Pointwise Expansion of Degenerating Immersions of Finite Total Curvature

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Abstract

Generalising classical result of Müller-Šverák (1995), we obtain a pointwise estimate of the conformal factor of sequences of conformal immersions from the unit disk of the complex plane of uniformly bounded total curvature and converging strongly outside of a concentration point towards a branched immersions for which the quantization of energy holds. We show that the multiplicity associated to the conformal parameter becomes eventually constant to an integer equal to the order of the branch point of the limiting branched immersion. Furthermore, we deduce a C^0 convergence of the normal unit in the neck regions. Finally, we show that these improved energy quantizations hold for Willmore surfaces of uniformly bounded energy and precompact conformal class, and for Willmore spheres arising as solutions of min-max problems in the viscosity method.

Contents

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1 Introduction

Let Σ be a Riemann surface (not necessarily closed) and $\vec{\Phi}:\Sigma\rightarrow\mathbb{R}^n$ be a smooth immersion. Denote by $g=\vec{\Phi}^*g_{\mathbb{R}^n}$ the induced metric on Σ. We say that $\vec{\Phi}$ has finite total curvature if

$$\int_\Sigma|\vec{\mathbb{I}}|^2d\mathrm{vol}_g<\infty,$$

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where $\vec{\mathbb{I}}$ is the second fundamental form of $\vec{\Phi}$. In 1994, T. Toro proved the surprising result that assuming only that $u \in W^{2,2}(\Sigma,\mathbb{R})$, the graph $\mathscr{S} = \mathbb{R}^3 \cap \{(x,y) : y = u(x) \text{ for some } x \in \Sigma\}$ admits a bi-Lipschitz parametrisation. The following year, Müller-Šverák extended this result and showed that immersed surfaces with finite total curvature are conformally equivalent to a punctured Riemann surface. Furthermore, they proved a pointwise estimate of the conformal parameter of immersions of finite total curvature at the ends. The result can be restated in terms of branched immersions of the disk, and this is this statement due to T. Rivière that we will now state ([29], Lemma A.5). Here, $B_1(0) \subset \mathbb{C}$ is the open unit ball of the complex plane.

Theorem (Müller-Šverák [23], Rivière [29])**.** *Let $n \geq 3$, and $\vec{\Phi} \in W^{2,2}_{\mathrm{loc}}(B_1(0),\mathbb{R}^n) \cap W^{1,2}(B_1(0),\mathbb{R}^n)$ be a conformal immersion of $B_1(0) \setminus \{0\}$ of* finite total curvature *and assume that*

$$\lambda = \frac{1}{2}\log|\nabla\vec{\Phi}| \in L^\infty_{\mathrm{loc}}(B_1(0) \setminus \{0\}).$$

Then $\vec{\Phi}$ can be extended to a Lipschitz conformal immersion of $B_1(0)$, and there exists a positive integer $\theta_0 \geq 1$ and $C > 0$ such that for all $z \in B_1(0)$

$$C(1-o(1))|z|^{\theta_0 - 1} \leq |\partial_z\vec{\Phi}| \leq C(1+o(1))|z|^{\theta_0-1}.$$

More precisely, there exists $\mu \in W^{2,1}(B_1(0))$ (so that $\mu \in C^0(B_1(0))$ in particular) and a harmonic function $\nu : B_1(0) \setminus \{0\} \to \mathbb{R}$ such that

$$\lambda = \mu + \nu,$$

and

$$\nu(z) = (\theta_0 - 1)\log|z| + h(z),$$

where $h : B_1(0) \to \mathbb{R}$ is a harmonic function. In particular, we have for some constant $C > 0$ depending only on $\vec{\Phi}$

$$\|\lambda - (\theta_0 - 1)\log|z|\|_{\mathrm{L}^\infty(B_1(0))} \leq C.$$

In the study of bubbling of sequences of Willmore immersions (or equivalently of the compactness of the moduli space), it is of great interest to understand the pointwise behaviour of degenerations of immersions of uniformly bounded Willmore energy, or equivalently finite total curvature and in the viscosity method (see [16] and [35]).

In the following theorem, we obtain a pointwise expansion of the conformal factor in the full neck region of an arbitrary sequence of immersions (not necessarily Willmore).

The following theorem shows that the *multiplicity* of weakly converging sequence of immersions becomes eventually constant to an integer. This is a significant improvement of the fundamental work of Müller-Šverák ([23]).

Theorem A. *Let $n \geq 3$ be a fixed integer. There exists a universal constant $C_0(n) > 0$ with the following property. Let $\{\vec{\Phi}_k\}_{k\in\mathbb{N}}$ be a sequence of smooth conformal immersions from the disk $B_1(0) \subset \mathbb{C}$ into $\mathbb{R}^n$ and $\{\rho_k\}_{k\in\mathbb{N}} \subset (0,1)$ be such that $\rho_k \underset{k\to\infty}{\longrightarrow} 0$, $\Omega_k = B_1 \setminus \overline{B}_{\rho_k}(0)$ and define for all $0 < \alpha < 1$ the sub-domain $\Omega_k(\alpha) = B_\alpha \setminus \overline{B}_{\alpha^{-1}\rho_k}(0)$. For all $k \in \mathbb{N}$, let*

$$\lambda_k = \log\left(\frac{|\nabla\vec{\Phi}_k|}{\sqrt{2}}\right)$$

be the conformal factor of $\vec{\Phi}_k$. Assume that

$$\sup_{k\in\mathbb{N}}\|\nabla\lambda_k\|_{\mathrm{L}^{2,\infty}(\Omega_k)} < \infty, \qquad \lim_{\alpha\to 0}\limsup_{k\to\infty}\int_{\Omega_k(\alpha)}|\nabla\vec{n}_k|^2dx = 0$$

and that there exists a $W^{2,2}_{\mathrm{loc}}(B_1(0)\setminus\{0\})\cap C^\infty(B_1(0)\setminus\{0\})$ *immersion* $\vec{\Phi}_\infty$ *such that*

$$\log|\nabla\vec{\Phi}_\infty|\in L^\infty_{\mathrm{loc}}(B_1(0)\setminus\{0\})$$

and $\vec{\Phi}_k \underset{k\to\infty}{\longrightarrow} \vec{\Phi}_\infty$ *in* $C^l_{\mathrm{loc}}(B_1(0)\setminus\{0\})$ *(for all* $l\in\mathbb{N}$*). Then, there exists an integer* $\theta_0\geq 1$*,* $\mu_k\in W^{1,(2,1)}(B_1(0))$ *such that*

$$\|\nabla\mu_k\|_{\mathrm{L}^{2,1}(B_1(0))}\leq C_0(n)\int_{\Omega_k}|\nabla\vec{n}_k|^2dx$$

and a harmonic function ν_k *on* Ω_k *such that* $\nu_k=\lambda_k$ *on* $\partial B_1(0)$*,* $\lambda_k=\mu_k+\nu_k$ *on* Ω_k *and such that for all* $0<\alpha<1$ *and for all* $k\in\mathbb{N}$ *sufficiently large, we have*

$$\|\nabla(\nu_k-(\theta_0-1)\log|z|)\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}\leq C_0(n)\left(\sqrt{\alpha}\,\|\nabla\lambda_k\|_{\mathrm{L}^{2,\infty}(\Omega_k)}+\int_{\Omega_k}|\nabla\vec{n}_k|^2dx\right).$$

Finally, we have for all $\rho_k\leq r_k\leq 1$ *and* k *large enough*

$$\frac{1}{2\pi}\int_{\partial B_{r_k}}*\,d\nu_k=\theta_0-1.$$

In particular, there exists a constant $C>0$ *independent of* $k\in\mathbb{N}$ *such that for all* $k\in\mathbb{N}$*, and for all* $z\in\Omega_k(1)=B_1\setminus\overline{B}_{\rho_k}(0)$

$$\frac{1}{C}|z|^{\theta_0-1}\leq e^{\lambda_k(z)}\leq C|z|^{\theta_0-1}.$$

Remark. Theorem A corresponds to Theorem 3.1.

This theorem has also been obtained recently by Nicolas Marque in the case of *minimal simple bubbling* ([17]). It constitutes a fundamental ingredient to show that in this special case„ there is an obstruction to the singularity of the limiting Willmore immersion at branch points (it is stated using the second residue, see [1]). As such, this result may be seen as a technical result aimed at providing new applications to the loss compactness of Willmore immersions and in particular an extension of Marque's main result to arbitrary codimension. This result also constitutes an improvement of Lemma V.3 of Bernard-Rivière ([2]) since it identifies the multiplicity d_k corresponding to $\vec{\Phi}_k$ to be the integer $\theta_0-1\geq 0$ eventually (*i.e.* for $k\in\mathbb{N}$ large enough), which also restricts the possibilities of bubbling of Willmore surfaces. If the limiting branched immersions has a branch point of order, then the bubble that appears at this point must have a branch point of the same order. Since the result also applies to the viscosity method, we expect that it should help shedding some light on the problem to determining the Morse index of branched Willmore spheres realising the *min-max sphere eversion* (see [35], [18], [20], [21], [19]).

More generally, an $L^{2,1}$ quantization of the energy permits to obtain a pointwise expansion of the conformal parameter by constructing—using by Hélein's methods ([10]) and their extension to Willmore immersions by T. Rivière ([28], [2])—a controlled $L^{2,1}$ Coulomb frame.

Theorem B. *Under the conditions of Theorem A, assume furthermore that the following strong* $L^{2,1}$ *no-neck energy holds*

$$\lim_{\alpha\to 0}\lim_{k\to\infty}\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}=0.$$

Then there exists $\alpha_0>0$ *such that for all* $k\in\mathbb{N}$ *large enough, there exists a moving frame* $(\vec{f}_{k,1},\vec{f}_{k,2})\in W^{1,(2,1)}(B_{\alpha_0}(0))\times W^{1,(2,1)}(B_{\alpha_0}(0))$ *and a universal constant* $C_1(n)$ *(independent of* k*) such that*

$$\left\|\nabla\vec{f}_{k,1}\right\|_{\mathrm{L}^{2,1}(B_{\alpha_0}(0))}+\left\|\nabla\vec{f}_{k,2}\right\|_{\mathrm{L}^{2,1}(B_{\alpha_0}(0))}\leq C_1(n)\left(1+\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha_0))}\right)\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha_0))}.$$

Furthermore, there exists a sequence of functions $\mu_k\in W^{2,1}(B_{\alpha_0}(0))$ *and a universal constant* $C_2(n)$ *such that*

$$\|\nabla^2\mu_k\|_{\mathrm{L}^1(B_{\alpha_0}(0))}+\|\nabla\mu_k\|_{\mathrm{L}^{2,1}(B_{\alpha_0}(0))}+\|\mu_k\|_{\mathrm{L}^\infty(B_{\alpha_0}(0))}\leq C_2(n)\int_{\Omega_k(\alpha_0)}|\nabla\vec{n}_k|^2dx$$

and there exists a sequence of holomorphic functions $\psi_k : B_{\alpha_0}(0) \to \mathbb{C}$ *and* $\chi_k : B_{\alpha_0}(0) \to \mathbb{C}$ *such that* $\chi_k(0) = 0$, $c \in \mathbb{C}$ *and* $\{c_k\}_{k\in\mathbb{N}} \subset \mathbb{C}$ *such that* $c_k \underset{k\to\infty}{\longrightarrow} c$ *and*

$$\psi_k(z) = e^{c_k} z^{\theta_0 - 1}\left(1 + \chi_k(z)\right) \tag{1.1}$$

and

$$e^{\lambda_k} = e^{\mu_k}|\psi_k(z)| = e^{\mathrm{Re}\,(c_k)}|z|^{\theta_0-1}\left(1 + o(1)\right), \qquad \text{for all } z \in \Omega_k(\alpha). \tag{1.2}$$

Finally, there exists $\vec{A}_0 \in \mathbb{C}^n$ *(satisfying* $\langle \vec{A}_0, \vec{A}_0\rangle = 0$*) and* $\{\vec{A}_{k,0}\}_{k\in\mathbb{N}} \in \mathbb{C}^n$ *such that* $\vec{A}_{k,0} \underset{k\to\infty}{\longrightarrow} \vec{A}_0$ *and for all* $z \in \Omega_k(\alpha_0)$*, we have the pointwise identities*

$$\begin{aligned}\partial_z \vec{\Phi}_k &= \frac{1}{2} e^{c_k + \mu_k(z)} z^{\theta_0 - 1}\left(1 + \chi_k(z)\right)\left(\vec{f}_{k,1} - i\vec{f}_{k,2}\right)\\ &= \vec{A}_{k,0} z^{\theta_0-1} + o\left(|z|^{\theta_0-1}\right).\end{aligned} \tag{1.3}$$

Remark. Theorem B corresponds to Theorem 3.5 below.

These two theorems have analogues in the case of multiple bubbles but we will not state them here for the sake of simplicity of presentation.

We also prove that this stronger quantization property holds for sequences of Willmore immersions of uniformly bounded Willmore energy and for Willmore spheres arising in min-max constructions in the viscosity method.

Theorem C. *Let* Σ *be a closed Riemann surface and assume that* $\{\vec{\Phi}_k\}_{k\in\mathbb{N}}$ *is a sequence of smooth Willmore immersions such that*

$$\limsup_{k\to\infty} W(\vec{\Phi}_k) < \infty.$$

Assume furthermore that the conformal class of $\{\vec{\Phi}_k^* g_{\mathbb{R}^n}\}_{k\in\mathbb{N}}$ *lies in a compact subset of the moduli space. Then for all* $0 < \alpha < 1$ *let* $\Omega_k(\alpha) = B_{\alpha R_k} \setminus \overline{B}_{\alpha^{-1}r_k}(0)$ *be a neck domain and* $\theta_0 \in \mathbb{N}$ *such that (by Theorem 3.1)*

$$\theta_0 - 1 = \lim_{\alpha\to 0}\lim_{k\to\infty}\int_{\partial B_{\alpha^{-1}r_k}(0)} \partial_\nu \lambda_k \, d\mathscr{H}^1, \tag{1.4}$$

and define

$$\Lambda = \sup_{k\in\mathbb{N}}\left(\|\nabla\lambda_k\|_{\mathrm{L}^{2,\infty}(\Omega_k(1))} + \int_{\Omega_k(1)} |\nabla \vec{n}_k|^2 dx\right).$$

Then there exist a universal constant $C_3 = C_3(n)$*, and* $\alpha_0 = \alpha_0(\{\vec{\Phi}_k\}_{k\in\mathbb{N}}) > 0$ *such that for all* $0 < \alpha < \alpha_0$ *and* $k \in \mathbb{N}$ *large enough,*

$$\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))} \leq C_3(n) e^{C_3(n)\Lambda}\left(1 + \|\nabla \vec{n}_k\|_{\mathrm{L}^2(\Omega_k(4\alpha))}\right)\|\nabla\vec{n}_k\|_{\mathrm{L}^2(\Omega_k(4\alpha))}. \tag{1.5}$$

In particular, we deduce by the $L^{2,1}$ *no-neck energy*

$$\lim_{\alpha\to 0}\limsup_{k\to\infty}\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))} = 0.$$

Remark. Theorem C corresponds to Theorem 4.1 below.

A similar result was proved by Lamm-Sharp ([12]) in the case of conformally invariant problems and in the more general setting introduced by Rivière ([27]) of elliptic systems with antisymmetric potentials, and by Changyou Wang in the case of harmonic maps ([37]).

Finally, we show that this hypothesis is indeed satisfied for sequences of Willmore immersions of precompact conformal class or in the viscosity method for spheres. The proof of such a result builds on the previous work of Rivière ([28], [32]), Bernard-Rivière ([1], [2]) and Laurain-Rivière ([13], [15], [14]) and on the general philosophy of integration by compensation and geometric analysis on surfaces (including [4], [36], [23], [10]). We refer to Theorem 4.1 and Theorem 6.2 for the precise (and somewhat technical) statement.

Corollary 1.4. *Let Σ be a closed Riemann surface and assume that $\{\vec{\Phi}_k\}_{k\in\mathbb{N}}$ is a sequence of Willmore imersions from Σ into $\mathbb{R}^n$ such that*

$$\limsup_{k\to\infty}\int_{\Sigma}|\vec{H}_{\vec{\Phi}_k}|^2 d\mathrm{vol}_{g_{\vec{\Phi}_k}}<\infty.$$

Assume furthermore that the conformal class of $\{\vec{\Phi}_k^{*}g_{\mathbb{R}^n}\}_{k\in\mathbb{N}}$ lies in a compact subset of the moduli space. Then there exists $\{a_1,\cdots,a_m\}\subset\Sigma$, sequences $\{x_k^{i,j}\}_{k\in\mathbb{N}}$, $1\leq i\leq n$, $1\leq j\leq m_i$ such that $x_k^{i,j}\underset{k\to\infty}{\longrightarrow}a_i$ for all i,j and branched Willmore immersions $\vec{\Phi}_\infty:\Sigma\to\mathbb{R}^n$, $\vec{\Phi}_\infty^{i,j}:S^2=\mathbb{C}\cup\{\infty\}\to\mathbb{R}^n$ and $\{\rho_k^{i,j}\}_{k\in\mathbb{N}}\subset(0,\infty)$ with $\rho_k^{i,j}\underset{k\to\infty}{\longrightarrow}0$ and for all $1\leq i\leq m$ and $1\leq j\neq j'\leq m$,

$$\lim_{k\to\infty}\max\left\{\frac{\rho_k^{i,j}}{\rho_k^{i,j'}}+\frac{\rho_k^{i,j'}}{\rho_k^{i,j}},\frac{|x_k^{i,j}-x_k^{i,j'}|}{\rho_k^{i,j}+\rho_k^{i,j'}}\right\}=\infty.$$

such that

$$\left\|\nabla\vec{n}_{\vec{\Phi}_k}-\nabla\vec{n}_{\vec{\Phi}_\infty}-\sum_{i=1}^{m}\sum_{j=1}^{m_i}\nabla\vec{n}_{\vec{\Phi}_\infty^{i,j}}((\rho_k^{i,j})^{-1}(\,\cdot\,-x_k^{i,j}))\right\|_{\mathrm{L}^{2,1}(\Sigma)}=0. \tag{1.6}$$

In particular, we have

$$\left\|\vec{n}_{\vec{\Phi}_k}-\vec{n}_{\vec{\Phi}_\infty}-\sum_{i=1}^{m}\sum_{j=1}^{m_i}\left(\vec{n}_{\vec{\Phi}_\infty^{i,j}}((\rho_k^{i,j})^{-1}(\,\cdot\,-x_k^{i,j}))-\vec{n}_{\vec{\Phi}_\infty^{i,j}}(\infty)\right)\right\|_{\mathrm{L}^{\infty}(\Sigma)}=0. \tag{1.7}$$

The proof of Corollary is found at the end of Section 4.

Remark. (1) The writing of (1.6) and (1.7), classical in concentration compactness theory, makes use of implicit cutoff functions (see [37]).

(2) This result is optimal in the $C^{l,\beta}$ topology since the $C^{0,\beta}$ norm for $\beta>0$ is not scaling invariant. For another C^0 theory for the blow-up of elliptic equations of order 2, see [24], [12] and [37].

More precisely, the C^0 energy quantization permits to link the values of the normal of the limiting immersion of the one of bubbles. Let us state the result in the case of a single bubble for simplicity.

Corollary 1.5. *Let Σ be a closed Riemann surface and assume that $\{\vec{\Phi}_k\}_{k\in\mathbb{N}}$ is a sequence of Willmore immersions from Σ in $\mathbb{R}^n$ such that*

$$\lim_{k\to\infty}\int_{\Sigma}|\vec{H}_{\vec{\Phi}_k}|^2 d\mathrm{vol}_{g_{\vec{\Phi}_k}}<\infty.$$

Assume furthermore that the conformal class of $\{\vec{\Phi}_k^{*}g_{\mathbb{R}^n}\}_{k\in\mathbb{N}}$ lies in a compact subset of the moduli space. Following [2]*, let $\vec{\Phi}_\infty:\Sigma\to\mathbb{R}^n$ be such that for some finite collection $\{a_1,\cdots,a_m\}\subset\Sigma$, we have*

$$\vec{\Phi}_k\underset{k\to\infty}{\longrightarrow}\vec{\Phi}_\infty\qquad\text{in } C^l_{\mathrm{loc}}(\Sigma\setminus\{a_1,\cdots,a_m\})\ \text{for all } l\in\mathbb{N}.$$

Let $1\leq i\leq n$ and assume that a single bubble $\vec{\Psi}_\infty^i:S^2\to\mathbb{R}^n$ forms at a_i. Then we have

$$\vec{n}_{\vec{\Phi}_\infty}(a_i)=\vec{n}_{\vec{\Psi}_\infty^i}(\infty). \tag{1.8}$$

In the case of bubbles over bubbles, normals at junctions coincide with the value of the normal at $N=\infty\in S^2$ of the bubble. The proof is exactly the same.

2 Uniform control of the conformal factor in necks

For the definitions related to Lorentz spaces, we refer the reader to the Appendix (Section 7.1).

In this section we obtain a refinement of Lemma V.3 of [2].

Theorem 2.1. *There exists a positive real numbers $\varepsilon_1 = \varepsilon_1(n) > 0$ and $\Gamma_0(n) > 0$ with the following property. Let $0 < 2^6 r < R < \infty$ be fixed radii and $\vec{\Phi} : \Omega = B_R \setminus \overline{B}_r(0) \to \mathbb{R}^n$ be a weak immersion of finite total curvature such that*

$$\|\nabla\vec{n}\|_{\mathrm{L}^{2,\infty}(\Omega)} \leq \varepsilon_1(n). \tag{2.1}$$

Fix some $\left(\frac{r}{R}\right)^{\frac{1}{3}} < \alpha < 1$, and define $\Omega_\alpha = B_{\alpha R} \setminus \overline{B}_{\alpha^{-1}r}(0)$. Then we have

$$\|\nabla(\lambda - d\log|z|)\|_{\mathrm{L}^{2,1}(\Omega_\alpha)} \leq \Gamma_0\left(\sqrt{\alpha}\,\|\nabla\lambda\|_{\mathrm{L}^{2,\infty}(\Omega)} + \int_\Omega |\nabla\vec{n}|^2 dx\right) \tag{2.2}$$

and for all $r \leq \rho < R$, we have

$$\left|d - \frac{1}{2\pi}\int_{\partial B_\rho} \partial_\nu \lambda \, d\mathscr{H}^1\right| \leq \Gamma_0\left(\int_{B_{\max\{\rho,2r\}}\setminus\overline{B}_r(0)} |\nabla\vec{n}|^2 dx + \frac{1}{\log\left(\frac{R}{\rho}\right)}\int_\Omega |\nabla\vec{n}|^2 dx\right) \tag{2.3}$$

In particular, there exists a universal constant $\Gamma_0' = \Gamma_0'(n)$ and $A_\alpha \in \mathbb{R}$ such that

$$\|\lambda - d\log|z| - A_\alpha\|_{\mathrm{L}^\infty(\Omega_\alpha)} \leq \Gamma_0'\left(\sqrt{\alpha}\,\|\nabla\lambda\|_{\mathrm{L}^{2,\infty}(\Omega)} + \int_\Omega |\nabla\vec{n}|^2 dx\right). \tag{2.4}$$

The proof relies on the strategy developed in [2] (and the lemmas from [13], [15] for the Lemmas 2.2 and 2.3) and the following two lemmas, which will allow us to move from a $L^{2,\infty}$ bound to a $L^{2,1}$ bound in a quantitative way.

Lemma 2.2. *Let $u : B_R \setminus \overline{B}_r(0) \to \mathbb{R}$ be a harmonic function such that for some $\rho_0 \in (r, R)$*

$$\int_{\partial B_{\rho_0}} \partial_\nu u \, d\mathscr{H}^1 = 0.$$

Then there exists a universal constant $\Gamma_1 > 0$ (independent of $0 < 4r < R < \infty$) such that for all $\left(\frac{r}{R}\right)^{\frac{1}{2}} < \alpha < \frac{1}{2}$, we have

$$\|\nabla u\|_{\mathrm{L}^2(B_{\alpha R}\setminus\overline{B}_{\alpha^{-1}r}(0))} \leq \Gamma_1\,\|\nabla u\|_{\mathrm{L}^{2,\infty}(B_R\setminus\overline{B}_r(0))}.$$

Proof. First, we show that for all $\alpha^{-1}r \leq \rho \leq \alpha R$, and for all $0 < \alpha < \frac{1}{2}$ we have

$$\|\nabla u\|_{\mathrm{L}^\infty(\partial B_\rho(0))} \leq \frac{4}{\log(2)}\sqrt{\frac{3}{\pi}\frac{1}{(1-\alpha)\rho}}\,\|\nabla u\|_{\mathrm{L}^{2,\infty}(B_{\alpha^{-1}\rho}\setminus\overline{B}_{\alpha\rho}(0))}. \tag{2.5}$$

By a slight abuse of notation, we will write r instead of ρ in the following estimates.

As $0 < \alpha < \frac{1}{2}$, we have for all $x \in \partial B_r(0)$, the inclusion $B_{(1-\alpha)r}(x) \subset B_{\alpha^{-1}r} \setminus \overline{B}_{\alpha r}(0)$. Therefore, thanks to the mean value property, we have for all $0 < \beta < (1-\alpha)r$

$$\nabla u(x) = \frac{1}{2\pi\beta}\int_{\partial B_\beta(x)} \nabla u(y)\, d\mathscr{H}^1(y). \tag{2.6}$$

Now, thanks to the co-area formula, we have $\left(\text{if } I_\alpha(r)=\left(\frac{(1-\alpha)}{2}r,(1-\alpha)r\right)\right)$

$$\begin{aligned}
&\int_{B_{(1-\alpha)r}\setminus\overline{B}_{(1-\alpha)r/2}(x)}|\nabla u(y)|dy=\int_{\frac{(1-\alpha)r}{2}}^{(1-\alpha)r}\left(\int_{\partial B_\beta(x)}|\nabla u(y)|\,d\mathscr{H}^1(y)\right)d\beta\\
&\geq\inf_{\beta\in I_\alpha(r)}\left(\beta\int_{\partial B_\beta(x)}|\nabla u(y)|d\mathscr{H}^1(y)\right)\int_{\frac{(1-\alpha)r}{2}}^{(1-\alpha)r}\frac{d\beta}{\beta}=\log(2)\inf_{\beta\in I_\alpha(r)}\left(\beta\int_{\partial B_\beta(x)}|\nabla u(y)|d\mathscr{H}^1(y)\right)
\end{aligned}$$

Therefore, there exists $\beta\in\left(\frac{(1-\alpha)r}{2},(1-\alpha)r\right)$ (notice that this shows that the limiting values $\rho=\alpha^{-1}r$ and $\rho=\alpha R$ are admissible) such that

$$\beta\int_{\partial B_\beta(x)}|\nabla u(y)|\,d\mathscr{H}^1(y)\leq\frac{1}{\log(2)}\int_{B_{(1-\alpha)r}\setminus\overline{B}_{(1-\alpha)r/2}(x)}|\nabla u(y)|dy$$

or

$$\frac{1}{2\pi\beta}\int_{\partial B_\beta(x)}|\nabla u(y)|d\mathscr{H}^1(y)\leq\frac{1}{2\pi\log(2)\beta^2}\int_{B_{(1-\alpha)r}\setminus\overline{B}_{(1-\alpha)r/2}(x)}|\nabla u(y)|dy. \tag{2.7}$$

Now, notice that

$$\begin{aligned}
\left\|\mathbf{1}_{B_{(1-\alpha)r}\setminus\overline{B}_{(1-\alpha)r/2}(x)}\right\|_{\mathrm{L}^{2,1}(\mathbb{R}^2)}&=4\int_0^\infty\left(\mathscr{L}^2(B_{(1-\alpha)r}\setminus\overline{B}_{(1-\alpha)r/2}(x)\cap\{x:1>t\}\right)^{\frac{1}{2}}dt\\
&=2\sqrt{3\pi}(1-\alpha)r. &(2.8)
\end{aligned}$$

Furthermore, as $\beta>\dfrac{(1-\alpha)r}{2}$, we have

$$\frac{1}{\beta^2}\leq\frac{4}{(1-\alpha)^2r^2}. \tag{2.9}$$

Therefore, we have by the mean value property (2.6), the inequalities (2.7), (2.8), (2.9) and the duality $L^{2,1}/L^{2,\infty}$

$$\begin{aligned}
|\nabla u(x)|&\leq\frac{1}{2\pi\beta}\int_{\partial B_\beta(x)}|\nabla u(y)|\,d\mathscr{H}^1(y)\\
&\leq\frac{2}{\pi\log(2)(1-\alpha)^2r^2}\left\|\mathbf{1}_{B_{(1-\alpha)r}\setminus\overline{B}_{(1-\alpha)r/2}(x)}\right\|_{\mathrm{L}^{2,1}(\mathbb{R}^2)}\|\nabla u\|_{\mathrm{L}^{2,\infty}(B_{(1-\alpha)r}\setminus\overline{B}_{(1-\alpha)r/2}(x))}\\
&\leq\frac{4}{\log(2)}\sqrt{\frac{3}{\pi}}\frac{1}{(1-\alpha)r}\|\nabla u\|_{\mathrm{L}^{2,\infty}(B_{\alpha^{-1}r}\setminus\overline{B}_{\alpha r}(0))}.
\end{aligned}$$

As $x\in\partial B_r(0)$ was arbitrary, this proves the inequality (2.5). Now, as u is harmonic, there exists $\{a_n\}_{n\in\mathbb{Z}}\subset\mathbb{C}$ such that

$$u(\rho,\theta)=a_0+d\log\rho+\sum_{n\in\mathbb{Z}^*}\left(a_n\rho^n+\overline{a_{-n}}\rho^{-n}\right)e^{in\theta},$$

which implies by the hypothesis that

$$0=\int_{\partial B_{\rho_0}}\partial_\nu u\,d\mathscr{H}^1=2\pi d \tag{2.10}$$

so that for all $r<\rho<R$

$$\int_{\partial B_\rho}\partial_\nu u=0.$$

Therefore, integrating by parts, we find

$$\begin{aligned}\int_{B_{\alpha R}\setminus\overline{B}_{\alpha^{-1}r}}|\nabla u(x)|^2dx&=\int_{\partial B_{\alpha R}}\partial_\nu u\, u\, d\mathscr{H}^1-\int_{\partial B_{\alpha^{-1}r}}\partial_\nu u\, u\, d\mathscr{H}^1\\&=\int_{\partial B_{\alpha R}}\partial_\nu u(u-\overline{u}_{\alpha R})d\mathscr{H}^1-\int_{B_{\alpha^{-1}r}}\partial_\nu u\,(u-\overline{u}_{\alpha^{-1}r})\,d\mathscr{H}^1\end{aligned}\tag{2.11}$$

where $\overline{u}_\rho=\fint_{\partial B_\rho}u\,d\mathscr{H}^1$ is the average of u on ρ, for all $r<\rho<R$.

Now, if $\Gamma_2=\Gamma_2(H^{\frac{1}{2}}(S^1),L^1(S^1))$ is the constant of the injection $H^{\frac{1}{2}}(S^1)\hookrightarrow L^1(S^1)$ (for the norm defined by the L^2 norm of the harmonic extension), we get by (2.5) for all $r<\rho<R$

$$\begin{aligned}\left|\int_{\partial B_\rho}\partial_\nu u\,(u-\overline{u}_\rho)\,d\mathscr{H}^1\right|&\leq\|\nabla u\|_{\mathrm{L}^\infty(\partial B_\rho)}\|u-\overline{u}_\rho\|_{\mathrm{L}^1(\partial B_\rho)}\\&\leq\frac{4}{\log(2)}\sqrt{\frac{3}{\pi}}\frac{1}{(1-\alpha)\rho}\|\nabla u\|_{\mathrm{L}^{2,\infty}(B_R\setminus\overline{B}_r(0))}\times\Gamma_2\rho\,\|u\|_{\mathrm{H}^{\frac{1}{2}}(\partial B_\rho)}\\&\leq\frac{4}{\log(2)}\sqrt{\frac{3}{\pi}}\frac{1}{(1-\alpha)}\Gamma_2\|\nabla u\|_{\mathrm{L}^{2,\infty}(B_R\setminus\overline{B}_r(0))}\|\nabla u\|_{\mathrm{L}^2(B_{\alpha R}\setminus\overline{B}_{\alpha^{-1}r}(0))}\end{aligned}$$

which implies by (2.11) that

$$\|\nabla u\|_{\mathrm{L}^2(B_{\alpha R}\setminus\overline{B}_{\alpha^{-1}r})}\leq\frac{8}{\log(2)}\sqrt{\frac{3}{\pi}}\frac{1}{(1-\alpha)}\Gamma_2\|\nabla u\|_{\mathrm{L}^{2,\infty}(B_R\setminus\overline{B}_r(0))}$$

and this concludes the proof of the Lemma. ☐

In the following Lemma we obtain a slight improvement from [15] and generalise it to a $W^{2,1}$ estimate, that will be used in the proof of Theorem 4.1.

Lemma 2.3. *Let $0<4r<R<\infty$ be fixed radii, and $u:\Omega=B_R\setminus\overline{B}_r(0)\to\mathbb{R}$ be a harmonic function such that for some $\rho_0\in(r,R)$*

$$\int_{\partial B_{\rho_0}}\partial_\nu u\,d\mathscr{H}^1=0.$$

Then for all $\left(\frac{r}{R}\right)^{\frac{1}{2}}<\alpha<1$, we have

$$\begin{aligned}\|\nabla u\|_{\mathrm{L}^{2,1}(B_{\alpha R}\setminus\overline{B}_{\alpha^{-1}r}(0))}&\leq 32\sqrt{\frac{2}{15}}\frac{\alpha}{1-\alpha}\|\nabla u\|_{\mathrm{L}^2(B_R\setminus\overline{B}_r(0))},\\\|\nabla^2u\|_{\mathrm{L}^1(B_{\alpha R}\setminus\overline{B}_{\alpha^{-1}r}(0))}&\leq 32\sqrt{\frac{\pi}{15}}\frac{\alpha}{1-\alpha}\|\nabla u\|_{\mathrm{L}^2(B_R\setminus\overline{B}_r(0))}.\end{aligned}$$

Proof. As u is harmonic on $B_R\setminus\overline{B}_r(0)$, there exists $\{a_n\}_{n\in\mathbb{Z}}\subset\mathbb{C}$ and $d\in\mathbb{R}$ such that

$$u(z)=a_0+d\log|z|+2\,\mathrm{Re}\left(\sum_{n\in\mathbb{Z}}a_nz^n\right).$$

Thanks to (2.10), we deduce that $d=0$. Furthermore, taking polar coordinates $z=\rho e^{i\theta}$, we have the identity

$$|\nabla u|^2=4|\partial_zu|^2=4\left|\sum_{n\in\mathbb{Z}^*}na_nz^{n-1}\right|^2=4\sum_{n,m\in\mathbb{Z}^*}nm\,a_n\overline{a}_m\rho^{n+m-2}e^{i(n-m)\theta}.\tag{2.12}$$

This implies by the inequality $0 < 4r < R < \infty$ that

$$\begin{aligned}\int_{B_R\setminus\overline{B}_r(0)} |\nabla u(x)|^2 dx =& 8\pi \sum_{n\in\mathbb{Z}^*}\int_r^R |n|^2|a_n|^2\rho^{2n-1}d\rho = 8\pi\sum_{n\in\mathbb{Z}^*}|n|^2\left(\frac{1}{2n}|a_n|^2\left(R^{2n}-r^{2n}\right)\right)\\ &= 4\pi\sum_{n\geq 1}|n||a_n|^2R^{2|n|}\left(1-\left(\frac{r}{R}\right)^{2|n|}\right)+4\pi\sum_{n\leq -1}|n||a_n|^2\frac{1}{r^{2|n|}}\left(1-\left(\frac{r}{R}\right)^{2|n|}\right)\\ &\geq \frac{15\pi}{4}\sum_{n\geq 1}|n|\left(|a_n|^2R^{2|n|}+|a_{-n}|^2\frac{1}{r^{2|n|}}\right). \end{aligned}\tag{2.13}$$

First $L^{2,1}$ estimate. Now, we have

$$\|1\|_{\mathrm{L}^{2,1}(B_R\setminus\overline{B}_r)} = 4\sqrt{\pi}\left(R^2-r^2\right)^{\frac{1}{2}} \leq 4\sqrt{\pi}R$$

while for all $m\geq 1$,

$$\||z|^m\|_{\mathrm{L}^{2,1}(B_R\setminus\overline{B}_r(0))} = 4\sqrt{\pi}r^m\left(R^2-r^2\right)^{\frac{1}{2}}+4\sqrt{\pi}\int_{r^m}^{R^m}(R^2-t^{\frac{2}{m}})^{\frac{1}{2}}\,dt \leq 4\sqrt{\pi}r^mR+4\sqrt{\pi}\int_{r^m}^{R^m}Rdt = 4\sqrt{\pi}R^{m+1}.$$

Likewise, for all $m\geq 2$

$$\left\|\frac{1}{|z|^m}\right\|_{\mathrm{L}^{2,1}(B_R\setminus\overline{B}_r(0))} \leq 4\sqrt{\pi}\int_0^{\frac{1}{r^m}}\left(\frac{1}{t^{\frac{2}{m}}}-r^2\right)^{\frac{1}{2}}dt \leq 4\sqrt{\pi}\int_0^{\frac{1}{r^m}}\frac{dt}{t^{\frac{1}{m}}} = 4\sqrt{\pi}\frac{m}{m-1}\frac{1}{r^{m-1}} \leq 8\sqrt{\pi}r^{-m+1}.$$

By (2.12), we have

$$|\nabla u| \leq 2\sum_{n\in\mathbb{Z}^*}|n||a_n|\rho^{n-1},$$

and the following estimates by Cauchy-Schwarz inequality

$$\begin{aligned}\|\nabla u\|_{\mathrm{L}^{2,1}(B_{\alpha R}\setminus\overline{B}_{\alpha^{-1}r}(0))} &\leq 16\sqrt{\pi}\left(\sum_{n\geq 1}|n||a_n|\left(\alpha R\right)^{|n|}+\sum_{n\geq 1}|n||a_{-n}|\left(\frac{\alpha}{r}\right)^{|n|}\right)\\ &\leq 16\sqrt{\pi}\left(\sum_{n\in\mathbb{Z}^*}|n|\alpha^{2|n|}\right)^{\frac{1}{2}}\left(\sum_{n\geq 1}|n||a_n|^2R^{2|n|}+|n||a_{-n}|^2\frac{1}{r^{2|n|}}\right)^{\frac{1}{2}}\\ &= 16\sqrt{2\pi}\frac{\alpha}{1-\alpha^2}\left(\sum_{n\geq 1}|n||a_n|^2R^{2|n|}+|n||a_{-n}|^2\frac{1}{r^{2|n|}}\right)^{\frac{1}{2}}. \end{aligned}\tag{2.14}$$

Combining (2.13) and (2.14) yields

$$\|\nabla u\|_{\mathrm{L}^{2,1}(B_{\alpha R}\setminus\overline{B}_{\alpha^{-1}r}(0))} \leq \frac{16\sqrt{2\pi}\alpha}{1-\alpha}\times\sqrt{\frac{4}{15\pi}}\|\nabla u\|_{\mathrm{L}^2(B_R\setminus\overline{B}_r(0))} = 32\sqrt{\frac{2}{15}}\frac{\alpha}{1-\alpha}\|\nabla u\|_{\mathrm{L}^2(B_R\setminus\overline{B}_r(0))},$$

which concludes the proof of the first part of the Lemma.

