Corrigendum:
Self-dual instantons and holomorphic curves

Stamatis Dostoglou
University of Missouri-Columbia

Dietmar A. Salamon
ETH–Zürich

7 September 2006

Abstract

We correct two mistakes in [1]. The first concerns the exponential decay in the proof of Theorem 7.4 and the second concerns the bubbling argument in the proof of Theorem 9.1.

1 Exponential decay

For Theorem 7.1: Replace the hypothesis $\|B_t\|_{L^\infty(\Omega\times\Sigma)} + \varepsilon \|C\|_{L^\infty(\Omega\times\Sigma)}$ on page 615 by the weaker assumption

$$\sup_{(s,t)\in\Omega} \|B_t(s,t)\|_{L^2(\Sigma)} + \varepsilon \sup_{(s,t)\in\Omega} \|C(s,t)\|_{L^2(\Sigma)} \le c_0. \tag{1}$$

All the estimates in the proof of Theorem 7.1 continue to hold under this assumption. To see this, use the inclusion $W^{1,2}(\Sigma) \hookrightarrow L^4(\Sigma)$ to obtain inequalities of the form

$$\|B_t\|_{L^4(\Sigma)} \|C\|_{L^4(\Sigma)} \le c\sqrt{u_0 v_0}, \qquad \|B_t\|^2_{L^4(\Sigma)} \le v_0 + c u_0,$$

where u_0, v_0 are as in the proof of Theorem 7.1.

Corollary 1.1. *Let $\Omega \subset \mathbb{C}$ be an open set and $K \subset \Omega$ be a compact subset. Then for every constant $c_0 > 0$, there exist constants $\varepsilon_0 > 0$ and $c > 0$ such that the following holds. If $0 < \varepsilon \le \varepsilon_0$ and $\Xi = A + \Phi\, ds + \Psi\, dt$ is a connection on $\Omega \times \Sigma$ that satisfies*

$$\begin{aligned} \partial_t A - \mathrm{d}_A \Psi + *_s(\partial_s A - \mathrm{d}_A \Phi - X_s(A)) &= 0, \\ \partial_t \Phi - \partial_s \Psi - [\Phi, \Psi] + \varepsilon^{-2} * F_A &= 0, \end{aligned} \tag{2}$$

and (1) then

$$\|B_t\|_{L^\infty(K\times\Sigma)} + \varepsilon \|C\|_{L^\infty(K\times\Sigma)} \le c \left(\|B_t\|_{L^2(\Omega\times\Sigma)} + \varepsilon \|C\|_{L^2(\Omega\times\Sigma)} \right).$$

Proof. By Theorem 7.1 (in the above strengthened form), the connection Ξ satisfies (7.4) in [1, page 615]. The assertion follows by taking $p = \infty$ and using [1, Lemma 7.6] with $p = 4$. $\square$

For Lemma 7.5: On page 620 replace the inequality (7.7) by

$$\begin{aligned} &\|\alpha\|^2 + \|\phi\|^2 + \|\psi\|^2 \\ &\quad \le c\Big(\|{*_s}\nabla_s\alpha - {*_s}\mathrm{d}X_s(A)\alpha - {*_s}\mathrm{d}_A\phi - \mathrm{d}_A\psi\|^2 \\ &\qquad + \varepsilon^2 \left\|\nabla_s\psi - \varepsilon^{-2}\mathrm{d}_A\alpha\right\|^2 + \varepsilon^2 \left\|\nabla_s *_s \phi + \varepsilon^{-2}\mathrm{d}_A *_s \alpha\right\|^2\Big). \end{aligned}$$

On page 621 replace the last two sentences in the proof of Lemma 7.5 by the following text.

Hence it follows from Lemma 7.3 and Lemma 7.4 in [10] that there exist constants $\varepsilon_0 > 0$, $\nu_0 \in \mathbb{N}$, and $c > 0$ such that the estimate (7.7) holds with $0 < \varepsilon \le \varepsilon_0$ and $A + \Phi\,\mathrm{d}s$ replaced by $A_\nu + \Phi_\nu\,\mathrm{d}s$ where $\nu \ge \nu_0$ (here the estimate for α follows from Lemma 7.4 and the estimate for ϕ and ψ from Lemma 7.3). With $\varepsilon = \varepsilon_\nu$ and $\nu > c$ this contradicts our assumption. $\square$

Proof of Theorem 7.4: The last displayed inequality on page 622 is correct as it stands, however its proof uses Corollary 1.1 above.

