1}\xi g, \qquad \partial_t\tilde{\xi} + [\mathrm{Re}(\tilde{k}^{-1}\partial_t\tilde{k}), \tilde{\xi}]$$

and show that $\tilde{\eta} = g\eta g^{-1}$. We must then show that the connection is Riemannian, i.e.

$$\partial_t \langle X, Y\rangle = \langle \nabla_t X, Y\rangle + \langle X, \nabla_t Y\rangle$$

for ant two vector fields along a curve $\gamma$, and that it is torsion free, i.e.

$$\nabla_s\partial_t\gamma = \nabla_t\partial_s\gamma$$

for any smooth map $\gamma:\mathbb{R}^2\to\mathrm{G}^c/\mathrm{G}$ of two variables. This follows easily by a direct calculation which is left to the reader.

**Step 2.** *We prove (i).*

A smooth curve $\gamma(t) = [k(t)]$ is a geodesic in $\mathrm{G}^c/\mathrm{G}$ if and only if $\nabla_t\dot{\gamma}\equiv 0$. By Step 1 this is equivalent to the differential equation

$$\partial_t\mathrm{Im}(k^{-1}\partial_t k) + [\mathrm{Re}(k^{-1}\partial_t k), \mathrm{Im}(k^{-1}\partial_t k)] = 0. \qquad (17)$$

A function $k:\mathbb{R}\to\mathrm{G}^c$ satisfies this equation if and only if it has the form $k(t) = k_0\exp(it\eta)g(t)$ for some $k_0\in\mathrm{G}^c$, $\eta\in\mathfrak{g}$, and $g:\mathbb{R}\to\mathrm{G}$.

**Step 3.** *We prove (ii).*

Choose maps $\gamma:\mathbb{R}^2\to\mathrm{G}^c$ and $\zeta:\mathbb{R}^2\to\mathfrak{g}^c$ and denote $\xi := k^{-1}\partial_s k$, $\eta := k^{-1}\partial_t$, and

$$\gamma := [k], \qquad X := [k\xi] = \partial_s\gamma, \qquad Y := [k\eta] = \partial_t\gamma, \qquad Z := [k\zeta].$$

Then $\partial_s\eta - \partial_t\xi + [\xi,\eta] = 0$ and

$$\begin{aligned}\nabla_s Z = [k\zeta_s], &\qquad \zeta_s := \partial_s\zeta + [\mathrm{Re}(\xi),\zeta],\\ \nabla_t Z = [k\zeta_t], &\qquad \zeta_t := \partial_t\zeta + [\mathrm{Re}(\eta),\zeta].\end{aligned}$$

Hence we obtain

$$R(X,Y)Z = \nabla_s\nabla_t Z - \nabla_t\nabla_s Z = [k\rho],$$

where

$$\begin{aligned}\rho &= \partial_s\zeta_t + [\mathrm{Re}(\xi),\zeta_t] - \partial_t\zeta_s - [\mathrm{Re}(\eta),\zeta_s]\\ &= [\mathrm{Re}(\partial_s\eta),\zeta] + [\mathrm{Re}(\xi),[\mathrm{Re}(\eta),\zeta]]\\ &\quad - [\mathrm{Re}(\partial_t\xi),\zeta] - [\mathrm{Re}(\eta),[\mathrm{Re}(\xi),\zeta]]\\ &= -[\mathrm{Re}([\xi,\eta]),\zeta] - [\zeta,[\mathrm{Re}(\xi),\mathrm{Re}(\eta)]]\\ &= [[\mathrm{Im}(\xi),\mathrm{Im}(\eta)],\zeta].\end{aligned}$$

Thus we have $R(X,Y)Z = ki\mathrm{Im}(\rho) = ki[[\mathrm{Im}(\xi),\mathrm{Im}(\eta)],\mathrm{Im}(\zeta)]$ and the sectional curvature is $\langle R(X,Y)Y,X\rangle = -\left|[\mathrm{Im}(\xi),\mathrm{Im}(\eta)]\right|^2 \le 0$. This proves the lemma. □

*Proof of Lemma 5.1.* We wish to prove that, under our assumptions, the map $\mathrm{G}\times\mathfrak{g}\to\mathrm{G}^c : (g,\eta)\mapsto \exp(i\eta)g$ is a diffeomorphism. The proof has four steps.