Second $W^{1,1}$ estimate. As $\Delta u = 0$, we have $|\nabla^2 u| = 4|\partial_z^2 u|$, and

$$\partial_z^2 u(z) = \sum_{n\in\mathbb{Z}^*}n(n-1)z^{n-2}.$$

Now, for all $m\in\mathbb{Z}\setminus\{-2\}$, we have

$$\||z|^m\|_{\mathrm{L}^1(B_{\alpha R}\setminus\overline{B}_{\alpha^{-1}r}(0))} = 2\pi\int_{\alpha^{-1}r}^{\alpha R}\rho^{m+1}d\rho = \frac{2\pi}{m+2}\left((\alpha R)^{m+2}-(\alpha^{-1}r)^{m+2}\right)$$

In particular, we have by the triangle inequality and Cauchy-Schwarz inequality

$$
\begin{aligned}
\left\|\partial_z^2 u\right\|_{\mathrm{L}^1(B_{\alpha R}\setminus\overline{B}_{\alpha^{-1}r}(0))} &\leq 2\pi \sum_{n\in\mathbb{Z}^*}\frac{|n||n-1|}{n}|a_n|\left((\alpha R)^n-\left(\alpha^{-1}r\right)^n\right)\\
&= 2\pi\sum_{n\geq 1}|n-1||a_n|(\alpha R)^{|n|}\left(1-\left(\frac{\alpha^2 r}{R}\right)^{|n|}\right)+2\pi\sum_{n\leq -1}|n-1||a_n|\left(\frac{\alpha}{r}\right)^{|n|}\left(1-\left(\frac{\alpha^2 r}{R}\right)^{|n|}\right)\\
&\leq 2\pi\sum_{n\geq 1}|n-1||a_n|(\alpha R)^{|n|}+\sum_{n\geq -1}|n-1||a_n|\left(\frac{\alpha}{r}\right)^{|n|}\\
&\leq 2\pi\left(\sum_{n\in\mathbb{Z}^*}\frac{|n-1|^2}{|n|}\alpha^{2|n|}\right)^{\frac{1}{2}}\left(\sum_{n\geq 1}|n||a_n|^2R^{2|n|}+\sum_{n\leq -1}|n||a_n|^2\frac{1}{r^{2|n|}}\right)^{\frac{1}{2}}.
\end{aligned}
$$

Now, notice that

$$
\sum_{n\in\mathbb{Z}*}\frac{|n-1|^2}{|n|}\alpha^{2|n|}=2\sum_{n\geq 1}\frac{n^2+1}{n}\alpha^{2n}=\frac{2\alpha^2}{(1-\alpha^2)^2}+2\log\left(\frac{1}{1-\alpha^2}\right)\leq\frac{4\alpha^2}{(1-\alpha^2)^2}.
$$

Recalling from (2.3) that

$$
\int_{B_R\setminus\overline{B}_{\alpha^{-1}r}(0)}|\nabla u(x)|^2dx\geq\frac{15\pi}{4}\sum_{n\geq 1}|n|\left(|a_n|^2R^{2|n|}+|a_{-n}|^2\frac{1}{r^{2|n|}}\right),
$$

we deduce that

$$
\left\|\partial_z^2 u\right\|_{\mathrm{L}^1(B_{\alpha R}\setminus\overline{B}_{\alpha^{-1}r}(0))}\leq\frac{4\pi\alpha}{(1-\alpha^2)}\times\sqrt{\frac{4}{15\pi}}\left\|\nabla u\right\|_{\mathrm{L}^2(B_R\setminus\overline{B}_r(0))}=8\sqrt{\frac{\pi}{15}}\frac{\alpha}{1-\alpha^2}\left\|\nabla u\right\|_{\mathrm{L}^2(B_R\setminus\overline{B}_r(0))}
$$

which concludes the proof as $|\nabla^2 u|=4|\partial_z^2 u|$. □

Remark 2.4. Notice that $\|\nabla\log|z|\|_{\mathrm{L}^2(B_R\setminus\overline{B}_r(0))}=\sqrt{2\pi}\sqrt{\log\left(\frac{R}{r}\right)}$ while

$$
\begin{aligned}
&\|\nabla\log|z|\|_{\mathrm{L}^{2,1}(B_{\alpha R}\setminus\overline{B}_{\alpha^{-1}r}(0))}=4\int_0^{\frac{1}{\alpha R}}\left(\mathscr{L}^2(B_{\alpha R}\setminus\overline{B}_{\alpha^{-1}r}(0))\right)^{\frac{1}{2}}dt+4\int_{\frac{1}{\alpha R}}^{\frac{1}{\alpha^{-1}r}}\left(\mathscr{L}^2\left(B_{\frac{1}{t}}\setminus\overline{B}_{\alpha^{-1}r}(0)\right)\right)^{\frac{1}{2}}dt\\
&=\frac{4\sqrt{\pi}}{\alpha R}\left(\alpha^2R^2-\alpha^{-2}r^2\right)^{\frac{1}{2}}+4\sqrt{\pi}\int_{\frac{1}{\alpha R}}^{\frac{1}{\alpha^{-1}r}}\frac{1}{t}\sqrt{1-\frac{r^2t^2}{\alpha^2}}dt=4\sqrt{\pi}\left(\log\left(\frac{\alpha^2R}{r}\right)+\log\left(1+\sqrt{1-\left(\frac{r}{\alpha^2R}\right)^2}\right)\right).
\end{aligned}
$$

In particular, for all fixed $0<\alpha<1$, if $\{R_k\}_{k\in\mathbb{N}},\{r_k\}_{k\in\mathbb{N}}\subset(0,\infty)$ are sequences chosen such that $\dfrac{R_k}{r_k}\underset{k\to\infty}{\longrightarrow}\infty$, we have

$$
\lim_{k\to\infty}\frac{\|\nabla\log|z|\|_{\mathrm{L}^{2,1}(B_{\alpha R_k}\setminus\overline{B}_{\alpha^{-1}r_k}(0))}}{\|\nabla\log|z|\|_{\mathrm{L}^2(B_{R_k}\setminus\overline{B}_{r_k}(0))}}=\infty.
$$

If the assumption $4r<R$ does not hold, observe that we get the estimate

$$
\|\nabla u\|_{\mathrm{L}^{2,1}(B_{\alpha R}\setminus\overline{B}_{\alpha^{-1}r}(0))}\leq\frac{8\sqrt{2}}{\sqrt{1-\left(\frac{r}{R}\right)^2}}\frac{\alpha}{1-\alpha^2}\|\nabla u\|_{\mathrm{L}^2(B_R\setminus\overline{B}_r(0))}.
$$

Proposition 2.5. *Let $0<2^6r<R<\infty$ be fixed radii, and $u:\Omega=B_R\setminus\overline{B}_r(0)\to\mathbb{R}$ be a harmonic function such that for some $\rho_0\in(r,R)$*

$$
\int_{\partial B_{\rho_0}}\partial_\nu u\,d\mathscr{H}^1=0.
$$

Then for all $\left(\frac{r}{R}\right)^{\frac{1}{3}} < \alpha < \frac{1}{4}$,

$$\|\nabla u\|_{\mathrm{L}^{2,1}(B_{\alpha R}\setminus\overline{B}_{\alpha^{-1}r}(0))} \leq 24\,\Gamma_1\sqrt{\alpha}\,\|\nabla u\|_{\mathrm{L}^{2,\infty}(B_R\setminus\overline{B}_r(0))}\,,$$

where Γ_1 *is given in Lemma 2.2.*

Proof. Let $\beta = \sqrt{\alpha}$. Then by Lemma 2.3, we have

$$\|\nabla u\|_{\mathrm{L}^{2,1}(B_{\beta^2 R}\setminus\overline{B}_{\beta^{-2}r}(0))} \leq \frac{12\beta}{1-\beta}\,\|\nabla u\|_{\mathrm{L}^{2}(B_{\beta R}\setminus\overline{B}_{\beta^{-1}r}(0))}\,.$$

Furthermore, by Lemma 2.3, we have

$$\|\nabla u\|_{\mathrm{L}^{2}(B_{\beta R}\setminus\overline{B}_{\beta^{-1}r}(0))} \leq \Gamma_1\,\|\nabla u\|_{\mathrm{L}^{2,\infty}(B_R\setminus\overline{B}_r(0))}$$

Therefore, as $\beta = \sqrt{\alpha} < 1/2$, we find

$$\|\nabla u\|_{\mathrm{L}^{2,1}(B_{\alpha R}\setminus\overline{B}_{\alpha^{-1}r}(0))} \leq \frac{12\sqrt{\alpha}}{1-\sqrt{\alpha}}\Gamma_1\,\|\nabla u\|_{\mathrm{L}^{2,\infty}(B_R\setminus\overline{B}_r(0))} \leq 24\,\Gamma_1\sqrt{\alpha}\,\|\nabla u\|_{\mathrm{L}^{2,\infty}(B_R\setminus\overline{B}_r(0))}\,,$$

which concludes the proof of the corollary. ☐

We will also need a quantitative estimate of the Lorentz-Sobolev embedding $W^{1,(2,1)}(\Omega) \to C^0(\Omega)$.

Lemma 2.6. *Let* $n \geq 2$, $\Omega \subset \mathbb{R}^n$ *be a bounded connected open set and* $u \in W^{1,(n,1)}(\Omega)$. *Then* $u \in C^0(\Omega)$ *and for all* $x, y \in \Omega$ *such that* $B_{2|x-y|}(x) \cup B_{2|x-y|}(y) \subset \Omega$, *we have*

$$|u(x) - u(y)| \leq \frac{2^{n+1}}{\alpha(n)^{\frac{1}{n}}}\,\|\nabla u\|_{\mathrm{L}^{n,1}(\Omega\cap B_{2|x-y|}(x))}\,. \tag{2.15}$$

Furthermore, if Ω *is a bounded Lipschitz open subset of* $\mathbb{R}^n$, *then there exists a constant* $C_4 = C_4(\Omega)$ *such that*

$$\|u - \overline{u}_\Omega\|_{\mathrm{L}^\infty(\Omega)} \leq C_4\,\|\nabla u\|_{\mathrm{L}^{n,1}(\Omega)}\,, \tag{2.16}$$

where $\overline{u}_\Omega = \fint_\Omega u\,d\mathscr{L}^n$ *is the mean of* u.

Remarks on the proof. The proof proceeds in a fairly standard way, using an estimate on averages, the $L^{n,1}/L^{\frac{n}{n-1},\infty}$ duality and Lebesgue differentiation theorem on $\mathbb{R}^n$. The extension to the case of domains is easily given by extension operators and interpolation theory to obtain a continue linear extension operator $W^{1,(n,1)}(\Omega) \to W^{1,(n,1)}(\mathbb{R}^n)$ (using the Stein-Weiss interpolation theorem).

Proof. Let $x \in \Omega$ and $d = \mathrm{dist}(x, \partial\Omega) > 0$. For all $0 < r < d$, let

$$u_{x,r} = \fint_{B_r(x)} u\,d\mathscr{L}^n = \frac{1}{\alpha(n)r^n}\int_{B_r(x)} u\,d\mathscr{L}^n.$$

Then for all $0 < r < d$, we have

$$u_{x,r} = \frac{1}{\alpha(n)}\int_{B_1(0)} u(x + r(y-x))dy$$

so that

$$\left|\frac{d}{dr}u_{x,r}\right| = \left|\fint_{B_1(0)} \nabla u(x + r(y-x))\cdot(y-x)dy\right| \leq \fint_{B_r(x)} |\nabla u| d\mathscr{L}^n. \tag{2.17}$$

Therefore, we have by Fubini theorem and the duality $L^{n,1}/L^{\frac{n}{n-1},\infty}$ (see the estimate (7.8)) for all $0<t\leq d$

$$\begin{aligned}
\int_0^t\left|\frac{d}{dr}u_{x,r}\right|dr&\leq\frac{1}{\alpha(n)}\int_0^t\frac{1}{r^n}\int_{B_r(x)}|\nabla u(y)|d\mathscr{L}^n(y)dr=\frac{1}{\alpha(n)}\int_0^t\int_{B_t(x)}\frac{1}{r^n}|\nabla u(y)|\mathbf{1}_{\{|x-y|<r\}}d\mathscr{L}^n(y)dr\\
&=\frac{1}{\alpha(n)}\int_{B_t(x)}|\nabla u(y)|\left(\int_{|x-y|}^d\frac{dr}{r^n}\right)d\mathscr{L}^n(y)\leq\frac{1}{(n-1)\alpha(n)}\int_{B_t(x)}\frac{|\nabla u(y)|}{|x-y|^{n-1}}d\mathscr{L}^n(y)\\
&\leq\frac{1}{n^2\alpha(n)}\|\nabla u\|_{\mathrm{L}^{n,1}(B_t(x))}\left\|\frac{1}{|x-\,\cdot\,|^{n-1}}\right\|_{\mathrm{L}^{\frac{n}{n-1},\infty}(B_t(x))}=\frac{1}{n\alpha(n)^{\frac{1}{n}}}\|\nabla u\|_{\mathrm{L}^{n,1}(B_t(x))}
\end{aligned}$$

as for all $x\in\mathbb{R}^n$

$$\left\|\frac{1}{|x-\,\cdot\,|^{n-1}}\right\|_{\mathrm{L}^{\frac{n}{n-1},\infty}(\mathbb{R}^n)}=n\alpha(n)^{\frac{n}{n-1}}. \tag{2.18}$$

Therefore, by the Sobolev embedding $W^{1,1}(\mathbb{R})\subset C^0(\mathbb{R})$, the function $(0,d]\to\mathbb{R},r\mapsto u_{x,r}$ is continuous, and for all $0<s<t\leq d$, we have

$$|u_{x,s}-u_{x,t}|\leq\int_s^t\left|\frac{d}{dr}u_{x,r}\right|dr\leq\frac{1}{n\alpha(n)^{\frac{1}{n}}}\|\nabla u\|_{\mathrm{L}^{n,1}(B_t(x))}. \tag{2.19}$$

Let $\{r_n\}_{n\in\mathbb{N}}\subset(0,\infty)$ such that $r_n\underset{n\to\infty}{\longrightarrow}0$. Then (2.19) implies that

$$|u_{x,r_n}-u_{x,r_m}|\leq\frac{1}{n\alpha(n)^{\frac{1}{n}}}\|\nabla u\|_{\mathrm{L}^{n,1}(B_{\max\{r_n,r_m\}}(x))}\underset{n,m\to\infty}{\longrightarrow}0$$

which implies that $\{u_{x,r_n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Now, recall that by the Lebesgue differentiation theorem, for $\mathscr{L}^n$ almost all $x\in\Omega$, we have

$$u(x)=\lim_{r\to0}u_{x,r}.$$

Therefore, for $\mathscr{L}^n$ almost all $x\in\Omega$ and for all $0<r<d(x)=\mathrm{dist}(x,\partial\Omega)$, we have

$$|u(x)-u_{x,r}|\leq\frac{1}{n\alpha(n)^{\frac{1}{n}}}\|\nabla u\|_{\mathrm{L}^{n,1}(B_r(x))}. \tag{2.20}$$

To prove that u is continuous, let $x,y\in\Omega$ such that (2.20) holds for x and y (the proof is an adaptation of the Hölder continuous embedding of Campanato spaces of the right indices). Furthermore, without loss of generality, we can assume that $x\neq y$, and $2|x-y|<\max\{d(x),d(y)\}$, so that

$$B_{2|x-y|}(x)\cup B_{2|x-y|}(y)\subset\Omega.$$

Therefore, if $r=|x-y|$ we have

$$\begin{aligned}
|u(x)-u(y)|&\leq|u(x)-u_{x,r}|+|u_{x,r}-u_{y,r}|+|u(y)-u_{y,r}|\\
&\leq\frac{1}{n\alpha(n)^{\frac{1}{n}}}\left(\|\nabla u\|_{\mathrm{L}^{n,1}(B_{|x-y|}(x))}+\|\nabla u\|_{\mathrm{L}^{n,1}(B_{|x-y|}(y))}\right)+|u_{x,r}-u_{y,r}|
\end{aligned} \tag{2.21}$$

so we need only estimate $|u_{x,r}-u_{y,r}|$, as

$$\|\nabla u\|_{\mathrm{L}^{n,1}(B_{|x-y|}(x))}+\|\nabla u\|_{\mathrm{L}^{n,1}(B_{|x-y|}(y))}\underset{y\to x}{\longrightarrow}0.$$

We have

$$\begin{aligned}
u_{x,r}-u_{y,r}&=\frac{1}{\alpha(n)r^n}\int_{B_r(x)}u(z_1)d\mathscr{L}^n(z)_1-\frac{1}{\alpha(n)r^n}\int_{B_r(y)}u(z_2)d\mathscr{L}^n(z_2)\\
&=\frac{1}{(\alpha(n)r^n)^2}\int_{B_r(x)\times B_r(y)}(u(z_1)-u(z_2))d\mathscr{L}^n(z_1)d\mathscr{L}^n(z_2)
\end{aligned}$$

$$= \frac{1}{(\alpha(n)r^n)^2}\int_{B_r(x)\times B_r(y)}\left(\int_0^1 \nabla u(z_2+t(z_1-z_2))\cdot(z_1-z_2)dt\right)d\mathscr{L}^n(z_1)d\mathscr{L}^n(z_2) \tag{2.22}$$

Furthermore, for all $t\in[0,1]$ and $(z_1,z_2)\in B_r(x)\times B_r(y)$, we have $z_2+t(z_1-z_2)\in B_{2r}(x)$ and $|z_1-z_2|\leq 2r$. Therefore, Fubini's theorem implies that (by (7.8))

$$\begin{aligned}
&\left|\int_{B_r(x)}\left(\int_0^1 \nabla u(z_2+t(z_1-z_2))\cdot(z_1-z_2)dt\right)d\mathscr{L}^n(z_1)\right|\\
&\leq \int_0^1\left(\int_{B_r(x)}\frac{|\nabla u(z_2+t(z_1-z_2))|}{|z_1-z_2|^{n-1}}|z_1-z_2|^n d\mathscr{L}^n(z_1)\right)dt\\
&\leq \frac{1}{n}2^nr^n\int_0^1 \|\nabla u(z_2+t(\,\cdot\,-z_2))\|_{\mathrm{L}^{n,1}(B_r(x))}\left\|\frac{1}{|\,\cdot\,-z_2|}\right\|_{\mathrm{L}^{\frac{n}{n-1},\infty}(B_r(x))}dt\\
&\leq 2^nr^n\alpha(n)^{\frac{n}{n-1}}\int_0^1\|\nabla u\|_{\mathrm{L}^{n,1}(B_{2r}(x))}dt = 2^nr^n\alpha(n)^{\frac{n}{n-1}}\|\nabla u\|_{\mathrm{L}^{n,1}(B_{2r}(x))}.
\end{aligned} \tag{2.23}$$

Therefore, by (2.22) and (2.23), we find

$$|u_{x,r}-u_{y,r}|\leq\frac{1}{\alpha(n)r^n}\int_{B_r(y)}\frac{2^n}{\alpha(n)^{\frac{1}{n}}}\|\nabla u\|_{\mathrm{L}^{n,1}(B_{2|x-y|}(x))}d\mathscr{L}^n(z_2)=\frac{2^n}{\alpha(n)^{\frac{1}{n}}}\|\nabla u\|_{\mathrm{L}^{n,1}(B_{2|x-y|}(x))}. \tag{2.24}$$

Furthermore, as the argument is symmetric in x and y notice that

$$|u_{x,r}-u_{y,r}|\leq\frac{2^n}{\alpha(n)^{\frac{1}{n}}}\min\left\{\|\nabla u\|_{\mathrm{L}^{n,1}(B_{2|x-y|}(x))},\|\nabla u\|_{\mathrm{L}^{n,1}(B_{2|x-y|}(y))}\right\}.$$

Finally, thanks to (2.21) and (2.24) we get

$$|u(x)-u(y)|\leq\frac{2^{n+1}}{\alpha(n)^{\frac{1}{n}}}\|\nabla u\|_{\mathrm{L}^{n,1}(B_{2|x-y|}(x))} \tag{2.25}$$

which implies that u is continuous, with modulus of continuity at x

$$r\mapsto\frac{2^{n+1}}{\alpha(n)^{\frac{1}{n}}}\|\nabla u\|_{\mathrm{L}^{n,1}(\Omega\cap B_{2r}(x))}.$$

Now, for the L^∞ bound, first consider the case $\Omega=\mathbb{R}^n$, and let $G:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}\cup\{\infty\}$ be the Green's function of the Laplacian on $\mathbb{R}^n$. Then

$$\nabla_yG(x,y)=\frac{1}{n\alpha(n)}\frac{1}{|x-y|^{n-1}}\in L^{\frac{n}{n-1},\infty}(\mathbb{R}^n)$$

and we have for all $x\in\mathbb{R}^n$

$$u(x)=\int_{\mathbb{R}^n}\Delta_yG(x,y)u(y)dy=-\int_{\mathbb{R}^n}\nabla_yG(x,y)\cdot\nabla u(y)dy$$

and (2.18) implies that

$$\begin{aligned}
\|u\|_{\mathrm{L}^\infty(\mathbb{R}^n)}&\leq\frac{n-1}{n^2}\|\nabla u\|_{\mathrm{L}^{n,1}(\mathbb{R}^n)}\|\nabla_yG(x,y)\|_{\mathrm{L}^{\frac{n}{n-1},\infty}(\mathbb{R}^n)}=\frac{(n-1)}{n^3\alpha(n)}\|\nabla u\|_{\mathrm{L}^{n,1}(\mathbb{R}^n)}\left\|\frac{1}{|x-\,\cdot\,|^{n-1}}\right\|_{\mathrm{L}^{\frac{n}{n-1},\infty}(\mathbb{R}^n)}\\
&=\frac{(n-1)}{n^2\alpha(n)^{\frac{1}{n}}}\|\nabla u\|_{\mathrm{L}^{n,1}(\mathbb{R}^n)}\leq\frac{1}{n\alpha(n)^{\frac{1}{n}}}\|\nabla u\|_{\mathrm{L}^{n,1}(\mathbb{R}^N)}
\end{aligned} \tag{2.26}$$

Now, (thanks to [3] IX.7) there exists a linear extension operator

$$P:\bigcup_{1\leq p<\infty}W^{1,p}(\Omega)\to\bigcup_{1\leq p<\infty}W^{1,p}(\mathbb{R}^n)$$

such that for $1 \leq p < \infty$ the restriction $P|W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$ be a continuous linear operator. Then by identifying $W^{1,p}(\Omega)$ with a closed subset of $L^p(\mathbb{R}^n)^{n+1}$, the Stein-Weiss interpolation theorem implies that for all P extends as a continuous linear operator $W^{1,(n,1)}(\Omega)$ into $W^{1,(n,1)}(\mathbb{R}^n)$, as the Sobolev embedding $L^n(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \leq q < \infty$ shows that $\nabla u \in L^{n,1}(\Omega)$ implies that $u \in L^{n,1}(\Omega)$. Therefore, by (2.26), for all $u \in W^{1,(n,1)}(\Omega)$, we have

$$\begin{aligned}\|u\|_{\mathrm{L}^\infty(\Omega)} &\leq \|\nabla Pu\|_{\mathrm{L}^\infty(\mathbb{R}^n)} \leq \frac{1}{n\alpha(n)^{\frac{1}{n}}}\|Pu\|_{\mathrm{L}^{n,1}(\mathbb{R}^n)} \leq \Gamma_3\left(\|u\|_{\mathrm{L}^{n,1}(\Omega)} + \|\nabla u\|_{\mathrm{L}^{n,1}(\Omega)}\right)\\ &\leq \Gamma_3'(\|u\|_{\mathrm{L}^n(\Omega)} + \|\nabla u\|_{\mathrm{L}^{n,1}(\Omega)}),\end{aligned} \tag{2.27}$$

where we have used in the last line the embedding $W^{1,n}(\Omega) \hookrightarrow L^{n,1}(\Omega)$.

Now, (2.27) implies by the classical Poincaré-Wirtinger inequality and the continuous embedding $L^{n,1}(\Omega) \hookrightarrow L^n(\Omega)$

$$\begin{aligned}\|u - \overline{u}_\Omega\|_{\mathrm{L}^\infty(\Omega)} &\leq \Gamma_3'(\|u - \overline{u}_\Omega\|_{\mathrm{L}^n(\Omega)} + \|\nabla u\|_{\mathrm{L}^{n,1}(\Omega)}) \leq \Gamma_3'\left(\Gamma_3''\|\nabla u\|_{\mathrm{L}^n(\Omega)} + \|\nabla u\|_{\mathrm{L}^{n,1}(\Omega)}\right)\\ &\leq C_4(\Omega)\|\nabla u\|_{\mathrm{L}^{n,1}(\Omega)}\end{aligned}$$

and this concludes the proof of the Lemma. □

Now, we will need to refine the L^∞ bound to obtain an estimate independent of the conformal class (bounded away from $-\infty$) of flat annuli in $\mathbb{R}^n$.

Proposition 2.7. *Let $0 < 2r < R < \infty$ and $\Omega = B_R \setminus \overline{B}_r(0) \subset \mathbb{R}^n$. Then there exists a universal constant $\Gamma_4 = \Gamma_4(n)$ such that for all $u \in W^{1,(n,1)}(\Omega)$, we have*

$$\|u - \overline{u}_\Omega\|_{\mathrm{L}^\infty(\Omega)} \leq \Gamma_4(n)\|\nabla u\|_{\mathrm{L}^{n,1}(\Omega)}. \tag{2.28}$$

Remarks on the proof. By scaling invariance of the inequality of Lemma 2.6, the constant $C_4(\Omega(r))$ inequality (2.16) for annuli $\Omega(r) = B_{2r}\setminus\overline{B}_r(0)$ is independent of $0 < r < \infty$, which allows one to introduce a dyadic decomposition of the annulus $\Omega = B_R \setminus \overline{B}_r(0)$ since the conformal class $\log\left(\frac{R}{r}\right) \geq \log(2)$ is bounded from below. Using once more the $L^{n,1}/L^{\frac{n}{n-1},\infty}$ duality and Fubini's theorem, we deduce that the various averages can be controlled by the $L^{n,1}$ norm of ∇u which finally permits after a suitable decomposition to obtain the inequality (2.28).

Proof. First, observe that the L^∞ norm and the $(n,1)$ norm of the gradient $\|\nabla\cdot\|_{\mathrm{L}^{n,1}(\Omega)}$ are scaling invariant (see (2.40) for the case $n = 2$). Therefore, the constant $C_4(\Omega)$ in Theorem 2.1 is scaling invariant. In particular, there exists a universal constant $C_4'(n) = C_4(B_2 \setminus B_1(0))$ such that for all $0 < r < \infty$ and $u \in W^{1,(n,1)}(B_{2r} \setminus \overline{B}_r(0))$, we have

$$\left\|u - \overline{u}_{B_{2r}\setminus\overline{B}_r(0)}\right\|_{\mathrm{L}^\infty(\overline{B}_{2r}\setminus\overline{B}_r(0))} \leq C_4'(n)\|\nabla u\|_{\mathrm{L}^{n,1}(B_{2r}\setminus\overline{B}_r(0))}. \tag{2.29}$$

Now, as $2r < R$ let $J \in \mathbb{N}$ such that

$$2^J r < R \leq 2^{J+1} r.$$

Then we have

$$\Omega = B_R \setminus B_{\frac{R}{2}}(0) \cup \bigcup_{j=0}^{J-1} B_{2^{j+1}r} \setminus \overline{B}_{2^j r}(0).$$

For the convenience of notation, let us write $\Omega_j = B_{2^{j+1}r} \setminus \overline{B}_{2^j r}$ for all $0 \leq j \leq J-1$. Thanks to (2.29) for all $0 \leq j \leq J$, we have

$$\|u - \overline{u}_j\|_{\mathrm{L}^\infty(\Omega_j)} \leq C_4'(n)\|\nabla u\|_{\mathrm{L}^{n,1}(\Omega_j)} \quad \text{where } \overline{u}_j = \fint_{\Omega_j} u\, d\mathscr{L}^n$$

$$\left\|u-\overline{u}_{B_R\setminus B_{R/2}(0)}\right\|_{\mathrm{L}^\infty(B_R\setminus B_{R/2}(0))}\leq C_4'(n)\left\|\nabla u\right\|_{\mathrm{L}^{n,1}(B_R\setminus B_{R/2}(0))}. \tag{2.30}$$

Now define for all $r<t<R$

$$u_t=\fint_{\partial B_t(0)}u\,d\mathscr{H}^{n-1}.$$

For all $r<t<R$, thanks to a similar argument as given in (2.17), we have

$$\left|\frac{d}{dt}u_t\right|\leq\fint_{\partial B_t}|\nabla u|d\mathscr{H}^{n-1}.$$

Furthermore, if $r\leq r_1<R$ is a fixed radius, thanks to the co-area formula, we have for $\mathscr{L}^1$ almost all $t\in(r_1,R)$

$$\int_{\partial B_t}|\nabla u|\,d\mathscr{H}^{n-1}=\frac{d}{dt}\int_{r_1}^t\left(\int_{\partial B_s}|\nabla u|d\mathscr{H}^{n-1}\right)d\mathscr{L}^1(s)=\frac{d}{dt}\int_{B_t\setminus\overline{B}_{r_1}(0)}|\nabla u|d\mathscr{L}^n.$$

Therefore, we have

$$\begin{aligned}\int_{r_1}^{r_2}\left|\frac{d}{dt}u_t\right|dt&\leq\frac{1}{n\alpha(n)}\int_{r_1}^{r_2}\frac{1}{t^{n-1}}\left(\int_{\partial B_t}|\nabla u|d\mathscr{H}^{n-1}\right)dt\\&=\frac{1}{n\alpha(n)}\left[\frac{1}{t^{n-1}}\int_{B_t\setminus B_{r_1}(0)}|\nabla u|d\mathscr{L}^n\right]_{r_1}^{r_2}+\frac{n-1}{n\alpha(n)}\int_{r_1}^{r_2}\frac{1}{t^n}\left(\int_{B_t\setminus B_{r_1}(0)}|\nabla u|d\mathscr{L}^n\right)dt\\&=\frac{1}{n\alpha(n)}\frac{1}{r_2^{n-1}}\int_{B_{r_2}\setminus\overline{B}_{r_1}(0)}|\nabla u|d\mathscr{L}^n+\frac{n-1}{n\alpha(n)}\int_{r_1}^{r_2}\int_{B_{r_2}\setminus\overline{B}_{r_1}(0)}\frac{|\nabla u(x)|}{t^n}1_{\{r_1\leq|x|\leq t\}}d\mathscr{L}^n(x)dt.\end{aligned} \tag{2.31}$$

Furthermore, observe that

$$\begin{aligned}\int_{B_{r_2}\setminus B_{r_1}(0)}\frac{|\nabla u(x)|}{|x|^{n-1}}d\mathscr{L}^n(x)&\leq\frac{1}{n}\left\|\nabla u\right\|_{\mathrm{L}^{n,1}(B_{r_2}\setminus\overline{B}_{r_1}(0))}\left\|\frac{1}{|x|^{n-1}}\right\|_{\mathrm{L}^{\frac{n}{n-1},\infty}(B_{r_2}\setminus\overline{B}_{r_1}(0))}\\&\leq\alpha(n)^{\frac{n}{n-1}}\left\|\nabla u\right\|_{\mathrm{L}^{n,1}(B_{r_2}\setminus\overline{B}_{r_1}(0))}\end{aligned} \tag{2.32}$$

while by Fubini's theorem

$$\begin{aligned}\int_{r_1}^{r_2}\int_{B_{r_2}\setminus\overline{B}_{r_1}(0)}\frac{|\nabla u(x)|}{t^n}1_{\{r_1\leq|x|\leq t\}}d\mathscr{L}^n(x)dt&=\int_{B_{r_2}\setminus\overline{B}_{r_1}(0)}|\nabla u(x)|\left(\int_{|x|}^{r_2}\frac{dt}{t^{n-1}}dt\right)d\mathscr{L}^n(x)\\&=\frac{1}{n-1}\int_{B_{r_2}\setminus\overline{B}_{r_1}(0)}|\nabla u(x)|\left(\frac{1}{|x|^{n-1}}-\frac{1}{r_2^{n-1}}\right).\end{aligned} \tag{2.33}$$

Finally, we get by (2.31), (2.32), (2.33), (2.34) and (2.18)

$$\begin{aligned}\int_{r_1}^{r_2}\left|\frac{d}{dt}u_t\right|dt&\leq\frac{1}{n\alpha(n)}\frac{1}{r_2^{n-1}}\int_{B_{r_2}\setminus\overline{B}_{r_1}(0)}|\nabla u|d\mathscr{L}^n+\frac{n-1}{n\alpha(n)}\int_{r_1}^{r_2}\int_{B_{r_2}\setminus\overline{B}_{r_1}(0)}\frac{|\nabla u(x)|}{t^n}1_{\{r_1\leq|x|\leq t\}}d\mathscr{L}^n(x)dt\\&=\frac{1}{n\alpha(n)}\frac{1}{r_2^{n-1}}\int_{B_{r_2}\setminus\overline{B}_{r_1}(0)}|\nabla u|d\mathscr{L}^n+\frac{1}{n\alpha(n)}\int_{B_{r_2}\setminus\overline{B}_{r_1}(0)}|\nabla u(x)|\left(\frac{1}{|x|^{n-1}}-\frac{1}{r_2^{n-1}}\right)\\&=\frac{1}{n\alpha(n)}\int_{B_{r_2}\setminus B_{r_1}(0)}\frac{|\nabla u(x)|}{|x|^{n-1}}d\mathscr{L}^n(x)\leq\frac{1}{n\alpha(n)^{\frac{1}{n}}}\left\|\nabla u\right\|_{\mathrm{L}^{n,1}(B_{r_2}\setminus\overline{B}_{r_1}(0))}\end{aligned} \tag{2.34}$$

Therefore, we have for all $r\leq r_1<r_2\leq R$

$$|u_{r_1}-u_{r_2}|\leq\frac{1}{n\alpha(n)^{\frac{1}{n}}}\left\|\nabla u\right\|_{\mathrm{L}^{n,1}(B_{r_2}\setminus\overline{B}_{r_1}(0))}. \tag{2.35}$$

Furthermore, recalling that $\beta(n)=\mathscr{H}^{n-1}(S^{n-1})=n\alpha(n)$ we obtain for all $r\leq s<t\leq R$, thanks to (2.35) that

$$\begin{aligned}
\fint_{B_t\setminus\overline{B}_s(0)}u\,d\mathscr{L}^n&=\frac{n}{\beta(n)(t^n-s^n)}\int_s^t\left(\int_{\partial B_\rho}u\,d\mathscr{H}^{n-1}\right)d\rho\\
&\leq\int_s^t\left(\frac{\rho^{n-1}}{t^{n-1}}\int_{\partial B_t}u\,d\mathscr{H}^{n-1}+\beta(n)\frac{\rho^{n-1}}{n\alpha(n)^{\frac{1}{n}}}\|\nabla u\|_{\mathrm{L}^{n,1}(B_t\setminus\overline{B}_s(0))}\right)\\
&=\frac{n}{\beta(n)(t^n-s^n)}\left(\frac{t^n-s^n}{n}\beta(n)\fint_{\partial B_t}u\,d\mathscr{H}^{n-1}+\frac{\beta(n)}{n}\frac{t^n-s^n}{n\alpha(n)^{\frac{1}{n}}}\|\nabla u\|_{\mathrm{L}^{n,1}(B_{r_2}\setminus\overline{B}_{r_1}(0))}\right)\\
&=\fint_{\partial B_t}u\,d\mathscr{H}^{n-1}+\frac{1}{n\alpha(n)^{\frac{1}{n}}}\|\nabla u\|_{\mathrm{L}^{n,1}(B_t\setminus\overline{B}_s(0))}
\end{aligned}$$

and the reverse inequality (given by (2.35))

$$\int_{\partial B_\rho}u\,d\mathscr{H}^{n-1}\geq\frac{\rho^{n-1}}{t^{n-1}}\int_{\partial B_t}u\,d\mathscr{H}^{n-1}-\frac{\beta(n)}{n\alpha(n)^{\frac{1}{n}}}\rho^{n-1}\|\nabla u\|_{\mathrm{L}^{n,1}(B_t\setminus\overline{B}_s(0))}\quad\text{for all }\ s<\rho<t$$

shows that for all $r\leq s<t\leq R$

$$\left|\fint_{B_t\setminus\overline{B}_s(0)}u\,d\mathscr{L}^n-\fint_{\partial B_t}u\,d\mathscr{H}^{n-1}\right|\leq\frac{1}{n\alpha(n)^{\frac{1}{n}}}\|\nabla u\|_{\mathrm{L}^{n,1}(B_t\setminus\overline{B}_s(0))}.$$

Therefore, by the triangle inequality we finally obtain that for all $0\leq j\leq J-1$,

$$\begin{aligned}
|u_j-\overline{u}_\Omega|&=\left|\fint_{B_{2^{j+1}r}\setminus B_{2^jr}(0)}u\,d\mathscr{L}^n-\fint_{B_R\setminus B_r(0)}u\,d\mathscr{L}^n\right|\leq\left|\fint_{B_{2^{j+1}r}\setminus B_{2^jr}(0)}u\,d\mathscr{L}^n-\fint_{\partial B_{2^{j+1}r}}u\,d\mathscr{H}^{n-1}\right|\\
&\quad+\left|\fint_{\partial B_{2^{j+1}r}}u\,d\mathscr{H}^{n-1}-\int_{\partial B_R}u\,d\mathscr{H}^{n-1}\right|+\left|\fint_{B_R\setminus B_r(0)}u\,d\mathscr{L}^n-\int_{\partial B_R}u\,d\mathscr{H}^{n-1}\right|\\
&\leq\frac{3}{n\alpha(n)^{\frac{1}{n}}}\|\nabla u\|_{\mathrm{L}^{n,1}(\Omega)},
\end{aligned}\tag{2.36}$$

and likewise,

$$\left|\overline{u}_{B_R\setminus B_{R/2}(0)}-\overline{u}_\Omega\right|\leq\frac{3}{n\alpha(n)^{\frac{1}{n}}}\|\nabla u\|_{\mathrm{L}^{n,1}(\Omega)}.\tag{2.37}$$

Finally, thanks to (2.30), (2.36) and (2.37), we have

$$\|u-\overline{u}_\Omega\|_{\mathrm{L}^\infty(\Omega)}\leq\left(C_4'(n)+\frac{3}{n\alpha(n)^{\frac{1}{n}}}\right)\|\nabla u\|_{\mathrm{L}^{n,1}(\Omega)}$$

and this concludes the proof of the Proposition. □

We now come back to the proof of Theorem 2.1.

Remarks on the proof. The proof closely follows the one of [2], using the $L^{2,1}$ estimate in lieu of the L^2 one, using the previous Lemma (2.5) to prove the inequality (2.2), and Proposition 2.7 for the inequality (2.3).