Replace the first displayed inequality on page 623 by

$$\|B_t\|^2 + \|C\|^2 \le c_3\Big(\|\nabla_s B_t - \mathrm{d}X_s(A)B_t - \mathrm{d}_A C\|^2 + \varepsilon^{-2}\|\mathrm{d}_A B_t\|\Big).$$

(The mistake in [1] is the factor ε^2 in front of $\|C\|^2$ in this inequality; it can be removed because of the improved inequality in Lemma 7.5.) Inspection of the formula for $f''(t)$ shows that this stronger estimate is needed to prove the inequality $f''(t) \ge \rho^2 f(t)$ for $t \ge 1$ (use the expression after the fourth equal sign in the formula for $f''(t)$ on page 622). $\square$

2 An a priori estimate

The following a priori estimate is an adaptation of [2, Lemma 9.1] to the present context. It is needed in the proof of Theorem 9.1.

Lemma 2.1. *There is a constant $\delta_0 > 0$ with the following significance. Let $\Omega \subset \mathbb{R}^2$ be an open set and $K \subset \Omega$ be a compact subset. Then, for every $c_0 > 0$ and every $p \ge 2$, there are positive constants ε_0 and c such that the following holds. If $0 < \varepsilon \le \varepsilon_0$ and the maps $A : \Omega \to \mathcal{A}(P)$ and $\Phi, \Psi : \Omega \to \Omega^0(\Sigma, \mathfrak{g}_P)$ satisfy (2) and*

$$\|\partial_t A - \mathrm{d}_A\Psi\|_{L^\infty(\Omega\times\Sigma)} \le c_0, \qquad \|F_A\|_{L^\infty(\Omega\times\Sigma)} \le \delta_0, \tag{3}$$

then

$$\int_K \Big(\|F_A\|^p_{L^2(\Sigma)} + \varepsilon^p \|\nabla_s F_A\|^p_{L^2(\Sigma)} + \varepsilon^p \|\nabla_t F_A\|^p_{L^2(\Sigma)}\Big) \le c\varepsilon^{2p}, \tag{4}$$

$$\sup_K \Big(\|F_A\|_{L^2(\Sigma)} + \varepsilon \|\nabla_s F_A\|_{L^2(\Sigma)} + \varepsilon \|\nabla_t F_A\|_{L^2(\Sigma)}\Big) \le c\varepsilon^{2-2/p}. \tag{5}$$

Proof. As in [1, Lemma 7.6] one can show that there exist constants $\delta_0 > 0$ and $c_1 > 0$ such that every $A \in \mathcal{A}(P)$ with $\|F_A\|_{L^\infty(\Sigma)} \le \delta_0$ satisfies the inequalities

$$\|\phi\| \le c_1 \|\mathrm{d}_A\phi\|,$$

$$\|\mathrm{d}_A (*_s\mathrm{d}X_s(A)\alpha + \dot{*}_s\alpha)\| \le c_1 (\|\alpha\| + \|\mathrm{d}_A\alpha\| + \|\mathrm{d}_A *_s \alpha\|)$$

for $s \in \mathbb{R}$, $\phi \in \Omega^0(\Sigma;\mathfrak{g}_P)$, and $\alpha \in \Omega^1(\Sigma;\mathfrak{g}_P)$. Here and in the following all norms are L^2-norms on Σ.

Now let A, Φ, Ψ satisy the hypotheses of the lemma and define

$$B_s := \partial_s A - \mathrm{d}_A\Phi, \quad B_t := \partial_t A - \mathrm{d}_A\Psi, \quad C := \partial_t\Phi - \partial_s\Psi - [\Phi,\Psi]. \tag{6}$$

Then the proof of [1, Theorem 7.1] shows that

$$\begin{aligned}\varepsilon^2 (\nabla_s\nabla_s C + \nabla_t\nabla_t C) &= \mathrm{d}_A^{*_s}\mathrm{d}_A C - 2*[B_t \wedge B_t] + *[*_s X_s(A) \wedge B_t] \\ &\quad - *\,\mathrm{d}_A (*_s\mathrm{d}X_s(A)B_t + \dot{*}_s B_t).\end{aligned}$$

