

# Energy Quantization of Willmore surfaces at the boundary of the Moduli Space

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## Abstract

We establish an energy quantization result for sequences of Willmore surfaces when the underlying sequence of Riemann surfaces is degenerating in the moduli space. We notably exhibit a new residue which quantifies the potential loss of energy in collar regions. Thanks to these residues, we also establish the compactness (modulo the action of the Möbius group of conformal transformations of  $\mathbb{R}^3 \cup \{\infty\}$ ) of the space of Willmore immersions of any arbitrary closed 2-dimensional oriented manifold into  $\mathbb{R}^3$  with uniformly bounded conformal class and energy below  $12\pi$ .

## Introduction

In the study of concentration compactness phenomena it is a central question to understand “where and in which quantity” the energy dissipates. The first step in analyzing such phenomena consists in looking for  $\epsilon$ -regularity properties. Such a property roughly says that, under a given threshold of energy, the convergence is strong. Once such a property is established, for conformally invariant problems in particular, a covering argument identifies the points where the energy concentrates. The question is then to understand if the whole energy concentrating at these points is given exclusively by the sum of the energies of the so called “bubbles” forming at these points or if there is some additional energy needed to “anchor” these bubbles to the rest of the solutions in the so called “neck region”. “Neck regions” are annuli of degenerating conformal classes separating the bubbles between themselves or separating the bubbles with the macroscopic solution.

The Willmore energy has been introduced in the XIX century in non linear mechanics as being the *ad hoc* modelization of the free energy of a bent two dimensional elastic membrane. It was then independently introduced in geometry by Wilhelm Blaschke around 1920 in an effort to merge minimal surfaces theory and conformal invariance. If  $\vec{\Phi}$  denotes the immersion of an abstract closed surface  $\Sigma$  into an euclidian space  $\mathbb{R}^m$ , the Willmore energy of such an immersion is given by

$$W(\vec{\Phi}) := \int_{\Sigma} |\vec{H}_{\vec{\Phi}}|^2 dvol_{g_{\vec{\Phi}}}$$

where  $g_{\vec{\Phi}}$  is the first fundamental form of the immersion (i.e. the induced metric by  $\vec{\Phi}$ )  $dvol_{g_{\vec{\Phi}}}$  is the associated volume form and  $\vec{H}_{\vec{\Phi}} := 2^{-1} \text{tr}_{g_{\vec{\Phi}}} \vec{\mathbb{I}}_{\vec{\Phi}}$  is the half of the trace of the second fundamental form  $\vec{\mathbb{I}}_{\vec{\Phi}}$  of  $\vec{\Phi}$ . Blaschke proved that the lagrangian  $W$  is invariant under conformal transformations for closed

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surfaces. That is to say, for any generic<sup>1</sup> element  $\Psi$  in  $\mathcal{M}(\mathbb{R}^m \cup \{\infty\})$ , the Möbius group of conformal transformations of  $\mathbb{R}^m \cup \{\infty\} \simeq S^m$

$$W(\Psi \circ \vec{\Phi}) = W(\vec{\Phi}) \quad .$$

During a long time the minimal surfaces and their conformal transformations were the only known critical points of  $W$ . One of the reasons for such a lack of examples and progresses during almost 45 years is possibly due to the fact that the Euler-Lagrange equation of  $W$  is a non-linear elliptic system of order 4 which made it difficult to be studied from an analyst perspective at a time where this high order PDE theory was not much developed<sup>2</sup>. The seminal paper of Tom Willmore ([42]) has relaunched the interest for the lagrangian to which his name has since then been given.

In the present work we are interested in sequences of immersions  $\vec{\Phi}_k$  of a given closed surface  $\Sigma$  into  $\mathbb{R}^m$  which are critical points of  $W$  and which are below a given energy level. It has been proven in [34] and [35] that critical points to  $W$  are satisfying an  $\epsilon$ -regularity property (see theorem 1.3) and that, under the assumption that the conformal class of the metrics  $g_{\vec{\Phi}_k}$  is controlled, the sequence, modulo the action of the Möbius group  $\mathcal{M}(\mathbb{R}^m \cup \{\infty\})$ , in a sequence of conformal parametrizations<sup>3</sup>, is compact in any  $C^l$  norm away from finitely many points. The iteration of this result at the various concentration points generates a bubble tree of Willmore surfaces.

In [22] the authors established an estimate on the Green function of the Laplace operator of any degenerating sequence of constant Gauss curvature metric which permits to extend the previously mentioned concentration compactness result for Willmore surfaces to the case where the underlying conformal classes degenerate. The difficult question to know whether or not some energy could “dissipate” in the “neck regions” of the limiting “Willmore bubble tree” was left open in this degenerating case. The following theorem, which is the main result of the present work, asserts that, if there is some loss of energy in a collar region, the amount of this loss has an explicit expression in terms of a residue and the hyperbolic length of the collar. It can be seen as a counterpart of a result of Zhu [43] for harmonic maps. Let us define this residue. Without loss of generality we can assume that the collar region<sup>4</sup> is conformally parametrized by  $\mathbb{D} \setminus B(0, e^{-1/l_k})$  where  $l_k$  is the length of shrinking geodesic which corresponds to the circle of radius  $e^{-1/2l_k}$ . For  $\vec{\Phi}_k : \mathbb{D} \setminus B(0, e^{-1/l_k}) \rightarrow \mathbb{R}^m$  a Willmore immersion we set, for  $e^{-1/l_k} < r < 1$ ,

$$2\pi\bar{c}^k = \int_{\partial B(0,r)} \partial_\nu \vec{H}_k - 3\pi_{\vec{n}_k}(\partial_\nu \vec{H}_k) - \star(\partial_\tau \vec{n}_k \wedge \vec{H}_k) d\sigma \quad (1)$$

where  $\nu$  and  $\tau$  are respectively a unit normal and a unit tangent to  $\partial B(0, r)$  such that  $(\nu, \tau)$  is oriented, and  $\vec{n}_k$  and  $\vec{H}_k$  are respectively the normal  $(m-2)$ -vector and the mean curvature vector of  $\vec{\Phi}_k$ . The operations  $\star, \lrcorner$  and  $\wedge$  are classical operations on multi-vectors that we recall in the first part of the paper. We also set

$$2\pi c_0^k = \int_{\gamma_k^i} -\langle (\partial_{\nu_k} \vec{H}_k - 3\pi_{\vec{n}_k}(\partial_{\nu_k} \vec{H}_k) - \star(\partial_{\tau_k} \vec{n}_k \wedge \vec{H}_k)), \vec{\Phi}_k \rangle d\sigma \quad (2)$$

and

$$2\pi\bar{c}_1^k = \int_{\gamma_k^i} -(\partial_{\nu_k} \vec{H}_k - 3\pi_{\vec{n}_k}(\partial_{\nu_k} \vec{H}_k) - \star(\partial_{\tau_k} \vec{n}_k \wedge \vec{H}_k)) \wedge \vec{\Phi}_k - (-1)^{m-1} 2(\star(\vec{n}_k \lrcorner \vec{H}_k)) \lrcorner \partial_{\tau_k} \vec{\Phi}_k d\sigma. \quad (3)$$

<sup>1</sup>“Generic” means that  $\Psi^{-1}(\infty) \cap \vec{\Phi}(\Sigma) = \emptyset$ .

<sup>2</sup>Indeed, in a conformal parametrization  $\vec{\Phi}$ , the Willmore functional may be recast as

$$W(\vec{\Phi}) = \frac{1}{4} \int_{\Sigma} |\Delta_{g_{\vec{\Phi}}} \vec{\Phi}|^2 dvol_{g_{\vec{\Phi}}} \quad ,$$

thereby giving rise to a fourth-order problem

<sup>3</sup>This sequence is arbitrary as long as  $\Sigma \neq S^2$  otherwise there is an *ad hoc* choice of gauge in  $\mathcal{M}(S^2)$  which has to be made (see [28]).

<sup>4</sup>see section 1.4 for details

The quantities  $\vec{c}$ ,  $c_0$  and  $\vec{c}_1$  are independent of  $r$  as shown in section 3.1. The quantity  $\vec{c}$  already appear in [34] and [3] as a residue which permits to erase true branch point of Willmore punctured disc( see also [19]). But it is not conformally invariant, even not scaling invariant. For instance  $\vec{c}$  is zero when compute on the closed geodesic of a catenoid, but not on the corresponding curve on the inverted catenoid, see Remark 1.1 [3]. This remark have a very important consequence in the bubbling phenomena, see remark 0.1 below and theorem 0.2. Moreover, it is hopeless to try to find a residue, like  $\vec{c}$ , which measures a defect of regularity and will be conformally invariant. Taking again the example of the inverted catenoid, we can blow-up it to a union of to plane, if we get a scaling invariant residue, it must vanishes on the inverted catenoid, since the blowdo-up is smooth. Hence this residue not detect the defect of regularity

But  $c_0$  and  $\vec{c}_1$  are clearly invariant under the composition by isometries and dilations. They can be considered as the Willmore analogue of flux for CMC-surfaces (see [26]). Since the Willmore equation is fourth order, it is not surprising to get two "fluxes". Moreover we can check, see section A for details, that there exit Willmore surfaces for which those residues are non zero. Such examples are provided by considering Willmore Hopf tori, see [30]. The cancellation of the residue  $\vec{c}_1$  forces the generating curve (an elastica on  $S^2$ ) to be a geodesic of  $S^2$ , hence the surface will be equivalent to a Clifford torus. Therefore the family of Hopf tori which are not necessary Clifford and produced by Pinkall in the above mentioned work, provides good examples. More generally, it is easy to prove that our residues vanishes on any minimal surfaces of  $S^m$ . Hence it provides a new way to detect non-minimal Willmore surfaces.

Those quantities being defined we can state our main result.

**Theorem 0.1.** *Let  $(\Sigma, h_k)$  a sequence of closed surfaces with fixed genus, constant curvature and normalized volume if needed. We assume that this sequence converges<sup>5</sup> to a nodal surface  $(\tilde{\Sigma}, \tilde{h})$  and we denote by  $\{\gamma_k^i\}$  the finite number of pinching geodesics. Then let  $\vec{\Phi}_k : (\Sigma, h_k) \rightarrow \mathbb{R}^m$  a sequence of conformal Willmore immersions with bounded energy, i.e.*

$$\limsup_{k \rightarrow +\infty} W(\vec{\Phi}_k) < +\infty$$

and such that, around every degenerating geodesic,

$$\lim_{k \rightarrow +\infty} \frac{\vec{c}_1^k}{\sqrt{l_k}} = 0.$$

Then, denoting  $(\tilde{\Sigma}^l)_{1 \leq l \leq q}$  the connected components of  $\tilde{\Sigma}$ , there exists  $q$  branched smooth immersions  $\vec{\Phi}_\infty^l : \tilde{\Sigma}^l \rightarrow \mathbb{R}^m$  and a finite number of possibly branched immersions  $\omega_j : S^2 \rightarrow \mathbb{R}^m$  and  $\zeta_t : S^2 \rightarrow \mathbb{R}^m$  which are all Willmore away from possibly finitely many points, and such that, up to a subsequence,

$$\lim_{k \rightarrow +\infty} W(\vec{\Phi}_k) = \sum_{l=1}^q W(\vec{\Phi}_\infty^l) + \sum_{j=1}^p W(\omega_j) + \sum_{t=1}^q (W(\zeta_t) - m_t 4\pi) \quad . \quad (4)$$

where  $m_t$  is the integer multiplicity of  $\zeta_t$  at the origin. □

**Remark 0.1.** *In [2] the second author and Bernard established the corresponding result but under the additional assumption that the conformal class induced by the sequence was pre-compact in the moduli space  $\mathcal{M}(\Sigma)$ . Under this much stronger assumptions the branched immersions  $\omega_j$  and  $\zeta_t$  are "true" Willmore surfaces in the sense that the Willmore equation is satisfied everywhere away from the branched points and moreover the first residue,  $\vec{c}$ , is zero on any curve surrounding these branched points. This excludes surfaces like the catenoid (or its inversion) in the bubble tree, but not the Enneper surface for instance.*

<sup>5</sup>The convergence holds in the classical sense of Mumford compactification recalled in [15].

This observation is a starting point for improving the classical bound of  $8\pi$  for having compactness, see theorem 0.2. In theorem 0.1 instead we cannot exclude a-priori the first residue to be non zero around the cuspidal point for both the  $\omega_j$  and the  $\zeta_t$   $\square$

The previous remark which excludes catenoid in the bubble trees, permits to increase the level under compactness holds true (Modulo the action of the Möbius group).

**Theorem 0.2.** *Let  $\Sigma$  be a closed surface of genus  $g \geq 1$  and  $\vec{\Phi}_k : \Sigma \rightarrow \mathbb{R}^3$  a sequence of conformal Willmore immersions such that  $[\vec{\Phi}_k^*(\xi)]$ , the conformal class of the pullback metric, remains in a compact set of the moduli space and*

$$\limsup_{k \rightarrow +\infty} W(\vec{\Phi}_k) < 12\pi.$$

*Then, There exists a diffeomorphism  $\Psi_k$  of  $\Sigma$  and an conformal transformation  $\Theta_k$  of  $\mathbb{R}^3 \cup \{\infty\}$ , such that  $\Theta_k \circ \Phi_k \circ \psi_k$ , up to a subsequence, converges to a smooth Willmore immersion  $\Phi_\infty : \Sigma \rightarrow \mathbb{R}^3$  in  $C^\infty(\Sigma)$ .  $\square$*

This result was already known, when  $\Sigma = S^2$ , in fact there is a complete classification of Willmore sphere in  $\mathbb{R}^3$ , see [6], and  $\mathbb{R}^4$ , see [27]. It is also known that we have compactness when  $\Phi_k$  is an embedding, see [23]. But nothing, was know when  $\Phi_k$  is an immersion with energy above  $8\pi$ .

Coming back to theorem 0.1, the new difficulty posed by the non compactness of the underlying conformal classes in comparison with the previous quantization result in [2] will come from the formation of *collars*. By definition, these *collars* are conformally equivalent to degenerating annuli. Unlike the *neck regions* from [2], which are also conformally equivalent to degenerating annuli, the solution is non extendable throughout the interior disc of the annuli. This lack of extendability is responsible for the presence of residues which were automatically zero in the Bernard-Rivière case. The main achievement of the present work is to derive a control of these *residues* in order to prove our main results.

The search for “energy quantization results” of the form above for conformally invariant problems is at the origin of numerous works in geometric analysis. For harmonic maps and it’s heat flow for instance we can quote [38], [40], [11], [29] or for Yang-Mills [31]. However, these problems are all of second-order elliptic or parabolic types. The novelty of [2] was to establish for the first time an energy quantization result and a full bubble-neck decomposition for a fourth-order problem. The proof of the “energy quantization” in [2] was using some of the *integrability compensation* lemma in interpolation spaces (mostly Lorentz spaces - see the subsection 1.3 below) coming from [21] where they have been originally conceived for proving the “energy quantization” property for general conformally invariant second order problems in two dimensions. The use of *integrability by compensation* in interpolation spaces for proving “energy quantization” properties goes back to a work of the second author in collaboration with Fanghua Lin (see [24]).

In [43], Miaomiao Zhu proved that for general sequences of harmonic maps from degenerating Riemann surfaces into a given manifold possessing *harmonic spheres*<sup>6</sup> “energy quantization” usually does not hold. Therefore our result above comes as a little surprise, since we where expected two residues for a fourth order problem, but we are able to prove that if there is some loss it can come only from the second residue  $\vec{c}_1$ .

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<sup>6</sup>Harmonic spheres in a manifold  $N^n$  are non constant harmonic maps from  $S^2$  into  $N^n$ . This space is non empty for instance if  $\pi_2(N^n) \neq 0$  but this condition is not necessary as the example  $N^n = S^3$  shows.

# 1 Preliminaries

## 1.1 Notations

Here we introduce some notations for vector calculus and differential calculus. Since we work on Riemann surfaces, it will be useful to have notation for the rotation of the gradient, so let denotes  $\nabla^\perp = J \circ \nabla$ , where  $J$  is the complex structure. Which gives for  $\mathbb{D}$ ,

$$\nabla^\perp f = -\frac{\partial f}{\partial y} \frac{\partial}{\partial x} + \frac{\partial f}{\partial x} \frac{\partial}{\partial y}$$

Let  $\vec{\Phi} : \Sigma \rightarrow \mathbb{R}^m$  a smooth immersion and  $\vec{e}$  the map which to a point  $p \in \Sigma$  assigns the oriented 2-plane given by the push forward by  $\vec{\Phi}$  of the oriented tangent space  $T_p\Sigma$ . Using a positive orthonormal basis  $(\vec{e}_1, \vec{e}_2)$  of  $\vec{\Phi}_*T_p\Sigma$ , we get

$$\vec{e} = \vec{e}_1 \wedge \vec{e}_2.$$

The Gauss map  $\vec{n}$  assigns the oriented  $m - 2$ -orthogonal plane to  $\vec{e}$ , that is to say

$$\vec{n} = \star \vec{e} = \vec{n}_1 \wedge \cdots \wedge \vec{n}_{m-2},$$

where  $\star$  is the Hodge operator on  $\mathbb{R}^m$ : if  $\alpha \in \bigwedge^p \mathbb{R}^m$  then  $\star \alpha \in \bigwedge^{m-p} \mathbb{R}^m$  such that for all  $\beta \in \bigwedge^{m-p} \mathbb{R}^m$  we get

$$\beta \wedge \star \alpha = \langle \beta, \alpha \rangle \star 1,$$

where  $\star 1$  is the canonical volume form of  $\mathbb{R}^m$ .

We will also need some other operator on  $\bigwedge \mathbb{R}^m$ . First, the contraction operator  $\lrcorner$  : for every choice of  $p$ -,  $q$ - and  $p - q$  vectors, respectively  $\alpha$ ,  $\beta$  and  $\gamma$  the following holds

$$\langle \alpha \lrcorner \beta, \gamma \rangle = \langle \alpha, \beta \wedge \gamma \rangle.$$

Thanks to this operator we can define the projection on the normal bundle as follow, for every  $\vec{w} \in \mathbb{R}^m$  we set

$$\pi_{\vec{n}}(\vec{w}) = (-1)^{m-1} \vec{n} \lrcorner (\vec{n} \lrcorner \vec{w}). \quad (5)$$

Then this operator can be generalised assigning to a pair of  $p$ - and  $q$ - vectors of  $\mathbb{R}^m$  a  $p + q - 2$ - vector of  $\mathbb{R}^m$  such that, for all  $\alpha \in \bigwedge^p \mathbb{R}^m$  and all  $\beta \in \bigwedge^q \mathbb{R}^m$ ,

$$\alpha \bullet \beta = \alpha \lrcorner \beta \quad (6)$$

and,  $\alpha \in \bigwedge^p \mathbb{R}^m$ ,  $\beta \in \bigwedge^q \mathbb{R}^m$  and  $\gamma \in \bigwedge^r \mathbb{R}^m$ ,

$$\alpha \bullet (\beta \wedge \gamma) = (\alpha \bullet \beta) \wedge \gamma + (-1)^{qr} (\alpha \bullet \gamma) \wedge \beta. \quad (7)$$

## 1.2 Weak immersions

Let  $\Sigma$  a smooth compact surface equipped with a reference smooth metric  $g_0$ . One defines the Sobolev spaces  $W^{k,p}(\Sigma, \mathbb{R}^m)$  of measurable maps from  $\Sigma$  into  $\mathbb{R}^m$  into the following way

$$W^{k,p}(\Sigma, \mathbb{R}^m) = \left\{ \vec{f} : \Sigma \rightarrow \mathbb{R}^m \text{ measurable s.t. } \sum_{l=0}^k \int_{\Sigma} |\nabla^l \vec{f}|_{g_0}^p dv_{g_0} < +\infty \right\}.$$

Since  $\Sigma$  is compact it is not difficult to see that this space is independent of the choice of  $g_0$  we have made.