Proof. (of Theorem 2.1) Thanks to Lemma IV.1 [2], there exists a universal constant $\Gamma_6=\Gamma_6(n)>0$ and an extension $\widetilde{\vec{n}}:B_R(0)\to\mathscr{G}_{n-2}(\mathbb{R}^n)$ of $\vec{n}$ such that

$$\begin{cases}\widetilde{\vec{n}}=\vec{n}\quad\text{on }\Omega=B_R\setminus\overline{B}_r(0)\\ \|\nabla\widetilde{\vec{n}}\|_{\mathrm{L}^{2,\infty}(B_R(0))}\leq\Gamma_6(n)\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}.\end{cases}\tag{2.38}$$

Therefore, by Lemma IV.3 of [2], there exists a universal constant $\Gamma_7=\Gamma_7(n)$ and a moving Coulomb frame $(\vec{e}_1,\vec{e}_2)\in W^{1,2}(B_R(0),S^{n-1})\times W^{1,2}(B_R(0),S^{n-1})$ such that

$$\begin{cases}\widetilde{\vec{n}}=\star\,(\vec{e}_1\wedge\vec{e}_2)\quad \mathrm{div}\,(\vec{e}_1\cdot\nabla\vec{e}_2)=0\\ \|\nabla\vec{e}_1\|^2_{\mathrm{L}^2(B_R(0))}+\|\nabla\vec{e}_2\|^2_{\mathrm{L}^2(B_R(0))}\leq\Gamma_7(n)\,\|\nabla\vec{n}\|^2_{\mathrm{L}^2(\Omega)}\,.\end{cases}\tag{2.39}$$

Furthermore, notice that for all $u\in W^{1,(2,1)}_{\mathrm{loc}}(\mathbb{R}^2)$, and for all $\rho>0$, we have

$$\begin{aligned}&\|\nabla u\|_{\mathrm{L}^{2,1}(B_\rho(0))}=4\int_0^\infty\left(\mathscr{L}^2(B_r0\cap\{x:|\nabla u(x)|>t\})\right)^{\frac{1}{2}}d\mathscr{L}^1(t)\\&=4\int_0^\infty\left(\int_{B_\rho(0)}1_{\{x:|\nabla u(x)|>t\}}d\mathscr{L}^2(x)\right)^{\frac{1}{2}}d\mathscr{L}^1(t)=4\int_0^\infty\left(\int_{B_1(0)}1_{\{y:|\nabla(u\circ\varphi_\rho)(y)|>\rho t\}}\rho^2d\mathscr{L}^2(y)\right)^{\frac{1}{2}}d\mathscr{L}^1(t)\\&=4\int_0^\infty\left(\int_{B_1(0)}1_{\{y:|\nabla(u\circ\varphi_\rho)(y)|>s\}}\rho^2d\mathscr{L}^2(y)\right)^{\frac{1}{2}}\rho^{-1}\,d\mathscr{L}^1(s)=\|\nabla(u\circ\varphi_\rho)\|_{\mathrm{L}^{2,1}(B_1(0))}\,.\end{aligned}\tag{2.40}$$

where $\varphi_\rho(y)=\rho y$. Now, if $\mu:B_R(0)\to\mathbb{R}$ is the unique solution of the system

$$\begin{cases}\Delta\mu=\nabla^\perp\vec{e}_1\cdot\nabla\vec{e}_2 & \text{in }\;B_R(0)\\ \quad\mu=0 & \text{on }\;\partial B_R(0)\end{cases}\tag{2.41}$$

then $\widetilde{\mu}=\mu\circ\varphi_R$ solves (with evident notations)

$$\begin{cases}\Delta\widetilde{\mu}=\nabla^\perp\widetilde{\vec{e}_1}\cdot\nabla\widetilde{\vec{e}_2} & \text{in }\;B_1(0)\\ \quad\widetilde{\mu}=0 & \text{on }\;S^1\end{cases}$$

Therefore, the improved Wente inequality ([10], 3.4.1) shows that there exists a universal constant $\Gamma_8>0$ such that

$$\begin{aligned}\|\nabla\mu\|_{\mathrm{L}^{2,1}(B_R(0))}=\|\nabla\widetilde{\mu}\|_{\mathrm{L}^{2,1}(B_1(0))}&\leq\Gamma_8\left\|\nabla\widetilde{\vec{e}_1}\right\|_{\mathrm{L}^2(B_1(0))}\left\|\nabla\widetilde{\vec{e}_2}\right\|_{\mathrm{L}^2(B_1(0))}=\Gamma_8\,\|\nabla\vec{e}_1\|_{\mathrm{L}^2(B_R(0))}\,\|\nabla\vec{e}_2\|_{\mathrm{L}^2(B_R(0))}\\&\leq\frac{1}{2}\Gamma_7(n)\Gamma_8\int_\Omega|\nabla\vec{n}|^2dx.\end{aligned}\tag{2.42}$$

Furthermore, notice that we also have the optimal inequality

$$\|\nabla\mu\|_{\mathrm{L}^2(B_R(0))}\leq\frac{1}{4}\sqrt{\frac{3}{\pi}}\,\|\nabla\vec{e}_1\|_{\mathrm{L}^2(B_R(0))}\,\|\nabla\vec{e}_2\|_{\mathrm{L}^2(B_R(0))}\leq\frac{1}{8}\sqrt{\frac{3}{\pi}}\Gamma_7(n)\int_\Omega|\nabla\vec{n}|^2dx.\tag{2.43}$$

Now, let $\upsilon=\lambda-\mu$ on $\Omega=B_R\setminus\overline{B}_r(0)$. Then υ is harmonic on Ω and $\upsilon=\lambda$ on $\partial B_R(0)$. Then as υ is harmonic, there exists $d\in\mathbb{R}$ and $\{a_k\}_{k\in\mathbb{Z}}\subset\mathbb{C}$ such that

$$\upsilon(\rho,\theta)=a_0+d\,\log\rho+\sum_{k\in\mathbb{Z}^\ast}\left(a_k\rho^k+\overline{a_{-k}}\rho^{-k}\right)e^{ik\theta}.$$

Now, noticing that for all $r<\rho<R$

$$d=\frac{1}{2\pi}\int_{\partial B_\rho}\partial_\nu\upsilon,\tag{2.44}$$

this implies that $\upsilon-d\log|z|$ satisfies the hypothesis of Proposition 2.5. Therefore, using the identity $\upsilon=\lambda-\mu$, the inequalities (2.43) and $\|\cdot\|_{\mathrm{L}^{2,\infty}(\,\cdot\,)}\leq 2\,\|\cdot\|_{\mathrm{L}^2(\,\cdot\,)}$, we have for all $\left(\dfrac{r}{R}\right)^{\frac{1}{3}}<\alpha<\dfrac{1}{4}$

$$\begin{aligned}&\|\nabla(\upsilon-d\log|z|)\|_{\mathrm{L}^{2,1}(B_{\alpha R}\setminus\overline{B}_{\alpha^{-1}r})}\leq 24\,\Gamma_1\sqrt{\alpha}\,\|\nabla(\upsilon-d\log|z|)\|_{\mathrm{L}^{2,\infty}(B_R\setminus\overline{B_r}(0))}\\&\leq 24\,\Gamma_1\sqrt{\alpha}\left(\|\nabla(\lambda-d\log|z|)\|_{\mathrm{L}^{2,\infty}(B_R\setminus\overline{B}_r(0))}+\|\nabla\mu\|_{\mathrm{L}^{2,\infty}(B_R\setminus\overline{B}_r(0))}\right)\end{aligned}$$

$$\leq 24\,\Gamma_1\sqrt{\alpha}\left(\|\nabla(\lambda-d\log|z|)\|_{\mathrm{L}^{2,\infty}(\Omega)}+2\,\|\nabla\mu\|_{\mathrm{L}^2(\Omega)}\right)$$
$$\leq 24\,\Gamma_1\sqrt{\alpha}\left(\|\nabla(\lambda-d\log|z|)\|_{\mathrm{L}^{2,\infty}(\Omega)}+\frac{1}{4}\sqrt{\frac{3}{\pi}}\Gamma_7(n)\int_{\Omega}|\nabla\vec{n}|^2dx\right). \tag{2.45}$$

Furthermore, notice that by the co-area formula, for all $s\in(r,R)$ such that $2s<R$, we have

$$\int_{B_{2s}\setminus\overline{B}_s(0)}|\nabla\upsilon(x)|dx=\int_s^{2s}\left(\rho\int_{\partial B_\rho}|\nabla\upsilon|d\mathscr{H}^1\right)\frac{d\rho}{\rho}\geq\log(2)\inf_{s<\rho<2s}\left(\rho\int_{\partial B_\rho}|\nabla\upsilon|d\mathscr{H}^1\right).$$

Therefore, there exists $\rho\in(s,2s)$ such that

$$\begin{aligned}\int_{\partial B_\rho}|\nabla\upsilon|d\mathscr{H}^1&\leq\frac{1}{\log(2)\rho}\int_{B_{2s}\setminus\overline{B}_s(0)}|\nabla\upsilon(x)|dx\leq\frac{1}{\log(2)\rho}\|1\|_{\mathrm{L}^{2,1}(B_{2s}\setminus\overline{B}_s)}\|\nabla\upsilon\|_{\mathrm{L}^{2,\infty}(B_{2s}\setminus\overline{B}_s(0))}\\&=\frac{1}{\log(2)\rho}4\sqrt{3\pi}s\,\|\nabla\upsilon\|_{\mathrm{L}^{2,\infty}(B_{2s}\setminus\overline{B}_s(0))}\leq\frac{4\sqrt{3\pi}}{\log(2)}\left(\|\nabla\lambda\|_{\mathrm{L}^{2,\infty}(\Omega)}+2\,\|\nabla\mu\|_{\mathrm{L}^2(\Omega)}\right)\\&\leq\frac{4\sqrt{3\pi}}{\log(2)}\left(\|\nabla\lambda\|_{\mathrm{L}^{2,\infty}(\Omega)}+\frac{1}{4}\sqrt{\frac{3}{\pi}}\Gamma_7(n)\int_\Omega|\nabla\vec{n}|^2dx\right).\end{aligned}$$

This implies by (2.44) that

$$|d|\leq\frac{2}{\log(2)}\sqrt{\frac{3}{\pi}}\left(\|\nabla\lambda\|_{\mathrm{L}^{2,\infty}(\Omega)}+\frac{1}{4}\sqrt{\frac{3}{\pi}}\Gamma_7(n)\int_\Omega|\nabla\vec{n}|^2dx\right). \tag{2.46}$$

As $\|\nabla\log|z|\|_{\mathrm{L}^{2,\infty}(\Omega)}=2\sqrt{\pi}$, by (2.45) and (2.46) there exists a universal constant $\Gamma_9=\Gamma_9(n)$ such that

$$\|\nabla(\upsilon-d\log|z|)\|_{\mathrm{L}^{2,1}(B_{\alpha R}\setminus\overline{B}_{\alpha^{-1}r})}\leq\Gamma_9(n)\sqrt{\alpha}\left(\|\nabla\lambda\|_{\mathrm{L}^{2,\infty}(\Omega)}+\int_\Omega|\nabla\vec{n}|^2dx\right). \tag{2.47}$$

Finally, putting together (2.42), (2.47) and recalling that $\lambda=\mu+\upsilon$, we have for all $\left(\frac{r}{R}\right)^{\frac{1}{4}}\leq\alpha<\frac{1}{4}$

$$\begin{aligned}&\|\nabla(\lambda-d\log|z|)\|_{\mathrm{L}^{2,1}(B_{\alpha R}\setminus\overline{B}_{\alpha^{-1}r}(0))}\leq\|\nabla(\upsilon-d\log|z|)\|_{\mathrm{L}^{2,1}(B_{\alpha R}\setminus\overline{B}_{\alpha^{-1}r})}+\|\nabla\mu\|_{\mathrm{L}^{2,1}(B_{\alpha R}\setminus B_{\alpha^{-1}}(0))}\\&\leq\Gamma_9(n)\sqrt{\alpha}\left(\|\nabla\lambda\|_{\mathrm{L}^{2,\infty}(\Omega)}+\int_\Omega|\nabla\vec{n}|^2dx\right)+\frac{1}{2}\Gamma_7(n)\Gamma_8\int_\Omega|\nabla\vec{n}|^2dx.\end{aligned} \tag{2.48}$$

Now, we estimate for $r\leq\rho<R$ the following quantity

$$\left|d-\frac{1}{2\pi}\int_{\partial B_\rho}\partial_\nu\lambda\,d\mathscr{H}^1\right|=\left|\frac{1}{2\pi}\int_{\partial B_\rho}\partial_\nu\mu\,d\mathscr{H}^1\right|.$$

We have, recalling that μ is well defined on $B_R(0)$ and satisfies (2.41), we find

$$\begin{aligned}0&=\int_{B_R\setminus B_\rho}\mu(x)\,\Delta\log\left(\frac{|x|}{R}\right)dx=-\log\left(\frac{R}{\rho}\right)\int_{\partial B_\rho}\partial_\nu\mu\,d\mathscr{H}^1+\int_{B_R\setminus B_\rho(0)}\Delta\mu\,\log\left(\frac{|x|}{R}\right)dx\\&=-\log\left(\frac{R}{\rho}\right)\int_{\partial B_\rho}\partial_\nu\mu\,d\mathscr{H}^1+\int_{B_R(0)}\Delta\mu\log\left(\frac{|x|}{R}\right)dx-\int_{B_\rho(0)}\left(\nabla^\perp\vec{e}_1\cdot\nabla\vec{e}_2\right)\log\left(\frac{|x|}{R}\right)dx.\end{aligned} \tag{2.49}$$

First, the previous estimate (2.42) yields

$$\begin{aligned}\left|\int_{B_R(0)}\Delta\mu\log\left(\frac{|x|}{R}\right)dx\right|&=\left|\int_{B_R(0)}\nabla\mu\cdot\nabla\log|x|dx\right|\leq\frac{1}{2}\|\nabla\mu\|_{\mathrm{L}^{2,1}(B_R(0))}\left\|\frac{1}{|x|}\right\|_{\mathrm{L}^{2,\infty}(B_R(0))}\\&\leq\frac{\sqrt{\pi}}{2}\Gamma_7(n)\Gamma_8\int_\Omega|\nabla\vec{n}|^2dx.\end{aligned} \tag{2.50}$$

Now, using once more Lemma IV.3 of [2], we see that exists a Coulomb moving frame $(\vec{f}_1,\vec{f}_2)\in W^{1,2}(B_\rho(0),S^{n-1})\times W^{1,2}(B_\rho(0),S^{n-1})$ such that

$$\widetilde{\vec{n}}=\star(\vec{f}_1\wedge\vec{f}_2)$$

and using the same inequalities as in (2.38) and (2.39)

$$\begin{aligned}&\left\|\nabla\vec{f}_1\right\|^2_{\mathrm{L}^2(B_\rho(0))}+\left\|\nabla\vec{f}_2\right\|^2_{\mathrm{L}^2(B_\rho(0))}\leq\Gamma_7(n)\left\|\nabla\widetilde{\vec{n}}\right\|^2_{\mathrm{L}^2(B_\rho(0))}=\Gamma_7(n)\int_{B_r(0)}|\nabla\widetilde{\vec{n}}|^2dx+\Gamma_7(n)\int_{B_\rho\setminus\overline{B}_r(0)}|\nabla\vec{n}|^2dx\\&\leq\Gamma_6(n)\Gamma_7(n)\int_{B_{2r}\setminus\overline{B}_r(0)}|\nabla\vec{n}|^2dx+\Gamma_7(n)\int_{B_\rho\setminus\overline{B}_r(0)}|\nabla\vec{n}|^2dx\leq(1+\Gamma_6(n))\Gamma_7(n)\int_{B_{\max\{\rho,2r\}}\setminus\overline{B}_r(0)}|\nabla\vec{n}|^2dx.\end{aligned}\tag{2.51}$$

Now, let ψ be the solution of

$$\begin{cases}\Delta\psi=\nabla^\perp\vec{f}_1\cdot\nabla\vec{f}_2 & \text{in } B_\rho(0)\\ \quad\psi=0 & \text{on } \partial B_\rho(0).\end{cases}$$

As in (2.51), we get

$$\|\nabla\psi\|_{\mathrm{L}^{2,1}(B_\rho(0))}\leq\frac{1}{2}\Gamma_7(n)\Gamma_8\int_{B_{\max\{\rho,2r\}}\setminus\overline{B}_r(0)}|\nabla\vec{n}|^2dx.\tag{2.52}$$

Furthermore, we have

$$\begin{aligned}\int_{B_\rho(0)}\left(\nabla^\perp\vec{e}_1\cdot\nabla\vec{e}_2\right)\log\left(\frac{|x|}{R}\right)dx&=\int_{B_\rho(0)}\left(\nabla^\perp\vec{f}_1\cdot\nabla\vec{f}_2\right)\log\left(\frac{|x|}{R}\right)dx=\int_{B_\rho(0)}\Delta\psi\,\log\left(\frac{|x|}{R}\right)dx\\&=-\log\left(\frac{R}{\rho}\right)\int_{\partial B_\rho}\partial_\nu\psi\,d\mathscr{H}^1-\int_{B_\rho}\nabla\psi\cdot\nabla\log|x|dx\end{aligned}\tag{2.53}$$

while by the Cauchy-Schwarz inequality

$$\left|\int_{\partial B_\rho}\partial_\nu\psi\,d\mathscr{H}^1\right|=\left|\int_{B_\rho(0)}\Delta\psi\,dx\right|=\left|\int_{B_\rho(0)}\nabla^\perp\vec{f}_1\cdot\nabla\vec{f}_2dx\right|\leq\frac{1}{2}(1+\Gamma_6(n))\Gamma_7(n)\int_{B_{\max\{\rho,2r\}}\setminus B_r(0)}|\nabla\vec{n}|^2dx.\tag{2.54}$$

We estimate as previously by (2.52)

$$\left|\int_{B_\rho}\nabla\psi\cdot\nabla\log|x|dx\right|\leq\frac{1}{2}\left\|\nabla\psi\right\|_{\mathrm{L}^{2,1}(B_\rho(0))}\left\|\frac{1}{|x|}\right\|_{\mathrm{L}^{2,\infty}(B_\rho(0))}\leq\frac{\sqrt{\pi}}{2}\Gamma_7(n)\Gamma_8\int_{B_{\max\{\rho,2r\}}\setminus\overline{B}_r(0)}|\nabla\vec{n}|^2dx.\tag{2.55}$$

Therefore, (2.53), (2.54) and (2.55) yield

$$\left|\int_{B_\rho}\left(\nabla^\perp\vec{e}_1\cdot\nabla\vec{e}_2\right)\log\left(\frac{|x|}{R}\right)dx\right|\leq\left(\frac{1}{2}(1+\Gamma_6(n))\Gamma_7(n))\log\left(\frac{R}{\rho}\right)+\frac{\sqrt{\pi}}{2}\Gamma_7(n)\Gamma_8\right)\int_{B_{\max\{\rho,2r\}}\setminus\overline{B}_r(0)}|\nabla\vec{n}|^2dx.\tag{2.56}$$

Finally, by (2.49), (2.50) and (2.56) we obtain for some universal constant $\Gamma_0=\Gamma_0(n)$

$$\frac{1}{2\pi}\left|\int_{\partial B_\rho}\partial_\nu\mu\,d\mathscr{H}^1\right|\leq\Gamma_0(n)\left(\int_{B_{\max\{\rho,2r\}}\setminus\overline{B}_r(0)}|\nabla\vec{n}|^2dx+\frac{1}{\log\left(\frac{R}{\rho}\right)}\int_\Omega|\nabla\vec{n}|^2dx\right)\tag{2.57}$$

which completes the proof of the theorem, up to the L^∞ estimate which is a direct consequence of the inequality $4r<R$ and of Proposition 2.7. ☐

3 Pointwise expansion of the conformal factor and of the immersion

3.1 Case of one bubbling domain

In the next Theorem, we obtain an integrality result for the multiplicity of a sequence of weak immersions from annuli converging strongly outside of the origin.

Theorem 3.1. *Let $\{\vec{\Phi}_k\}_{k\in\mathbb{N}}$ be a sequence of smooth conformal immersions from the disk $B_1(0)\subset\mathbb{C}$ into $\mathbb{R}^n$, let*

$$e^{\lambda_k}=\frac{1}{\sqrt{2}}|\nabla\vec{\Phi}_k|$$

be the conformal factor of $\vec{\Phi}_k$, and $\{\rho_k\}_{k\in\mathbb{N}}\subset(0,1)$ be such that $\rho_k\underset{k\to\infty}{\longrightarrow}0$, $\Omega_k=B_1\setminus\overline{B}_{\rho_k}(0)$ and assume that

$$\sup_{k\in\mathbb{N}}\int_{B_1(0)\setminus\overline{B}_{\rho_k}(0)}|\nabla\vec{n}_k|^2dx\leq\varepsilon_1(n),\qquad\sup_{k\in\mathbb{N}}\|\nabla\lambda_k\|_{\mathrm{L}^{2,\infty}(\Omega_k)}<\infty$$

where $\varepsilon_1(n)$ is given by the proof of Theorem 2.1. Define for all $0<\alpha<1$ and $k\in\mathbb{N}$ large enough $\Omega_k(\alpha)=B_\alpha\setminus\overline{B}_{\alpha^{-1}\rho_k}(0)$, and assume that

$$\lim_{\alpha\to 0}\limsup_{k\to\infty}\int_{\Omega_k(\alpha)}|\nabla\vec{n}_k|^2dx=0$$

and that there exists a $W^{2,2}_{\mathrm{loc}}(B_1(0)\setminus\{0\})\cap C^\infty(B_1(0)\setminus\{0\})$ immersion $\vec{\Phi}_\infty$ such that

$$\log|\nabla\vec{\Phi}_\infty|\in L^\infty_{\mathrm{loc}}(B_1(0)\setminus\{0\})$$

and $\vec{\Phi}_k\underset{k\to\infty}{\longrightarrow}\vec{\Phi}_\infty$ in $C^l_{\mathrm{loc}}(B_1(0)\setminus\{0\})$ (for all $l\in\mathbb{N}$). Then, there exists an integer $\theta_0\geq 1$, $\mu_k\in W^{1,(2,1)}(B_1(0))$ such that

$$\|\nabla\mu_k\|_{\mathrm{L}^{2,1}(B_1(0))}\leq\frac{1}{2}\Gamma_7(n)\Gamma_8\int_{\Omega_k}|\nabla\vec{n}_k|^2dx$$

and a harmonic function ν_k on Ω_k such that $\nu_k=\lambda_k$ on $\partial B_1(0)$, $\lambda_k=\mu_k+\nu_k$ on Ω_k and such that for all $0<\alpha<1$ and such that for all $k\in\mathbb{N}$ sufficiently large

$$\|\nabla(\nu_k-(\theta_0-1)\log|z|)\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}\leq\Gamma_{10}\left(\sqrt{\alpha}\,\|\nabla\lambda_k\|_{\mathrm{L}^{2,\infty}(\Omega_k)}+\int_{\Omega_k}|\nabla\vec{n}_k|^2dx\right)$$

for some universal constant $\Gamma_{10}=\Gamma_{10}(n)$. Furthermore, we have for all $\rho_k\leq r_k\leq 1$ and k large enough

$$\frac{1}{2\pi}\int_{\partial B_{r_k}}*\,d\nu_k=\theta_0-1.$$

Remarks on the proof. In **Step 1**, we first use the classical fact that branch points of Willmore surfaces are positive integers, Theorem 2.1 and the strong convergence outside of 0 to show that the multiplicity d_k converges towards a non-negative integer.

In **Step 2**, as in [32] (see Lemma A.2, A.3 and A.5), we construct a moving frame that allows us to obtain a precise expansion of $\partial_z\vec{\Phi}_k$ in the annular region and show how the existence of a holomorphic function implies in virtue of the first step that for k large enough, the multiplicity must be an integer.

Proof. First, applying Lemma A.5 of [32], we deduce that there exists an integer $\theta_0\geq 1$ and $\vec{A}_0\in\mathbb{C}^n\setminus\{0\}$ such that

$$\vec{\Phi}_\infty(z)=\mathrm{Re}\left(\vec{A}_0z^{\theta_0}\right)+o(|z|^{\theta_0})$$

$$\partial_z\vec{\Phi}_\infty(z)=\frac{\theta_0}{2}\vec{A}_0z^{\theta_0-1}+o(|z|^{\theta_0-1}). \tag{3.1}$$

Step 1. Asymptotic integrality.

First, define $\Omega_k(\alpha)=B_\alpha\setminus\overline{B}_{\alpha^{-1}\rho_k}(0)$ and recall that by Theorem 2.1, we have (applying the inequality on Ω_α) for all $\alpha^{-1}\rho_k<\rho<\alpha$

$$\left|d_k-\frac{1}{2\pi}\int_{\partial B_\rho}\partial_\nu\lambda\,d\mathscr{H}^1\right|\leq\Gamma_0\left(\int_{B_{\max\{\rho,2\alpha^{-1}\rho_k\}}\setminus B_{\alpha^{-1}\rho_k}(0)}|\nabla\vec{n}|^2dx+\frac{1}{\log\left(\frac{\alpha^2}{\rho_k}\right)}\int_{\Omega_k(\alpha)}|\nabla\vec{n}|^2dx\right).$$

Now, taking $\rho=\alpha^2$, we get

$$\left|d_k-\frac{1}{2\pi}\int_{\partial B_{\alpha^2}}\partial_\nu\lambda_k\,d\mathscr{H}^1\right|\leq\Gamma_0\left(\int_{B_{\alpha^2}\setminus B_{\alpha^{-1}\rho_k}}|\nabla\vec{n}_k|^2dx+\frac{1}{\log\left(\frac{\alpha^2}{\rho_k}\right)}\int_{\Omega_k(\alpha)}|\nabla\vec{n}_k|^2dx\right).$$

Therefore, the no-neck energy (see [2])

$$\lim_{\alpha\to 0}\limsup_{k\to\infty}\int_{\Omega_k(\alpha)}|\nabla\vec{n}_k|^2dx=0$$

implies that

$$\lim_{\alpha\to 0}\limsup_{k\to\infty}\left|d_k-\frac{1}{2\pi}\int_{\partial B_{\alpha^2}}\partial_\nu\lambda_k\,d\mathscr{H}^1\right|=0.$$

Furthermore, as $\vec{\Phi}_\infty$ has a branch point of order $\theta_0-1\geq 0$ at $z=0$, we have the expansion for some $\beta\in\mathbb{R}$

$$\lambda_\infty(z)=(\theta_0-1)\log|z|+\beta+O(|z|)$$

we have by the strong convergence

$$\frac{1}{2\pi}\int_{\partial B_{\alpha^2}}\partial_\nu\lambda_k\,d\mathscr{H}^1\underset{k\to\infty}{\longrightarrow}\frac{1}{2\pi}\int_{\partial B_{\alpha^2}}\partial_\nu\lambda_\infty\,d\mathscr{H}^1=\theta_0-1+O(\alpha^2).$$

Finally, this implies that

$$\lim_{\alpha\to 0}\limsup_{k\to\infty}|d_k-(\theta_0-1)|=0. \tag{3.2}$$

Now, recalling that d_k is *independent* of $\alpha>0$ (as it corresponds to the coefficient in front of the logarithm of the associated harmonic function ν_k on $B_1\setminus\overline{B}_{\alpha^{-1}\rho_k}(0)$), we deduce that (3.2) implies that

$$d_k\underset{k\to\infty}{\longrightarrow}\theta_0-1. \tag{3.3}$$

Step 2: Moving frames and integrality.

As in the proof of the forthcoming Theorem 2.1, we introduce an extension of $\widetilde{\vec{n}_k}:B_1(0)\to\mathscr{G}_{n-2}(\mathbb{R}^n)$ of $\vec{n}_k:\Omega_k=B_1\setminus\overline{B}_{\rho_k}(0)\to\mathscr{G}_{n-2}(\mathbb{R}^n)$ such that

$$\begin{cases}\widetilde{\vec{n}_k}=\vec{n}\qquad\text{on }\;\Omega_k=B_1\setminus B_{\rho_k}(0)\\ \left\|\nabla\widetilde{\vec{n}_k}\right\|_{\mathrm{L}^2(B_1(0))}\leq\Gamma_6(n)\left\|\nabla\vec{n}_k\right\|_{\mathrm{L}^2(\Omega_k)}.\end{cases}$$

Therefore, by Lemma IV.3 of [2], there exists a constant $\Gamma_7(n)$ and a Coulomb moving frame $(\vec{f}_{k,1},\vec{f}_{k,2})\in W^{1,2}(B_1(0),S^{n-1})\times W^{1,2}(B_1(0),S^{n-1})$ of $\widetilde{\vec{n}_k}$ such that

$$\begin{cases}\widetilde{\vec{n}_k}=\star(\vec{f}_{k,1}\wedge\vec{f}_{k,2})\qquad\mathrm{div}\left(\vec{f}_{k,1}\cdot\nabla\vec{f}_{k,2}\right)=0\\ \left\|\nabla\vec{f}_{k,1}\right\|_{\mathrm{L}^2(B_1(0))}+\left\|\nabla\vec{f}_{k,2}\right\|_{\mathrm{L}^2(B_1(0))}\leq\Gamma_7(n)\left\|\nabla\vec{n}_k\right\|_{\mathrm{L}^2(\Omega_k)}.\end{cases} \tag{3.4}$$

Now, define for all $j=1,2$ $\vec{e}_{k,j}=e^{-\lambda_k}\partial_{x_j}\vec{\Phi}_k$. As $\vec{\Phi}_k$ is conformal, $(\vec{e}_{k,1},\vec{e}_{k,2})$ is a Coulomb frame of $\vec{n}_k$ on Ω_k. Furthermore, as $\widetilde{\vec{n}_k}=\vec{n}_k$ on Ω_k, both $(\vec{f}_{k,1},\vec{f}_{k,2})$ and $(\vec{e}_{k,1},\vec{e}_{k,2})$ are Coulomb frames of $\vec{n}_k$ on Ω_k, so there exists a rotation $e^{i\theta_k}$ such that

$$(\vec{f}_{k,1}+i\vec{f}_{k,2})=e^{i\theta_k}\left(\vec{e}_{k,1}+i\vec{e}_{k,2}\right). \tag{3.5}$$

Now, we let $f_{k,1},f_{k,2}$ be the vector fields such that

$$d\vec{\Phi}_k(f_{k,j})=\vec{f}_{k,j}\qquad \text{for all } j=1,2. \tag{3.6}$$

Then observe as $\vec{\Phi}_k$ is conformal that

$$\delta_{i,j}=\langle\vec{f}_{k,i},\vec{f}_{k,j}\rangle=\langle d\vec{\Phi}_k(f_{k,i}),d\vec{\Phi}_k(f_{k,j})\rangle=e^{2\lambda_k}\langle f_{k,i},f_{k,j}\rangle$$

so we have

$$\langle f_{k,i},f_{k,j}\rangle=e^{-2\lambda_k}\delta_{i,j}.$$

Likewise, if $(f_{k,1}^{*},f_{k,j}^{*})$ is the dual framing, we deduce that

$$|f_{k,j}^{*}|=e^{\lambda_k}\qquad \text{for all } j=1,2. \tag{3.7}$$

Now, let μ_k the unique solution of

$$\begin{cases}\Delta\mu_k=\nabla^{\perp}\vec{f}_{k,1}\cdot\nabla\vec{f}_{k,2} & \text{in } B_1(0)\\ \mu_k=0 & \text{on } \partial B_1(0).\end{cases}$$

Furthermore, introduce the notation $\nu_k=\lambda_k-\mu_k$. Then ν_k is harmonic, and by **Step 1**, we have

$$d_k=\frac{1}{2\pi}\int_{\partial B_{\rho_k}}*\,d\nu_k\underset{k\to\infty}{\longrightarrow}\theta_0-1.$$

As $\vec{f}_{k,1}\cdot\partial_{\nu}\vec{f}_{k,2}=0$ on $\partial B_1(0)$, we also have

$$d\mu_k=*(\vec{f}_{k,1}\cdot d\vec{f}_{k,2}) \tag{3.8}$$

Then we compute with $\mathbb{Z}_2$ indices for all $j\in\{1,2\}$

$$d\mu_k\wedge f_{k,j}^{*}=(*d\mu_k)\wedge(*\vec{f}_{k,j}^{*})=(-1)^j(\vec{f}_{k,1}\cdot d\vec{f}_{k,2})\wedge\vec{f}_{k,j+1}.$$

Likewise, as in [32], we compute

$$df_{k,j}^{*}=(-1)^j\left(\vec{f}_{k,1}\cdot d\vec{f}_{k,2}\right)\wedge f_{k,j+1}^{*}.$$

Therefore, we have

$$d\left(e^{-\mu_k}f_{k,j}^{*}\right)=0\qquad \text{in } \Omega_k\qquad \text{for } j=1,2.$$

In particular, by Stokes theorem, we have for all $\rho_k\leq r_1<r_2\leq 1$

$$0=\int_{B_{r_2}\setminus\overline{B}_{r_1}(0)}d\left(e^{-\mu_k}f_{k,j}^{*}\right)=\int_{\partial B_{r_2}}e^{-\mu_k}f_{k,j}^{*}-\int_{\partial B_{r_1}}e^{-\mu_k}f_{k,j}^{*}.$$

Therefore, we introduce the constants $c_j\in\mathbb{R}$ defined for all $\rho_k\leq\rho\leq 1$ by

$$c_{k,j}=\int_{\partial B_{\rho}}e^{-\mu_k}f_{k,j}^{*}.$$

Now, introduce the complex valued 1-forms

$$f^*_{k,z} = f^*_{k,1} + i f^*_{k,2} \qquad f^*_{k,\overline{z}} = f^*_{k,1} - i f^*_{k,2},$$

so that

$$f^*_{k,1} = \frac{1}{2}\left(f^*_{k,z} + f^*_{k,\overline{z}}\right) \qquad f^*_{k,2} = \frac{1}{2i}\left(f^*_{k,z} - f^*_{k,\overline{z}}\right).$$

Notice also that

$$\ast\, df^*_{k,z} = -i f^*_{k,z} \qquad \text{and} \quad \ast\, df^*_{k,\overline{z}} = i f_{k,\overline{z}}.$$

Furthermore, if

$$\begin{cases} f_{k,z} = \dfrac{1}{2}\left(f_{k,1} - i f_{k,2}\right) \\ f_{k,\overline{z}} = \dfrac{1}{2}\left(f_{k,1} + i f_{k,2}\right), \end{cases} \tag{3.9}$$

then for all smooth function $\varphi : \Omega_k \to \mathbb{C}$, we have

$$\begin{aligned} d\varphi &= d\varphi \cdot f_{k,1}\, f^*_{k,1} + d\varphi \cdot f_{k,2}\, f^*_{k,2} \\ &= d\varphi \cdot f_{k,z}\, f^*_{k,z} + d\varphi \cdot f_{k,z} f^*_{k,\overline{z}}. \end{aligned}$$

Now, we introduce the differential form $\alpha \in \Omega^1(\mathbb{R}^2 \setminus \{0\})$

$$\begin{aligned} \alpha &= \frac{1}{2\pi} \ast\, d\log|z| = \frac{1}{2\pi} \ast\, \left(\nabla \log|z| \cdot f_{k,z}\, f^*_{k,z} + \nabla \log|z| \cdot f_{k,\overline{z}}\, f^*_{k,\overline{z}}\right) \\ &= \frac{1}{2\pi i}\nabla \log|z| \cdot f_{k,z}\, f^*_{k,z} - \frac{1}{2\pi i}\nabla \log|z| \cdot f_{k,\overline{z}}\, f^*_{k,\overline{z}}. \end{aligned}$$

In particular, notice that

$$\alpha + \frac{1}{2\pi i}\nabla \log|z| \cdot f_{k,\overline{z}} f^*_{k,\overline{z}} = \frac{1}{2\pi i}\nabla \log|z| \cdot f_{k,z} f^*_{k,z}. \tag{3.10}$$

As log is harmonic on $\mathbb{R}^2 \setminus \{0\}$, the differential form α is closed on Ω_k and we deduce that the 1-form

$$\omega_{k,j} = e^{-\mu_k} f^*_{k,j} - c_{k,j}\alpha$$

is also closed. Furthermore, as

$$\int_{\partial B_{\rho_k}} \omega_{k,j} = 0,$$

we deduce by Poincaré lemma that there exists $(\sigma_{k,1}, \sigma_{k,2}) \in W^{1,2}(\Omega_k, \mathbb{R}^2)$ such that

$$d\sigma_{k,j} = \omega_{k,j} = e^{-\mu_k} f^*_{k,j} - c_{k,j}\alpha \qquad \text{for } j = 1, 2.$$

Therefore, we deduce if $c_k = c_{k,1} + i c_{k,2}$ and $\sigma_k = \sigma_{k,1} + i\sigma_{k,2}$ that

$$\begin{aligned} d\sigma_k &= e^{-\mu_k}\left(f^*_{k,1} + i f^*_{k,2}\right) - c_k\alpha \\ &= \left(e^{-\mu_k} - \frac{c_k}{2\pi i}\nabla \log|z| \cdot f_{k,z}\right)\left(f^*_{k,1} + f^*_{k,2}\right) f^*_{k,z} + \frac{c_k}{2\pi i}\nabla \log|z| \cdot f_{k,\overline{z}}\, f^*_{k,\overline{z}}. \end{aligned}$$

This implies by (3.10) that

$$d\left(\sigma_k - \frac{c_k}{2\pi i}\log|z|\right) = \left(e^{-\mu_k} - \frac{c_k}{\pi i}\nabla \log|z| \cdot f_{k,z}\right) f^*_{k,z}. \tag{3.11}$$

Therefore, the function

$$\tau_k = \sigma_k - \frac{c_k}{2\pi i}\log|z|$$

is holomorphic. Now, let $\left(\frac{\partial}{\partial\tau_{k,1}},\frac{\partial}{\partial\tau_{k,2}}\right)$ be the dual basis of $(\tau_{k,1},\tau_{k,2})$, where $\tau_k=\tau_{k,1}+i\tau_{k,2}$. Then we define

$$\varphi=e^{-\mu_k}-\frac{c_k}{\pi i}\nabla\log|z|\cdot f_{k,z}, \tag{3.12}$$

and we notice that (3.11) implies that

$$\begin{aligned}d(\tau_{k,1}+i\tau_{k,2})&=(\mathrm{Re}\,(\varphi)+i\,\mathrm{Im}\,(\varphi))\left(f_{k,1}^*+if_{k,2}^*\right)\\&=\left(\mathrm{Re}\,(\varphi)f_{k,1}^*-\mathrm{Im}\,(\varphi)f_{k,2}^*\right)+i\left(\mathrm{Im}\,(\varphi)f_{k,1}^*+\mathrm{Re}\,(\varphi)f_{k,2}^*\right)\end{aligned}$$

Therefore, we deduce that

$$\begin{pmatrix}d\tau_{k,1}\\d\tau_{k,2}\end{pmatrix}=\begin{pmatrix}\mathrm{Re}\,(\varphi)&-\mathrm{Im}\,(\varphi)\\\mathrm{Im}\,(\varphi)&\mathrm{Re}\,(\varphi)\end{pmatrix}\begin{pmatrix}f_{k,1}^*\\f_{k,2}^*.\end{pmatrix}$$