Hence, with $\Delta := \partial^2/\partial s^2 + \partial^2/\partial t^2$ the standard Laplacian, we have

$$\begin{aligned}\Delta \|C\|^2 &= 2\|\nabla_s C\|^2 + 2\|\nabla_t C\|^2 + 2\langle \nabla_s\nabla_s C + \nabla_t\nabla_t C, C\rangle \\ &= 2\varepsilon^{-4}\|\mathrm{d}_A *_s B_t\|^2 + 2\varepsilon^{-4}\|\mathrm{d}_A B_t\|^2 + 2\varepsilon^{-2}\|\mathrm{d}_A C\|^2 \\ &\quad - 4\varepsilon^{-2}\langle C, *[B_t\wedge B_t]\rangle + 2\varepsilon^{-2}\langle C, *[*_s X_s(A)\wedge B_t]\rangle \\ &\quad - 2\varepsilon^{-2}\langle C, *\mathrm{d}_A (*_s\mathrm{d}X_s(A)B_t + \dot{*}_s B_t)\rangle \\ &\ge \frac{\delta}{\varepsilon^2}\|C\|^2 - \frac{c}{\varepsilon^2}\|C\|.\end{aligned}$$

The last inequality holds for $\varepsilon \le \varepsilon_0$, with ε_0 sufficiently small, and suitable positive constants δ and c, depending only on δ_0, c_0, and c_1 (as well as the metrics on Σ and the vector fields X_s). Since $2\Delta\|C\|^p \ge p\|C\|^{p-2}\Delta\|C\|^2$ for $p \ge 2$, this implies

$$\|C\|^p \le \frac{c}{\delta}\|C\|^{p-1} + \frac{2\varepsilon^2}{p\delta}\Delta\|C\|^p.$$

Using the inequality $ab \le a^p/p + b^q/q$ with $1/p + 1/q = 1$, $a := c/\delta$ and $b := \|C\|^{p-1}$ we obtain $b^q = \|C\|^p$, and hence

$$\|C\|^p \le \frac{c^p}{\delta^p} + \frac{2\varepsilon^2}{\delta}\Delta\|C\|^p. \tag{7}$$

By [2, Lemma 9.2], this implies that

$$\int_{B_R(z)} \|C\|^p \le \frac{\pi(R+r)^2 c^p}{\delta^p} + \frac{8\varepsilon^2}{r^2\delta}\int_{B_{R+r}(z)} \|C\|^p.$$

for every $z \in \mathbb{C}$ and every pair of positive real numbers R and r such that $B_{R+r}(z) \subset \Omega$. Now observe that $\varepsilon^2\|C\| = \|F_A\| \le \delta_0\mathrm{Vol}(\Sigma)$ and use the last inequality repeatedly, with R replaced by $R+r, R+2r, \ldots, R+(p-1)r$, to obtain the estimate $\int_{B_R(z)} \|C\|^p \le c_p$ for every $z \in \mathbb{C}$ such that $B_{R+pr}(z) \subset \Omega$. Now choose R and r such that $B_{R+pr}(z) \subset \Omega$ for every $z \in K$. Cover K by finitely many balls of radius R to obtain

$$\int_K \|F_A\|^p = \varepsilon^{2p} \int_K \|C\|^p \le c_{K,p}\varepsilon^{2p}. \tag{8}$$

It follows from (7) that the function $z \mapsto \|C(z)\|^p + c^p |z - z_0|^2 / 8\delta^{p-1}\varepsilon^2$ is subharmonic in Ω for every $z_0 \in \mathbb{C}$. Hence, by the mean value inequality and (8), we have

$$\sup_K \|F_A\| = \varepsilon^2 \sup_K \|C\| \le c_{K,p}\varepsilon^{2-2/p} \tag{9}$$

