The Singular Set of 1-1 Integral Currents.

Tristan Rivière* and Gang Tian†

Abstract : *We prove that 2 dimensional integer multiplicity 2 dimensional rectifiable currents which are almost complex cycles in an almost complex manifold admitting locally a compatible positive symplectic form are smooth surfaces aside from isolated points and therefore are $J-$holomorphic curves.*

I Introduction

Let (M^{2p}, J) be an almost complex manifold. Let $k \in \mathbb{N}$, $k \leq p$. We shall adopt classical notations from Geometric Measure Theory [Fe]. We say that a $2k-$current C in (M^{2p}, J) is an almost complex integral cycle whenever it fulfills the following three conditions

i) *rectifiability :* there exists an at most countable union of disjoint oriented C^1 $2k-$submanifolds $\mathcal{C} = \cup_i N_i$ and an integer multiplicity $\theta \in L^1_{loc}(\mathcal{C})$ such that for any smooth compactly supported in M $2k-$form ψ one has

$$C(\psi) = \sum_i \int_{N_i} \theta \ \psi \quad .$$

ii) *closedness :* C is a cycle

$$\partial C = 0 \quad \text{i. e. } \forall \alpha \in \mathcal{D}^{2k-1}(M) \quad C(d\alpha) = 0$$

*D-Math, ETH Zentrum, 8049 Zürich, Switzerland.
†Department of Mathematics, Princeton University, Princeton NJ 08544, USA.

iii) *almost complex :* for $\mathcal{H}^{2k}$ almost every point x in $\mathcal{C}$ the approximate tangent plane T_x to the rectifiable set $\mathcal{C}$ is invariant under the almost complex structure J i.e.

$$J(T_x) = T_x$$

In this work we address the question of the regularity of such a cycle : Does there exist a *smooth* almost complex manifold (Σ^{2k}, j) without boundary and a *smooth* $j - J$-holomorphic map u ($\forall x \in \Sigma$ and $\forall X \in T_x\Sigma$ $du_x j \cdot X = J \cdot du_x X$) such that u would realise an *embeding* in M^{2p} aside from a locally finite $2k-2$ measure closed subset of M and such that $C = u_*[\Sigma^{2k}]$, i.e. $\forall \psi \in C_0^\infty(\wedge^{2k} M)$

$$C(\psi) = \int_\Sigma u^* \psi$$

In the very particular case where the almost complex structure J is integrable, this regularity result is optimal (C is the integral over multiples of algebraic subvarieties of M) and was established in [HS] and [Ale]. There are numerous motivations for studiying the general case of arbitrary almost complex structures J. First, as explained in [RT], the above regularity question for rectifiable almost complex cycles is directly connected to the regularity question of $J-$holomorphic maps into complex projective spaces. It is conjectured, for instance, that the singular set of $W^{1,2}(M^{2p}, N)$ $J-$holomorphic maps between almost complex manifolds M and N should be of finite $(2p-4)$-Hausdorff measure. The resolution of that question leads, for instance, to the caracterisation of stable bundle almost complex structures over almost Kähler manifolds via Hermite-Einstein Structures and extends Donaldson, Uhlenbeck-Yau characterisation in the integrable case (see [Do], [UY]) to the non-integrable one. Another motivation for studying the regularity of almost complex rectifiable cycles is the following. In [Li] and [Ti] it is explained how the loss of compactness of solutions to geometric PDEs having a given conformal invariant dimension q (a dimension at which the PDE is invariant under conforormal transformations - $q = 2$ for harmonic maps, $q = 4$ for Yang-Mills Fields...etc) arises along $m - q$ rectifiable cycles (if m denotes the dimension of the domain). These cycles happen sometimes to be almost complex (see more details in [Ri1]).

By trying to produce in $(\mathbb{R}^{2p}, J)$ an almost complex graph of real dimension $2k$ in a neighborhood of a point $x_0 \in \mathbb{R}^{2p}$ as a perturbation of a complex one ($J_{x_0}-$holomorphic), one realises easily that, for generic almost complex stuctures J, the problem is overdetermined whenever $k > 1$ and well posed for $k = 1$. Therefore the case of $2-$dimensional integer rectifiable almost complex cycles is the generic one from the existence point of view. We shall restrict to that important case in the present paper. After complexification of the tangent bundle to M^{2p} a classical result asserts that a $2-$plane is invariant under J if and only if it has a $1-1$ tangent $2-$vector. Therefore we shall also speak about $1-1$ integral cycles for the almost complex $2-$dimensional integral cycles. In the present work we consider the locally symplectic case : We say that (M^{2p}, J) has the *locally symplectic property* if at a neighborhood of each point x_0 in M^{2p} there exists a positive symplectic structure compatible with J : there exists a neighborhood U of x_0 and a smooth closed 2-form ω such that $\omega(\cdot, J\cdot)$ defines a scalar product. It was proved in [RT] that arbitrary 4-dimensional almost complex manifolds satisfies the *locally symplectic property.* This is no more the case in larger dimension : one can find an almost complex structure in S^6 which admits no compatible positive symplectic form even locally see [Br].

Our main result is the following.

Theorem I.1 *Let (M^{2p}, J) be an almost complex manifold satisfying the locally symplectic property above. Let C be an integral 2 dimensional almost complex cycle. Then, there exists J-holomorphic curve Σ in M, smooth aside from isolated points, and a smooth integer valued function θ on Σ such that, for any 2 form $\psi \in C^\infty_0(M)$,*

$$C(\psi) = \int_\Sigma \theta\ \psi \quad .$$

In the "locally symplectic case" being an almost-complex 2 cycle is equivalent for a 2-cycle of being calibrated by the local symplectic form ω for the local metric $\omega(\cdot, J\cdot)$. Therefore the regularity question for almost complex cycles is embeded into the problem of calibrated current and hence the theory of area minimizing rectifiable 2-cycles. Therefore our result appears to be a consequence of the "Big Regularity Paper" of F.Almgren [Alm] combined with the PhD thesis of his student S.Chang [Ch]. Our attempt here was to present an alternative proof independent of Almgren's monumental work [Alm] and adapted to the case we are interrested with. The motivation is to give a proof that could be modified in order to solve the general case (non locally symplectic one) which cannot be "embeded" in the theory of area minimizing cycles anymore.

The attempt of writing a proof for the regularity of almost complex cycle in the locally symplectic case, independent of the regularity theory for area minimizing surfaces, was also the main purpose of the work Gr$\Longrightarrow$SW of C.Taubes [Ta] for $p = 2$. In this work a proof of theorem I.1 was given in the particular case where $p = 2$. Some argument happened to be incomplete in that proof, and [RT] fills the missing steps in [Ta]. Theorem I.1 has to be seen as the generalisation to higher dimension ($p > 2$) of these works.

One of the main difficulties arising in dimension ($p > 2$) is the non-necessary existence of $J-$holomorphic foliations transverse to our almost complex current C in a neighborhood of a point. This prevents then describing the current as a $Q-$multivalued graph from D^2 into $\mathbb{C}^{p-1}$, $\{(a_i^k(z))_{k=1^{p-1}}\}_{i=1\cdots Q}$ in a neighborhood of a point of density N solving locally an equation of the form

$$\partial_{\overline{z}} a_i^k = \sum_{l=1}^{p-1} A(z, a_i)_l^k \cdot \nabla a_i^l + \alpha^k(a_i, z) \quad , \tag{I.1}$$

where A and α are small in C^2 norm, as we did for $p = 2$ in [RT]. What we can only ensure instead is to describe the current C, in a neighborhood of a point of multiplicity Q, as a "algebraic $Q-$valued graph" from D^2 into $\mathbb{C}^{p-1}$: that is a family of points in $\mathbb{C}^{p-1}$, $\{a_1(z), \cdots, a_P(z), b_1(z), \cdots, b_N(z)\}$ where only $P - N = Q$ is independent on z (neither P nor N are a-priori independent on z), a_i are the positive intersection points and b_j are the negative ones. This "algebraic $Q-$valued graph" solves locally a much less attractive equation than (I.1)

$$\partial_{\overline{z}} a_i^k = \sum_{l=1}^{p-1} A_k^l(z, a_i, \nabla a_i) \cdot \nabla a_i^l + \sum_{l=1}^{p-1} B_k^l(z, a_i) \cdot \nabla a_i^l + C^k(z, a_i) \quad . \tag{I.2}$$

where $A(z, a, p)$, $B(z, a)$ and $C(z, a)$ are also small in C^2 norm but the dependence in p in $A(z, a, p)$ is linear and therefore as ∇a_i gets bigger, which can happen, the right-hand-side of (I.2) can not be handled as a perturbation of the left-hand one in steps such as the "unique continuation argument". This was used in [RT] for proving that singularities of mutiplicity Q cannot have accumulation point in the carrier $\mathcal{C}$ of C.

The strategy of the proof goes as follows. A classical blow-up analysis tells us that, for an arbitrary point x_0 of the manifold M^{2p}, the limiting density $\theta(x_0) = lim_{r\to 0}r^{-2}M(C\llcorner B_r(x_0))$ - Here M denotes the mass of a current and $\llcorner$ is the restriction operator - equals π times an integer Q. Since the density function $r \to r^{-2}M(C\llcorner B_r(x_0))$ at every point is a monotone increasing function, the complement of the set $\mathcal{C}_Q := \{x \in M\ ;\ \theta(x) \leq Q\}$ is closed in M and this permits to perform an inductive proof of theorem I.1 restricting the current to $\mathcal{C}_Q$ and considering increasing integers Q. A point of multiplicity Q is called a singular point of C if it is in the closure of points of non zero multiplicity strictly less than Q. The goal of the proof is then to show that singularities of multiplicity less than Q are isolated. We assume this fact for $Q-1$ and the paper is devoted to the proof that this then holds for Q itself. From a now classical result of B. White (see [Wh]), the dilated currents at a point x_0 of density $Q \neq 0$ converge in flat norm to a sum of Q flat $J_{x_0}-$holomorphic disks, moreover, for any $\varepsilon > 0$ and r sufficiently small $C\llcorner B_r(x_0)$ is supported in the cones whose axis are the limiting disks and angle ε. For $Q > 1$, if two of these limiting disks are different it is then easy to observe that x_0 cannot be an accumulation point of singularities of multiplicity Q, this is the so called "easy case". If the limiting disks are all identical, equal to D_0, then we are in the "difficult case" and much more work has to be done in order to reach the same statement. Contrary to the special case of dimension 4 (p=2) considered by the authors before in [RT], we could not find nice coordinates that would permit to write C as a $Q-$valued graph over the limiting disk D_0. Considering then some $J_{x_0}-$complex coordinates $(z, w_1, \cdots, w_{p-1})$ in a neighborhood $B^{2p}_{\rho_{x_0}}(x_0)$ such that $\cap_i w_i^{-1}\{0\}$ corresponds to D_0, by the mean of the "lower-epiperimetric inequality" proved by the first author in [Ri2], one can construct a Whitney-Besicovitch covering, $\{B^2_{\rho_i}(z_i)\}_{i\in I}$, of the orthogonal projection on D_0 of the points in $B^{2p}_{\rho_{x_0}}(x_0)$ having a positive density strictly less than Q. This covering is such that for every $i \in I$ there exists $x_i = (z_i, w_i) \in B^{2p}_{\rho_{x_0}}(x_0)$ verifying that the restriction of C to the tube $B^2_{\rho_i}(z_i) \times B^{2p-2}_{\rho_{x_0}}(0)$ is in fact supported in the ball $B^{2p}_{2\rho_i}(x_i)$ of radius $2\rho_i$, two times the width of the tube. Moreover if one looks inside $B^{2p}_{\rho_i}(x_i)$, C is "split" : this last word means that C restricted to $B^{2p}_{\rho_i}(x_i)$ is at a flat distance comparable to ρ_i^3 from the Q multiple of any graph over $B^2_{\rho_i}(z_i)$ - this comes from the fact that the density ratio $\rho_i^{-2}M(B_{\rho_i}(x_i))$ is strictly less than πQ minus a constant α depending only on p, Q, J and ω . We then construct an average curve for C. In the 4-dimensional case since C was a $Q-$valued graph over D_0 we took simply the average of the Q points over any point in D_0. Here, in arbitrary dimension, the construction of the average curve is more delicate and uses the covering. We first approximate $C \llcorner B^{2p}_{\rho_i}(x_i)$ by a $J_{x_i}-$holomorphic graph C_i using a technique introduced in [Ri3], and choosing a $J_{x_i}-$holomorphic disk D_i approximating D_0 we can express C_i as a $Q-$valued graph over D_i for which we take the average $\tilde{C}_i$ that happens to be Lipschitz with a uniformly bounded Lipschitz constant. Therefore the $J_{x_i}-$holomorphic curve $\tilde{C}_i$ can be viewed as a graph $\tilde{a}_i$ over $B^2_{\rho_i}(z_i)$. Patching the $\tilde{a}_i$ together we get a graph $\tilde{a}$ that extends over the whole $B^2_{\rho_{x_0}}(0)$ as a $C^{1,\alpha}$ graph for any $\alpha < 1$ which is almost $J-$holomorphic and which passes through all the $B^{2p}_{\rho_i}(x_i)$. The fact that the average curve is more regular than the $J-$holomorphic cycle C from which it is produced is clear in the integrable case (since it it holomorphic) - $(z, \pm\sqrt{z})$ is a $C^{0,\frac{1}{2}}$ 2-valued graph whereas its average $(z, 0)$ is smooth. This was extended in the non-integrable case in the particular case of the 4 dimension in [ST]. The points of multiplicity Q in C are contained in the average curve $\tilde{a}$. We then show, by the mean of a unique continuation argument in the spirit of the one developed in [Ta] in 4 dimension, that the points where C get to coincide with $\tilde{a}$ are either isolated or coincide with the whole curve $\tilde{a}$. We have then showed that any point x_0 of multiplicity Q is either surrounded by points of multiplicity Q only, and in $B^{2p}_{\rho_{x_0}}$, C coincides with Q time a smooth graph over D_0 or x_0 is not an accumulation point of points of multiplicity Q and is surrounded in $B^{2p}_{\rho_{x_0}}$ by points of multiplicity strictly less than Q. It remains at the end to show that it cannot be an accumulation point of singularities of lower density. This is obtained again using an approximation argument by holomorphic curves introduced in [Ri3].

The paper is organised as follows. In chapter II we establish preliminaries, introduce notations and give the main statement, assertion $\mathcal{P}_Q$, we are going to prove by induction in the rest of the paper. In chapter III, with the help of the "upper-epiperimetric inequality" of B. White, we establish the uniqueness of the tangent cone and a quantitative version of it, see lemma III.2. In chapter IV we prove the relative Lipschitz estimate together with a tilting control of the tangent cones of density Q points in a neighborhood of a density Q point - see lemma IV.2. In chapter V we proceed to the covering argument lemma V.3 which is based on the "splitting before tilting" lemma - see lemma V.1 - proved in [Ri2]. In chapter VI we construct the approximated average curve and prove the $C^{1,\alpha}$ estimate for this curve -

lemma VI.3. In chapter VII we perform the unique continuation argument that shows that singularities of multiplicity Q cannot be an accumulation point of singularities of multiplicity Q. In chapter VIII we show that singularities of multiplicity Q cannot be an accumulation point of singularities of multiplicity less than Q either.

II Preliminaries

Notations : We shall adopt standard notations from the Geometric Measure Theory [Fe] such as $M(A)$ for the Mass of a current A, $\mathcal{F}(A)$ for it's flat norm, $A\llcorner E$ for it's restriction to a measurable subset E...etc, we refer the reader to [Fe].

Preliminaries : Since our result is a local one we shall work in a neighborhood U of a point x_0, use a symplectic form ω compatible with J. We denote by g the metric generated by J and ω : $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$. We also introduce normal coordinates $(x_1, x_2, \cdots, x_{2p-1}, x_{2p})$ about x_0 in U which can be chosen such that

$$\text{at } x_0 \qquad J_{x_0}\cdot\frac{\partial}{\partial x_{2i+1}} = \frac{\partial}{\partial x_{2i+2}} \qquad \text{for } i = 0\cdots p-1 \quad . \tag{II.1}$$

Since C is a calibrated current in (U, ω, J), it is an area minimizing current and it's generalized mean curvature vanishes (see [All] or [Si]). One may isometrically embed (U, g) into an euclidian space $\mathbb{R}^{2p+k}$ and the generalised mean curvature of C in $\mathbb{R}^{2p+k}$ coincides with the mean curvature of the embeding of (U, g) and is therefore a bounded function. Combining this fact together with the monotonicity formula (17.3) of [Si] we get that

$$\frac{M(C\llcorner B_r(x_0))}{r^2} = f(r) + O(r) \quad , \tag{II.2}$$

where $f(r)$ is an increasing function, M denotes the Mass of a current and $C\llcorner B_r(x_0)$ is the restriction of C to the geodesic ball of center x_0 and radius r. There exists in fact a constant α depending only on g such that $e^{\alpha r}\frac{M(C\llcorner B_r(x_0))}{r^2}$ is an increasing function in r (see [Si]). The factor $e^{\alpha r}$ is a perturbation of an order which will have no influence on the analysis below, therefore, by an abuse of notation we will often omit to write it and consider straight that $\frac{M(C\llcorner B_r(x_0))}{r^2}$ is an increasing function.

By the mean of the coordinates $(x_1 \cdots x_{2p})$ we shall identify U with a subdomain in $\mathbb{R}^{2p}$ and use the same notation C for the push forward of C in $\mathbb{R}^{2p}$ by this chart. For small radii r we introduce the dilation function $\lambda^{r,x_0}(x) = \frac{x-x_0}{r}$, and we introduce the following dilation of C about x_0 with rate r as being the following current in $\mathbb{R}^{2p}$

$$C_{r,x_0} := (\lambda^{r,x_0}{}_*C)\llcorner B_1^{2p}(0) \quad . \tag{II.3}$$

Observe that $r^2 M_0(C_{r,x_0}) = M_0(C\llcorner B_r^{2p}(0))$ where M_0 denotes the mass in for the flat metric g_0 in $\mathbb{R}^{2p}$. Since $g = g_0 + O(r^2)$, we deduce from (II.2) that $M_0(C_{r,x_0})$ is uniformly bounded as r tends to zero. Again since g and g_0 coincide up to the second order, it does not hurt in the analysis below if one mixes the notations for the two masses M and M_0 and speaks only about M. Since now C is a cycle in U, $\partial C_{r,x_0}\llcorner B_1^{2p}(0) = 0$ and we can apply Federer-Fleming compactness theorem to deduce that, from any sequence $r_i \to 0$ one can extract a subsequence $r_{i'}$ such that $C_{r_{i'},x_0}$ converges in Flat norm to a limiting current C_{0,x_0} called a tangent cone of C at x_0. One of the purpose of the next section will be to establish that C_{0,x_0} is independent of the subsequence and that the tangent cone is unique. The lower semi-continuity of the mass under weak convergence implies that

$$\lim_{r\to 0} \frac{M(C\llcorner B_r(x_0))}{r^2} = \lim_{r\to 0} M(C_{r,x_0}) \geq M(C_{0,x_0}) \tag{II.4}$$

In fact, from the fact that C is calibrated by ω we deduce now that the inequality (II.4) is an equality. Indeed

$$M(C_{r,x_0}) = r^{-2} C\llcorner B_r^{2p}(0)(\omega) = C_{r,x_0}\left(r^2(\lambda^{r,x_0})^*\omega\right)$$

It is clear that $\lim_{r\to 0} \|r^2(\lambda^{r,x_0})^*\omega - \omega_0\|_\infty$ where $\omega_0 = \sum_{i=1}^p dx_{2i-1}\wedge dx_{2i}$ therefore $C_{r,x_0}(r^2(\lambda^{r,x_0})^*\omega - \omega_0) \longrightarrow 0$ and we get that

$$\lim_{r\to 0} M(C_{r,x_0}) = \lim_{i'\to+\infty} C_{r_{i'},x_0}(\omega_0) = C_{0,x_0}(\omega_0) \tag{II.5}$$

Since the comass of ω_0 is equal to 1, $C_{0,x_0}(\omega_0) \leq M(C_{0,x_0})$. Combining this last fact with (II.4) and (II.5) we have established that

$$\lim_{r\to 0} M(C_{r,x_0}) = M(C_{0,x_0}) = C_{0,x_0}(\omega_0) \tag{II.6}$$

which means in particular that C_{0,x_0} is calibrated by the Kähler form ω_0 in $(R^{2p}, J_0) \simeq \mathbb{C}^p$ which is equivalent to the fact that C_{0,x_0} is J_0-holomorphic. Using the explicit form of the monotonicity formula (see [Si] page 202), one observes that for any $s \in \mathbb{R}_+^*$

$$C_{0,x_0} = \lambda_*^s C_{0,x_0}$$

which means that $\mathcal{H}^2$ almost everywhere on the carrier $\mathcal{C}_{0,x_0}$ of C_{0,x_0}, $\frac{\partial}{\partial r}$ is in the approximate tangent plane to C_{0,x_0}, in other words, C_{0,x_0} is a cone. Since it is J_0-holomorphic, $\mathcal{H}^2-$a.e. x in $\mathcal{C}_{0,x_0}$, the approximate tangent cone is given by

$$T_x C_{0,x_0} = \text{Span}\left\{\frac{\partial}{\partial r}, J_0\frac{\partial}{\partial r}\right\}$$

Integral curves of $J_0\frac{\partial}{\partial r}$ are great-circles, fibers of the Hopf fibration

$$(z_1 = x_1 + ix_2, \cdots, z_p) \longrightarrow [z_1, \cdots, z_p]$$

therefore we deduce that C_{0,x_0} is the sum of the integrals over radial extensions of such great circles $\Gamma_1 \cdots \Gamma_Q$ in S^{2p-1} which is the integral over a sum of Q flat holomorphic disks. We adopt the following notation (in fact identical to the one used in [Wh]) for the radial extensions in $B_1^{2p}(0)$ of currents supported in $\partial B_1^{2p}(0)$

$$C_{0,x_0} = \oplus_{i=1}^Q 0 \sharp \Gamma_i \quad .$$

Then we deduce that

$$\lim_{r\rightarrow 0} M(C_{r,x_0}) = \pi Q \in \pi\mathbb{Z} \quad . \tag{II.7}$$

For any $x \in U$ one denotes Q_x the integer such that

$$\lim_{r\rightarrow 0} \frac{M(C \llcorner B_r(x))}{\pi r^2} = Q_x \quad .$$

Using the monotonicity formula, it is straightforward to deduce that for any $Q \in \mathbb{N}$

$$\mathcal{C}_Q = \{x \in U \quad \text{s. t.} \quad 0 < Q_x \leq Q\}$$

is an open subset of $\mathcal{C}_* = \{x \in U \quad \text{s. t.} \quad 0 < Q_x\}$. For $Q > 1$, let us also denote

$$Sing^Q = \{x \in \mathcal{C}_* \quad \text{s. t.} \quad Q_x = Q \quad \text{and } x \text{ is an acc. point of } \mathcal{C}_{Q-1}\}$$

Observe that, from Allard's theorem, it is clear that $C\llcorner(U\setminus\cup_Q Sing^Q)$ is the integral along a smooth surface with a smooth integer multiplicity. Although we won't make use of Allard's theorem this justifies a-priori our notation. The whole purpose of our paper is to show that $\cup_Q Sing^Q$ is made of isolated points. As we said, we won't make use of Allard's paper below since the relative Lipschitz estimate we establish in lemma IV.2 gives Allard's result in our case which is more specific. Because of this nice stratification of C ($\mathcal{C}_Q$ is open in $\mathcal{C}_*$) we can argue by induction on Q . Let $\mathcal{P}_Q$ be the following assertion

$$\mathcal{P}_Q \quad : \quad \cup_{q\le Q} Sing^q \text{ is made of isolated points} \quad . \tag{II.8}$$

From the begining of chapter IV until chapter VII we will assume either $Q=2$ or that $\mathcal{P}_{Q-1}$ holds and the goal will be to establish $\mathcal{P}_Q$.