This implies that

$$\begin{pmatrix}\frac{\partial}{\partial\tau_{k,1}}\\\frac{\partial}{\partial\tau_{k,2}}\end{pmatrix}=\frac{1}{|\varphi|^2}\begin{pmatrix}\mathrm{Re}\,(\varphi)&\mathrm{Im}\,(\varphi)\\-\mathrm{Im}\,(\varphi)&\mathrm{Re}\,(\varphi)\end{pmatrix}\begin{pmatrix}f_{k,1}\\f_{k,2}\end{pmatrix}. \tag{3.13}$$

Now, defining

$$\frac{\partial}{\partial\tau_k}=\frac{1}{2}\left(\frac{\partial}{\partial\tau_{k,1}}-i\frac{\partial}{\partial\tau_{k,2}}\right),$$

we compute thanks to (3.6) and (3.13)

$$\begin{aligned}\frac{\partial\vec{\Phi}_k}{\partial\tau_k}&=\frac{1}{2|\varphi|^2}d\vec{\Phi}_k\cdot(\mathrm{Re}\,(\varphi)f_{k,1}+\mathrm{Im}\,(\varphi)f_{k,2}-i\,(\mathrm{Im}\,(\varphi)f_{k,1}-\mathrm{Re}\,(\varphi)f_{k,2}))\\&=\frac{1}{2|\varphi|^2}\left(\mathrm{Re}\,(\varphi)\vec{f}_{k,1}+\mathrm{Im}\,(\varphi)\vec{f}_{k,2}-i\left(\mathrm{Im}\,(\varphi)\vec{f}_{k,1}-\mathrm{Re}\,(\varphi)\vec{f}_{k,2}\right)\right)\\&=\frac{1}{2|\varphi|^2}\left((\mathrm{Re}\,(\varphi)+i\mathrm{Im}\,(\varphi))\,\vec{f}_{k,1}+(\mathrm{Im}\,(\varphi)-i\mathrm{Re}\,(\varphi))\,\vec{f}_{k,2}\right)\\&=\frac{\varphi}{2|\varphi|^2}\left(\vec{f}_{k,1}-i\vec{f}_{k,2}\right)=\frac{1}{2\overline{\varphi}}\left(\vec{f}_{k,1}-i\vec{f}_{k,2}\right).\end{aligned}$$

Therefore, we deduce that

$$\frac{e^{\lambda_k}}{2}\left(\vec{e}_{k,1}-i\vec{e}_{k,2}\right)=\partial_z\vec{\Phi}_k=\frac{\partial\vec{\Phi}_k}{\partial\tau_k}\frac{\partial\tau_k}{\partial z}=\frac{\tau_k'(z)}{2\overline{\varphi}}\left(\vec{f}_{k,1}-i\vec{f}_{k,2}\right) \tag{3.14}$$

Now, recall by (3.5) that there exists a rotation $e^{i\theta_k}$ (beware that the function θ_k is multi-valued) such that

$$\vec{f}_{k,1}+i\vec{f}_{k,2}=e^{i\theta_k}\left(\vec{e}_{k,1}+i\vec{e}_{k,2}\right).$$

Therefore, (3.14), (3.14) and $\lambda_k=\mu_k+\nu_k$ imply that

$$e^{\lambda_k}\overline{\varphi}=e^{\nu_k}+\frac{\overline{c_k}e^{\lambda_k}}{\pi i}\nabla\log|z|\cdot f_{k,\overline{z}}=\tau_k'(z)e^{-i\theta_k}. \tag{3.15}$$

Recalling that

$$d_k=\frac{1}{2\pi}\int_{\partial B_{\rho_k}}*\,d\nu_k\underset{k\to\infty}{\longrightarrow}\theta_0-1,$$

we will now show that $d_k=\theta_0-1$ for k large enough. First, recall that there exists a rotation $e^{i\theta_k}$ such that

$$(\vec{f}_{k,1}+i\vec{f}_{k,2})=e^{i\theta_k}\left(\vec{e}_{k,1}+i\vec{e}_{k,2}\right), \tag{3.16}$$

and that there exists vector fields $f_{k,1}, f_{k,2}$ such that

$$d\vec{\Phi}_k(f_{k,j}) = \vec{f}_{k,j} \qquad \text{for all } \; j = 1, 2. \tag{3.17}$$

To simplify the notations, we will now delete the subscript k in the following formulas. Now, rewrite (3.16) as

$$\vec{f}_1 + i\vec{f}_2 = e^{i\theta}(\vec{e}_1 + i\vec{e}_2) = \cos(\theta)\vec{e}_1 - \sin(\theta)\vec{e}_2 + i\left(\sin(\theta)\vec{e}_1 + \cos(\theta)\vec{e}_2\right),$$

so that

$$\begin{cases} \vec{f}_1 = \cos(\theta)\vec{e}_1 - \sin(\theta)\vec{e}_2 \\ \vec{f}_2 = \sin(\theta)\vec{e}_1 + \cos(\theta)\vec{e}_2 \end{cases}$$

Now, write $f_1 = (f_1^1, f_1^2)$, $f_2 = (f_2^1, f_2^2)$, and observe that

$$\begin{aligned} d\vec{\Phi}(f_1) &= e^{\lambda} f_1^1 \vec{e}_1 + e^{\lambda} f_1^2 \vec{e}_2 = \vec{f}_1 = \cos(\theta)\vec{e}_1 - \sin(\theta)\vec{e}_2 \\ d\vec{\Phi}(f_2) &= e^{\lambda} f_2^1 \vec{e}_1 + e^{\lambda} f_2^2 \vec{e}_2 = \vec{f}_2 = \sin(\theta)\vec{e}_1 + \cos(\theta)e_2 \end{aligned}$$

implies that

$$\begin{cases} f_1 = e^{-\lambda}(\cos(\theta), -\sin(\theta)) \\ f_2 = e^{-\lambda}(\sin(\theta), \cos(\theta)). \end{cases}$$

Therefore, we deduce that

$$\begin{cases} f_1^* = e^{\lambda}\cos(\theta)dx_1 - e^{\lambda}\sin(\theta)dx_2 \\ f_2^* = e^{\lambda}\sin(\theta)dx_1 + e^{\lambda}\cos(\theta)dx_2. \end{cases} \tag{3.18}$$

Recall the definitions (from (3.9))

$$c_j = \int_{\partial B_\rho} e^{-\mu} f_j^* \qquad j = 1, 2, \qquad f_{\overline{z}} = \frac{1}{2}(f_1 + i f_2).$$

Introducing

$$c = -\frac{1}{2\pi i}(c_1 - i c_2),$$

we have for some holomorphic function χ on Ω_k and for all $z \in \Omega_k$ (in the preceding notations, we have $\chi = \tau_k'$ in the previous notations) by (3.15)

$$e^{\nu} = \chi(z)e^{-i\theta} + 2c\, e^{\lambda}\nabla \log|z| \cdot f_{\overline{z}} \tag{3.19}$$

Notice that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ implies that

$$\begin{aligned} f_{\overline{z}} &= \frac{e^{-\lambda}}{2}\left((\cos(\theta), -\sin(\theta)) + i(\sin(\theta), \cos(\theta))\right) \\ &= \frac{e^{\lambda}}{2}\left(\cos(\theta) + i\sin(\theta), i\cos(\theta) - \sin(\theta)\right) = \frac{e^{-\lambda}}{2}\left(\cos(\theta) + i\sin(\theta), i(\cos(\theta) + i\sin(\theta))\right) \\ &= \frac{e^{-\lambda + i\theta}}{2}\left(1, i\right), \end{aligned} \tag{3.20}$$

Therefore, recalling the notation $z = x_1 + ix_2$, (3.19) and (3.20) imply that

$$\begin{aligned} e^{\nu} &= \chi(z)e^{-i\theta} + ce^{i\theta}\nabla \log|z| \cdot (1, i) = \chi(z)e^{-i\theta} + ce^{i\theta}\left(\frac{x_1}{|z|^2}, \frac{x_2}{|z|^2}\right) \cdot (1, i) \\ &= \chi(z)e^{-i\theta} + ce^{i\theta}\frac{x_1 + ix_2}{|z|^2} = \chi(z)e^{-i\theta} + ce^{i\theta}\frac{z}{|z|^2} \end{aligned}$$

$$= \chi(z)e^{-i\theta} + \frac{ce^{i\theta}}{\overline{z}} \tag{3.21}$$

Now, as the left hand-side of (3.21) is *real*, taking imaginary parts of the right hand-side, we find that

$$\chi(z)e^{-i\theta} - \overline{\chi(z)}e^{i\theta} + \frac{ce^{i\theta}}{\overline{z}} - \frac{\overline{c}e^{-i\theta}}{z} = 0.$$

Multiplying this identity by $e^{i\theta}$, we deduce that

$$e^{2i\theta}\left(-\overline{\chi(z)} + \frac{c}{\overline{z}}\right) + \chi(z) - \frac{\overline{c}}{z} = 0.$$

This implies that

$$e^{2i\theta} = \frac{\chi(z) - \frac{\overline{c}}{z}}{\overline{\chi(z)} - \frac{c}{\overline{z}}} = \left(\frac{\chi(z) - \frac{\overline{c}}{z}}{\left|\chi(z) - \frac{\overline{c}}{z}\right|}\right)^2.$$

Finally, as $e^{\nu} > 0$, we deduce thanks to (3.21) that

$$e^{i\theta} = \frac{\chi(z) - \frac{\overline{c}}{z}}{\left|\chi(z) - \frac{\overline{c}}{z}\right|}.$$

Letting now ψ be the *holomorphic* function such that

$$\psi(z) = \chi(z) - \frac{\overline{c}}{z},$$

we deduce that

$$e^{i\theta} = \frac{\psi(z)}{|\psi(z)|}. \tag{3.22}$$

This implies readily that

$$d\theta = \mathrm{Im}\left(\frac{\partial\psi}{\psi}\right) = \mathrm{Im}\left(\frac{\psi'(z)}{\psi(z)}dz\right). \tag{3.23}$$

Indeed, we have formally (in other words, the following expression must be understood as the equality of two multi-valued functions, *i.e.* modulo $2\pi i$)

$$i\theta = \log\left(\frac{\psi(z)}{|\psi(z)|}\right).$$

Therefore, we have

$$\begin{aligned} i\partial\theta &= \frac{|\psi(z)|}{\psi(z)}\left\{\frac{\psi'(z)}{|\psi(z)|} - \frac{1}{2}\psi(z)\psi'(z)\overline{\psi}(z)|\psi(z)|^{-3}\right\}dz = \frac{|\psi(z)|}{\psi(z)}\left\{\frac{\psi'(z)}{|\psi(z)|} - \frac{1}{2}\frac{\psi'(z)}{|\psi(z)|}\right\}dz \\ &= \frac{1}{2}\frac{\psi'(z)}{\psi(z)}dz = \frac{1}{2}\frac{\partial\psi}{\psi}. \end{aligned} \tag{3.24}$$

As θ is real, we deduce that

$$i\overline{\partial}\theta = \overline{-i\partial\theta} = -\frac{1}{2}\overline{\left(\frac{\partial\psi}{\psi}\right)}. \tag{3.25}$$

Using that $d = \partial + \overline{\partial}$, we deduce from (3.24) and (3.25) that

$$d\theta = \partial\theta + \overline{\partial}\theta = \frac{1}{2i}\left(\frac{\partial\psi}{\psi} - \overline{\left(\frac{\partial\psi}{\psi}\right)}\right) = \mathrm{Im}\left(\frac{\partial\psi}{\psi}\right).$$

Finally, we deduce from (3.23) that

$$\int_{\partial B_\rho} d\theta \in 2\pi\mathbb{Z},$$

Now, a classical computation shows that

$$* d\nu = d\theta.$$

This can be directly checked using the Coulomb condition, but as we have already used it to obtain the closedness of $e^{-\mu}f_1^*$ and $e^{-\mu}f_2^*$, we can also check this property with these 1-forms. Recall that thanks to (3.18)

$$\begin{cases} e^{-\mu}f_1^* = e^{\nu}\cos(\theta)dx_1 - e^{\nu}\sin(\theta)dx_2 \\ e^{-\mu}f_2^* = e^{\nu}\sin(\theta)dx_1 + e^{\nu}\cos(\theta)dx_2. \end{cases}$$

Therefore, that $e^{-\mu}f_1^*$ be closed is equivalent to

$$0 = (\partial_{x_2}\nu)\, e^{\nu}\cos(\theta) - (\partial_{x_2}\theta)\, e^{\nu}\sin(\theta) + (\partial_{x_1}\nu)e^{\nu}\sin(\theta) + (\partial_{x_1}\theta)e^{\nu}\cos(\theta)$$

or (writing scripts for partial derivatives)

$$(\nu_2 + \theta_1)\cos(\theta) + (\nu_1 - \theta_2)\sin(\theta) = 0. \tag{3.26}$$

Likewise, the closedness of $e^{-\mu}f_2^*$ is equivalent to

$$(-\nu_1 + \theta_2)\cos(\theta) + (\nu_2 + \theta_1)\sin(\theta) = 0. \tag{3.27}$$

Therefore, (3.26) and (3.27) are equivalent to the system

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix} \begin{pmatrix} \nu_2 + \theta_1 \\ \nu_1 - \theta_2 \end{pmatrix} = 0.$$

As

$$\det \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{pmatrix} = -\cos^2(\theta) - \sin^2(\theta) = -1 \neq 0,$$

we deduce that

$$\begin{cases} \nu_2 + \theta_1 = 0 \\ \nu_1 - \theta_2 = 0 \end{cases} \tag{3.28}$$

In other words, (3.28) is equivalent to $\nabla\nu = \nabla^{\perp}\theta$, or

$$* d\nu = d\theta. \tag{3.29}$$

Therefore, thanks to (3.1) and (3.28), we deduce that for k large enough

$$\frac{1}{2\pi}\int_{\partial B_\rho} * d\nu_k = \theta_0 - 1.$$

This argument concludes the proof of the Proposition. □

We are now going to improve the expansion of the conformal parameter to obtain a pointwise estimate of $\nabla\vec{\Phi}_k$.

We first need an extension lemma which is a refinement of Lemma IV.1 of [2]. For the sake of completeness, we add all details.

Lemma 3.2. *Let $0<r<1$ and $\vec{n}\in W^{1,(2,1)}(B_{2r}\setminus \overline{B}_r(0),\mathscr{G}_{n-2}(\mathbb{R}^n))$. There exists $\varepsilon_2(n)>0$ with the following property. Assume that*

$$\|\nabla\vec{n}\|_{\mathrm{L}^{2,1}(B_{2r}\setminus\overline{B}_r(0))}\leq \varepsilon_3(n).$$

Then there exists an extension $\widetilde{\vec{n}}\in W^{1,(2,1)}(B_{2r}(0),\mathscr{G}_{n-2}(\mathbb{R}^n))$ such that $\widetilde{\vec{n}}=\vec{n}$ on $B_{2r}\setminus\overline{B}_r(0)$ and a universal constant $C_5(n)$ such that

$$\left\|\nabla\widetilde{\vec{n}}\right\|_{\mathrm{L}^{2,1}(B_{2r})}\leq C_5(n)\,\|\nabla\vec{n}\|_{\mathrm{L}^{2,1}(B_{2r}\setminus\overline{B}_r(0))}$$

Proof. First, as in [5] 3.2.28, we view $\mathscr{G}_{n-2}(\mathbb{R}^n)$ as a submanifold of $\mathbb{R}^{N(n)}$ for some (large) $N(n)$. Thanks to the Sobolev embedding $W^{1,(2,1)}(B_{2r}\setminus\overline{B}_r(0))\subset C^0(B_{2r}\setminus\overline{B}_r(0))$ and scaling invariance, there exists $\vec{p}\in\mathscr{G}_{n-2}(\mathbb{R}^n)\subset\mathbb{R}^{N(n)}$ and a universal constant $\Gamma_{11}(n)>0$ independent of $r>0$ such that

$$\|\vec{n}-\vec{p}\|_{\mathrm{L}^\infty(B_{2r}\setminus\overline{B}_r(0))}\leq\Gamma_{11}(n)\,\|\nabla\vec{n}\|_{\mathrm{L}^{2,1}(B_{2r}\setminus\overline{B}_r(0))}\leq\Gamma_{11}(n)\varepsilon_3(n). \tag{3.30}$$

As $\mathscr{G}_{n-2}(\mathbb{R}^n)$ is a compact smooth submanifold, its injectivity radius is strictly positive, there exists $\varepsilon_3(n)>0$ independent of $\vec{p}\in\mathscr{G}_{n-2}(\mathbb{R}^n)$ such that (3.30) implies that $\vec{n}(B_{2r}\setminus\overline{B}_r(0))$ is included in a geodesic ball of $\mathscr{G}_{n-2}(\mathbb{R}^n)$. Therefore, we deduce that there exists $\delta=\delta(n)>0$ such that $\vec{n}(B_{2r}\setminus\overline{B}_r(0))\subset B_\delta(\vec{p})$ global coordinates $\varphi:B_\delta(\vec{p})\to\varphi(B_\delta(\vec{p}))\subset\mathbb{R}^{m(n)}$ (where $m(n)=\dim\mathscr{G}_{n-2}(\mathbb{R}^n)$). Once more, by compactness, we can assume that $\delta=\delta(n)$ has been fixed independently of $\vec{p}$ and such that

$$\left\|\nabla\varphi^{-1}\right\|_{\mathrm{L}^\infty(\varphi(B_{\delta(n)}(\vec{p})))}<\infty \tag{3.31}$$

depends only on n. Furthermore, we can assume without loss of generality that $\varphi(B_\delta(\vec{p}))=B_\delta^{\mathbb{R}^{m(n)}}(0)=B_\delta^m(0)$ is the standard geodesics ball in $\mathbb{R}^m$ of radius $\delta>0$. Now, apply the extension Theorem 7.2 to the composition $\vec{n}_\varphi=\varphi\circ\vec{n}:B_{2r}\setminus\overline{B}_r(0)\to\mathbb{R}^{m(n)}$ to find an extension $\widetilde{\vec{n}}_\varphi:B_{2r}(0)\to\mathbb{R}^{m(n)}$ such that

$$\left\|\widetilde{\vec{n}}_\varphi\right\|_{\mathrm{W}^{1,(2,1)}(B_{2r}(0))}\leq\Gamma_{12}(n)\,\|\vec{n}_\varphi\|_{\mathrm{W}^{1,(2,1)}(B_{2r}(0))}.$$

We deduce by the Poincaré-Wirtinger inequality that

$$\begin{aligned}\left\|\nabla\widetilde{\vec{n}}_\varphi\right\|_{\mathrm{L}^{2,1}(B_{2r}(0))}&\leq\Gamma_{12}(n)\left(\|\nabla\vec{n}_\varphi\|_{\mathrm{L}^{2,1}(B_{2r}\setminus\overline{B}_r(0))}+\left\|\vec{n}_\varphi-\overline{\vec{n}_\varphi}_{B_{2r}\setminus\overline{B}_r}\right\|_{\mathrm{L}^{2,1}(B_{2r}\setminus\overline{B}_r(0))}\right)\\&\leq\Gamma_{13}(n)\,(1+r)\,\|\nabla\vec{n}_\varphi\|_{\mathrm{L}^{2,1}(B_{2r}\setminus\overline{B}_r(0))}\\&\leq 2\,\Gamma_{13}\,\|\nabla\vec{n}_\varphi\|_{\mathrm{L}^{2,1}(B_{2r}\setminus\overline{B}_r(0))}\end{aligned}$$

Taking $\widetilde{\vec{n}}=\varphi^{-1}\circ\widetilde{\vec{n}}_\varphi$ finishes the proof of the theorem by the previous remark in (3.31). ☐

The next lemma is an easy consequence of Lemme (5.1.4) of [10] (see also Lemma IV.3 of [2]).

Lemma 3.3. ($W^{1,(2,1)}$-controlled Coulomb frame) *Let $0<r<\dfrac{1}{2}$ and $\vec{n}\in W^{1,(2,1)}(B_1\setminus\overline{B}_r(0))\to\mathscr{G}_{n-2}(\mathbb{R}^n)$. Then there exists $0<\varepsilon_3(n)<\varepsilon_2(n)$ with the following property. Assume that*

$$\|\nabla\vec{n}\|_{\mathrm{L}^{2,1}(B_1\setminus\overline{B}_r(0))}\leq\varepsilon_3(n).$$

Then there exists $(\vec{e}_1,\vec{e}_2)\in W^{1,(2,1)}(B_1(0))\times W^{1,(2,1)}(B_1(0))\to\mathbb{R}^n$ which is a Coulomb frame on $B_1\setminus\overline{B}_r(0)$ associated to $\vec{n}$ such that

$$\vec{n}=*\,(\vec{e}_1\wedge\vec{e}_2)\;\;\text{in }B_1\setminus\overline{B}_r(0)\qquad\text{and}\qquad\begin{cases}\operatorname{div}(\vec{e}_1\cdot\nabla\vec{e}_2)=0 & \text{in }B_1(0)\\ \vec{e}_1\cdot\partial_\nu\vec{e}_2=0 & \text{on }\partial B_1(0),\end{cases}$$

and there exists a universal constant $C_6(n)>0$ such that

$$\begin{aligned}&\|\nabla\vec{e}_1\|^2_{\mathrm{L}^2(B_1(0))}+\|\nabla\vec{e}_2\|^2_{\mathrm{L}^2(B_1(0))}\leq\frac{1}{4}C_5(n)^2\,\|\nabla\vec{n}\|^2_{\mathrm{L}^{2,1}(B_1\setminus\overline{B}_r(0))}\\&\|\nabla\vec{e}_1\|_{\mathrm{L}^{2,1}(B_1(0))}+\|\nabla\vec{e}_2\|_{\mathrm{L}^{2,1}(B_1(0))}\leq C_6(n)\left(1+\|\nabla\vec{n}\|_{\mathrm{L}^{2,1}(B_1\setminus\overline{B}_r(0))}\right)\|\nabla\vec{n}\|_{\mathrm{L}^{2,1}(B_1\setminus\overline{B}_r(0))}.\end{aligned} \tag{3.32}$$

Remark 3.4. Notice that we do *not* have in general $\vec{e}_1\cdot\partial_\nu\vec{e}_2=0$ on $\partial B_r(0)$.

Proof. First, as $\varepsilon_3(n)<\varepsilon_2(n)$, we have

$$\|\nabla\vec{n}\|_{\mathrm{L}^{2,1}(B_{2r}\setminus\overline{B}_r(0))}\leq\varepsilon_2(n).$$

Therefore, by Lemma 3.2, there exists an extension $\widetilde{\vec{n}}:B_1(0)\to\mathscr{G}_{n-2}(\mathbb{R}^n)$ (such that $\widetilde{\vec{n}}=\vec{n}$ on $B_1\setminus\overline{B}_r(0)$) and satisfying (up to replacing $C_5(n)$ by $\max\{1,C_5(n)\}$ in Lemma 3.2)

$$\begin{aligned}\left\|\nabla\widetilde{\vec{n}}\right\|_{\mathrm{L}^{2,1}(B_1(0))}&\leq C_5(n)\left\|\nabla\vec{n}\right\|_{\mathrm{L}^{2,1}(B_{2r}\setminus\overline{B}_r(0))}+\|\nabla\vec{n}\|_{\mathrm{L}^{2,1}(B_1\setminus\overline{B}_{2r}(0))}\\&\leq C_5(n)\left\|\nabla\vec{n}\right\|_{\mathrm{L}^{2,1}(B_1\setminus\overline{B}_r(0))}\leq C_5(n)\varepsilon_3(n).\end{aligned}\tag{3.33}$$

By the inequality $\|\cdot\|_{\mathrm{L}^2}\leq\dfrac{1}{2\sqrt{2}}\|\cdot\|_{\mathrm{L}^{2,1}}$ (see the Appendix (7.6)), we deduce by (3.33) that

$$\left\|\nabla\widetilde{\vec{n}}\right\|_{\mathrm{L}^2(B_1(0))}\leq\frac{1}{2\sqrt{2}}\left\|\nabla\widetilde{\vec{n}}\right\|_{\mathrm{L}^{2,1}(B_1(0))}\leq\frac{1}{2\sqrt{2}}C_5(n)\varepsilon_3(n)$$

so taking

$$0<\varepsilon_3(n)<\frac{2\sqrt{2}}{C_5(n)}\frac{8\pi}{3},$$

we deduce by Lemme 5.1.4 of [10] that there exists a Coulomb frame $(\vec{e}_1,\vec{e}_2)\in W^{1,2}(B_1(0))\times W^{1,2}(B_1(0))\to\mathbb{R}^n$ such that

$$\widetilde{\vec{n}}=*\left(\vec{e}_1\wedge\vec{e}_2\right)\qquad\text{and}\qquad\begin{cases}\operatorname{div}\left(\vec{e}_1\cdot\nabla\vec{e}_2\right)=0 & \text{in } B_1(0)\\ \vec{e}_1\cdot\partial_\nu\vec{e}_2=0 & \text{on } \partial B_1(0),\end{cases}$$

and (by [10], (5.23), (5.24) p.244) and the elementary inequality

$$1-\sqrt{1-t}\leq t\qquad\text{for all } t\in[0,1],$$

we deduce that

$$\|\nabla\vec{e}_1\|^2_{\mathrm{L}^2(B_1(0))}+\|\nabla\vec{e}_2\|^2_{\mathrm{L}^2(B_1(0))}\leq\frac{16\pi}{3}\left(1-\sqrt{1-\frac{3}{8\pi}\int_{B_1(0)}|\nabla\widetilde{\vec{n}}|^2dx}\right)\leq 2\int_{B_1(0)}|\nabla\widetilde{\vec{n}}|^2dx.\tag{3.34}$$

Now, let $\mu:B_1(0)\to\mathbb{R}$ be the unique solution of

$$\begin{cases}\Delta\mu=\nabla^{\perp}\vec{e}_1\cdot\nabla\vec{e}_1 & \text{in } B_1(0)\\ \mu=0 & \text{on } \partial B_1(0)\end{cases}$$

Then by the generalised Wente inequality (or [4] and the Sobolev embedding $W^{2,1}(\mathbb{R}^2)\hookrightarrow W^{1,(2,1)}(\mathbb{R}^n)$), we have

$$\|\nabla\mu\|_{\mathrm{L}^{2,1}(B_1(0))}\leq 2\Gamma_0\left\|\nabla\vec{e}_1\right\|_{\mathrm{L}^2(B_1(0))}\left\|\nabla\vec{e}_2\right\|_{\mathrm{L}^2(B_1(0))}\leq\Gamma_0\left\|\nabla\widetilde{\vec{n}}\right\|^2_{\mathrm{L}^2(B_1(0))}\tag{3.35}$$

Now recall the identity ([10], (5.39), p. 247)

$$|\nabla\vec{e}_1|^2+|\nabla\vec{e}_2|^2=2|\nabla\mu|^2+|\nabla\widetilde{\vec{n}}|^2.\tag{3.36}$$

Therefore, we have

$$|\nabla\vec{e}_1|+|\nabla\vec{e}_2|\leq\sqrt{2}\sqrt{|\nabla\vec{e}_1|^2+|\nabla\vec{e}_2|^2}\leq 2|\nabla\mu|+\sqrt{2}|\nabla\widetilde{\vec{n}}|.\tag{3.37}$$

The identity (3.37) and the estimates (3.33), (3.34) and (3.35) yield

$$\begin{aligned}\|\nabla\vec{e}_1\|_{\mathrm{L}^{2,1}(B_1(0))}+\|\nabla\vec{e}_2\|_{\mathrm{L}^{2,1}(B_1(0))}&\leq 2\|\nabla\mu\|_{\mathrm{L}^{2,1}(B_1(0))}+\sqrt{2}\left\|\nabla\widetilde{\vec{n}}\right\|_{\mathrm{L}^{2,1}(B_1(0))}\\&\leq\Gamma_0\left\|\nabla\widetilde{\vec{n}}\right\|^2_{\mathrm{L}^{2}(B_1(0))}+\sqrt{2}\left\|\nabla\widetilde{\vec{n}}\right\|_{\mathrm{L}^{2,1}(B_1(0))}\\&\leq\frac{1}{8}\Gamma_0\left\|\nabla\widetilde{\vec{n}}\right\|^2_{\mathrm{L}^{2,1}(B_1(0))}+\sqrt{2}\left\|\nabla\widetilde{\vec{n}}\right\|_{\mathrm{L}^{2,1}(B_1(0))}\\&\leq\frac{1}{8}\Gamma_0C_5(n)^2\|\nabla\vec{n}\|_{\mathrm{L}^{2,1}(B_1\setminus\overline{B}_r(0))}+\sqrt{2}\,C_5(n)\|\nabla\vec{n}\|_{\mathrm{L}^{2,1}(B_1\setminus\overline{B}_r(0))}\\&\leq C_6(n)\left(1+\|\nabla\vec{n}\|_{\mathrm{L}^{2,1}(B_1\setminus\overline{B}_r(0))}\right)\|\nabla\vec{n}\|_{\mathrm{L}^{2,1}(B_1\setminus\overline{B}_r(0))},\end{aligned}\tag{3.38}$$

where

$$C_6(n)=\max\left\{\frac{1}{8}\Gamma_0C_5(n)^2,\sqrt{2}\,C_5(n)\right\}.$$

The estimate (3.38) finishes the proof of the lemma. □

We can finally state the precise pointwise estimate.

Theorem 3.5. *Under the conditions of Theorem 3.1, assume furthermore that the following strong $L^{2,1}$ no-neck energy holds*

$$\lim_{\alpha\to 0}\lim_{k\to\infty}\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}=0.\tag{3.39}$$

Then, there exists $\alpha_0>0$ such that for all $k\in\mathbb{N}$ large enough, there exists a moving frame $(\vec{f}_{k,1},\vec{f}_{k,2})\in W^{1,(2,1)}(B_{\alpha_0}(0))\times W^{1,(2,1)}(B_{\alpha_0}(0))$ and a universal constant $C_7(n)$ (independent of k) such that

$$\left\|\nabla\vec{f}_{k,1}\right\|_{\mathrm{L}^{2,1}(B_{\alpha_0}(0))}+\left\|\nabla\vec{f}_{k,2}\right\|_{\mathrm{L}^{2,1}(B_{\alpha_0}(0))}\leq C_7(n)\left(1+\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha_0))}\right)\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha_0))}.$$

Furthermore, there exists a sequence of functions $\mu_k\in W^{2,1}(B_{\alpha_0}(0))$ and a universal constant $C_8(n)$ such that

$$\left\|\nabla^2\mu_k\right\|_{\mathrm{L}^{1}(B_{\alpha_0}(0))}+\|\nabla\mu_k\|_{\mathrm{L}^{2,1}(B_{\alpha_0}(0))}+\|\mu_k\|_{\mathrm{L}^{\infty}(B_{\alpha_0}(0))}\leq C_8(n)\left(1+\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha_0))}\right)\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha_0))}$$

and there exists a sequence of holomorphic functions $\psi_k:B_{\alpha_0}(0)\to\mathbb{C}$ and $\chi_k:B_{\alpha_0}(0)\to\mathbb{C}$ such that $\chi_k(0)=0$, $c\in\mathbb{C}$ and $\{c_k\}_{k\in\mathbb{N}}\subset\mathbb{C}$ such that $c_k\underset{k\to\infty}{\longrightarrow}c$ and

$$\psi_k(z)=e^{c_k}z^{\theta_0-1}\left(1+\chi_k(z)\right)\tag{3.40}$$

and

$$e^{\lambda_k}=e^{\mu_k}|\psi_k(z)|=e^{\mathrm{Re}\,(c_k)}|z|^{\theta_0-1}\left(1+o(1)\right),\qquad \textit{for all } z\in\Omega_k(\alpha).\tag{3.41}$$

Finally, there exists $\vec{A}_0\in\mathbb{C}^n$ (such that $\langle\vec{A}_0,\vec{A}_0\rangle=0$) and $\left\{\vec{A}_{k,0}\right\}_{k\in\mathbb{N}}\in\mathbb{C}^n$ such that $\vec{A}_{k,0}\underset{k\to\infty}{\longrightarrow}\vec{A}_0$ and for all $z\in\Omega_k(\alpha_0)$, we have the pointwise identities

$$\partial_z\vec{\Phi}_k=\frac{1}{2}e^{c_k+\mu_k(z)}z^{\theta_0-1}\left(1+\chi_k(z)\right)\left(\vec{f}_{k,1}-i\vec{f}_{k,2}\right)=\vec{A}_{k,0}z^{\theta_0}+o(|z|^{\theta_0-1})\tag{3.42}$$

Proof. **Step 1: Expansion of $\nabla\vec{\Phi}_k$ in the neck region.** By, fix $\alpha_0>0$ such that for all $k\in\mathbb{N}$ large enough

$$\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha_0))}\leq\varepsilon_3(n)\tag{3.43}$$

where $\varepsilon_3(n) > 0$ is given by Lemma 3.3. Then we define as in the proof of Theorem 3.1 for all $j = 1, 2$ $\vec{e}_k, j = e_k^{-\lambda}\partial_{x_j}\vec{\Phi}_k$, and by Lemma 3.2, Lemma 3.3, (3.42) and (3.43), there exists a controlled extension $\widetilde{\vec{n}_k} : B_{\alpha_0}(0) \to \mathscr{G}_{n-2}(\mathbb{R}^n)$ if $\vec{n}_k : \Omega_k(\alpha_0) \to \mathscr{G}_{n-2}(\mathbb{R}^n)$ such that

$$\begin{cases}\widetilde{\vec{n}_k} = \vec{n}_k & \text{in } \Omega_k(\alpha_0) = B_{\alpha_0} \setminus B_{\alpha_0^{-1}\rho_k}(0)\\ \left\|\nabla\widetilde{\vec{n}_k}\right\|_{\mathrm{L}^{2,1}(B_{\alpha_0}(0))} \leq C_5(n)\left\|\nabla\vec{n}_k\right\|_{\mathrm{L}^{2,1}(\Omega_k)}.\end{cases}$$

and a Coulomb frame $(\vec{f}_{k,1}, \vec{f}_{k,2}) \in W^{1,(2,1)}(B_{\alpha_0}(0), S^{n-1}) \times W^{1,(2,1)}(B_{\alpha_0}(0), S^{n-1})$ associated to $\widetilde{\vec{n}_k}$ such that

$$\widetilde{\vec{n}_k} = \star(\vec{f}_{k,1} \wedge \vec{f}_{k,2}) \quad \text{in } B_{\alpha_0}(0) \qquad \text{and} \qquad \begin{cases}\mathrm{div}\left(\vec{f}_{k,1}\cdot\nabla\vec{f}_{k,2}\right) = 0 & \text{in } B_{\alpha_0}(0)\\ \vec{f}_{k,1}\cdot\partial_\nu\vec{f}_{k,2} & \text{on } \partial B_{\alpha_0}(0)\end{cases} \tag{3.44}$$

and

$$\left\|\nabla\vec{f}_{k,1}\right\|_{\mathrm{L}^{2,1}(B_{\alpha_0}(0))} + \left\|\nabla\vec{f}_{k,2}\right\|_{\mathrm{L}^{2,1}(B_{\alpha_0}(0))} \leq C_6(n)\left(1 + \|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha_0))}\right)\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha_0))}. \tag{3.45}$$

Finally, we introduce the rotation θ_k (which is a multivalued function on $\Omega_k(\alpha)$) such that

$$\left(\vec{f}_{k,1} + i\vec{f}_{k,2}\right) = e^{i\theta_k}\left(\vec{e}_{k,1} + i\vec{e}_{k,2}\right) \qquad \text{on } \Omega_k(\alpha_0) \tag{3.46}$$

As previously, let μ_k the unique solution of

$$\begin{cases}\Delta\mu_k = \nabla^\perp\vec{f}_{k,1}\cdot\nabla\vec{f}_{k,2} & \text{in } B_{\alpha_0}(0)\\ \mu_k = 0 & \text{on } \partial B_{\alpha_0}(0).\end{cases}$$

Then we have by the improved Wente inequality $\mu_k \in W^{1,(2,1)}(B_{\alpha_0}(0)) \cap C^0(B_{\alpha_0}(0))$ and (3.45) for some universal constant $C_9(n)$

$$\left\|\nabla^2\mu_k\right\|_{\mathrm{L}^1(B_{\alpha_0}(0))} + \|\nabla\mu_k\|_{\mathrm{L}^{2,1}(B_{\alpha_0}(0))} + \|\mu_k\|_{\mathrm{L}^\infty(B_{\alpha_0}(0))} \leq C_9(n)\left(1 + \|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha_0))}\right)\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha_0))} \tag{3.47}$$

Furthermore, introduce the notation $\nu_k = \lambda_k - \mu_k$. Then ν_k is harmonic, and implies that for k large enough

$$\frac{1}{2\pi}\int_{\partial B_{\rho_k}} *\,d\nu_k \underset{k\to\infty}{\longrightarrow} = \theta_0 - 1.$$

Indeed, recall that by the proof of Theorem 3.1, $*\,d\nu_k = d\theta_k$ and that there exists a holomorphic function $\psi_k : \Omega_k(\alpha_0) \to \mathbb{C}$ such that

$$e^{i\theta_k} = \frac{\psi_k}{|\psi_k|}. \tag{3.48}$$

In particular, a computation of the proof of Theorem 3.1 shows that

$$d\theta_k = \mathrm{Im}\left(\frac{\partial\psi_k}{\psi_k}\right), \tag{3.49}$$

so that for all $\alpha_0^{-1}\rho_k < \rho < \alpha_0$

$$\frac{1}{2\pi}\int_{\partial B_\rho} *d\nu_k = \frac{1}{2\pi}\int_{\partial B_\rho} d\theta_k = \frac{1}{2\pi}\mathrm{Im}\int_{\partial B_\rho} \in \mathbb{Z}.$$

As

$$d_k = \frac{1}{2\pi}\int_{\partial B_\rho} *d\nu_k \underset{k\to\infty}{\longrightarrow} \theta_0 - 1,$$

we deduce that

$$\frac{1}{2\pi} * d\nu_k \int_{\partial B_\rho} = \frac{1}{2\pi}\int_{\partial B_\rho} d\theta_k = \theta_0 - 1$$

for all k large enough. In other words ν_k satisfies as in (3.28)

$$\begin{cases} \partial_{x_2}\nu_k + \partial_{x_1}\theta_k = 0 \\ \partial_{x_1}\nu_k - \partial_{x_2}\theta_k = 0 \end{cases}. \tag{3.50}$$

Therefore, we deduce by (3.50) that

$$\partial_z \nu_k = \frac{1}{2}\left(\partial_{x_1}\nu_k - i\partial_{x_2}\nu_k\right) = \frac{1}{2}\left(\partial_{x_2}\theta_k + i\,\partial_{x_1}\theta_k\right) = \frac{i}{2}\left(\partial_{x_1}\theta_k - i\,\partial_{x_2}\theta_k\right) = i\,\partial_z\theta_k. \tag{3.51}$$