for a suitable constant $c_{K,p}$. It follows from (8) and (9) that every connection $\Xi = A + \Phi\, ds + \Psi\, dt$ on $\Omega \times P$ that satisfies (2) and (3) also satisfies (1) in every compact subset of Ω and hence, by Corollary 1.1, satisfies the hypotheses of [1, Theorem 7.1]. Hence it follows from [1, Theorem 7.1] with $p = \infty$ that, for every open set U with $\mathrm{cl}(U) \subset \Omega$, there is a constant c_U such that every conection Ξ on $\Omega \times P$ that satisfies (2) and (3) also satisfies the estimates

$$\begin{aligned} \varepsilon \|\nabla_s B_t\|_{L^\infty(U\times\Sigma)} + \varepsilon \|\nabla_t B_t\|_{L^\infty(U\times\Sigma)} &\le c_U, \\ \varepsilon \|C\|_{L^\infty(U\times\Sigma)} + \varepsilon^2 \|\nabla_s C\|_{L^\infty(U\times\Sigma)} + \varepsilon^2 \|\nabla_t C\|_{L^\infty(U\times\Sigma)} &\le c_U, \\ \|C\|_{L^2(U\times\Sigma)} + \varepsilon \|\nabla_s C\|_{L^2(U\times\Sigma)} + \varepsilon \|\nabla_t C\|_{L^2(U\times\Sigma)} &\le c_U. \end{aligned} \tag{10}$$

Note that the last inequality is equivalent to (4) for $p = 2$.

Now consider the function $u : U \to \mathbb{R}$ defined by

$$u(s,t)^2 := \frac{1}{2}\left(\|C(s,t)\|^2 + \varepsilon^2 \|\nabla_s C(s,t)\|^2 + \varepsilon^2 \|\nabla_t C(s,t)\|^2\right)$$

Again all norms are L^2-norms on Σ. In the following we shall assume, for simplicity, that the Hodge $*$-operator $*_s = *$ is independent of s and that $X_s = 0$ for all s. Then, as in the proof of [1, Theorem 7.1], we have

$$\begin{aligned} \Delta u^2 \;=\; & \varepsilon^{-2} \|\mathrm{d}_A C\|^2 + \|\nabla_s C\|^2 + \|\nabla_t C\|^2 + \|\mathrm{d}_A \nabla_s C\|^2 + \|\mathrm{d}_A \nabla_t C\|^2 \\ & + \varepsilon^2 \|\nabla_s \nabla_s C\|^2 + \varepsilon^2 \|\nabla_t \nabla_t C\|^2 + 2\varepsilon^2 \|\nabla_s \nabla_t C\|^2 \\ & - 2\varepsilon^2 \langle C, [\nabla_s C, \nabla_t C]\rangle - 2\varepsilon^{-2} \langle C, *[B_t \wedge B_t]\rangle \\ & - 4\langle \nabla_s C, *[B_t \wedge \nabla_s B_t]\rangle - 4\langle \nabla_t C, *[B_t \wedge \nabla_t B_t]\rangle \\ & + \langle \mathrm{d}_A \nabla_s C, [B_s, C]\rangle + \langle \mathrm{d}_A \nabla_t C, [B_t, C]\rangle \\ & - \langle \nabla_s C, *[B_s \wedge *\mathrm{d}_A C]\rangle - \langle \nabla_t C, *[B_t \wedge *\mathrm{d}_A C]\rangle. \end{aligned}$$

For ε sufficiently small it follows that

$$\Delta u^2 \ge \frac{\delta}{\varepsilon^2} u^2 - \frac{c}{\varepsilon^2} u$$

with suitable positive constants δ and c. To see this examine the last eight terms in the formula for Δu^2 and use (10). Now it follows as in (7) that

$$u^p \le \frac{c}{\delta} u^{p-1} + \frac{2\varepsilon^2}{p\delta} \Delta u^p$$

for $p \geq 2$. By (9) and (10), we have $u \leq c'/\varepsilon$ for some constant c'. Hence we can argue as above to show that, for every compact subset $K \subset U$, there is a constant $c_{K,p} > 0$ such that $\int_K u^p \leq c_{K,p}$ and $\sup_K u^p \leq c_{K,p}\varepsilon^{-2}$. This proves the lemma. □

3 Bubbling analysis

The assertion on page 634 that the limit connection Ξ_0 represents a **nonconstant** holomorphic sphere $S^2 \to \mathcal{M}(P)$ does not seem to follow from the argument in [1]. A modified bubbling argument does result in a nonconstant holomorphic sphere but only proves a weaker estimate, i.e. we must weaken the assertion of Theorem 9.1 and the assumption of Theorem 8.1. Then Theorem 9.2 remains valid.