**Step 1.** *If $\eta\in\mathfrak{g}$ and $\exp(i\eta)\in\mathrm{G}$ then $[\xi,\eta]=0$ for every $\xi\in\mathrm{G}$.*

Define $\gamma:\mathbb{R}^2\to\mathrm{G}^c/\mathrm{G}$ by

$$\gamma(s,t) := [\exp(is\xi)\exp(it\eta)].$$

By Lemma 5.2 the curve $t\mapsto\gamma(s,t)$ is a geodesic for every $s$, and by assumption it is periodic with period 1. Denote

$$X(s,t) := \partial_s\gamma(s,t)\in T_{\gamma(s,t)}\mathrm{G}^c/\mathrm{G}.$$

Since $t\mapsto\gamma(s,t)$ is a geodesic for every $s$ we have that $X$ satisfies the Jacobi equation (16) Since $X(s,t+1) = X(s,t)$ we may obtain as in the proof of Theorem 4.2 that $X$ satisfies

$$\int_0^1 \left(|\nabla_t X|^2 - \langle R(X,\partial_t\gamma)\partial_t\gamma, X\rangle\right)\,dt = 0.$$

Since $G^c/G$ has nonpositive sectional curvature, by Lemma 5.2, we deduce that the function $\langle R(X, \partial_t\gamma)\partial_t\gamma, X\rangle$ vanishes identically. With $s = t = 0$ we have $X(0,0) = [i\xi]$ and $\partial_t\gamma(0,0) = [i\eta]$ and hence

$$0 = \langle R(i\xi, i\eta)i\eta, i\xi\rangle = |[\xi,\eta]|^2 .$$

This proves Step 1.

**Step 2.** *If* $\eta \in \mathfrak{g}$ *and* $\exp(i\eta) \in G$ *then* $\eta = 0$.

This is the only place in the proof where we use the fact that G is a maximal compact subgroup of $G^c$. Suppose by contradiction that $\eta \neq 0$. Then $\exp(it\eta) \notin G$ for small $t$ and hence

$$0 < \lambda := \inf\{t > 0 \,|\, \exp(it\eta) \in G\} \leq 1.$$

By Step 1 we have $[\xi, \eta] = 0$ for every $\xi \in \mathfrak{g}$ and, since G is connected, this implies $g^{-1}\eta g = \eta$ for every $g \in G$. Since $G^c$ is a complex Lie group we obtain $g^{-1}i\eta g = i\eta$ for every $g \in G$ and hence

$$\exp(it\eta)g = g\exp(it\eta)$$

for every $t \in \mathbb{R}$. In particular, this holds for $t = \lambda/2$ and so the element

$$h := \exp(i\lambda\eta/2) \in G^c \setminus G$$

commutes with every element of G. Hence $H := G \cup hG$ is a compact subgroup of $G^c$, contradicting our assumption. This proves Step 2.

**Step 3.** $G^c/G$ *is simply connected.*

Suppose not. Then, by the usual variational argument, we can find a nonconstant geodesic $\gamma : [0,1] \to G^c/G$ based at $\gamma(0) = \gamma(1) = [1]$. By Lemma 5.2 the geodesic has the form $\gamma(t) = \exp(it\eta)$ for some $\eta \in \mathfrak{g}$. Since $\gamma(1) = [1]$ we have $\exp(i\eta) \in G$ and hence, by Step 2, $\eta = 0$. Thus the geodesic is constant after all, a contradiction. This proves Step 3.