As $d\theta_k = \partial\theta_k + \overline{\partial}\theta_k$, (3.49) implies that

$$i\,\partial\theta_k = \frac{1}{2}\frac{\partial\psi_k}{\psi_k} = \frac{1}{2}\frac{\partial_z\psi_k}{\psi_k}dz = \partial\log|\psi_k|, \tag{3.52}$$

as $\partial_{\overline{z}}\psi_k = 0$ implies that $\partial_z\overline{\psi_k} = \overline{\partial_{\overline{z}}\psi_k} = 0$ and

$$\partial_z\log|\psi_k| = \frac{1}{2}\log\left(\psi_k(z)\overline{\psi_k(z)}\right) = \frac{1}{2}\frac{\psi_k'(z)\overline{\psi_k(z)}}{\psi_k(z)\overline{\psi_k(z)}} = \frac{1}{2}\frac{\psi_k'(z)}{\psi_k(z)}.$$

Therefore, (3.51) and (3.52) show that

$$\partial\Big(\nu_k - \log|\psi_k|\Big) = 0.$$

So the function $\nu_k - \log|\psi_k|$ is anti-holomorphic and *real*, so it must be constant by the maximum principle as $\Omega_k(\alpha_0) = B_{\alpha_0}\setminus\overline{B}_{\alpha_0^{-1}\rho_k}$ is connected. Therefore, there exists $\gamma_k \in \mathbb{R}$ such that

$$\nu(z) = \gamma_k + \log|\psi_k(z)|, \tag{3.53}$$

or

$$e^{\nu_k(z)} = e^{\gamma_k}|\psi_k(z)|. \tag{3.54}$$

Now, as $\widetilde{\psi_k} = e^{\gamma_k}\psi_k$ is holomorphic and satisfies

$$\frac{1}{2\pi}\mathrm{Im}\int_{\partial B_\rho}\frac{\partial\widetilde{\psi_k}}{\widetilde{\psi_k}} = \frac{1}{2\pi}\mathrm{Im}\int_{\partial B_\rho}\frac{\partial\psi_k}{\psi_k} = \theta_0 - 1, \tag{3.55}$$

we can assume without loss of generality that $\gamma_k = 0$. Furthermore, (3.55) shows that the holomorphic 1-form $\frac{\partial\psi_k}{\psi_k}$ on $\Omega_k(\alpha_0)$ admits the expansion

$$\frac{\partial\psi_k}{\psi_k} = (\theta_0 - 1)\frac{dz}{z} + \xi_k(z)dz,$$

where ξ_k admits a holomorphic extension on $B_{\alpha_0}(0)$. In particular, ψ_k admits a Laurent series expansion

$$\psi_k(z) = \sum_{m=\theta_0-1}^{\infty} a_m z^m,$$

where $a_{\theta_0-1}\neq 0$. Therefore, ψ_k extends *holomorphically* in $B_{\alpha_0}(0)$, and letting $c_k\in\mathbb{C}$ be such that

$$e^{c_k}=a_{\theta_0-1},$$

there exists a holomorphic function $\chi_k:B_{\alpha_0}(0)\to\mathbb{C}$ such that $\chi_k(0)=0$ and

$$\psi_k(z)=e^{c_k}z^{\theta_0-1}\left(1+\chi_k(z)\right), \tag{3.56}$$

where we have explicitly

$$\chi_k(z)=\sum_{m=\theta_0}^{\infty}e^{-c_k}a_m z^{m-(\theta_0-1)}.$$

Notice in particular as $\lambda_k=\mu_k+\nu_k$ that

$$e^{\lambda_k}=e^{\mu_k}|\psi_k(z)|, \tag{3.57}$$

where ψ_k is holomorphic and admits the expansion (3.56). Now, we come back to the identity (3.46) to observe that

$$\partial_z\vec{\Phi}_k=\frac{1}{2}\left(\partial_{x_1}\vec{\Phi}_k-i\partial_{x_2}\vec{\Phi}_k\right)=\frac{1}{2}e^{\lambda_k}\left(\vec{e}_{k,1}-i\,\vec{e}_{k,2}\right)=\frac{1}{2}e^{\lambda_k}e^{i\theta_k}\left(\vec{f}_{k,1}-i\vec{f}_{k,2}\right). \tag{3.58}$$

Now, observe that by (3.48) and (3.57)

$$e^{\lambda_k}e^{i\theta_k}=e^{\mu_k}|\psi_k(z)|\times\frac{\psi_k(z)}{|\psi_k(z)|}=e^{\mu_k}\psi_k(z). \tag{3.59}$$

Therefore, (3.58), (3.59) and (3.56) finally yield the expansion

$$\begin{aligned}\partial_z\vec{\Phi}_k&=\frac{1}{2}e^{\mu_k}\psi_k(z)\left(\vec{f}_{k,1}-i\,\vec{f}_{k,2}\right)\\&=\frac{1}{2}e^{c_k+\mu_k}z^{\theta_0-1}\left(1+\chi_k(z)\right)\left(\vec{f}_{k,1}-i\,\vec{f}_{k,2}\right).\end{aligned} \tag{3.60}$$

By (3.45) and (3.47), $e^{\mu_k}\left(\vec{f}_{k,1}-i\,\vec{f}_{k,2}\right)\in W^{1,(2,1)}\cap C^0(B_{\alpha_0}(0),S^{n-1})$ and

$$\begin{aligned}&\left\|e^{\mu_k}\left(\vec{f}_{k,1}-i\,\vec{f}_{k,2}\right)\right\|_{\mathrm{L}^\infty(B_{\alpha_0}(0))}\leq e^{C_9(n)\left(1+\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha_0))}\right)\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha_0))}}\\&\left\|\nabla\left(e^{\mu_k}\left(\vec{f}_{k,1}-i\,\vec{f}_{k,2}\right)\right)\right\|_{\mathrm{L}^{2,1}(B_{\alpha_0}(0))}\\&\leq\left(C_6(n)+C_9(n)\right)\left(1+\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha_0))}\right)\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha_0))}\,e^{\left(1+\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha_0))}\right)\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha_0))}}\end{aligned} \tag{3.61}$$

In particular, if

$$\frac{1}{2}e^{c_k+\mu_k(0)}\left(\vec{f}_{k,1}-i\vec{f}_{k,2}\right)(0)=\vec{A}_{k,0}\in\mathbb{C}^n\setminus\{0\}$$

then (notice that $\vec{A}_{k,0}\neq 0$ as $\vec{\Phi}_k$ is an immersion) (3.60) becomes

$$\partial_z\vec{\Phi}_k=\vec{A}_{k,0}z^{\theta_0-1}+o\left(|z|^{\theta_0-1}\right)\qquad\text{for all }\ z\in\Omega_k(\alpha_0).$$

Furthermore by the strong convergence of $\vec{\Phi}_k$ towards $\vec{\Phi}_\infty$ in $C^l_{\mathrm{loc}}(B_1(0)\setminus\{0\})$ (for all $l\in\mathbb{N}$) which satisfies

$$\partial_z\vec{\Phi}_\infty=\vec{A}_0z^{\theta_0-1}+o\left(|z|^{\theta_0-1}\right),$$

we deduce that

$$\vec{A}_{k,0}\underset{k\to\infty}{\longrightarrow}\vec{A}_0.$$

This concludes the proof of the theorem. ☐

3.2 General case

Theorem 3.6. *Let $\{\vec{\Phi}_k\}_{k\in\mathbb{N}}$ be a sequence of smooth conformal immersions from the disk $B_1(0)\subset\mathbb{C}$ into $\mathbb{R}^n$. Let $m\in\mathbb{N}$, and for all $1\leq j\leq m$, let $\{a_k^j\}_{k\in\mathbb{N}}\subset B_1(0)$, $\{\rho_k^j\}_{k\in\mathbb{N}}\subset(0,\infty)$ and define for $0<\alpha<1$ and k large enough*

$$\Omega_k=B_1(0)\setminus\bigcup_{j=1}^m\overline{B}_{\rho_k^j}(a_k^j),\qquad \Omega_k(\alpha)=B_\alpha(0)\setminus\bigcup_{j=1}^m\overline{B}_{\alpha^{-1}\rho_k^j}(a_k^j).$$

Assume that for all $1\leq j\neq j'\leq m$, and all $0<\alpha<1$ we have $B_{\alpha^{-1}\rho_k^j}(a_k^j)\cap B_{\alpha^{-1}\rho_k^{j'}}(a_k^{j'})=\varnothing$ for k large enough, and

$$\rho_k^j\underset{k\to\infty}{\longrightarrow}0,\qquad a_k^j\underset{k\to\infty}{\longrightarrow}0.$$

Furthermore, assume that

$$\sup_{k\in\mathbb{N}}\int_{\Omega_k}|\nabla\vec{n}_k|^2dx\leq\varepsilon_1(n),\qquad \sup_{k\in\mathbb{N}}\|\nabla\lambda_k\|_{\mathrm{L}^{2,\infty}(\Omega_k)}<\infty,$$

where $\varepsilon_1(n)$ is given by the proof of Theorem 2.1. Finally, assume that

$$\lim_{\alpha\to0}\limsup_{k\to\infty}\int_{\Omega_k(\alpha)}|\nabla\vec{n}_k|^2dx=0$$

and that there exists a $W^{2,2}_{\mathrm{loc}}(B_1(0)\setminus\{0\})\cap C^\infty(B_1(0)\setminus\{0\})$ immersion $\vec{\Phi}_\infty$ such that

$$\log|\nabla\vec{\Phi}_\infty|\in L^\infty_{\mathrm{loc}}(B_1(0)\setminus\{0\})$$

and $\vec{\Phi}_k\underset{k\to\infty}{\longrightarrow}\vec{\Phi}_\infty$ in $C^l_{\mathrm{loc}}(B_1(0)\setminus\{0\})$. For all $k\in\mathbb{N}$, let

$$e^{\lambda_k}=\frac{1}{\sqrt{2}}|\nabla\vec{\Phi}_k|$$

be the conformal factor of $\vec{\Phi}_k$. Then, there exists a positive integer $\theta_0\geq1$, and for all $k\in\mathbb{N}$ integers $\theta_k^1,\cdots,\theta_k^m\in\mathbb{N}$ such that for all $k\in\mathbb{N}$ large enough

$$\sum_{j=1}^m\theta_k^j=\theta_0-1,$$

and for all $k\in\mathbb{N}$, there exists $1/2<\alpha_k<1$ and $A_k\in\mathbb{R}$ such that

$$\left\|\lambda_k-\sum_{j=1}^j\theta_k^j\log|z-a_k^j|-A_k\right\|_{\mathrm{L}^\infty(\Omega_k(\alpha_k))}\leq\Gamma_{14}\left(\|\nabla\lambda_k\|_{\mathrm{L}^{2,\infty}(\Omega_k)}+\int_{\Omega_k}|\nabla\vec{n}_k|^2dx\right)\tag{3.62}$$

for some universal constant $\Gamma_{14}=\Gamma_{14}(n)$. Furthermore, we have for all $0<\rho_k\leq1$ such that

$$\bigcup_{j=1}^m\overline{B}_{\rho_k^j}(a_k^j)\subset B_{\rho_k}(0).$$

and for all $k\in\mathbb{N}$ large enough

$$\frac{1}{2\pi}\int_{\partial B_{\rho_k}(0)}*\,d\nu_k=\theta_0-1.$$

Finally, for all $k\in\mathbb{N}$ and $j\in\{1,\cdots,m\}$, we have

$$\frac{1}{2\pi}\int_{\partial B_{\rho_k^j}(a_k^j)}*\,d\nu_k=\theta_k^j\in\mathbb{Z}.$$

Proof. Indeed, the same argument shows that there exists a holomorphic function φ_k on Ω_k and $c_k^1,\cdots,c_k^m\in\mathbb{C}$ such that

$$e^{\nu_k}=\varphi_k(z)e^{-i\theta_k}+\sum_{j=1}^{m}\frac{c_k^j e^{i\theta_k}}{\overline{z-a_k^j}}$$

and the same computation shows if

$$\psi_k(z)=\varphi_k(z)-\sum_{j=1}^{m}\frac{\overline{c_k^j}}{z-a_k^j}$$

that

$$e^{i\theta_k}=\frac{\psi_k(z)}{|\psi_k(z)|}.$$

Therefore, we have

$$d\theta_k=\mathrm{Im}\left(\frac{\partial\psi_k}{\psi_k}\right)$$

and for all $1\leq j\leq m$

$$\int_{\partial B_{\rho_k^j}(a_k^j)}d\theta_k\in 2\pi\mathbb{Z}.$$

Furthermore, we have

$$\lim_{k\to\infty}\int_{\partial B_1(0)}d\theta_k=2\pi(\theta_0-1)\geq 0. \tag{3.63}$$

In particular, if $\{\rho_k\}_{k\in\mathbb{N}}\subset(0,\infty)$ is such that $\rho_k\underset{k\to\infty}{\longrightarrow}0$ and

$$\bigcup_{j=1}^{m}\overline{B}_{\rho_k^j}(a_k^j)\subset B_{\alpha^{-1}\rho_k}(0),$$

then we also have for $k\in\mathbb{N}$ large enough

$$\frac{1}{2\pi}\mathrm{Im}\int_{\partial B_{\rho_k}(0)}\frac{\partial\psi_k}{\psi_k}=\theta_0-1\geq 0,$$

which implies that ψ_k admits a holomorphic extension on $B_1(0)$. Analytic continuation then implies that for all $1\leq j\leq m$

$$\sum_{j=1}^{m}\theta_k^j=\frac{1}{2\pi}\mathrm{Im}\int_{\partial B_{\rho_k}(0)}\frac{\partial\psi_k}{\psi_k}=\theta_k^j\geq 0.$$

Therefore, we have by (3.63) for k large enough

$$\frac{1}{2\pi}\sum_{j=1}^{m}\int_{\partial B_{\rho_k^j}(a_k^j)}d\theta_k=\theta_0-1.$$

Then, we deduce by the argument of Lemma V.3 of [2] that there exists a universal constant $\Gamma_{15}(n)=\Gamma_{15}(n)$ such that for all $k\in\mathbb{N}$ there exists $1/2<\alpha_k<1$ such that for all $k\in\mathbb{N}$ large enough

$$\left\|\nu_k-\sum_{j=1}^{m}\theta_k^j\log|z-a_j|-A_k\right\|_{\mathrm{L}^\infty(\Omega_k(\alpha_k))}\leq\Gamma_{15}(n)\left(\|\nabla\lambda_k\|_{\mathrm{L}^{2,\infty}(\Omega_k)}+\int_{\Omega_k}|\nabla\vec{n}_k|^2dx\right), \tag{3.64}$$

In particular, as $\mu_k \in L^\infty(B_1(0))$ we get the estimate (3.62) from (3.64) and $\|\mu_k\|_{\mathrm{L}^\infty(B_1(0))} \leq \Gamma_{16}$ for some universal $\Gamma_{16} = \Gamma_{16}(\Lambda, n)$ (thanks to Wente's estimate), we deduce that there exists a universal constant $C = C(n, \Lambda)$, where

$$\Lambda = \sup_{k\in\mathbb{N}} \left(\|\nabla \lambda_k\|_{\mathrm{L}^{2,\infty}(\Omega_k)} + \int_{\Omega_k} |\nabla \vec{n}_k|^2 dx \right)$$

such that for all k large enough and $z \in \Omega_k(1/2)$ (noticing that A_k is bounded by the strong convergence outside of 0)

$$\frac{1}{C} \leq \frac{e^{\lambda_k(z)}}{\displaystyle\prod_{j=1}^{m} |z - a_k^j|^{\theta_k^j}} \leq C. \tag{3.65}$$

This additional remarks completes the proof of the Proposition. □

Remarks 3.7. (1) The integers θ_k^j *a priori* depend on k, but we will see in the case of interest of bubbling of Willmore immersions, they must stabilise for k large enough.

(2) The reader will notice that we do not need the limiting immersion to be smooth, but merely $C^{1,\alpha}$ for some $0 < \alpha < 1$ (this allows one to define branch points, [8]). As in the application we restrict to Willmore immersions, we automatically get the smoothness of the limiting immersion outside of the point of concentration.

Theorem 3.5 also has an analogue in this setting, but we will not state it for the sake of brevity of the paper.

4 Improved energy quantization for Willmore immersions

In this section, we build on [2] to obtain an improved no-neck energy.

Theorem 4.1. *Let Σ be a closed Riemann surface and assume that $\{\vec{\Phi}_k\}_{k\in\mathbb{N}}$ is a sequence of smooth Willmore immersions such that*

$$\limsup_{k\to\infty} W(\vec{\Phi}_k) < \infty.$$

Assume furthermore that the conformal class of $g_k = \vec{\Phi}_k^* g_{\mathbb{R}^n}$ is precompact in the moduli space. Then for all $0 < \alpha < 1$ let $\Omega_k(\alpha) = B_{\alpha R_k} \setminus \overline{B}_{\alpha^{-1} r_k}(0)$ be a neck domain and $\theta_0 \in \mathbb{N}$ such that (by Theorem 3.1)

$$\theta_0 - 1 = \lim_{\alpha\to 0} \lim_{k\to\infty} \int_{\partial B_{\alpha^{-1} r_k}(0)} \partial_\nu \lambda_k \, d\mathscr{H}^1, \tag{4.1}$$

and define

$$\Lambda = \sup_{k\in\mathbb{N}} \left(\|\nabla \lambda_k\|_{\mathrm{L}^{2,\infty}(\Omega_k(1))} + \int_{\Omega_k(1)} |\nabla \vec{n}_k|^2 dx \right).$$

Then there exist a universal constant $\Gamma_{17} = \Gamma_{17}(n)$, and $\alpha_0 = \alpha_0(\{\vec{\Phi}_k\}_{k\in\mathbb{N}}) > 0$ such that for all $0 < \alpha < \alpha_0$ and $k \in \mathbb{N}$ large enough,

$$\|\nabla \vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))} \leq \Gamma_{17}(n) e^{\Gamma_{17}(n)\Lambda} \left(1 + \|\nabla \vec{n}_k\|_{\mathrm{L}^2(\Omega_k(4\alpha))} \right) \|\nabla \vec{n}_k\|_{\mathrm{L}^2(\Omega_k(4\alpha))}. \tag{4.2}$$

In particular, we deduce by the L^2 no-neck energy

$$\lim_{\alpha\to 0} \limsup_{k\to\infty} \|\nabla \vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))} = 0.$$

Proof. **Step 1: $L^{2,1}$-quantization of the mean curvature.** Here, we will prove that

$$\lim_{\alpha\to 0}\limsup_{k\to\infty}\left(\left\|e^{\lambda_k}\vec{H}_k\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}+\left\|e^{\lambda_k}\nabla\vec{H}_k\right\|_{\mathrm{L}^1(\Omega_k(\alpha))}\right)=0.$$

This statement is a consequence of the following lemma.

Theorem 4.2. *There exists constants $R_0(n),\varepsilon_4(n)>0$ with the following property. Let $0<100r<R\leq R_0(n)$, and $\vec{\Phi}:B_R(0)\to\mathbb{R}^n$ be a weak conformal Willmore immersion of finite total curvature, such that*

$$\sup_{r<s<R/2}\int_{B_{2s}\setminus\overline{B}_s(0)}|\nabla\vec{n}|^2dx\leq\varepsilon_4(n).$$

Set $\Omega=B_R\setminus\overline{B}_r(0)$, and

$$\Lambda=\|\nabla\lambda\|_{\mathrm{L}^{2,\infty}(\Omega)}+\int_{\Omega}|\nabla\vec{n}|^2dx,$$

where λ is the conformal parameter of $\vec{\Phi}$. Then there exists a universal constant $\Gamma_{18}=\Gamma_{18}(n)$ such that for all $\left(\dfrac{4r}{5R}\right)^{\frac{1}{3}}<\alpha<\dfrac{1}{5}$, we have

$$\left\|e^{\lambda}\vec{H}\right\|_{\mathrm{L}^{2,1}(\Omega_\alpha)}+\left\|e^{\lambda}\nabla\vec{H}\right\|_{\mathrm{L}^1(\Omega_\alpha)}\leq\Gamma_{18}(n)\left(1+\Lambda\right)e^{\Gamma_{18}(n)\Lambda}\left(1+\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\right)\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}. \tag{4.3}$$

Remarks on the proof. The proof closely follows the proof in [2]. In **Step 1**, we use the previous results to obtain the $L^{2,1}\cap W^{1,1}$ control for the harmonic parts of tensors, and the Wente inequality for the part with Dirichlet boundary conditions.

In **Step 2**, we use a structural property of the unit normal $\vec{n}$ to transfer the $L^{2,1}$ control of $e^{\lambda}\vec{H}$ into a $L^{2,1}$ control of $\nabla\vec{n}$. The proof uses other results on moving frames from [10], and the rest follows again by classical Calderón-Zygmund estimates, Wente inequality, and an averaging lemma. The proof is quite lengthy but globally straightforward.

Remark 4.3. Notice that by $L^{2,1}/L^{2,\infty}$ duality, we have

$$\left\|\nabla(e^{\lambda}\vec{H})\right\|_{\mathrm{L}^1(\Omega_\alpha)}\leq\left\|(\nabla\lambda)e^{\lambda}\vec{H}\right\|_{\mathrm{L}^1(\Omega_\alpha)}+\left\|e^{\lambda}\vec{H}\right\|_{\mathrm{L}^1(\Omega_\alpha)}\leq\|\nabla\lambda\|_{\mathrm{L}^{2,\infty}(\Omega_\alpha)}\left\|e^{\lambda}\vec{H}\right\|_{\mathrm{L}^{2,1}(\Omega_\alpha)}+\left\|e^{\lambda}\nabla\vec{H}\right\|_{\mathrm{L}^1(\Omega_\alpha)}$$

Proof. Define for all $\left(\dfrac{r}{R}\right)^{\frac{1}{2}}<\alpha<1$ the open subset $\Omega_\alpha=B_{\alpha R}\setminus\overline{B}_{\alpha^{-1}r}$ of Ω. We follow step by steps the proof of Lemma VI.6 of [2]. First, the pointwise estimate on $\nabla\vec{n}$ is identical and we find that there exists $\Gamma_{19}=\Gamma_{19}(n),\Gamma_{19}'=\Gamma_{19}'(n)>0$ such that for all $z\in B_{4R/5}\setminus\overline{B}_{5r/4}(0)$

$$|\nabla\vec{n}(z)|\leq\frac{\Gamma_{19}}{|z|^2}\int_{B_{2|z|}\setminus\overline{B}_{|z|/2}(0)}|\nabla\vec{n}|^2d\mathscr{L}^2\leq\frac{\Gamma_{19}'\sqrt{\varepsilon_4(n)}}{|z|} \tag{4.4}$$

so that

$$\|\nabla\vec{n}\|_{\mathrm{L}^{2,\infty}(\Omega)}\leq\sqrt{\pi}\Gamma_{19}'\sqrt{\varepsilon_4(n)}$$

and we can choose $\varepsilon_4(n)=\dfrac{\varepsilon_1(n)^2}{\sqrt{\pi}\Gamma_{19}'(n)}$. Therefore, thanks to Theorem 2.1, there exists $d\in\mathbb{R}$ such that

$$|d|\leq\Gamma_{20}(n)\Lambda.$$

and for all $\left(\dfrac{5}{4}\right)^{\frac{2}{3}}\left(\dfrac{r}{R}\right)^{\frac{1}{3}}=\left(\dfrac{\frac{5r}{4}}{\frac{4R}{5}}\right)^{\frac{1}{3}}<\dfrac{5\alpha}{4}<\dfrac{1}{4}$, there exists $A_\alpha\in\mathbb{R}$ such that

$$\|\lambda-d\log|z|-A_\alpha\|_{\mathrm{L}^\infty(\Omega_\alpha)}\leq\Gamma_0'(n)\sqrt{\frac{5\alpha}{4}}\Lambda+\Gamma_0'(n)\leq\Gamma_0''(n)\left(\sqrt{\alpha}\Lambda+\int_{\Omega}|\nabla\vec{n}|^2dx\right). \tag{4.5}$$

As $\vec{\Phi}$, is Willmore, the following 1-form is closed :

$$\vec{\alpha} = \mathrm{Im}\left(\partial\vec{H} + |\vec{H}|^2\partial\vec{\Phi} + g^{-1}\otimes\langle\vec{H},\vec{h}_0\rangle\otimes\overline{\partial}\vec{\Phi}\right).$$

As $\vec{\Phi}$ is well-defined on $B_R(0)$, the the Poincaré lemma, implies that there exists $\vec{L}: B_R(0)\rightarrow\mathbb{R}^n$ such that

$$2i\,\partial\vec{L} = \partial\vec{H} + |\vec{H}|^2\partial\vec{\Phi} + g^{-1}\otimes\langle\vec{H},\vec{h}_0\rangle\otimes\overline{\partial}\vec{\Phi}. \tag{4.6}$$

Now, introduce for $0<s<R/2$

$$\delta(s) = \left(\frac{1}{s^2}\int_{B_{2s}\setminus\overline{B}_{s/2}(0)}|\nabla\vec{n}|^2dx\right)^{\frac{1}{2}}.$$

Then we have trivially for all $2r<s<R/2$

$$s\delta(s)\leq\left(\int_{\Omega}|\nabla\vec{n}|^2dx\right)^{\frac{1}{2}} = \|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)} \tag{4.7}$$

and Fubini's theorem implies that for all $r\leq r_1<r_2\leq R/2$

$$\begin{aligned}\int_{r_1}^{r_2}s\delta(s)^2ds &= \int_{r_1}^{r_2}\frac{1}{s}\left(\int_{B_{2s}\setminus\overline{B}_{s/2}(0)}|\nabla\vec{n}(x)|^2dx\right)ds = \int_{r_1}^{r_2}\int_{B_{2r_2}\setminus\overline{B}_{r_1/2}(0)}\frac{|\nabla\vec{n}(x)|^2}{s}\mathbf{1}_{\{s/2\leq|x|\leq 2s\}}dxds\\ &= \log(4)\int_{B_{2r_2}\setminus\overline{B}_{r_1}(0)}|\nabla\vec{n}|^2dx.\end{aligned} \tag{4.8}$$

Now, (4.4) shows that for some $C_{10}=C_{10}(n)$

$$\max\left\{e^{\lambda(z)}|\vec{H}(z)|, e^{\lambda(z)}|\vec{H}_0(z)|\right\}\leq|\nabla\vec{n}(z)|\leq C_{10}\delta(|z|). \tag{4.9}$$

Furthermore, the same argument of [2] (see [1] for more details) using a Theorem from [7] implies that there exists a constant $C_{11}=C_{11}(n)$ such that

$$e^{\lambda(z)}|\nabla\vec{H}(z)|\leq C_{11}\frac{\delta(|z|)}{|z|}\qquad\text{for all }\; z\in\Omega_{1/2} \tag{4.10}$$

Therefore, we have thanks to (4.6), (4.9) and (4.10)

$$|\nabla\vec{L}(z)| = 2|\partial\vec{L}(z)|\leq e^{-\lambda(z)}\left(C_{11}\frac{\delta(z)}{|z|}+2C_{10}\delta(z)^2\right). \tag{4.11}$$

Now assume for simplicity that $\alpha=1/2$ (then we do not need to use the precised form (4.5) and we can use instead Lemma V.3 from [2]). Denoting for all $r<s<R$

$$\vec{L}_s = \fint_{\partial B_s}\vec{L}\,d\mathscr{H}^1,$$

we deduce from (4.10) that for all $z\in\Omega_{1/2}$ (taking $\alpha=1/2$ in (4.5))

$$\begin{aligned}|\vec{L}(z)-\vec{L}_{|z|}| &\leq \int_{\partial B_{|z|}}|\nabla\vec{L}|d\mathscr{H}^1 \leq 2\pi e^{2\Gamma_1\Lambda}e^{-\lambda(|z|)}\left(C_{11}\delta(|z|)+2C_{10}|z|\delta(|z|)\cdot\delta(|z|)\right)\\ &\leq 2\pi e^{2\Gamma_1\Lambda}e^{-\lambda(|z|)}\left(C_{11}+2C_{10}\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\right)\delta(|z|).\end{aligned} \tag{4.12}$$

Then we get

$$\int_{\Omega_{1/2}}|\vec{L}(z)-\vec{L}_{|z|}|^2e^{2\lambda(z)}|dz|^2\leq 2\pi e^{2\Gamma_1\Lambda}\left(C_{11}+2C_{10}\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\right)\int_{\Omega}\delta(|z|)^2|dz|^2$$

$$
\begin{aligned}
&= 4\pi^2 e^{2\Gamma_1\Lambda}\left(C_{11}+2C_{10}\,\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\right)\int_{2r}^{R/2} s\delta^2(s)ds\\
&= 4\pi^2\log(4)e^{2\Gamma_1\Lambda}\left(C_{11}+2C_{10}\,\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\right)\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\,. \qquad (4.13)
\end{aligned}
$$

Now, we continue the proof in an exact same way to obtain the pointwise estimate (for some universal constant $C_{12}=C_{12}(n)$)

$$
e^{\lambda(z)}|z||\vec{L}_{|z|}|\leq C_{12}e^{2\Gamma_1\Lambda}\left(1+\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\right)\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\,. \qquad (4.14)
$$

Therefore, we get

$$
\left\|e^{\lambda(z)}\vec{L}_{|z|}\right\|_{\mathrm{L}^{2,\infty}(\Omega_{1/2})}\leq 2\sqrt{\pi}C_{12}e^{2\Gamma_1\Lambda}\left(1+\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\right)\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\,. \qquad (4.15)
$$

Combining (4.13) and (4.15) implies as $\|\cdot\|_{\mathrm{L}^{2,\infty}(\,\cdot\,)}\leq\|\cdot\|_{\mathrm{L}^2(\,\cdot\,)}$ that

$$
\begin{aligned}
\left\|e^{\lambda}\vec{L}\right\|_{\mathrm{L}^{2,\infty}(\Omega_{1/2})}&\leq\left(8\pi^2\log(4)+2\sqrt{\pi}C_{12}\right)e^{2\Gamma_1\Lambda}\left(1+\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\right)\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\\
&=C_{13}(n)e^{2\Gamma_1\Lambda}\left(1+\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\right)\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\,.
\end{aligned}
$$

The estimates (4.12), (4.14) and (4.7) imply that for all $z\in\Omega_{1/2}$

$$
\begin{aligned}
e^{\lambda(z)}|\vec{L}(z)|&\leq\left(2\pi\max\left\{C_{11}(n),2C_{10}(n)\right\}+C_{13}(n)\right)e^{2\Gamma_1\Lambda}\left(1+\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\right)\left(\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}+|z|\delta(|z|)\right)|z|^{-1}\\
&\leq 2\left(2\pi\max\left\{C_{11}(n),2C_{10}(n)\right\}+C_{13}(n)\right)e^{2\Gamma_1\Lambda}\left(1+\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\right)\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\;|z|^{-1}\\
&=C_{14}(n)e^{2\Gamma_1\Lambda}\left(1+\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\right)\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\,\frac{1}{|z|}. \qquad (4.16)
\end{aligned}
$$

Now, recall that there exists $S:B_R(0)\to\mathbb{R}$ and $\vec{R}:B_R(0)\to\Lambda^2\mathbb{R}^n$ such that

$$
\begin{cases}
\nabla S=\vec{L}\cdot\nabla\vec{\Phi}\\
\nabla\vec{R}=\vec{L}\wedge\nabla\vec{\Phi}+2\,\vec{H}\wedge\nabla^{\perp}\vec{\Phi},
\end{cases}
$$

we trivially obtain from the pointwise inequality (4.16), (4.9) and (4.7) for all $z\in\Omega_{1/2}$

$$
|\nabla S(z)|\leq 2C_{14}(n)e^{2\Gamma_1\Lambda}\left(1+\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\right)\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\,\frac{1}{|z|}
$$

and

$$
\begin{aligned}
|\nabla\vec{R}(z)|&\leq 2C_{14}(n)e^{2\Gamma_1\Lambda}\left(1+\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\right)\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\,\frac{1}{|z|}+4C_{10}(n)\delta(|z|)\\
&\leq 2\left(C_{14}(n)+2C_{10}(n)\right)e^{2\Gamma_1\Lambda}\left(1+\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\right)\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\,\frac{1}{|z|}.
\end{aligned}
$$

Therefore, if $C_{15}(n)=4\sqrt{\pi}(C_{14}(n)+2C_{10}(n))>0$, we deduce that

$$
\begin{aligned}
&\|\nabla S\|_{\mathrm{L}^{2,\infty}(\Omega_{1/2})}\leq C_{15}(n)e^{2\Gamma_1\Lambda}\left(1+\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\right)\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\\
&\left\|\nabla\vec{R}\right\|_{\mathrm{L}^{2,\infty}(\Omega_{1/2})}\leq C_{15}(n)e^{2\Gamma_1\Lambda}\left(1+\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\right)\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\,. \qquad (4.17)
\end{aligned}
$$

Now, define for all $2r\leq\rho<\dfrac{R}{2}$

$$
S_\rho=\fint_{\partial B_\rho(0)}S\,d\mathscr{H}^1,\qquad \vec{R}_\rho=\fint_{\partial B_\rho(0)}\vec{R}\,d\mathscr{H}^1.
$$

Following the exact same steps as [2], we find that for some universal constant $C_{16} = C_{16}(n)$

$$\left|\frac{dS_\rho}{d\rho}\right| + \left|\frac{d\vec{R}_\rho}{d\rho}\right| \leq C_{16}(n)e^{2\Gamma_1\Lambda}\left(1+\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\right)\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\,\delta(\rho). \tag{4.18}$$

Therefore, (4.8) and (4.18) imply that

$$\int_{2r}^{\frac{R}{2}}\left(\left|\frac{dS_\rho}{d\rho}\right|^2 + \left|\frac{d\vec{R}_\rho}{d\rho}\right|^2\right)\rho\, d\mathscr{L}^1(\rho) \leq C_{16}(n)^2 e^{4\Gamma_1\Lambda}\left(1+\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\right)^2\|\nabla\vec{n}\|^3_{\mathrm{L}^2(\Omega)}. \tag{4.19}$$

We will now use a precised version of Lemma VI.2 of [2] (proved in [13], see also [15]).

Lemma 4.4. *There exists a universal constant $R_0 > 0$ with the following property. Let $0 < 4r < R < R_0$, $\Omega = B_R \setminus \overline{B}_r(0) \to \mathbb{R}$, $a, b : \Omega \to \mathbb{R}$ such that $\nabla a \in L^{2,\infty}(\Omega)$ and $\nabla b \in L^2(\Omega)$, and $u : \Omega \to \mathbb{R}$ be a solution of*

$$\Delta\varphi = \nabla a \cdot \nabla^{\perp} b \qquad \text{in } \Omega.$$

Furthermore, define for $r \leq \rho \leq R$

$$\overline{\varphi}_\rho = \fint_{\partial B_\rho(0)}\varphi\, d\mathscr{H}^1 = \frac{1}{2\pi\rho}\int_{\partial B_\rho(0)}\varphi\, d\mathscr{H}^1.$$

Then $\nabla\varphi \in L^2(\Omega)$, and there exists a positive constant $\Gamma_{20} > 0$ independent of $0 < 4r < R$ such that for all $\left(\frac{r}{R}\right)^{\frac{1}{2}} < \alpha < \frac{1}{2}$

$$\|\nabla\varphi\|_{\mathrm{L}^2(B_{\alpha R}\setminus\overline{B}_{\alpha^{-1}r})} \leq \Gamma_{20}\left(\|\nabla a\|_{\mathrm{L}^{2,\infty}(\Omega)}\|\nabla b\|_{\mathrm{L}^2(\Omega)} + \|\nabla\overline{\varphi}_r\|_{\mathrm{L}^2(\Omega)} + \|\nabla\varphi\|_{\mathrm{L}^{2,\infty}(\Omega)}\right).$$

Proof. Let $\widetilde{a} : B_R(0) \to \mathbb{R}$ and $\widetilde{b} : B_R(0) \to \mathbb{R}$ the extensions of a and b given by Theorem 7.2. As $0 < 4r < R$ and scaling invariance of the $L^{2,\infty}$ and the L^2 norm of the gradient, we deduce that there exists a universal constant $\Gamma_{20} >> 0$ such that

$$\begin{aligned}
&\|\nabla\widetilde{a}\|_{\mathrm{L}^{2,\infty}(B_R(0))} \leq \Gamma_{20}\left(\|\nabla a\|_{\mathrm{L}^{2,\infty}(\Omega)} + \|a\|_{\mathrm{L}^{2,\infty}(\Omega)}\right)\\
&\left\|\nabla\widetilde{b}\right\|_{\mathrm{L}^2(B_R(0))} \leq \Gamma_{20}\left(\|\nabla b\|_{\mathrm{L}^2(\Omega)} + \|b\|_{\mathrm{L}^2(\Omega)}\right).
\end{aligned}$$

Thanks to Poincaré-Wirtinger inequality, and as $\widetilde{a} = a$ and $\widetilde{b} = b$ on Ω, we deduce that

$$\begin{aligned}
\|\nabla\widetilde{a}\|_{\mathrm{L}^{2,\infty}(B_R(0))} &\leq \Gamma_{20}\left(\|\nabla a\|_{\mathrm{L}^{2,\infty}(\Omega)} + \left\|a - \overline{\overline{a}}_{B_R(0)}\right\|_{\mathrm{L}^{2,\infty}(\Omega)}\right) = \Gamma_{20}\left(\|\nabla a\|_{\mathrm{L}^{2,\infty}(B_R(0))} + \left\|\widetilde{a} - \overline{\overline{a}}_{B_R(0)}\right\|_{\mathrm{L}^{2,\infty}(\Omega)}\right)\\
&\leq \Gamma_{20}\left(\|\nabla a\|_{\mathrm{L}^{2,\infty}(\Omega)} + \left\|\widetilde{a} - \overline{\overline{a}}_{B_R(0)}\right\|_{\mathrm{L}^{2,\infty}(B_R(0))}\right)\\
&\leq \Gamma_{20}\|\nabla a\|_{\mathrm{L}^{2,\infty}(\Omega)} + \Gamma_{20}C_{PW}(L^{2,\infty})R\,\|\nabla\widetilde{a}\|_{\mathrm{L}^{2,\infty}(B_R(0))}.
\end{aligned}$$

Therefore, if $\Gamma_{20}C_{PW}(L^{2,\infty})R_0 \leq \frac{1}{2}$, we find

$$\|\nabla\widetilde{a}\|_{\mathrm{L}^{2,\infty}(B_R(0))} \leq 2\Gamma_{20}\|\nabla a\|_{\mathrm{L}^{2,\infty}(\Omega)},$$

and likewise, provided $\Gamma_{20}C_{PW}(L^2)R_0 \leq \frac{1}{2}$, we find

$$\left\|\nabla\widetilde{b}\right\|_{\mathrm{L}^2(B_R(0))} \leq 2\Gamma_{20}\|\nabla b\|_{\mathrm{L}^2(\Omega)}.$$

Now, let $u : B_R(0) \to \mathbb{R}$ be the solution of

$$\begin{cases}\Delta u = \nabla \widetilde{a} \cdot \nabla^{\perp} \widetilde{b} & \text{in } B_R(0)\\ \quad u = 0 & \text{on } \partial B_R(0).\end{cases}$$

Then the improved Wente inequality of Bethuel ([10], 3.3.6) and the scaling invariance shows that there exists a universal constant $\Gamma_{21} >> 0$ such that

$$\|\nabla u\|_{\mathrm{L}^2(B_R(0))} \leq \Gamma_{21} \|\nabla \widetilde{a}\|_{\mathrm{L}^{2,\infty}(B_R(0))} \left\|\nabla \widetilde{b}\right\|_{\mathrm{L}^2(B_R(0))} \leq 4\Gamma_{20}^2 \Gamma_{21} \|\nabla a\|_{\mathrm{L}^{2,\infty}(\Omega)} \|\nabla b\|_{\mathrm{L}^2(\Omega)}.$$

Now, let $v = \varphi - u - \overline{(\varphi - u)}_r$. Then v is a harmonic function such that for all $r < \rho < R$

$$\int_{\partial B_\rho} \partial_\nu v \, d\mathscr{H}^1 = 0.$$

Therefore, Lemma 2.2 implies that

$$\|\nabla v\|_{\mathrm{L}^2(B_{\alpha R}\setminus \overline{B}_{\alpha^{-1}r}(0))} \leq \Gamma_1 \|\nabla v\|_{\mathrm{L}^{2,\infty}(\Omega)} \leq \Gamma_1' \left(\|\nabla a\|_{\mathrm{L}^{2,\infty}(\Omega)} \|\nabla b\|_{\mathrm{L}^2(\Omega)} + \|\nabla \varphi_r\|_{\mathrm{L}^2(\Omega)} + \|\nabla \varphi\|_{\mathrm{L}^{2,\infty}(\Omega)}\right)$$

which concludes the proof. □

Now, recall that the following system holds

$$\begin{cases}\Delta S = - * \nabla \vec{n} \cdot \nabla^{\perp} \vec{R}\\ \Delta \vec{R} = (-1)^n * \left(\nabla \vec{n} \,\lrcorner\, \nabla^{\perp} \vec{R}\right) + *\nabla \vec{n} \cdot \nabla^{\perp} S.\end{cases}$$

First, thanks to Lemma IV.1 of [2], we extend the restriction $\vec{n} : B_R \setminus \overline{B}_r(0) \to \mathscr{G}_{n-2}(\mathbb{R}^n)$ to a map $\widetilde{\vec{n}} : B_R(0) \to \mathscr{G}_{n-2}(\mathbb{R}^n)$ such that

$$\left\|\nabla \widetilde{\vec{n}}\right\|_{\mathrm{L}^2(B_R(0))} \leq C_0(n) \|\nabla \vec{n}\|_{\mathrm{L}^2(\Omega)}.$$

In particular we have

$$\begin{cases}\Delta S = - * \nabla \widetilde{\vec{n}} \cdot \nabla^{\perp} \vec{R} & \text{in } \Omega\\ \Delta \vec{R} = (-1)^n * \left(\nabla \widetilde{\vec{n}} \llcorner \nabla^{\perp} \vec{R}\right) + *\nabla \widetilde{\vec{n}} \cdot \nabla^{\perp} S & \text{in } \Omega.\end{cases} \tag{4.20}$$

Therefore, applying the proof of Lemma 4.4 by using the already constructed extension of $\vec{n}$, we deduce thanks to (4.17) and (4.19) that

$$\|\nabla S\|_{\mathrm{L}^2(\Omega_{1/4})} + \left\|\nabla \vec{R}\right\|_{\mathrm{L}^2(\Omega_{1/4})} \leq C_{17}(n) e^{2\Gamma_1 \Lambda} \left(1 + \|\nabla \vec{n}\|_{\mathrm{L}^2(\Omega)}\right) \|\nabla \vec{n}\|_{\mathrm{L}^2(\Omega)}.$$

As in [2], we obtain readily

$$\begin{aligned}&\|\nabla S\|_{\mathrm{L}^2(\Omega_{1/2})} + \left\|\nabla \vec{R}\right\|_{\mathrm{L}^2(\Omega_{1/2})} \leq C_{18}(n) e^{2\Gamma_1 \Lambda} \left(1 + \|\nabla \vec{n}\|_{\mathrm{L}^2(\Omega)}\right) \|\nabla \vec{n}\|_{\mathrm{L}^2(\Omega)}\\ &\|S\|_{\mathrm{L}^\infty(\Omega_{1/2})} + \left\|\vec{R}\right\|_{\mathrm{L}^\infty(\Omega_{1/2})} \leq C_{18}(n) e^{2\Gamma_1 \Lambda} \left(1 + \|\nabla \vec{n}\|_{\mathrm{L}^2(\Omega)}\right) \|\nabla \vec{n}\|_{\mathrm{L}^2(\Omega)}.\end{aligned} \tag{4.21}$$

Now, introduce the following slight variant from a Lemma of [13].