For Theorem 8.1: The assertion of Theorem 8.1 in [1, page 623] continues to hold if the hypothesis (8.1) is replaced by the weaker inequality

$$\varepsilon^{-1} \left\|F_A\right\|_{L^\infty} + \left\|\partial_t A - \mathrm{d}_A\Psi\right\|_{L^\infty} \leq c_0 \tag{11}$$

To see this, replace the last inequality on page 625 by $\left\|C^\nu\right\|_{L^p} \leq c\varepsilon_\nu^{2/p-1}$ or, equivalently,

$$\left\|F_{A_\nu}\right\|_{L^p} \leq c\varepsilon_\nu^{1+2/p}.$$

For $p = 2$ this follows from the first inequality in Step 2 on page 625, for $p = \infty$ it holds by assumption, and for $2 \leq p \leq \infty$ it follows by interpolation. Now replace the constant ε_ν^2 by $\varepsilon_\nu^{1+2/p}$ in the following places.

- In the inequality (8.4) on page 626.
- Replace the inequality $\left\|A' - A\right\|_{L^p} \leq c_2\varepsilon^2$ by $\left\|A' - A\right\|_{L^p} \leq c_2\varepsilon^{1+2/p}$ in the middle of page 626.
- In the first two inequalities after (8.9), in the first inequality after (8.10), and in the first inequality in the proof of Step 5 (page 628).
- In the first inequality on page 629 and in the last inequality before (8.11).

The next lemma is a local version of Theorem 8.1; it is needed in the proof of Theorem 9.1. Let $\Omega_\nu \subset \mathbb{C}$ be an exhausting sequence of open sets and s_ν, $\varepsilon_\nu > 0$, $\delta_\nu > 0$ be seqences of real numbers such that $s_\nu \to s_0$, $\varepsilon_\nu \to 0$, $\delta_\nu \to 0$. Abbreviate $*_{\nu s} := *_{s_\nu+\delta_\nu s}$ and $X_{\nu s} := \delta_\nu X_{s_\nu+\delta_\nu s}$.

Lemma 3.1. *Let $\Xi_\nu = A_\nu + \Phi_\nu\,\mathrm{d}s + \Psi_\nu\,\mathrm{d}t$ be a sequence of solutions of the equation (2), with $(*_s, X_s)$ replaced by $(*_{\nu s}, X_{\nu s})$, on $\Omega_\nu \times P$ such that*

$$\sup_\nu \left(\varepsilon_\nu^{-1} \left\|F_{A_\nu}\right\|_{L^2(\Omega_\nu\times\Sigma)} + \left\|\partial_t A_\nu - \mathrm{d}_{A_\nu}\Psi_\nu\right\|_{L^2(\Omega_\nu\times\Sigma)}\right) < \infty, \tag{12}$$

$$\sup_\nu \left(\varepsilon_\nu^{-1} \left\|F_{A_\nu}\right\|_{L^\infty(\Omega_\nu\times\Sigma)} + \left\|\partial_t A_\nu - \mathrm{d}_{A_\nu}\Psi_\nu\right\|_{L^\infty(\Omega_\nu\times\Sigma)}\right) < \infty.$$

Then there is a subsequence, still denoted by Ξ_ν, a sequence of gauge transformations $g_\nu : \Omega_\nu \to \mathcal{G}(P)$, and a connection $\Xi_0 = A_0 + \Phi_0 \, \mathrm{d}s + \Psi_0 \, \mathrm{d}t$ on $\mathbb{C} \times P$ such that

$$\partial_t A_0 - \mathrm{d}_{A_0}\Psi_0 + *_{s_0}(\partial_s A_0 - \mathrm{d}_{A_0}\Phi_0) = 0, \qquad F_{A_0} = 0,$$

$$\lim_{\nu\to\infty}\left(\|g_\nu^* A_\nu - A_0\|_{L^\infty(K\times\Sigma)} + \sup_{(s,t)\in K} \left\|g_\nu^{-1} B_{\nu t} g_\nu - B_{0t}\right\|_{L^2(\Sigma)} \right) = 0$$

for every compact set $K \subset \mathbb{C}$; here $B_{\nu t} := \partial_t A_\nu - \mathrm{d}_{A_\nu}\Psi_\nu$, $B_{0t} := \partial_t A_0 - \mathrm{d}_{A_0}\Psi_0$.