Let  $\vec{\Phi} \in W^{1,\infty}(\Sigma, \mathbb{R}^m)$ , we define  $g_{\vec{\Phi}}$  to be the following symmetric bilinear form

$$g_{\vec{\Phi}}(X, Y) = \langle d\vec{\Phi}(X), d\vec{\Phi}(Y) \rangle,$$

and we shall assume that there exists  $C_{\vec{\Phi}} > 1$  such that

$$C_{\vec{\Phi}}^{-1} g_0(X, X) \leq g_{\vec{\Phi}}(X, X) \leq C_{\vec{\Phi}} g_0(X, X). \quad (8)$$

For such a map, we can define the Gauss map as being the following measurable map in  $L^\infty(\Sigma)$  taking values into the Grassmannian of oriented  $m - 2$ -planes of  $\mathbb{R}^m$ ,

$$\vec{n}_{\vec{\Phi}} = \star \frac{\frac{\partial \vec{\Phi}}{\partial x} \wedge \frac{\partial \vec{\Phi}}{\partial y}}{\left| \frac{\partial \vec{\Phi}}{\partial x} \wedge \frac{\partial \vec{\Phi}}{\partial y} \right|}.$$

We then introduce the space  $\mathcal{E}_\Sigma$  of weak immersions of  $\Sigma$  with bounded second fundamental form as follow:

$$\mathcal{E}_\Sigma = \left\{ \begin{array}{l} \vec{\Phi} \in W^{1,\infty}(\Sigma) \text{ which satisfies (8) for some } C_{\vec{\Phi}} > 0 \\ \text{and } \int_\Sigma |d\vec{n}_{\vec{\Phi}}|_{g_{\vec{\Phi}}}^2 dvol_{\vec{\Phi}} < +\infty \end{array} \right\},$$

where  $g_{\vec{\Phi}} = \vec{\Phi}^* \xi$ .

It is proved in [32] that any weak immersion defines a smooth conformal structure on  $\Sigma$ . Let  $\vec{\Phi} \in \mathcal{E}_\Sigma$ , we denote by  $\pi_{\vec{n}_{\vec{\Phi}}}$  the orthonormal projection of vector in  $\mathbb{R}^m$  onto the  $m - 2$ -plane given by  $\vec{n}_{\vec{\Phi}}$ . With these notations the second fundamental form of the immersion at  $p$  is given by

$$\forall X, Y \in T_p \Sigma \quad \vec{\mathbb{I}}_{\vec{\Phi}}(X, Y) = \pi_{\vec{n}_{\vec{\Phi}}} d^2 \vec{\Phi}(X, Y),$$

and the mean curvature vector of the immersion at  $p$  is given by

$$\vec{H}_{\vec{\Phi}} = \frac{1}{2} \text{tr}_{g_{\vec{\Phi}}}(\vec{\mathbb{I}}_{\vec{\Phi}}).$$

A natural quantity while considering such immersions is the Lagrangian given by the  $L^2$ -norm of the second fundamental form :

$$E(\vec{\Phi}) = \int_\Sigma |\vec{\mathbb{I}}_g|^2 dv_g.$$

An elementary computation, using Gauss-Bonnet formula, gives

$$E(\vec{\Phi}) = \int_\Sigma |\vec{\mathbb{I}}_{\vec{\Phi}}|_{g_{\vec{\Phi}}}^2 dvol_{g_{\vec{\Phi}}} = \int_\Sigma |d\vec{n}_{\vec{\Phi}}|_{g_{\vec{\Phi}}}^2 dvol_{g_{\vec{\Phi}}} = 4W(\vec{\Phi}) - 4\pi\chi(\Sigma),$$

where  $\chi(\Sigma)$  is the Euler characteristic and

$$W(\vec{\Phi}) = \int_\Sigma |\vec{H}_{\vec{\Phi}}|_{g_{\vec{\Phi}}}^2 dvol_{g_{\vec{\Phi}}},$$

is the so called Willmore energy.

### 1.3 Lorentz spaces

Here we recall some classical facts about Lorentz spaces, see [12] for details.

**Definition 1.1.** Let  $\Omega$  be a domain of  $\mathbb{R}^k$ ,  $p \in (1, +\infty)$  and  $q \in [1, +\infty]$ . The Lorentz space  $L^{p,q}(\Omega)$  is the set of measurable functions  $f : \Omega \rightarrow \mathbb{R}$  such that

$$\|f\|_{p,q} := \left( \int_0^{+\infty} \left( t^{\frac{1}{p}} f^*(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < +\infty \text{ if } q < +\infty$$

or

$$\|f\|_{p,\infty} := \sup \left( t^{\frac{1}{p}} f^*(t) \right) \text{ if } q = +\infty$$

where  $f^*$  the decreasing rearrangement of  $|f|$ .

$\|\cdot\|_{p,q}$  happens to be a quasi norm equivalent to a norm for which  $L^{p,q}$  is a Banach space. Each  $L^{p,q}$  may be seen as a deformation of  $L^p$ . For instance, we have the strict inclusions

$$L^{p,1} \subset L^{p,q'} \subset L^{p,q''} \subset L^{p,\infty},$$

if  $1 < q' < q''$ . Moreover,

$$L^{p,p} = L^p.$$

Furthermore, if  $|\Omega|$  is finite, we have that for all  $q$  and  $q'$ ,

$$p > p' \Rightarrow L^{p,q} \subset L^{p',q'}.$$

Finally, for  $p \in (1, +\infty)$  and  $q \in [1, +\infty]$ ,  $L^{\frac{p}{p-1}, \frac{q}{q-1}}$  is the dual of  $L^{p,q}$ .

**Important remarks:** Using the fact that  $f^*(t) = \inf\{s > 0 \text{ s.t. } d_f(s) \leq t\}$  where  $d_f$  is the distribution function of  $|f|$ , we see that the  $L^{2,\infty}$  norm of  $f$  is finite if and only if  $\sup_{t>0} t^2 |\{x \in D \mid |f(x, \cdot)| \geq t\}|$  is finite. Hence we easily get the following important facts:

$$\frac{1}{\rho} \in L^{2,\infty} \tag{9}$$

and, there exists  $C > 0$  such that

$$\frac{|\log(r)|}{C} \leq \left\| \frac{1}{\rho} \right\|_{L^{2,1}(\mathbb{D} \setminus B(0,r))} \leq C |\log(r)|. \tag{10}$$

We will often needs some estimates on the mean of some functions. Thanks to (9) we get the following estimate. If  $f$  is radial then

$$\int_r^1 |f| d\rho \leq \|f\|_{L^{2,1}(\mathbb{D} \setminus B(0,r))} \left\| \frac{1}{\rho} \right\|_{L^{2,\infty}(\mathbb{D} \setminus B(0,r))} = O\left(\|f\|_{L^{2,1}(\mathbb{D} \setminus B(0,r))}\right). \tag{11}$$

### 1.4 Degenerating Riemann surfaces

Here we recall some aspects of the Deligne-Mumford's description of the loss of compactness of the conformal class for a sequence of Riemann surfaces with fixed topology, see [15] for details.

Let  $(\Sigma, c_k)$  a sequence of closed Riemann surface of fixed genus  $g$ . If  $g = 0$  then the conformal class is fixed since there is only one conformal class on the sphere. If  $g = 1$  then, we know that,  $(\Sigma, c_k)$  is conformally equivalent to  $\mathbb{R}^2 / \left( \frac{1}{\sqrt{\Im(v_k)}} \mathbb{Z} \times \frac{v_l}{\sqrt{\Im(v_k)}} \mathbb{Z} \right)$  where  $v_k$  lies in the fundamental domain  $\{z \in \mathbb{C} \text{ s.t. } |\Re(z)| \leq 1/2 \text{ and } |z| \geq 1\}$  of  $\mathbb{H}/\text{PSL}_2(\mathbb{Z})$ , and we say that  $c_k$  degenerates if  $|v_k| \rightarrow +\infty$ .

If  $g \geq 1$ , let  $h_k$  the hyperbolic metric associated with  $c_k$ , then  $(\Sigma, c_k)$  degenerates if there exists a closed geodesic whose length goes to zero. In that case, up to a subsequence, there exists

1. an integer  $N \in \{1, \dots, 3g - 3\}$ ,
2. a sequence  $\mathcal{L}_k = \{\Gamma_k^i; i = 1 \dots N\}$  of finitely many pairwise disjoint simple closed geodesics of  $(\Sigma, h_k)$  with length converging to zero,
3. a closed Riemann surfaces  $(\bar{\Sigma}, \bar{c})$ ,
4. a complete hyperbolic surface  $(\tilde{\Sigma}, \tilde{h})$  with  $2N$  cusps  $\{(q_1^i, q_2^i); i = 1 \dots N\}$  such that  $\tilde{\Sigma}$  has been obtained topologically after removing the geodesic of  $\mathcal{L}_k$  to  $\Sigma$  and after closing each component of the boundary of  $\Sigma \setminus \mathcal{L}_k$  by adding a puncture  $q_l^i$  at each of these component. Moreover  $\bar{\Sigma}$  is topologically equal to  $\tilde{\Sigma}$  and the complex structure defined by  $\tilde{h}$  on  $\tilde{\Sigma} \setminus \{q_l^i\}$  extends uniquely to  $\bar{c}$ . We can also equip  $\bar{\Sigma}$  with a metric  $\bar{h}$  with constant curvature, but not necessarily hyperbolic since the genus of  $\bar{\Sigma}$  can be lower than the one of  $\Sigma$ .

$(\tilde{\Sigma}, \tilde{h})$  is called the nodal surface of the converging sequence and  $(\bar{\Sigma}, \bar{c})$  is its renormalization. These objects are related, in the sense that, there exists a diffeomorphism  $\psi_k : \tilde{\Sigma} \setminus \{q_l^i\} \rightarrow \Sigma \setminus \mathcal{L}_k$  such that  $\tilde{h}_k = \psi_k^* h_k$  converge in  $C_{loc}^\infty$  topology to  $\tilde{h}$ .

## 1.5 Previous results : $\varepsilon$ -regularity and global control of the conformal factor.

The first result has to do with the fact that any weak immersion with  $L^2$ -bounded second fundamental form defines a unique conformal structure, see [32].

**Theorem 1.1.** *Let  $\vec{\Phi}$  be a weak immersion from a surface  $\Sigma$  into  $\mathbb{R}^m$  with  $L^2$ -bounded second fundamental form. Then there exists a constant Gauss curvature metric  $h$  on  $\Sigma$  and a bilipschitz homeomorphism  $\Psi$  of  $\Sigma$  such that  $\vec{\Phi} \circ \Psi$  is a conformal bilipschitz immersion from  $(\Sigma, h)$  into  $\mathbb{R}^m$ . The induced metric  $g_{\vec{\Phi}} := (\vec{\Phi} \circ \Psi)^* g_{\mathbb{R}^m}$  is continuous, moreover this immersion  $\vec{\Phi} \circ \Psi$  is in  $W^{2,2}(\Sigma, \mathbb{R}^m)$  and its Gauss map is in  $W^{1,2}(\Sigma, Gr_{m-2}(\mathbb{R}^m))$ .  $\square$*

Assuming  $\vec{\Phi}$  is conformal from the disc  $\mathbb{D}$  into  $\mathbb{R}^m$  we will denote by  $\lambda$  the conformal factor, i.e.

$$e^\lambda = \left| \frac{\partial \vec{\Phi}}{\partial x_1} \right| = \left| \frac{\partial \vec{\Phi}}{\partial x_2} \right|,$$

and we will denote also by  $\{\vec{e}_1, \vec{e}_2\}$  the orthogonal basis of  $T_{\vec{\Phi}(z)}\Sigma$  given by

$$\vec{e}_i = e^{-\lambda} \frac{\partial \vec{\Phi}}{\partial x_i}.$$

The existence of a conformal structure is a consequence the local estimate established in [13].



**Theorem 1.2.** *There exists a constant  $\varepsilon_0 > 0$  depending only on  $m$  such that for any  $\vec{\Phi}$  weak conformal immersion from the two dimensional disc  $\mathbb{D}$  into  $\mathbb{R}^m$  satisfying*

$$\int_{\mathbb{D}} |\nabla \vec{n}|^2 dz \leq \varepsilon_0$$

then

$$\|\lambda - \bar{\lambda}\|_{L^\infty(D(0,1/2))} \leq C \left( \int_{\mathbb{D}} |\nabla \vec{n}|^2 + \|d\lambda\|_{2,\infty} \right) dz,$$

where  $\bar{\lambda} = \frac{1}{\pi} \int_{\mathbb{D}} \lambda dz$ . □

As mentioned in the introduction the starting result in the analysis of conformally invariant problems is the so called  $\varepsilon$ -regularity. In the present situation this has been proved in [34] (see theorem I.5).

**Theorem 1.3** ( $\varepsilon$ -regularity). *There exists a constant  $\varepsilon_0 > 0$  depending only on  $m$  and for any  $A > 0$  a sequence of positive numbers  $C_l(A) > 0$  for  $l \in \mathbb{N}^*$  such that for any weak conformal immersion  $\vec{\Phi} : \mathbb{D} \rightarrow \mathbb{R}^m$  satisfying*

$$\int_{\mathbb{D}} |\nabla \vec{n}|^2 dz \leq \varepsilon_0$$

and  $\|d\lambda\|_{L^{2,\infty}(\mathbb{D})} \leq A$  then

$$\|\nabla^l \vec{n}_{\vec{\Phi}}\|_{L^\infty(D(0,1/2))} \leq C_l(A) \left( \int_{\mathbb{D}} |\nabla \vec{n}_{\vec{\Phi}}|^2 dz \right)^{\frac{1}{2}}. \quad (12)$$

□

The following result is a consequence of the  $\varepsilon$ -regularity

**Theorem 1.4** (Theorem I.5 of [34]). *There exists a constant  $\varepsilon_0 > 0$  and some constant  $C_m$  for  $m \in \mathbb{N}$ , such that, if  $\vec{\Phi}_k$  is a sequence of conformal Willmore immersion of  $\mathbb{D}$  into  $\mathbb{R}^m$  which satisfies*

$$\int_{\mathbb{D}} |\nabla \vec{n}_k|^2 dz \leq \varepsilon_0 \quad \text{and} \quad \limsup_{k \rightarrow +\infty} \|d\lambda\|_{L^{2,\infty}(\mathbb{D})} < +\infty$$

where  $\vec{n}_k$  is normal associated to  $\vec{\Phi}_k$ , then up to a dilation in the image,  $\vec{\Phi}_k$  converges in  $C_{loc}^2(\mathbb{D})$  to a conformal Willmore immersion  $\vec{\Phi}_\infty$ . □

An other ingredient consists in controlling the conformal factor independently of the conformal class when the  $L^2$ -norm of the second fundamental form is bounded. This correspond to theorem 3.1 of Laurain-Rivière [22].

**Theorem 1.5.** *Let  $(\Sigma, c_k)$  be a sequence of closed Riemann surface of fixed genus greater than one. Let denote  $h_k$  the metric with constant curvature <sup>7</sup> in  $c_k$  and  $\vec{\Phi}_k$  a sequence of weak conformal immersion of  $\Sigma$  into  $\mathbb{R}^m$ , i.e.*

$$\vec{\Phi}_k^* g_{\mathbb{R}^m} = e^{2u_k} h_k,$$

where  $u_k \in L^\infty(\Sigma)$ . Then there exists a finite conformal atlas  $(U_i, \psi_i)$  independent of  $k$  and a positive constant  $C$  depending only on the genus of  $\Sigma$ , such that

$$\|d\lambda_k^i\|_{L^{2,\infty}(V_i)} \leq C W(\vec{\Phi}_k),$$

where  $\lambda_k^i$  is the conformal factor of  $\vec{\Phi}_k \circ \psi_i^{-1}$  in  $V_i = \psi_i(U_i)$ , i.e.  $\lambda_k^i = \frac{1}{2} \log \left| \frac{\partial \vec{\Phi}_k \circ \psi_i^{-1}}{\partial x} \right| = \frac{1}{2} \log \left| \frac{\partial \vec{\Phi}_k \circ \psi_i^{-1}}{\partial y} \right|$ .

□

<sup>7</sup>equal to  $-1, 0$  or  $1$  and volume equal to one in the torus case

## 2 Detecting Bubbles, Necks and Collars.

We consider a sequence  $\vec{\Phi}_k$  in  $\mathcal{E}_\Sigma$  critical points of  $W$  and bounded energy, i.e.

$$\limsup_k W(\vec{\Phi}_k) < +\infty.$$

Let denote  $c_k$  the conformal class defined by  $\vec{\Phi}_k$ . Let  $h_k$  be a constant curvature metric such that  $g_{\vec{\Phi}_k} = e^{\alpha_k} h_k$ . Moreover in the case of genus one, we normalize the area of  $(\Sigma, h_k)$  to be 1.

In the sphere case there is only one conformal class, we can directly apply the main result of Bernard and Rivière [2].

So we don't consider the case  $\Sigma = S^2$  and we shall decompose our surface into thin and thick parts. On the thick parts the metric converges smoothly and the classical theory of [2] applies. The thin parts are conformally equivalent to long cylinders. Then we need an equivalent result to the classical bubble tree decomposition in this context.

### 2.1 Bubble Tree lemma

In this section, we generalized the bubble extraction made in section III of [2] to a collar region.

**Lemma 2.1.** *Let  $l_k \rightarrow +\infty$  and  $\vec{\Phi}_k : S^1 \times [0, l_k] \rightarrow \mathbb{R}^m$  a sequence of Willmore immersions with  $L^2$ -bounded second fundamental form. We assume that there is no concentration at the boundary that is to say for every  $R > 0$ ,  $\vec{\Phi}_k$  and  $\vec{\Phi}_k(\theta, l_k - t)$  converge in  $C^2(S^1 \times [0, R])$ . Then, either*

$$\lim_{R \rightarrow +\infty} \lim_{n \rightarrow +\infty} \sup_{t \in [R, l_k - R]} \int_{S^1 \times [t, t+1]} |\nabla \vec{n}_k|^2 d\theta dt = 0$$

or there exist  $p > 0$ ,  $2p$  sequences of real  $0a_k^1 \leq b_k^1, a_k^2 \leq b_k^2, \dots, a_k^p \leq b_k^p$ , such that

- $\lim_{k \rightarrow +\infty} b_k^i - a_k^i > 1$  for all  $1 \leq i \leq p$ ,
- $\lim_{k \rightarrow +\infty} a_k^{i+1} - b_k^i = +\infty$  or all  $1 \leq i \leq p-1$ ,
- $\lim_{k \rightarrow +\infty} \frac{b_k^i - a_k^i}{l_k} = 0$
- $\lim_{R \rightarrow +\infty} \lim_{n \rightarrow +\infty} \sup_{t \in [b_k^i + R, a_k^{i+1} - R]} \int_{S^1 \times [t, t+1]} |\nabla \vec{n}_k|^2 d\theta dt = 0$  for all  $1 \leq i \leq p-1$

and  $\vec{\Phi}_k^i(\theta, t) = \vec{\Phi}_k\left(\theta, \frac{a_k^i + b_k^i}{2} + t\right)$  satisfies a non trivial<sup>8</sup> energy identity<sup>9</sup> on  $\left[\frac{a_k^i - b_k^i}{2}, \frac{b_k^i - a_k^i}{2}\right]$ . □

**Proof of lemma 2.1:**

Let  $\Gamma = \lim_{R \rightarrow +\infty} \lim_{k \rightarrow +\infty} \sup_{t \in [R, l_k - R]} \int_{S^1 \times [t, t+1]} |\nabla \vec{n}_k|^2 dz$ . Either  $\Gamma = 0$  and there is nothing to extract or  $\Gamma > 0$ . Hence we pick  $t_k \in (0, l_k)$  such that

<sup>8</sup>In the sens that there is at least one bubble.