III The uniqueness of the tangent cone.

The uniqueness of the tangent cone means that the limiting cone C_{0,x_0}, obtained in the previous section while dilating at a point following a subsequence of radii $r_{i'}$, is independent of the subsequence and is unique. Since our calibrated two dimensional rectifiable cycle is area minimizing, this fact is a consequence of B. White upper-epiperimetric inequality in [Wh] (see also [Ri2] for the justification of the prefix "upper"). We need, however, a more quantitative version of this uniqueness of the tangent cone and express how far we are from the unique tangent cone in terms of the closedness of the density of area $M(C\llcorner B_r(x_0))/\pi r^2$ to the limiting density Q. Precisely the goal of this section is to prove the following lemma

Lemma III.1 (Uniqueness of the tangent cone.) *For any $\varepsilon>0$ and $Q\in\mathbb{N}$ there exists $\delta>0$ and $\rho_\varepsilon\le 1$ such that, for any compatible pair (J,ω) almost complex structure-symplectic form over $B_1^{2p}(0)$ satisfying $J(0)=J_0(0)$, $\omega(0)=\omega_0(0)$*

$$\|J-J_0\|_{C^2(B_1)}+\|\omega-\omega_0\|_{C^2(B_1)}\le\delta \quad , \tag{III.1}$$

for any $J-$holomorphic integral 2-cycle C in $B_1(0)$ such that $Q_0=Q$, if

$$M(C\llcorner B_1^{2p}(0))\le\pi Q+\delta$$

then, there exists Q J_0 holomorphic flat discs $D_1 \cdots D_Q$ passing through 0, intersection of holomorphic lines of $\mathbb{C}^p$ with $B_1^{2p}(0)$, such that, for any $\rho \le \rho_\varepsilon$

$$\mathcal{F}(C_{\rho,0} - \oplus_{i=1}^{Q} D_i) \le \varepsilon \tag{III.2}$$

and for any $\psi \in C_0^\infty(B^1 \setminus \{x \in B^1 \ ; \ dist(x, \cup_i D_i) \le \varepsilon |x|\})$,

$$C_{\rho,0}(\psi) = 0 \tag{III.3}$$

Before proving lemma III.1, we first establish the following intermediate result

Lemma III.2 *For any $\varepsilon > 0$ and $Q \in \mathbb{N}$ there exists $\delta > 0$ such that the following is true. If (J, ω) is a compatible pair of almost complex structure-symplectic form over $B_1^{2p}(0)$ satisfying $J(0) = J_0(0)$, $\omega(0) = \omega_0(0)$*

$$\|J - J_0\|_{C^2(B_1)} + \|\omega - \omega_0\|_{C^2(B_1)} \le \delta \quad ,$$

for any $J-$holomorphic integer rectifiable 2-cycle C such that $Q_x = Q$, if

$$M(C \llcorner B_1^{2p}(0)) \le \pi Q + \delta$$

then, there exist Q J_0 holomorphic flat discs $D_1 \cdots D_Q$ passing through 0, intersection of holomorphic lines of $\mathbb{C}^p$ with $B_1^{2p}(0)$, such that,

$$\mathcal{F}(C \llcorner B_1(0) - \oplus_{i=1}^{Q} D_i) \le \varepsilon$$

Remark III.1 *Lemma III.2 give much less information than lemma III.1. Since a-priori in lemma III.2 the disks D_i may vary a lot as one dilate C about 0, whereas lemma III.1 controls such a tilting as one dilates the current further.*

Proof of lemma III.2 : We prove lemma III.2 by contradiction. Assume there exists $\varepsilon_0 > 0$, $\delta_n \to 0$, compatible J_n and ω_n and C_n such that

i)

$$\|J_n - J_0\|_{C^2} + \|\omega_n - \omega_0\|_{C^2} \le \delta_n$$

ii)

$$\lim_{r \to 0} \pi^{-1} r^{-2} M(C_n \llcorner B_r(0)) = Q$$

iii)

$$M(C_n \llcorner B_1) \leq \pi Q + \delta_n$$

iv)

$$\inf \left\{ \mathcal{F}(C_n \llcorner B_1 - \oplus_{i=1}^{Q} D_i) \ \text{ s.t. } D_i \text{ flat holom discs}, 0 \in D_i \right\} \geq \varepsilon_0 \tag{III.4}$$

Since $\partial C_n \llcorner B_1 = 0$ and since the mass of C_n is uniformly bounded, one may assume, modulo extraction of a subsequence if necessarily, that C_n converges to a limiting rectifiable cycle C_∞. Exactly like in section III we have the fact that for any $0 < r \leq 1$

$$\lim_{n \to +\infty} M(C_n \llcorner B_r) = M(C_\infty \llcorner B_r) = C_\infty \llcorner B_r(\omega_0) \tag{III.5}$$

We deduce then that C_∞ is calibrated by ω_0 and is therefore a J_0-holomorphic cycle. Using ii) we deduce also that

$$\lim_{r \to 0} \pi^{-1} r^{-2} M(C_\infty \llcorner B_r(0)) = Q$$

and finally, from iii) and the lower semicontinuity of the mass, we have that $M(C_\infty \llcorner B_1) = \pi Q$ Thus, since $\pi^{-1} r^{-2} M(C_\infty \llcorner B_r(0))$ is an increasing function, we have established that on $[0, 1]$

$$\pi^{-1} r^{-2} M(C_\infty \llcorner B_r(0)) \equiv Q \tag{III.6}$$

Let, for almost every r, $S^r_\infty =< C_\infty, dist(\cdot, 0), r >$ be the slice current obtained by slicing C_∞ with $\partial B_r(0)$ (see [Fe] 4.2.1). By Fubini, we have that for a.e. $0 < r < 1$

$$M(< C_\infty, dist(\cdot, 0), r >) \leq 2\pi Q r$$

Let $0 \sharp S^r_\infty$ be the radial extension of S^r_∞ in $B_r(0)$.

$$M(0 \sharp S^r_\infty) = \frac{r}{2} M(S^r_\infty) = \pi Q r^2 = M(C_\infty \llcorner B_r(0))$$

Since $\partial(C_\infty \llcorner B_r(0) - 0 \sharp S^r_\infty) = 0$, and since $C_\infty \llcorner B_r(0)$ is area minimizing we have that $0 \sharp S^r_\infty$ is also area minimizing. Let α such that $d\alpha = \omega$.

$$M(0 \sharp S^r_\infty) = M(C_\infty \llcorner B_r(0)) = C_\infty \llcorner B_r(0)(\omega_0) = S^r_\infty(\alpha) = (0 \sharp S^r_\infty)(\omega_0)$$

Therefore $0\sharp S^r_\infty$ is an holomorphic cone which is a cycle. So we deduce like in section II that $0\sharp S^r_\infty$ is a sum of flat holomorphic disk for any r. Thus C_∞ is also a sum

$$C_\infty \llcorner B_1(0) = \sum_{i=1}^{Q} D_i$$

where each D_i is the intersection of a complex straight line in $\mathbb{C}^p$ with B_1^{2p}. From Federer-Fleming compactness theorem we have the fact that the weak convergence of C_n to C_∞ holds in flat norm

$$\mathcal{F}(C_n \llcorner B_1 - \sum_{i=1}^{Q} D_i) \longrightarrow 0$$

which contradicts iv) and lemma III.2 is proved. ■

Proof of lemma III.1 :

We first prove assertion (III.2). We first recall Brian White's upper-epiperimetric inequality adapted to our present context : Brian White's upper-epiperimetric inequality was proved for area minimizing surfaces in $\mathbb{R}^{2p}$. Here, in the present situation, we are dealing with area minimizing currents which are $J-$holomorphic for a metric $g = \omega(\cdot, J\cdot)$ which get's as close as we want to the standard one because of assumption (III.1). Therefore very minor changes have to be provided to adapt B.White theorem to the present context. An adaptation of the epiperimetric inequality for ambient non flat metric is also given in [Ch] Appendix A. So we have the following result.

Given an integer Q, there exists a positive number $\varepsilon_Q > 0$, such that, for any compatible pair ω, J in $B_2^{2p}(0)$ satisfying $\|\omega - \omega_0\|_{C^2(B_2)} + \|J - J_0\|_{C^2(B_2)} \leq \varepsilon_Q$ and for any C $J-$holomorphic 2-rectifiable integral current in $B_2^{2p}(0)$, satisfying $\partial C \llcorner B_2^{2p} = 0$, assuming there exist Q flat holomorphic disks $D_1 \ldots D_Q$ in $(B_1^{2p}(0), J) \simeq \mathbb{C}^p \cap B_1^{2p}(0)$ passing through the origin such that

$$\mathcal{F}\left(C_{2,0} \llcorner B_1(0) - \sum_{i=1}^{Q} D_i\right) \leq \varepsilon_Q \qquad \text{(III.7)}$$

(where we used a common notation for the oriented 2-disks D_i and the corresponding 2-currents) then

$$M(C\llcorner B_1^{2p}) - \pi Q \leq (1-\varepsilon_Q)\ \left(\frac{1}{2}M\left(\partial(C\llcorner B_1^{2p})\right) - \pi Q\right) \tag{III.8}$$

Remark III.2 *Observe that in the statement of the epiperimetric property in Definition 2 of [Wh] the epiperimetric constant ε_Q may a-priori also depend of the cone $\sum_{i=1}^Q D_i$. It is however elementary to observe that this space of cones made of the intersection of Q holomorphic straight lines passing through the origin with $B_1^{2p}(0)$ is compact for the flat distance. Now using a simple finite covering argument for this space of cones by balls (for the flat distance) permits one to obtain a constant $\varepsilon_Q > 0$ for which the epiperimetric property holds independently of the cone $\sum_{i=1}^Q D_i$.*

Once again we shall ignore the factor $e^{\alpha r}$ in front of $r^{-2}M(C\llcorner B_r)$ which induces lower order perturbations and argue as $r^{-2}M(C\llcorner B_r)$ itself would be an increasing function (observe also that α may be taken arbitrarily small because J and ω are chosen as close as we want to J_0 and ω_0).

Let then $\varepsilon > 0$ such that $\varepsilon < \varepsilon_Q$ and let $\delta > 0$ given by lemma III.2 for that ε. Assuming then $M(C\llcorner B_1) \leq \pi Q + \delta$ implies from the monotonicity formula that for any $r < 1$ $r^{-2}M(C\llcorner B_r(0)) = M(C_{r,0}) \leq (\pi Q+\delta)$. Applying then Lemma III.2 to $C_{2r,0}$ for $r < 1/2$ we deduce the existence of Q flat disks $D_1 \cdots D_Q$ such that

$$\mathcal{F}(C_{2r,0} - \sum_{i=1}^Q D_i) \leq \varepsilon \tag{III.9}$$

We can then apply the epiperimetric inequality to $C_{r,0}$ and we get, after rescaling, that

$$M(C\llcorner B_r(0)) - \pi Q r^2 \leq (1-\varepsilon_Q)\left(\frac{r}{2}M(\partial(C\llcorner B_r(0))) - \pi Q r^2\right) \tag{III.10}$$

Denote $f(r) = M(C\llcorner B_r(0)) - \pi Q r^2$. $f'(r) \geq M(\partial(C\llcorner B_r(0))) - 2\pi Q r$. Therefore (III.10) implies

$$\frac{1-\varepsilon_Q}{2} r\ f'(r) \geq f(r) \quad .$$

Integrating this differential inequality between s and σ $(1/2 > s > \sigma)$, $f(s) \geq \left(\frac{s}{\sigma}\right)^{\frac{2}{1-\varepsilon_Q}} f(\sigma)$. Let $\nu = \frac{2}{1-\varepsilon} - 2 > 0$, we then have

$$\frac{f(s)}{s^2} \geq \left(\frac{s}{\sigma}\right)^{\nu} \frac{f(\sigma)}{\sigma^2} \quad . \tag{III.11}$$

Let $F(x) = \frac{x}{|x|}$. We have

$$M(F_*\left(C \llcorner B_s(0) \setminus B_\sigma(0)\right)) = \int_{B_S(0)\setminus B_\sigma(0)} \frac{1}{|x|^3} \left|\tau \wedge \frac{x}{|x|}\right| \ |\theta| \ d\mathcal{H}^2 \llcorner \mathcal{C}$$

where τ denotes the unit $2-$vector associated to the oriented approximate tangent plane to C and defined $\mathcal{H}^2-$a.e. along the carrier $\mathcal{C}$ of the rectifiable current, θ is the $L^1(\mathcal{C})$ integer-valued multiplicity of C (i.e using classical GMT notations :$C =< \mathcal{C}, \theta, \tau >$) and $d\mathcal{H}^2 \llcorner \mathcal{C}$ is the restriction to $\mathcal{C}$ of the $2-$dimensional Hausdorff measure. Using Cauchy-Schwarz and 5.4.3 (2) of [Fe] (the explicit formulation of the monotonicity formula) and (III.11), we have

$$\begin{aligned} & M(F_*\left(C \llcorner B_s(0) \setminus B_\sigma(0)\right)) \\ & \leq \left[\frac{M(C \llcorner B_s(0))}{s^2} - \frac{M(C \llcorner B_\sigma(0))}{\sigma^2}\right]^{\frac{1}{2}} \left[\frac{M(C \llcorner B_s(0))}{\sigma^2}\right]^{\frac{1}{2}} \\ & \quad \leq \left[\frac{M(C \llcorner B_s(0))}{s^2} - \pi Q\right]^{\frac{1}{2}} \left[\frac{M(C \llcorner B_s(0))}{\sigma^2}\right]^{\frac{1}{2}} \\ & \quad \leq \left[\frac{f(s)}{s^2}\right]^{\frac{1}{2}} \left[\frac{s^2}{\sigma^2}\frac{M(C \llcorner B_s(0))}{s^2}\right]^{\frac{1}{2}} \leq K \ s^{\frac{\nu}{2}} \ \frac{s}{\sigma} \end{aligned} \tag{III.12}$$

Let $r < \rho < 1/2$, applying (III.12) for $s = 2^{-k}\rho$ and $\sigma = 2^{-k-1}\rho$ for $k \leq \log_2 \frac{\rho}{r}$ and summing over k we get

$$M(F_*\left(C \llcorner B_\rho(0) \setminus B_r(0)\right)) \leq C\rho^{\frac{\nu}{2}} \tag{III.13}$$

Observe that $\partial(F_*\left(C \llcorner B_\rho(0) \setminus B_r(0)\right)) = \partial C_{\rho,0} - \partial C_{r,0}$. Therefore we deduce

$$\mathcal{F}\left((C_{\rho,0} - C_{r,0}) \llcorner B_1(0) \setminus B_{\frac{1}{2}}(0)\right) \leq C\rho^{\frac{\nu}{2}} \tag{III.14}$$

Since

$$\mathcal{F}\left((C_{\rho,0}-C_{r,0})\llcorner B_{\frac{1}{2}}(0)\setminus B_{\frac{1}{4}}(0)\right)\le\left(\frac{1}{3}\right)^{\frac{1}{3}}\mathcal{F}\left((C_{\frac{\rho}{2},0}-C_{\frac{r}{2},0})\llcorner B_1(0)\setminus B_{\frac{1}{2}}(0)\right)$$

applying then (III.14) for ρ, r replaced by $2^{-k}\rho$, $2^{-k}r$ and summing over $k=1,\cdots\infty$ we finally obtain

$$\mathcal{F}\left((C_{\rho,0}-C_{r,0})\llcorner B_1(0)\right)\le C\rho^{\frac{\nu}{2}} \tag{III.15}$$

which is the desired inequality (III.2).

It remains to show (III.3) in order to finish the proof of lemma III.1. We argue by contradiction. Assume there exists $\varepsilon_0>0$, $\rho_n\to 0$ and $\psi_n\in C^\infty_0(\wedge^2 B_1)$ such that

$$\operatorname{supp}\psi_n\subset E_0=\{x\in B^1\ ;\ \operatorname{dist}(x,\cup_i D_i)\le\varepsilon_0|x|\}\quad ,$$

where $C_{0,0}=\oplus_{i=1}^Q D_i$, and

$$C_{\rho_n,0}(\psi_n)\neq 0$$

This later fact implies in particular that there exists $x^n\in E_0$ such that $\lim_{r\to 0}M(C_{r,x_n})\neq 0$. Using the monotonicity formula we deduce then that

$$M(C_{\rho_n|x_n|,0}\llcorner B_{\varepsilon_0/2}(\frac{x_n}{|x_n|}))\ge\frac{\pi}{4}\varepsilon_0^2$$

We may then extract a subsequence such that $\frac{x_n}{|x_n|}\to x_\infty$. Thus we have

$$M(C_{\rho_n|x_n|,0}\llcorner B_{3\varepsilon_0/4}(x_\infty))\ge\frac{\pi}{4}\varepsilon_0^2$$

We have

$$M(C_{\rho_n|x_n|,0}\llcorner B_{3\varepsilon_0/4}(x_\infty))=C_{\rho_n|x_n|,0}\llcorner B_{3\varepsilon_0/4}(x_\infty)\left(\frac{x}{\rho_n|x_n|}^*\omega_n\right)$$

Since $\|\omega_n-\omega_0\|_{C^2}\to 0$ and since $\omega_n(0)=\omega_0(0)$ we clearly have that

$$\|\frac{x}{\rho_n|x_n|}^*\omega_n-\omega_0\|_\infty\longrightarrow 0$$

Therefore

$$\begin{aligned}
&\left| C_{\rho_n|x_n|,0} \llcorner B_{3\varepsilon_0/4}(x_\infty) \left(\frac{x}{\rho_n|x_n|}^* \omega_n - \omega_0 \right) \right| \\
\leq \quad &\leq M(C_{\rho_n|x_n|,0} \llcorner B_{3\varepsilon_0/4}(x_\infty)) \ \|\tfrac{x}{\rho_n|x_n|}^* \omega_n - \omega_0\|_\infty \longrightarrow 0 \quad ,
\end{aligned}$$

Thus

$$C_{0,0} \llcorner B_{3\varepsilon_0/4}(x_\infty)(\omega_0) = \lim_{n \to +\infty} C_{\rho_n|x_n|,0} \llcorner B_{3\varepsilon_0/4}(x_\infty)(\omega_0) \geq \frac{\pi}{4}\varepsilon_0^2$$

which contradicts the fact that $B_{3\varepsilon_0/4}(x_\infty) \subset E_0$. Therefore (III.3) holds and lemma III.1 is proved. ■

IV Consequences of lemma III.1 : No accumulation of points in $Sing^Q$ in the easy case - The relative Lipschitz estimate in the difficult case.

In this section we expose two important consequences of lemma III.1. Before explaining them we first observe that proving the implication $\mathcal{P}_{Q-1} \Longrightarrow \mathcal{P}_Q$, will require considering two cases separately. The first case (the easy one) is the case where the tangent cone at the point x_0 of multiplicity Q (i.e. $\pi^{-1}r^{-2}M(B_r(x_0)) \to Q$) is not made of Q times the same disk. The second case is the case where the tangent case is made of Q times the same disk. In the first case we will deduce almost straight from lemma III.1 that such an x_0 cannot be an accumulation point of points of multiplicity Q also, see lemma IV.1 below. In the second case, much more analysis will be needed to reach the same statement and this is the purpose of chapters IV through IX. We can nevertheless deduce in this chapter an important consequence of our quantitative version of the uniqueness of the tangent cone (lemma III.1) for the difficult case : this is the so called "relative lipshitz estimate" (see lemma IV.2 below). This property says that, given a point x_0 of multiplicity Q whose tangent cone is Q times a flat disk and given an $\varepsilon > 0$, there exists a radius $r_{\varepsilon,x_0} > 0$ such that given any two points of $\mathcal{C}_* \cap B_{r_{\varepsilon,x_0}}(x_0)$, one of the two being also of multiplicity Q, the slope they realise relative to the tangent cone of x_0 is less than ε. The condition that one of the two points has multiplicity Q (this could be x_0 itself for instance) is a crucial assumption. It is indeed straightforward to find counterexamples to any Lipschitz estimates of multivalued graphs of holomorphic curves : take for instance $w^2 = z$ in $\mathbb{C}^2 \simeq \{(z,w)\ z,w \in \mathbb{C}\}$ viewed as a 2-valued graph over the line $\{w = 0\}$, all points have multiplicity 1, $(0,0)$ included of course, but the best possible estimate is a Hölder one $C^{0,\frac{1}{2}}$. We cannot exclude that such a configuration exists as we dilate at a point x_0 of mupltiplicity $Q > 1$.

We first then prove the following consequence of lemma III.1

Lemma IV.1 (no accumulation - the easy case) *Let $Q \in \mathbb{N}$, $Q \geq 2$. Let x_0 be a point in $\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$ (i.e.$\pi^{-1}r^{-2}M(B_r(x_0)) \to Q$ as $r \to 0$). Assume that the tangent cone at x_0, C_{0,x_0}, contains at least two different flat $J_{x_0}-$holomorphic disks (i.e. $C_{0,x_0} \neq Q\,D$ for the single flat $J_{x_0}-$holomorphic disk D). Then, there exists $r > 0$ sucht that*

$$B_r(x_0) \cap (\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}) = \{x_0\} \quad .$$

Proof of lemma IV.1.