Lemma 4.5. *Let $R_0 > 0$ be the constant of Lemma 4.4. Let $0 < 16r < R < R_0$, $\Omega = B_R \setminus \overline{B}_r(0) \to \mathbb{R}$, $a, b : \Omega \to \mathbb{R}$ such that $\nabla a \in L^2(\Omega)$ and $\nabla b \in L^2(\Omega)$, and $\varphi : \Omega \to \mathbb{R}$ be a solution of*

$$\Delta \varphi = \nabla a \cdot \nabla^{\perp} b \quad \text{in } \Omega.$$

Assume that $\|\varphi\|_{\mathrm{L}^\infty(\partial\Omega)} < \infty$. Then there exists a universal constant $\Gamma_{22} > 0$ such that for all $\left(\frac{r}{R}\right)^{\frac{1}{2}} < \alpha < \frac{1}{4}$,

$$\|\varphi\|_{\mathrm{L}^\infty(\Omega)} + \|\nabla \varphi\|_{\mathrm{L}^{2,1}(B_{\alpha R}\setminus \overline{B}_{\alpha^{-1}r}(0))} + \left\|\nabla^2 \varphi\right\|_{\mathrm{L}^1(B_{\alpha R}\setminus \overline{B}_{\alpha^{-1}r}(0))} \leq \Gamma_{22} \left(\|\nabla a\|_{\mathrm{L}^2(\Omega)} \|\nabla b\|_{\mathrm{L}^2(\Omega)} + \|\varphi\|_{\mathrm{L}^\infty(\partial\Omega)}\right).$$

Proof. As in the proof of Lemma 4.4, introduce extensions $\widetilde{a}: B_R(0) \to \mathbb{R}$ and $\widetilde{b}: B_R(0) \to \mathbb{R}$ of a and b, such that

$$
\begin{aligned}
&\|\nabla \widetilde{a}\|_{\mathrm{L}^2(B_R(0))} \leq 2\Gamma_{20} \|\nabla a\|_{\mathrm{L}^2(\Omega)} \\
&\|\nabla \widetilde{b}\|_{\mathrm{L}^2(B_R(0))} \leq 2\Gamma_{20} \|\nabla b\|_{\mathrm{L}^2(\Omega)}.
\end{aligned}
$$

Now, let $v: B_R(0) \to \mathbb{R}$ be the solution of

$$
\begin{cases}
\Delta v = \nabla \widetilde{a} \cdot \nabla^{\perp} \widetilde{b} & \text{in } B_R(0) \\
v = 0 & \text{on } \partial B_R(0).
\end{cases}
$$

Then the improved Wente inequality and the Coifman-Lions-Meyer-Semmes estimate ([4]) shows (by scaling invariance of the different norms considered) that

$$
\|v\|_{\mathrm{L}^\infty(B_R(0))} + \|\nabla v\|_{\mathrm{L}^{2,1}(B_R(0))} + \left\|\nabla^2 v\right\|_{\mathrm{L}^1(B_R(0))} \leq \Gamma_{22} \|\nabla a\|_{\mathrm{L}^2(\Omega)} \|\nabla b\|_{\mathrm{L}^2(\Omega)}. \tag{4.22}
$$

Now let $u = \varphi - v$. Then u is harmonic, and let $d \in \mathbb{R}$, $\{a_n\}_{n \in \mathbb{Z}} \subset \mathbb{C}$ such that

$$
u(z) = a_0 + d \log|z| + \mathrm{Re}\left(\sum_{n \in \mathbb{Z}^*} a_n z^n\right).
$$

Then we have by the maximum principle for all $r \leq \rho \leq R$

$$
|a_0 + d \log \rho| = \left|\frac{1}{2\pi} \int_0^{2\pi} u(\rho e^{i\theta}) d\theta\right| \leq \|u\|_{\mathrm{L}^\infty(\partial\Omega)}.
$$

Therefore, we have

$$
|d| \log\left(\frac{R}{r}\right) = |a_0 + d \log R - (a_0 + d \log r)| \leq |a_0 + d \log R| + |a_0 + d \log r| \leq 2 \|u\|_{\mathrm{L}^\infty(\partial\Omega)}. \tag{4.23}
$$

Now, recall that

$$
\begin{aligned}
&\|\nabla \log|z|\|_{\mathrm{L}^{2,1}(B_R \setminus \overline{B}_r(0))} = 4\sqrt{\pi}\left(\log\left(\frac{R}{r}\right) + \log\left(1 + \sqrt{1 - \left(\frac{r}{R}\right)^2}\right)\right) \\
&\left\|\nabla^2 \log|z|\right\|_{\mathrm{L}^1(B_R \setminus \overline{B}_r(0))} = 4\left\|\partial_z^2 \log|z|\right\|_{\mathrm{L}^1(B_R \setminus \overline{B}_r(0))} = 4\pi \log\left(\frac{R}{r}\right).
\end{aligned} \tag{4.24}
$$

Therefore, as $R > 4r$, (4.23) and (4.24) imply that

$$
\begin{aligned}
&\|\nabla (d \log|z|)\|_{\mathrm{L}^{2,1}(B_R \setminus \overline{B}_r(0))} \leq 4\sqrt{\pi}\left(\log\left(\frac{R}{r}\right) + \log(2)\right) |d| \leq 16\sqrt{\pi} \|u\|_{\mathrm{L}^\infty(\partial\Omega)} \\
&\left\|\nabla^2 (d \log|z|)\right\|_{\mathrm{L}^1(B_R \setminus \overline{B}_r(0))} \leq 8\pi \|u\|_{\mathrm{L}^\infty(\partial\Omega)}.
\end{aligned} \tag{4.25}
$$

These estimates (4.23) imply by Lemmas 2.3 and 4.5 imply that

$$
\begin{aligned}
&\|\nabla u\|_{\mathrm{L}^{2,1}(B_{\alpha R} \setminus \overline{B}_{\alpha^{-1} r}(0))} + \left\|\nabla^2 u\right\|_{\mathrm{L}^1(B_{\alpha R} \setminus \overline{B}_{\alpha^{-1} r}(0))} \leq 64 \frac{\sqrt{2} + \sqrt{\pi}}{\sqrt{15}} (2\alpha) \|\nabla (u - d \log|z|)\|_{\mathrm{L}^2(B_{R/2} \setminus \overline{B}_{r/2}(0))} \\
&+ 24\pi \|u\|_{\mathrm{L}^\infty(\partial\Omega)} \\
&\leq 128 \frac{\sqrt{2} + \sqrt{\pi}}{\sqrt{15}} \alpha \|\nabla u\|_{\mathrm{L}^2(B_{R/2} \setminus \overline{B}_{2r}(0))} + \left(24\pi + 256\pi \frac{\sqrt{2} + \sqrt{\pi}}{\sqrt{15}} \alpha\right) \|u\|_{\mathrm{L}^\infty(\partial\Omega)}.
\end{aligned} \tag{4.26}
$$

Now, recall that the mean value formula and the maximum principle ([9] 1.10) imply that for all $x \in B_R \setminus \overline{B}_r(0)$, and $0 < \rho < \mathrm{dist}(x, \partial\Omega)$,

$$
|\nabla u(x)| \leq \frac{2}{\rho} \|u\|_{\mathrm{L}^\infty(\partial B_\rho(x))} \leq \frac{2}{\rho} \|u\|_{\mathrm{L}^\infty(\partial\Omega)}. \tag{4.27}
$$

As

$$\|\nabla u\|^2_{\mathrm{L}^2(B^2_{R/2}\setminus\overline{B}_{2r}(0))}=\int_{\partial B_{R/2}(0)}u\,\partial_\nu u\,d\mathscr{H}^1-\int_{\partial B_{2r}(0)}u\,\partial_\nu u\,d\mathscr{H}^1$$
$$\leq\|u\|_{\mathrm{L}^\infty(\partial\Omega)}\left(\int_{\partial B_{R/2}(0)}|\nabla u|d\mathscr{H}^1+\int_{\partial B_{2r}(0)}|\nabla u|d\mathscr{H}^1\right)\tag{4.28}$$

the estimate (4.27) shows that

$$\int_{\partial B_{R/2}(0)}|\nabla u|d\mathscr{H}^1+\int_{\partial B_{2r}(0)}|\nabla u|d\mathscr{H}^1\leq 4\int_{\partial B_{R/2}(0)}\frac{\|u\|_{\mathrm{L}^\infty(\partial\Omega)}}{R}d\mathscr{H}^1+\int_{\partial B_{2r}(0)}\frac{\|u\|_{\mathrm{L}^\infty(\partial\Omega)}}{r}d\mathscr{H}^1$$
$$=8\pi\,\|u\|_{\mathrm{L}^\infty(\partial\Omega)}\,.\tag{4.29}$$

Therefore, we have by (4.28) and (4.29)

$$\|\nabla u\|_{\mathrm{L}^2(B_{R/2}\setminus\overline{B}_{2r}(0))}\leq 2\sqrt{2\pi}\,\|u\|_{\mathrm{L}^\infty(\partial\Omega)}\,.\tag{4.30}$$

Combining (4.26) and (4.30) shows that

$$\|\nabla u\|_{\mathrm{L}^{2,1}(B_{\alpha R}\setminus\overline{B}_{\alpha^{-1}r}(0))}+\left\|\nabla^2u\right\|_{\mathrm{L}^1(B_{\alpha R}\setminus\overline{B}_{\alpha^{-1}r}(0))}\leq\left(256\frac{2\sqrt{\pi}+\pi\sqrt{2}}{\sqrt{15}}\alpha+24\pi+256\pi\frac{\sqrt{2}+\sqrt{\pi}}{\sqrt{15}}\alpha\right)\|u\|_{\mathrm{L}^\infty(\partial\Omega)}$$
$$\leq\left(24\pi+512\pi\frac{\sqrt{2}+\sqrt{\pi}}{\sqrt{15}}\alpha\right)\|u\|_{\mathrm{L}^\infty(\partial\Omega)}\,.\tag{4.31}$$

Combining the maximum principle and inequalities (4.22), (4.31) yields the expected estimate. □

Now, apply Lemma 4.5 to the estimates (4.21) shows by using the previous extension $\widetilde{\vec{n}}$ of $\vec{n}$ that

$$\|\nabla S\|_{\mathrm{L}^{2,1}(\Omega_{1/2})}+\left\|\nabla\vec{R}\right\|_{\mathrm{L}^{2,1}(\Omega_{1/2})}\leq C_{19}(n)e^{4\Gamma_1\Lambda}\left(1+\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\right)\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}$$
$$\left\|\nabla^2S\right\|_{\mathrm{L}^1(\Omega_{1/2})}+\left\|\nabla^2\vec{R}\right\|_{\mathrm{L}^1(\Omega_{1/2})}\leq C_{19}(n)e^{4\Gamma_1\Lambda}\left(1+\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\right)\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\,.\tag{4.32}$$

As (see [28] for the definition of the restriction operator $\llcorner$ between a 2-vector and a vector)

$$e^{2\lambda}\vec{H}=\frac{1}{4}\nabla^\perp S\cdot\nabla\vec{\Phi}-\frac{1}{4}\nabla\vec{R}\llcorner\nabla^\perp\vec{\Phi},\tag{4.33}$$

we trivially have

$$\left\|e^\lambda\vec{H}\right\|_{\mathrm{L}^{2,1}(\Omega_{1/2})}\leq\|\nabla S\|_{\mathrm{L}^{2,1}(\Omega_{1/2})}+\left\|\nabla\vec{R}\right\|_{\mathrm{L}^{2,1}(\Omega_{1/2})}\leq 2C_{19}(n)e^{4\Gamma_1\Lambda}\left(1+\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\right)\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\,.\tag{4.34}$$

Now, (4.33) implies that

$$2(\partial_z\lambda)e^{2\lambda}\vec{H}+e^{2\lambda}\partial_z\vec{H}=\frac{1}{4}\nabla^\perp\left(\partial_zS\right)\cdot\nabla\vec{\Phi}+\frac{1}{4}\nabla^\perp S\cdot\nabla(\partial_z\vec{\Phi})-\frac{1}{4}\nabla(\partial_z\vec{R})\llcorner\nabla^\perp\vec{\Phi}-\frac{1}{4}\nabla\vec{R}\llcorner\nabla^\perp\left(\partial_z\vec{\Phi}\right),$$

so that

$$e^\lambda\partial_z\vec{H}=-2(\partial_z\lambda)e^\lambda\vec{H}+\frac{1}{4}\nabla^\perp\left(\partial_zS\right)\cdot e^{-\lambda}\nabla\vec{\Phi}+\frac{1}{4}\nabla^\perp S\cdot e^{-\lambda}\nabla(\partial_z\vec{\Phi})-\frac{1}{4}\nabla(\partial_z\vec{R})\llcorner e^{-\lambda}\nabla^\perp\vec{\Phi}$$
$$-\frac{1}{4}\nabla\vec{R}\llcorner e^{-\lambda}\nabla^\perp\left(\partial_z\vec{\Phi}\right).$$

As $\nabla\lambda\in L^{2,\infty}$, $e^\lambda\vec{H}\in L^{2,1}$ and $e^{-\lambda}\nabla^2\vec{\Phi}\in L^{2,\infty}$, we deduce by (4.32) and (4.34) that

$$\left\|e^\lambda\nabla\vec{H}\right\|_{\mathrm{L}^1(\Omega_{1/2})}\leq C_{20}(n)\left(1+\Lambda\right)e^{4\Gamma_1\Lambda}\left(1+\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\right)\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega)}\,,$$

and this concludes the proof of the Theorem. □

For all neck region of the form $\Omega_k = B_{R_k} \setminus \overline{B}_{r_k}(0)$, define for all $0 < \alpha < 1$

$$\Omega_k(\alpha) = B_{\alpha R_k} \setminus \overline{B}_{\alpha^{-1} r_k}(0).$$

The estimate (4.3) implies that

$$\left\| e^{\lambda_k} \vec{H}_k \right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))} + \left\| e^{\lambda_k} \nabla \vec{H}_k \right\|_{\mathrm{L}^1(\Omega_k(\alpha))} \leq C_{21}(n)\,(1+\Lambda)\, e^{C_{21}\Lambda} \left(1 + \|\nabla \vec{n}_k\|_{\mathrm{L}^2(\Omega_k(2\alpha))}\right) \|\nabla \vec{n}_k\|_{\mathrm{L}^2(\Omega_k(2\alpha))}, \tag{4.35}$$

where

$$\Lambda = \sup_{k\in\mathbb{N}} \left(\|\nabla \lambda_k\|_{\mathrm{L}^{2,\infty}(\Omega_k)} + \|\nabla \vec{n}_k\|_{\mathrm{L}^2(\Omega_k)} \right) < \infty,$$

is finite by hypothesis. Therefore, the no-neck energy

$$\lim_{\alpha\to 0} \limsup_{k\to\infty} \|\nabla \vec{n}_k\|_{\mathrm{L}^2(\Omega_k(\alpha))} = 0 \tag{4.36}$$

implies by (4.35) that

$$\lim_{\alpha\to 0} \limsup_{k\to\infty} \left(\left\| e^{\lambda_k} \vec{H}_k \right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))} + \left\| e^{\lambda_k} \nabla \vec{H}_k \right\|_{\mathrm{L}^1(\Omega_k(\alpha))} \right) = 0.$$

Step 2: $L^{2,1}$-quantization of the Weingarten tensor The proof relies on an algebraic computation first given in [28] (II.10). We will give its easy derivation in codimension 1.

Algebraic identity in codimension 1. Let $\vec{\Phi} : B_1(0) \to \mathbb{R}^3$ be a conformal immersion, and $\vec{n} : B_1(0) \to S^2$ be its unit normal. If $e^{\lambda} = \dfrac{1}{\sqrt{2}} |\nabla \vec{\Phi}|$ is the associate conformal parameter and $\vec{e}_j = e^{-\lambda} \partial_{x_j} \vec{\Phi}$ for $j = 1, 2$, we have by definition

$$\vec{n} = \vec{e}_1 \times \vec{e}_2,$$

where $\times$ is the vector product. Recall the Grassmann identity valid for all $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$

$$(\vec{u} \times \vec{v}) \times \vec{w} = \langle \vec{u}, \vec{w} \rangle \vec{v} - \langle \vec{v}, \vec{w} \rangle \vec{u}.$$

Therefore, we deduce that

$$\begin{cases} \vec{n} \times \vec{e}_1 = (\vec{e}_1 \times \vec{e}_2) \times \vec{e}_1 = \langle \vec{e}_1, \vec{e}_1 \rangle \vec{e}_2 - \langle \vec{e}_2, \vec{e}_1 \rangle \vec{e}_1 = \vec{e}_2 \\ \vec{n} \times \vec{e}_2 = -\vec{e}_1. \end{cases} \tag{4.37}$$

As $|\vec{n}| = 1$, we have for all $j = 1, 2$

$$\partial_{x_j} \vec{n} = \langle \nabla_{\partial_{x_j}} \vec{n}, \vec{e}_1 \rangle \vec{e}_1 + \langle \nabla_{\partial_{x_j}} \vec{n}, \vec{e}_2 \rangle \vec{e}_2 = -\mathbb{I}_{1,j} \vec{e}_1 - \mathbb{I}_{2,j} \vec{e}_2.$$

This implies that

$$\nabla \vec{n} = \left(-\mathbb{I}_{1,1} \vec{e}_1 - \mathbb{I}_{1,2} \vec{e}_2, -\mathbb{I}_{1,2} \vec{e}_1 - \mathbb{I}_{2,2} \vec{e}_2 \right) \tag{4.38}$$

and (4.37) combined with the identity $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$ (valid for all $\vec{u}, \vec{v} \in \mathbb{R}^3$) yield

$$\partial_{x_j} \vec{n} \times \vec{n} = -\mathbb{I}_{1,j} \vec{e}_1 \times \vec{n} - \mathbb{I}_{2,j} \vec{e}_2 \times \vec{n} = \mathbb{I}_{1,j} \vec{e}_2 - \mathbb{I}_{2,j} \vec{e}_1.$$

Therefore, we deduce that

$$\nabla^{\perp} \vec{n} \times \vec{n} = \left(\partial_{x_2} \vec{n} \times \vec{n}, -\partial_{x_1} \vec{n} \times \vec{n} \right) = \left(-\mathbb{I}_{2,2} \vec{e}_1 + \mathbb{I}_{1,2} \vec{e}_2, -\mathbb{I}_{1,1} \vec{e}_2 + \mathbb{I}_{1,2} \vec{e}_1 \right). \tag{4.39}$$

As

$$e^{\lambda} H = \frac{1}{2} \left(\mathbb{I}_{1,1} + \mathbb{I}_{2,2} \right), \tag{4.40}$$

the identities (4.39) and (4.40) show that

$$\nabla^{\perp}\vec{n}\times\vec{n}+2H\nabla\vec{\Phi}=(\mathbb{I}_{1,1}\vec{e}_1+\mathbb{I}_{1,2}\vec{e}_2,\mathbb{I}_{1,2}\vec{e}_1+\mathbb{I}_{2,2}\vec{e}_2). \tag{4.41}$$

Comparing (4.41) and (4.38), we deduce that

$$\nabla\vec{n}=\vec{n}\times\nabla^{\perp}\vec{n}-2H\,\nabla\vec{\Phi}. \tag{4.42}$$

Taking the divergence of this equation we find

$$\Delta\vec{n}=\nabla\vec{n}\times\nabla^{\perp}\vec{n}-2\,\mathrm{div}(H\nabla\vec{\Phi}).$$

Argument in arbitrary codimension. Then we can find a trivialisation of $\vec{n}$ such that $\vec{n}=\vec{n}_1\wedge\vec{n}_2\wedge\cdots\wedge\vec{n}_{n-2}$ satisfying the Coulomb condition

$$\mathrm{div}\left(\nabla\vec{n}_\beta\cdot\vec{n}_\gamma\right)=0\qquad\text{for all }\ 1\leq\beta,\gamma\leq n-2. \tag{4.43}$$

Furthermore, recall that for all $1\leq\beta\leq n-2$, [28] implies that (using (4.43) for the second condition)

$$\nabla\vec{n}_\beta=-*\left(\vec{n}\wedge\nabla^{\perp}\vec{n}_\beta\right)+\sum_{\gamma=1}^{n-2}\langle\nabla\vec{n}_\beta,\vec{n}_\gamma\rangle\cdot\vec{n}_\gamma-2\,H_\beta\nabla\vec{\Phi} \tag{4.44}$$

Taking the divergence of this equation yields by the Coulomb condition (4.43)

$$\Delta\vec{n}_\beta=-*\left(\nabla\vec{n}\wedge\nabla^{\perp}\vec{n}_\beta\right)+\sum_{\gamma=1}^{n-2}\langle\nabla\vec{n}_\beta,\vec{n}_\gamma\rangle\cdot\nabla\vec{n}_\gamma-2\,\mathrm{div}\left(H_\beta\nabla\vec{\Phi}\right). \tag{4.45}$$

Now, as in (2.39) (recall that this comes from Lemma IV.1. in [2]), construct for small enough $\alpha>0$ and k large enough (thanks to the no-neck property) an extension $\widetilde{\vec{n}}_k:B_{\alpha R_k}(0)\to\mathscr{G}_{n-2}(\mathbb{R}^n)$ of $\vec{n}_k:\Omega_k(\alpha)=B_{\alpha R_k}(0)\setminus\overline{B}_{\alpha^{-1}r_k}(0)\to\mathscr{G}_{n-2}(\mathbb{R}^n)$ such that for some universal constant $C_{22}=C_{22}(n)>0$

$$\left\|\nabla\widetilde{\vec{n}}_k\right\|_{\mathrm{L}^2(B_{\alpha R_k}(0))}\leq C_{22}\left\|\nabla\vec{n}_k\right\|_{\mathrm{L}^2(\Omega_k(\alpha))}. \tag{4.46}$$

Furthermore, as in Lemma IV.1 of [2] (see also [10] 4.1.3 – 4.1.7) we can construct extensions $\widetilde{\vec{n}}_k^{\beta}$ of $\vec{n}_k^{\beta}$ on $B_{\alpha R_k}(0)$ such that

$$\widetilde{\vec{n}}_k=\widetilde{\vec{n}}_k^{1}\wedge\cdots\wedge\widetilde{\vec{n}}_k^{n-2}\qquad\text{on }\ B_{\alpha R_k}(0)$$

satisfying the Coulomb condition for all $1\leq\beta,\gamma\leq n-2$

$$\begin{cases}\mathrm{div}\left(\nabla\widetilde{\vec{n}}_k^{\beta}\cdot\widetilde{\vec{n}}_k^{\gamma}\right)=0 & \text{for all }\ B_{\alpha R_k}(0)\\ \partial_\nu\widetilde{\vec{n}}_k^{\beta}\cdot\widetilde{\vec{n}}_k^{\gamma}=0 & \text{on on }\ \partial B_{\alpha R_k}(0)\end{cases} \tag{4.47}$$

and for all $1\leq\beta\leq n-2$ (by (4.46) for the second inequality)

$$\left\|\nabla\widetilde{\vec{n}}_k^{\beta}\right\|_{\mathrm{L}^2(B_{\alpha R_k}(0))}\leq\widetilde{C_{22}}(n)\left\|\nabla\widetilde{\vec{n}}_k\right\|_{\mathrm{L}^2(B_{\alpha R_k}(0))}\leq C_{22}'(n)\left\|\nabla\vec{n}_k\right\|_{\mathrm{L}^2(\Omega_k(\alpha))}. \tag{4.48}$$

Furthermore, using [10] 4.1.7, we have the estimate for all $1\leq\beta\leq n-2$

$$\left\|\nabla\widetilde{\vec{n}}_k^{\beta}\cdot\widetilde{\vec{n}}_k^{\gamma}\right\|_{\mathrm{L}^{2,1}(B_{\alpha R_k}(0))}\leq C_{22}''(n)\left\|\nabla\vec{n}_k\right\|^2_{\mathrm{L}^2(\Omega_k(\alpha))}. \tag{4.49}$$

Let us recall the argument for this crucial step. By (4.47), there exists $\vec{A}_{\beta,\gamma}:B_{\alpha R_k}(0)\to\mathbb{R}^n$ such that

$$\nabla^{\perp}\vec{A}_{\beta,\gamma}=\nabla\widetilde{\vec{n}}_k^{\beta}\cdot\widetilde{\vec{n}_k}^{\gamma}. \tag{4.50}$$

Furthermore, the boundary conditions of (4.47) implies that we can choose $\vec{A}_{\beta\gamma}$ such that $\vec{A}_{\beta,\gamma}=0$ on $\partial B_{\alpha R_k}(0)$. Therefore, we have

$$\begin{cases}\Delta\vec{A}_{\beta,\gamma}=\nabla\widetilde{\vec{n}}_k^{\beta}\cdot\nabla^{\perp}\widetilde{\vec{n}}_k^{\gamma} & \text{in } B_{\alpha R_k}(0)\\ \vec{A}_{\beta,\gamma}=0 & \text{on } \partial B_{\alpha R_k}(0)\end{cases}\tag{4.51}$$

Therefore, we get by the improved Wente estimate and (4.48)

$$\left\|\nabla\vec{A}_{\beta,\gamma}\right\|_{\mathrm{L}^{2,1}(B_{\alpha R_k}(0))}\leq C_{22}''(n)\left\|\nabla\widetilde{\vec{n}}_k^{\beta}\right\|_{\mathrm{L}^2(B_{\alpha R_k}(0))}\left\|\nabla\widetilde{\vec{n}}_k^{\gamma}\right\|_{\mathrm{L}^2(B_{\alpha R_k}(0))}\leq C_{22}''(n)C_{22}'(n)\left\|\nabla\vec{n}_k\right\|_{\mathrm{L}^2(\Omega_k(\alpha))}^2.\tag{4.52}$$

Combining the pointwise identity (4.50) with (4.52) yields (4.49).

Now fix some $1\leq\beta\leq n-2$ and let $\vec{u}_k:B_{\alpha R_k}(0)\to\mathbb{R}^n$ be the unique solution of

$$\begin{cases}\Delta\vec{u}_k=-*\left(\nabla\widetilde{\vec{n}}_k\wedge\nabla^{\perp}\widetilde{\vec{n}}_k^{\beta}\right)+\displaystyle\sum_{\gamma=1}^{n-2}\langle\nabla\widetilde{\vec{n}}_k^{\beta},\widetilde{\vec{n}}_k^{\gamma}\rangle\cdot\nabla\widetilde{\vec{n}}_k^{\beta} & \text{in } B_{\alpha R_k}(0)\\ \vec{u}_k=0 & \text{in } \partial B_{\alpha R_k}(0).\end{cases}\tag{4.53}$$

Now, thanks to (4.43), we can apply [4], scaling invariance and (4.46) to find that there exists $C_{23}=C_{23}(n)>0$ such that

$$\left\|\nabla^2\vec{u}_k\right\|_{\mathrm{L}^1(B_{\alpha R_k}(0))}\leq C_1\left\|\nabla\widetilde{\vec{n}}_k\right\|_{\mathrm{L}^2(B_{\alpha R_k}(0))}^2\leq C_{22}^2C_{23}\left\|\nabla\vec{n}_k\right\|_{\mathrm{L}^2(\Omega_k(\alpha))}^2.$$

Furthermore, as $\vec{u}_k=0$ on $\partial B_{\alpha R_k}(0)$, and scaling invariance (of $\|u_k\|_{\mathrm{L}^\infty(B_{\alpha R_k}(0))}$, $\|\nabla\vec{u}_k\|_{\mathrm{L}^{2,1}(B_{\alpha R_k}(0))}$ and $\left\|\nabla^2u_k\right\|_{\mathrm{L}^1(B_{\alpha R_k}(0))}$) and Sobolev embedding, there exists $C_{24}=C_{24}(n)>0$ such that

$$\|\vec{u}_k\|_{\mathrm{L}^\infty(B_{\alpha R_k}(0))}+\|\nabla\vec{u}_k\|_{\mathrm{L}^{2,1}(B_{\alpha R_k}(0))}+\left\|\nabla^2\vec{u}_k\right\|_{\mathrm{L}^1(B_{\alpha R_k}(0))}\leq C_{24}\left\|\nabla\vec{n}_k\right\|_{\mathrm{L}^2(\Omega_k(\alpha))}^2.\tag{4.54}$$

Now, by Theorem (4.2), $H_k^{\beta}\nabla\vec{\Phi}_k\in L^{2,1}(\Omega_k(\alpha))$. Furthermore, as

$$\lim_{k\to\infty}\frac{R_k}{r_k}=\infty,\qquad\limsup_{k\to\infty}R_k<\infty,$$

there exists by Theorem 7.2 an extension $\vec{F}_k:B_{\alpha R_k}(0)\to\mathbb{R}^n$ of $H_k^{\beta}\nabla\vec{\Phi}_k$ such that for all k large enough

$$\left\|\vec{F}_k\right\|_{\mathrm{L}^{2,1}(B_{\alpha R_k}(0))}\leq C_{25}(n)\left\|H_k^{\beta}\nabla\vec{\Phi}_k\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))},$$

where $C_{25}(n)>0$ is independent of k large enough and $0<\alpha<\alpha_0(n)$ fixed (small enough with respect to some $\alpha_0(n)>0$). Now, let $\vec{v}_k:\Omega_k(\alpha)\to\mathbb{R}^n$ be the solution of the system

$$\begin{cases}\Delta\vec{v}_k=-2\,\mathrm{div}\left(\vec{F}_k\right) & \text{in } B_{\alpha R_k}(0)\\ \vec{v}_k=0 & \text{on } \partial B_{\alpha R_k}(0).\end{cases}$$

As we trivially have

$$\left\|\mathrm{div}(\vec{F}_k)\right\|_{\mathrm{W}^{-1,(2,1)}(B_{\alpha R_k}(0))}\leq\left\|\vec{F}_k\right\|_{\mathrm{L}^{2,1}(B_{\alpha R_k}(0))},$$

scaling invariance and standard Calderón-Zygmund estimates show that there exits a universal constant $C_{26}=C_{26}(n)$ such that

$$\|\nabla\vec{v}_k\|_{\mathrm{L}^{2,1}(B_{\alpha R_k}(0))}\leq C_{26}(n)\left\|\vec{F}_k\right\|_{\mathrm{L}^{2,1}(B_{\alpha R_k}(0))}\leq C_{25}(n)C_{26}(n)\left\|H_k^{\beta}\nabla\vec{\Phi}_k\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}$$

$$\leq 2C_{25}(n)C_{26}(n)\left\|e^{\lambda_k}\vec{H}_k\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}. \tag{4.55}$$

Furthermore, the Sobolev embedding shows that for some universal constant $\Gamma_{23}>0$

$$\|\vec{v}_k\|_{\mathrm{L}^\infty(B_{\alpha R_k}(0))}\leq \Gamma_{23}\|\nabla\vec{v}_k\|_{\mathrm{L}^{2,1}(B_{\alpha R_k}(0))}\leq \Gamma_{23}C_{25}(n)C_{26}(n)\left\|e^{\lambda_k}\vec{H}_k\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}. \tag{4.56}$$

Finally, let $\vec{\varphi}_k=\vec{n}_k^\beta-\vec{u}_k-\vec{v}_k$. The $\vec{\varphi}_k:\Omega_k(\alpha)\to\mathbb{R}^n$ is harmonic and

$$\begin{cases}\Delta\vec{\varphi}_k=0 & \text{in } \Omega_k(\alpha)\\ \varphi_k=\vec{n}_k^\beta & \text{on } \partial B_{\alpha R_k}(0)\\ \varphi_k=\vec{n}_k^\beta-\vec{u}_k-\vec{v}_k & \text{on } \partial B_{\alpha^{-1}r_k}(0).\end{cases}$$

In particular, as $\vec{u}_k,\vec{v}_k,\vec{n}_k^\beta\in L^\infty(\Omega_k(\alpha))$ (as $|\vec{n}_k^\alpha|=1$ and using the bounds (4.54) and (4.69)), if $\vec{d}_k\in\mathbb{R}$ and $\{\vec{a}_n\}_{n\in\mathbb{Z}}\subset\mathbb{C}^n$ are such that

$$\vec{\varphi}_k(z)=\vec{a}_0+\vec{d}_k\log|z|+\mathrm{Re}\left(\sum_{n\in\mathbb{Z}^*}\vec{a}_nz^n\right),$$

then

$$|\vec{d}_k|\leq\frac{\|\vec{\varphi}_k\|_{\mathrm{L}^\infty(\partial\Omega_k(\alpha))}}{\log\left(\frac{\alpha^2R_k}{r_k}\right)}\leq\frac{2}{\log\left(\frac{\alpha^2R_k}{r_k}\right)}\left(1+C_{27}(n)\|\nabla\vec{n}_k\|^2_{\mathrm{L}^2(\Omega_k(\alpha))}+C_{27}(n)\left\|e^{\lambda_k}\vec{H}_k\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}\right), \tag{4.57}$$

so that by the proof of Lemma 4.5

$$\begin{aligned}&\|\nabla\vec{\varphi}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha/2))}\leq 16\sqrt{\pi}+C_{28}(n)\left(\left(1+\|\nabla\vec{n}_k\|_{\mathrm{L}^2(\Omega_k(\alpha))}\right)\|\nabla\vec{n}_k\|_{\mathrm{L}^2(\Omega_k(\alpha))}+\left\|e^{\lambda_k}\vec{H}_k\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}\right)\\ &\left\|\nabla^2\vec{\varphi}_k\right\|_{\mathrm{L}^1(\Omega_k(\alpha/2))}\leq 8\pi+C_{28}(n)\left(\left(1+\|\nabla\vec{n}_k\|_{\mathrm{L}^2(\Omega_k(\alpha))}\right)\|\nabla\vec{n}_k\|_{\mathrm{L}^2(\Omega_k(\alpha))}+\left\|e^{\lambda_k}\vec{H}_k\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}\right).\end{aligned} \tag{4.58}$$

Finally, we have by (4.54), (4.55), (4.58) and Theorem 4.2 for some $C_{29}(n)>0$

$$\begin{aligned}\left\|\nabla\vec{n}_k^\beta\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha/2))}&\leq\|\nabla\varphi_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha^2))}+\|\nabla\vec{u}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha^2))}+\|\nabla\vec{v}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha^2))}\\ &\leq 16\sqrt{\pi}+C_{29}(n)\left(1+\Lambda\right)\left(1+\|\nabla\vec{n}_k\|_{\mathrm{L}^2(\Omega_k(2\alpha))}\right)\|\nabla\vec{n}_k\|_{\mathrm{L}^2(\Omega_k(2\alpha))}.\end{aligned} \tag{4.59}$$

Therefore, the no-neck energy yields for all $1\leq\beta\leq n-2$

$$\limsup_{\alpha\to 0}\limsup_{k\to\infty}\left\|\nabla\vec{n}_k^\beta\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}\leq 16\sqrt{\pi}. \tag{4.60}$$