**Step 4.** *The map* $G \times \mathfrak{g} \to G^c : (g, \eta) \mapsto \exp(i\eta)g$ *is a diffeomorphism.*

This follows from Hadamard's theorem. Namely the exponential map (the Riemannian and Lie group meanings of the term coincide in this case)

$$T_{[1]}G^c/G \to G^c/G : [i\eta] \mapsto [\exp(i\eta)]$$

is a diffeomorphism by Theorem 4.2, and this is equivalent to Step 4. This proves the lemma. □

*Proof of Theorems 1.4 and 1.6.* Let G be a compact connected Lie group. By Theorem 3.5, there is an embedding $\iota : \mathrm{G} \to \mathrm{G}^c$ into a complex connected Lie group (diffeomorphic to $\mathrm{G} \times \mathfrak{g}$) that satisfies condition (ii) in Theorem 1.4. We shall prove below that (ii) implies (i) in Theorem 1.4. When this is established, we know that the embedding $\iota : \mathrm{G} \to \mathrm{G}^c$ constructed in Theorem 3.5 satisfies both (i) and (ii) in Theorem 1.4 and hence is a complexification, which proves Theorem 1.6. Secondly, any two embeddings of G into a complex Lie group that satisfy (i) in Theorem 1.4 are isomorphic by definition. And since we know that at least one of them also satisfies (ii) it follows that they all do. Thus it remains to prove that (ii) implies (i) in Theorem 1.4.

Suppose that $\mathrm{G}^c$ is complex connected Lie group that contains G as a maximal compact subgroup and such that its Lie algebra $\mathfrak{g}^c := \mathrm{Lie}(\mathrm{G}^c)$ is equal to $\mathfrak{g}^c = \mathfrak{g} \oplus i\mathfrak{g}$, where $\mathfrak{g} := \mathrm{Lie}(\mathrm{G}) \subset \mathfrak{g}^c$. (In the notation of Theorem 1.4 the map $\iota$ is the inclusion of G into $\mathrm{G}^c$.) Let H be a complex Lie group with Lie algebra $\mathfrak{h} := \mathrm{Lie}(\mathrm{H})$ and $\rho : \mathrm{G} \to \mathrm{H}$ be a Lie group homomorphism. We use the following two basic facts to construct the homomorphism $\rho^c : \mathrm{G}^c \to \mathrm{H}$ that extends $\rho$.

**Fact 1.** *Since* $\mathrm{G}^c$ *is connected there exists, for every* $a \in \mathrm{G}^c$*, a smooth path* $\alpha : [0,1] \to \mathrm{G}^c$ *such that* $\alpha(0) \in \mathrm{G}$ *and* $\alpha(1) = a$*.*

**Fact 2.** *Since* G *is connected and* $\mathrm{G}^c/\mathrm{G}$ *is simply connected, any two smooth paths* $\alpha_0, \alpha_1 : [0,1] \to \mathrm{G}^c$ *as in Fact 1 can be connected by a smooth homotopy* $\{\alpha_s\}_{0 \le s \le 1}$ *satisfying the same conditions (with fixed endpoint* $\alpha_\lambda(1) = a$*).*

We define $\rho^c$ as follows. Let $\Phi := d\rho(1) : \mathfrak{g} \to \mathfrak{h}$ be the induced Lie algebra homomorphism and define

$$\Phi^c : \mathfrak{g}^c \to \mathfrak{h}$$

as the complexification of $\Phi$. Given an element $a \in \mathrm{G}^c$ choose $\alpha$ as in Fact 1. Then define $\beta : [0,1] \to \mathrm{H}$ as the unique solution of the differential equation

$$\beta^{-1}\dot{\beta} = \Phi^c(\alpha^{-1}\dot{\alpha}), \qquad \beta(0) = \rho(\alpha(0)). \tag{18}$$

Define

$$\rho^c(a) := \beta(1).$$

We prove first that $\rho^c$ is well defined, i.e. that the endpoint $\beta(1)$ does not depend on the choice of the path $\alpha$. By Fact 2 any two paths $\alpha_0$ and $\alpha_1$ can be connected by a smooth homotopy

$$[0,1]^2 \to \mathrm{G}^c : (s,t) \mapsto \alpha_s(t) = \alpha(s,t).$$

Define $\beta : [0,1] \to \mathrm{H}$ by

$$\beta^{-1}\partial_t\beta = \Phi^c(\alpha^{-1}\partial_t\alpha), \qquad \beta(s,0) = \rho(\alpha(s,0)).$$