Proof. For every compact set $K \subset \mathbb{C}$ there is a constant $\nu_K > 0$ such that, for every $(s,t) \in K$ and every $\nu \geq \nu_K$, there is a unique section $\eta_\nu(s,t) \in \Omega^0(\Sigma, \mathfrak{g}_P)$ such that

$$F_{A'_\nu} = 0, \qquad A'_\nu := A_\nu + *_{\nu s}\mathrm{d}_{A_\nu}\eta_\nu,$$

and

$$\|d_{A_\nu}\eta_\nu\|_{L^\infty(\Sigma)} \leq c_1 \|F_{A_\nu}\|_{L^\infty(\Sigma)} \leq c_2 \varepsilon_\nu \tag{13}$$

(see Lemma 8.2 in [1]). Choose $\Phi'_\nu(s,t), \Psi'_\nu(s,t) \in \Omega^0(\Sigma, \mathfrak{g}_P)$ such that

$$\mathrm{d}_{A'_\nu} *_{\nu s} \left(\partial_s A'_\nu - \mathrm{d}_{A'_\nu}\Phi'_\nu - X_{\nu s}(A'_\nu)\right) = \mathrm{d}_{A'_\nu} *_{\nu s} \left(\partial_t A'_\nu - \mathrm{d}_{A'_\nu}\Psi'_\nu\right) = 0.$$

Note that the sequence $\Xi'_\nu = A'_\nu + \Phi'_\nu \, \mathrm{d}s + \Psi'_\nu \, \mathrm{d}t$ depends only on ν and not on the compact set K in question. One proves exactly as in [1, pages 626–627] that the sequence Ξ'_ν satisfies the estimates

$$\begin{aligned} \|\Xi'_\nu - \Xi_\nu\|_{1,p,\varepsilon;K} &\leq c_{K,p}\varepsilon_\nu^{1+2/p}, &(14)\\ \|B'_{\nu t}\|_{L^\infty(K\times\Sigma)} &\leq c_K, &(15)\\ \|B'_{\nu t} + *_{\nu s}(B'_{\nu s} - X_{\nu s}(A'_\nu))\|_{L^p(K\times\Sigma)} &\leq c_{K,p}\varepsilon_\nu^{1+2/p}, &(16)\end{aligned}$$

for every compact set $K \subset \mathbb{C}$ and every $p \geq 2$, with suitable positive constants c_K and $c_{K,p}$. In addition we wish to prove the estimate

$$\sup_K \|B'_{\nu t} - B_{\nu t}\|_{L^2(\Sigma)} \leq c_K \sqrt{\varepsilon_\nu}. \tag{17}$$

To see this we use the identities

$$\begin{aligned} B'_t - B_t &= \mathrm{d}_{A'}(\Psi' - \Psi) + *_s\mathrm{d}_A\nabla_t\eta + *_s[B_t, \eta],\\ \mathrm{d}_A *_s \mathrm{d}_A(\Psi' - \Psi) &= \mathrm{d}_A *_s B_t - [\mathrm{d}_A B_t, \eta] - [F_A, \nabla_t\eta]\\ &\quad -[(A' - A)\wedge([\mathrm{d}_A\nabla_t\eta + [B_t, \eta]])] \qquad (18)\\ \mathrm{d}_A *_s \mathrm{d}_A\nabla_t\eta &= -\mathrm{d}_A B_t - [\mathrm{d}_A\nabla_t\eta \wedge \mathrm{d}_A\eta] - [[B_t, \eta]\wedge d_A\eta]\\ &\quad -2[B_t \wedge *_s\mathrm{d}_A\eta] - [\mathrm{d}_A *_s B_t, \eta]\end{aligned}$$