Let x_0 be as in the statement of the lemma : $x_0 \in \mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$ and $C_{0,x_0} = \oplus_{i=1}^K Q_i\ D_i$ (where $D_i \neq D_j$ for $i \neq j$ and $K > 1$). Let $\varepsilon > 0$ be a positive number smaller than 1/4Q times the maximal angle α between the various diks $D_1, \cdots, D_K$ in the tangent cones in such a way that there exists $i \neq j$ such that

$$E_\varepsilon(D_i) \cap E_\varepsilon(D_j) = \{x_0\} \quad ,$$

where we are using the following notation

$$E_\varepsilon(D_i)\{x \in \mathbb{R}^{2p}\ ;\ \mathrm{dist}(x, D_i) \leq \varepsilon|x - x_0|\}) \quad .$$

By taking ε as small as above, we even have ensured that $\cup E_\varepsilon(D_i) \setminus x_0$ has at least two connected components whose intersections with $\partial B_1(x_0)$ are at a distance larger than $\alpha/2$. We prove now lemma IV.1 by contradiction : we assume there exists $x_n \in \mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$ such that $x_n \to x_0$ and $x_n \neq x_0$. Let $\delta > 0$ given by lemma III.1 for ε chosen just above. Let $\rho > 0$ such that

$\rho^{-2}M(C\llcorner B_\rho(x_0)) \leq \pi Q + \delta/2$. For any $x \in B_{\rho\frac{\delta}{4\pi Q}}(x_0)$ we have

$$
\begin{aligned}
M(C\llcorner B_{\rho(1-\frac{\delta}{4\pi Q})}(x)) &\leq M(C\llcorner B_\rho(x_0)) \leq \rho^2\ \left(\pi Q + \tfrac{\delta}{2}\right) \\
&\leq (\rho(1-\tfrac{\delta}{4\pi Q}))^2\ (1-\tfrac{\delta}{4\pi Q})^{-2}\left(\pi Q + \tfrac{\delta}{2}\right) \\
&\leq (\rho(1-\tfrac{\delta}{4\pi Q}))^2\ (\pi Q + \delta)
\end{aligned}
\tag{IV.1}
$$

Choose then $x_n \in B_{\rho\frac{\delta}{8\pi Q}}(x_0)$. Applying (III.3) for x_0 we know that x_n is contained in one of the $E_\varepsilon(D_i)$, say $E_\varepsilon(D_1)$. Denote E_1 the connected component of $\cup E_\varepsilon(D_i)\setminus\{x_0\}$ that contains $E_\varepsilon(D_1)$. ε has been chosen small enough such that $\cup E_\varepsilon(D_i)$ has at least two connected components. Therefore we can chose D_j such that $E_\varepsilon(D_j)$ is disjoint from the component containing $E_\varepsilon(D_1)\setminus\{x_0\}$. Let α be the angular distance, relative to x_0, from $E_\varepsilon(D_j)$ and the component containing $E_\varepsilon(D_1)\setminus\{x_0\}$. al is clearly bounded from below by a positive number as one choses ε smaller and smaller. Applying lemma III.1 this time to x_n, we know that in $B_{4|x_n|}(x_n)\setminus B_{|x_n|}(x_n)$ the support of C is at the $\varepsilon|x_n|$ distance from a union of flat disks passing through x_n (the tangent cone at x_n). This implies that the angular distance between the tangent cone at x_n and D_1 is less than $C_Q\varepsilon$, where C_Q depends on Q only . Therefore

$$
\text{supp}(C\llcorner B_{2|x_n|}(x_n)\setminus B_{|x_n|}(x_n)) \subset \tilde{E}_1 = \{x\ ;\ dist(x, D_1) \leq C_Q\varepsilon|x_n|\} \tag{IV.2}
$$

Observe that $dist\{E_\varepsilon(D_j)\cap\left(B_{|x_n|}(x_0)\setminus B_{|x_n|/2}(x_0)\right); \tilde{E}_1\} \geq \alpha/4$. This later fact combined with (IV.2) contradicts (III.2). Lemma IV.1 is then proved. ∎

From now on until the begining of chapter X we will be dealing with the difficult case only : the case where the point x_0 of multiplicity Q has a tangent cone which is made of Q time the same disk. As we have been doing since chapter II, we will work in a neighborhood of x_0 where a compatible simplectic form ω for J exists, and we shall use normal coordinates for $g(\cdot, cdot) = \omega(\cdot, J\cdot)$ about x_0, compatible with J_{x_0} at x_0, satisfying (II.1), and we can also assume that the tangent cone at x_0 is

$$
C_{0,x_0}\llcorner B_1(0) = Q[D_0] \tag{IV.3}
$$

where D_0 is the flat oriented disk whose tangent 2−vector is $\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_2}$. From now on we will also use the following notations for complex coordinates about

x_0

$$z = x_1 + ix_2 \quad \text{ and } \quad w_i = x_{2k+1} + ix_{2k+2} \text{ for } k = 1 \cdots p-1 \quad . \tag{IV.4}$$

We will also denote $w = (w_1, \cdots, w_{p-1})$. A second consequence to lemma III.1 is the following result :

Lemma IV.2 (the relative Lipschitz estimate) *: Let x_0 be a point of multiplicity Q (i.e. $x_0 \in \mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$), assume the tangent cone $C_{0,x_0} \llcorner B_1(0)$ at x_0 is Q times a flat disk (i.e. of the form (IV.3)). Let $\varepsilon > 0$, then there exists r_{ε,x_0} such that for any $r \leq r_{\varepsilon,x_0}$*

$$\forall x \in B_r(x_0) \cap (\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}) \qquad \mathcal{F}((C_{0,x} - C_{0,x_0}) \llcorner B_1(0)) \leq \varepsilon \tag{IV.5}$$

and $\forall x = (z, w) \in B_r(x_0) \cap (\mathcal{C}_Q \setminus \mathcal{C}_{Q-1})$ and $x' = (z', w') \in B_r(x_0) \cap \mathcal{C}_*$ we have

$$|w - w'| \leq \varepsilon |z - z'| \tag{IV.6}$$

Proof of lemma IV.2. Let $\varepsilon > 0$ and $\delta > 0$ given by lemma III.1. Choose r_1 such that

$$M(C \llcorner B_{r_1}(x_0)) \leq r^2 \left(\pi Q + \frac{\delta}{2} \right)$$

This implies in particular that for any $r < r_1$

$$\mathcal{F}((C_{r,x_0} - C_{0,x_0}) \llcorner B_1(0)) \leq \varepsilon \tag{IV.7}$$

As in the proof of lemma IV.1, see (IV.1) we have that for any $x \in B_{r_1 \frac{\delta}{4\pi Q}}(x_0)$ and $r < r_1(1 - \frac{\delta}{4\pi Q})$

$$M(C \llcorner B_r(x)) \leq r^2 (\pi Q + \delta)$$

Let then $x \in B_{r_1 \frac{\delta}{4\pi Q}}(x_0) \cap (\mathcal{C}_Q \setminus \mathcal{C}_{Q-1})$, applying lemma III.1, we have then for $r < r_1(1 - \frac{\delta}{4\pi Q}) = r_2$

$$\mathcal{F}((C_{r,x} - C_{0,x}) \llcorner B_1(0)) \leq \varepsilon \tag{IV.8}$$

Choose now $x \in B_{\varepsilon r_1}(x_0) \cap (\mathcal{C}_Q \setminus \mathcal{C}_{Q-1})$. $C_{r_2,x}$ is fast eine $\varepsilon-$translation from C_{r_1,x_0}, therefore, since $M(C_{r,x_0}) \leq 2\pi Q$, we have

$$\mathcal{F}((C_{r_2,x_0} - C_{r_2,x}) \llcorner B_1(0)) \leq 2\pi Q \tag{IV.9}$$

Take $r_{\varepsilon,x_0} = \min\{\varepsilon r_1, \frac{\delta}{4\pi Q} r_1\}$. Combining (IV.7), (IV.8) and (IV.9), we deduce that

$$\forall x \in B_{r_{\varepsilon,x_0}}(x_0) \cap (\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}) \qquad \mathcal{F}((C_{0,x} - C_{0,x_0}) \llcorner B_1(0)) \leq (2 + 2\pi Q)\varepsilon \tag{IV.10}$$

It remains to check (IV.6) which is in fact an almost direct consequence of (III.3) and (IV.5). Lemma IV.2 is then proved. ■

V The covering argument.

Let x_0 a point of multiplicity $Q > 1$ whose tangent cone C_{0,x_0} is Q times the integral over the flat disk D_0 given by $w_i = 0$ for $i \cdots Q-1$ (we use the system of coordinates introduced in the begining of chapter IV) . The purpose of this section is to construct a Whitney-Besicovitch covering $B^2_{r_i}(z_i)$ of $\Pi(\mathcal{C}_{Q-1}) \cap B^2_\rho(x_0)$ where Π is the projection on D_0 which gives the first complex coordinate of each point ($\Pi(z, w_1 \cdots w_{p-1}) = z$), for some radius ρ, small enough depending on x_0. This covering will be chosen in such a way the following striking facts hold : first $C \llcorner \Pi^{-1}(B^2_{r_i}(z_i))$ is in fact supported in a ball of radius $2r_i$, $B^{2p}_{2r_i}(x_i)$, moreover $C \llcorner B^{2p}_{2r_i}(x_i)$ is "split". This last word means that the flat distance between $C \llcorner \Pi^{-1}(B^2_{r_i}(z_i))$ and the Q multiple of any single valued graph over D_0 is larger than $K\, r_i^3$ where K only depends on p, J and ω. This will come from the fact that r_i may be chosen i such a way that $r_i^{-2} M(B^{2p}_{r_i}(x_i) \leq \pi - K'$ where again $K' > 0$ only depends on p, Q, J and ω. The existence of such a covering is a consequence of the "splitting before tilting" lemma proved in [Ri2].

Let α be given by lemma V.1 and let $\varepsilon > 0$ to be chosen small enough, compare to α later. Let r_{ε,x_0} be the radius given by lemma IV.2. We may chose also r_{ε,x_0} small enough in such a way that

$$\forall r \leq r_{\varepsilon,x_0} \qquad M(C_{r,x_0} \llcorner B_1(0)) \leq \pi Q + \varepsilon^2 \quad . \tag{V.1}$$

Using the proof of lemma III.1 (from (III.12) until the end of the proof), we deduce that

$$\forall r \leq r_{\varepsilon,x_0} \qquad \mathcal{F}((C_{r,x_0} - C_{0,x_0}) \llcorner B_1(0)) \leq K\varepsilon \quad . \tag{V.2}$$

(In fact $\delta = O(\varepsilon^2)$ works in the statement of lemma III.1). In one hand, as in the proof of lemma IV.1, see (IV.1) we have that for any $x \in B_{\varepsilon^2 r_{\varepsilon,x_0}}(x_0)$

and $r \leq r_{\varepsilon,x_0}(1-\varepsilon^2)$

$$M(C \llcorner B_r(x)) \leq r^2\ (\pi Q + \varepsilon^2) \quad , \tag{V.3}$$

and, on the other hand, arguing like in the proof of lemma IV.2, between (IV.8) and (IV.10), we have, using also (V.23)

$$\forall x \in B_{\varepsilon^2}(x_0) \qquad \mathcal{F}(C_{\frac{r_{\varepsilon,x_0}}{2},x_0} \llcorner B_1(0) - \pi Q\ [D_0]) \leq K\varepsilon \quad . \tag{V.4}$$

Having chosen $K\varepsilon < \alpha$ we are in a position to apply the “Splitting before tilting” lemma of [Ri2] which is a key step in our proof of the regularity of 1-1 rectifiable cycles.

Lemma V.1 (splitting before tilting) *[Ri2] There exists $\alpha > 0$ such that for any $x_0 \in \mathcal{C}_{Q-1}$ and for any radius $0 < \rho < \alpha$ satisfying*

$$\mathcal{F}(C_{2\rho,x_0} \llcorner B_1(0) - Q\ [D_0]) \leq \alpha \tag{V.5}$$

where D_0 is a flat $J_{x_0}-$holomorphic disk passing through x_0. Then, for any $r < \rho$ and any $J_{x_0}-$holomorphic flat disk, D_1, passing through x_0 and satisfying

$$\mathcal{F}([D_0] - [D_1]) \geq \frac{1}{4} \tag{V.6}$$

we have

$$\mathcal{F}(C_{r,x_0} \llcorner B_1(0) - Q[D_1]) \geq \alpha \quad . \tag{V.7}$$

Moreover, there exists $r_0 < \rho$ and K_0 a constant depending only on $\|\omega\|_{C^1}$ and $\varepsilon_Q^{\pm}$, the epiperimetric constants, such that

$$M(C_{r_0,x_0} \llcorner B_1(0)) = \pi Q - K_0\alpha \quad , \tag{V.8}$$

$$\mathcal{F}(C_{r_0,x_0} \llcorner B_1(0) - Q[D_0]) \leq K\ \sqrt{\alpha} \tag{V.9}$$

for some constant K depending also only on $\|\omega\|_{C^1}$ and $\varepsilon_Q^{\pm}$. Finally, for any $J_{x_0}-$holomorphic disk D passing through 0

$$\forall r \leq r_0 \qquad \mathcal{F}(C_{r,x_0} \llcorner B_1(0) - Q\ [D]) \geq \alpha \quad . \tag{V.10}$$

For any $x \in \mathcal{C}_{p-1} \cap B_{\varepsilon^2 r_{\varepsilon,x_0}}(x_0)$ we denote by r_x the radius r_0 given by the lemma. We then have

$$M(C_{r_x,x} \llcorner B_1(0)) = \pi Q - K_0 \alpha \quad , \tag{V.11}$$

$$\mathcal{F}(C_{r_x,x} \llcorner B_1(0) - \pi Q[D_0]) \leq K\ \sqrt{\alpha} \tag{V.12}$$

for some constant K depending also only on $\|\omega\|_{C^1}$ and $\varepsilon_Q^{\pm}$. Moreover, for α chosen small enough and ε small enough compare to α, the following lemma holds

Lemma V.2 *Under the above notations we have that for any $x \in \mathcal{C}_{p-1} \cap B_{\varepsilon r_{\varepsilon,x_0}}(x_0)$*

$$supp\left(C \llcorner \Pi^{-1}(B^2_{r_x}(\Pi(x)) \cap B_{r_{\varepsilon,x_0}}(x_0)\right) \subset B^2_{r_x}(\Pi(x)) \times B^{2p-2}_{r_x}(0) \quad , \tag{V.13}$$

and that

$$supp\left(C \llcorner \Pi^{-1}(B^2_{r_x}(\Pi(x)) \cap B_{r_{\varepsilon,x_0}}(x_0)\right) \subset \mathcal{C}_{p-1} \tag{V.14}$$

$\blacksquare$

Proof of lemma V.2. We claim that for any r between $\frac{r_{\varepsilon,x_0}}{2}$ and r_x one has

$$supp(C_{r,x} \llcorner B_1(0)) \subset E(\alpha^{\frac{1}{16}}) \tag{V.15}$$

where we use the notation

$$E(\lambda) = \{y = (z,w) \in B_1(0) \quad \text{s. t.} \quad |w| \leq \lambda\} \tag{V.16}$$

We show (V.15) arguing by contradiction. First of all from the proof of lemma V.1 that we apply to x we have the fact that for any $r \in [r_x, \frac{r_{\varepsilon,x_0}}{2}]$

$$\mathcal{F}(C_{r,x} \llcorner B_1(0) - \pi Q[D_0]) \leq K\ \sqrt{\alpha} \quad . \tag{V.17}$$

Let $\omega_\alpha = \chi_\alpha\ \omega = \chi\left(\frac{|w|}{\alpha^{\frac{1}{4}}}\right)\omega$ where χ is a smooth cut-off function on $\mathbb{R}_+$ satisfying $\chi \equiv 1$ on $[0, 1/2]$ and $\chi \equiv 0$ in $[1, +\infty)$. Let S and R be a 3 and a 2−current satisfying $(C_{r,x} \llcorner B_1(0) - \pi Q[D_0]) = \partial S + R$ with $M(S) + M(R) \leq 2K\sqrt{\alpha}$. We have

$$|(C_{r,x} \llcorner B_1(0) - \pi Q[D_0])(\omega_\alpha)| = |S(\omega \wedge d\chi_\alpha) + R(\omega_\alpha)|$$

$$\leq \|\nabla\chi_\alpha\|_\infty\ \|\omega_\infty\|\ M(S) + \|\omega_\alpha\|_\infty\ M(R) \leq K\alpha^{\frac{1}{4}} \quad .$$

Thus we get in particular

$$\pi Q - K\alpha^{\frac{1}{4}} \le |C_{r,x} \llcorner B_1(0)(\omega_\alpha)| \le M(C_{r,x} \llcorner B_1(0) \cap E(\alpha^{\frac{1}{4}}))\|\omega_\alpha\|_\infty$$

$$\le M(C_{r,x} \llcorner B_1(0) \cap E(\alpha^{\frac{1}{4}})) \tag{V.18}$$

Assuming now there exists $y \in (\mathcal{C}_{r,x})_* \cap B_1(0) \cap (\mathbb{R}^{2p} \setminus E(\alpha^{\frac{1}{16}}))$. From the monotonicity formula we deduce that

$$M(C \llcorner B_{\frac{\alpha^{\frac{1}{16}}}{2} r}(y)) \ge \frac{\pi}{4} \alpha^{\frac{1}{8}}\, r^2 \quad . \tag{V.19}$$

Combining (V.18) and (V.19), we obtain that

$$M(C \llcorner B_r(x)) \ge r^2 \left(\pi Q - K\alpha^{\frac{1}{4}} + \frac{\pi}{4}\alpha^{\frac{1}{8}}\right) \tag{V.20}$$

For α small enough (V.20) contradicts (V.11) and (V.15) holds true for any $r \in [r_x, \frac{r_{\varepsilon,x_0}}{2}]$. From this later fact one deduces (V.13).

It remains to prove (V.14). Again we argue by contradiction. Assume there exists $y \in (\mathcal{C}_p \setminus \mathcal{C}_{p-1}) \cap \Pi^{-1}(B^2_{r_x}(\Pi(x)) \cap B_{r_{\varepsilon,x_0}}(x_0)$. Because of (V.3) and since $y \in B_{r_{x_0,\varepsilon}}(x_0)$ we can apply lemma IV.2 to y in order to deduce that $\mathcal{C}_\star \cap B_{r_x}(x)$ is included in a cone of center y, axis parallel to D_0 and angle ε. This cone of course contains x and then we can deduce that

$$supp(C_{r_x,x} \llcorner B_1(0)) \subset E(4\varepsilon) \quad . \tag{V.21}$$

(The notation $E(\lambda)$ is introduced in (V.16)). We have $\partial C_{r,x} \llcorner B_1(0) = 0$ moreover, because of (V.17), for α small enough we deduce that the intersection number of $C \llcorner B^2_{r_x}(\Pi(x)) \times B^{2p-2}_{r_x}(0)$ with any vertical current $\Pi^{-1}(z)$ for $z \in B^2_{r_x}$ is Q. Combining this fact with (V.21), using Fubini, one deduces that

$$M(C_{r_x,x} \llcorner B_1(0)) \ge \pi Q - O(\varepsilon^2) \quad . \tag{V.22}$$

For ε small enough compare to α we get a contradiction while comparing (V.22) and (V.11) and (V.14) is proved. This concludes the proof of lemma V.2. ■

In the following second lemma of this chapter, we show that the covering $(B^2_{r_x}(\Pi(x)))_{x \in \mathcal{C}_{p-1} \cap B^2_{\varepsilon^2 r_{\varepsilon,x_0}}(x_0)}$ of $\Pi(\mathcal{C}_{p-1} \cap B^{2p}_{\varepsilon^2 r_{\varepsilon,x_0}}(x_0)$ has the "Whitney" property : two balls intersecting each-other have comparable size. From now on we adopt the following notation granting the fact that α and ε are fixed small enough for the constraint mentionend above to be fulfilled :

$$\rho_{x_0} := \varepsilon^2 r_{\varepsilon,x_0} \quad . \tag{V.23}$$

Precisely we have.

Lemma V.3 (Whitney property of the covering.) *There exists $\gamma > 0$ depending only on Q such that, given $x_0 \in \mathcal{C}_p \setminus \mathcal{C}_{p-1}$ whose tangent cone is $Q[D_0]$ and let $(B^2_{r_x}(\Pi(x)))$ for $x \in \mathcal{C}_{p-1} \cap B^2_{\rho_{x_0}}(x_0)$ the covering of* $\Pi(\mathcal{C}_{p-1} \cap B^{2p}_{\rho_{x_0}}(x_0)$ *described above, assuming for some $x, y \in \mathcal{C}_{p-1} \cap B^2_{\rho_{x_0}}(x_0)$*

$$B^2_{r_x}(x) \cap B^2_{r_y}(y) \neq \emptyset \quad ,$$

then

$$r_x \geq \alpha^\gamma r_y \quad . \tag{V.24}$$

$\blacksquare$

Proof of lemma V.3. This lemma is again a consequence of the upper and lower-epiperimetric inequalities. Assume for instance that $r_x \leq r_y$. From (V.10) we have

$$\mathcal{F}(C_{r_y,y} \llcorner B_1(0) - Q\ [D_0]) \geq \alpha \quad . \tag{V.25}$$

Which implies that for all $r \leq r_y$

$$\mathcal{F}(C \llcorner B^2_r(z_y) \times B^{2p}_{\rho_{x_0}}(0) - Q[B^2_r(z_y) \times \{0\}]) \geq \alpha r^3 \tag{V.26}$$

where $y = (z_y, w_y)$. Since $|z_x - z_y| \leq 2\max\{r_x, r_y\} = 2r_y$ and $B_{r_y}(z_y) \subset B_{3r_y}(z_x)$, (V.26) implies that

$$\mathcal{F}(C \llcorner B^2_{4r_y}(z_x) \times B^{2p}_{\rho_{x_0}}(0) - Q[B^2_{4r_y}(z_x) \times \{w_y\}]) \geq \frac{\alpha}{3} r_y^3 \tag{V.27}$$

This passage from (V.26) to (V.27) is obtained by applying some Fubini type argument. Indeed, let $A = C \llcorner B^2_{4r_y}(z_x) \times B^{2p}_{\rho_{x_0}}(0) - Q[B^2_{4r_y}(z_x) \times \{w_y\}]$ and let S and R such that $A = \partial S + R$ and $M(S) + M(R)^{\frac{3}{2}} \leq 2\mathcal{F}(A)$. For almost every r in $[r_y/2, r_y]$ we have

$$\partial(S \llcorner B^2_r(z_y) \times B^{2p}_{\rho_{x_0}}(0)) = \partial S \llcorner B^2_r(z_y) \times B^{2p}_{\rho_{x_0}}(0)) + \langle S, dist(\cdot, \{z = z_y\}, r\rangle$$

where $\langle S, dist(\cdot,\{z=z_y\}), r\rangle$ is the slice current between S and the boundary of the cylinder $B^2_r(z_y)\times B^{2p}_{\rho_{x_0}}(0)$ and $dist(\cdot,\{z=z_y\})$ denotes the distance function to the axis to this cylinder (see [Fe] 4.2.1 pages 395...). Thus

$$
\begin{aligned}
&A\llcorner B^2_r(z_y)\times B^{2p}_{\rho_{x_0}}(0)\\
&=\partial(S\llcorner B^2_r(z_y)\times B^{2p}_{\rho_{x_0}}(0))-\langle S, dist(\cdot,\{z=z_y\}, r\rangle+R\llcorner B^2_r(z_y)\times B^{2p}_{\rho_{x_0}}(0)
\end{aligned}
\tag{V.28}
$$

We have, see [Fe] 4.2.1 page 395,

$$
\int_{\frac{r_y}{2}}^{r_y} M\left(\langle S, dist(\cdot,\{z=z_y\}, r\rangle\right)\le M\left(S\llcorner(B^2_{r_y}\setminus B^2_{\frac{r_y}{2}})\times B^{2p}_{\rho_{x_0}}(0)\right) \tag{V.29}
$$