Now, as

$$|\nabla\vec{n}_k|=\left|\sum_{\beta=1}^{n-2}\vec{n}_k\wedge\cdots\wedge\nabla\vec{n}_k^\beta\wedge\cdots\wedge\vec{n}_k^{n-2}\right|\leq\sum_{\beta=1}^{n-2}|\nabla\vec{n}_k^\beta|, \tag{4.61}$$

we deduce from (4.60) that

$$\limsup_{\alpha\to 0}\limsup_{k\to\infty}\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}\leq 16\sqrt{\pi}(n-2)<\infty \tag{4.62}$$

Now, define $\overline{\vec{n}}_k:B_{\alpha R_k}(0)\setminus\overline{B}_{\alpha^{-1}r_k}(0)$ such that for all $z\in\Omega_k(\alpha)$ such that $|z|=r$

$$\overline{\vec{n}}_k(z)=\fint_{\partial B_r(0)}\vec{n}_k\,d\mathscr{H}^1.$$

We will prove that for certain universal constants $C_{30}(n)$

$$\left\|\nabla\overline{\vec{n}}_k\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}\leq C_{30}(n)e^{\Gamma_2(n)\Lambda}\left(1+\|\nabla\vec{n}_k\|_{\mathrm{L}^2(\Omega_k(2\alpha))}\right)\|\nabla\vec{n}_k\|_{\mathrm{L}^2(\Omega_k(2\alpha))}\,, \tag{4.63}$$

and this will finish the proof of the Theorem by using Lemmas 4.4, 4.5. Indeed, notice that the following lemma imply by (4.54) and (4.55) that

$$\begin{aligned}&\left\|\nabla\overline{\vec{u}_k}\right\|_{\mathrm{L}^{2,1}(B_{\alpha R_k}(0))}\leq C_{31}(n)\,\|\nabla\vec{n}_k\|^2_{\mathrm{L}^2(\Omega_k(\alpha))}\\ &\left\|\nabla\overline{\vec{v}_k}\right\|_{\mathrm{L}^{2,1}(B_{\alpha R_k}(0))}\leq C_{31}(n)\left\|e^{\lambda_k}\vec{H}_k\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}.\end{aligned} \tag{4.64}$$

Lemma 4.6. *Let $n\geq 2$, $0<r<R<\infty$, $\Omega=B_R\setminus\overline{B}_r(0)\subset\mathbb{R}^n$, $1\leq p<\infty$ and assume that $u\in W^{1,p}(B_R\setminus\overline{B}_r(0))$. Define $\overline{u}:\Omega\to\mathbb{R}$ to be the radial function such that for all $r<t<R$ if $t=|x|$, then*

$$\overline{u}(x)=u_t=\fint_{\partial B_t(0)}u\,d\mathscr{H}^{n-1}=\frac{1}{\beta(n)t^{n-1}}\int_{\partial B_t(0)}u\,d\mathscr{H}^{n-1}.$$

Then $\overline{u}\in W^{1,p}(\Omega)$ and

$$\|\nabla\overline{u}\|_{\mathrm{L}^p(\Omega)}\leq\|\nabla u\|_{\mathrm{L}^p(\Omega)}\,.$$

Furthermore, for all $1<p<\infty$, and $1\leq q\leq\infty$, there exists a constant $C(p,q)$ independent of $0<r<R<\infty$ such that for all $u\in W^{1,(p,q)}(\Omega)$, $\overline{u}\in W^{1,(p,q)}(\Omega)$ and

$$\|\nabla\overline{u}\|_{\mathrm{L}^{p,q}(\Omega)}\leq C(p,q)\,\|\nabla u\|_{\mathrm{L}^{p,q}(\Omega)}\,.$$

Proof. First, assume that $u\in W^{1,p}(\Omega)$ for some $1\leq p<\infty$. Recall that by the proof of Proposition 2.7, we have

$$\left|\frac{d}{dt}u_t\right|\leq\fint_{\partial B_t(0)}|\nabla u|\,\mathscr{H}^{n-1}. \tag{4.65}$$

Therefore, as $\overline{u}$ is radial, we have by the co-area formula

$$\|\nabla\overline{u}\|^p_{\mathrm{L}^p(\Omega)}=\beta(n)\int_r^R\left|\frac{d}{dt}u_t\right|^p t^{n-1}dt. \tag{4.66}$$

Furthermore, by Hölder's inequality and (4.65)

$$\left|\frac{d}{dt}u_t\right|^p\leq\frac{1}{(\beta(n)t^{n-1})^p}\left|\int_{\partial B_t(0)}|\nabla u|d\mathscr{H}^{n-1}\right|^p\leq\frac{1}{(\beta(n)t^{n-1})^p}\int_{\partial B_t(0)}|\nabla u|^p\,d\mathscr{H}^{n-1}\left(\beta(n)t^{n-1}\right)^{\frac{p}{p'}} \tag{4.67}$$

$$=\frac{1}{\beta(n)t^{n-1}}\int_{\partial B_t(0)}|\nabla u|^p\,d\mathscr{H}^{n-1}. \tag{4.68}$$

Putting together (4.66) and (4.67), we find by a new application of the co-area formula

$$\|\nabla\overline{u}\|^p_{\mathrm{L}^p(\Omega)}\leq\int_r^R\left(\int_{\partial B_t(0)}|\nabla u|^p d\mathscr{H}^{n-1}\right)dt=\int_{B_R\setminus\overline{B}_r(0)}|\nabla u|^p d\mathscr{L}^n=\|\nabla u\|^p_{\mathrm{L}^p(\Omega)}\,.$$

The last statement comes from the Stein-Weiss interpolation theorem ([10], 3.3.3). ☐

Now, in order to obtain (4.63), recall the algebraic equation on $\Omega_k(\alpha)$ from (4.44)

$$\nabla\vec{n}_k^\beta=-*\left(\vec{n}_k\wedge\nabla^\perp\vec{n}_k^\beta\right)+\sum_{\gamma=1}^{n-2}\langle\nabla\vec{n}_k^\beta,\vec{n}_k^\gamma\rangle-2\,H_k^\beta\nabla\vec{\Phi}_k.$$

To simplify notations, let

$$\vec{G}_k=\sum_{\beta=1}^{n-2}\langle\nabla\vec{n}_k^{\beta},\vec{n}_k^{\gamma}\rangle-2\,H_k^{\beta}\nabla\vec{\Phi}_k.$$

Then (4.49) implies that

$$\left\|\vec{G}_k\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}\leq C_{32}(n)\left\|\nabla\vec{n}_k\right\|_{\mathrm{L}^2(\Omega_k(\alpha))}^2+4\left\|e^{\lambda_k}\vec{H}_k\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}. \tag{4.69}$$

We have

$$\left|\frac{d}{dt}\vec{n}_{k,t}^{\beta}\right|\leq\left|\fint_{\partial B_t(0)}\vec{n}_k\wedge\partial_{\tau}\vec{n}_k^{\beta}\,d\mathscr{H}^1\right|+\left|\frac{d}{dt}\vec{G}_{k,t}\right|=\left|\fint_{\partial B_t(0)}\left(\vec{n}_k-\overline{\vec{n}}_{k,t}\right)\wedge\partial_{\tau}\vec{n}_k^{\beta}\,d\mathscr{H}^1\right|+\left|\frac{d}{dt}\vec{G}_{k,t}\right| \tag{4.70}$$

Furthermore, by (4.69) and Lemma 4.6, we have (as $\overline{\vec{G}}_k$ is radial)

$$\left\|\frac{d}{dt}\vec{G}_{k,t}\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}=\left\|\nabla\overline{\vec{G}}_k\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}\leq C_{33}(n)\left(\left\|\nabla\vec{n}_k\right\|_{\mathrm{L}^2(\Omega_k(\alpha))}^2+\left\|e^{\lambda_k}\vec{H}_k\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}\right). \tag{4.71}$$

Now, the ε-regularity ([28] I.5) combined with the small L^2 norm of $\nabla\vec{n}_k$ in $\Omega_k(2\alpha)$ implies that there exists a universal constant $C_{34}(n)$ such that

$$\left\|\nabla\vec{n}_k\right\|_{\mathrm{L}^{\infty}(\partial B_t)}\leq\frac{C_{34}(n)}{t}\left(\int_{B_{2t}\setminus\overline{B}_{t/2}(0)}|\nabla\vec{n}_k|^2dx\right)^{\frac{1}{2}}$$

so that

$$\left\|\vec{n}_k-\overline{\vec{n}}_{k,t}\right\|_{\mathrm{L}^{\infty}(\partial B_t(0))}\leq\int_{\partial B_t(0)}|\nabla\vec{n}_k|\,d\mathscr{H}^1\leq 2\pi C_{34}(n)\left\|\nabla\vec{n}_k\right\|_{\mathrm{L}^2(B_{2t}\setminus\overline{B}_{t/2}(0))}.$$

Therefore,

$$\left|\fint_{\partial B_t(0)}\left(\vec{n}_k-\overline{\vec{n}}_{k,t}\right)\wedge\partial_{\tau}\vec{n}_k^{\beta}\,d\mathscr{H}^1\right|\leq 2\pi C_{34}(n)\left\|\nabla\vec{n}_k\right\|_{\mathrm{L}^2(\Omega_k(2\alpha))}\fint_{\partial B_t(0)}|\nabla\vec{n}_k^{\beta}|\,d\mathscr{H}^1. \tag{4.72}$$

The proof of Lemma 4.6 now implies by (4.72) that

$$\left\|\fint_{\partial B_t(0)}\left(\vec{n}_k-\overline{\vec{n}}_{k,t}\right)\wedge\partial_{\tau}\vec{n}_k^{\beta}\,d\mathscr{H}^1\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}\leq C_{35}(n)\left\|\nabla\vec{n}_k\right\|_{\mathrm{L}^2(\Omega_k(2\alpha))}\left\|\nabla\vec{n}_k^{\beta}\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}. \tag{4.73}$$

Finally, thanks to (4.70), (4.71) and (4.73), we find

$$\left\|\nabla\overline{\vec{n}}_k^{\beta}\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}\leq C_{33}(n)\left(\left\|\nabla\vec{n}_k\right\|_{\mathrm{L}^2(\Omega_k(\alpha))}^2+\left\|e^{\lambda_k}\vec{H}_k\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}\right)+C_{35}(n)\left\|\nabla\vec{n}_k\right\|_{\mathrm{L}^2(\Omega_k(2\alpha))}\left\|\nabla\vec{n}_k^{\beta}\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}. \tag{4.74}$$

Therefore, (4.59) and (4.74) imply that

$$\begin{aligned}\left\|\nabla\overline{\vec{n}}_k^{\beta}\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}&\leq C_{33}(n)\left(\left\|\nabla\vec{n}_k\right\|_{\mathrm{L}^2(\Omega_k(\alpha))}^2+\left\|e^{\lambda_k}\vec{H}_k\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}\right)\\&\quad+C_{35}(n)\left(16\sqrt{\pi}+C_9(n)\,(1+\Lambda)\left(1+\left\|\nabla\vec{n}_k\right\|_{\mathrm{L}^2(\Omega_k(2\alpha))}\right)\left\|\nabla\vec{n}_k\right\|_{\mathrm{L}^2(\Omega_k(2\alpha))}\right)\left\|\nabla\vec{n}_k\right\|_{\mathrm{L}^2(\Omega_k(2\alpha))}\\&\leq C_{36}(n)\,(1+\Lambda)^2\,e^{4\Gamma_1(n)\Lambda}\left(1+\left\|\nabla\vec{n}_k\right\|_{\mathrm{L}^2(\Omega_k(2\alpha))}\right)\left\|\nabla\vec{n}_k\right\|_{\mathrm{L}^2(\Omega_k(2\alpha))}.\end{aligned} \tag{4.75}$$

Therefore, (4.64) and (4.75) imply that

$$\left\|\nabla\overline{\vec{\varphi}_k}\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}\leq\left\|\nabla\overline{\vec{n}_k}\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}+\left\|\nabla\overline{\vec{u}_k}\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}+\left\|\nabla\vec{v}_k\right\|_{\mathrm{L}^2(\Omega_k(\alpha))}$$

$$\leq C_{37}(n)e^{\Gamma_2(n)\Lambda}\left(1+\|\nabla\vec{n}_k\|_{\mathrm{L}^2(\Omega_k(2\alpha))}\right)\|\nabla\vec{n}_k\|_{\mathrm{L}^2(\Omega_k(2\alpha))} \tag{4.76}$$

We can now use Lemma 2.3 (or equivalently Proposition 2.5) and Lemma 4.6 to get for all $0<\beta<1$

$$\begin{aligned}
\left\|\nabla\left(\vec{\varphi}_k-\overline{\vec{\varphi}_k}\right)\right\|_{\mathrm{L}^{2,1}(\Omega_k(\beta\alpha))} &\leq 24\beta\left\|\nabla\left(\vec{\varphi}_k-\overline{\vec{\varphi}_k}\right)\right\|_{\mathrm{L}^2(\Omega_k(\alpha))}\leq 48\beta\|\nabla\vec{\varphi}_k\|_{\mathrm{L}^2(\Omega_k(\alpha))}\\
&\leq 48\beta\left(\|\nabla\vec{n}_k\|_{\mathrm{L}^2(\Omega_k(\alpha))}+\|\nabla\vec{u}_k\|_{\mathrm{L}^2(\Omega_k(\alpha))}+\|\nabla\vec{v}_k\|_{\mathrm{L}^2(\Omega_k(\alpha))}\right)\\
&\leq C_{38}(n)\beta\, e^{\Gamma_2(n)\Lambda}\left(1+\|\nabla\vec{n}_k\|_{\mathrm{L}^2(\Omega_k(2\alpha))}\right)\|\nabla\vec{n}_k\|_{\mathrm{L}^2(\Omega_k(2\alpha))}\,.
\end{aligned}\tag{4.77}$$

Therefore, taking $\beta=1/2$ in (4.77), we get by (4.76) and (4.77) show that

$$\|\nabla\vec{\varphi}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha/2))}\leq C_{39}(n)\,e^{\Gamma_2(n)\Lambda}\left(1+\|\nabla\vec{n}_k\|_{\mathrm{L}^2(\Omega_k(2\alpha))}\right)\|\nabla\vec{n}_k\|_{\mathrm{L}^2(\Omega_k(2\alpha))}\,. \tag{4.78}$$

Finally, by (4.54), (4.55) and (4.56) we obtain the expected estimate for $\vec{n}_k^{\beta}=\vec{u}_k+\vec{v}_k+\vec{\varphi}_k$ on $\Omega_k(\alpha/2)$, and for $\vec{n}_k$ by the algebraic inequality (4.61). ☐

Remark 4.7. Observe that for the mean curvature, we have the improved (because of the Sobolev embedding $W^{1,1}(\mathbb{R}^2)\hookrightarrow L^{2,1}(\mathbb{R}^2)$) no-neck energy

$$\lim_{\alpha\to 0}\limsup_{k\to\infty}\left\|e^{\lambda_k}\nabla\vec{H}_k\right\|_{\mathrm{L}^1(\Omega_k(\alpha))}=0$$

but this is not completely clear if this also holds for $\nabla^2\vec{n}_k$ (here, $\Omega_k(\alpha)=B_{\alpha R_k}\setminus\overline{B}_{\alpha^{-1}r_k}(0)$). However, notice that (4.51) implies that

$$\left\|\nabla^2\vec{A}_{\beta,\gamma}\right\|_{\mathrm{L}^1(B_{\alpha R_k}(0))}\leq C(n)\,\|\nabla\vec{n}\|^2_{\mathrm{L}^2(\Omega_k(\alpha))}$$

and as $\nabla^{\perp}\vec{A}_{\beta,\gamma}=\nabla\widetilde{\vec{n}}_k^{\beta}\cdot\widetilde{\vec{n}}_k^{\gamma}$, we deduce that

$$\nabla^2\widetilde{\vec{n}}_k^{\beta}\cdot\widetilde{\vec{n}}_k^{\gamma}+\nabla\widetilde{\vec{n}}_k^{\beta}\cdot\nabla\widetilde{\vec{n}}_k^{\gamma}\in L^1(B_{\alpha R_k}(0)),$$

and by the Cauchy-Schwarz inequality, this implies that for all $1\leq\beta,\gamma\leq n-2$

$$\left\|\nabla^2\widetilde{\vec{n}}_k^{\beta}\cdot\widetilde{\vec{n}}_k^{\gamma}\right\|_{\mathrm{L}^1(B_{\alpha R_k}(0))}\leq C'(n)\,\|\nabla\vec{n}_k\|^2_{\mathrm{L}^2(\Omega_k(\alpha))}\,.$$

Therefore, we deduce as $\widetilde{\vec{n}}_k^{\beta}=\vec{n}_k^{\beta}$ on $\Omega_k(\alpha)$ that

$$\lim_{\alpha\to 0}\limsup_{k\to\infty}\left\|\pi_{\vec{n}_k}(\nabla^2\vec{n}_k)\right\|_{\mathrm{L}^1(\Omega_k(\alpha))}=0,$$

(where $\pi_{\vec{n}_k}$ is the projection on the normal bundle) but this is not completely clear how one may obtain the same result for the tangential part of $\nabla^2\vec{n}_k$.

We finish this section by the proof of Corollary 1.4.

Proof of Corollary 1.4. Introduce for all $\alpha>0$ small enough the domain decomposition of [2]:

$$\Sigma=\left(\Sigma\setminus\bigcup_{i=1}^{m}\overline{B}_{\alpha}(a_i)\right)\cup\Omega_k(\alpha)\cup\sum_{i=1}^{m}\sum_{j=1}^{m_i}B(i,j,\alpha,k),$$

where

$$\Omega_k(\alpha)=\left(\bigcup_{i=1}^{m}B_{\alpha}(a_i)\setminus\bigcup_{j=1}^{m_i}B_{\alpha^{-1}\rho_k^{i,j}}(x_k^{i,j})\right)\bigcup_{i=1}^{m}\bigcup_{j=1}^{m_i}\bigcup_{j'\in I^{i,j}}\left(B_{\alpha\rho_k^{i,j}}(x_k^{i,j'})\setminus\bigcup_{j''\in I^{i,j}}B_{\alpha^{-1}\rho_k^{i,j''}}(x_k^{i,j''})\right)$$

$$= \bigcup_{i=1}^{m} \Omega_k^i(\alpha) \bigcup_{i=1}^{m} \bigcup_{j=1}^{m_i} \bigcup_{j' \in I^{i,j}} \Omega_k^{i,j,j'}(\alpha)$$

and

$$B(i,j,\alpha,k) = B_{\alpha^{-1}\rho_k^{i,j}}(x_k^{i,j}) \setminus \bigcup_{j' \in I^{i,j}} B_{\alpha\rho_k^{i,j}}(x_k^{i,j'})$$

and for all $1 \leq i \leq m$, for all $1 \leq j \leq m_i$, we have

$$\frac{\rho_k^{i,j}}{\rho_k^{i,j'}} \underset{k\to\infty}{\longrightarrow} \infty.$$

Thanks to the no-neck property, we have

$$\lim_{\alpha\to 0} \limsup_{k\to\infty} \|\nabla \vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))} = 0.$$

By the strong convergence, for all $0 < \alpha \leq \alpha_0$, we have

$$\limsup_{k\to\infty} \|\vec{n}_k - \vec{n}_\infty\|_{\mathrm{L}^\infty(\Sigma_\alpha)} = 0,$$

where $\Sigma_\alpha = \Sigma \setminus \bigcup_{i=1}^m \overline{B}_\alpha(a_i)$. This implies by Proposition 2.7 that there exists sequences of constants $\{\vec{c}_k^{\,i}(\alpha)\}_{k\in\mathbb{N}}, \{\vec{c}_k^{\,i,j,j'}(\alpha)\}_{k\in\mathbb{N}} \subset \Lambda^{n-2}\mathbb{R}^n$ such that for all i,j

$$\begin{aligned}
&\lim_{\alpha\to 0} \limsup_{k\to\infty} \left\|\vec{n}_k - \vec{c}_k^{\,i}(\alpha)\right\|_{\mathrm{L}^\infty(\Omega_k^i(\alpha))} = 0 \\
&\lim_{\alpha\to 0} \limsup_{k\to\infty} \left\|\vec{n}_k - \vec{c}_k^{\,i,j,j'}(\alpha)\right\|_{\mathrm{L}^\infty(\Omega_k^{i,j,j'}(\alpha))} = 0.
\end{aligned}$$

Since $|\vec{n}_k| = 1$, we deduce that up to a subsequence $\vec{c}_k^{\,i}(\alpha_0) \underset{k\to\infty}{\longrightarrow} \vec{c}_\infty^{\,i}(\alpha_0)$ such that $|\vec{c}_\infty^{\,i}(\alpha_0)| = 1$. Likewise, there exists $\{\alpha_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ such that $\vec{c}_\infty^{\,i}(\alpha_k) \to \vec{c}_\infty^{\,i}$ where $|\vec{c}_\infty^{\,i}| = 1$. Therefore, we deduce that there exists $\vec{c}_\infty^{\,i}, \vec{c}_\infty^{\,i,j,j'} \in S^{n-1}$ such that

$$\begin{aligned}
&\lim_{\alpha\to 0} \limsup_{k\to\infty} \left\|\vec{n}_k - \vec{c}_\infty^{\,i}\right\|_{\mathrm{L}^\infty(\Omega_k^i(\alpha))} = 0 \\
&\lim_{\alpha\to 0} \limsup_{k\to\infty} \left\|\vec{n}_k - \vec{c}_\infty^{\,i,j,j'}\right\|_{\mathrm{L}^\infty(\Omega_k^{i,j,j'}(\alpha))} = 0.
\end{aligned}$$

Finally, in a bubble domain $B(i,j,\alpha,k)$, there exists a sequence $\{\mu_k^{i,j}\}_{k\in\mathbb{N}} \subset \mathbb{R}$ such that the function

$$\begin{aligned}
\vec{\Phi}_k^{i,j} : B_{\alpha^{-1}}(x_k^{i,j'}) \setminus \bigcup_{j'' \in I^{i,j}} B_\alpha(x_k^{i,j''}) &\to \mathbb{R}^n \\
z &\mapsto e^{-\mu_k^{i,j}}\left(\vec{\Phi}_k(\rho_k^{i,j} z) - \vec{\Phi}_k(x_k^{i,j'})\right)
\end{aligned}$$

converges smoothly towards to the branched Willmore sphere $\vec{\Phi}_\infty^{i,j} : \mathbb{C} \to \mathbb{R}^n$. Since

$$\vec{n}_{\vec{\Phi}_k^{i,j}}(z) = \vec{n}_{\vec{\Phi}_k}(\rho_k z),$$

we deduce that for all $0 < \alpha < \alpha_0$,

$$\left\|\nabla \vec{n}_k - \nabla \vec{n}_{\vec{\Phi}_\infty^{i,j}}((\rho_k^{i,j})^{-1}\,\cdot)\right\|_{\mathrm{L}^{2,1}(B(i,j,\alpha,k))} = 0.$$

This implies that there exists $\{\vec{d}_k^{\,i,j}(\alpha)\}_{k\in\mathbb{N}} \subset \Lambda^{n-2}\mathbb{R}^n$ such that

$$\lim_{\alpha\to 0} \limsup_{k\to\infty} \left\|\vec{n}_k - \vec{n}_{\vec{\Phi}_\infty^{i,j}}((\rho_k^{i,j})^{-1}\,\cdot)) - \vec{d}_k^{\,i,j}(\alpha)\right\|_{\mathrm{L}^\infty(B(i,j,\alpha,k))} = 0$$

Since $\vec{n}_k$ and $\vec{n}_{\vec{\Phi}_\infty^{i,j}}$ are unitary, we deduce that

$$\lim_{\alpha\to 0}\lim_{k\to\infty}|\vec{d}_k^{i,j}(\alpha)|=0,$$

so that

$$\lim_{\alpha\to 0}\limsup_{k\to\infty}\left\|\vec{n}_k-\vec{n}_{\vec{\Phi}_\infty^{i,j}}((\rho_k^{i,j})^{-1}\,\cdot\,)\right\|_{\mathrm{L}^\infty(B(i,j,\alpha,k))}=0.$$

Notice that the function $\vec{n}_k-\vec{n}_{\vec{\Phi}_\infty^{i,j}}((\rho_k^{i,j})^{-1}\,\cdot\,)$ is independent of α and that $B(i,j,\alpha,k)\subset B(i,j,\beta,k)$ for $\alpha<\beta$, which implies that

$$\lim_{k\to\infty}\left\|\vec{n}_k-\vec{n}_{\vec{\Phi}_\infty^{i,j}}((\rho_k^{i,j})^{-1}\,\cdot\,)\right\|_{\mathrm{L}^\infty(B(i,j,\alpha_0,k))}=0.$$

Now, using the proof of Proposition 2.7, we deduce that we can take

$$\vec{c}_k^{\,i}(\alpha)=\fint_{\partial B^{i,1}_{\alpha^{-1}\rho_k}(x_k^{i,1})}\vec{n}_k\,d\mathscr{H}^1$$

and since $\vec{\Phi}_\infty^{i,1}:\mathbb{C}\to\mathbb{R}^n$ extends to an immersion $S^2\to\mathbb{R}^n$, the normal has a continuous extension and identifying $N=(0,0,1)\in S^2$ and $\infty\in\mathbb{C}\cup\{\infty\}$, we deduce that

$$\lim_{\alpha\to 0}\lim_{k\to\infty}\left\|\vec{n}_k-\vec{n}_{\vec{\Phi}_\infty^{i,1}}(N)\right\|_{\mathrm{L}^\infty(\Omega_k^i(\alpha))}=0,$$

and likewise for all $1\leq i\leq m$ and $1\leq j\leq m_i$, we have

$$\lim_{\alpha\to 0}\limsup_{k\to\infty}\left\|\vec{n}_k-\vec{n}_{\vec{\Phi}_\infty^{i,j}}(N)\right\|_{\mathrm{L}^\infty(\Omega_k^{i,j}(\alpha))}=0.$$

Which completes the proof of the theorem. □

In the next section we recall basic facts on the viscosity method for the Willmore energy, and then in the following section we show the improved $L^{2,1}$ quantization in this setting.

5 The viscosity method for the Willmore energy

We first introduce for all weak immersion $\vec{\Phi}:S^2\to\mathbb{R}^n$ of finite total curvature the associated metric $g=\vec{\Phi}^*g_{\mathbb{R}^n}$ on S^2. By the uniformisation theorem, there exists a function $\omega:S^2\to\mathbb{R}$ such that

$$g=e^{2\omega}g_0,$$

where g_0 is a metric of constant Gauss curvature 4π and unit volume on S^2. Furthermore, in all fixed chart $\varphi:B_1(0)\to S^2$, we define $\mu:B_1(0)\to\mathbb{R}$ such that

$$\lambda=\omega+\mu,$$

where in the given chart

$$g=e^{2\lambda}|dz|^2.$$

For technical reasons, we will have to make a peculiar choice of ω (see [35], Definition III.2).

Definition 5.1. Under the preceding notations, we say that a choice (ω,φ) of a map $\omega:S^2\to\mathbb{R}$ and of a diffeomorphism $\varphi:S^2\to S^2$ is an Aubin gauge if

$$\varphi^*g_0=\frac{1}{4\pi}g_{S^2}\qquad\text{and}\qquad\int_{S^2}x_je^{2\omega\circ\varphi(x)}d\mathrm{vol}_{g_{S^2}}(x)=0\qquad\text{for all }\;j=1,2,3,$$

where g_{S^2} is the standard metric on S^2.

We also recall that the limiting maps arise from a sequence of critical point of the following regularisation of the Willmore energy (see [35] for more details) :

$$
\begin{aligned}
W_\sigma(\vec{\Phi}) &= W(\vec{\Phi}) + \sigma^2 \int_{S^2} \left(1+|\vec{H}|^2\right)^2 d\mathrm{vol}_g \\
&\quad + \frac{1}{\log\left(\frac{1}{\sigma}\right)}\left(\frac{1}{2}\int_{S^2}|d\omega|_g^2 d\mathrm{vol}_g + 4\pi\int_{S^2}\omega\, e^{-2\omega} d\mathrm{vol}_g - 2\pi\log\int_{S^2} d\mathrm{vol}_g\right)
\end{aligned}
$$

where $\omega : S^2 \to \mathbb{R}$ is as above.

We need a refinement of a standard estimate (see [10], 3.3.6).

Lemma 5.2. *Let Ω be a open subset of $\mathbb{R}^2$ whose boundary is a finite union of C^1 Jordan curves. Let $f \in L^1(\Omega)$ and let u be the solution of*

$$
\begin{cases} \Delta u = f & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.1}
$$

Then $\nabla u \in L^{2,\infty}(\Omega)$, and

$$
\|\nabla u\|_{\mathrm{L}^{2,\infty}(\Omega)} \leq 3\sqrt{\frac{2}{\pi}}\,\|f\|_{\mathrm{L}^1(\Omega)}\,.
$$

Remark 5.3. We need an estimate independent of the domain for a sequence of annuli of conformal class diverging to ∞, but the argument applies to a general domain (although some regularity conditions seem to be necessary).

Proof. First assume that $f \in C^{0,\alpha}(\overline{\Omega})$ for some $0<\alpha<1$. Then by Schauder theory, $u \in C^{2,\alpha}(\overline{\Omega})$, and by Stokes theorem ([11] 1.2.1), we find as $u=0$ on $\partial\Omega$ that for all $z \in \Omega$

$$
\partial_z u(z) = \frac{1}{2\pi i}\int_\Omega \frac{\partial_{\overline{z}}\left(\partial_z u(\zeta)\right)}{\zeta - z} d\zeta \wedge d\overline{\zeta}. \tag{5.2}
$$

As $\Delta u = 4\,\partial^2_{z\overline{z}} u$ and $|d\zeta|^2 = \dfrac{d\overline{\zeta}\wedge d\zeta}{2i}$, the pointwise estimate (5.2) implies that

$$
\partial_z u(z) = -\frac{1}{4\pi}\int_\Omega \frac{\Delta u(\zeta)}{\zeta - z}|d\zeta|^2 = -\frac{1}{4\pi}\int_\Omega \frac{f(\zeta)}{\zeta - z}|d\zeta|^2. \tag{5.3}
$$

Now, define $\overline{f} \in L^1(\mathbb{R}^2)$ by

$$
\overline{f}(z) = \begin{cases} f(z) & \text{for all } z \in \Omega \\ 0 & \text{for all } z \in \mathbb{R}^2 \setminus \Omega. \end{cases}
$$

and $U : \mathbb{R}^2 \to \mathbb{C}$ by

$$
U(z) = -\frac{1}{4\pi}\int_{\mathbb{R}^2}\frac{\overline{f}(\zeta)}{\zeta - z}|d\zeta|^2 = -\frac{1}{4\pi}\left(\left(\zeta \mapsto \frac{1}{\zeta}\right) * \overline{f}\right)(z), \tag{5.4}
$$

where $*$ indicates the convolution on $\mathbb{R}^2$. Now, recall that for all $1 \leq p < \infty$ and $g \in L^p(\mathbb{R}^2, \mathbb{C})$, we have

$$
\|\overline{f} * g\|_{\mathrm{L}^p(\mathbb{R}^2)} \leq \|\overline{f}\|_{\mathrm{L}^1(\mathbb{R}^2)}\|g\|_{\mathrm{L}^p(\mathbb{R}^2)}\,.
$$

Interpolating between L^1 and L^p for all $p > 2$ shows by the Stein-Weiss interpolation theorem ([10] 3.3.3) that for all $g \in L^{2,\infty}(\mathbb{R}^2, \mathbb{C})$

$$
\|\overline{f} * g\|_{\mathrm{L}^{2,\infty}(\mathbb{R}^2)} \leq \sqrt{2}\left(\frac{2\times 1}{2-1} + \frac{p\cdot 1}{p-2}\right)\|\overline{f}\|_{\mathrm{L}^1(\mathbb{R}^2)}\|g\|_{\mathrm{L}^{2,\infty}(\mathbb{R}^2)} = \sqrt{2}\left(2 + \frac{p}{p-2}\right)\|\overline{f}\|_{\mathrm{L}^1(\mathbb{R}^2)}\|g\|_{\mathrm{L}^{2,\infty}(\mathbb{R}^2)}\,.
$$

Taking the infimum in $p > 2$ (that is, $p \to \infty$) shows that for all $g \in L^{2,\infty}(\mathbb{R}^2)$,

$$\left\|\overline{f} * g\right\|_{\mathrm{L}^{2,\infty}(\mathbb{R}^2)} \leq 3\sqrt{2}\left\|\overline{f}\right\|_{\mathrm{L}^1(\mathbb{R}^2)}\|g\|_{\mathrm{L}^{2,\infty}(\mathbb{R}^2)}. \tag{5.5}$$

Therefore, we deduce from (5.3) and (5.5) that

$$\|U\|_{\mathrm{L}^{2,\infty}(\mathbb{R}^2)} \leq \frac{3\sqrt{2}}{4\pi}\left\|\overline{f}\right\|_{\mathrm{L}^1(\mathbb{R}^2)}\left\|\frac{1}{|\,\cdot\,|}\right\|_{\mathrm{L}^{2,\infty}(\mathbb{R}^2)} = \frac{3}{\sqrt{2\pi}}\|f\|_{\mathrm{L}^1(\Omega)}.$$

Now, as $U = \partial_z u$ on Ω and $2|\partial_z u| = |\nabla u|$, we finally deduce that

$$\|\nabla u\|_{\mathrm{L}^{2,\infty}(\Omega)} \leq 3\sqrt{\frac{2}{\pi}}\|f\|_{\mathrm{L}^1(\Omega)}. \tag{5.6}$$

In the general case $f \in L^1(\Omega)$, by density of $C_c^\infty(\Omega)$ in $L^1(\Omega)$, let $\{f_k\}_{k\in\mathbb{N}} \subset C_c^\infty(\Omega)$ such that

$$\|f_k - f\|_{\mathrm{L}^1(\Omega)} \underset{k\to\infty}{\longrightarrow} 0. \tag{5.7}$$

Then $u_k \in C^\infty(\overline{\Omega})$ (defined to be the solution of the system (5.1) with f replaced by f_k and the same boundary conditions) so for all $k \in \mathbb{N}$, $\nabla u_k \in L^{2,\infty}(\Omega)$ and

$$\|\nabla u_k\|_{\mathrm{L}^{2,\infty}(\Omega)} \leq 3\sqrt{\frac{2}{\pi}}\|f_k\|_{\mathrm{L}^1(\Omega)}. \tag{5.8}$$

As $\left\{\|f_k\|_{\mathrm{L}^1(\Omega)}\right\}_{k\in\mathbb{N}}$ is bounded, up to a subsequence $u_k \underset{k\to\infty}{\rightharpoonup} u_\infty$ in the weak topology of $W^{1,(2,\infty)}(\Omega)$. Therefore, (5.7) and (5.8) yield

$$\|\nabla u_\infty\|_{\mathrm{L}^{2,\infty}(\Omega)} \leq \liminf_{k\to\infty}\|\nabla u_k\|_{\mathrm{L}^{2,\infty}(\Omega)} \leq 3\sqrt{\frac{2}{\pi}}\|f\|_{\mathrm{L}^1(\Omega)}.$$

Furthermore, as $f_k \underset{k\to\infty}{\longrightarrow} f$ in $L^1(\Omega)$, we have $\Delta u_\infty = f$ in $\mathscr{D}'(\Omega)$, so we deduce that $u_\infty = u$ and this concludes the proof of the lemma. □

Finally, recall the following Lemma from [2] (se also [6]).

Lemma 5.4. *Let Ω be a Lipschitz bounded open subset of $\mathbb{R}^2$, $1 < p < \infty$ and $1 \leq q \leq \infty$, and $(a, b) \in W^{1,(p,q)}(B_1(0)) \times W^{1,(2,\infty)}(B_1(0))$. Let $u : B_1(0) \to \mathbb{R}$ be the solution of*

$$\begin{cases} \Delta u = \nabla a \cdot \nabla^\perp b & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

Then there exists a constant $C_{p,q}(\Omega) > 0$ such that

$$\|\nabla u\|_{\mathrm{L}^{p,q}(\Omega)} \leq C_{p,q}(\Omega)\|\nabla a\|_{\mathrm{L}^{p,q}(\Omega)}\|\nabla b\|_{\mathrm{L}^{2,\infty}(\Omega)}.$$

Remark 5.5. Notice that by scaling invariance, we have for all $R > 0$ if $\Omega_R = B_R(0)$

$$\|\nabla u\|_{\mathrm{L}^{2,1}(B_R(0))} \leq C_{2,1}(B_1(0))\|\nabla a\|_{\mathrm{L}^{2,1}(B_R(0))}\|\nabla b\|_{\mathrm{L}^{2,\infty}(B_R(0))}.$$

6 Improved energy quantization in the viscosity method

The viscosity method ([22], [34], [33], [26], [25], [35], [31], [21]) developed by the T. Rivière and collaborators aims at constructing solutions of min-max problems for functionals that do not satisfy the Palais-Smale condition or defined on spaces that are not Banach manifolds. Here, we will be focusing on the viscosity method for Willmore surfaces ([35]). Let us recall a couple of definitions

Definition 6.1. Let $\mathcal{M} = W_\iota^{2,4}(S^2, \mathbb{R}^n)$ be the space of $W^{2,4}$ immersions from the sphere S^2 into $\mathbb{R}^n$. We say that a family $\mathcal{A} \subset \mathcal{P}(\mathcal{M})$ is an admissible family if for every homeomorphism Ψ of $\mathscr{M}$ isotopic to the identity, we have

$$\forall A \in \mathcal{A}, \quad \Psi(A) \in \mathcal{A}.$$

Now fix some admissible family $\mathscr{A} \subset \mathcal{P}(W_\iota^{2,4}(S^2, \mathbb{R}^n))$ and define

$$\beta_0 = \inf_{A \in \mathscr{A}} \sup W(A).$$

For all $\sigma > 0$ and all smooth immersion $\vec{\Phi} : S^2 \to \mathbb{R}^n$, recall the definition

$$W_\sigma(\vec{\Phi}) = W(\vec{\Phi}) + \sigma^2 \int_\Sigma \left(1 + |\vec{H}|^2\right)^2 d\mathrm{vol}_g + \frac{1}{\log\left(\frac{1}{\sigma}\right)} \mathscr{O}(\vec{\Phi}),$$

where $\mathscr{O}$ is the Onofri energy (see above or [35] for more details), and define

$$\beta(\sigma) = \inf_{A \in \mathscr{A}} \sup W_\sigma(A).$$

We can now introduce the main result of this section.