(see (8.5), (8.7), and (8.8) in [1]). Here we have dropped the subscript ν. Since

$$\mathrm{d}_A B_t = \nabla_t F_A, \qquad \mathrm{d}_A *_s B_t = \mathrm{d}_A B_s = \nabla_s F_A$$

we obtain from Lemma 2.1 with $p = 2$ that, for every compact set $K \subset \mathbb{C}$, there is a constant $c'_K > 0$ such that

$$\sup_K \left(\|\mathrm{d}_A B_t\|_{L^2(\Sigma)} + \|\mathrm{d}_A *_s B_t\|_{L^2(\Sigma)} \right) \le c'_K \sqrt{\varepsilon}.$$

Hence it follows from (13) and the last equation in (18) that

$$\sup_K \|\mathrm{d}_A \nabla_t \eta\|_{L^2(\Sigma)} \le c''_K \sqrt{\varepsilon}.$$

Using this estimate and the second equation in (18) we obtain

$$\sup_K \|\mathrm{d}_A (\Psi' - \Psi)\|_{L^2(\Sigma)} \le c'''_K \sqrt{\varepsilon}.$$

Combining the last two estimates with the first equation in (18) we obtain (17). Now Ξ'_ν descends to a sequence

$$\bar{u}'_\nu : K \to \mathcal{M}(P)$$

of approximate holomorphic curves (see (16)) with uniformly bounded derivatives (see (15)). We must prove that the sequence $\bar{u}'_\nu$ is bounded in $W^{2,p}$ for some $p > 2$. By the elliptic bootstrapping analysis for holomorphic curves (see [3, Appendix B]), this is equivalent to a $W^{1,p}$-bound on $\bar{\partial}_J(\bar{u}'_\nu)$. To obtain such a bound we examine the following formula from [1, page 627]:

$$\begin{aligned} B'_t + *_s(B'_s - X_s(A')) \quad &= \quad *_s \dot{*}_s \mathrm{d}_A \eta - [X_s(A), \eta] - *_s(X_s(A') - X_s(A)) \\ &\quad + [(A' - A), \nabla_s \eta] - *_s[(A' - A), \nabla_t \eta] \\ &\quad - \mathrm{d}_{A'}(\Psi' - \Psi + \nabla_s \eta) - *_s \mathrm{d}_{A'}(\Phi' - \Phi - \nabla_t \eta). \end{aligned} \tag{19}$$

To begin with observe that, by Lemma 2.1, we have estimates of the form

$$\int_K \left(\|\mathrm{d}_A B_t\|^p_{L^2(\Sigma)} + \|\mathrm{d}_A *_s B_t\|^p_{L^2(\Sigma)} \right) \le c_{K,p} \varepsilon^p.$$

Carrying the argument in the proof of Lemma 2.1 one step further we obtain estimates for the second derivatives of the curvature and hence

$$\int_K \left(\|\mathrm{d}_A \nabla_s B_t\|^p_{L^2(\Sigma)} + \|\mathrm{d}_A *_s \nabla_s B_t\|^p_{L^2(\Sigma)} \right) \le c_{K,p};$$

similarly for ∇_t. Differentiate the identities in (18) to obtain

$$\int_K \left(\|\mathrm{d}_A \nabla_s \nabla_s \eta\|^p_{L^2(\Sigma)} + \|\mathrm{d}_A \nabla_t \nabla_t \eta\|^p_{L^2(\Sigma)} + \|\mathrm{d}_A \nabla_s \nabla_t \eta\|^p_{L^2(\Sigma)} \right) \le c_{K,p},$$

$$\int_K \left(\|\mathrm{d}_A \nabla_s (\Psi' - \Psi)\|^p_{L^2(\Sigma)} + \|\mathrm{d}_A \nabla_t (\Psi' - \Psi)\|^p_{L^2(\Sigma)} \right) \le c_{K,p}.$$

Combining these estimates with (19) we obtain

$$\int_K \|\nabla_s (B'_t + *_s(B'_s - X_s(A')))\|^p_{L^2(\Sigma)} \le c_{K,p},$$

and similarly for ∇_t. This is the required $W^{1,p}$-estimate for $\bar{\partial}_J(\bar{u}'_\nu)$. It follows that $\bar{u}'_\nu$ is bounded in $W^{2,p}$ and hence has a C^1-convergent subsequence. The limit of this subsequence is the required holomorphic curve in $\mathcal{M}(P)$. The assertion of the lemma now follows from (17) and the C^1-convergence of $\bar{u}'_\nu$. □