<sup>9</sup>In the sens of the main theorem of [2].

$$\lim_{k \rightarrow +\infty} t_k = \lim_{k \rightarrow +\infty} l_k - t_k = +\infty$$

and

$$\lim_{k \rightarrow +\infty} \int_{S^1 \times [t_k, t_k+1]} |\nabla \vec{n}_k|^2 dz = \Gamma > 0.$$

Then we consider  $\vec{\Phi}_k^1(\theta, t) = \vec{\Phi}_k(\theta, t_k + t)$ . Since the energy is finite, there is a finite number of concentration points, where the energy identity is satisfied since locally we can apply the main result of Bernard Rivière. So all the points of concentration of  $\vec{\Phi}_k^1$  are contained in  $[-R_1, R_1]$  for some  $R_1 > 0$  and  $\vec{\Phi}_k^1$  converge to some bubble in  $C_{loc}^2((S^1 \times \mathbb{R}) \setminus \{\text{concentration points}\})$ .

Hence, setting  $a_k^1 = t_k - R_1$  and  $b_k^1 = t_k + R_1$ , the boundary hypothesis of the lemma on  $S^1 \times [0, a_k^1]$  and  $S^1 \times [b_k^1, l_k]$  are satisfied and we have an energy identity on  $S^1 \times [a_k^1, b_k^1]$ . Then we apply the process recursively to  $S^1 \times [0, a_k^1]$  and  $S^1 \times [b_k^1, l_k]$ . The process has to stop, since at each step the central cylinder contains at least  $\varepsilon_0$  of energy since either there is concentration or one converges in  $C_{loc}^2(S^1 \times \mathbb{R})$  to a nontrivial bubble.  $\square$

## 2.2 Choosing the thin part

### 2.2.1 The torus case

The theorem 0.2 of [22] ensures that a torus, which is isometric to the cylinder  $C_k = \frac{1}{\sqrt{2\pi l_k}} (S^1 \times [0, l_k])$  with the standard identification of its boundary components, admits the following chart

$$\begin{aligned} \psi_k : \mathbb{D} \setminus B(0, e^{-l_k}) &\rightarrow C_l \\ (\theta, r) &\mapsto \left( \frac{\cos(\theta)}{\sqrt{2\pi l_k}}, \frac{\sin(\theta)}{\sqrt{2\pi l_k}}, \frac{-\log(r)}{\sqrt{2\pi l_k}} \right), \end{aligned}$$

such that, the conformal factor  $u_k$  of  $\vec{\Phi}_k = \vec{\Phi}_k \circ \psi_k$ , i.e.  $\vec{\Phi}_k^*(\xi) = e^{2u_k} dz^2$ , satisfies

$$\|\nabla u_k\|_{L^2, \infty} \leq C.$$

Moreover we can choose the place where we "cut" the torus into a cylinder in a way that there is no concentration near the boundary.

Indeed there is a finite number of  $t^i \in [-\frac{l_k}{2}, \frac{l_k}{2}]$  such that  $\lim_{k \rightarrow +\infty} \int_{S^1 \times [t^i, t^i+1]} |\nabla \vec{n}_k|^2 dt d\theta \geq \frac{\varepsilon_0}{2}$ , then we pick  $t_k$  such that  $\lim_{k \rightarrow +\infty} |t_k - t^i| = +\infty$ . Hence, setting  $\tilde{\vec{\Phi}}_k = \vec{\Phi}_k(\cdot + t_k)$ , thanks to  $\varepsilon$ -regularity, we have the convergence of  $\tilde{\vec{\Phi}}_k$  in  $C_{loc}^2$  to some (possibly trivial) bubble. Hence cutting the torus at  $t^k$  provide a cylinder with no concentration near the boundary.

### 2.2.2 The hyperbolic case

Thanks to the collar lemma we know that choosing  $\delta < \sinh(1)$ , the thin part  $\{x \in (\Sigma, h_k) \mid inj(x) < \delta\}$  consists of a finite number of collars. Up to extraction, this number is fixed for  $k$  large enough, but in order to simplify notations in the rest of this part we will assume that there is only one collar. This collar contains a smallest closed geodesic and is conformal to an hyperbolic cylinder of the form

$$A_l = \left\{ z = re^{i\vec{\Phi}} \in \mathbb{H} : 1 \leq r \leq e^l, \arctan\left(\sinh\left(\frac{l}{2}\right)\right) < \vec{\Phi} < \pi - \arctan\left(\sinh\left(\frac{l}{2}\right)\right) \right\},$$

where the geodesic correspond to  $\{re^{i\frac{\pi}{2}} \in \mathbb{H} : 1 \leq r \leq e^l\}$  and the line  $\{r = 1\}$  and  $\{r = e^l\}$  are identified via  $z \mapsto e^l z$ . This is the **collar region**. It is sometimes easier to consider the following cylindrical parametrization, i.e.

$$P_l = \left\{ (t, \theta) : \frac{2\pi}{l} \arctan \left( \sinh \left( \frac{l}{2} \right) \right) < t < \frac{2\pi}{l} \left( \pi - \arctan \left( \sinh \left( \frac{l}{2} \right) \right) \right), 0 \leq \theta \leq 2\pi \right\}$$

in this parametrization the constant scalar curvature metric reads

$$ds^2 = \left( \frac{l}{2\pi \sin(\frac{lt}{2\pi})} \right)^2 (dt^2 + d\theta^2),$$

where the geodesic corresponds to  $\{t = \frac{\pi^2}{l}\}$  and the line  $\{\theta = 0\}$  and  $\{\theta = 2\pi\}$  are identified.

Then, as  $l_k$ , the length of the degenerating geodesic, goes to zero,  $P_{l_k} = [0, T_k] \times S^1$  becomes a long cylinder.

Let  $\psi_k$  be the chart from the cylinder to the collar and keep denoting  $\vec{\Phi}_k = \vec{\Phi}_k \circ \psi_k$ . Choosing  $\delta$  small enough, we can assume that there is no concentration near the boundary of the collar, i.e., for every  $R > 0$ ,  $\vec{\Phi}_k$  and  $\vec{\Phi}_k(T_k - t, \theta)$  converges in  $C_{loc}^2([0, R] \times S^1)$ .

### 2.3 Extraction of necks inside the collars

From the last section, we reduce the study of the thin part to a long cylinder without concentration near the boundary. A priori, the metric is not flat, but since the energy is invariant by conformal change of metric, we can apply directly the bubble tree lemma. Indeed, lemma 2.1 permits us to split our cylinder in parts where we have energy identities, the  $S^1 \times [a_k^i, b_k^i]$ , and necks, the  $S^1 \times [b_k^i, a_k^{i+1}]$ . Then in the rest of the paper we will concentrate on a specific neck region. In order to prove that there is no energy in such a neck we will use the  $L^{2,\infty} - L^{2,1}$  duality. As remark in Laurain-Rivi re [22], this norm are no more invariant by conformal change (beside simple dilatations and isometries of course). So we are going to fix the chart once for all. But thanks to Laurain-Rivi re [22] we know that there is an appropriate choice to make use of this theory. We make it precise in the next section.

### 2.4 $L^{2,\infty}$ estimate in the neck

Thanks to theorem 1.5, we know that we can choose our chart such that the  $L^{2,\infty}$  norm of the gradient of the conformal factor is uniformly bounded. Moreover, every collar (resp. long and thin cylinders) has a chart of annular type, i.e.

$$A_k = \mathbb{D} \setminus B(0, e^{-l_k}),$$

where  $l_k \rightarrow +\infty$ . In this setting the result corresponding to lemma VII.1 of [2] holds. Precisely we have the following lemma

**Lemma 2.2.** *Let  $m \geq 3$  and  $\vec{\Phi}_k : \mathbb{D} \setminus B(0, \varepsilon_k) \rightarrow \mathbb{R}^m$  be a sequence of conformal Willmore immersion, with  $\varepsilon_k \rightarrow 0$ , satisfying*

$$\|\nabla \lambda_k\|_{L^{2,\infty}(\mathbb{D} \setminus B(0, \varepsilon_k))} + \|\nabla \vec{n}_{\vec{\Phi}_k}\|_{L^2(\mathbb{D} \setminus B(0, \varepsilon_k))} \leq C,$$

where  $\lambda_k$  is the conformal factor, and such that

$$\lim_{R \rightarrow +\infty} \lim_{k \rightarrow +\infty} \sup_{R\varepsilon_k < \rho < \frac{1}{R}} \int_{B(0, 2\rho) \setminus B(0, \rho)} |\nabla \vec{n}_{\vec{\Phi}_k}|^2 dz = 0. \quad (13)$$

Then for every  $\varepsilon > 0$ , there exists  $R > 0$ , depending only on  $C$  and  $\varepsilon$ , such that

$$\lim_{k \rightarrow +\infty} \sup_{R\varepsilon_k < |z| < \frac{1}{R}} |z| |\nabla \vec{n}_{\vec{\Phi}_k}| < \varepsilon.$$

in particular

$$\lim_{R \rightarrow +\infty} \lim_{k \rightarrow +\infty} \|\nabla \vec{n}_{\vec{\Phi}_k}\|_{L^{2,\infty}(B(0, \frac{1}{R}) \setminus B(0, R\varepsilon_k))} = 0. \quad (14)$$

□

## 2.5 Behaviour of the conformal factor in a neck region

First of all, let us remind the following lemma which permits to control the behaviour of the conformal factor in an annulus.

**Lemma 2.3** (lemma V.3 of [2]). *There exists  $\varepsilon_0 > 0$  with the following property. Let  $0 < r < \frac{1}{4}$ . If  $\vec{\Phi}$  is any conformal weak immersion of  $\mathbb{D} \setminus B(0, r)$  into  $\mathbb{R}^m$  with  $L^2$ -bounded second fundamental form, and satisfying*

$$\|\nabla \vec{n}\|_{L^{2,\infty}} \leq \varepsilon_0,$$

then there exist  $d, A \in \mathbb{R}$  such that

$$\|\lambda(x) - d \operatorname{Log}(|x|) - A\|_{L^\infty(B(0, \frac{1}{2}) \setminus B(0, 2r))} \leq C \left( \|\nabla \lambda\|_{L^{2,\infty}} + \int_{\mathbb{D} \setminus B(0, r)} |\nabla \vec{n}|^2 dz \right),$$

where  $d$  satisfies

$$\left| 2\pi d - \int_{\partial B_r} \frac{\partial \lambda}{\partial \rho} d\sigma \right| \leq C \left( \int_{B(0, 2r) \setminus B(0, r)} |\nabla \vec{n}|^2 dz + \frac{1}{\operatorname{Log}(1/r)} \left( \|\nabla \lambda\|_{2,\infty} + \int_{\mathbb{D} \setminus B(0, r)} |\nabla \vec{n}|^2 dz \right) \right),$$

where  $C$  depends only on  $m$  and  $\lambda$  is as in lemma 36. □

As we will see later, we need to prevent  $d$  to be close to  $-1$ , which correspond to the parametrization of a long thin cylinder. Hence in the end of this paragraph we exclude the possibility for a sequence of  $\vec{\Phi}_k$  satisfying the hypothesis of lemma 2.3 and (13) to have its corresponding  $d_k$  that converge to  $-1$ .

Let  $\vec{\Phi}_k$  satisfying the hypothesis of lemma 2.2. We set  $\vec{\Psi}_k = \frac{\vec{\Phi}_k(\sqrt{\varepsilon_k}z) - c_k}{L_k}$  where  $c_k$  and  $L_k$  are respectively the center of mass and the length of  $\theta \mapsto \vec{\Phi}_k(\sqrt{\varepsilon_k}e^{i\theta})$ .

Let  $K$  a compact set of  $\mathbb{C} \setminus \{0\}$ , for  $k$  large enough,  $\vec{\Psi}_k$  is well defined on  $K$  and its normal satisfies, thanks to lemma 2.2, the following estimate

$$|\nabla \vec{n}_{\vec{\Psi}_k}| = o(1).$$

Moreover, since the energy does not concentrate our new conformal factor satisfies an Harnarck inequality, see Lemma V.2 of [2]. Using lemma 2.3, we can write it as  $\lambda_{\vec{\Psi}_k} = d_k \ln(\rho) + A_k + B_k$ , where  $d_k$  and  $A_k$  are some real constant and  $B_k$  is a uniformly bounded function. Since,  $\vec{\Psi}_k$  is conformal and the length of the image of  $S^1$  is one, then  $A_k$  is uniformly bounded. Moreover,  $d_k$  is also uniformly bounded by Harnarck inequality.

Finally  $\vec{\Psi}_k$ , up to an extraction, converge to  $\vec{\Psi}_\infty$  a conformal parametrization of a piece of plane in  $C_{loc}^2(\mathbb{C} \setminus \{0\})$ , and its conformal factor satisfies

$$\frac{1}{C}\rho^{d_\infty} \leq e^\lambda \leq C\rho^{d_\infty}, \quad (15)$$

for some  $C > 0$ .

Up to orientation, we can consider  $\vec{\Psi}_\infty$  as an holomorphic map from  $\mathbb{C} \setminus \{0\}$  to  $\mathbb{C}$ , we note it simply  $\psi$  from now. Then  $\psi' dz$  is an holomorphic one form on  $\mathbb{C} \setminus \{0\}$ . Thanks to (15), it is in fact rational form on  $\hat{\mathbb{C}}$ . Then we get that  $d_\infty \in \mathbb{Z}$  and, up to multiply  $\psi$  by a non null complex number,  $\psi'(z) = z^{d_\infty}$ . Of course the case  $d_\infty = -1$  is excluded since  $\frac{dz}{z}$  has no primitive on  $\mathbb{C} \setminus \{0\}$ .

**Remark 2.1.** *When considering a sequence of Willmore immersion, in all neck we have the decomposition of the conformal factor as in lemma 2.2, moreover we can assume that  $|d_k + 1| > \frac{1}{2}$ .*

### 3 Preliminary Estimates in Collar Regions.

We shall very often omit to write explicitly the subscript  $k$  when there is no ambiguity for understanding the argument.

#### 3.1 $L^{2,1}$ -estimate on degenerating annuli

Thanks to the conclusion of the section 2.4, in order to prove the quantification of the energy in the collar it suffices to prove an  $L^{2,1}$ -estimate on  $\nabla \vec{n}$  in degenerating annuli. This is however not exactly what we shall establish. Here is the main difference with case where the conformal class stays bounded : our annuli are parts of the collars and cannot be filled inside by the solution which extends to the whole disc. Bernard and the second author were facing a somehow similar situation in their study of Willmore surfaces near branched points (see [3]). In order to overcome the impossibility to extend the solution throughout the middle region at each steps of the derivation of the conservation laws issued from the application of Noether theorem (see [32]) residues will show up. The main task of the present work will be to control these residues.

Let  $\vec{\Phi} : \mathbb{D} \setminus B(0, r) \rightarrow \mathbb{R}^m$  a conformal Willmore immersion. Thanks to theorem 1.1 of [34], denoting

$$2\pi \vec{c} := \int_{\partial \mathbb{D}} \vec{\nu} \cdot \left( \nabla \vec{H} - 3\pi_{\vec{n}}(\nabla \vec{H}) + \star(\nabla^\perp \vec{n} \wedge \vec{H}) \right) d\sigma \quad (16)$$

there exists  $\vec{L} : \mathbb{D} \setminus B(0, r) \rightarrow \mathbb{R}^m$  such that

$$\nabla^\perp \vec{L} = \nabla \vec{H} - 3\pi_{\vec{n}}(\nabla \vec{H}) + \star(\nabla^\perp \vec{n} \wedge \vec{H}) - \vec{c} \nabla \log(\rho) \quad (17)$$

and following [34] we observe that

$$\begin{cases} \operatorname{div}(\langle \vec{L}, \nabla^\perp \vec{\Phi} \rangle - \langle \vec{c}, \vec{\Phi} \rangle \nabla \log(\rho)) = 0 & , \\ \operatorname{div}(\vec{L} \wedge \nabla^\perp \vec{\Phi} + (-1)^{m-1} 2(\star(\vec{n} \lrcorner \vec{H}) \lrcorner \nabla^\perp \vec{\Phi} - \vec{c} \wedge \vec{\Phi} \nabla \log(\rho))) = 0 & . \end{cases}$$

Let

$$2\pi c_0 := - \int_{\partial \mathbb{D}} \langle \vec{L}, \partial_\tau \vec{\Phi} \rangle d\sigma - \int_{\partial \mathbb{D}} \langle \vec{c}, \vec{\Phi} \rangle d\sigma \quad (18)$$

and

$$2\pi\vec{c}_1 := \int_{\partial\mathbb{D}} \left( -\vec{L} \wedge \partial_\tau \vec{\Phi} - (-1)^{m-1} 2(\star(\vec{n} \lrcorner \vec{H}) \lrcorner \partial_\tau \vec{\Phi}) \right) d\sigma - \int_{\partial\mathbb{D}} \vec{c} \wedge \vec{\Phi} d\sigma, \quad (19)$$

where  $\partial_\tau = \frac{1}{\rho} \frac{\partial}{\partial\theta}$  and  $d\sigma = \rho d\theta$ .

Then, thanks to Poincaré lemma, there exists  $S : \mathbb{D} \setminus B(0, r) \rightarrow \mathbb{R}$  and  $\vec{R} : \mathbb{D} \setminus B(0, r) \rightarrow \mathbb{R}^m$  such that

$$\nabla^\perp S = \langle \vec{L}, \nabla^\perp \vec{\Phi} \rangle - C_0 \nabla \log(\rho) \quad (20)$$

and

$$\nabla^\perp \vec{R} = \vec{L} \wedge \nabla^\perp \vec{\Phi} + (-1)^{m-1} 2(\star(\vec{n} \lrcorner \vec{H}) \lrcorner \nabla^\perp \vec{\Phi}) - \vec{C}_1 \nabla \log(\rho). \quad (21)$$

where  $C_0 = c_0 + \langle \vec{c}, \vec{\Phi} \rangle$  and  $\vec{C}_1 = \vec{c}_1 + \vec{c} \wedge \vec{\Phi}$ .