Using Fubini theorem we may then find $r=r_1\in[r_y/2,r_y]$ such that

$$
M\left(\langle S, dist(\cdot,\{z=z_y\}, r_1\rangle\right)\le\frac{2}{r_y}M\left(S\llcorner(B^2_{r_y}\setminus B^2_{\frac{r_y}{2}})\times B^{2p}_{\rho_{x_0}}(0)\right)\le\frac{2}{r_y}M(S) \tag{V.30}
$$

Combining (V.28), (V.29) and (V.30) we deduce that

$$
\begin{aligned}
&\mathcal{F}(C\llcorner B^2_{r_1}(z_y)\times B^{2p}_{\rho_{x_0}}(0)-Q[B^2_{r_1}(z_y)\times\{0\}])=\mathcal{F}(A\llcorner B^2_r(z_y)\times B^{2p}_{\rho_{x_0}}(0))\\
&\quad\le M(S)+(M(R)+\frac{2}{r_y}M(S))^{\frac{3}{2}}\quad .
\end{aligned}
\tag{V.31}
$$

Thus, combining (V.26) for $r=r_1$ and (V.31), we have $M(S)+(M(R)+\frac{2}{r_y}M(S))^{\frac{3}{2}}\ge\alpha r_y^3$ and since $\mathcal{F}(C\llcorner B^2_{4r_y}(z_x)\times B^{2p}_{\rho_{x_0}}(0)-Q[B^2_{4r_y}(z_x)\times\{w_y\}])\ge\frac{1}{2}\left[M(S)+M(R)^{\frac{3}{2}}\right]$, we obtain (V.27). Therefore we deduce that

$$
\mathcal{F}(C_{4r_y,x}\llcorner B_1(0)-Q\ [D_0])\ge\frac{1}{3\times 4^3}\alpha\quad . \tag{V.32}
$$

Let $\frac{\rho_{x_0}}{\varepsilon^2}>s_x>r_x$ such that

$$
M(C_{s_x,x}\llcorner B_1(0))=\pi Q\quad . \tag{V.33}
$$

Because of (V.3), arguing like in the prof of lemma IV.2, between IV.8) and (IV.10), we have

$$
\forall s_x\le r\le\frac{\rho_{x_0}}{\varepsilon^2}\qquad \mathcal{F}(C_{r,x}\llcorner B_1(0)-Q\ [D_0])\le K\varepsilon\quad . \tag{V.34}
$$

Assuming, as we did above that $\alpha >> \varepsilon$, comparing (V.32) and (V.26) we deduce that $s_x > 4r_y$. Let $\lambda = 4\frac{r_x}{r_y}$. From the proof of lemma V.1, in fact from (V.9) precisely, for any $r \in [r_x, s_x]$ we have

$$\mathcal{F}(C_{2r,x} \llcorner B_1(0) - \pi Q[D_0]) \leq K\ \sqrt{\alpha} \leq \varepsilon_Q^- \quad , \tag{V.35}$$

which means in particular that we are in the position to apply the lower-epiperimetric inequality and the differential inequality (III.27) in [Ri2] deduced from it. Integrating then this inequality between r_x and $4r_y$ we have

$$\begin{aligned} \pi Q - M(C_{4r_y,x} \llcorner B_1(0)) &\leq \left(\frac{1}{2^{2\varepsilon_Q^-}}\right)^{\log_2 \lambda} [\pi Q - M(C_{r_x,x} \llcorner B_1(0))] \\ &= \left(\frac{1}{2^{2\varepsilon_Q^-}}\right)^{\log_2 \lambda} \alpha \quad . \end{aligned} \tag{V.36}$$

Using now (III.28) from [Ri2], we deduce from (V.36)

$$\begin{aligned} \mathcal{F}((C_{s_x,x} - C_{4r_y,x}) \llcorner B_1(0)) &\leq \sqrt{\pi Q - M(C_{4r_y,x} \llcorner B_1(0))} \\ &= \left(\frac{1}{2^{\varepsilon_Q^-}}\right)^{\log_2 \lambda} \alpha^{\frac{1}{2}} \quad . \end{aligned} \tag{V.37}$$

Combining (V.34) and (V.37) one gets that

$$\mathcal{F}(C_{4r_y,x} \llcorner B_1(0) - Q[D_0]) \leq \left(\frac{1}{2^{\varepsilon_Q^-}}\right)^{\log_2 \lambda} \alpha^{\frac{1}{2}} + K\varepsilon \quad . \tag{V.38}$$

Comparing (V.32) and (V.38) we obtain

$$\frac{1}{3 \times 4^3}\alpha \leq \left(\frac{1}{2^{\varepsilon_Q^-}}\right)^{\log_2 \lambda} \alpha^{\frac{1}{2}} \quad . \tag{V.39}$$

Since $K\varepsilon \leq \frac{1}{6\times 4^3}\alpha$ we have $\frac{1}{6\times 4^3}\alpha \leq \left(\frac{1}{2^{\varepsilon_Q^-}}\right)^{\log_2 \lambda} \alpha^{\frac{1}{2}}$. Taking the log of this last inequality we obtain

$$\frac{\log \lambda}{\log 2}\varepsilon_Q^- \log \frac{1}{2} + \frac{1}{2} \log \alpha \geq \log \alpha - \log(6 \times 4^3)$$

Thus

$$\frac{1}{2}\log\frac{1}{\alpha}+\log(6\times 4^3)\geq\varepsilon_Q^-\log\lambda$$

By taking α small enough, we may always assume that $\log\frac{1}{\alpha}\geq 4\times\log(6\times 4^3)$ and we finally get that

$$\frac{1}{\varepsilon_Q^-}\log\frac{1}{\alpha}\geq\log\lambda$$

Which leads to the desired inequality (V.24) and lemma V.3 is proved. ∎

Constructing a partition of unity adapted to the covering.

From the covering $(B^2_{r_x}(x))$ for $x\in\mathcal{C}_{p-1}\cap B_{\rho_{x_0}}(x_0)$ of $\Pi(\mathcal{C}_{p-1}\cap B_{\rho_{x_0}}(x_0))$ we extract a Besicovitch covering $(B^2_{r_{x_i}}(x_i))$ for $i\in I$ (I is a countable set) of $\Pi(\mathcal{C}_{p-1}\cap B_{\rho_{x_0}}(x_0))$ that is a covering such that

$$\forall z\in B^2_{\rho_{x_0}}(x_0)\qquad Card\left\{i\in I\quad\text{s. t.}\quad z\in B^2_{r_{x_i}}(x_i)\right\}\leq n\quad,\tag{V.40}$$

where N is some universal number (see [Fe]). To simplify the notation we will simply write r_i for r_{x_i}. Remark that since balls intersecting each-other have comparable size (see lemma V.3), each ball $B^2_{r_i}(z_i)$ intersects a uniformly bounded number of other balls : there exists $N_{Q,\alpha}$ such that

$$\forall i\in I\qquad\text{Card}\left\{j\in I\quad\text{s. t.}\quad B^2_{r_j}(z_j)\cap B^2_{r_i}(z_i)\neq\emptyset\right\}\leq N_{Q,\alpha}\quad.\tag{V.41}$$

We now construct a partition of unity adapted to a slightly modified covering. Considering the covering $(B^2_{r_{z_i}}(z_i))$ for $i\in I$ (I is a countable set) of $\Pi(\mathcal{C}_{p-1}\cap B^{2p}_{\rho_{x_0}}(x_0))$, we can apply lemma A.1 and obtain δ depending on α and Q such that (A.3) holds true for some $P\in\mathbb{N}$. Let $i\in I$ we can deduce from (A.3) and (V.24) that the radii r_j of balls $B^2_{r_j}(z_j)$ intersecting $B^2_{r_i(1+\delta)}(z_i)$ satisfy $\alpha^{\gamma^P}r_i\leq r_j\leq\alpha^{-\gamma^P}r_i$. From this later fact we deduce that there exists a number $M\in\mathbb{N}$ depending only on α and Q such that

$$\text{Card}\left\{j\in I\quad\text{s. t. }B_{r_i}(z_i)\cap B_{(1+\delta)r_j}(z_j)\neq\emptyset\right\}\leq M\quad.\tag{V.42}$$

Indeed, assuming $B_{r_i}(z_i)\cap B_{r_j}(z_j)=\emptyset$, if $B_{r_i}(z_i)\cap B_{(1+\delta)r_j}(z_j)$ we just have seen that $\alpha^{\gamma^P}r_j\leq r_i\leq\alpha^{-\gamma^P}r_j$: the two radii have comparable size which is of course also comparable with the distance $|z_i-z_j|$. From (V.40) it is then clear that the number of such ball $B_{r_j}(z_j)$ is bounded by a constant depending only of the variables α and Q. It is now not difficult to deduce that $(B^2_{r_i(1+\frac{\delta}{2})})_{i\in I}$ realizes a locally finite covering of $\Pi(\mathcal{C}_{p-1}\cap B^{2p}_{\rho_{x_0}}(x_0))$ satisfying

$$\forall i\in I \qquad \mathrm{Card}\left\{j\in I \quad \text{s. t. } B_{(1+\frac{\delta}{2})r_i}(z_i)\cap B_{(1+\frac{\delta}{2})r_j}(z_j)\neq\emptyset\right\}\le M+P \quad . \tag{V.43}$$

Indeed assuming for instance that $r_i\ge r_j$, then $B_{(1+\frac{\delta}{2})r_i}(z_i)\cap B_{(1+\frac{\delta}{2})r_j}(z_j)\neq\emptyset$ implies clearly that $B_{(1+\delta)r_i}(z_i)\cap B_{r_j}(z_j)\neq\emptyset$ and the numer of such a j is controled by P (see A.3), whereas if $r_i\le r_j$, $B_{(1+\frac{\delta}{2})r_i}(z_i)\cap B_{(1+\frac{\delta}{2})r_j}(z_j)\neq\emptyset$ implies clearly that $B_{r_i}(z_i)\cap B_{r_j(1+\delta)}(z_j)\neq\emptyset$ and the numer of such a j is controled by M (see V.42). Thus (V.43) holds true. And we shall use from now on the notation

$$\forall i\in I \qquad \rho_i:=r_i\left(1+\frac{\delta}{2}\right) \quad . \tag{V.44}$$

For any $i\in I$ we define χ_i to be a smooth non-negative function satisfying

i)
$$\chi_i\equiv 1 \quad \text{in } B^2_{r_i}(z_i) \quad . \tag{V.45}$$

ii)
$$\chi_i\equiv 0 \quad \text{in } \mathbb{R}^2\setminus B^2_{(1+\frac{\delta}{2}r_i)}(z_i) \tag{V.46}$$

iii)
$$\forall k\in\mathbb{N} \qquad \|\nabla^k\chi_i\|_\infty\le\frac{K_k}{r_i^k} \quad , \tag{V.47}$$

where K_k depends only on k and Q.

We define now
$$\varphi_i:=\frac{\chi_i}{\sum_{i\in I}\chi_i} \quad . \tag{V.48}$$

It is clear that (φ_i) defines a partition of unity adapted to $B^2_{(1+\frac{\delta}{2}r_i)}(z_i)$ and satisfying the following estimates

$$\forall k\in\mathbb{N} \qquad \|\nabla^k\varphi_i\|_\infty\le\frac{K_k}{r_i^k} \quad , \tag{V.49}$$

where K_k depends only on k and Q.

VI The approximated average curve.

This chapter is another step towards the proof that $\mathcal{P}_{Q-1} \Longrightarrow \mathcal{P}_Q$ which goes until chapter VIII. We thus assume that $\mathcal{P}_{Q-1}$ holds (or that $Q = 1$). Again in this part we consider the difficult case which is the case where we are blowing-up the current at a point x_0 of multiplicity $Q > 1$ whose tangent cone C_{0,x_0} is Q times the integral over the flat disk D_0 given by $w_i = 0$ for $i \cdots Q-1$ (we use the system of coordinates introduced in the begining of chapter II) and where x_0 belongs to the closure of $\mathcal{C}_{Q-1}$. The purpose of this chapter is to approximate first our current over each ball of the covering introduced in the previous section $C \llcorner \Pi^{-1}(B^2_{r_i}(z_i))$ by a $Q-$valued graph $\{a^k_i\}_{k=1\cdots Q}$ over $B^2_{r_i}(z_i)$ which is almost $J-$holomorphic ($J_{x_i}-$holomorphic in fact where $x_i \in \mathcal{C}_*$ and $\Pi(x_i) = z_i$) and glueing the average curves $\tilde{a}_i = \frac{1}{Q}\sum_{k=1}^{Q} a^k_i$ of each of these $J_{x_i}-$holomorphic $Q-$valued graphs together we shall produce a single-valued graph $\tilde{a}$ over $B^2_{\rho_{x_0}}(x_0)$ which approximates C and for which we will study regularity properties that will be used in the following chapter VII devoted to the unique continuation argument . Finally in the second subsection of this chapter we construct new coordinates adapted to the average curve.

VI.1 Constructing the average curve.

Let ρ_{x_0} given by (V.23) and let $(B^2_{\rho_i}(z_i))_{i\in I}$ be the Besicovitch-Whitney covering of $\Pi(\mathcal{C}_{Q-1} \cap B_{\rho_{x_0}}(x_0))$ obtained at the end of the previous chapter. As we have seen above, for any $i \in I$, $\mathcal{C}_* \cap \Pi^{-1}(B^2_{\rho_i}(z_i)) \subset \mathcal{C}_{Q-1} \cap B^2_{\rho_i}(z_i) \times B^{2p-2}_{2\rho_i}(w_i)$ where $x_i = (z_i, w_i)$ is in $\mathcal{C}_*$ (see lemma V.2). For convenience we shall adopt the following notation

$$N^i_r := B^2_r(z_i) \times B^{2p-2}_{2\rho_i}(w_i) \quad . \tag{VI.1}$$

Assuming $\mathcal{P}_{Q-1}$, $C \llcorner N_{2\rho_i}$ is a $J-$holomorphic curve : there exists a smooth Riemannian surface and a smooth $J-$holomorphic map

$$\begin{aligned} \Psi_i \ : \ \Sigma_{2,i} \ &\longrightarrow \ N^i_{2\rho_i} \\ \xi \ &\longrightarrow \ \Psi_i(\xi) \end{aligned} \tag{VI.2}$$

such that $\Psi_*[\Sigma_{2,i}] = C \llcorner N_{2\rho_i}$. Let $H^0_\pm(\Sigma_{2,i})$ be the sets respectively of holomorphic and antiholomorphic functions on $\Sigma_{2,i}$. We introduce now η_i

the map from $\Sigma_{2,i}$ into $\mathbb{R}^{2p-2}$ chosen such that the perturbation $\Psi_i + \eta_i$ is $J_{x_i}-$holomorphic, precisely η_i is given by

$$\left\{\begin{array}{ll} \dfrac{\partial}{\partial\xi_1}(\Psi_i+\eta_i) + J_{x_i}\dfrac{\partial}{\partial\xi_2}(\Psi_i+\eta_i) = 0 & \text{in } \Sigma_{2,i} \\ \forall h \in H(\Sigma_{2,i}) \qquad \displaystyle\int_{\partial\Sigma_{2,i}} h\ d\eta_i = 0 & \end{array}\right. \tag{VI.3}$$

where (ξ_1, ξ_2) are local coordinates on $\Sigma_{2,i}$ compatible with the complex structure. The existence of η_i is justified in a few lines below. To this aim we shall make use of the following notations. Since J is smooth the inverse function theorem gives the existence of a smooth map $\Lambda\ :\ B^{2p}_{\rho_{x_0}}(x_0) \times \mathbb{R}^{2p} \longrightarrow \mathbb{R}^{2p}$ - for ρ_{x_0} chosen small enough such that

i)

$$\Lambda_x := \Lambda(x,.) \qquad \text{is a linear isomorphism of } \mathbb{R}^{2p} \quad , \tag{VI.4}$$

ii)

$$\Lambda_{x_0} = id \quad , \tag{VI.5}$$

iii)

$$\forall x \in B^{2p}_{\rho_{x_0}}(x_0) \qquad J_{x_0} = \begin{pmatrix} 0 & -1 & 0\cdots & 0 & 0 \\ 1 & 0 & 0\cdots & 0 & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & .\cdots & 0 & -1 \\ 0 & 0 & .\cdots & 1 & 0 \end{pmatrix} = \Lambda_x\ J_x\ \Lambda_x^{-1} \tag{VI.6}$$

We shall denote by $(z^i = x_1^i + ix_2^i, w_1^i = x_3^i + ix_4^i, \cdots, w_p^i = x_{2p-1}^i + ix_{2p}^i)$ the following complex coordinates in $N_{2\rho_i}$

$$
\begin{pmatrix} x_1^i \\ \\ x_2^i \\ \\ . \\ \\ . \\ \\ . \\ \\ x_{2p-1}^i \\ \\ x_{2p}^i \end{pmatrix} = \Lambda_{x_i} \cdot \begin{pmatrix} x_1 \\ \\ x_2 \\ \\ . \\ \\ . \\ \\ . \\ \\ x_{2p-1} \\ \\ x_{2p} \end{pmatrix} - x_i \tag{VI.7}
$$

We will also denote by Π_i the map that assign to any point x in $N_{\rho_i}^i$ the complex coordinate z^i and by D^i we denote the $J_{x_i}-$holomorphic 2-disk

$$
D^i := \left\{ x \ ; \ \forall k = 1 \cdots p-1 \quad w_k^i = 0 \right\} = \Lambda_{x_i}^{-1} D_0 \tag{VI.8}
$$

Using these complex coordinates in $N_{2\rho_i}^i$, (VI.3) means

$$
\left\{
\begin{array}{l}
\overline{\partial}\eta_i = -\overline{\partial}\Psi_i \quad \text{ in } \Sigma_{2,i} \\
\\
\forall h \in H(\Sigma_{2,i}) \qquad \displaystyle\int_{\partial\Sigma_{2,i}} h \ d\eta_i = 0 \quad .
\end{array}
\right. \tag{VI.9}
$$

The existence and uniqueness of η_i is given by proposition A.3 of [Ri3]. Since Ψ_i is $J-$holomorphic we have $\partial_{\xi_1}\Psi_i + J(\Psi_i(\xi))\partial_{\xi_2}\Psi_i = 0$, thus $|\partial_{\xi_1}\Psi_i + J(x_i)\partial_{\xi_2}\Psi_i| \le |J(\Psi_i(\xi)) - J(x_i)| \ |\nabla\psi|$. Combining this fact with the second part of proposition A.3 (i.e. estimate (A.13) of [Ri3]) we obtain

$$
\int_{\Sigma_{2,i}} |\nabla\eta_i|^2 \le K r_i^2 \int_{\Sigma_{2,i}} |\nabla\Psi_i|^2 \le K r_i^4 \quad . \tag{VI.10}
$$

For $\lambda \le 2$, we denote by $\Sigma_{\lambda,i}$ the surface $\Sigma_{\lambda,i} = \Sigma_{2,i} \cap \Psi_i^{-1}(N_{\lambda\rho_i}^i)$ so that

$$
\Psi_{i*}[\Sigma_i] = C \llcorner B_{\rho_i}^2(z_i) \times B_{\rho_{x_0}}^{n-2}(0) \quad . \tag{VI.11}
$$

We then prove in [Ri3] the following lemma

Lemma VI.1 *Under the above notations one has*

$$\|\eta_i\|_{L^\infty(\Sigma_{\frac{3}{2},i})} \le K r_i^2 \tag{VI.12}$$

where K is a constant depending only on $\|\nabla J\|_\infty$ and the choice of α made in the previous chapter.

Consider now the $J_{x_i}-$holomorphic curve C^i given by the image by $\Psi_i + \eta_i$ of $\Sigma_{\frac{3}{2},i}$. Since $\partial\Psi_{i*}[\Sigma_{\frac{3}{2},i}]$ is supported in $\Pi^{-1}(\partial B^2_{\frac{3}{2}\rho_i})$, we know from Lemma VI.1 that $|\eta_i|_\infty \le C r_i^2$ therefore $\partial\Psi_i + \eta_{i*}[\Sigma_{\frac{3}{2},i}]$ is supported in an r_i^2 neighborhood of $\Pi^{-1}(\partial B^2_{\frac{3}{2}\rho_i}(0))$ (for r_i small enough : that holds if ε has been chosen small enough in section VI). Thus we have that $\Psi_{i*}[\Sigma_{\frac{3}{2},i}])$ is a cycle in $\Pi_i^{-1}(B^2_{\frac{5}{4}\rho_i}(0))$ and the part of the image of $\Sigma_{\frac{3}{2},i}$ included in $\Pi_i^{-1}(B^2_{\frac{5}{4}\rho_i}(0))$ by $\Psi_i+\eta_i$ is a $J_{x_i}-$holomorphic cycle and therefore it is a $Q-$valued graph over D^i for the complex coordinates given by (z^i, w^i). We denote by $\{a^i_k\}_{k=1\cdots Q}$ this $Q-$valued graph (i.e. $a^i_k(z^i_0)$ are the w^i coordinates, in the chart (z^i,w^i), of the Q intersection points between the $J_{x_i}-$holomorphic curve $\Psi_i+\eta_i(\Sigma_{\frac{3}{2},i})$ and the $J_{x_i}-$holomorphic submanifold given by $z^i = z^i_0$). We now define $\tilde{C}^i$ to be the J_{x_i} holomorphic curve in $\Pi_i^{-1}(B^2_{\rho_i})$ given by

$$\tilde{C}^i := \left\{ x = \Lambda^{-1}_{x_i}\left((z^i, \tilde{a}^i_i(z_i) = \frac{1}{Q}\sum_{k=1}^{Q} a^i_k(z^i)) + x_i \right) \quad \forall z^i \in B^2_{\frac{5}{4}\rho_i}(0) \right\} \quad . \tag{VI.13}$$

Observe that

$$\frac{\partial}{\partial z^i}\tilde{a}^i_i = 0 \qquad \text{in } \mathcal{D}'(B^2_{\frac{5}{4}\rho_i}(0)) \quad . \tag{VI.14}$$

Moreover, the conformal invariance of the Dirichlet energy gives

$$\begin{aligned}\int_{B^2_{\frac{5}{4}\rho_i}(0)} \sum_{k=1}^{Q} |\nabla a^i_k|^2(z^i)\ dz^i\wedge d\overline{z}^i \\ = \int_{(\Psi_i+\eta_i)^{-1}(C_i\cap\Pi_i^{-1}(B^2_{\frac{5}{4}\rho_i}(0)))} |\nabla(\Psi_i+\eta_i)|^2(\xi)\ d\xi \le K r_i^2 \quad .\end{aligned} \tag{VI.15}$$

We then deduce that

$$\int_{B^2_{\frac{5}{4}\rho_i}(0)} |\nabla\tilde{a}^i_i|^2(z^i)\ dz^i \le K r_i^2 \quad . \tag{VI.16}$$

Combining (VI.15) and (VI.16) and using standard elliptic estimates we get that for any $l \in \mathbb{N}$

$$\|\nabla^l \tilde{a}_i^i\|_{L^\infty(B^2_{\frac{6}{5}\rho_i}(0))} \le K_l r_i^{-l+1} \quad . \tag{VI.17}$$

The subscript i in the notation $\tilde{a}_i^i$ is here to recall that we express $\tilde{C}^i$ as a graph in the (z^i, w^i) coordinates. The same $J_{x_i}-$holomorphic curve $\tilde{C}^i$ can also, due to (VI.17), be expressed as a graph in a neighborhing system of coordinate (z^j, w^j) where $B_{\rho_i}(z_i) \cap B_{\rho_j}(z_j) \neq \emptyset$ (indeed the passage from (z^i, w^i) to (z^j, w^j) is given by a transformation matrix in $\mathbb{R}^{2p}$ close to the identity at a distance of the order r_i). In such system of coordinates (z^j, w^j), we shall denote $\tilde{a}_j^i(z^j)$ the graph corresponding to $\tilde{C}^i$.