For Theorem 9.1: On Page 630 replace the estimate in the assertion of Theorem 9.1 by (11) above. In the proof on page 631 replace the factor ε_ν^{-2} in (9.1) and (9.2) by ε_ν^{-1}. Replace the next displayed formula by

$$c_\nu = c_\nu(w_\nu) = \varepsilon_\nu^{-1} \left\| F_{A_\nu(w_\nu)} \right\|_{L^2(\Sigma)} + \left\| \partial_t A_\nu(w_\nu) - \mathrm{d}_{A_\nu(w_\nu)} \Psi_\nu(w_\nu) \right\|_{L^2(\Sigma)}.$$

On page 633 the assertion that the limits $A_\infty(\theta)$ and $\Phi_\infty(\theta)$ exist can be proved by a similar argument as in [2, Proposition 11.1]. Alternatively, one can use the beautiful and elegant argument in [4] for a direct proof of the energy identity.

On page 634 replace the second displayed inequality by

$$\sup_{|w|\le\rho_\nu c_\nu} \left(\frac{1}{\varepsilon_\nu c_\nu} \left\| F_{\widetilde{A}_\nu(w)} \right\|_{L^2(\Sigma)} + \left\| \partial_t \widetilde{A}_\nu(w) - \mathrm{d}_{\widetilde{A}_\nu(w)} \widetilde{\Psi}_\nu(w) \right\|_{L^2(\Sigma)} \right) \le 2.$$

We prove that the limit connection Ξ_0 represents a nonconstant holomorphic sphere. First, note that

$$\frac{1}{\varepsilon_\nu c_\nu} \left\| F_{\widetilde{A}_\nu(0)} \right\|_{L^2(\Sigma)} + \left\| \partial_t \widetilde{A}_\nu(0) - \mathrm{d}_{\widetilde{A}_\nu(0)} \widetilde{\Psi}_\nu(0) \right\|_{L^2(\Sigma)} = 1$$

and use Corollary 1.1 with ε replaced by $\widetilde{\varepsilon}_\nu := \varepsilon_\nu c_\nu \to 0$ to deduce that the functions $\partial_t \widetilde{A}_\nu - \mathrm{d}_{\widetilde{A}_\nu} \widetilde{\Psi}_\nu$ and $(\varepsilon_\nu c_\nu)^{-1} F_{\widetilde{A}_\nu}$ are uniformly bounded on every compact subset of $\mathbb{C} \times \Sigma$. Second, use Lemma 3.1 to deduce that the sequence $\widetilde{\Xi}_\nu = \widetilde{A}_\nu + \widetilde{\Phi}_\nu \mathrm{d}s + \widetilde{\Psi}_\nu \,\mathrm{d}t$ has a C^1 convergent subsequence (after gauge transformation). Third, use Lemma 2.1 to deduce that $(\varepsilon_\nu c_\nu)^{-1} \| F_{\widetilde{A}_\nu(0)} \|_{L^2(\Sigma)} \to 0$ and hence

$$\left\| \partial_t A_0(0) - \mathrm{d}_{A_0(0)} \Psi_0(0) \right\|_{L^2(\Sigma)} = \lim_{\nu\to\infty} \left\| \partial_t \widetilde{A}_\nu(0) - \mathrm{d}_{\widetilde{A}_\nu(0)} \widetilde{\Psi}_\nu(0) \right\|_{L^2(\Sigma)} = 1.$$

References

[1] S. Dostoglou and D.A. Salamon, Self-dual instantons and holomorphic curves, *Annals of Mathematics* **139** (1994), 581–640.

[2] A.R. Gaio, D.A. Salamon, Gromov–Witten invariants of symplectic quotients and adiabatic limits, Preprint ETH–Zürich, June 2001. Submitted. http://www.math.ethz.ch/ salamon

[3] D. McDuff, D. Salamon, *J-holomorphic Curves and Symplectic Topolpogy*, AMS Colloquium Publications, Volume **52**, 2004.

[4] K. Wehrheim, Energy identity for anti-self-dual instantons on $\mathbb{C} \times \Sigma$, *Math. Res. Lett.* **13** (2006), 161–166.