The rest of this section consists essentially in estimating the *residues*  $C_0$  and  $\vec{C}_1$  independently of the conformal class of the annuli. This is the novelty of this paper, those *residues* do not appear when the neck regions is not part of collar since the solution is extendable throughout the internal small disc. Before doing so, we generalize the estimate (IV.24) of [2], to our setting. We propose in fact here a shorter argument than the one in [2].

### 3.1.1 Pointwise estimate of $\vec{L}$

We are going to use the lemma 2.3 to derive a pointwise control of  $\vec{L}$ . Precisely we have.

**Lemma 3.1.** *There exists  $C > 0$ , independent of  $r$ , such that*

$$e^{\lambda_\rho} \|\vec{L}\|_{L^\infty(\partial B(0, \rho))} \leq \frac{C}{\rho} \quad \text{for all } \rho \in (2r, 1/2), \quad (22)$$

where  $\lambda_\rho = \sup_{|x|=\rho} \lambda(x)$ . □

#### Proof of lemma 3.1.

First of all, let us remind the pointwise estimate on  $\nabla \vec{H}$ . From (VI.12) of [2] and the Harnack inequality satisfies by the conformal factor, see lemma V.2 of [2], we get that

$$|\nabla \vec{H}(x)| \leq C e^{-\lambda_{|x|}} \frac{1}{|x|^2} \left( \int_{B(0, 2|x|) \setminus B(0, \frac{|x|}{2})} |\nabla \vec{n}|^2 dz \right)^{\frac{1}{2}} \quad \text{for all } x \in B(0, 1/2) \setminus B(0, 2r), \quad (23)$$

where  $C$  depends only on  $\|\nabla \lambda\|_{2, \infty}$ . Then we set  $\vec{L}_\rho = \frac{1}{2\pi} \int_0^{2\pi} \vec{L}(\rho, \theta) d\theta$ . Thanks to (16), (17), (12) and (23), we get, for all  $\rho \in (2r, 1/2)$ ,

$$|\vec{L} - \vec{L}_\rho| \leq \int_0^{2\pi} |\nabla \vec{L}| \rho d\theta \leq C \frac{e^{-\lambda_\rho}}{\rho} \left( \int_{B(0, 2\rho) \setminus B(0, \frac{\rho}{2})} |\nabla \vec{n}|^2 dz \right)^{\frac{1}{2}}.$$

Then we estimate  $\vec{L}_\rho$ . First of all, Thanks to (22), (12) and (23) we have

$$\left| \frac{d\vec{L}_\rho}{d\rho} \right| \leq C(|\nabla \vec{H}| + |\nabla \vec{n}| |\vec{H}|) \leq C \frac{e^{-\lambda_\rho}}{\rho^2} \left( \int_{B(0, 2\rho) \setminus B(0, \frac{\rho}{2})} |\nabla \vec{n}|^2 dz \right)^{\frac{1}{2}}.$$

But, since  $\vec{L}$  is defined up to a constant, we can assume that  $\vec{L}_{2r} = 0$  (or  $\vec{L}_{1/2} = 0$ , this will be make clear later), which gives

$$\begin{aligned} |\vec{L}_\rho| &\leq C \int_{2r}^\rho \frac{e^{-\lambda_t}}{t^2} \left( \int_{B(0,2t) \setminus B(0, \frac{t}{2})} |\nabla \vec{n}|^2 dz \right)^{\frac{1}{2}} dt \\ &\leq C e^{-\lambda_\rho} \int_{2r}^\rho \frac{e^{\lambda_\rho - \lambda_t}}{t^2} \left( \int_{B(0,2t) \setminus B(0, \frac{t}{2})} |\nabla \vec{n}|^2 dz \right)^{\frac{1}{2}} dt. \end{aligned}$$

Thanks to lemma 2.3 we have  $\lambda_\rho - \lambda_t = d \log \left( \frac{\rho}{t} \right) + B$ , where  $B$  is uniformly bounded, hence, if  $d+1 \leq 0$ ,

$$\begin{aligned} |\vec{L}_\rho| &\leq C e^{-\lambda_\rho} \int_{2r}^\rho \frac{1}{t^2} \left( \frac{\rho}{t} \right)^d \left( \int_{B(0,2t) \setminus B(0, \frac{t}{2})} |\nabla \vec{n}|^2 dz \right)^{\frac{1}{2}} dt \\ &\leq C \frac{e^{-\lambda_\rho}}{\rho} \left( \frac{\rho}{2r} \right)^{d+1} \int_{2r}^\rho \frac{1}{t^2} \left( \int_{B(0,2t) \setminus B(0, \frac{t}{2})} |\nabla \vec{n}|^2 dz \right)^{\frac{1}{2}} t dt \\ &\leq C \frac{e^{-\lambda_\rho}}{\rho} \left( \frac{\rho}{2r} \right)^{d+1} \end{aligned}$$

which gives the desired estimate if  $d+1 \leq 0$ . Then if  $d+1 \geq 0$  it suffies to assume that  $\vec{L}_{\frac{1}{2}} = 0$  and to integrate between  $\rho$  and  $1/2$ .  $\square$

### 3.1.2 Estimating the first residue $C_0$ .

**Lemma 3.2.** *There exists  $C > 0$ , independent of  $r$ , such that*

$$\left\| \frac{C_0}{\rho} \right\|_{L^{2,1}(B(0, \frac{1}{2}) \setminus B(0, 2r))} \leq C. \quad (24)$$

$\square$

**Proof of lemma 3.2.** Integrating by part (18) and using Stokes's theorem, we get, for every  $z \in B(0, \frac{1}{2}) \setminus B(0, 2r)$ ,

$$2\pi C_0(z) = \int_{\partial B(0, |z|)} \langle \partial_\tau \vec{L}, \vec{\Phi} \rangle d\sigma - \int_{\partial B(0, |z|)} \left\langle \frac{\vec{c}}{|z|}, \vec{\Phi} \right\rangle d\sigma + 2\pi \langle \vec{c}, \vec{\Phi} \rangle$$

Thanks to (17), we have the fact that  $\partial_\tau \vec{L} = -\vec{T}_\nu + \frac{\vec{c}}{|z|}$  where  $\vec{T} = \nabla \vec{H} - 3\pi_{\vec{n}}(\nabla \vec{H}) + \star(\nabla^\perp \vec{n} \wedge \vec{H})$ . Then

$$\begin{aligned} 2\pi C_0(z) &= \int_{\partial B(0, |z|)} -\langle \vec{T}_\nu, \vec{\Phi} \rangle d\sigma + \int_{\partial B(0, |z|)} \left\langle \frac{\vec{c}}{|z|}, \vec{\Phi} \right\rangle d\sigma - \int_{\partial B(0, |z|)} \left\langle \frac{\vec{c}}{|z|}, \vec{\Phi} \right\rangle d\sigma + 2\pi \langle \vec{c}, \vec{\Phi} \rangle \\ &= \int_{\partial B(0, |z|)} \langle \vec{T}_\nu(y), \vec{\Phi}(z) - \vec{\Phi}(y) \rangle d\sigma \end{aligned} \quad (25)$$

On the one hand

$$\vec{T}_\nu = \frac{\partial \vec{H}}{\partial \rho} - 3\pi_{\vec{n}} \left( \frac{\partial \vec{H}}{\partial \rho} \right) - \star \left( \frac{\partial \vec{n}}{\partial \tau} \wedge \vec{H} \right) = \pi_T \left( \frac{\partial \vec{H}}{\partial \rho} \right) - 2\pi_{\vec{n}} \left( \frac{\partial \vec{H}}{\partial \rho} \right) - \star \left( \frac{\partial \vec{n}}{\partial \tau} \wedge \vec{H} \right), \quad (26)$$



where  $\pi_T$  is the projection onto the tangent plane. Writing, locally,  $\vec{H} = \sum_{\alpha=1}^{m-2} H_\alpha \vec{n}^\alpha$ , we easily get that

$$\left| \pi_T \left( \frac{\partial \vec{H}}{\partial \rho} \right) \right| \leq |H| |\nabla \vec{n}|. \quad (27)$$

On the other hand, let  $\Gamma_{y,z}$  the direct circular arc from  $y$  to  $z$ , we get

$$\begin{aligned} \left| \left\langle \pi_{\vec{n}} \left( \frac{\partial \vec{H}}{\partial \rho} \right) (y), \vec{\Phi}(z) - \vec{\Phi}(y) \right\rangle \right| &= \left| \left\langle \pi_{\vec{n}} \left( \frac{\partial \vec{H}}{\partial \rho} \right), \int_{\Gamma_{y,z}} d\vec{\Phi} \right\rangle \right| \\ &= \left| \int_{\Gamma_{y,z}} \left\langle \pi_{\vec{n}(y)} \left( \frac{\partial \vec{H}}{\partial \rho} \right) (y) - \pi_{\vec{n}(x)} \left( \frac{\partial \vec{H}}{\partial \rho} \right) (y), d\vec{\Phi}(x) \right\rangle \right| \\ &\leq |z|^2 \|\nabla \pi_{\vec{n}}\|_{L^\infty(\partial B(0,|z|))} \left\| \frac{\partial \vec{H}}{\partial \rho} \right\|_{L^\infty(\partial B(0,|z|))} e^{\lambda|z|}. \end{aligned}$$

Thanks to (5), we have, for all  $\vec{w} \in \mathbb{R}^m$ , that

$$\nabla \pi_{\vec{n}}(\vec{w}) = (-1)^{m-1} (\nabla \vec{n}) \lrcorner (\vec{n} \lrcorner \vec{w}) + (-1)^{m-1} \vec{n} \lrcorner ((\nabla \vec{n}) \lrcorner \vec{w}).$$

Hence we have

$$\left\| \left\langle \pi_{\vec{n}} \left( \frac{\partial \vec{H}}{\partial \rho} \right) (y), \vec{\Phi}(z) - \vec{\Phi}(y) \right\rangle \right\|_{L^\infty(\partial B(0,|z|))} \leq |z|^2 \|\nabla \vec{n}\|_{L^\infty(\partial B(0,|z|))} \left\| \nabla \vec{H} \right\|_{L^\infty(\partial B(0,|z|))} e^{\lambda|z|}. \quad (28)$$

Combining (25), (27), (26) and (28) we get

$$\begin{aligned} |2\pi C_0(z)| &\leq C \left( |z| \|\nabla \vec{n}\|_{L^\infty(\partial B(0,|z|))} \|\vec{\Phi} - \vec{\Phi}(z)\|_{L^\infty(\partial B(0,|z|))} \right. \\ &\quad \left. + |z|^3 \|\nabla \vec{n}\|_{L^\infty(\partial B(0,|z|))} \left\| \nabla \vec{H} \right\|_{L^\infty(\partial B(0,|z|))} e^{\lambda|z|} \right). \end{aligned}$$

Then using (12) and (23), we get

$$|2\pi C_0(z)| \leq C \|\nabla \vec{n}\|_{L^2(B(0,2|z|) \setminus B(0, \frac{|z|}{2}))}^2.$$

Hence, we finally get

$$|2\pi C_0(z)| \leq C \left( \int_{B(0,2|z|) \setminus B(0, \frac{|z|}{2})} |\nabla \vec{n}|^2 dz \right) \text{ for all } z \in B(0, 1/2) \setminus B(0, 2r)$$

Then, applying the following lemma, we get the desired result.  $\square$

**Lemma 3.3.** *Let  $s : B(0, 1/2) \setminus B(0, 2r) \rightarrow \mathbb{R}_+$  such that*

$$s(z) = \frac{1}{|z|} \left( \int_{B(0,2|z|) \setminus B(0, |z|/2)} |\nabla \vec{n}|^2 dz \right)^{\frac{1}{2}}$$

*Then  $s^2(z)|z|$  is uniformly bounded in  $L^{2,1}$  independently of  $r$ .*  $\square$

**Proof of lemma 3.3.**

$$\frac{\partial(s^2(z)|z|)}{\partial\rho} = -\frac{1}{|z|^2} \int_{B(0,2|z|) \setminus B(0,|z|/2)} |\nabla \vec{n}|^2 dz + \frac{1}{|z|} \int_{\partial B(0,2|z|) \cup \partial B(0,|z|/2)} |\nabla \vec{n}|^2 dz$$

In order to control the second term we can apply (12), which gives

$$\frac{1}{|z|} \int_{\partial B(0,2|z|) \cup \partial B(0,|z|/2)} |\nabla \vec{n}|^2 dz \leq C \frac{1}{|z|^2} \int_{B(0,4|z|) \setminus B(0,|z|/4)} |\nabla \vec{n}|^2 dz$$

Hence

$$\int_{4r}^{1/4} \left| \frac{\partial(s^2(z)|z|)}{\partial|z|} \right| |z| d|z| \leq C \int_{B(0,1) \setminus B(0,r)} |\nabla \vec{n}|^2 dz,$$

which gives the desired estimate thanks to the standard embedding of  $W^{1,1}$  into  $L^{2,1}$ , see theorem 3.3.10 of [13].  $\square$

### 3.1.3 Estimating the second residue $\vec{C}_1$ .

**Lemma 3.4.** *There exists  $C > 0$ , independent of  $r$  and  $k$ , such that*

$$\left\| \frac{\vec{C}_1}{\rho} \right\|_{L^2(B(0, \frac{1}{2}) \setminus B(0, 2r))} \leq C. \quad (29)$$

$\square$

**Proof of lemma 3.4.**

Integrating by parts (19) and using Stokes's theorem, we get for every  $\rho \in (\varepsilon, 1)$

$$\begin{aligned} 2\pi \vec{C}_1(z) &= \int_{\partial B(0,\rho)} \partial_\tau \vec{L} \wedge \vec{\Phi} - (-1)^{m-1} 2(\star(\vec{n} \lrcorner \vec{H})) \lrcorner \partial_\tau \vec{\Phi} d\sigma - \frac{1}{\rho} \int_{\partial B(0,\rho)} \vec{c} \wedge \vec{\Phi} d\sigma + 2\pi \vec{c} \wedge \vec{\Phi} \\ &= \int_{\partial B(0,\rho)} -\vec{T}_\nu \wedge \vec{\Phi} + \frac{\vec{c} \wedge \vec{\Phi}}{\rho} - (-1)^{m-1} 2(\star(\vec{n} \lrcorner \vec{H})) \lrcorner \partial_\tau \vec{\Phi} d\sigma - \frac{1}{\rho} \int_{\partial B(0,\rho)} \vec{c} \wedge \vec{\Phi} d\sigma + 2\pi \vec{c} \wedge \vec{\Phi} \\ &= \int_{\partial B(0,\rho)} \vec{T}_\nu \wedge (\vec{\Phi}(z) - \vec{\Phi}(y)) - (-1)^{m-1} 2(\star(\vec{n} \lrcorner \vec{H})) \lrcorner \partial_\tau \vec{\Phi} d\sigma, \end{aligned} \quad (30)$$

where, as in the previous section,

$$\vec{T}_\nu = \frac{\partial \vec{H}}{\partial \rho} - 3\pi_{\vec{n}} \left( \frac{\partial \vec{H}}{\partial \rho} \right) - \star \left( \frac{\partial \vec{n}}{\partial \tau} \wedge \vec{H} \right) = \pi_T \left( \frac{\partial \vec{H}}{\partial \rho} \right) - 2\pi_{\vec{n}} \left( \frac{\partial \vec{H}}{\partial \rho} \right) - \star \left( \frac{\partial \vec{n}}{\partial \tau} \wedge \vec{H} \right).$$

Then

$$2\pi \vec{C}_1(z) = \int_{\partial B(0,\rho)} \left( \pi_T \left( \frac{\partial \vec{H}}{\partial \rho} \right) - 2\pi_{\vec{n}} \left( \frac{\partial \vec{H}}{\partial \rho} \right) - \star \left( \frac{\partial \vec{n}}{\partial \tau} \wedge \vec{H} \right) \right) \wedge (\vec{\Phi}(z) - \vec{\Phi}(y)) - (-1)^{m-1} 2(\star(\vec{n} \lrcorner \vec{H})) \lrcorner \partial_\tau \vec{\Phi} d\sigma.$$

Then, we easily check that except the second term and the last term, i.e.

$$I = -2 \int_{\partial B(0,\rho)} \pi_{\vec{n}} \left( \frac{\partial \vec{H}}{\partial \rho} \right) \wedge (\vec{\Phi}(z) - \vec{\Phi}(y)) d\sigma$$

and

$$II = -(-1)^{m-1} 2 \int_{\partial B(0,\rho)} (\star(\vec{n} \lrcorner \vec{H})) \lrcorner \partial_\tau \vec{\Phi} d\sigma,$$

all the other terms can be controlled by  $2\pi\rho e^{\lambda\rho} \|\nabla \vec{n}\|_{L^\infty(\partial B(0,\rho))} \|\vec{H}\|_{L^\infty(\partial B(0,\rho))}$  which is controlled by  $\left( \int_{B(0,2\rho) \setminus B(0,\frac{\rho}{2})} |\nabla \vec{n}|^2 dz \right)$ , as already done in the previous section for  $C_0$ .

For  $I$  and  $II$ , we won't be able to derive a similar estimate than the one for  $C_0$ . The best we can do, applying (12) and (23), is

$$I + II \leq C \left( \int_{B(0,2\rho) \setminus B(0,\frac{\rho}{2})} |\nabla \vec{n}|^2 dz \right)^{\frac{1}{2}}.$$

Hence we get

$$|\vec{C}_1(z)| \leq C \left( \int_{B(0,2|z|) \setminus B(0,\frac{|z|}{2})} |\nabla \vec{n}|^2 dz \right)^{\frac{1}{2}} \quad \text{for all } z \in B(0,1/2) \setminus B(0,2r), \quad (31)$$

which implies that the desired estimate.  $\square$

## 4 Proof of theorem 0.1.

After deriving some equations relating  $\vec{H}$  together with  $S$  and  $\vec{R}$ , we prove that an appropriate combination of those quantities is bounded in  $L^{2,1}$ . This will be done using some additional conservation laws combined with results from integrability by compensation theory from [21] or proved in the appendix. Finally combining those estimates with some algebraic fact, we prove that  $\vec{H}$  converges to 0 in  $L^2$  norm in the neck region.