Since $\tilde{C}^i$ is a graph over $w^i = 0$ given by $(z^i, \tilde{a}_i^i(z^i))$ whose gradient is bounded (see (VI.17)), and since the passage from the (z, w) coordinates to (z^i, w^i) coordinates is given by a transformation Λ_{x_i} whose distance to the identity is bounded by $|x_i|$ that tends to zero, $\tilde{C}^i$ is then also realised by a graph over $B^2_{\frac{7}{6}\rho_i}(\Pi(x_i))$ that we shall now denote $(z, \tilde{a}_i(z))$:

$$\tilde{C}^i \llcorner \Pi^{-1}(B^2_{\frac{7}{6}\rho_i}(\Pi(x_i))) = (z, \tilde{a}_i(z))_*[B^2_{\frac{7}{6}\rho_i}(\Pi(x_i))] \quad . \tag{VI.18}$$

Consider now i and j such that $B_{\rho_i}(z_i) \cap B_{\rho_j}(z_j) \neq \emptyset$. We shall compare $\tilde{a}_i$ and $\tilde{a}_j$ in $\Pi^{-1}(B_{\rho_i}(z_i) \cap B_{\rho_j}(z_j))$. Precisely we have the following lemma

Lemma VI.2 *Under the above notations one has*

$$\forall l \in \mathbb{N} \qquad \|\nabla^l(\tilde{a}_i - \tilde{a}_j)\|_{L^\infty(B^2_{\rho_i}(z_i) \cap B^2_{\rho_j}(z_j))} \le K_l \rho_i^{2-l} \quad . \tag{VI.19}$$

Proof of lemma VI.2.

First of all we compare C_i and C_j in $\Pi^{-1}(B_{\rho_i}(z_i) \cap B_{\rho_j}(z_j))$. We can always assume that $\Sigma_{2,i}$ and $\Sigma_{2,j}$ are part of a same Riemannian surface Σ with a joint parametrization $\Psi = \Psi_i$ on $\Sigma_{2,i}$ and $\Psi = \Psi_j$ on $\Sigma_{2,j}$ and such that $\Psi_*[\Sigma] = C \llcorner N^i_{2\rho_i} \cup N^j_{2\rho_j}$. Let $\Sigma^{ij} := \Psi^{-1}(supp(C) \cap N^i_{2\rho_i} \cap N^j_{2\rho_j}$. We consider the following mapping

$$\begin{array}{rl} \Xi^{ij} \quad \Sigma^{ij} \times [0,1] \longrightarrow & N^i_{3\rho_i} \cap N^j_{3\rho_j} \\ (\xi, t) \longrightarrow & \Psi(\xi) + t\eta_j(\xi) + (1-t)\eta_i(\xi) \quad . \end{array} \tag{VI.20}$$

Clearly for any $\lambda < \frac{3}{2}$

$$\partial \Xi^{ij}{}_*[\Sigma^{ij}] \times [0,1] \llcorner N^i_{\lambda\rho_i} \cap N^j_{\lambda\rho_j} = C^j - C^i \llcorner N^i_{\lambda\rho_i} \cap N^j_{\lambda\rho_j} \quad . \tag{VI.21}$$

We have

$$M({\Xi^{ij}}_*[\Sigma^{ij}]\times[0,1])=\int_0^1\int_{\Sigma^{ij}}J_3\Xi^{ij}\quad, \tag{VI.22}$$

where $(J_3\Xi^{ij})^2$ is the sum of the squares of the determinants of the 3×3 submatrices of $D\Xi^{ij}$. Clearly

$$|J_3\Xi^{ij}|(\xi,t)\le K\ \left[\|\eta_i\|_\infty+\|\eta_j\|_\infty\right]\ \left[|\nabla\Psi|^2(\xi)+|\nabla\eta_i|^2(\xi)+|\nabla\eta_j|^2(\xi)\right]\quad. \tag{VI.23}$$

Combining Lemma V.3, Lemma VI.1, (VI.22) and (VI.23), we get that for any $\lambda<\frac{3}{2}$

$$M({\Xi^{ij}}_*[\Sigma^{ij}]\times[0,1]\llcorner N^i_{\lambda\rho_i}\cap N^j_{\lambda\rho_j})\le K\ r_i^4\quad. \tag{VI.24}$$

Therefore, combining (VI.21) and (VI.24), using a standard slicing and Fubini type argument, we may find $\lambda\in(\frac{5}{4},\frac{3}{2})$ such that

$$\mathcal{F}((C^i-C^j)\llcorner N^i_{\lambda\rho_i}\cap N^j_{\lambda\rho_j})\le Kr_i^4\quad. \tag{VI.25}$$

We shall now compare $\tilde{C}^i$ and $\tilde{C}^j$. Denote $(x_1^t,x_2^t\cdots x_{2p}^t)$ the coordinates given by $(x_1^t,x_2^t\cdots x_{2p}^t)^T=\Lambda_{x_t}\cdot[(x_1,x_2\cdots x_{2p})^T-x_i]$ where we keep denoting $(x_1,x_2\cdots x_{2p})$ our original normal coordinates vanishing at the center x_0 introduced in (II.1) and Λ_{x_t} is the transformation matrix introduced in (VI.4). Observe that with these notations $(x_1^0,x_2^0\cdots x_{2p}^0)=(x_1^i,x_2^i\cdots x_{2p}^i)$ that $(x_1^1,x_2^1\cdots x_{2p}^1)=(x_1^j,x_2^j\cdots x_{2p}^j)+\Lambda_{x^j}\cdot(x_i-x_j)$ and that $(x_1^t,x_2^t\cdots x_{2p}^t)$ has been chosen in order to vanish at a fixed point x_i. Observe also that

$$\left|\frac{d}{dt}\Lambda_{x^t}\right|\le Kr_i\quad,\quad \|\frac{d}{dt}x^t\|_{L^\infty(B^{2p}_{4r_i}(x_i))}+\|\frac{d}{dt}y^t\|_{L^\infty(B^{2p}_{4r_i}(x_i))}\le Kr_i^2\quad. \tag{VI.26}$$

We also adopt the notations $z^t:=x_1^t+ix_2^t$ and for $k=1\cdots p-1$ $w^t:=x^t_{2k+1}+ix^t_{2k+2}$. Observe then that z^t =constant or w^t_k =constant are $J_{x^t}-$holomorphic $2p-2$ submanifolds, or simply complex variety in $(\mathbb{R}^{2p},J_{x^t})$. In order to compare $\tilde{C}^i$ and $\tilde{C}^j$ we shall perturb Ξ^{ij} in the following way : denote first Ψ^t, η^t_i and η^t_j the maps Ψ,η_i and η_j expressed in the coordinates (z^t,w^t), and consider the map $s^t\ :\ (\Sigma')^{ij}\to\mathbb{C}^p$ solving

$$\begin{aligned}&\partial_{\overline{\xi}}s^t=\partial_{\overline{\xi}}(\Psi^t+t\eta^t_j+(1-t)\eta^t_i)\quad \text{in } (\Sigma')^{ij}\\ &\forall h\in H(\Sigma^{ij})\qquad \int_{\partial\Sigma^{ij}}s^t\,dh=0\quad.\end{aligned} \tag{VI.27}$$

where $\Sigma^{ij} := \Psi^{-1}(supp(C)) \cap N^i_{\frac{3}{2}\rho_i} \cap N^j_{\frac{3}{2}\rho_j}$. The existence and uniqueness of s^t is given by proposition A.3 of [Ri3] We shall now replace the map Ξ^{ij} on $(\Sigma')^{ij}$ by the map

$$
\begin{aligned}
(\Xi')^{ij} \; : \; (\Sigma')^{ij} \times [0,1] &\longrightarrow \; N^i_{3\rho_i} \cap N^j_{3\rho_j} \\
(\xi, t) &\longrightarrow \; \Lambda^{-1}_{x^t} \cdot [\Psi^t(\xi) + t\eta^t_j(\xi) + (1-t)\eta^t_i(\xi) - s^t] + x_i \quad .
\end{aligned}
\tag{VI.28}
$$

Observe that for each $t \in [0,1]$ the map $\Xi'^{ij}(\cdot, t)$ is a $J_{x^t}-$holomorphic curve. Observe also that, for $t = 0$ $s^t = 0$ and that for $t = 1$, $\partial_{\overline{\xi}}(\Psi^1 + \eta^1_j) = 0$, since $\Psi + \eta_j$ is $J_{x^j}-$holomorphic and (z^1, w^1) are J_{x^j} coordinates, thus we have also $s^1 = 0$. One can easily verify, like for proving (VI.10) that for all $t \in [0,1]$

$$
\int_{(\Sigma')^{ij}} |\partial_{\overline{\xi}}(\Psi^t + t\eta^t_j + (1-t)\eta^t_i)|^2 \leq K r_i^4 \quad ,
$$

and using lemma VI.1 we have

$$
\|s^t\|_{L^\infty((\Sigma'')^{ij})} \leq K r_i^2 \quad ,
$$

where

$$
(\Sigma'')^{ij} := \Psi^{-1}(supp(C)) \cap N^i_{\frac{5}{4}\rho_i} \cap N^j_{\frac{5}{4}\rho_j} \quad .
$$

Therefore for any $\lambda < \frac{6}{5}$ we have

$$
\partial (\Xi')^{ij}{}_*[(\Sigma')^{ij}] \times [0,1] \llcorner N^i_{\lambda\rho_i} \cap N^j_{\lambda\rho_j} = C^j - C^i \llcorner N^i_{\lambda\rho_i} \cap N^j_{\lambda\rho_j} \quad . \tag{VI.29}
$$

We consider now the following interpolation between $\tilde{C}^i$ and $\tilde{C}^j$: let $\tilde{\Xi}^{ij}$ be the following map

$$
\begin{aligned}
\tilde{\Xi}^{ij} \quad \Pi(\Psi((\Sigma'')^{ij})) \times [0,1] &\longrightarrow \quad N^i_{3\rho_i} \cap N^j_{3\rho_j} \\
(z,t) &\longrightarrow \quad \Lambda^{-1}_{x^t} \cdot [(z, \tilde{a}^t(z)] + x_i \quad ,
\end{aligned}
\tag{VI.30}
$$

where the $p-1$ complex components are given by the slices of $C^t := (\Xi')^{ij}{}_*[(\Sigma')^{ij}] \times \{t\}$ by $z^t = \xi$ evaluated on the functions w^t_k. Precisely using the notations of [Fe] 4.3

$$
\tilde{a}^t_k(z) := \left\langle C^t, z^t, z \right\rangle (w^t_k) \quad . \tag{VI.31}
$$

It is clear that for any $\lambda < \frac{6}{5}$

$$\tilde{\Xi}^{ij}_*[\Pi_i(\Psi((\Sigma')^{ij}))] \times [0,1] \llcorner N^i_{\lambda\rho_i} \cap N^j_{\lambda\rho_j} = \tilde{C}^j - \tilde{C}^i \llcorner N^i \lambda\rho_i \cap N^j_{\lambda\rho_j} \quad . \tag{VI.32}$$

In order to get a bound for $\mathcal{F}(\tilde{C}^j - \tilde{C}^i \llcorner N^i_{\lambda\rho_i} \cap N^j_{\lambda\rho_j})$, it remains to evaluate the mass of $\tilde{\Xi}^{ij}_*[\Pi_i(\Psi((\Sigma')^{ij}))] \times [0,1] \llcorner N^i_{\lambda\rho_i} \cap N^j_{\lambda\rho_j}$ for $\lambda = \frac{6}{5}$ for instance. We have

$$|J_3 \tilde{\Xi}^{ij}|(z,t) \le \left| \frac{\partial}{\partial t} \Lambda^{-1}_{x^t} \cdot [(z, \tilde{a}^t(z)] \right| (z,t) \ \left[1 + |\nabla_z \tilde{a}^t(z)|^2\right] \quad . \tag{VI.33}$$

Because of the same arguments developed to prove (VI.17), since $\tilde{a}^t(\xi)$ is holomorphic, we have

$$\|\nabla_z \tilde{a}^t(z)\|_{L^\infty((\Sigma'')^{ij})} \le K \quad . \tag{VI.34}$$

Thus

$$\begin{aligned} M(\tilde{\Xi}^{ij}_*[\Pi\Psi((\Sigma')^{ij})] \times [0,1] \llcorner N^i \lambda\rho_i \cap N^j_{\lambda\rho_j}) &\le \int_0^1 \int_{\Pi(\Psi((\Sigma'')^{ij}))} |J_3 \tilde{\Xi}^{ij}| \\ \le K \int_0^1 \int_{\Pi(\Psi((\Sigma'')^{ij}))} \left| \frac{\partial}{\partial t} \Lambda^{-1}_{x^t} \cdot [(z, \tilde{a}^t(z)] \right| \, dz \wedge d\overline{z} \wedge dt & \\ \le K \int_0^1 \int_{\Pi(\Psi((\Sigma'')^{ij}))} \left[r_i |(z, \tilde{a}^t(z))| + \left| \frac{\partial \tilde{a}^t}{\partial t} \right| \right] \quad . & \end{aligned} \tag{VI.35}$$

On the one hand, since $\Pi_i(\Psi((\Sigma'')^{ij}))\ |(z, \tilde{a}^t(z))| \le K\ r_i$, we have

$$\int_0^1 \int_{\Pi(\Psi((\Sigma'')^{ij}))} r_i |(z, \tilde{a}^t(z))| \le K r_i^4 \quad . \tag{VI.36}$$

On the other hand

$$\int_0^1 \int_{\Pi(\Psi((\Sigma'')^{ij}))} \left| \frac{\partial \tilde{a}^t}{\partial t} \right| = \lim_{N \to +\infty} \frac{1}{N} \sum_{l=1}^{N-1} \int_{\Pi(\Psi((\Sigma'')^{ij}))} N\, |\tilde{a}^{\frac{l}{N}}(z) - \tilde{a}^{\frac{l+1}{N}}(z)|\, dz \wedge d\overline{z} \quad . \tag{VI.37}$$

We have

$$|\tilde{a}_k^{\frac{l}{N}}(z)-\tilde{a}_k^{\frac{l+1}{N}}(z)| = |< C^{\frac{l}{N}}, z^{\frac{l}{N}}, z > (w_k^{\frac{l}{N}}) - < C^{\frac{l+1}{N}}, z^{\frac{l+1}{N}}, z > (w_k^{\frac{l+1}{N}})|$$

$$\le |< C^{\frac{l}{N}} - C^{\frac{l+1}{N}}, z^{\frac{l}{N}}, z > (w_k^{\frac{l}{N}})|$$

$$+|< C^{\frac{l+1}{N}}, z^{\frac{l}{N}}, z > (w_k^{\frac{l}{N}}) - < C^{\frac{l+1}{N}}, z^{\frac{l+1}{N}}, z > (w_k^{\frac{l}{N}})|$$

$$+|< C^{\frac{l+1}{N}}, z^{\frac{l+1}{N}}, z > (w_k^{\frac{l}{N}} - w_k^{\frac{l+1}{N}})| \tag{VI.38}$$

We have to control the sum over l of the integral over $\Pi(\Psi((\Sigma'')^{ij}))$ of the 3 absolute values in the right-hand-side of (VI.38) one by one. For the first term of the r.h.s. of (VI.38) we have, using [Fe] 4.3.1, since $\|w_k^{\frac{l}{N}}\|_\infty + \|dw_k^{\frac{l}{N}}\|_\infty \le 1$,

$$\begin{aligned}
&\int_{\Pi(\Psi((\Sigma'')^{ij}))} |< C^{\frac{l}{N}} - C^{\frac{l+1}{N}}, z^{\frac{l}{N}}, z > (w_k^{\frac{l}{N}})|\ dz\wedge d\overline{z}\\
&\le Lip(z^{\frac{k}{N}})\ \mathcal{F}_{N^i_{\lambda\rho_i}\cap N^j_{\lambda\rho_j}}(C^{\frac{k}{N}} - C^{\frac{k+1}{N}}) \le K\ \mathcal{F}_{N^i_{\lambda\rho_i}\cap N^j_{\lambda\rho_j}}(C^{\frac{k}{N}} - C^{\frac{k+1}{N}})
\end{aligned} \tag{VI.39}$$

Similarly as the way we have established estimate (VI.25) we can show that

$$\mathcal{F}_{N^i_{\lambda\rho_i}\cap N^j_{\lambda\rho_j}}(C^{\frac{k}{N}} - C^{\frac{k+1}{N}}) \le K\ \frac{1}{N}\ r_i^4 \quad . \tag{VI.40}$$

Thus

$$\lim_{N\rightarrow+\infty}\sum_{l=1}^{N-1}\int_{\Pi(\Psi((\Sigma'')^{ij}))} |< C^{\frac{l}{N}} - C^{\frac{l+1}{N}}, z^{\frac{l}{N}}, z > (w_k^{\frac{l}{N}})| \le K\ r_i^4 \quad . \tag{VI.41}$$

For the second term of the r.h.s. of (VI.38) we use 4.3.9 (3) of [Fe] and we get

$$\begin{aligned}
&\int_{\Pi(\Psi((\Sigma'')^{ij}))} |< C^{\frac{l+1}{N}}, z^{\frac{l}{N}}, z > (w_k^{\frac{l}{N}}) - < C^{\frac{l+1}{N}}, z^{\frac{l+1}{N}}, z > (w_k^{\frac{l}{N}})| \\
&K \int_{\frac{l}{N}}^{\frac{l+1}{N}} dt \int_{(z^t)^{-1}(\Pi(\Psi((\Sigma'')^{ij})))} |z^{\frac{l}{N}} - z^{\frac{l+1}{N}}| \; d\|C^{\frac{l+1}{N}}\| \\
&K \ \frac{r_i^2}{N} M(C^{\frac{l+1}{N}} \llcorner N^i \lambda\rho_i \cap N^j_{\lambda\rho_j}) \\
&\le K \frac{r_i^4}{N}
\end{aligned} \qquad \text{(VI.42)}$$

where we have used (VI.26). Therefore we obtain

$$\lim_{l\to+\infty} \sum_l^{N-1} \int_{\Pi(\Psi((\Sigma'')^{ij}))} |< C^{\frac{l+1}{N}}, z^{\frac{l}{N}}, z > (w_k^{\frac{l}{N}}) - < C^{\frac{l+1}{N}}, z^{\frac{l+1}{N}}, z > (w_k^{\frac{l}{N}})| \le K r_i^4 \quad . \qquad \text{(VI.43)}$$

Finally for the second term of the r.h.s. of(VI.38), we use again (VI.26) and 4.3.2 (2) of [Fe] to obtain that

$$\begin{aligned}
&\int_{\Pi(\Psi((\Sigma'')^{ij}))} |< C^{\frac{l+1}{N}}, z^{\frac{l+1}{N}}, z > (w_k^{\frac{l}{N}} - w_k^{\frac{l+1}{N}})| \\
&\le M(C^{\frac{l+1}{N}} \llcorner N^i \lambda\rho_i \cap N^j_{\lambda\rho_j}) \ \frac{r_i^2}{N} \quad \le K \ \frac{r_i^4}{N} \quad .
\end{aligned} \qquad \text{(VI.44)}$$

Combining (VI.35), (VI.36), (VI.37), (VI.38), (VI.41), (VI.43) and (VI.44), we obtain that

$$M(\tilde{\Xi}^{ij}_*[\Pi\Psi((\Sigma')^{ij})] \times [0,1] \llcorner N^i_{\lambda\rho_i} \cap N^j_{\lambda\rho_j}) \le r_i^4 \quad . \qquad \text{(VI.45)}$$

Combining this last inequality with (VI.32) and a Fubini type argument we obtain that there exists $\lambda \in [\frac{7}{6}, \frac{6}{5}]$ such that

$$\mathcal{F}((\tilde{C}_i - \tilde{C}_j) \llcorner N^i \lambda\rho_i \cap N^j_{\lambda\rho_j}) \le r_i^4 \quad . \qquad \text{(VI.46)}$$

From this fact we then deduce, since $\tilde{C}_i$ and $\tilde{C}_j$ are single valued graphs with uniformly bounded gradients

$$\int_{\Pi_i(N^i \frac{7}{6}\rho_i \cap N^j_{\frac{7}{6}\rho_j})} |\tilde{a}^i_j(z^i) - \tilde{a}^j_j| \le K r_i^4 \quad . \qquad \text{(VI.47)}$$

Using the notations introduced in (VI.4) and (VI.13), we have that for any z there exists ξ such that

$$(z - z_i, \tilde{a}_i(z) - w_i) = \Lambda_{x_i}^{-1}(\xi, \tilde{a}_i^i(\xi)) \quad , \tag{VI.48}$$

where $|\Lambda_{x_i} - id| \leq K\ |x_i| \leq K\ \rho_{x_0}$. Let $z' := \rho_i^{-1}(z - z_i)$ and $\hat{a}_i(z') := \rho_i^{-1}(\tilde{a}_i(z) - w_i)$. Let also $\xi' := \rho_i^{-1}\xi$ and $\hat{a}_i^i(\xi') := \tilde{a}_i^i(\xi)$. Since $\tilde{a}_i^i$ is holomorphic (see (VI.14), $\hat{a}_i^i$ is also clearly holomorphic and since $\|\hat{a}_i^i\|_{L^\infty(B_{\frac{3}{2}}(0))} \leq K$, we have that for any $l \in \mathbb{N}$

$$\|\nabla^l \hat{a}_i^i\|_{L^\infty(B_{\frac{5}{4}}(0))} \leq K_l \quad . \tag{VI.49}$$

Using the above notations we have

$$(z', \hat{a}_i(z')) = \Lambda_{x_i}^{-1}(\xi', \hat{a}_i^i(\xi')) \quad . \tag{VI.50}$$

From the inverse function theorem, since $|\Lambda_{x_i} - id| \leq K\ |x_i| \leq K\ \rho_{x_0}$ can be taken as small as we want by taking ρ_{x_0} small enough, we have that for all $l \in \mathbb{N}$ there exists K_l such that

$$\|\nabla_{z'}^l \xi'\|_\infty \leq K_l \quad . \tag{VI.51}$$

Therefore, combining (VI.49), (VI.50 and (VI.51, we obtain

$$\|\nabla_{z'}^l \hat{a}_i(z')\|_\infty \leq K_l \quad . \tag{VI.52}$$

From that estimate we then deduce

$$\|\nabla_z^l \tilde{a}_i\|_{L^\infty(B^2_{\frac{7}{6}\rho_i}(z_i))} \leq K_l\ r_i^{l-1} \quad . \tag{VI.53}$$