Theorem 6.2. *Let $\{\sigma_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ be such that $\sigma_k \underset{k\to\infty}{\longrightarrow} 0$ and let $\{\vec{\Phi}_k\}_{k\in\mathbb{N}} : S^2 \to \mathbb{R}^n$ be a sequence of critical points associated to W_{σ_k} such that*

$$\begin{cases} W_{\sigma_k}(\vec{\Phi}_k) = \beta(\sigma_k) \underset{k\to\infty}{\longrightarrow} \beta_0 \\ W_{\sigma_k}(\vec{\Phi}_k) - W(\vec{\Phi}_k) = o\left(\dfrac{1}{\log\left(\frac{1}{\sigma_k}\right)\log\log\left(\frac{1}{\sigma_k}\right)}\right) \end{cases}. \tag{6.1}$$

Let $\{R_k\}_{k\in\mathbb{N}}, \{r_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ be such that

$$\lim_{k\to\infty} \frac{R_k}{r_k} = 0, \qquad \limsup_{k\to\infty} R_k < \infty,$$

and for all $0 < \alpha < 1$ and $k \in \mathbb{N}$, let $\Omega_k(\alpha) = B_{\alpha R_k} \setminus \overline{B}_{\alpha^{-1} r_k}(0)$ be a neck region, i.e. such that

$$\lim_{\alpha\to 0} \lim_{k\to\infty} \sup_{2\alpha^{-1} r_k < s < \alpha R_k/2} \int_{B_{2s}\setminus\overline{B}_{s/2}(0)} |\nabla \vec{n}_k|^2 dx = 0.$$

Then we have

$$\lim_{\alpha\to 0} \limsup_{k\to\infty} \|\nabla \vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))} = 0.$$

Remarks on the proof. The proof is in the same spirit of the proof of the no-neck energy for the $L^{2,1}$ norm in the case of Willmore immersions, up to the need to introduce more conversation laws and derive more estimates to obtain the $L^{2,1}$ estimates.

Proof. As in [35], we give the proof in the special case $n = 3$. By Theorem 4.1 this is not restrictive.

$$\Lambda = \sup_{k\in\mathbb{N}} \left(\|\nabla \lambda_k\|_{\mathrm{L}^{2,\infty}(B_1(0))} + \int_{B_1(0)} |\nabla \vec{n}_k|^2 dx \right) < \infty$$

and

$$l(\sigma_k) = \frac{1}{\log\left(\frac{1}{\sigma_k}\right)}, \qquad \widetilde{l}(\sigma_k) = \frac{1}{\log\log\left(\frac{1}{\sigma_k}\right)}.$$

Furthermore, the entropy condition (6.1) and the improved Onofri inequality show (see [2] III.2)

$$
\begin{aligned}
&\frac{1}{\log\left(\frac{1}{\sigma_k}\right)}\,\|\omega_k\|_{\mathrm{L}^\infty(B_1(0))}=o\left(\frac{1}{\log\log\left(\frac{1}{\sigma_k}\right)}\right)\\
&\frac{1}{\log\left(\frac{1}{\sigma_k}\right)}\int_{S^2}|d\omega_k|_{g_k}^2d\mathrm{vol}_{g_k}=o\left(\frac{1}{\log\log\left(\frac{1}{\sigma_k}\right)}\right)\\
&\frac{1}{\log\left(\frac{1}{\sigma_k}\right)}\left(\frac{1}{2}\int_{S^2}|d\omega_k|_{g_k}^2d\mathrm{vol}_{g_k}+4\pi\int_{S^2}\omega_ke^{-2\omega_k}d\mathrm{vol}_{g_k}-2\pi\log\int_{S^2}d\mathrm{vol}_{g_k}\right)=o\left(\frac{1}{\log\log\left(\frac{1}{\sigma_k}\right)}\right).
\end{aligned}
\tag{6.2}
$$

Thanks to [35], we already have

$$
\lim_{\alpha\to 0}\limsup_{k\to\infty}\|\nabla\vec{n}_k\|_{\mathrm{L}^2(\Omega_k(\alpha))}=0.
$$

Therefore, as in Lemma IV.1 in [2] (and using the same argument as in Lemma 4.4), there exists a controlled extension $\widetilde{\vec{n}}_k:B_{\alpha R_k}(0)\to\mathscr{G}_{n-2}(\mathbb{R}^n)$ such that $\widetilde{\vec{n}}_k=\vec{n}_k$ on $\Omega_k(\alpha)=B_{\alpha R_k}(0)\setminus\overline{B}_{\alpha^{-1}r_k}(0)$ and

$$
\begin{aligned}
&\left\|\nabla\widetilde{\vec{n}}_k\right\|_{\mathrm{L}^2(B_{\alpha R_k}(0))}\leq\kappa_0(n)\,\|\nabla\vec{n}_k\|_{\mathrm{L}^2(\Omega_k(\alpha))}\\
&\left\|\nabla\widetilde{\vec{n}}_k\right\|_{\mathrm{L}^{2,1}(B_{\alpha R_k}(0))}\leq\kappa_0(n)\,\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))},
\end{aligned}
\tag{6.3}
$$

in all equations involving $\vec{n}_k$ on $B_{\alpha R_k}(0)$, we replace $\vec{n}_k$ by $\widetilde{\vec{n}}_k$ as one need only obtain estimates on $\Omega_k(\alpha)$, where $\widetilde{\vec{n}}_k=\vec{n}_k$. Likewise, $\vec{H}_k$ can be replaced by a controlled extension using Lemma B.4 in [15] (see also the Appendix).

Now, by [35], let $\vec{L}_k:B_1(0)\to\mathbb{R}^3$ be such that

$$
\begin{aligned}
d\vec{L}_k&=\ast\, d\left(\vec{H}_k+2\sigma_k^2(1+|\vec{H}_k|^2)\vec{H}_k\right)-2\left(1+2\sigma_k^2(1+|\vec{H}_k|^2)\right)H_k\,\ast d\vec{n}_k\\
&+\left(-\left(|\vec{H}_k|^2+\sigma_k^2(1+|\vec{H}_k|^2)^2\right)+\frac{1}{\log\left(\frac{1}{\sigma_k}\right)}\left(\frac{1}{2}|d\omega_k|_{g_k}^2-2\pi\omega_ke^{-2\omega_k}+\frac{2\pi}{\mathrm{Area}(\vec{\Phi}_k(S^2))}\right)\right)\ast d\vec{\Phi}_k\\
&-\frac{1}{\log\left(\frac{1}{\sigma_k}\right)}\langle d\vec{\Phi}_k,d\omega_k\rangle_{g_k}\ast d\omega_k+\frac{1}{\log\left(\frac{1}{\sigma_k}\right)}\vec{\mathbb{I}}_k\llcorner_{g_k}(\ast\, d\omega_k).
\end{aligned}
\tag{6.4}
$$

Then following [35], we have

$$
e^{\lambda_k(z)}|\vec{L}_k(z)|\leq\left(\kappa_1(n)\,(1+\Lambda)\,e^{\kappa_1(n)\Lambda}\,\|\nabla\vec{n}\|_{\mathrm{L}^2(\Omega_k(\alpha))}+\widetilde{l}(\sigma_k)\right)\frac{1}{|z|}\qquad\text{for all }\ z\in\Omega_k(\alpha/2),
$$

so that

$$
\left\|e^{\lambda_k}\vec{L}_k\right\|_{\mathrm{L}^{2,\infty}(\Omega_k(\alpha/2))}\leq 2\sqrt{\pi}\left(\kappa_1(n)\,(1+\Lambda)\,e^{\kappa_1(n)\Lambda}\,\|\nabla\vec{n}_k\|_{\mathrm{L}^2(\Omega_k(\alpha))}+\widetilde{l}(\sigma_k)\right).
$$

Now let $Y_k:B_{\alpha R_k}(0)\to\mathbb{R}$ (see [35], VI.21) be the solution of

$$
\begin{cases}\Delta Y_k=-4e^{2\lambda_k}\sigma_k^2\left(1-H_k^4\right)-2l(\sigma_k)K_{g_0}\omega_ke^{2\mu_k}+8\pi\,l(\sigma_k)e^{2\lambda_k}\mathrm{Area}(\vec{\Phi}(S^2))^{-1}\quad&\text{in }\ B_{\alpha R_k}(0)\\ Y_k=0&\text{on }\ \partial B_{\alpha R_k}(0).\end{cases}
\tag{6.5}
$$

Then we have (recall that $K_{g_0}=4\pi$ by the chosen normalisation in Definition 5.1)

$$
\|\Delta Y_k\|_{\mathrm{L}^1(B_{\alpha R_k}(0))}\leq 4\sigma_k^2\int_{B_{\alpha R_k}(0)}\left(1+H_k^4\right)d\mathrm{vol}_{g_k}+8\pi\,l(\sigma_k)\,\|\omega_k\|_{\mathrm{L}^\infty(B_{\alpha_k}(0))}\int_{B_{\alpha R_k}(0)}e^{2\mu_k}dx
$$

$$+ 8\pi\, l(\sigma_k) \frac{\mathrm{Area}(\vec{\Phi}_k(B_{\alpha R_k}(0)))}{\mathrm{Area}(\vec{\Phi}_k(S^2))} = o(\widetilde{l}(\sigma_k)). \tag{6.6}$$

Therefore, Lemma 7.9 implies by (6.6) that

$$\|\nabla Y_k\|_{\mathrm{L}^{2,\infty}(B_{\alpha R_k}(0))} \leq 3\sqrt{\frac{2}{\pi}}\, \|\Delta Y_k\|_{\mathrm{L}^1(B_{\alpha R_k}(0))} = o(\widetilde{l}(\sigma_k)) \leq \widetilde{l}(\sigma_k) \tag{6.7}$$

for k large enough. Now, let $\vec{v}_k : B_{\alpha R_k}(0) \to \mathbb{R}^3$ be the solution of

$$\begin{cases} \Delta \vec{v}_k = \nabla \widetilde{\vec{n}}_k \cdot \nabla^\perp Y_k & \text{in } B_{\alpha R_k}(0) \\ \quad \vec{v}_k = 0 & \text{on } \partial B_{\alpha R_k}(0). \end{cases} \tag{6.8}$$

By scaling invariance and the inequality of Lemma 5.4, we deduce by (6.7) that for some universal constant $\kappa_2 > 0$

$$\begin{aligned} \|\nabla \vec{v}_k\|_{\mathrm{L}^{2,1}(B_{\alpha R_k}(0))} &\leq \kappa_2 \left\|\nabla \widetilde{\vec{n}}_k\right\|_{\mathrm{L}^{2,1}(B_{\alpha R_k}(0))} \|\nabla Y_k\|_{\mathrm{L}^{2,\infty}(B_{\alpha R_k}(0))} \\ &\leq \kappa_2 \kappa_0(n)\, \|\nabla \vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))} \|\nabla Y_k\|_{\mathrm{L}^{2,\infty}(B_{\alpha R_k}(0))} \leq \widetilde{l}(\sigma_k)\; \|\nabla \vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}. \end{aligned} \tag{6.9}$$

Furthermore, we have by Lemma 5.4 and scaling invariance

$$\begin{aligned} \|\nabla \vec{v}_k\|_{\mathrm{L}^2(B_{\alpha R_k}(0))} &\leq \kappa_3 \left\|\nabla \widetilde{\vec{n}}_k\right\|_{\mathrm{L}^2(B_{\alpha R_k}(0))} \|\nabla Y_k\|_{\mathrm{L}^{2,\infty}(B_{\alpha R_k}(0))} \leq \kappa_3 \kappa_0(n)\, \|\nabla \vec{n}_k\|_{\mathrm{L}^2(\Omega_k(\alpha))}\, o(\widetilde{l}(\sigma_k)) \\ &\leq \widetilde{l}(\sigma_k)\, \|\nabla \vec{n}_k\|_{\mathrm{L}^2(\Omega_k(\alpha))} \end{aligned} \tag{6.10}$$

Now, recall that the Codazzi identity ([35], III.58) implies that

$$\mathrm{div}\left(e^{-2\lambda_k} \sum_{j=1}^{2} \mathbb{I}_{2,j} \partial_{x_j} \vec{\Phi}_k, -e^{-2\lambda_k} \sum_{j=1}^{2} \mathbb{I}_{1,j} \partial_{x_j} \vec{\Phi}_k\right) = 0 \qquad \text{in } B_{\alpha R_k}(0) \tag{6.11}$$

Therefore, by the Poincaré Lemma, there exists $\vec{D}_k : B_{\alpha R_k}(0) \underset{k\to\infty}{\longrightarrow} \mathbb{R}^3$ such that

$$\nabla \vec{D}_k = \left(e^{-2\lambda_k} \sum_{j=1}^{2} \mathbb{I}_{1,j} \partial_{x_j} \vec{\Phi}_k, e^{-2\lambda_k} \sum_{j=1}^{2} \mathbb{I}_{2,j} \partial_{x_j} \vec{\Phi}_k\right).$$

Notice that we have the trivial estimate

$$\left\|\nabla \vec{D}_k\right\|_{\mathrm{L}^2(B_{\alpha_k}(0))} \leq 2\, \|\nabla \vec{n}_k\|_{\mathrm{L}^2(B_{\alpha_k}(0))} \leq 2\sqrt{\Lambda}. \tag{6.12}$$

Furthermore,

$$l(\sigma_k) \left\|\nabla \vec{D}_k\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))} \leq 2\, l(\sigma_k)\; \|\nabla \vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}. \tag{6.13}$$

Now, let $\vec{E}_k : B_{\alpha R_k}(0) \to \mathbb{R}^3$ be the solution of

$$\begin{cases} \Delta \vec{E}_k = 2\, \nabla(l(\sigma_k)\omega_k) \cdot \nabla^\perp \vec{D}_k & \text{in } B_{\alpha R_k}(0) \\ \quad \vec{E}_k = 0 & \text{on } \partial B_{\alpha R_k}(0). \end{cases}. \tag{6.14}$$

The improved Wente estimate, the scaling invariance and the estimates (6.1) and (6.12) imply that

$$\begin{aligned} &\left\|\nabla \vec{E}_k\right\|_{\mathrm{L}^{2,1}(B_{\alpha R_k}(0))} \leq 2\kappa_0\, l(\sigma_k)\, \|\nabla \omega_k\|_{\mathrm{L}^2(B_{\alpha R_k}(0))} \left\|\nabla \vec{D}_k\right\|_{\mathrm{L}^2(B_{\alpha R_k}(0))} \leq 4\kappa_0 \sqrt{\Lambda}\, o(\sqrt{l(\sigma_k)}) \leq \sqrt{l(\sigma_k)} \\ &\left\|\nabla \vec{E}_k\right\|_{\mathrm{L}^2(B_{\alpha R_k}(0))} \leq \frac{1}{2}\sqrt{\frac{3}{\pi}}\, l(\sigma_k)\, \|\nabla \omega_k\|_{\mathrm{L}^2(B_{\alpha R_k}(0))} \left\|\nabla \vec{D}_k\right\|_{\mathrm{L}^2(B_{\alpha R_k}(0))} \leq \sqrt{l(\sigma_k)}. \end{aligned} \tag{6.15}$$

Now, let $\vec{F}_k : B_{\alpha R_k}(0) \to \mathbb{R}^3$ be such that

$$2\omega_k\, l(\sigma_k)\, \nabla^{\perp}\vec{D}_k = \nabla^{\perp}\vec{F}_k + \nabla\vec{E}_k.$$

Combining (6.13), (6.15), and recalling that $l(\sigma_k)\,\|\omega_k\|_{\mathrm{L}^{\infty}(B_{\alpha R_k}(0))} = o(\widetilde{l}(\sigma_k))$ (by (6.1)), we deduce that

$$\begin{aligned}\left\|\vec{F}_k\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))} &\leq 2\, l(\sigma_k)\,\|\omega_k\|_{\mathrm{L}^{\infty}(\Omega_k(\alpha))}\left\|\nabla\vec{D}_k\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))} + \left\|\nabla\vec{E}_k\right\|_{\mathrm{L}^{2,1}(B_{\alpha R_k}(0))}\\ &\leq \widetilde{l}(\sigma_k)\,\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))} + \sqrt{l(\sigma_k)}. \end{aligned} \tag{6.16}$$

Finally, let $\vec{w}_k : B_{\alpha R_k}(0) \to \mathbb{R}^3$ be the solution of

$$\begin{cases} \Delta\vec{w}_k = \nabla\widetilde{\vec{n}}_k \cdot \nabla^{\perp}\left(\vec{v}_k - \vec{E}_k\right) & \text{in } B_{\alpha R_k}(0)\\ \vec{w}_k = 0 & \text{on } \partial B_{\alpha R_k}(0). \end{cases}$$

As previously, the improved Wente implies that

$$\begin{aligned}\|\nabla\vec{w}_k\|_{\mathrm{L}^{2,1}(B_{\alpha R_k}(0))} &\leq \kappa_0\left\|\nabla\widetilde{\vec{n}}_k\right\|_{\mathrm{L}^2(B_{\alpha R_k}(0))}\left\|\nabla(\vec{v}_k - \vec{E}_k)\right\|_{\mathrm{L}^2(B_{\alpha R_k}(0))}\\ &\leq \kappa_0\left\|\nabla\widetilde{\vec{n}}_k\right\|_{\mathrm{L}^2(\Omega_k(\alpha))}\left(\|\nabla\vec{v}_k\|_{\mathrm{L}^2(B_{\alpha R_k}(0))} + \left\|\nabla\vec{E}_k\right\|_{\mathrm{L}^2(B_{\alpha R_k}(0))}\right)\\ &\leq \kappa_0\kappa(n)\,\|\nabla\vec{n}_k\|_{\mathrm{L}^2(\Omega_k(\alpha))}\left(\widetilde{l}(\sigma_k)\,\|\nabla\vec{n}_k\|_{\mathrm{L}^2(\Omega_k(\alpha))} + \sqrt{l(\sigma_k)}\right)\\ &\leq \kappa_0\kappa(n)\sqrt{\Lambda}\left(\widetilde{l}(\sigma_k)\sqrt{\Lambda} + \sqrt{l(\sigma_k)}\right) \leq \widetilde{l}(\sigma_k) \end{aligned} \tag{6.17}$$

for k large enough. Finally, if $\vec{Z}_k : \Omega_k(\alpha) \to \mathbb{R}^3$ satisfies

$$\nabla^{\perp}\vec{Z}_k = \vec{n}_k \times \nabla^{\perp}\left(\vec{v}_k - \vec{E}_k\right) - \nabla\vec{w}_k,$$

the estimates (6.9), (6.15), (6.17) show that (as $\widetilde{\vec{n}}_k = \vec{n}_k$ on $\Omega_k(\alpha)$)

$$\left\|\nabla\vec{Z}_k\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))} \leq \widetilde{l}(\sigma_k)\,\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))} + \sqrt{l(\sigma_k)} + \widetilde{l}(\sigma_k). \tag{6.18}$$

Finally, following constants and using the controlled extension $\widetilde{\vec{n}}_k$ of $\vec{n}_k$, we deduce as in [35] (see (VI.75)) that

$$\begin{aligned}&\left\|2\left(1 + 2\sigma_k^2\left(1 + H_k^2\right) - l(\sigma_k)\omega_k\right)e^{\lambda_k}\vec{H}_k + \left(\nabla\vec{v}_k + \nabla^{\perp}\left(\vec{F}_k + \vec{Z}_k\right)\right) \times \nabla\vec{\Phi}_k\, e^{-\lambda_k} + l(\sigma_k)\nabla^{\perp}\vec{D}_k \cdot \nabla\vec{\Phi}_k\, e^{-\lambda_k}\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}\\ &\leq \kappa_4(n)e^{\kappa_4(n)\Lambda}\,\|\nabla\vec{n}_k\|_{\mathrm{L}^2(\Omega_k(2\alpha))}. \end{aligned} \tag{6.19}$$

Furthermore, as $l(\sigma_k)\,\|\omega_k\|_{\mathrm{L}^{\infty}(\Omega_k(\alpha))} = o(\widetilde{l}(\sigma_k))$, we have $2(1 + 2\sigma_k^2(1 + H_k^2) - l(\sigma_k)\omega_k) \geq 1$ for k large enough and by the estimates (6.9), (6.13), (6.16), (6.18), (6.19), we deduce that

$$\begin{aligned}&\left\|e^{\lambda_k}\vec{H}_k\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))} \leq \kappa_4(n)e^{\kappa_4(n)\Lambda}\,\|\nabla\vec{n}_k\|_{\mathrm{L}^2(\Omega_k(2\alpha))} + \widetilde{l}(\sigma_k)\,\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}\\ &+ \widetilde{l}(\sigma_k)\,\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))} + \sqrt{l(\sigma_k)} + \widetilde{l}(\sigma_k)\,\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))} + \sqrt{l(\sigma_k)} + \widetilde{l}(\sigma_k) + 2\, l(\sigma_k)\,\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}\\ &\leq \kappa_4(n)e^{\kappa_4(n)\Lambda}\,\|\nabla\vec{n}_k\|_{\mathrm{L}^2(\Omega_k(2\alpha))} + 5\,\widetilde{l}(\sigma_k)\,\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))} + 3\,\widetilde{l}(\sigma_k). \end{aligned} \tag{6.20}$$

Thanks to the proof of Theorem 3.1 and (6.20), we have

$$\begin{aligned}\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha/2))} &\leq \kappa_5(n)e^{\kappa_5(n)\Lambda}\left(\|\nabla\vec{n}_k\|_{\mathrm{L}^2(\Omega_k(\alpha))} + \left\|e^{\lambda_k}\vec{H}_k\right\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))}\right)\\ &\leq \kappa_6(n)e^{\kappa_6(n)\Lambda}\,\|\nabla\vec{n}_k\|_{\mathrm{L}^2(\Omega_k(2\alpha))} + 5\,\widetilde{l}(\sigma_k)\,\|\nabla\vec{n}_k\|_{\mathrm{L}^{2,1}(\Omega_k(\alpha))} + 3\,\widetilde{l}(\sigma_k). \end{aligned} \tag{6.21}$$

Furthermore, thanks to the ε-regularity ([28]), we obtain

$$\begin{aligned}
&\|\nabla\vec{n}_k\|_{L^{2,1}(B_{\alpha R_k}(0)\setminus\overline{B}_{\alpha R_k/2}(0))} \leq \kappa_7(n)\,\|\nabla\vec{n}_k\|_{L^2(B_{2\alpha R_k}\setminus\overline{B}_{\alpha R_k/4}(0))}\\
&\|\nabla\vec{n}_k\|_{L^{2,1}(B_{2\alpha^{-1}r_k}\setminus\overline{B}_{\alpha^{-1}r_k}(0))} \leq \kappa_7(n)\,\|\nabla\vec{n}_k\|_{L^2(B_{4\alpha^{-1}r_k}\setminus\overline{B}_{\alpha^{-1}r_k/2}(0))}\,.
\end{aligned} \tag{6.22}$$

Finally, by (6.21) and (6.22), we have

$$\|\nabla\vec{n}_k\|_{L^{2,1}(\Omega_k(\alpha))} \leq \kappa_8(n)e^{\kappa_8(n)\Lambda}\,\|\nabla\vec{n}_k\|_{L^2(\Omega_k(2\alpha))} + 5\,\widetilde{l}(\sigma_k)\,\|\nabla\vec{n}_k\|_{L^{2,1}(\Omega_k(\alpha))} + 3\,\widetilde{l}(\sigma_k),$$

which directly implies as $\widetilde{l}(\sigma_k) \underset{k\to\infty}{\longrightarrow} 0$ that for k large enough

$$\|\nabla\vec{n}_k\|_{L^{2,1}(\Omega_k(\alpha))} \leq 2\kappa_8(n)e^{\kappa_8(n)\Lambda}\,\|\nabla\vec{n}_k\|_{L^2(\Omega_k(2\alpha))}$$

and the improved no-neck energy

$$\lim_{\alpha\to 0}\limsup_{k\to\infty}\|\nabla\vec{n}_k\|_{L^{2,1}(\Omega_k(\alpha))} = 0.$$

This concludes the proof of the Theorem. ☐

We close this article with a short appendix concerning Lorentz spaces.

7 Appendix

7.1 Some basic properties of Lorentz spaces

Fix a measured space (X,μ). Define for all $0 < t < \infty$ the measurable function f_* on $(0,\infty)$ by

$$f_*(t) = \inf\left\{\lambda > 0 : \mu(X\cap\{x : |f(x)| > \lambda\}) \leq t\right\}$$

and recall that for all $\lambda > 0$

$$\mathscr{L}^1\left((0,\infty)\cap\{t : f_*(t) > \lambda\}\right) = \mu\left(X\cap\{x : |f(x)| > \lambda\}\right).$$

In particular, using twice the usual slicing formula (valid for an arbitrary measure μ that need not be σ-finite), we find

$$\begin{aligned}
\|f\|_{L^p(X,\mu)} &= p\int_0^\infty \lambda^p\mu\left(X\cap\{x : |f(x)| > \lambda\}\right)\frac{d\lambda}{\lambda} = p\int_0^\infty \lambda^p\mathscr{L}^1\left((0,\infty)\cap\{t : f_*(t) > \lambda\}\right)\frac{d\lambda}{\lambda}\\
&= \int_0^\infty f_*^p(t)dt = \|f_*\|_{L^p((0,\infty),\mathscr{L}^1)}\,.
\end{aligned}$$

To simplify notations we will often remove the reference to the measure μ. This motivates the introduction of the following quasi-norm for $1 < p < \infty$ and $1 \leq q < \infty$

$$|f|_{L^{p,q}(X)} = \left(\int_0^\infty t^{\frac{q}{p}} f_*^q(t)\frac{dt}{t}\right)^{\frac{1}{q}}. \tag{7.1}$$

If we define $f_{**}(t) = \dfrac{1}{t}\displaystyle\int_0^t f_*(s)ds$, then the associated norm to $L^{p,q}$ is

$$\|f\|_{L^{p,q}(X)} = \left(\int_0^\infty t^{\frac{q}{p}} f_{**}^q(t)\frac{dt}{t}\right)^{\frac{1}{q}}, \tag{7.2}$$

and $(L^{p,q}(X,\mu),\|\cdot\|_{L^{p,q}(X)})$ is a Banach space for all $1 < p < \infty$ and $1 \leq q < \infty$. Now, we have by Fubini's theorem for all $f \in L^{p,q}(X,\mu)$

$$\|f\|_{L^{p,1}(X)} = \int_0^\infty t^{\frac{1}{p}} f_{**}(t)\frac{dt}{t} = \int_0^\infty\int_0^\infty t^{\frac{1}{p}-2}f_*(s)\mathbf{1}_{\{0<s<t\}}dsdt = \int_0^\infty f_*(s)\left(\int_0^\infty t^{\frac{1}{p}-2}\mathbf{1}_{\{0<s<t\}}dt\right)ds$$

$$= \int_0^\infty f_*(s) \left(\int_s^\infty t^{\frac{1}{p}-2} dt \right) ds = \frac{p}{p-1} \int_0^\infty s^{\frac{1}{p}-1} f_*(s) ds = \frac{p}{p-1} |f|_{\mathrm{L}^{p,1}(X)} .$$

Therefore, $|\cdot|_{\mathrm{L}^{p,1}(X)}$ is a norm for all $1 < p < \infty$. Furthermore, notice that Fubini's theorem also shows ([30]) that

$$|f|_{\mathrm{L}^{p,q}(X)} = p^{\frac{1}{q}} \left(\int_0^\infty \lambda^q \mu \left(X \cap \{x : |f(x)| > \lambda\} \right)^{\frac{q}{p}} \frac{d\lambda}{\lambda} \right)^{\frac{1}{q}} . \tag{7.3}$$

In particular, for $q = 1$ each of the quantities (7.1), (7.2) and (7.3) defines a norm on $L^{p,1}(X, \mu)$. Finally, for $q = \infty$, we define the quasi-norm

$$|f|_{\mathrm{L}^{p,\infty}(X)} = \sup_{\lambda > 0} t \left(\mu \left(X \cap \{x : |f(x)| > \lambda\} \right) \right)^{\frac{1}{p}} = \sup_{t > 0} t^{\frac{1}{p}} f_*(t)$$

and the norm

$$\|f\|_{\mathrm{L}^{p,\infty}(X)} = \sup_{t > 0} t^{\frac{1}{p}} f_{**}(t)$$

makes $\left(L^{p,\infty}(X), \|\cdot\|_{\mathrm{L}^{p,\infty}(X)}\right)$ a Banach space (they are the classical Marcinkiewicz weak L^p spaces). Notice however that $L^{1,\infty}$ is *not* a Banach space. We have the general inequality for all $1 < p < \infty$

$$|f|_{\mathrm{L}^{p,\infty}(X)} \leq \|f\|_{\mathrm{L}^{p,\infty}(X)} \leq \frac{p}{p-1} |f|_{\mathrm{L}^{p,\infty}(X)} .$$

The norms are related as follows (see [30]).

Lemma 7.1. *For all $1 < p < \infty$ and $1 \leq q \leq r < \infty$, and for all $f \in L^{p,q}(X, \mu)$ we have*

$$|f|_{\mathrm{L}^{p,\infty}(X)} \leq \left(\frac{q}{p} \right)^{\frac{1}{q}} |f|_{\mathrm{L}^{p,q}(X)}$$
$$|f|_{\mathrm{L}^{p,r}(X)} \leq \left(\frac{q}{p} \right)^{\frac{r-q}{rq}} |f|_{\mathrm{L}^{p,q}(X)} .$$

Proof. As f_* is decreasing, we have for all $0 < t < \infty$

$$\begin{aligned} t^{\frac{1}{p}} f_*(t) &= \left(\frac{q}{p} \int_0^t s^{\frac{q}{p}} f_*^q(t) \frac{ds}{s} \right)^{\frac{1}{q}} \leq \left(\frac{q}{p} \int_0^t s^{\frac{q}{p}} f_*^q(s) \frac{ds}{s} \right)^{\frac{1}{q}} \\ &\leq \left(\frac{q}{p} \right)^{\frac{1}{q}} |f|_{\mathrm{L}^{p,q}(X)} , \end{aligned}$$

which implies that for all $1 \leq q < \infty$

$$\|f\|_{\mathrm{L}^{p,\infty}(X)} \leq \left(\frac{q}{p} \right)^{\frac{1}{q}} |f|_{\mathrm{L}^{p,q}(X)} . \tag{7.4}$$

Now, assume that $1 \leq q < r < \infty$. Then (7.4) implies that

$$\begin{aligned} |f|_{\mathrm{L}^{p,r}(X)} &= \left(\int_0^\infty t^{\frac{r}{p}} f_*^r(t) \frac{dt}{t} \right)^{\frac{1}{r}} = \left(\int_0^\infty t^{\frac{q}{p}} f_*^q(t) t^{\frac{r-q}{p}} f_*(t)^{r-q} \frac{dt}{t} \right)^{\frac{1}{r}} \leq |f|_{\mathrm{L}^{p,\infty}(X)}^{\frac{r-q}{r}} |f|_{\mathrm{L}^{p,q}(X)}^{\frac{q}{r}} \\ &\leq \left(\frac{q}{p} \right)^{\frac{r-q}{rq}} |f|_{\mathrm{L}^{p,q}(X)} . \end{aligned} \tag{7.5}$$

This concludes the proof of the Lemma. $\square$

In particular, if $q=1$, as $\|f\|_{\mathrm{L}^{p,1}(X)}=\frac{p}{p-1}\,|f|_{\mathrm{L}^{p,1}(X)}$, we deduce by (7.5) that

$$|f|_{\mathrm{L}^{p,r}(X)}\leq\frac{p-1}{p}\left(\frac{1}{p}\right)^{\frac{r-1}{r}}\|f\|_{\mathrm{L}^{p,1}(X)}=\frac{p-1}{p^{2-\frac{1}{r}}}\|f\|_{\mathrm{L}^{p,1}(X)}\leq\|f\|_{\mathrm{L}^{p,1}(X)}\,.$$

In particular as $|\cdot|_{\mathrm{L}^{p,p}(X)}=\|\cdot\|_{\mathrm{L}^{p}(X)}$, we have

$$\|f\|_{\mathrm{L}^{p}(X)}\leq\frac{p-1}{p^{2-\frac{1}{p}}}\|f\|_{\mathrm{L}^{p,1}(X)}$$

Notice that for $p=2$. this yields

$$\|f\|_{\mathrm{L}^{2}(X)}\leq\frac{1}{2\sqrt{2}}\|f\|_{\mathrm{L}^{2,1}(X)}\,. \tag{7.6}$$

Finally, recall the inequality

$$\left|\int_X fg d\mu\right|\leq\int_0^\infty f_\ast(t)g_\ast(t)dt.$$

It implies that for all $1<p<\infty$

$$\int_0^\infty f_\ast(t)g_\ast(t)dt=\int_0^\infty t^{\frac{1}{p}}f_\ast(t)t^{p'}g_\ast(t)\frac{dt}{t}\leq|g|_{\mathrm{L}^{p',\infty}(X)}\int_0^\infty t^{\frac{1}{p}}f_\ast(t)\frac{dt}{t}=|f|_{\mathrm{L}^{p,1}(X)}\,|g|_{\mathrm{L}^{p',\infty}(X)}\,,$$

while for all $1<p<\infty$ and $1\leq q<\infty$, we have by Hölder's inequality (applied to the Haar measure $\nu=\dfrac{dt}{t}$ on $(0,\infty)$)

$$\int_0^\infty f_\ast(t)g_\ast(t)dt=\int_0^\infty t^{\frac{1}{p}}f_\ast(t)t^{\frac{1}{p'}}g_\ast(t)\frac{dt}{t}\leq\left(\int_0^\infty t^{\frac{p}{q}}f_\ast^q(t)\frac{dt}{t}\right)^{\frac{1}{q}}\left(\int_0^\infty t^{\frac{q'}{p'}}g_\ast^{q'}(t)dt\right)^{\frac{1}{q'}}=|f|_{\mathrm{L}^{p,q}(X)}\,|g|_{\mathrm{L}^{p',q'}(X)}\,.$$

Therefore, we have for all $1<p<\infty$ and $1\leq q\leq\infty$

$$\left|\int_X fg\,d\mu\right|\leq|f|_{\mathrm{L}^{p,q}(X)}\,|g|_{\mathrm{L}^{p',q'}(X)}\leq\|f\|_{\mathrm{L}^{p,q}(X)}\,\|g\|_{\mathrm{L}^{p',q'}(X)} \tag{7.7}$$

and one shows that for all $1<p<\infty$ and $1\leq q<\infty$, the dual space of $L^{p,q}(X,\mu)$ is $L^{p',q'}(X,\mu)$. In particular, (7.7) implies that for all $1<p<\infty$

$$\|f\|_{\mathrm{L}^{p,1}(X)}=\frac{p^2}{p-1}\int_0^\infty\mu\left(X\cap\{x:|f(x)|>t\}\right)^{\frac{1}{p}}dt$$

The main case of interest in this article is the $L^{2,1}$ norm, which now can be defined as

$$\|f\|_{\mathrm{L}^{2,1}(X)}=4\int_0^\infty\left(\mu\left(X\cap\{x:|f(x)|>t\}\right)\right)^{\frac{1}{2}}dt,$$

and

$$\left|\int_X fg\,d\mu\right|\leq\frac{1}{2}\|f\|_{\mathrm{L}^{2,1}(X)}\,|g|_{\mathrm{L}^{2,\infty}(X)}\leq\|f\|_{\mathrm{L}^{2,1}(X)}\,\|g\|_{\mathrm{L}^{2,\infty}(X)}\,.$$

As

$$\frac{1}{n}\left\|\frac{1}{|x-y|^{n-1}}\right\|_{\mathrm{L}^{\frac{n}{n-1},\infty}(\mathbb{R}^n,\mathscr{L}^n)}=\left|\frac{1}{|x-y|^{n-1}}\right|_{\mathrm{L}^{\frac{n}{n-1},\infty}(\mathbb{R}^n,\mathscr{L}^n)}=\alpha(n)^{\frac{n}{n-1}},$$

we have for all open subset $\Omega\subset\mathbb{R}^n$ and $f\in L^{n,1}(\Omega)$, for all $y\in\mathbb{R}^n$

$$\int_\Omega\frac{|f(x)|}{|x-y|^{n-1}}d\mathscr{L}^n(x)\leq\frac{n-1}{n^2}\|f\|_{\mathrm{L}^{2,1}(\Omega)}\left\|\frac{1}{|x-y|^{n-1}}\right\|_{\mathrm{L}^{\frac{n}{n-1},\infty}(\mathbb{R}^n)}=\frac{n-1}{n}\alpha(n)^{\frac{n-1}{n}}\|f\|_{\mathrm{L}^{n,1}(\Omega)}\,. \tag{7.8}$$

In particular, if $\Omega\subset\mathbb{R}^2$, we have

$$\int_\Omega\frac{|f(x)|}{|x-y|}d\mathscr{L}^2(x)\leq\frac{\sqrt{\pi}}{2}\|f\|_{\mathrm{L}^{2,1}(\Omega)}\,. \tag{7.9}$$

7.2 Extension operators on annuli

The following result was used in [2] and [15].

Lemma 7.2. *Let $n \geq 2$, $\varepsilon > 0$ and $1+\varepsilon < R < \infty$ and $\Omega_R = B_R \setminus \overline{B}_1(0)$ be the associated annulus. Then there exists a linear extension operator*

$$T : \bigcup_{1\leq p<\infty} W^{1,p}(\Omega_R) \to \bigcup_{1\leq p<\infty} W^{1,p}(B_R(0))$$

such that for all $1 \leq p < \infty$, there exists a universal constant $C_1(n,\varepsilon) > 0$ (independent of $R > 1+\varepsilon$) such that for all $1 \leq p < \infty$

$$\|Tu\|_{\mathrm{W}^{1,p}(B_R(0))} \leq C_1(n,\varepsilon)\,\|u\|_{\mathrm{W}^{1,p}(\Omega_R)}\,.$$

Furthermore, for all $1 < p < \infty$ and $1 \leq q \leq \infty$, T extends as a linear operator $W^{1,(p,q)}(\Omega_R) \to W^{1,(p,q)}(B_R(0))$ such that for some universal constant $C_2(n,p,q,\varepsilon)$

$$\|Tu\|_{\mathrm{W}^{1,(p,q)}(B_R(0))} \leq C_2(n,p,q,\varepsilon)\,\|u\|_{\mathrm{W}^{1,(p,q)}(\Omega_R)}\,.$$

Proof. The second assertion follows directly from the Stein-Weiss interpolation theorem ([10], 3.3.3). For the first part, construct by [3], IX.7 a linear extension operator $\widetilde{T}$ such that for all $u \in W^{1,p}(B_{1+\varepsilon}\setminus\overline{B}_1(0))$, $\widetilde{T}u \in W^{1,p}(B_{1+\varepsilon}(0))$ and such that

$$\left\|\widetilde{T}u\right\|_{\mathrm{W}^{1,p}(B_{1+\varepsilon}(0))} \leq C(n,\varepsilon)\,\|u\|_{\mathrm{W}^{1,p}(B_{1+\varepsilon}(0))}\,. \tag{7.10}$$

Now, if $u \in W^{1,p}(B_R(0))$, just consider the restriction $\overline{u}|B_{1+\varepsilon}(0)\setminus\overline{B}_1(0)$, and define

$$Tu(x) = \begin{cases} u(x) & \text{if } x \in B_R \setminus \overline{B}_{1+\varepsilon}(0) \\ \widetilde{T}u(x) & \text{if } x \in B_{1+\varepsilon}(0). \end{cases}$$

As $\widetilde{T}u = u$ on $B_{1+\varepsilon}\setminus\overline{B}_1(0)$, T satisfies the claimed properties by (7.10). ☐

Remark 7.3. Although the norm of the norm of the operator $T : W^{1,p}(\Omega_R) \to W^{1,p}(B_R(0))$ does not depend on $1 < p < \infty$, the norm of $T : W^{1,(p,q)}(\Omega_R) \to W^{1,(p,q)}(B_R(0))$ depends *a priori* on $1 < p < \infty$ and $1 \leq q \leq \infty$, as the constant of the Stein-Weiss interpolation theorem depends on these parameters.

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