### 4.1 Equations satisfied by $\vec{H}$ , $\vec{R}$ and $S$ .

Equation (21) gives

$$\nabla \vec{R} \lrcorner \nabla^\perp \vec{\Phi} = (\vec{L} \wedge \nabla \vec{\Phi}) \lrcorner \nabla^\perp \vec{\Phi} + (-1)^{m-1} 2 (\star(\vec{n} \lrcorner \vec{H})) \lrcorner \nabla \vec{\Phi} \lrcorner \nabla^\perp \vec{\Phi} + \vec{C}_1 \nabla^\perp \log(\rho) \lrcorner \nabla^\perp \vec{\Phi},$$

which simplify into

$$\nabla \vec{R} \lrcorner \nabla^\perp \vec{\Phi} = \langle \vec{L}, \nabla^\perp \vec{\Phi} \rangle \nabla \vec{\Phi} + (-1)^{m-1} 2 (\star(\vec{n} \lrcorner \vec{H})) \lrcorner \nabla \vec{\Phi} \lrcorner \nabla^\perp \vec{\Phi} + \frac{1}{\rho} \vec{C}_1 \lrcorner \frac{\partial \vec{\Phi}}{\partial \rho}.$$

Since, see X.210 of [36], for any normal vector  $\vec{N}$ ,

$$\star(\vec{n} \lrcorner \vec{N}) = (-1)^{m-1} \vec{e}_1 \wedge \vec{e}_2 \wedge \vec{N}, \quad (32)$$

we deduce that

$$(-1)^{m-1} 2 (\star(\vec{n} \lrcorner \vec{H})) \lrcorner \nabla \vec{\Phi} \lrcorner \nabla^\perp \vec{\Phi} = -4 e^{2\lambda} \vec{H}.$$

Moreover (20) gives

$$\langle \vec{L}, \nabla^\perp \vec{\Phi} \rangle = \nabla^\perp S + C_0 \nabla \log(\rho),$$

which finally implies

$$4 e^{2\lambda} \vec{H} = -\nabla \vec{R} \lrcorner \nabla^\perp \vec{\Phi} + \nabla^\perp S \cdot \nabla \vec{\Phi} + \frac{1}{\rho} C_0 \frac{\partial \vec{\Phi}}{\partial \rho} + \frac{1}{\rho} \vec{C}_1 \lrcorner \frac{\partial \vec{\Phi}}{\partial \rho}. \quad (33)$$

We now compute an autonomous system satisfies by  $\vec{R}$  and  $S$ . From (20) and (21), we have

$$\begin{cases} \nabla^\perp S = \langle \vec{L}, \nabla^\perp \vec{\Phi} \rangle - C_0 \nabla \log(\rho) \\ \nabla^\perp \vec{R} = \vec{L} \wedge \nabla^\perp \vec{\Phi} + (-1)^{m-1} 2(\star(\vec{n} \lrcorner \vec{H}) \lrcorner \nabla^\perp \vec{\Phi}) - \vec{C}_1 \nabla \log(\rho) \end{cases} \quad (34)$$

But we get, for any normal vector  $\vec{N}$ , as a consequence of (32),

$$(-1)^{m-1} (\star(\vec{n} \lrcorner \vec{N})) \lrcorner \nabla \vec{\Phi} = -\nabla^\perp \vec{\Phi} \wedge \vec{N}. \quad (35)$$

Then, combining (21), (35) and the definition of  $\bullet$ , see (6) and (7), we deduce that

$$\vec{n} \bullet \nabla^\perp \vec{R} = (\vec{n} \lrcorner \pi_{\vec{n}}(\vec{L})) \wedge \nabla^\perp \vec{\Phi} - 2(\vec{n} \lrcorner \vec{H}) \wedge \nabla \vec{\Phi} - \vec{n} \bullet \vec{C}_1 \nabla \log(\rho). \quad (36)$$

From X.215 of [36], if  $\vec{N}$  is a normal vector field, we get

$$\star((\vec{n} \lrcorner \vec{N}) \wedge \nabla^\perp \vec{\Phi}) = (-1)^{m-1} \nabla \vec{\Phi} \wedge \vec{N} \quad (37)$$

which gives, combining it with (36),

$$\star(\vec{n} \bullet \nabla^\perp \vec{R}) = (-1)^m \pi_{\vec{n}}(\vec{L}) \wedge \nabla \vec{\Phi} + (-1)^{m-1} 2 \nabla^\perp \vec{\Phi} \wedge \vec{H} - \star(\vec{n} \bullet \vec{C}_1 \nabla \log(\rho)). \quad (38)$$

Moreover, combining (21) and (35), we get

$$\begin{aligned} \nabla \vec{R} &= \vec{L} \wedge \nabla \vec{\Phi} + 2(\star(\vec{n} \lrcorner \vec{H}) \lrcorner \nabla \vec{\Phi}) + \vec{C}_1 \nabla^\perp \log(\rho) \\ &= \pi_{\vec{n}}(\vec{L}) \wedge \nabla \vec{\Phi} - 2 \nabla^\perp \vec{\Phi} \wedge \vec{H} + \pi_T(\vec{L}) \wedge \nabla \vec{\Phi} + \vec{C}_1 \nabla^\perp \log(\rho) \end{aligned}$$

Then thanks to (38), we get

$$(-1)^m \star(\vec{n} \bullet \nabla^\perp \vec{R}) = \nabla \vec{R} - \pi_T(\vec{L}) \wedge \nabla \vec{\Phi} - \vec{C}_1 \nabla^\perp \log(\rho) - (-1)^m \star(\vec{n} \bullet \vec{C}_1 \nabla \log(\rho)). \quad (39)$$

We easily check, using (20), that

$$\pi_T(\vec{L}) \wedge \nabla \vec{\Phi} = \nabla^\perp S \star \vec{n} + C_0 \nabla \log(\rho) \star \vec{n},$$

which finally gives with (39)

$$\begin{aligned} \nabla \vec{R} &= (-1)^m \star(\vec{n} \bullet \nabla^\perp \vec{R}) + \nabla^\perp S \star \vec{n} + C_0 \nabla \log(\rho) \star \vec{n} \\ &\quad + \vec{C}_1 \nabla^\perp \log(\rho) + (-1)^m \star(\vec{n} \bullet \vec{C}_1 \nabla \log(\rho)). \end{aligned} \quad (40)$$

Now, taking the scalar product again with  $\star \vec{n}$  we get

$$\nabla S = -\langle \nabla^\perp \vec{R}, \star \vec{n} \rangle + C_0 \nabla^\perp \log(\rho) - \langle \vec{C}_1 \nabla \log(\rho), \star \vec{n} \rangle, \quad (41)$$

here we used the fact that

$$\langle \star(\vec{n} \bullet \nabla^\perp \vec{R}), \star \vec{n} \rangle = -\langle \star(\vec{n} \bullet \vec{C}_1 \nabla \log(\rho)), \star \vec{n} \rangle$$

since, in the right hand side of (38), excepted the term involving  $\vec{C}_1$ , all terms are a linear combination of wedges of tangent and normal vectors.

Taking divergence we finally get

$$\begin{cases} \Delta \vec{R} = (-1)^m \star (\nabla \vec{n} \bullet \nabla^\perp \vec{R}) + \nabla^\perp S \nabla \star \vec{n} + \operatorname{div}(C_0 \nabla \log(\rho) \star \vec{n}) \\ \quad + \operatorname{div}(\vec{C}_1 \nabla^\perp \log(\rho)) + (-1)^m \operatorname{div}(\star(\vec{n} \bullet \vec{C}_1 \nabla \log(\rho))) \\ \Delta S = -\langle \nabla^\perp \vec{R}, \nabla \star \vec{n} \rangle + \operatorname{div}(C_0 \nabla^\perp \log(\rho)) \\ \quad - \operatorname{div}(\langle \vec{C}_1 \nabla \log(\rho), \star \vec{n} \rangle) \end{cases} \quad (42)$$

## 4.2 Estimate I of $\vec{C}_1$

**Lemma 4.1.** *There exists  $C > 0$ , independent of  $r$ , such that*

$$\|\langle \vec{C}_1 \nabla \log(\rho), \star \vec{n} \rangle\|_{L^{2,1}(B(0, \frac{1}{2}) \setminus B(0, 2r))} \leq C. \quad (43)$$

□

**Proof of lemma 4.1.** Proceeding as in lemma 3.4, we easily control every term except the one involving  $I$  and  $II$ . That is to say, for all  $z \in \partial B(0, \rho)$  with  $2r < \rho < 1/2$ , we get

$$\begin{aligned} |\langle \vec{C}_1(z), \star \vec{n}(z) \rangle| &= O \left( \left| \int_{\partial B(0, \rho)} \left\langle \star \vec{n}(z), \left( 2\pi_{\vec{n}} \left( \frac{\partial \vec{H}}{\partial \rho} \right) \wedge (\vec{\Phi}(z) - \vec{\Phi}(y)) + 2(\star(\vec{n} \perp \vec{H})) \perp \partial_\tau \vec{\Phi} \right) \right\rangle d\sigma_z \right| \right) \\ &\quad + O \left( \int_{B(0, 2\rho) \setminus B(0, \frac{\rho}{2})} |\nabla \vec{n}|^2 dz \right) \\ &= O \left( \left| \int_{\partial B(0, \rho)} \left\langle \star(\vec{n}(z) - \vec{n}(y)), \left( 2\pi_{\vec{n}} \left( \frac{\partial \vec{H}}{\partial \rho} \right) \wedge (\vec{\Phi}(z) - \vec{\Phi}(y)) + 2(-1)^m \frac{\partial \vec{\Phi}}{\partial \rho} \wedge \vec{H} \right) \right\rangle d\sigma_z \right| \right) \\ &\quad + O \left( \left| \int_{\partial B(0, \rho)} \left\langle \star \vec{n}(y), \left( 2\pi_{\vec{n}} \left( \frac{\partial \vec{H}}{\partial \rho} \right) \wedge (\vec{\Phi}(z) - \vec{\Phi}(p)) + 2(-1)^m \frac{\partial \vec{\Phi}}{\partial \rho} \wedge \vec{H} \right) \right\rangle d\sigma_z \right| \right) \\ &\quad + O \left( \int_{B(0, 2\rho) \setminus B(0, \frac{\rho}{2})} |\nabla \vec{n}|^2 dz \right) \end{aligned} \quad (44)$$

Here we used (35) to simplify the last term of the first line. Proceeding as in the estimate of  $C_0$ , we get

$$\begin{aligned}
|(\vec{C}_1(z), \star \vec{n}(z))| &= O\left(|z|^2 \|\nabla \vec{n}\|_{L^\infty(\partial B(0,\rho))} \left[|z| \|\nabla \vec{H}\|_{L^\infty(\partial B(0,\rho))} + \|\vec{H}\|_{L^\infty(\partial B(0,\rho))}\right] e^{\lambda\rho}\right) \\
&+ O\left(\left|\int_{\partial B(0,\rho)} \left\langle \star \vec{n}(y), \left(2\pi_{\vec{n}} \left(\frac{\partial \vec{H}}{\partial \rho}\right) \wedge (\vec{\Phi}(z) - \vec{\Phi}(y)) + 2(-1)^m \frac{\partial \vec{\Phi}}{\partial \rho} \wedge \vec{H}\right)\right\rangle d\sigma_z\right|\right) \\
&+ O\left(\int_{B(0,2\rho) \setminus B(0,\frac{\rho}{2})} |\nabla \vec{n}|^2 dz\right) \\
&= O\left(\int_{B(0,2\rho) \setminus B(0,\frac{\rho}{2})} |\nabla \vec{n}|^2 dz\right).
\end{aligned} \tag{45}$$

Here we use the fact that

$$\left\langle \star \vec{n}(y), \left(2\pi_{\vec{n}} \left(\frac{\partial \vec{H}}{\partial \rho}\right) \wedge (\vec{\Phi}(z) - \vec{\Phi}(y)) + 2(-1)^m \frac{\partial \vec{\Phi}}{\partial \rho} \wedge \vec{H}\right)\right\rangle = 0,$$

since  $\star \vec{n} = \vec{e}_1 \wedge \vec{e}_2$  and that  $\pi_{\vec{n}} \left(\frac{\partial \vec{H}}{\partial \rho}\right) \wedge (\vec{\Phi}(z) - \vec{\Phi}(y)) + 2(-1)^m \frac{\partial \vec{\Phi}}{\partial \rho} \wedge \vec{H}$  can easily be rewritten as a linear combination of  $\vec{n}^\alpha \wedge \vec{n}^\beta$  and  $\vec{n}^\alpha \wedge \vec{e}_i$ . Finally, using (45), we conclude as in the proof of lemma 3.2.  $\square$

### 4.3 Estimate II of $\vec{C}_1$ .

**Lemma 4.2.** *There exists  $C > 0$ , independent of  $r$ , such that*

$$\|\nabla \vec{C}_1\|_{L^2(B(0,\frac{1}{2}) \setminus B(0,2r))} \leq C. \tag{46}$$

$\square$

**Proof of lemma 4.2.** First we observe that  $\nabla \vec{C}_1 = \vec{c} \wedge \nabla \vec{\Phi}$ , hence using the definition of  $\vec{c}$ , we get

$$\begin{aligned}
|\nabla \vec{C}_1(z)| &\leq C|z|e^{\lambda|z|} (|\nabla H| + e^{-\lambda|z|} |\nabla \vec{n}|^2) \\
&\leq C \frac{1}{|z|} \left( \int_{B(0,2|z|) \setminus B(0,\frac{|z|}{2})} |\nabla \vec{n}|^2 dz \right)^{\frac{1}{2}}
\end{aligned} \tag{47}$$

The first inequality is a consequence of (16) and the Harnack estimate on the conformal factor, while the second uses (12), (23) and the fact that the norm on the dyadic annuli are uniformly bounded. As already observe the right hand side is uniformly bounded into  $L^2$ .  $\square$

**Lemma 4.3.** *There exists  $C > 0$ , independent of  $r$ , such that*

$$|\Delta \vec{C}_1(z)| \leq \frac{C}{|z|^2} \left( \int_{B(0,2|z|) \setminus B(0,\frac{|z|}{2})} |\nabla \vec{n}|^2 dz \right) \text{ for all } z \in B(0,1/2) \setminus B(0,2r), \tag{48}$$

*In particular,  $\Delta \vec{C}_1$  is uniformly bounded in  $L^1$ .*  $\square$

**Proof of lemma 49.** We have  $\Delta \vec{C}_1 = \vec{c} \wedge \Delta \vec{\Phi} = 2e^{2\lambda} \vec{c} \wedge \vec{H}$ , which gives, for all  $z \in B(0,1/2) \setminus B(0,2r)$ ,

$$|\Delta \vec{C}_1(z)| \leq C \left| e^{2\lambda} \vec{H}(z) \wedge \int_{\partial B(0,|z|)} \left( \partial_\nu \vec{H} - 3\pi_{\vec{n}}(\partial_\nu \vec{H}) - \star(\partial_\tau \vec{n} \wedge \vec{H}) \right) d\sigma \right|$$

Remembering (27), we have

$$|\Delta \vec{C}_1(z)| \leq C \left| e^{2\lambda} \vec{H}(z) \right|_{L^\infty(\partial B(0,|z|))} |z| \left( \|\nabla \vec{H}\|_{L^\infty(\partial B(0,|z|))} + \|\nabla \vec{n}\|_{L^\infty(\partial B(0,|z|))} \|\vec{H}\|_{L^\infty(\partial B(0,|z|))} \right).$$

Combining this with (12) and (23), we get

$$|\Delta \vec{C}_1(z)| \leq \frac{C}{|z|^2} \left( \int_{B(0,2|z|) \setminus B(0, \frac{|z|}{2})} |\nabla \vec{n}|^2 dz \right).$$

□

**Lemma 4.4.** *There exists  $C > 0$ , independent of  $r$ , such that*

$$\|\nabla \vec{C}_1\|_{L^2(B(0, \frac{1}{2}) \setminus B(0, 2r))} \leq C \sup_{\rho \in (r, 1)} \left( \int_{B(0, 2\rho) \setminus B(0, \frac{\rho}{2})} |\nabla \vec{n}|^2 dz \right)^{\frac{1}{4}} \quad (49)$$

□

**Proof of lemma 4.4.** A simple integration by part gives

$$\|\nabla \vec{C}_1\|_2^2 \leq \left| \int_{B(0, \frac{1}{2}) \setminus B(0, 2r)} \langle \vec{C}_1, \Delta \vec{C}_1 \rangle dz + \int_{\partial B(0, \frac{1}{2}) \cup \partial B(0, 2r)} \langle \vec{C}_1, \partial_\nu \vec{C}_1 \rangle d\sigma \right|$$

Now using (31), (47) and lemma 4.3 we obtain the desired result. □

#### 4.4 $L^{2, \infty}$ -bounds on $\nabla \vec{R}$ and $\nabla S$ .

Using (20) and (21) we get

$$|\nabla S| = O \left( |\vec{L}| e^{\lambda_\rho} + \left| \frac{C_0}{\rho} \right| \right) \text{ on } \mathbb{D} \setminus B(0, r) \quad (50)$$

and

$$|\nabla \vec{R}| = O \left( |\vec{L}| e^{\lambda_\rho} + |\vec{H}| e^{\lambda_\rho} + \left| \frac{\vec{C}_1}{\rho} \right| \right) \text{ on } \mathbb{D} \setminus B(0, r). \quad (51)$$

Now, combining (50), (51), (22), (23), lemma 3.2, lemma 3.4 and (9), we get

$$\|\nabla S\|_{L^{2, \infty}(\mathbb{D} \setminus B(0, r))} + \|\nabla \vec{R}\|_{L^{2, \infty}(\mathbb{D} \setminus B(0, r))} \leq C, \quad (52)$$

with  $C$  independent of  $r$ .

#### 4.5 $L^2$ -bounds on $\nabla S$ and $\nabla \vec{R}$ .

Let denote  $S_\rho = \frac{1}{|\partial B(0, \rho)|} \int_{\partial B(0, \rho)} S d\sigma$  and  $\vec{R}_\rho = \frac{1}{|\partial B(0, \rho)|} \int_{\partial B(0, \rho)} \vec{R} d\sigma$ . Thanks to equation (42) and the following lemma it suffices to control  $\int_r^1 \left| \frac{dS_\rho}{d\rho} \right|^2 \rho d\rho$  and  $\int_r^1 \left| \frac{d\vec{R}_\rho}{d\rho} \right|^2 \rho d\rho$  to prove that  $\nabla S$  and  $\nabla \vec{R}$  are uniformly bounded in  $L^2(B(0, 1/2) \setminus B(0, 2r))$ .