Since $\tilde{C}^i$ is $J_{x_i}-$holomorphic, we have the existence of $\lambda_1^i, \mu_1^i, \lambda_2^i, \mu_2^i$ such that

$$\left\{ \begin{array}{l} J_{x_i} \cdot \begin{pmatrix} 1 \\ 0 \\ \dfrac{\partial \tilde{a}_i}{\partial x} \end{pmatrix} = \lambda_1^i \begin{pmatrix} 1 \\ 0 \\ \dfrac{\partial \tilde{a}_i}{\partial x} \end{pmatrix} + \mu_1^i \begin{pmatrix} 0 \\ 1 \\ \dfrac{\partial \tilde{a}_i}{\partial y} \end{pmatrix} \\ J_{x_i} \cdot \begin{pmatrix} 0 \\ 1 \\ \dfrac{\partial \tilde{a}_i}{\partial y} \end{pmatrix} = \lambda_2^i \begin{pmatrix} 1 \\ 0 \\ \dfrac{\partial \tilde{a}_i}{\partial x} \end{pmatrix} + \mu_2^i \begin{pmatrix} 0 \\ 1 \\ \dfrac{\partial \tilde{a}_i}{\partial y} \end{pmatrix} \end{array} \right. \tag{VI.54}$$

Writing $J_{x_i} = J_0 + \delta(x_i)$, we first observe that

$$|\delta(x_i)| \le \|J\|_{C^1} |x_i| \le K \rho_{x_0} \quad . \tag{VI.55}$$

Using this notation we deduce from (VI.54)

$$\left\{\begin{array}{l} \lambda_1^i = \delta_{1,1}(x_i) + \displaystyle\sum_{l=3}^{2p} \delta_{1,l}(x_i) \frac{\partial \tilde{a}_i^l}{\partial x} \\ \mu_1^i = 1 + \delta_{2,1}(x_i) + \displaystyle\sum_{l=3}^{2p} \delta_{2,l}(x_i) \frac{\partial \tilde{a}_i^l}{\partial x} \\ \lambda_2^i = -1 + \delta_{1,2}(x_i) + \displaystyle\sum_{l=3}^{2p} \delta_{1,l}(x_i) \frac{\partial \tilde{a}_i^l}{\partial y} \\ \mu_2^i = \delta_{2,2}(x_i) + \displaystyle\sum_{l=3}^{2p} \delta_{2,l}(x_i) \frac{\partial \tilde{a}_i^l}{\partial y} \end{array}\right. \tag{VI.56}$$

Therefore the equation solved by $\tilde{a}_i$ is for any $k = 1 \cdots p-1$

$$\left\{\begin{array}{rl} \dfrac{\partial \tilde{a}_i^{2k+1}}{\partial x} - \dfrac{\partial \tilde{a}_i^{2k+2}}{\partial y} & = \left[\delta_{1,1}(x_i) + \displaystyle\sum_{l=3}^{2p} \delta_{1,l}(x_i) \frac{\partial \tilde{a}_i^l}{\partial x}\right] \dfrac{\partial \tilde{a}_i^{2k+2}}{\partial x} \\ & \quad + \left[\delta_{2,1}(x_i) + \displaystyle\sum_{l=3}^{2p} \delta_{2,l}(x_i) \frac{\partial \tilde{a}_i^l}{\partial x}\right] \dfrac{\partial \tilde{a}_i^{2k+2}}{\partial y} \\ & \quad - \delta_{2k+2,1}(x_i) - \displaystyle\sum_{l=3}^{2p} \delta_{2k+2,l} \frac{\partial \tilde{a}_i^l}{\partial x} \\ \dfrac{\partial \tilde{a}_i^{2k+1}}{\partial y} + \dfrac{\partial \tilde{a}_i^{2k+2}}{\partial x} & = \left[\delta_{1,2}(x_i) + \displaystyle\sum_{l=3}^{2p} \delta_{1,l}(x_i) \frac{\partial \tilde{a}_i^l}{\partial y}\right] \dfrac{\partial \tilde{a}_i^{2k+2}}{\partial x} \\ & \quad + \left[\delta_{2,2}(x_i) + \displaystyle\sum_{l=3}^{2p} \delta_{2,l}(x_i) \frac{\partial \tilde{a}_i^l}{\partial y}\right] \dfrac{\partial \tilde{a}_i^{2k+2}}{\partial y} \\ & \quad - \delta_{2k+2,1}(x_i) - \displaystyle\sum_{l=3}^{2p} \delta_{2k+2,l}(x_i) \frac{\partial \tilde{a}_i^l}{\partial y} \end{array}\right. \tag{VI.57}$$

Then we deduce that there exists a linear map

$$A(x_i, \cdot) \ : \ \mathbb{R}^2 \otimes \mathbb{R}^{2p-2} \longrightarrow \mathbb{C}^{p-1} \otimes_{\mathbb{R}} \left(\mathbb{R}^2 \otimes \mathbb{R}^{2p-2}\right)^* \quad ,$$

and an element

$$B(x_i,\cdot)\in\mathbb{C}^{p-1}\otimes_{\mathbb{R}}\left(\mathbb{R}^2\otimes\mathbb{R}^{2p-2}\right)^*\quad ,$$

such that $\tilde{a}_i$ solves

$$\frac{\partial\tilde{a}_i}{\partial\overline{z}}=A(x_i,\nabla\tilde{a}_i)\cdot\nabla\tilde{a}_i+B(x_i,\nabla\tilde{a}_i)+D(x_i,\tilde{a}_i)\quad . \tag{VI.58}$$

Observe also that the dependence of A and B in $B_{\rho_{x_0}}(x_0)$ is smooth and that $A(x_0,\cdot)=0$, $B(x_0,\cdot)=0$, $D(x_0)=0$ and because of (VI.55) we have an estimate of the sort

$$\forall p\in\mathbb{R}^2\otimes\mathbb{R}^{2p}\qquad |A(x_i,p)|+|B(x_i,p)|\le K\ |x_i|(1+|p|)\quad . \tag{VI.59}$$

Consider now i and j such that $B^2_{\rho_i}(z_i)\cap B^2_{\rho_j}(z_j)\neq\emptyset$. On $B^2_{\frac{7}{6}\rho_i}(z_i)\cap B^2_{\frac{7}{6}\rho_j}(z_j)$ $\tilde{a}_i-\tilde{a}_j$ solves the following equation

$$\begin{aligned}\partial_{\overline{z}}(\tilde{a}_i-\tilde{a}_j)\ &=A(x_i,\nabla\tilde{a}_i)\cdot\nabla\tilde{a}_i+B(x_i,\nabla\tilde{a}_i)\\ &-A(x_j,\nabla\tilde{a}_j)\cdot\nabla\tilde{a}_j-B(x_j,\nabla\tilde{a}_j)\\ &=C(x_i,\nabla\tilde{a}_i,\nabla\tilde{a}_j)\cdot\nabla(\tilde{a}_i-\tilde{a}_j)\\ &+E(x_i,\nabla\tilde{a}_j)-E(x_j,\nabla\tilde{a}_j)\quad ,\end{aligned} \tag{VI.60}$$

where

$$\begin{aligned}C(x_i,\nabla\tilde{a}_i,\nabla\tilde{a}_j)\cdot\nabla(\tilde{a}_i-\tilde{a}_j)\ &:=A(x_i,\nabla\tilde{a}_i)\cdot\nabla\tilde{a}_i-A(x_i,\nabla\tilde{a}_j)\cdot\nabla\tilde{a}_j\\ &+B(x_i,\nabla\tilde{a}_i)-B(x_i,\nabla\tilde{a}_j)\quad ,\end{aligned} \tag{VI.61}$$

(where we have used the linear dependance in p of $A(x_i,p)$ and $B(x_i,p)$), and where

$$E(x,p):=A(x,p)\cdot p+B(x,p)+D(x)\quad . \tag{VI.62}$$

Observe, on the one hand, that $C(x,p,q)$ has a linear dependence in p and q in $\mathbb{R}^2\otimes\mathbb{R}^{2p-2}$, that

$$|C(x,p,q)|\le K\ |x|\ (1+|p|+|q|)\quad , \tag{VI.63}$$

and that following estimates hold for $D(x,p)$, forall $l\in\mathbb{N}$

$$|\nabla^l_x E(x,p)|\le K_l\ (1+|p|^2)\quad . \tag{VI.64}$$

On $B^2_{\frac{7}{6}}(\rho_i^{-1}z_i)\cap B^2_{\frac{7}{6}}(\rho_i^{-1}z_j)$ the function $f(z'):=a_i(\rho_i z')-a_j(\rho_i z')$ solves

$$\partial_{\overline{z'}}f-C(z')\cdot\nabla f=g(z')\quad , \tag{VI.65}$$

where

$$C(z'):=C(x_i,\nabla\tilde{a}_i,\nabla\tilde{a}_j)(\rho_i z')\quad ,$$

and

$$g(z'):=\rho_i\left[D(x_i,\nabla\tilde{a}_j(\rho_i z'))-D(x_j,\nabla\tilde{a}_j(\rho_i z'))\right]\quad .$$

Using (VI.53), observe that for any $l\in\mathbb{N}$

$$\begin{array}{l}\|\nabla^l_{z'}(C(x_i,(\nabla_z\tilde{a}_i)(\rho_i z'),(\nabla_z\tilde{a}_j)(\rho_i z')))\|_\infty\\[3mm] \le K\ |x_i|\ \rho_i^l\left[\|\nabla^{l+1}_z\tilde{a}_i\|_\infty+\|\nabla^{l+1}_z\tilde{a}_j\|_\infty\right]\le\ K_l\ \rho_{x_0}\quad .\end{array} \tag{VI.66}$$

Therefore, for ρ_{x_0} small enough, $L:=\partial_{\overline{z'}}-C(z')\cdot\nabla_{z'}$ is an elliptic coercive first order operator with smooth coefficients whose derivatives are uniformly bounded. Observe also that, using again (VI.53),

$$\begin{array}{l}\|\nabla^l_{z'}\rho_i\left[D(x_i,\nabla\tilde{a}_j(\rho_i z'))-D(x_j,\nabla\tilde{a}_j(\rho_i z'))\right]\|_\infty\\[3mm] \le K\rho_i^2\ \left[\rho_i^l\displaystyle\sum_{s=0}^{[l/2]}\|\nabla^{s+1}_z\tilde{a}_j\|_\infty\ \|\nabla^{l-s+1}_z\tilde{a}_j\|_\infty+\rho_i^l\ \|\nabla^{l+1}_{z'}\tilde{a}_j\|_\infty\right]\end{array} \tag{VI.67}$$

Then we have

$$\|\nabla^l_{z'}g\|_\infty\le K_l\ \rho_i^2\quad . \tag{VI.68}$$

From (VI.46) we deduce that

$$\int_{B^2_{\frac{7}{6}}(\rho_i^{-1}z_i)\cap B^2_{\frac{7}{6}}(\rho_i^{-1}z_j)}|f|\le K\ \rho_i^2\quad . \tag{VI.69}$$

Thus combining (VI.65)...(VI.69) and using standard elliptic estimates we obtain that for any $l\in\mathbb{N}$

$$\|\nabla^l f\|_{L^\infty(B^2_1(\rho_i^{-1}z_i)\cap B^2_1(\rho_i^{-1}z_j))}\le K_l\ \rho_i\quad , \tag{VI.70}$$

which yields, going back to the original scale, the estimate (VI.19) and lemma VI.2 is proved. ■

Definition of the approximated average curve. On $B^2_{\rho_{x_0}}(0) = \Pi(B_{\rho_{x_0}}(x_0))$ we define the approximated average curve as follows. Let φ be the partition of unity of $\Pi(\mathcal{C}_{Q-1}\cap B_{\rho_{x_0}}(x_0))$ defined in (V.48). We denote

$$\left\{\begin{array}{ll}\tilde{a}(z_0) := \displaystyle\sum_{i\in I}\varphi(z_0)\tilde{a}_i(z_0) & \forall z_0\in\Pi(\mathcal{C}_{Q-1}\cap B_{\rho_{x_0}}(x_0)) \quad . \\ \tilde{a}(z_0) :=< C, z, z_0 > (w) & \forall z_0\in\Pi((\mathcal{C}_Q\setminus\mathcal{C}_{Q-1})\cap B_{\rho_{x_0}}(x_0))\end{array}\right. \tag{VI.71}$$

Observe that, because of lemma IV.2, for any $z_0\in\Pi((\mathcal{C}_Q\setminus\mathcal{C}_{Q-1})\cap B_{\rho_{x_0}}(x_0))$ the slice $< C, z, z_0 >$ consists of exactly one point and $\tilde{a}(z_0)$ is simply the w coordinates of that point. The following estimates for $\tilde{a}$ holds :

Lemma VI.3 *Under the above notations, for any $q<+\infty$ there exists a constant K_q independent of ρ_{x_0} such that*

$$\int_{B^2_{\frac{\rho_{x_0}}{2}}(0)}|\nabla^2\tilde{a}|^q\le K_q\ \rho_{x_0}^2 \quad . \tag{VI.72}$$

■

Proof of lemma VI.3.

We claim first that $\tilde{a}$ is a Lipschitz map over $B^2_{\rho_{x_0}}(0)$. In $\Pi(\mathcal{C}_{Q-1}\cap B_{\rho_{x_0}}(x_0))$, by the assumptions of the inductive procedure, we know that $\tilde{a}$ is smooth and using both (VI.53) and (VI.19), because also of (V.49), we have

$$\|\nabla\tilde{a}\|_{L^\infty(\Pi(\mathcal{C}_{Q-1}\cap B_{\rho_{x_0}}(x_0)))}\le K \quad . \tag{VI.73}$$

Consider now two arbitrary points x and y of $B^2_{\rho_{x_0}}(0)$, either the segment $[x,y]$ in $B^2_{\rho_{x_0}}(0)$ is included in $\Pi(\mathcal{C}_{Q-1}\cap B_{\rho_{x_0}}(x_0))$ and then we can integrate (VI.73) all along that segment to get

$$|\tilde{a}(x)-\tilde{a}(y)|\le K|x-y| \quad , \tag{VI.74}$$

or there exists $z\in[x,y]\cap\Pi((\mathcal{C}_Q\setminus\mathcal{C}_{Q-1})\cap B_{\rho_{x_0}}(x_0))$ and using this time (IV.6) we also get (VI.74), which proves the desired claim. Using the equation (VI.58) solved by the $\tilde{a}_i$s we obtain that, in $\Pi(\mathcal{C}_{Q-1}\cap B_{\rho_{x_0}}(x_0))$, $\tilde{a}$ is a solution of

$$\partial_{\overline{z}}\tilde{a}-A((z,\tilde{a}(z)),\nabla\tilde{a})\cdot\nabla\tilde{a}-B((z,\tilde{a}(z)),\nabla\tilde{a})-D(z,\tilde{a}(z))=\zeta(z) \quad , \tag{VI.75}$$

where

$$\begin{aligned}\zeta(z) \quad &:= \sum_{i\in I} \varphi_i \ \ [A(x_i,\nabla\tilde{a}_i)\cdot\nabla\tilde{a}_i - A((z,\tilde{a}(z)),\nabla\tilde{a})\cdot\nabla\tilde{a}] \\ &+\sum_{i\in I} \ [B(x_i,\nabla\tilde{a}_i) - B((z,\tilde{a}(z)),\nabla\tilde{a})] \\ &+\sum_{i\in I} \partial_{\overline{z}}\varphi_i \ \ [\tilde{a}_i - \tilde{a}] \quad .\end{aligned} \tag{VI.76}$$

Observe that $\sum_{i\in I} \partial_{\overline{z}}\varphi_i\tilde{a}$ was added at the end because this quantity vanishes. Since for any x_i and r_i, (V.10) holds, since also for any $z \in \Pi((\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}) \cap B_{\rho_{x_0}}(x_0))$ we have for an ε as small as we want (recall α was fixed independently of ε), the relative Lipschitz estimate (IV.6) holds and granting the fact that $\Pi((\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}) \cap B_{\rho_{x_0}}(x_0))$ is a compact subset of $B_{\rho_{x_0}}(x_0)$, it is clear that

$$\begin{aligned}&\forall\eta>0 \quad \exists\delta>0 \qquad \text{s. t. } \forall i\in I \\ &dist(B_{\rho_i}(z_i),\Pi((\mathcal{C}_Q\setminus\mathcal{C}_{Q-1}))) \le \delta \quad \Longrightarrow \quad |\rho_i| \le \eta \quad .\end{aligned} \tag{VI.77}$$

Observe that

$$\begin{aligned}|\zeta(z)| \quad &\le K \sum_{i,j\in I} \varphi_i(z)\varphi_j(z) \ |\tilde{a}_j - w_i| \\ &\sum_{i,j\in I} \ \varphi_i(z)\varphi_j(z) \ |\nabla(\tilde{a}_i - \tilde{a}_j)| \\ &\sum_{i,j\in I} \ |\nabla\varphi_i(z)|\varphi_j(z) \ |\tilde{a}_i(z) - \tilde{a}_j(z)| \quad ,\end{aligned} \tag{VI.78}$$

using (VI.19), we have that

$$|\zeta(z)| \le \ K\max\left\{r_i \ i\in I; \ \ \text{s. t. } z\in B^2_{\rho_i}(z_i)\right\} \le K \ dist(z,\Pi((\mathcal{C}_Q\setminus\mathcal{C}_{Q-1}))) \quad . \tag{VI.79}$$

Combining (VI.77) and (VI.79), we have that $\zeta(z)$ converge uniformly to 0 as z tends to $\Pi((\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}))$. We then extend ζ by 0 in $\Pi((\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}))$. ζ is now a continuous function in $B^2_{\rho_{x_0}}(x_0)$ and we claim that

$$\partial_{\overline{z}}\tilde{a} - A((z,\tilde{a}(z)),\nabla\tilde{a})\cdot\nabla\tilde{a} - B((z,\tilde{a}(z)),\nabla\tilde{a}) = \zeta(z) \qquad \text{in } \mathcal{D}'(B^2_{\rho_{x_0}}(x_0)) \quad . \tag{VI.80}$$

Since $\tilde{a}$ is a Lipschitz function in $B^2_{\rho_{x_0}}(x_0)$, $\partial_{\overline{z}}\tilde{a} - A((z,\tilde{a}(z)),\nabla\tilde{a})\cdot\nabla\tilde{a} - B((z,\tilde{a}(z)),\nabla\tilde{a})$ is a bounded function in $B^2_{\rho_{x_0}}(x_0)$ and therefore, in order to prove (VI.80), it suffices to prove that for $\mathcal{H}^2$ almost every z in $\Pi((\mathcal{C}_Q\setminus\mathcal{C}_{Q-1}))\cap B^2_{\rho_{x_0}}(x_0)$

$$\partial_{\overline{z}}\tilde{a} - A((z,\tilde{a}(z)),\nabla\tilde{a})\cdot\nabla\tilde{a} - B((z,\tilde{a}(z)),\nabla\tilde{a}) - D(z,\tilde{a}(z)) = 0 \quad . \tag{VI.81}$$

This later equality, because of the computations leaded between (VI.54)...(VI.58), is just equivalent to the fact that the tangent plane of the graph $(z,\tilde{a}(z))$ at that point is $J-$holomorphic, which is the case at every point of $\Pi((\mathcal{C}_Q\setminus\mathcal{C}_{Q-1}))\cap B^2_{\rho_{x_0}}(x_0)$ due to lemma IV.2. Thus (VI.80) is proved. Using again (VI.19) we observe that

$$\begin{array}{rl}|\nabla\zeta(z)| & \le K\sum_{i,j\in I}\varphi_i(z)\varphi_j(z)\ \left[|\nabla\tilde{a}_j| + |\nabla\tilde{a}_j|^3\right] \\ & \sum_{i,j\in I}\ \varphi_i(z)\varphi_j(z)\ |\nabla\tilde{a}_i|(z)\ |\nabla^2(\tilde{a}_i-\tilde{a}_j)|(z) \\ & \sum_{i,j\in I}\ \left[|\nabla^2\varphi_i(z)|\varphi_j(z) + |\nabla\varphi_i|(z)|\nabla\varphi_j(z)|\right]\ |\tilde{a}_i(z)-\tilde{a}_j(z)| + |\tilde{a}_j - w_i| \quad , \\ & \le K\end{array} \tag{VI.82}$$

We claim now that ζ is Lipschitz in $B^2_{\rho_{x_0}}(x_0)$. Indeed, arguing like for $\tilde{a}$, given x and y in $B^2_{\rho_{x_0}}(x_0)$, if the segment $[x,y]$ has no intersection with $\Pi((\mathcal{C}_Q\setminus\mathcal{C}_{Q-1}))$, then, integrating (VI.78) on that segment gives

$$|\zeta(x)-\zeta(y)|\le K\ |x-y| \quad . \tag{VI.83}$$

Otherwise, if there exist $z\in[x,y]\cap\Pi((\mathcal{C}_Q\setminus\mathcal{C}_{Q-1}))$, then, (VI.79) gives

$$|\zeta(x)| + |\zeta(y)| \le K\ |x-z| + |y-z| \quad ,$$

which gives (VI.83) and the claim is proved. We claim now that $\tilde{a}\in W^{2,q}(B^2_{\rho_{x_0}/2}(0))$ for any $q<+\infty$. Let e be a unit vector in $B^2_{\rho_{x_0}/2}(0)$. For small h we denote $\tilde{a}_h(z):=\tilde{a}(z+h)$ and $\zeta_h(z):=\zeta(z+h)$. We have, using the linear dependencies of

$$\begin{aligned}\partial_{\overline{z}}(\tilde{a}-\tilde{a}_h) \quad & -A((z,\tilde{a}),\nabla\tilde{a})\cdot\nabla(\tilde{a}-\tilde{a}_h)-A((z,\tilde{a}),\nabla(\tilde{a}-\tilde{a}_h))\cdot\nabla\tilde{a}_h\\ & -B((z,\tilde{a}),\nabla(\tilde{a}-\tilde{a}_h))\\ & =\zeta-\zeta_h+A((z,\tilde{a}),\nabla\tilde{a}_h)\cdot\nabla\tilde{a}_h-A((z+h,\tilde{a}_h),\nabla\tilde{a}_h)\cdot\nabla\tilde{a}_h\\ & B((z,\tilde{a}),\nabla\tilde{a}_h)-B((z+h,\tilde{a}_h),\nabla\tilde{a}_h)+D(z,\tilde{a})-D(z+h,\tilde{a}_h) \quad .\end{aligned} \tag{VI.84}$$

Denote Δ_h the right-hand side of (VI.84) and observe that there exists a constant K such that

$$|\Delta_h|\le K\ h \quad . \tag{VI.85}$$

Let $\chi_{\rho_{x_0}}$ be a cut-off function such that $\chi_{\rho_{x_0}}\equiv 1$ in $B^2_{\frac{\rho_{x_0}}{2}}(0)$ and $\chi_{\rho_{x_0}}\equiv 0$ in $\mathbb{R}^2\setminus B^2_{\rho_{x_0}}(0)$. Let $f_h:=\chi_{\rho_{x_0}}\ (\tilde{a}-\tilde{a}_h)$, and let L_h be the operator such that

$$L_h f:=\partial_{\overline{z}}f-A((z,\tilde{a}),\nabla\tilde{a})\cdot\nabla f-A((z,\tilde{a}),\nabla f)\cdot\nabla\tilde{a}_h-B((z,\tilde{a}),\nabla f) \quad , \tag{VI.86}$$

we have

$$\begin{aligned}L_h f_h \quad & =\chi_{\rho_{x_0}}\Delta_h+(\tilde{a}-\tilde{a}_h)\partial_{\overline{z}}\chi_{\rho_{x_0}}-A((z,\tilde{a}),\nabla\tilde{a})\cdot(\tilde{a}-\tilde{a}_h)\nabla\chi_{\rho_{x_0}}\\ & -A((z,\tilde{a}),(\tilde{a}-\tilde{a}_h)\nabla\chi_{\rho_{x_0}})\cdot\nabla\tilde{a}_h-B((z,\tilde{a}),(\tilde{a}-\tilde{a}_h)\nabla\chi_{\rho_{x_0}}) \quad .\end{aligned} \tag{VI.87}$$

Since $\tilde{a}$ is Lipschitz, using (VI.85) and (VI.59), we have that

$$|L_h f_h|\le K\ h \quad . \tag{VI.88}$$

Observe that $|L_h f_h|\ge|\partial_{\overline{z}}f_h|-K\rho_{x_0}|\nabla f_h|$. Since $f_h=0$ on $\partial B^2_{\rho_{x_0}}(0)$, for any $p<+\infty$ we have that

$$\int_{B^2_{\rho_{x_0}}(0)}|\nabla f_h|^q\le K_q\int_{B^2_{\rho_{x_0}}(0)}|\partial_{\overline{z}}f_h|^q\le K_q\int_{B^2_{\rho_{x_0}}(0)}|L_h f_h|^q+K\rho_{x_0}^q|\nabla f_h|^q \quad . \tag{VI.89}$$

Dividing by h^q and making h tend to zero, we get that for ρ_{x_0} small enough inequality (VI.72) holds and lemma VI.3 is proved. ■

VI.2 Constructing adapted coordinates to C in a neighborhood of $x_0 \in \mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$.