We claim that

$$\beta^{-1}\partial_s\beta = \Phi^c(\alpha^{-1}\partial_s\alpha) \tag{19}$$

and hence, in particular, $\partial_s\beta(s,1) = 0$. To prove this we denote

$$\xi := \alpha^{-1}\partial_s\alpha, \qquad \eta := \alpha^{-1}\partial_t\alpha, \qquad \xi' := \beta^{-1}\partial_s\beta, \qquad \eta' := \beta^{-1}\partial_t\beta.$$

Then

$$\partial_t\xi' = \partial_s\eta' + [\xi', \eta'], \qquad \partial_t\Phi^c(\xi) = \partial_s\Phi^c(\eta) + [\Phi^c(\xi), \Phi^c(\eta)].$$

Moreover, when $t = 0$ we have $d\rho(\alpha)\alpha\xi = \rho(\alpha)\Phi\xi$ and hence

$$\xi'(s,0) = \beta(s,0)^{-1}\partial_s\beta(s,0) = \Phi(\alpha(s,0)^{-1}\partial_s\alpha(s,0)) = \Phi(\xi(s,0)).$$

Hence both functions $t \mapsto \xi'(s,t)$ and $t \mapsto \Phi^c(\xi(s,t)$ satisfy the same initial value problem and hence agree. This proves that $\rho^c$ is well defined.

Next we prove that $\rho^c$ is a group homomorphism. This is done in two steps. First one proves the identity

$$\Phi^c(a^{-1}\xi a) = \rho^c(a)^{-1}\Phi^c(\xi)\rho^c(a) \tag{20}$$

for $a \in \mathrm{G}^c$ and $\xi \in \mathfrak{g}^c$. This follows by choosing $\alpha$ and $\beta$ as in the definition of $\rho^c(a)$ and then introducing the functions

$$\eta(t) := \alpha(t)^{-1}\xi\alpha(t), \qquad \eta'(t) := \beta(t)^{-1}\Phi^c(\xi)\beta(t).$$

Then one shows again that $\Phi^c(\eta)$ and $\eta'$ satsify the same initial value problem $\dot{\eta}' + [\beta^{-1}\dot{\beta}, \eta'] = 0$ and hence have the same endpoints. The second step is to prove that

$$\rho^c(a_1a_2) = \rho^c(a_1)\rho^c(a_2) \tag{21}$$

for $a_1, a_2 \in \mathrm{G}^c$. This follows by choosing $\alpha_j$ and $\beta_j$ as in the definition of $\rho^c(a_j)$ for $j = 1, 2$. Then define $\alpha := \alpha_1\alpha_2$ and $\beta := \beta_1\beta_2$. To verify that $\beta^{-1}\dot{\beta} = \Phi^c(\alpha^{-1}\dot{\alpha})$ one then needs equation (20). This proves that $\rho^c$ is a group homomorphism.

That $\rho^c$ is smooth follows from the commutative diagram

$$\begin{array}{ccc} \mathrm{G}^c & \xrightarrow{\rho^c} & \mathrm{H}\ , \\ \approx\uparrow & \nearrow & \\ \mathrm{G}\times\mathfrak{g} & & \end{array}$$

where the map $\mathrm{G}\times\mathfrak{g} \to \mathrm{G}^c$ is the diffeomorphism of Lemma 5.1 and the map $\mathrm{G}\times\mathfrak{g} \to \mathrm{H}$ is given by $(g,\eta) \mapsto \exp(i\Phi^c(\eta))\rho(g)$ and hence is smooth. That the differential of $\rho^c$ at 1 is equal to $\Phi^c$ follows also from this diagram. This proves the theorem. $\square$