**Lemma 4.5** (lemma 2.4 [21]). *Let  $a, b \in L^2(B_1)$ ,  $0 < \varepsilon < \frac{1}{4}$ , assume that  $\nabla a \in L^{2,\infty}(B_1)$  and that  $\nabla b \in L^2(B_1)$ , let  $\varphi \in W^{1,(2,\infty)}(B_1 \setminus B_\varepsilon)$  a solution of*

$$\Delta \varphi = a_x b_y - a_y b_x \text{ on } B_1 \setminus B_\varepsilon, \quad (53)$$

Denote, for  $\varepsilon \leq r \leq 1$ ,  $\varphi_\rho := (2\pi\rho)^{-1} \int_{\partial B(0,\rho)} \varphi \, d\sigma$  and assume

$$\int_\varepsilon^1 \left| \frac{d\varphi_\rho}{d\rho} \right|^2 r \, dr < +\infty. \quad (54)$$

Then, for  $0 \leq \delta < 1$ , there exists a positive constant  $C(\delta) > 0$  independent of  $\varepsilon$  and  $\varphi$  such that

$$\begin{aligned} \|\nabla \varphi\|_{L^2(B_\delta \setminus B_{\delta-1\varepsilon})} &\leq C(\delta) \left( \|\nabla a\|_{2,\infty} \|\nabla b\|_2 + \|\nabla \varphi_\rho\|_{L^2(B_1 \setminus B_\varepsilon)} \right. \\ &\quad \left. + \|\nabla \varphi\|_{L^{2,\infty}(B_1 \setminus B_\varepsilon)} \right). \end{aligned} \quad (55)$$

□

Thanks to (41) and (40) we get

$$\frac{dS_\rho}{d\rho} = \int_0^{2\pi} \left\langle \frac{1}{\rho} \frac{\partial \vec{R}}{\partial \theta}, \star \vec{n} \right\rangle d\theta - \int_0^{2\pi} \left\langle \star \vec{n}, \frac{\vec{C}_1}{\rho} \right\rangle d\theta \quad (56)$$

and

$$\frac{d\vec{R}_\rho}{d\rho} = - \int_0^{2\pi} (-1)^m \star \left( \vec{n} \bullet \frac{1}{\rho} \frac{\partial \vec{R}}{\partial \theta} \right) d\theta - \int_0^{2\pi} \frac{1}{\rho} \frac{\partial S}{\partial \theta} \star \vec{n} \, d\theta + \int_0^{2\pi} \frac{C_0 \star \vec{n}}{\rho} d\theta + \int_0^{2\pi} (-1)^m \star \left( \vec{n} \bullet \frac{\vec{C}_1}{\rho} \right) d\theta. \quad (57)$$

On the one hand

$$\left| \int_0^{2\pi} \left\langle \frac{1}{\rho} \frac{\partial \vec{R}}{\partial \theta}, \star \vec{n} \right\rangle d\theta \right| = \left| \int_0^{2\pi} \left\langle \frac{1}{\rho} \frac{\partial \vec{R}}{\partial \theta}, \star (\vec{n}(z) - \vec{n}(|z|, 0)) \right\rangle d\theta \right| \leq \rho \|\nabla \vec{n}\|_{L^\infty(\partial B(0,\rho))} \|\nabla \vec{R}\|_{L^\infty(\partial B(0,\rho))}.$$

On the other hand, thanks to (21),

$$\|\nabla \vec{R}\|_{L^\infty(\partial B(0,\rho))} \leq e^{\lambda\rho} \left( \|\vec{L}\|_{L^\infty(\partial B(0,\rho))} + \|\vec{H}\|_{L^\infty(\partial B(0,\rho))} \right) + \left\| \frac{\vec{C}_1}{\rho} \right\|_{L^\infty(\partial B(0,\rho))}. \quad (58)$$

Then combining (12), (22), (31), (56), (58) we finally get

$$\left| \frac{dS_\rho}{d\rho} \right|^2 \leq \frac{C}{\rho^2} \int_{B(0,2\rho) \setminus B(0,\rho/2)} |\nabla \vec{n}|^2 \, dz$$

which gives

$$\int_r^1 \left| \frac{dS_\rho}{d\rho} \right|^2 \rho d\rho \leq C \int_{\mathbb{D} \setminus B(0,r)} |\nabla \vec{n}|^2 \, dz.$$

Mutatis mutandis, we get

$$\int_r^1 \left| \frac{d\vec{R}_\rho}{d\rho} \right|^2 \rho d\rho \leq C \int_{\mathbb{D} \setminus B(0,r)} |\nabla \vec{n}|^2 \, dz.$$

Now applying lemma 4.5, we get that

$$\|\nabla(S - \psi_S)\|_{L^2(B(0,\frac{1}{2}) \setminus B(0,2r))} \leq C \|\nabla \vec{n}\|_2 \|\nabla \vec{R}\|_{2,\infty}$$



where  $\psi_S \in W_0^{1,1}(\mathbb{D} \setminus B(0,r))$  satisfies

$$\Delta \psi_S = \operatorname{div}(C_0 \nabla^\perp \log(\rho)) - \operatorname{div}(\langle \vec{C}_1 \nabla \log(\rho), \star \vec{n} \rangle),$$

and thanks to standard elliptic theory, we have

$$\int_r^1 \left| \frac{d}{d\rho}(\psi_S)_\rho \right|^2 \rho \, d\rho \leq \|\nabla \psi_S\|_2^2 \leq C \left( \left\| \frac{C_0}{\rho} \right\|_{L^2(\mathbb{D} \setminus B(0,r))}^2 + \left\| \frac{\vec{C}_1}{\rho} \right\|_{L^2(\mathbb{D} \setminus B(0,r))}^2 \right)$$

which is bounded thanks to lemmas 3.2 and 3.4.

We also get

$$\begin{aligned} \|\nabla(\vec{R} - \psi_{\vec{R}})\|_{L^2(B(0,\frac{1}{2}) \setminus B(0,2r))} &\leq C \left( \|\nabla \vec{n}\|_2 (\|\nabla \vec{R}\|_{L^{2,\infty}(\mathbb{D} \setminus B(0,r))} + \|\nabla S\|_{L^{2,\infty}(\mathbb{D} \setminus B(0,r))}) \right) \\ &\quad + \left\| \frac{C_0}{\rho} \right\|_{L^2(\mathbb{D} \setminus B(0,r))} + \left\| \frac{\vec{C}_1}{\rho} \right\|_{L^2(\mathbb{D} \setminus B(0,r))} \end{aligned}$$

where  $\psi_{\vec{R}} \in W_0^{1,1}(\mathbb{D} \setminus B(0,r))$  satisfies

$$\Delta \psi_{\vec{R}} = \operatorname{div}(C_0 \nabla \log(\rho) \star \vec{n}) + \operatorname{div}(\vec{C}_1 \nabla^\perp \log(\rho)) + \operatorname{div}((-1)^m \star (\vec{n} \bullet \vec{c}_1 \nabla \log(\rho))),$$

and thanks to standard elliptic theory, we have

$$\int_r^1 \left| \frac{d}{d\rho}(\psi_{\vec{R}})_\rho \right|^2 \rho \, d\rho \leq \|\nabla \psi_{\vec{R}}\|_2^2 \leq C \left\| \frac{C_0}{\rho} \right\|_{L^2(\mathbb{D} \setminus B(0,r))}^2 + \left\| \frac{\vec{C}_1}{\rho} \right\|_{L^2(\mathbb{D} \setminus B(0,r))}^2$$

which is bounded thanks to lemma 3.2 and lemma 3.4.

Finally, using (52), we can conclude that

$$\|\nabla S\|_{L^2(B(0,\frac{1}{2}) \setminus B(0,2r))} + \|\nabla \vec{R}\|_{L^2(B(0,\frac{1}{2}) \setminus B(0,2r))} \leq C, \quad (59)$$

with  $C$  independent of  $r$ .

#### 4.6 $L^{2,1}$ -bounds on $\nabla S$ .

First, integrating (41), we get

$$\begin{aligned} \frac{dS_\rho}{d\rho} &= \int_0^{2\pi} \left\langle \frac{\partial \vec{R}}{\rho \partial \theta}, \star \vec{n}(z) \right\rangle d\theta - \int_0^{2\pi} \left\langle \star \vec{n}, \vec{C}_1 \frac{1}{\rho} \right\rangle d\theta \\ &= \int_0^{2\pi} \left\langle \frac{\partial \vec{R}}{\rho \partial \theta}, \star (\vec{n}(z) - \vec{n}((0,\rho))) \right\rangle d\theta - \int_0^{2\pi} \left\langle \star \vec{n}, \vec{C}_1 \frac{1}{\rho} \right\rangle d\theta \\ &\leq \left( \int_0^{2\pi} |\vec{n}(z) - \vec{n}((0,\rho))|^2 d\theta \right)^{\frac{1}{2}} \left( \int_0^{2\pi} |\nabla \vec{R}|^2 d\theta \right)^{\frac{1}{2}} + \left| \int_0^{2\pi} \left\langle \star \vec{n}, \vec{C}_1 \frac{1}{\rho} \right\rangle d\theta \right| \\ &\leq \left( \rho \int_{\partial B(0,\rho)} |\nabla \vec{n}|^2 d\sigma \right)^{\frac{1}{2}} \left( \frac{1}{\rho} \int_{\partial B(0,\rho)} |\nabla \vec{R}|^2 d\sigma \right)^{\frac{1}{2}} \\ &\quad + O \left( \frac{1}{\rho} \int_{B(0,2\rho) \setminus B(0,\frac{\rho}{2})} |\nabla \vec{n}|^2 dz \right) \end{aligned} \quad (60)$$

Here we used (45) to estimate the last term of third line. Finally, we get

$$\int_r^1 \left| \frac{dS_\rho}{d\rho} \right| d\rho \leq C \int_{\mathbb{D} \setminus B(0,r)} |\nabla \vec{n}|^2 dz \leq C$$

Since both  $S$  and  $\vec{R}$  are defined up to an additive constant, we can assume that  $S_r = 0$  and  $\vec{R}_r = 0$ , which gives

$$|S_\rho| \leq C \quad \forall \rho \in (r, 1).$$

Now let us recall the following lemma from [21].

**Lemma 4.6.** *Let  $a, b \in W^{1,2}(B_1)$ ,  $0 < \varepsilon < \frac{1}{4}$ , and  $\varphi \in W^{1,1}(B_1 \setminus B_\varepsilon)$  a solution of*

$$\begin{cases} \Delta \varphi = a_x b_y - a_y b_x \text{ on } B_1 \setminus B_\varepsilon \\ \int_{\partial B_\varepsilon} \varphi d\sigma = 0, \\ \left| \int_{\partial B_1} \varphi d\sigma \right| \leq K, \end{cases} \quad (61)$$

where  $K$  is a constant independent of  $\varepsilon$ . Then there exists a positive constant  $C$  independent of  $\varepsilon$  such that

$$\|\nabla \varphi\|_{L^{2,1}(B(0,1/2) \setminus B(0,2\varepsilon))} \leq C(\|\nabla a\|_2 \|\nabla b\|_2 + \|\nabla \varphi\|_2 + 1) \quad .$$

□

Thanks to lemma 3.2 and lemma 4.1, we easily get that the solution  $\psi_S \in W_0^{1,1}(\mathbb{D} \setminus B(0,r))$  of

$$\Delta \psi_S = \operatorname{div}(C_0 \nabla^\perp \log(\rho) \star \vec{n}) - \operatorname{div}((\vec{C}_1 \nabla \log(\rho), \star \vec{n})),$$

satisfies the following estimate

$$\|\psi_S\|_\infty \leq C \|\nabla \psi_S\|_{L^{2,1}(\mathbb{D} \setminus B(0,r))} \leq C \left( \|C_0 \nabla^\perp \log(\rho) \star \vec{n}\|_{L^{2,1}(\mathbb{D} \setminus B(0,r))} + \left\| \frac{\langle \vec{c}_1, \star \vec{n} \rangle}{\rho} \right\|_{L^{2,1}(\mathbb{D} \setminus B(0,r))} \right) \leq C, \quad (62)$$

with  $C$  independent of  $r$ . Here we have used the injection  $W_0^{1,(2,1)} \subset L^\infty$ , see theorem 3.3.4 of [13]. Applying lemma 4.6, we get

$$\|\nabla(S - \psi_S)\|_{L^{2,1}(B(0,\frac{1}{2}) \setminus B(0,2r))} \leq C(\|\nabla \vec{n}\|_2(\|\nabla \vec{R}\|_2 + \|\nabla S\|_2 + 1))$$

Finally using again (62), we get that  $\|\nabla S\|_{L^{2,1}(B(0,\frac{1}{2}) \setminus B(0,2r))}$  is uniformly bounded.

## 4.7 Final estimate

Thanks to (33), (24) and the previous paragraph, we get that

$$\left\| 4e^\lambda \vec{H} + e^{-\lambda} \nabla \vec{R} \lrcorner \nabla^\perp \vec{\Phi} - e^{-\lambda} \frac{1}{\rho} \vec{C}_1 \lrcorner \frac{\partial \vec{\Phi}}{\partial \rho} \right\|_{L^{2,1}(B(0,\frac{1}{2}) \setminus B(0,2r))} \leq C, \quad (63)$$

with  $C$  independent of  $r$ .

But (40) gives that

$$\nabla \vec{R} - (-1)^m \star (\vec{n} \bullet \vec{C}_1 \nabla \log(\rho)) = (-1)^m \star (\vec{n} \bullet \nabla^\perp \vec{R}) + \nabla^\perp S \star \vec{n} + C_0 \nabla \log(r) \star \vec{n} + \vec{C}_1 \nabla^\perp \log(\rho).$$

Let  $f, g \in W_0^{1,1}(\mathbb{D} \setminus B(0, r))$  such

$$\Delta \vec{f} = \nabla \vec{C}_1 \nabla^\perp \log(\rho) \quad (64)$$

and

$$\Delta \vec{g} = -(-1)^m \star (\nabla^\perp (\vec{n} \bullet \vec{C}_1) \nabla \log(\rho)).$$

Thanks to lemma B.4, lemma 4.4 and (31), we have

$$\|\nabla f\|_2 + \|\nabla g\|_2 \leq C \sup_{\rho \in (r, 1)} \left( \int_{B(0, 2\rho) \setminus B(0, \frac{\rho}{2})} |\nabla \vec{n}|^2 dz \right)^{\frac{1}{4}} \quad (65)$$

with  $C$  independent of  $r$ .

Then we set  $\vec{X} = \nabla \vec{R} - (-1)^m \star (\vec{n} \bullet \vec{C}_1 \nabla \log(\rho)) - \nabla f - \nabla^\perp g$  which satisfies

$$\begin{cases} \operatorname{div}(\vec{X}) = (-1)^m \star (\nabla \vec{n} \bullet \nabla^\perp \vec{R}) + \nabla^\perp S \nabla \star \vec{n} + \operatorname{div}(C_0 \nabla \log(r) \star \vec{n}) \\ \operatorname{curl}(\vec{X}) = 0 \end{cases}$$

Using lemma B.2 and lemma 3.2 and lemma 4.2, we get that there exist  $\vec{\alpha}$  and  $\vec{\beta}$  such that

$$\|\vec{X} - \vec{\alpha} \nabla \log(\rho) - \vec{\beta} \nabla^\perp \log(\rho)\|_{L^{2,1}(B(0, 1/2) \setminus B(0, 2r))} \leq C. \quad (66)$$

Proceeding as in (60), for all  $\rho \in (2r, 1/2)$ , we get

$$\left| \frac{d\vec{R}_\rho}{d\rho} - \frac{1}{2\pi} \int_0^{2\pi} (-1)^m \star (\vec{n} \bullet \frac{\vec{C}_1}{\rho}) \right| \leq C \left( \rho \int_{\partial B(0, \rho)} |\nabla \vec{n}|^2 d\sigma \right)^{\frac{1}{2}} \left( \frac{1}{\rho} \int_{\partial B(0, \rho)} |\nabla \vec{R}|^2 + |\nabla S|^2 d\sigma \right)^{\frac{1}{2}} + \left| \frac{C_0}{\rho} \right| \quad (67)$$

In particular, using paragraph 4.5 and lemma 3.2, we get that

$$\int_{2r}^{1/2} \left| \frac{d\vec{R}_\rho}{d\rho} - \frac{1}{2\pi} \int_0^{2\pi} (-1)^m \star \left( \vec{n} \bullet \frac{\vec{C}_1}{\rho} \right) \right| d\rho \leq C, \quad (68)$$

with  $C$  independent of  $r$ . Then, setting  $\vec{f}_\rho = \frac{1}{|\partial B(0, \rho)|} \int_{\partial B(0, \rho)} \vec{f} d\sigma$ , since

$$\frac{d\vec{f}_\rho}{d\rho} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial \vec{f}}{\partial \rho}(r, \theta) d\theta,$$

we get

$$\int_{2r}^{1/2} \left| \frac{\partial \vec{f}}{\partial \rho} \right| d\rho \leq C \int_{B(0, 1) \setminus B(0, r)} |\nabla f| dz \leq C \|\nabla f\|_2 \leq C. \quad (69)$$

Combining (68) and (69), we have

$$\int_{2r}^{1/2} \left| \frac{d\vec{R}_\rho}{d\rho} - \frac{1}{2\pi} \int_0^{2\pi} (-1)^m \star \left( \vec{n} \bullet \frac{\vec{C}_1}{\rho} \right) - \frac{d\vec{f}_\rho}{d\rho} \right| d\rho \leq C, \quad (70)$$

Thanks to (66) and the convexity of the norm we get

$$\left\| \frac{d\vec{R}_\rho}{d\rho} - \frac{1}{2\pi} \int_0^{2\pi} (-1)^m \star \left( \vec{n} \bullet \frac{\vec{C}_1}{\rho} \right) - \frac{df_\rho}{d\rho} - \frac{\vec{\alpha}}{\rho} \right\|_{L^{2,1}(B(0,1/2) \setminus B(0,2r))} \leq C \quad (71)$$

Using (9) and the last estimate, we get that

$$\begin{aligned} & \int_{2r}^{1/2} \left| \frac{d\vec{R}_\rho}{d\rho} - \frac{1}{2\pi} \int_0^{2\pi} (-1)^m \star \left( \vec{n} \bullet \frac{\vec{C}_1}{\rho} \right) - \frac{\vec{\alpha}}{\rho} \right| d\rho \\ & \leq \left\| \frac{d\vec{R}_\rho}{d\rho} - \frac{1}{2\pi} \int_0^{2\pi} (-1)^m \star \left( \vec{n} \bullet \frac{\vec{C}_1}{\rho} \right) - \frac{\vec{\alpha}}{\rho} \right\|_{L^{2,1}(B(0,1/2) \setminus B(0,2r))} \left\| \frac{1}{\rho} \right\|_{L^{2,\infty}(B(0,1/2) \setminus B(0,2r))} \leq C, \end{aligned} \quad (72)$$

with  $C$  independent of  $r$ . Then using (70) and (72), we get

$$\|\vec{\alpha} \nabla \log(\rho)\|_{L^{2,1}(B(0,1/2) \setminus B(0,2r))} \leq C.$$

Hence (66) gives

$$\|\vec{X} - \vec{\beta} \nabla^\perp \log(\rho)\|_{L^{2,1}(B(0,1/2) \setminus B(0,2r))} \leq C, \quad (73)$$

with  $C$  independent of  $r$ . Then we set  $\vec{X} = \vec{X}_1 \frac{\partial}{\partial \rho} + \vec{X}_2 \frac{\partial}{\partial \theta}$ , we get

$$\left\| \vec{X}_2 - \frac{\vec{\beta}}{\rho} \right\|_{L^{2,1}(B(0,1/2) \setminus B(0,2r))} \leq C,$$

with  $C$  independent of  $r$ . Using once more the norm convexity we get from (66) that

$$\left\| \frac{\vec{\beta}}{\rho} \right\|_{L^{2,1}(B(0,1/2) \setminus B(0,2r))} = \left\| (\vec{X}_2)_\rho - \frac{\vec{\beta}}{\rho} \right\|_{L^{2,1}(B(0,1/2) \setminus B(0,2r))} \leq C,$$

which leads, with (73), to

$$\|\nabla \vec{R} - (-1)^m \star (\vec{n} \bullet \vec{C}_1 \nabla \log(\rho)) - \nabla f - \nabla^\perp g\|_{L^{2,1}(B(0,1/2) \setminus B(0,2r))} \leq C,$$

with  $C$  independent of  $r$ . Hence

$$\|e^{-\lambda} (\nabla \vec{R} - (-1)^m \star (\vec{n} \bullet \vec{C}_1 \nabla \log(\rho))) \lrcorner \nabla^\perp \vec{\Phi} - e^{-\lambda} (\nabla f + \nabla^\perp g) \lrcorner \nabla^\perp \vec{\Phi}\|_{L^{2,1}(B(0,1/2) \setminus B(0,2r))} \leq C. \quad (74)$$

To conclude we need to make a last algebraic computation. In order to do so, we set

$$\vec{C}_1 = \sum_{\alpha < \beta} c_{\alpha\beta} \vec{n}_\alpha \wedge \vec{n}_\beta + \sum_\alpha c_\alpha^1 \vec{n}_\alpha \wedge \vec{e}_1 + \sum_\alpha c_\alpha^2 \vec{n}_\alpha \wedge \vec{e}_2 + c^3 \vec{e}_1 \wedge \vec{e}_2.$$