We consider a point x_0 in the support of our $J-$holomorphic current C and we assume, as above, that the multiplicity at x_0 is Q and that the tangent cone C_{0,x_0} is Q times a $J_{x_0}-$holomorphic disk D. We start with the coordinates $(z, w_1, \cdots, w_{p-1})$ chosen in (II.1) such that $C_{0,x_0} = Q[D]$ is Q times the "horizontal" disk given by $w_i = 0$ for $i = 1, \cdots, p-1$ and we work in the ball $B^{2p}_{\rho_{x_0}}(x_0)$ whose radius ρ_{x_0} is given by (V.23). The purpose of this subsection is to construct new coordinates $(\xi, \lambda_1, \cdots, \lambda_{p-1})$ in $B^{2p}_{\rho_{x_0}}(x_0)$ such that the set $\lambda_i = 0$ for $i = 1, \cdots, p-1$ coincides with the graph of the average map $\tilde{a}$ constructed in the previous subsection.

On the graph $\tilde{A}(z) := (z, \tilde{a}(z))$ we consider the complex structure j given by the metric induced by $g := \omega(J\cdot, \cdot)$. Let X be a vector tangent to $\tilde{A}$ at $(z, \tilde{a}(z))$. We compare jX and JX in $\mathbb{R}^{2p}$. Let $n(z, \tilde{a}(z))$ be the $2p-2$-unit vector normal to $T_{(z,\tilde{a}(z))}\tilde{A}$, making the identification between $2p-1$-vectors and vector given by the ambient metric, we have

$$jX := n \wedge X \quad .$$

We have seen that

$$|Jn - n|(z, \tilde{a}(z)) \le r_{(z,\tilde{a}(z))} \quad .$$

Therefore

$$|jJX - JjX| \le |n \wedge JX - J(n \wedge X)| \le |n \wedge JX - Jn \wedge JX| \le r_{(z,\tilde{a}(z))}|X| \quad .$$

Thus we get that $|(J-j)(J+j)X| \le r_{(z,\tilde{a}(z))}|X|$ where we extend j to the normal bundle to $\tilde{A}$ again by the mean of the induced metric. Since $|(J+j)X|$ and $|X|$ are comparable independent of X, we have

$$\forall X \in T_{(z,\tilde{a}(z))}\tilde{A} \qquad |(J-j)X| \le K r_{(z,\tilde{a}(z))}|X| \quad . \tag{VI.90}$$

We choose now coordinates $\xi = (\xi_1, \xi_2)$ on $\tilde{A}$ compatible with j (i.e. $j\frac{\partial}{\partial \xi_1} = \frac{\partial}{\partial \xi_2}$). Let $(z', \hat{a}(z')) := (\rho_{x_0}^{-1}, \rho_{x_0}^{-1}\tilde{a}(\rho_{x_0} z'))$ and in $B^{2p}_1(0)$ we consider the metric $\hat{g}(z', w') := \rho_{x_0}^{-2}(\rho_{x_0} z', \rho_{x_0} w')^* g$ where g in $B^{2p}_{\rho_{x_0}}(0)$ is the original metric $g(\cdot, \cdot) = \omega(J\cdot, \cdot)$. After this scaling we have, using (VI.72)

$$\int_{B^2_1} |\nabla^2 \hat{a}|^p \, dz' \le K_q \rho_{x_0}^q \quad , \tag{VI.91}$$

and

$$\hat{g}^{ij} = \delta^{ij} + h^{ij} \quad \text{where } h^{ij}(0,0) = 0 \quad \text{and } \|\nabla h^{ij}\|_\infty \le K\rho_{x_0} \quad . \tag{VI.92}$$

We look for isothermal coordinates (ξ'_1, ξ'_2) in $\hat{A} = \{(z', \hat{a}(z')) \quad z' \in B_1^2(0)\}$ of the form $\xi' = z' + \delta(z')$ where δ will be small in $W^{2,p}$. On $B_1^2(0)$ we consider the metric $\hat{k} = (z', \hat{a}(z'))^* \hat{g} = (1 + k_{11})(dx'_1)^2 + 2k_{12} dx'_1 dx'_2 + (1 + k_{22})(dx'_2)^2$. From the estimates above we have for any $q > 0$, (since $\nabla \hat{a}(0,0) = 0$ and $\|\nabla^2 \hat{a}\|_q \le \rho_{x_0}$ we have $\|\nabla \hat{a}\|_\infty \le \rho_{x_0}$)

$$\int_{B_1^2} |\nabla k|^q \le K_q \rho_{x_0}^{2q} \quad . \tag{VI.93}$$

Following [DNF] page 110-111 it suffices to find δ_1 solving

$$-\frac{\partial}{\partial x'_1}\left[\frac{(1+k_{11})\frac{\partial \delta_1}{\partial x'_1} - k_{12}\frac{\partial \delta_1}{\partial x'_2}}{\sqrt{(1+k_{11})(1+k_{22}) - k_{12}^2}}\right] - \frac{\partial}{\partial x'_2}\left[\frac{(1+k_{22})\frac{\partial \delta_1}{\partial x'_2} - k_{12}\frac{\partial \delta_1}{\partial x'_1}}{\sqrt{(1+k_{11})(1+k_{22}) - k_{12}^2}}\right] = 0 \quad . \tag{VI.94}$$

Taking $\delta_1 = 0$ on $\partial B_1^2(0)$ we get a well-posed elliptic problem and we obtain the existence of δ_1 satisfying

$$\sum_{i=1}^{2} a_{ij} \frac{\partial^2 \delta_1}{\partial x'_i \partial x'_j} = F \cdot \nabla \delta_1 \quad ,$$

where a_{ij} are Hölder continuous $\|a_{ij} - \delta_{ij}\|_{C^{0,\alpha}(B_1^2)} \le K_\alpha \rho^2$ and $F \in L^q$ with $\int |F|^q \le K_p \rho_{x_0}^{2q}$. Standard elliptic estimates give then

$$\|\delta_1\|_{W^{2,q}(B_1^2)} \le K\rho_{x_0}^2 \quad . \tag{VI.95}$$

Therefore, going back to the original scale, we have found coordinates $\xi_i = x_i + \rho_{x_0}\delta_i(\rho_{x_0}^{-1} z) = x_i + \alpha_i(z)$ such that

$$\|\nabla \alpha\|_\infty \le K\rho_{x_0}^2 \quad \text{and} \quad j\frac{\partial}{\partial \xi_1} = \frac{\partial}{\partial \xi_2} \quad . \tag{VI.96}$$

We translate these coordinates in such a way that $\alpha(0,0) = (0,0)$.

Inside $Gl(\mathbb{R}^{2p})$, the space of invertible $2p \times 2p$ matrices with real coefficients we denote $U(p)$ the subspace of matrices M which commute with J_0.

$U(p)$ is a compact submanifold of $Gl(\mathbb{R}^{2p})$ and for some metric in $Gl(\mathbb{R}^{2p})$ we denote $\pi_{U(p)}$ the orthogonal projection from a neighborhood of $U(p)$ onto $U(p)$. We consider $M(z)$ the matrix which is given by

$$M(z) := \Lambda^{-1}_{(z,\tilde{a}(z))}\pi_{U(p)}(\Lambda_{(z,\tilde{a}(z))}) \quad , \tag{VI.97}$$

where we recall that Λ_x is given by (VI.6). We have clearly, since $\|\nabla\tilde{a}\| \le K$ and $\int_{B^2_{\rho_{x_0}}(0)} |\nabla^2\tilde{a}|^q \le K_q\rho^2_{x_0}$ for any $p < +\infty$,

$$\|\nabla M(z)\|_{L^\infty(B^2_{\rho_{x_0}}(0))} \le K \quad \text{and} \quad \|\nabla^2 M(z)\|_{L^q(B^2_{\rho_{x_0}}(0))} \le K_q\rho_{x_0}^{\frac{2}{q}} \quad . \tag{VI.98}$$

We keep denoting $e_1, e_2, \cdots, e_{2p}$ the canonical basis of $\mathbb{R}^{2p}$. Let

$$\varepsilon_k(z) := M(z)\cdot e_i \quad . \tag{VI.99}$$

We have for all $i = 1\cdots p$ that

$$\begin{aligned} J((z,\tilde{a}(z))\cdot\varepsilon_{2i-1}(z) &= J((z,\tilde{a}(z))\cdot\Lambda^{-1}_{(z,\tilde{a}(z))}\pi_{U(p)}(\Lambda_{(z,\tilde{a}(z))})\cdot e_{2i-1} \\ &= \Lambda^{-1}_{(z,\tilde{a}(z))}J_0\pi_{U(p)}(\Lambda_{(z,\tilde{a}(z))})\cdot e_{2i-1} \\ &= \Lambda^{-1}_{(z,\tilde{a}(z))}\pi_{U(p)}(\Lambda_{(z,\tilde{a}(z))})J_0\cdot e_{2i-1} = \Lambda^{-1}_{(z,\tilde{a}(z))}\pi_{U(p)}(\Lambda_{(z,\tilde{a}(z))})\cdot e_{2i} = \varepsilon_{2i}(z) \quad . \end{aligned} \tag{VI.100}$$

In $B^{2p}_{\rho_{x_0}}(x_0)$ we consider the new coordinates (ξ,λ) given by

$$\Psi \ : (\xi,\lambda) \longrightarrow \Psi(\xi,\lambda) := (z(\xi),\tilde{a}(z(\xi))) + \sum_{l=1}^{2p}\lambda_l\varepsilon_{l+2}(z(\xi)) \quad . \tag{VI.101}$$

Let $\tilde{J}$ be the expression of the almost complex structure in these coordinates (i.e. $\tilde{J}_{(\xi,\lambda)}\cdot X := d\Psi^{-1}J_{\Psi(\xi,\lambda)}\cdot d\Psi\cdot X$), we shall now estimate $|\tilde{J}-J_0|$ for points satisfying $|\lambda| \le r_{\Psi(\xi,\lambda)}$ - recall that r_x was defined in the beginning of chapter VI (see (V.11) and (V.12)) - which corresponds in $B^{2p}_{\rho_{x_0}}(x_0)$ to a neighborhood of $\tilde{A}(z) = (z,\tilde{a}(z))$ containing the support of C. We have first for $i = 1,2$, using (VI.90),

$$\begin{aligned} d\Psi\tilde{J}_{(\xi,0)}e_i = J_{\Psi(\xi,0)}\cdot\tfrac{\partial}{\partial\xi_i} &= j_{\Psi(\xi,0)}\cdot\tfrac{\partial}{\partial\xi_i} + (J_{\Psi(\xi,0)} - j_{\Psi(\xi,0)})\cdot\tfrac{\partial}{\partial\xi_i} \\ &= (-1)^{i+1}\frac{\partial}{\partial\xi_{i+1}} + O(r_\Psi(\xi,0)) \quad , \end{aligned} \tag{VI.102}$$

where we are using the convention $\frac{\partial}{\partial \xi_{i+1}} = \frac{\partial}{\partial \xi_{i-1}}$. We have then for $1 < l \le p$

$$d\Psi \tilde{J}_{(\xi,0)} e_{2l} = J_{\Psi(\xi,0)} \frac{\partial}{\partial \lambda_{2l}} = J_{\Psi(\xi,0)} \varepsilon_{2l} = -\varepsilon_{2l-1} = \frac{\partial}{\partial \xi_{2l+1}} \quad . \tag{VI.103}$$

For $|\lambda| \le r_{\Psi(\xi,\lambda)}$ we have for $i = 1, 2$

$$\begin{aligned} d\Psi \tilde{J}_{(\xi,\lambda)} e_i &= J_{\Psi(\xi,\lambda)} \cdot d\Psi_{(\xi,\lambda)} e_i = J_{\Psi(\xi,0)} \cdot d\Psi_{(\xi,0)} e_i \\ &+ J_{\Psi(\xi,0)} \cdot \left[d\Psi_{(\xi,\lambda)} e_i - d\Psi_{(\xi,0)} e_i \right] + \left[J_{\Psi(\xi,\lambda)} - J_{\Psi(\xi,0)} \right] \cdot d\Psi_{(\xi,\lambda)} e_i \quad . \end{aligned} \tag{VI.104}$$

Using the fact that $|J_{\Psi(\xi,\lambda)} - J_{\Psi(\xi,0)}| \le \|J\|_{C^1} \ |\Psi(\xi,\lambda) - \Psi(\xi,0)| \le K r_{\Psi(\xi,\lambda)}$ and that $d\Psi_{(\xi,\lambda)} e_i - d\Psi_{(\xi,0)} e_i = \frac{\partial \Psi}{\partial \xi_i}(\xi,\lambda) - \frac{\partial \Psi}{\partial \xi_i}(\xi,0) = \sum_{l=1} \lambda_l \partial_{\xi_i} \varepsilon_{l+2}$, we have then

$$|d\Psi \tilde{J}_{(\xi,\lambda)} e_1 - J_{\Psi(\xi,0)} \cdot d\Psi_{(\xi,0)} e_1| \le O(r_{\Psi(\xi,\lambda)}) \tag{VI.105}$$

Using (VI.90) again, we have $|[J_{\Psi(\xi,0)} - j(\Psi(\xi,0))] \cdot d\Psi_{(\xi,0)} e_i| \le O(r_{\Psi(\xi,0)})$, thus, since Ψ is Lipschitz $r_{\Psi(\xi,\lambda)}$ and $r_{\Psi(\xi,0)}$ from lemma V.3 are comparable and (VI.105) implies

$$|d\Psi \tilde{J}_{(\xi,\lambda)} e_1 - d\Psi_{(\xi,\lambda)} \cdot e_2| \le |d\Psi \tilde{J}_{(\xi,\lambda)} e_1 - d\Psi_{(\xi,0)} \cdot e_2| + |d\Psi_{(\xi,0)} \cdot e_2 - d\Psi_{(\xi,\lambda)} \cdot e_2| \tag{VI.106}$$

and using again the fact that $|d\Psi_{(\xi,0)} \cdot e_2 - d\Psi_{(\xi,\lambda)} \cdot e_2| = |\frac{\partial \Psi}{\partial \xi_2}(\xi,\lambda) - \frac{\partial \Psi}{\partial \xi_2}(\xi,0)| = |\sum_{l=1} \lambda_l \partial_{\xi_2} \varepsilon_{l+2}| \le O(r_{\Psi(\xi,\lambda)})$, we have finally that

$$|d\Psi \tilde{J}_{(\xi,\lambda)} e_1 - d\Psi_{(\xi,\lambda)} \cdot e_2| \le O(r_{\Psi(\xi,\lambda)}) \quad . \tag{VI.107}$$

Finally, we have for $1 < l \le p$

$$\begin{aligned} d\Psi \tilde{J}_{(\xi,\lambda)} e_{2l} &= J_{\Psi(\xi,\lambda)} \cdot d\Psi_{(\xi,\lambda)} e_{2l} = -d\Psi_{(\xi,\lambda)} e_{2l-1} \\ &+ [d\Psi_{(\xi,\lambda)} - d\Psi_{(\xi,0)}] e_{2l-1} + J_{\Psi(\xi,0)} \cdot [d\Psi_{(\xi,\lambda)} - d\Psi_{(\xi,0)} e_{2l}] \\ &+ [J_{\Psi(\xi,\lambda)} - J_{\Psi(\xi,0)}] \cdot d\Psi_{(\xi,\lambda)} \quad . \end{aligned} \tag{VI.108}$$

Using the estimates we used in the above lines, (VI.108) becomes for $1 < l \le p$

$$|d\Psi \tilde{J}_{(\xi,\lambda)} e_{2l} + d\Psi_{(\xi,\lambda)} e_{2l-1}| \le O(r_{\Psi(\xi,\lambda)}) \quad . \tag{VI.109}$$

Thus combining (VI.107) and (VI.109), we obtain that

$$\forall (\xi,\lambda) \qquad \text{s. t. } |\lambda| \le r_{\Psi(\xi,\lambda)} \quad |\tilde{J}_{(\xi,\lambda)} - J_0| \le K \ r_{\Psi(\xi,\lambda)} \quad . \tag{VI.110}$$

VII The unique continuation argument.

In this part we show that, assuming $\mathcal{P}_{Q-1}$ (recall that the definition is given by (II.8)), a point x_0 in $\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$ is isolated in $\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$ unless all points in $\mathcal{C}_*$ in a neighborhood from x_0 ar of multiplicity Q. This fact has been already proved in chapter IV in the case where C_{0,x_0} was not Q times the same flat holomorphic disk (the easy case). Here we assume that we are in the difficult case $C_{0,x_0} = Q[D_0]$ where D_0 is the horizontal unit disk as before. We adopt the coordinate system about x_0 constructed in section VIII.2. We denote by Π the map that assigns the first complex coordinate $\xi = \xi_1 + i\xi_2$. Assuming there exists a sequence of points $x_n \in \mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$ different from x_0 and converging to x_0, the goal of this chapter is to show that C in a neighborhood is a Q times the same graph. The strategy is inspired by [Ta] chapter 1 : in our coordinates, the points in $\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$ are contained in the disk $\lambda_i = 0$ for $i = 1, \cdots, 2p-2$ and we shall use a unique continuation argument based on the proof of some Carleman estimate to show that our assumption implies that the whole cycle in the neighborhood of x_0 is included in that disk. Let $(\xi_n, 0)$ be the coordinates of $x_n \to x_0$. We can always extract a subsequence such that $|\xi_{n+1}| \leq |\xi_n|^2$. We then introduce the function $g_N(\xi) := \prod_{j=1}^{N}(\xi - \xi_n)$. Because of the speed of convergence of our sequence ξ_n to zero it is not difficult to check that there exists a constant K independent of N such that for any $\xi \in B^2_{\rho_{x_0}}$ the following holds

$$\frac{K^{-1}}{|\xi|^{N-N_\xi}}\frac{1}{|\xi-\xi_{N_\xi}|}\prod_{j=1}^{N_\xi-1}\frac{1}{|\xi_j|} \leq |g_N^{-1}|(\xi) \leq \frac{K}{|\xi|^{N-N_\xi}}\frac{1}{|\xi-\xi_{N_\xi}|}\prod_{j=1}^{N_\xi-1}\frac{1}{|\xi_j|} \tag{VII.1}$$

where N_ξ is the index less than N such that $|\xi - \xi_{N_\xi}|$ is minimal among the $|\xi - \xi_n|$. It is also straightforward to check that

$$\begin{aligned}|\nabla g_N^{-1}|(\xi) \leq & \frac{K(N-N_\xi)}{|\xi|^{N-N_\xi+1}}\frac{1}{|\xi-\xi_{N_\xi}|}\prod_{j=1}^{N_\xi-1}\frac{1}{|\xi_j|}\\ & +\frac{K}{|\xi|^{N-N_\xi}}\frac{1}{|\xi-\xi_{N_\xi}|^2}\prod_{j=1}^{N_\xi-1}\frac{1}{|\xi_j|}+\frac{K}{|\xi|^{N-N_\xi}}\frac{1}{|\xi-\xi_{N_\xi}|}\sum_{l=1}^{N_\xi-1}\frac{1}{|\xi_l|}\prod_{j=1}^{N_\xi-1}\frac{1}{|\xi_j|}\end{aligned} \tag{VII.2}$$

and we have a corresponding estimate for $|\nabla^k g_N^{-1}|(\xi)$ for arbitrary k in general. Let $\xi \in \Pi(\mathcal{C}_{Q-1})$. ξ belong to some $B^2_{\rho_i}(\xi_i)$ of the covering constructed in chapter VI. To every such a i we assign k_i an index such that $|\xi_{k_i}|+\rho_{k_i}\le|\xi_i|-\rho_i$ and such that $|\xi_{k_i}-\xi_i|\le K\rho_i$ and such that ρ_i and ρ_{k_i} are comparable :

$$K^{-1}\rho_i\le\rho_{k_i}\le K\rho_i\quad . \tag{VII.3}$$

This is always possible due to the Whitney-Besicovitch nature of our covering, moreover for every k there exists a uniformly bounded number of i such that $k_i=k$. Observe also, because of the relative Lipshitz estimate (IV.6) with constant ε and because of the "splitting stage" of C_{ξ_i,ρ_i} characterised by (V.10) we have that for any $\delta>0$ one may choose ε small enough compared to α defined in chapter V such that for any $\xi\in\Pi(\mathcal{C}_{Q-1})$

$$dist\,(\xi,\Pi(\mathcal{C}_Q\setminus\mathcal{C}_{Q-1}))\ge\delta^{-1}\rho_i\quad , \tag{VII.4}$$

where $\xi\in B^2_{\rho_i}(\xi_i)$. Combining (VII.3) and (VII.4) we get that

$$\forall i\in I\quad\forall\xi\in B^2_{\rho_i}(\xi_i)\quad\forall\zeta\in B^2_{\rho_{k_i}}(\xi_{k_i})$$

$$\frac{1}{2}dist\,(\xi,\Pi(\mathcal{C}_Q\setminus\mathcal{C}_{Q-1}))\le dist\,(\zeta,\Pi(\mathcal{C}_Q\setminus\mathcal{C}_{Q-1}))\le 2dist\,(\xi,\Pi(\mathcal{C}_Q\setminus\mathcal{C}_{Q-1})) \tag{VII.5}$$