On the one hand

$$\langle \vec{C}_1 \lrcorner \vec{e}_1, \vec{n}_\alpha \rangle = -c_\alpha^1, \quad (75)$$

on the other hand, we get

$$\begin{aligned} \vec{n} \bullet \vec{C}_1 &= \sum_{\alpha < \beta=1}^{m-2} c_{\alpha\beta} ((\vec{n} \lrcorner \vec{n}_\alpha) \wedge \vec{n}_\beta - (\vec{n} \lrcorner \vec{n}_\beta) \wedge \vec{n}_\alpha) + \sum_{\alpha=1}^{m-2} c_\alpha^1 (\vec{n} \lrcorner \vec{n}_\alpha) \wedge \vec{e}_1 + \sum_{\alpha=1}^{m-2} c_\alpha^2 (\vec{n} \lrcorner \vec{n}_\alpha) \wedge \vec{e}_2 \\ &= \sum_{\alpha=1}^{m-2} c_\alpha^1 \left( (-1)^{\alpha-1} \bigwedge_{\beta \neq \alpha} \vec{n}_\beta \right) \wedge \vec{e}_1 + \sum_{\alpha=1}^{m-2} c_\alpha^2 \left( (-1)^{\alpha-1} \bigwedge_{\beta \neq \alpha} \vec{n}_\beta \right) \wedge \vec{e}_2. \end{aligned}$$

Then

$$\star(\vec{n} \bullet \vec{C}_1) = \sum_{\alpha=1}^{m-2} c_\alpha^1 (-1)^{m-1} \vec{e}_2 \wedge \vec{n}_\alpha + \sum_{\alpha=1}^{m-2} c_\alpha^2 (-1)^m \vec{e}_1 \wedge \vec{n}_\alpha .$$

Finally we have

$$\langle \star(\vec{n} \bullet \vec{C}_1) \lrcorner \vec{e}_2, \vec{n}_\alpha \rangle = (-1)^{m-1} c_\alpha^1 . \quad (76)$$

Hence (75) and (76) give

$$\left\langle e^{-\lambda} (-1)^m \star \left( \vec{n} \bullet \vec{C}_1 \frac{1}{\rho} \right) \lrcorner \frac{\partial \vec{\Phi}}{\partial \theta}, \vec{n}^\alpha \right\rangle = \left\langle e^{-\lambda} \vec{C}_1 \lrcorner \frac{\partial \vec{\Phi}}{\partial \rho}, \vec{n}^\alpha \right\rangle \quad (77)$$

To get our final estimate it suffies to prove that

$$\|e^\lambda H^\alpha\|_{L^2(B(0,1/2) \setminus B(0,2r))} \leq C \sqrt{\sup_{\rho \in (r,1)} \left( \int_{B(0,2\rho) \setminus B(0,\frac{\rho}{2})} |\nabla \vec{n}|^2 dz \right)^{\frac{1}{2}}}$$

with  $C$  independent of  $r$  and  $k$ . But taking the scalar product of (63) and  $\vec{n}^\alpha$ , we get

$$\left\| 4e^\lambda H^\alpha + \left\langle e^{-\lambda} \left( \nabla \vec{R} - \nabla^\perp \log(\rho) \vec{C}_1 \right) \lrcorner \nabla^\perp \vec{\Phi}, \vec{n}^\alpha \right\rangle \right\|_{L^{2,1}(B(0,1/2) \setminus B(0,2r))} \leq C. \quad (78)$$

Moreover (74) gives

$$\left\| \left\langle e^{-\lambda} (\nabla \vec{R} - (-1)^m \star(\vec{n} \bullet \vec{C}_1 \nabla \log(\rho)) - (\nabla f + \nabla^\perp g)) \lrcorner \nabla^\perp \vec{\Phi}, \vec{n}^\alpha \right\rangle \right\|_{L^{2,1}(B(0,1/2) \setminus B(0,2r))} \leq C. \quad (79)$$

Combining (77), the last inequality gives

$$\left\| \left\langle e^{-\lambda} \left( \nabla \vec{R} + \nabla^\perp \log(\rho) \vec{C}_1 - (\nabla f + \nabla^\perp g) \right) \lrcorner \nabla^\perp \vec{\Phi}, \vec{n}^\alpha \right\rangle \right\|_{L^{2,1}(B(0,1/2) \setminus B(0,2r))} \leq C, \quad (80)$$

with  $C$  independent of  $r$ . Then, combining (78) and (80) we have that

$$\left\| 4e^\lambda H^\alpha - e^{-\lambda} \left\langle (2\vec{C}_1 \nabla^\perp \log(\rho) - (\nabla f + \nabla^\perp g)) \lrcorner \nabla^\perp \vec{\Phi}, \vec{n}^\alpha \right\rangle \right\|_{L^{2,1}(B(0,1/2) \setminus B(0,2r))} \leq C. \quad (81)$$

Hence using the duality between  $L^{2,\infty}$  and  $L^{2,1}$ , (65), we see that it suffies to estimate

$$e^{-\lambda} \left\langle (\vec{C}_1 \nabla^\perp \log(\rho)) \lrcorner \nabla^\perp \vec{\Phi}, \vec{n}^\alpha \right\rangle.$$

But,

$$\begin{aligned} e^{-\lambda} \left\langle (\vec{C}_1 \nabla^\perp \log(\rho)) \lrcorner \nabla^\perp \vec{\Phi}, \vec{n}^\alpha \right\rangle &= \frac{e^{-\lambda}}{\rho} \left\langle \vec{C}_1 \lrcorner \frac{\partial \vec{\Phi}}{\partial \rho}, \vec{n}^\alpha \right\rangle = \frac{1}{\rho} \left\langle \vec{C}_1 \lrcorner \vec{e}_1, \vec{n}^\alpha \right\rangle = \frac{1}{\rho} \left\langle \vec{C}_1, \vec{e}_1 \wedge \vec{n}^\alpha \right\rangle \\ &= \frac{1}{\rho} \left\langle \star \vec{C}_1, \star(\vec{e}_1 \wedge \vec{n}^\alpha) \right\rangle = (-1)^{\alpha+1} \frac{1}{\rho} \left\langle \star \vec{C}_1, \vec{e}_2 \bigwedge_{\beta \neq \alpha} \vec{n}^\beta \right\rangle \\ &= (-1)^{\alpha+1} \frac{e^{-\lambda}}{\rho} \left\langle \star \vec{C}_1, \frac{\partial \vec{\Phi}}{\rho \partial \theta} \bigwedge_{\beta \neq \alpha} \vec{n}^\beta \right\rangle \\ &= (-1)^{\alpha+1} \frac{e^{-\lambda}}{\rho} \left( \frac{\partial \left\langle \star \vec{C}_1, \vec{\Phi} \bigwedge_{\beta \neq \alpha} \vec{n}^\beta \right\rangle}{\rho \partial \theta} - \left\langle \star \frac{\partial \vec{C}_1}{\rho \partial \theta}, \vec{\Phi} \bigwedge_{\beta \neq \alpha} \vec{n}^\beta \right\rangle \right) \\ &\quad - \left\langle \star \vec{C}_1, \vec{\Phi} \frac{\partial}{\rho \partial \theta} \left( \bigwedge_{\beta \neq \alpha} \vec{n}^\beta \right) \right\rangle. \end{aligned}$$

But

$$\left\langle \star(\bar{c} \wedge \bar{\Phi}), \bar{\Phi} \wedge \left( \bigwedge_{\beta \neq \alpha} \bar{n}^\beta \right) \right\rangle = 0. \quad (82)$$

Finally, using (82), we get

$$\begin{aligned} e^{-\lambda} \left\langle (\bar{C}_1 \nabla^\perp \log(\rho)) \lrcorner \nabla^\perp \bar{\Phi}, \bar{n}^\alpha \right\rangle &= (-1)^{\alpha+1} \frac{e^{-\lambda}}{\rho} \left( \frac{\partial \left\langle \star \bar{c}_1, \bar{\Phi} \bigwedge_{\beta \neq \alpha} \bar{n}^\beta \right\rangle}{\rho \partial \theta} \right. \\ &\quad \left. - \left\langle \star \frac{\partial \bar{C}_1}{\rho \partial \theta}, \bar{\Phi} \bigwedge_{\beta \neq \alpha} \bar{n}^\beta \right\rangle - \left\langle \star \bar{C}_1, \bar{\Phi} \frac{\partial}{\rho \partial \theta} \left( \bigwedge_{\beta \neq \alpha} \bar{n}^\beta \right) \right\rangle \right) \\ &= (-1)^{\alpha+1} \frac{e^{-\lambda}}{\rho} \left\langle \star \bar{c}_1, \frac{\partial \bar{\Phi}}{\rho \partial \theta} \bigwedge_{\beta \neq \alpha} \bar{n}^\beta \right\rangle + (-1)^{\alpha+1} \frac{e^{-\lambda}}{\rho} \left\langle \star \bar{c}_1, \bar{\Phi} \frac{\partial}{\rho \partial \theta} \bigwedge_{\beta \neq \alpha} \bar{n}^\beta \right\rangle \\ &\quad - (-1)^{\alpha+1} \frac{e^{-\lambda}}{\rho} \left\langle \star \frac{\partial \bar{C}_1}{\rho \partial \theta}, \bar{\Phi} \bigwedge_{\beta \neq \alpha} \bar{n}^\beta \right\rangle - (-1)^{\alpha+1} \frac{e^{-\lambda}}{\rho} \left\langle \star \bar{C}_1, \bar{\Phi} \frac{\partial}{\rho \partial \theta} \left( \bigwedge_{\beta \neq \alpha} \bar{n}^\beta \right) \right\rangle \end{aligned}$$

Moreover, up to a translation we can assume that  $\int_{\partial B(0,r)} \bar{\Phi} d\sigma = 0$ , then thanks to lemma 2.3 and remark 2.1, we easily get that

$$\left| \frac{1}{|\partial B(0,\rho)|} \int_{\partial B(0,\rho)} \bar{\Phi} d\sigma \right| \leq C\rho e^\lambda,$$

Hence  $\frac{\bar{\Phi}}{\rho} e^{-\lambda}$  is uniformly bounded.

Then, the first term of the right hand side have its  $L^2$  norm control by  $\lim_{k \rightarrow \infty} \frac{\bar{c}_1^k}{\sqrt{I_k}}$ , the  $L^2$ -norm control of the second and fourth terms are controlled by  $\bar{c}_1$  and  $\|\bar{C}_1\|_\infty$ , which goes to zero when the conformal class degenerates, see section 3.1.3, Finally the  $L^2$ -norm of the third term also goes to zero when the conformal class degenerates thanks to lemma 4.4. This achieves the proof of the main theorem. ■

## 5 Proof of theorem 0.2.

We are exactly under the assumption of the result of Bernard and Rivière, see [2], that we remind for the sake of completeness.

**Theorem 5.1.** *Let  $(\Sigma, h_k)$  a sequence of surfaces with fixed genus, constant curvature and normalized volume if needed. We assume that this sequence converges into the moduli space. Then let  $\bar{\Phi}_k : (\Sigma, h_k) \rightarrow \mathbb{R}^m$  a sequence of conformal Willmore immersions with bounded energy, i.e.*

$$\limsup_{k \rightarrow +\infty} W(\bar{\Phi}_k) < +\infty.$$

*Then, there exists a branched smooth immersion  $\bar{\Phi}_\infty : \Sigma \rightarrow \mathbb{R}^m$  and a finite number of possibly branched immersions  $\omega_j : S^2 \rightarrow \mathbb{R}^m$  and  $\zeta_t : S^2 \rightarrow \mathbb{R}^m$  which are all Willmore away from possibly finitely many points, and such that, up to a subsequence,*

$$\lim_{k \rightarrow +\infty} W(\bar{\Phi}_k) = W(\bar{\Phi}_\infty) + \sum_{j=1}^p W(\omega_j) + \sum_{t=1}^q (W(\zeta_t) - m_t 4\pi) \quad . \quad (83)$$

where  $m_t$  is the integer multiplicity of  $\zeta_t$  at the origin.  $\square$

We want to prove that there is no bubble. Let pick a point where a possible bubble concentrates and we choose some conformal parametrization around this point. That is to say, we assume that  $\Phi_k : \mathbb{D} \rightarrow \mathbb{R}^3$  is a conformal Willmore immersion and concentrates only at 0. Then by some classical scheme of bubble extraction, we can find  $z_k \in \mathbb{D}$ ,  $\lambda_k \in \mathbb{R}_+^*$  and  $\theta_0 \in \mathbb{R}$  such that

$$z_k \rightarrow 0$$

$$\lambda_k \rightarrow 0$$

and

$$\tilde{\Phi}_k = \frac{\Phi_k(z_k + \lambda_k z) - \Phi_k(z_k)}{|\Phi_k(z_k) - \Phi_k(z_k + \lambda_k e^{i\theta_0})|} \text{ converges to } \tilde{\Phi}_\infty \text{ in } C_{loc}^2(\mathbb{R}^2)$$

where  $\tilde{\Phi}_\infty$  is a conformal Willmore immersion from  $\mathbb{R}^2$  to  $\mathbb{R}^3$ , which is not a plane.

**Claim 1 :  $\tilde{\Phi}_\infty$  is bounded at infinity.**

If  $\tilde{\Phi}_\infty$  is unbounded at infinity, hence considering an appropriate inversion, we can close its image. The closed surface is Lipschitz with a possible branched point, see Lemma A.5 of [33], but for topological reason, the order of this branched point is odd, see lemma C.1 in the appendix. If the branching order is bigger than 3, then the energy will be close to  $12\pi$ , hence the closing point is not branched. Moreover the closing point is regular since the residue vanishes (every loop is homotopic to a point), see theorem I.2 in [3]. So we have a Bryant sphere whose energy is below  $12\pi$  that is to say a round sphere, hence it was a plan before inversion, which is a contradiction.  $\square$

**Claim 2 :  $\tilde{\Phi}_\infty(\mathbb{R}^2)$  is a round sphere.**

Then,  $\tilde{\Phi}_\infty$  is bounded at infinity, using Lemma A.5 of [33], it is easy to see that it comes from a Willmore immersion from  $S^2$  to  $S^3$  with at most one branched point at infinity. Hence the first non trivial order is 3, see lemma C.1, which would insure that this surface has at least  $12\pi$  of energy, contradicting our hypothesis. Hence, we can assume there is no branched point at infinity, and it is smooth since the first residue must vanish, so we have a Bryant whose energy is below  $12\pi$  that is to say a round sphere.  $\square$

Then, we are going to study the neck region joining this sphere to our limiting surface. Let

$$l_k(r) = \int_0^{2\pi} \left| \frac{\partial \Phi_k}{\partial \theta} \right| d\theta.$$

Let  $r_k$  such that  $l(r_k) = \inf_{r \in [\lambda_k, 1/2]} l(r)$ .

**Claim 3:  $\lambda_k \ll r_k \ll 1$  for  $k$  big enough and  $l(r_k) \rightarrow 0$ .**

Indeed, the infimum can't be achieved at the scale  $\lambda_k$  of the blown sphere, since we can we can always reduce the length choosing a bigger radius, that it is to say shrinking the curve to the pole. And it is neither achieved at the scale of the limit surface, since this one is closed once the singularity at 0 is erased. Moreover  $l(r_k) \leq l(r)$  for all  $r \in (0, 1/2)$  which achieved the proof of the claim.  $\square$

Let  $\theta_1$  such that no concentration of the energy occurs at  $e^{i\theta_1}$  for  $\frac{\Phi_k(z_k + r_k z)}{l(r_k)}$ . It is always possible to find such an angle, since there is always only a finite number of points of concentration. Then, following

our classical scheme of blow-up, we have that

$$\hat{\Phi}_k = \frac{\Phi_k(z_k + rz) - \Phi_k(z_k + e^{i\theta_1} r_k)}{l(r_k)} \text{ converges to } \hat{\Phi}_\infty \text{ in } C_{loc}^2(\mathbb{R}^2 \setminus S),$$

where  $S$  is a finite set of possible concentration points. A priori, the conformal factor can diverge. But, since it must satisfies some harnack estimate, if it diverges then it diverges uniformly. If it goes to  $-\infty$ , we pick  $r > 0$  such that there is no concentration on  $\partial B(0, r)$ , hence the  $\Phi_k(\partial B(0, r))$  will be very small with respect to  $\Phi_k(\partial B(0, 1))$ , contradicting the definition of  $r_k$ . It can't neither go to  $+\infty$  since  $\Phi_k(\partial B(0, 1))$  is of length one.

**Claim 4 :**  $\hat{\Phi}_\infty(\mathbb{R}^2 \setminus S)$  has at least two ends.

We are going to prove that there is one end at 0, we will obtain the second one at infinity considering an inversion of the parametrization. Indeed, if  $\hat{\Phi}_\infty$  were bounded around zero, then we will be able to erase the singularity because the branching order is necessary one and the first residue vanishes. Hence for  $r > 0$  small enough and  $k$  big enough the length of  $\Phi_k(\partial B(0, r))$  will be smaller than  $1/2$ , which will be a contradiction with the definition of  $r_k$ .  $\square$

Now considering an appropriate inversion of  $\hat{\Phi}_\infty$ , we can close this surface and erase all potential singularities as above. Hence the result is a Willmore sphere which is not a round sphere, else the surface would have only one end before inversion. Thanks to the classification of Bryant, it implies that  $\hat{\Phi}_\infty$  have at least four ends.

The end corresponding to the infinity is joined to the limiting surface and provides at least  $\beta_g$  of energy<sup>10</sup>. The three others are given by 0 and two other points of concentration. Because they need to be "closed", they contribute to at least  $4\pi$  each. It can be seen, thanks to the following monotonicity formula, see [32], let  $\Sigma$  a smooth surface with boundary and  $\Phi : \Sigma \rightarrow \mathbb{R}^m$  be a weak immersion, then

$$4\pi \leq \int_{\Sigma} |\vec{H}|^2 dv_g + 2 \frac{\mathcal{H}^1(\partial\Sigma)}{d(\partial\Sigma, \Sigma)},$$

where  $g = \Phi^*(\xi)$ ,  $\mathcal{H}^1$  the 1-dimensional Hausdorff measure, and  $d(A, B) = \sup_{p \in A} \inf_{q \in B} |p - q| + \sup_{q \in B} \inf_{p \in A} |p - q|$ .

Hence applying this monotonicity formula around the three points of concentration that provided an end, we get that total energy of the surface is as close as we want to  $\beta_g + 12\pi$ . This excludes the existence of a concentration point and the proof of the theorem is completed.  $\square$

## A Residues of Willmore Hopf Tori.

In this section, we briefly explain that the residue  $\vec{c}_1$ , which notably appears in our main theorem, is a non trivial invariant. Here is the main result

**Proposition A.1.** *Let  $\vec{\Phi} : [0, L/2] \times S^1 \rightarrow S^3$  a Willmore Hopf tori, then the residue  $\vec{c}_1$  compute on any  $\{t\} \times S^1$  vanishes if and only if the tori is equivalent to the Clifford tori.*

**Proof proposition A.1.** let  $\gamma : [0, L/2] \rightarrow S^2$  an elastic curves such that  $|\dot{\gamma}| = 2$ , i.e a critical point of

$$\int_0^{L/2} (1 + k^2(s)) ds,$$

<sup>10</sup> $\beta_g = \inf\{W(\Phi) \mid \Phi : \Sigma \rightarrow \mathbb{R}^3 \text{ immersion and } \text{genus}(\Sigma) = g\}$



where  $k$  is the geodesic curvature of  $\gamma$  into  $S^2$ . We consider a lift  $\tilde{\gamma} : [0, L/2] \rightarrow S^3$  of  $\gamma$ , via the Hopf fibration  $\pi : S^3 \rightarrow S^2$ , such that  $\tilde{\gamma}$  cuts the fiber of  $\pi$  orthogonally. Then we set

$$\vec{\Phi}(s, \theta) = e^{i\theta} \tilde{\gamma}(s),$$

where  $S^3$  is seen as the set of unit quaternions, see [30]. Then  $\vec{\Phi}$  is an isometric immersion, and since  $\gamma$  is an elastica, the image of  $\vec{\Phi}$  is Willmore in  $S^3$ , i.e. a critical point of

$$\int (1 + H^2) d\sigma,$$

which implies that  $\vec{\Phi} : [0, L/2] \times S^1 \rightarrow \mathbb{R}^4$  is Willmore. In this setting, the normal is equal to

$$\vec{N} = \vec{n} \wedge \vec{\Phi},$$

where  $\vec{n}$  is the normal of  $\vec{\Phi}$  into  $S^3$ . We easily check that  $\vec{n} = i\partial_s \vec{\Phi}$ . The mean curvature vector is equal to

$$\vec{H} = k(s)\vec{n} - \vec{\Phi}.$$

Hence

$$\vec{c}_1 = \int_0^{2\pi} -(\partial_s \vec{H} - 3\pi_{\vec{N}}(\partial_s \vec{H}) - \star(\partial_\theta \vec{N} \wedge \vec{H})) \wedge \vec{\Phi} + 2(\star(\vec{N} \lrcorner \vec{H}) \lrcorner \partial_\theta \vec{\Phi}) d\theta.$$

Let compute each term separately.

$$\begin{aligned} I &= - \int_0^{2\pi} -\partial_s \vec{H} \wedge \vec{\Phi} d\theta \\ &= - \int_0^{2\pi} \partial_s k \vec{N} - k \partial_s \vec{n} \wedge \vec{\Phi} + \partial_s \vec{\Phi} \wedge \vec{\Phi} d\theta \end{aligned}$$

we check that

$$\int_0^{2\pi} \vec{N} d\theta = \int_0^{2\pi} \vec{n} \wedge \vec{\Phi} d\theta = \pi(\vec{n}(s, 0) \wedge \vec{\Phi}(s, 0) - \partial_s \vec{\Phi}(s, 0) \wedge \partial_\theta \vec{\Phi}(s, 0))$$

and

$$\int_0^{2\pi} \partial_s \vec{n} \wedge \vec{\Phi} d\theta = -2k \int_0^{2\pi} \partial_s \vec{\Phi} \wedge \vec{\Phi} d\theta - \int_0^{2\pi} \partial_\theta \vec{\Phi} \wedge \vec{\Phi} d\theta$$

where we used the fact that  $\partial_s \vec{n} = -2k\partial_s \vec{\Phi} - \partial_\theta \vec{\Phi}$ . But

$$\int_0^{2\pi} \partial_s \vec{\Phi} \wedge \vec{\Phi} d\theta = \pi(\partial_s \vec{\Phi}(s, 0) \wedge \vec{\Phi}(s, 0) + \vec{n}(s, 0) \wedge \partial_\theta \vec{\Phi}(s, 0))$$

and

$$\int_0^{2\pi} \partial_\theta \vec{\Phi} \wedge \vec{\Phi} d\theta = 2\pi(\partial_\theta \vec{\Phi}(s, 0) \wedge \vec{\Phi}(s, 0)).$$

Finally

$$\begin{aligned} I &= -\pi(\partial_s k(\vec{n}(s, 0) \wedge \vec{\Phi}(s, 0) - \partial_s \vec{\Phi}(s, 0) \wedge \partial_\theta \vec{\Phi}(s, 0)) - k(-2k(\partial_s \vec{\Phi}(s, 0) \wedge \vec{\Phi}(s, 0) + \vec{n}(s, 0) \wedge \partial_\theta \vec{\Phi}(s, 0)) \\ &\quad - 2(\partial_\theta \vec{\Phi}(s, 0) \wedge \vec{\Phi}(s, 0))) + (\partial_s \vec{\Phi}(s, 0) \wedge \vec{\Phi}(s, 0) + \vec{n}(s, 0) \wedge \partial_\theta \vec{\Phi}(s, 0))). \end{aligned}$$

Smilingly we have,

$$II = \int_0^{2\pi} 3\pi_{\vec{N}}(\partial_s \vec{H}) \wedge \vec{\Phi} d\theta = 3\pi \partial_s k(\vec{n}(s, 0) \wedge \vec{\Phi}(s, 0) - \partial_s \vec{\Phi}(s, 0) \wedge \partial_\theta \vec{\Phi}(s, 0)) \\ + 3\pi k(-2k(\partial_s \vec{\Phi}(s, 0) \wedge \vec{\Phi}(s, 0) + \vec{n}(s, 0) \wedge \partial_\theta \vec{\Phi}(s, 0)) - 2(\partial_\theta \vec{\Phi}(s, 0) \wedge \vec{\Phi}(s, 0)))$$

$$III = \int_0^{2\pi} \star(\partial_\theta \vec{N} \wedge \vec{H}) \wedge \vec{\Phi} d\theta = \int_0^{2\pi} k \partial_\theta \vec{\Phi} \wedge \vec{\Phi} - \partial_s \vec{\Phi} \wedge \vec{\Phi} d\theta \\ = 2\pi k(\partial_\theta \vec{\Phi}(s, 0) \wedge \vec{\Phi}(s, 0)) - \pi(\partial_s \vec{\Phi}(s, 0) \wedge \vec{\Phi}(s, 0) + \vec{n}(s, 0) \wedge \partial_\theta \vec{\Phi}(s, 0)),$$

and

$$IV = \int_0^{2\pi} 2(\star(\vec{N} \lrcorner \vec{H}) \lrcorner \partial_\theta \vec{\Phi}) d\theta = \int_0^{2\pi} 2k \partial_\theta \vec{\Phi} \wedge \vec{n} - 2\partial_\theta \vec{\Phi} \wedge \vec{\Phi} d\theta \\ = 2\pi k(\partial_\theta \vec{\Phi}(s, 0) \wedge \vec{n}(s, 0) + \vec{\Phi}(s, 0) \wedge \partial_s \vec{\Phi}(s, 0)) - 4\pi(\partial_\theta \vec{\Phi}(s, 0) \wedge \vec{\Phi}(s, 0)).$$

All the right hand side are orthogonal to  $\vec{n}(s, 0) \wedge \vec{\Phi}(s, 0)$ , except the first term of  $I$ , i.e.  $-\pi \partial_s k(\vec{n}(s, 0) \wedge \vec{\Phi}(s, 0))$ . Hence, if  $\vec{c}_1 = 0$ , we necessarily get  $\partial_s k = 0$ , but since the equation of an elastica is given by

$$\partial_{ss} k + \frac{1}{2}(k^3 + k) = 0.$$

We get  $k = 0$ , that is to say  $\gamma$  must be a great circle, which implies that  $\vec{\Phi}$  parametrizes a Clifford Tori. Reciprocally, it is clear that if  $k = 0$ , then  $\vec{c}_1 = 0$ .

## B Integrability by compensation results.

First of all, we give a more precise version of lemma A.1 of [21].

**Lemma B.1.** *Let  $0 < \varepsilon < \frac{1}{2}$  and  $f : \mathbb{D} \setminus B(0, \varepsilon) \rightarrow \mathbb{R}$  an harmonic function. Then, there exists  $d \in \mathbb{R}$  and a positive constant  $C$  independent of  $\varepsilon$  and  $f$  such that*

$$\left\| \nabla f - \frac{d}{\rho} \right\|_{L^{2,1}(B(0,1/2) \setminus B(0,2\varepsilon))} \leq C \|\nabla f\|_2.$$

□

**Proof of lemma B.1.** We start by decomposing  $f$  as a Fourier series, which gives

$$f(\rho, \theta) = c_0 + d_0 \ln(\rho) + \sum_{n \in \mathbb{Z}^*} (c_n \rho^n + d_n \rho^{-n}) e^{in\theta}.$$

Then we get

$$f(\rho, \theta) - c_0 - d_0 \ln(\rho) = \sum_{n \in \mathbb{Z}^*} (c_n \rho^n + d_n \rho^{-n}) e^{in\theta}.$$

Then we estimate the gradient as follows

$$\left| \nabla f(\rho, \theta) - \frac{d_0}{\rho} \right| \leq 2 \sum_{n \in \mathbb{Z}^*} (|n c_n| \rho^{n-1} + |n d_n| \rho^{-n-1}).$$

Then, we estimate the  $L^{2,1}$ -norm of the  $f_m(z) = |z|^m$  on  $B(0, 1/2) \setminus B(0, 2\varepsilon)$ , for  $m \in \mathbb{Z} \setminus \{-1\}$ , which gives

$$\begin{aligned} \|f_m\|_{L^{2,1}(B(0,1/2) \setminus B(0,2\varepsilon))} &\leq \sqrt{\pi} \int_0^{(2\varepsilon)^m} t^{\frac{1}{m}} dt \leq 2\sqrt{\pi}(2\varepsilon)^{m+1} \text{ for } m < -1 \\ \text{and} & \\ \|f_m\|_{L^{2,1}(B(0,1/2) \setminus B(0,2\varepsilon))} &\leq \frac{\sqrt{\pi}}{2^{m+1}} \text{ for } m \geq 0. \end{aligned} \tag{84}$$

Hence we get

$$\|\nabla f\|_{L^{2,1}(B(1,2) \setminus B(0,2\varepsilon))} \leq 4\sqrt{\pi} \left( \sum_{n>0} (|n c_n| + |n d_{-n}|) 2^{-n} + \sum_{n<0} (|n c_n| + |n d_{-n}|) (2\varepsilon)^n \right).$$

Hence, thanks to the Cauchy-Scharwz, we get

$$\|\nabla f\|_{L^{2,1}(B(0,1/2) \setminus B(0,2\varepsilon))} \leq 8\sqrt{\pi} \left( \sum_{n \neq 0} |n| 2^{-2|n|} \right)^{\frac{1}{2}} \left( \sum_{n>0} (|n| |c_n|^2 + |n| |d_{-n}|^2) + \sum_{n<0} (|n| |c_n|^2 + |n| |d_{-n}|^2) \varepsilon^{2n} \right)^{\frac{1}{2}}.$$

That is to say

$$\|\nabla f\|_{L^{2,1}(B(0,1/2) \setminus B(0,2\varepsilon))} \leq 8\sqrt{\pi} \left( \sum_{n \neq 0} |n| 2^{-2|n|} \right)^{\frac{1}{2}} \left( \sum_{n>0} (|n| |c_n|^2 + |n| |d_n|^2 \varepsilon^{-2n}) + \sum_{n<0} |n| |c_n|^2 \varepsilon^{2n} + |n| |d_n|^2 \right)^{\frac{1}{2}}.$$

Finally we compute the  $L^2$ -norm of  $\nabla f$ ,

$$\begin{aligned} \|\nabla f\|_2 &= |d_0| (\log(1/\varepsilon))^{\frac{1}{2}} + \left( 2\pi \int_{\varepsilon}^1 \sum_{n \neq 0} (|n c_n|^2 \rho^{2n-2} + |n d_n|^2 \rho^{-2n-2} + 2n^2 \Re(c_n \bar{d}_n) \rho^{-2}) \rho d\rho \right)^{\frac{1}{2}} \\ &\geq \left( 2\pi \sum_{n \neq 0} \frac{|n c_n|^2}{2n} (1 - \varepsilon^{2n}) - \frac{|n d_n|^2}{2n} (1 - \varepsilon^{-2n}) + 2n^2 \Re(c_n \bar{d}_n) \log\left(\frac{1}{\varepsilon}\right) \right)^{\frac{1}{2}} \\ &\geq C \left( \sum_{n<0} |n| |c_n|^2 \varepsilon^{2n} + |n| |d_n|^2 + \sum_{n>0} |n| |d_n| \varepsilon^{-2n} + |n| |c_n|^2 \right)^{\frac{1}{2}} \end{aligned}$$

which completes the proof of lemma B.1 . □

**Lemma B.2.** *Let  $0 < r < 1/2$  and  $a, b, c, d \in W^{1,2}(\mathbb{D} \setminus B(0, r))$ . Let  $\vec{X} \in L^1(\mathbb{D} \setminus B(0, r), \mathbb{R}^2)$  satisfying*

$$\begin{cases} \operatorname{div}(\vec{X}) = a_x b_y - a_y b_x \\ \operatorname{curl}(\vec{X}) = c_x d_y - c_y d_x \end{cases}.$$

*Then there exists  $\vec{\alpha}, \vec{\beta} \in \mathbb{R}^2$  and  $C > 0$  a constant independent of  $r$ , such that*

$$\|\vec{X} - \vec{\alpha} \nabla \log(\rho) - \vec{\beta} \nabla^\perp \log(\rho)\|_{L^{2,1}(B(0,1/2) \setminus B(0,2r))} \leq C(\|\nabla a\|_2 \|\nabla b\|_2 + \|\nabla c\|_2 \|\nabla d\|_2).$$

□

**Proof of lemma B.2.** Let  $f, g \in W^{1,1}(\mathbb{D} \setminus B(0, r))$  and  $\vec{\alpha}', \vec{\beta}' \in \mathbb{R}^2$  such that

$$\vec{X} = \nabla f + \vec{\alpha}' \nabla \log(\rho) + \nabla^\perp g + \vec{\beta}' \nabla^\perp \log(\rho).$$

Then  $f$  satisfies

$$\Delta f = a_x b_y - a_y b_x.$$

Hence we can apply lemma B.1, and we get that there exists  $\vec{\alpha} \in \mathbb{R}^2$  and  $C > 0$  a constant independent of  $r$ , such that

$$\|\nabla f - \vec{\alpha} \nabla \log(\rho)\|_{L^{2,1}(B(0,1/2) \setminus B(0,2r))} \leq C(\|\nabla a\|_2 \|\nabla b\|_2).$$

Then we proceed exactly the same way for  $g$ , which achieves the proof.  $\square$

An improvement of the classical Wente inequality was obtain by Bethuel [4] using a duality argument.

**Lemma B.3.** *Let  $a$  and  $b$  be two measurable functions such that  $\nabla a \in L^{2,\infty}(B(0,1))$  and  $\nabla b \in L^2(B(0,1))$ . Let  $\varphi \in W_0^{1,1}(B_1)$  be the solution of*

$$\Delta \varphi = a_x b_y - a_y b_x \text{ on } B_1$$

*Then there exists a constant  $C$  independent of  $\varphi$  such that*

$$\|\nabla \varphi\|_2 \leq C \|\nabla a\|_{2,\infty} \|\nabla b\|_2. \quad (85)$$

$\square$

**Lemma B.4.** *Let  $0 < \varepsilon < 1/2$ ,  $a$  and  $b$  be two measurable functions such that  $\nabla a \in L^{2,\infty}(B(0,1) \setminus B(0,\varepsilon))$  and  $\nabla b \in L^2(B(0,1) \setminus B(0,\varepsilon))$ . Let  $\varphi \in W_0^{1,1}(B(0,1) \setminus B(0,\varepsilon))$  be the solution of*

$$\Delta \varphi = a_x b_y - a_y b_x \text{ on } B_1$$

*Then there exists a constant  $C$  independent of  $\varphi$  and  $\varepsilon$  such that*

$$\|\nabla \varphi\|_2 \leq C \|\nabla a\|_{2,\infty} \|\nabla b\|_2. \quad (86)$$

$\square$

**Proof of lemma B.4.** Let  $\tilde{a}$  and  $\tilde{b}$  be Whitney extensions of  $a$  and  $b$  in  $B(0,\varepsilon)$  such that

$$\|\nabla \tilde{a}\|_{2,\infty} \leq C \|\nabla a\|_{2,\infty}$$

and

$$\|\nabla \tilde{b}\|_2 \leq C \|\nabla b\|_2.$$

Then, if  $\tilde{\varphi} \in W_0^{1,1}(B(0,1))$  satisfies

$$\Delta \tilde{\varphi} = \tilde{a}_x \tilde{b}_y - \tilde{a}_y \tilde{b}_x \text{ on } B(0,1)$$

we have from the previous lemma that

$$\|\nabla \tilde{\varphi}\|_2 \leq C \|\nabla a\|_{2,\infty} \|\nabla b\|_2. \quad (87)$$

Moreover  $\varphi - \tilde{\varphi}$  is nothing else than the harmonic extension of  $-\tilde{\varphi}$  in  $B(0,1) \setminus B(0,\varepsilon)$ , hence the  $L^2$  of its gradient does not exceed the one of  $\nabla \tilde{\varphi}$ , which permits to conclude.  $\square$

## C Branched points are of odd order

We give a short proof of the fact in  $\mathbb{R}^3$ , branched point obtain as limit of immersion are odd. It can also be found in [20], see lemma 3.6.

**Lemma C.1.** *Let  $\vec{\Phi}_k : \mathbb{D} \rightarrow \mathbb{R}^3$  a sequence of immersion which converge to  $\vec{\Phi}_\infty : \mathbb{D} \rightarrow \mathbb{R}^3$  in  $C_{loc}^2(\mathbb{D} \setminus \{0\})$ . If  $\vec{\Phi}_\infty$  have an isolated branched point of finite order at 0, then its order is odd.*

**Proof of lemma C.1.** Up to, Rotate and reparametrize  $\vec{\Phi}_\infty$  we can assume that

$$\vec{\Phi}_\infty(z) = \begin{pmatrix} z^m \\ 0 \end{pmatrix} + o(|z|^m).$$

Let  $r > 0$   $M_k : \partial B(0, r) \rightarrow SO(3)$  define by

$$M_k(x) = (\vec{e}_1^k, \vec{e}_2^k, \vec{e}_1^k \wedge \vec{e}_2^k),$$

where  $\vec{e}_1^k = \frac{(\vec{\Phi}_k)_x}{\|(\vec{\Phi}_k)_x\|}$  and  $\vec{e}_2^k = \frac{(\vec{\Phi}_k)_y - \langle (\vec{\Phi}_k)_y, (\vec{\Phi}_k)_x \rangle}{\|(\vec{\Phi}_k)_y - \langle (\vec{\Phi}_k)_y, (\vec{\Phi}_k)_x \rangle\|}$ .

Since  $\vec{\Phi}_k$  is an immersion,  $M_k$  must be in the same connected component than the loop

$$\Gamma_1(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

But, taking  $r > 0$  small enough and  $k$  large enough,  $M_k$  is clearly homotopic to

$$\Gamma_m(\theta) = \begin{pmatrix} \cos(m\theta) & -\sin(m\theta) & 0 \\ \sin(m\theta) & \cos(m\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which impose that  $m \equiv 1[2]$ , since the fundamental group of  $SO(3)$  posses exactly two elements: the odd loops and even loops.  $\square$

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