From (VII.1), (VII.2) and (VII.5) we get that for any $N\in\mathbb{N}$, for any $\xi\in B^2_{\rho_i}(\xi_i)$ and for any $\zeta\in B^2_{\rho_{k_i}}(\xi_{k_i})$

$$|g_N^{-1}|(\xi)\le K|g_N^{-1}|(\zeta)\quad . \tag{VII.6}$$

and

$$\rho_i\frac{|\nabla g_N|}{|g_N|^2}(\xi)+\rho_i^2\frac{|\nabla^2 g_N|}{|g_N|^2}(\xi)\le K\frac{1}{|g_N|}(\zeta)\quad . \tag{VII.7}$$

Let $\chi_{\rho_{x_0}}$ be a cut-off function identically equal to 1 in $B^2_{\rho_{x_0}/2}(0)$ and equal to 0 outside $B^2_{\rho_{x_0}}$. In $B^2_{\rho_{x_0}}(z_0)\times\mathbb{R}^{2p-2}$ we introduce the cycle C^{g_N} which is given by

$$\forall\xi\in B^2_{\rho_{x_0}}\qquad\langle C^{g_N},\Pi,\xi\rangle:=g_N^{-1}(\xi)_*\,\langle C,\Pi,\xi\rangle \tag{VII.8}$$

In other words if $\Psi\ :\ \Sigma\to B^2_{\rho_{x_0}}(0)\times\mathbb{R}^{2p-2}$ is a parametrisation of a piece of C, a parametrisation of the corresponding piece in C^{g_N} is given by $(\Psi_\xi,g_N^{-1}\circ\Pi\circ\Psi\ \Psi_\lambda)$ where (Ψ_ξ,Ψ_λ) are the coordinates of Ψ. Since C^{g_N} is a cycle in $B^2_{\rho_{x_0}}(z_0)\times\mathbb{R}^{2p-2}$, we have, denoting $\Omega:=\sum_{l=1}^{p-1}d\lambda_{2l-1}\wedge d\lambda_{2l})$

$$C^{g_N}(\chi_{\rho_{x_0}}\circ\Pi\ \Omega)=-C^{g_N}(d\chi_{\rho_{x_0}}\circ\Pi\wedge\sum_{l=1}^{p-1}\lambda_{2l-1}d\lambda_{2l})\quad . \tag{VII.9}$$

We split $C^{g_N}(\chi_{\rho_{x_0}}\circ\Pi\ \Omega)=C^{g_N}\llcorner B^2_{\rho_{x_0}/2}\times\mathbb{R}^{2p-2}(\Omega)+C^{g_N}\llcorner(B^2_{\rho_{x_0}}\setminus B^2_{\rho_{x_0}/2})\times\mathbb{R}^{2p-2}(\chi_{\rho_{x_0}}\circ\Pi\ \Omega)$ and we have

$$C^{g_N}\llcorner B^2_{\rho_{x_0}/2}\times\mathbb{R}^{2p-2}(\Omega)=\sum_{i\in I}C^{g_N}\llcorner B^2_{\rho_{x_0}/2}\times\mathbb{R}^{2p-2}(\varphi_i\circ\Pi\sum_{l=1}^{p-1}\Omega)\quad,\tag{VII.10}$$

where we recall that the partition of unity was constructed in (V.48) adapted to the covering $B^2_{\rho_i}(\xi_i)$. Let $\Psi_i\ :\ \Sigma_i\rightarrow B^2_{2\rho_i}(\xi_i)\times B^{2p-2}_{2\rho_i}(0)$ be a smooth parametrisation of $C\llcorner B^2_{2\rho_i}(\xi_i)\times B^{2p-2}_{\rho_{x_0}}(0)$ and denote by η_i the map from Σ_i into $\mathbb{R}^{2p}$ given by [Ri3] - Proposition A.3 - such that $\Psi_i+\eta_i$ is J_0-holomorphic. Since J in $B^{2p}_{2\rho_i}((\xi_i,0))$ is closed to J_0 at a distance comparable to ρ_i - see (VI.110) - we have

$$\|\nabla\eta_i\|_{L^2(\Sigma_{\frac{3}{2},i})}\le\rho_i^2\tag{VII.11}$$

where we recall that $\Sigma_{\frac{3}{2},i}=\Sigma_i\cap\Psi_i^{-1}(B^2_{\frac{3\rho_i}{2}}(\xi_i)\times\mathbb{R}^2)$. Using now lemma II.2 of [Ri3], which does not require J to be C^1 in these coordinates but just the metric g to be close to the flat one, one has

$$\|\eta_i\|_{L^2(\Sigma_{\frac{4}{3},i})}\le\rho_i^3\tag{VII.12}$$

Using the parametrisation $\Psi_i=(\Psi_{i,\xi},\Psi_{i,\lambda})$, we have

$$\begin{aligned}&C^{g_N}\llcorner B^2_{\rho_{x_0}/2}\times\mathbb{R}^{2p-2}(\varphi_i\circ\Pi\ \Omega)\\&=\int_{\Sigma_i}\varphi(\Psi_{i,\xi})\sum_{l=1}^{p-1}d\left[\frac{\Psi^{2l-1}_{i,\lambda}}{g(\Psi_{i,\xi})}\right]\wedge d\left[\frac{\Psi^{2l}_{i,\lambda}}{g(\Psi_{i,\xi})}\right]\end{aligned}\tag{VII.13}$$

We compare this quantity with

$$\begin{aligned}&C_i^{g_N}\llcorner(\varphi_i\circ\Pi\ \Omega)\\&:=\int_{\Sigma_i}\varphi(\Psi_{i,\xi})\sum_{l=1}^{p-1}d\left[\frac{\Psi^{2l-1}_{i,\lambda}+\eta^{2l-1}_{i,\lambda}}{g(\Psi_{i,\xi}+\eta_{i,\xi})}\right]\wedge d\left[\frac{\Psi^{2l}_{i,\lambda}+\eta^{2l}_{i,\lambda}}{g(\Psi_{i,\xi}+\eta_{i,\xi})}\right]\quad.\end{aligned}\tag{VII.14}$$

One has

$$
\begin{aligned}
&|(C_i^{g_N} - C^{g_N})(\varphi_i \circ \Pi \sum_{l=1}^{p-1} d\lambda_{2l-1} \wedge d\lambda_{2l})| \le \\
&K \int_{\Sigma_i} \left| \nabla \left[\frac{\Psi_{i,\lambda} + \eta_{i,\lambda}}{g(\Psi_{i,\xi} + \eta_{i,\xi})} - \frac{\Psi_{i,\lambda}}{g(\Psi_{i,\xi})} \right] \right| \; \left| \nabla \left[\frac{\Psi_{i,\lambda} + \eta_{i,\lambda}}{g(\Psi_{i,\xi} + \eta_{i,\xi})} \right] \right| \\
&\le \delta K \int_{\Sigma_i} \varphi_{k_i} \left| \nabla \left[\frac{\Psi_{i,\lambda} + \eta_{i,\lambda}}{g(\Psi_{i,\xi} + \eta_{i,\xi})} \right] \right|^2 \\
&+ \frac{K}{\delta} \int_{\Sigma_i} \left| \nabla \left[\frac{\Psi_{i,\lambda} + \eta_{i,\lambda}}{g(\Psi_{i,\xi} + \eta_{i,\xi})} - \frac{\Psi_{i,\lambda}}{g(\Psi_{i,\xi})} \right] \right|^2
\end{aligned}
\tag{VII.15}
$$

We have

$$
\begin{aligned}
&\int_{\Sigma_i} \left| \nabla \left[\frac{\Psi_{i,\lambda} + \eta_{i,\lambda}}{g(\Psi_{i,\xi} + \eta_{i,\xi})} - \frac{\Psi_{i,\lambda}}{g(\Psi_{i,\xi})} \right] \right|^2 \\
&\le \int_{\Sigma_i} \left| \nabla \left[\Psi_i \left(\frac{1}{g(\Psi_{i,\xi} + \eta_{i,\xi})} - \frac{1}{g(\Psi_{i,\xi})} \right) \right] \right|^2 \\
&+ \int_{\Sigma_i} |\nabla \eta_i| sup_{\xi \in B^2_{\rho_i}(\xi_i)} \frac{1}{|g(\xi)|^2} + \int_{\Sigma_i} |\eta_i|^2 \; sup_{\xi \in B^2_{\rho_i}(\xi_i)} \frac{|\nabla g|^2}{|g|^4}(\xi) \quad .
\end{aligned}
\tag{VII.16}
$$

Let $f_0(\xi)$ be the flat norm of the slice of C^{g_N} minus the average curve, $\Psi_\lambda = 0$, by $\Pi^{-1}(\xi)$,

$$
f_N(\xi) = \mathcal{F}(< C^{g_N}, \Pi, \xi > - Q\delta_0) \quad .
$$

Using (V.8),(observe that the difference of the densities for the metrics $\omega(\cdot, J\cdot)$ and $\omega_0(\cdot, J_0\cdot)$ is as small as we want for ρ_{x_0} chosen small enough, using also (VII.11), (VII.12), (VII.6) and (VII.7), we have

$$
\begin{aligned}
&\int_{\Sigma_i} |\nabla \eta_i|^2 sup_{\xi \in B^2_{\rho_i}(\xi_i)} \frac{1}{|g(\xi)|^2} + \int_{\Sigma_i} |\eta|^2 \; sup_{\xi \in B^2_{\rho_i}(\xi_i)} \frac{|\nabla g_N|^2}{|g_N|^4}(\xi) \\
&\le K \int_{B^2_{\rho_{k_i}}(\xi_{k_i})} |f_N|^2
\end{aligned}
\tag{VII.17}
$$

where we have also used the fact that $\int_{\Sigma_{k_i}} \varphi_{k_i} |\Psi_{k_i,\lambda}|^2 \ge K\rho_i^4$. This lower bound is a crucial point in our paper it comes from the fact that C restricted to $B_{\rho_{k_i}}(x_{k_i})$ is split : we have that $\rho_{k_i}^{-2}M(C\llcorner B_{\rho_{k_i}}(x_{k_i}))$ is less that $\pi Q - K_0\alpha$ (see (V.11)) where K_0 and α only depend on p, Q, J and ω. If $\Psi_{k_i,\lambda}$ would have been too close to 0 in L^2 norm, since the intersection number between $(\Psi_{k_i})_*[\Sigma_{k_i}]$ and the $2p-2$-planes $\Pi^{-1}(\xi)$ for $\xi \in B^2_{\rho_{k_i}}(\xi_{k_i})$ is Q, $\rho_{k_i}^{-2}M(C\llcorner B_{\rho_{k_i}}(x_{k_i}))$ would have been too large would have contradicts the upper bound (V.11). The first term in the right-hand-side of (VII.16) can be bounded as follows

$$
\begin{aligned}
&\int_{\Sigma_i}\left|\nabla\left[\Psi_i\left(\frac{1}{g_N(\Psi_{i,\xi}+\eta_{i,\xi})}-\frac{1}{g_N(\Psi_{i,\xi})}\right)\right]\right|^2\\
&\le\int_{\Sigma_i}\left|\frac{1}{g_N(\Psi_{i,\xi}+\eta_{i,\xi})}-\frac{1}{g_N(\Psi_{i,\xi})}\right|^2+\int_{\Sigma_i}\rho_i^2\left|\nabla\left(\frac{1}{g_N(\Psi_{i,\xi}+\eta_{i,\xi})}-\frac{1}{g_N(\Psi_{i,\xi})}\right)\right|^2\\
&\le K\int_{\Sigma_i}\rho_i^4\left|\sup_{\xi\in B^2_{\rho_i}(\xi_i)}\frac{|\nabla g_N|}{|g_N|^2}\right|^2(\xi)\\
&+K\int_{\Sigma_i}\rho_i^2|\nabla\eta_i|^2\left|\sup_{\xi\in B^2_{\rho_i}(\xi_i)}\frac{|\nabla g_N|}{|g_N|^2}\right|^2(\xi)\\
&+K\int_{\Sigma_i}\rho_i^2|\eta_i|^2\sup_{\xi\in B^2_{\rho_i}(\xi_i)}\left|\frac{|\nabla g_N|^2}{|g_N|^3}\right|^2(\xi)\\
&+K\int_{\Sigma_i}\rho_i^2|\eta_i|^2\sup_{\xi\in B^2_{\rho_i}(\xi_i)}\left|\frac{|\nabla^2 g_N|}{|g_N|^2}\right|^2(\xi)
\end{aligned}
\tag{VII.18}
$$

Using (VII.6) and (VII.7) like above and combining (VII.15)...(VII.18), we have finally for every $\delta > 0$

$$
\begin{aligned}
&|(C_i^{g_N}-C^{g_N})(\varphi_i\circ\Pi\sum_{l=1}^{p-1}d\lambda_{2l-1}\wedge d\lambda_{2l})|\le\\
&\delta K\int_{\Sigma_i}\varphi_i\left|\nabla\left[\frac{\Psi_{i,\lambda}+\eta_{i,\lambda}}{g(\Psi_{i,\xi}+\eta_{i,\xi})}\right]\right|^2+\frac{K}{\delta}K\int_{B^2_{\rho_{k_i}}(\xi_{k_i})}|f_N|^2
\end{aligned}
\tag{VII.19}
$$

Since $\Psi_i+\eta_i/g\circ\Pi\circ\Psi_i+\eta_i$ is an holomorphic map into $\mathbb{C}^p$, the $\lambda-$coordinates of it, $\Psi_{i,\lambda}+\eta_{i,\lambda}/g\circ\Pi\circ\Psi_i+\eta_i$ is also a holomorphic map but into $\mathbb{C}^{p-1}$ and one has

$$\frac{\Psi_{i,\lambda}+\eta_{i,\lambda}}{g(\Psi_{i,\xi}+\eta_{i,\xi})}^*(\sum_{l=1}^{p-1} d\lambda_{2l-1}\wedge d\lambda_{2l}) = \frac{1}{2}(\Psi_{i,\lambda}+\eta_{i,\lambda})^*\left(\sum_{l=1}^{p-1} d\Lambda_l\wedge d\overline{\Lambda_l}\right)$$
$$= \left|\nabla\frac{\Psi_{i,\lambda}+\eta_{i,\lambda}}{g(\Psi_{i,\xi}+\eta_{i,\xi})}\right|^2 \ d\zeta\wedge d\overline{\zeta} \tag{VII.20}$$

where ζ denotes local complex coordinates on Σ_i, and Λ_l is the complex coordinate $\Lambda_l=\lambda_{2l-1}+i\lambda_{2l}$. Therefore, combining (VII.19) and (VII.20) we have for δ chosen such that $\delta K<\frac{1}{2}$

$$\begin{aligned} &C^{g_N}(\varphi_i\circ\Pi\sum_{l=1}^{p-1} d\lambda_{2l-1}\wedge d\lambda_{2l})\ge \\ &\frac{1}{2}\int_{\Sigma_i}\varphi_i\left|\nabla\frac{\Psi_{i,\lambda}+\eta_{i,\lambda}}{g(\Psi_{i,\xi}+\eta_{i,\xi})}\right|^2-\frac{K}{\delta}\int_{B^2_{\rho_{k_i}}(\xi_{k_i})}|f_N|^2 \end{aligned} \tag{VII.21}$$

Let $\{a_i^l(\xi)\}_{l=1\cdots Q}$ be the holomorphic $Q-$valued graph realised by $(\Psi_{i,\xi}+\eta_{i,\xi}, g_N^{-1}(\Psi_{i,\xi}+\eta_{i,\xi})\Psi_{i,\lambda}+\eta_{i,\lambda})$, we have that

$$\int_{\Sigma_i}\varphi_i\left|\nabla\frac{\Psi_{i,\lambda}+\eta_{i,\lambda}}{g(\Psi_{i,\xi}+\eta_{i,\xi})}\right|^2=\int_{B^2_{\rho_i}}\varphi_i\sum_{l=1}^{p-1}|\nabla a_i^l|^2(\xi)\ d\xi\wedge d\overline{\xi} \tag{VII.22}$$

Clearly this quantity is larger than $\int_{B^2_{\rho_i}}\varphi_i|\nabla\tilde{f}_N|^2$ where $\tilde{f}_N(\xi)$ is the Flat norm of the slice by $\Pi^{-1}(\xi)$ of the difference between$C_i^{g_N}$ and the average curve. Replacing $\tilde{f}_N$ by f_N itself, the slice by $\Pi^{-1}(\xi)$, of C^{g_N} minus the average curve, in the integral $\int_{B^2_{\rho_i}}\varphi_i|\nabla\tilde{f}_N|^2$, induces error terms which can be controlled by $\int_{B^2_{\rho_{k_i}}(\xi_{k_i})}|f_N|^2$ like in the computation of the error between C^{g_N} and $C_i^{g_N}$ above. Therefore we have

$$C^{g_N}(\varphi_i\circ\Pi\ \Omega)\ge\frac{1}{2}\int_{\Sigma_i}\varphi_i\ |\nabla f_N|^2-\frac{K}{\delta}\int_{B^2_{\rho_{k_i}}(\xi_{k_i})}|f_N|^2 \tag{VII.23}$$

Because of the relative Lipschitz estimate, f_N extends as a $W^{1,2}$ function on all of $B^2_{\rho_{x_0}}(0)$. Standard Poincaré estimates yield

$$\int_{B^2_{\rho_{x_0}}}|\chi_{\rho_{x_0}}f_N|^2\le K\rho_{x_0}^2\int_{B^2_{\rho_{x_0}}}|\nabla(\chi_{\rho_{x_0}}f_N)|^2 \quad . \tag{VII.24}$$

Taking ρ_{x_0} small enough we can ensure that $K\rho_{x_0}^2 > O(\delta)$ and combining (VII.9), (VII.23) and (VII.24) we finally get that

$$\int_{B^2_{\rho_{x_0}/2}} |f_N|^2 \leq K_1 \int_{B^2_{\rho_{x_0}} \setminus B^2_{\rho_{x_0}/2}} |f_N|^2 + |\nabla f_N|^2 + C^{g_N}(d\chi_{\rho_{x_0}} \circ \Pi \wedge \sum_{l=1}^{p-1} \lambda_{2l-1} d\lambda_{2l}) \tag{VII.25}$$

Where K_1 is a constant independent of N. By taking the sequence ξ_n such that the largest $|\xi_1|$ satisfies $|\xi_1| \leq \frac{\rho_{x_0}}{4}$, if C does not coincide with the average curve (f_N is not identically zero) near the origin we would have that $\left(\frac{\rho_{x_0}}{4}\right)^{2N} \int_{B^2_{\rho_{x_0}/2}} |f_N|^2$ would tend to infinity, whereas, it is not difficult to check that the right-hand-side of (VII.25) which involves quantities supported in $B^2_{\rho_{x_0}} \setminus B^2_{\rho_{x_0}/2}$ is bounded by $K_{\rho_{x_0}} N^2 \left(\frac{\rho_{x_0}}{2}\right)^{-2N}$. The multiplication of it by $\left(\frac{\rho_{x_0}}{4}\right)^{2N}$ tends clearly to zero as N tend to infinity. We have then obtained a contradiction and we have proved that any point inside $\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$ is surrounded in $\mathcal{C}_*$ by points which are all in $\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$ or by points which are all in $\mathcal{C}_{Q-1}$. It remains to show that a point in $\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$ is not an accumulation point of $\cup_{q \leq Q-1} Sing^q$. This is the purpose of the next chapter.

VIII Points in $\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$ are not accumulation points of $\cup_{q \leq Q-1} Sing^q$.

In this chapter we prove, Assuming $\mathcal{P}_{Q-1}$, that points in $\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$ are not accumulation points of $\cup_{q \leq Q-1} Sing^q$ and combining this fact with the result in the previous chapter we will have proved $\mathcal{P}_Q$.

Let then $x_0 \in \mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$, and assume then that x_0 is an accumulation point of $\mathcal{C}_{Q-1}$, which means, using the monotonicity formula, lemma IV.1 together with the result obtained in the previous chapter, that there exists a radius ρ such that $\mathcal{C}_* \cap B_\rho(x_0) \subset \mathcal{C}_Q$ and that $(\mathcal{C}_Q \setminus \mathcal{C}_{Q-1}) \cap B_\rho(x_0) = \{x_0\}$. From the assumed hypothesis $\mathcal{P}_{Q-1}$, we have then that there exists a Riemann surface Σ and a smooth $J-$holomorphic map Ψ such that $C \llcorner B_{r_0}(x_0) = \Psi_*[\Sigma]$. The goal is to show that Σ has a finite topology and that it is a closed Riemann Surface. The idea is to perturb Ψ by finding $\eta \in L^\infty(\Sigma)$ such that $\Psi + \eta$ is J_0-holomorphic and $(\Psi + \eta)_*[\Sigma]$ is a cycle.

For any $r < r_0$, we denote Σ_r the finite Riemann surface obtained by taking $\Sigma\cap\Psi^{-1}(B_\rho(x_0)\setminus B_r(x_0))$ and we shall denote Γ_r the part of the boundary of Σ_r which is disjoint from $\partial\Sigma\subset(|\Psi-x_0|)^{-1}(r_0)$. On Σ_r we consider η_r the map which is given by Proposition A.3 in [Ri3]. It satisfies in particular, using the complex coordinates induced by J_0,

$$
\begin{aligned}
&\overline{\partial}(\Psi+\eta_r)=0 \qquad \text{in } \Sigma_r\\
&\forall r\le r_0 \qquad \int_{\Sigma_r}|\nabla\eta_r|^2\le\int_{\Sigma_r}|J(\Psi)-J_0|^2|\nabla\Psi|^2\le K r_0^4 \quad ,
\end{aligned}
\tag{VIII.1}
$$

where we have used that, for the induced metric by Ψ on Σ, (Ψ is an isometry) we have $\int_\Sigma|\nabla\Psi|^2=M(C\llcorner B_{r_0}(x_0))\le Kr_0^2$. Using local ξ_1 ξ_2 coordinates in Σ_r, we have for all $k=1\cdots 2p$

$$
\frac{\partial\Psi_i^k}{\partial\xi_1}=-\sum_{l=1}^{2p}J_l^k(\Psi_i)\frac{\partial\Psi^l}{\partial\xi_2}\qquad\text{and}\qquad\frac{\partial\Psi_i^k}{\partial\xi_2}=\sum_{l=1}^{2p}J_l^k(\Psi_i)\frac{\partial\Psi^l}{\partial\xi_1}
$$

Taking respectively the ξ_1 derivative and the ξ_2 derivative of these two equations we obtain

$$
\forall k=1\cdots 2p\quad \Delta_{\Sigma_r}\Psi_i^k=*\left(\sum_{l=1}^{2p}d(J_l^k(\Psi_i))\wedge d\Psi_i^l\right)\quad . \tag{VIII.2}
$$

From (VIII.1) we deduce that $\Delta_{\Sigma_r}(\Psi+\eta_r)=0$ therefore this yields

$$
\forall k=1\cdots 2p\quad \Delta_{\Sigma_r}\eta_r^k=-*\left(\sum_{l=1}^{2p}d(J_l^k(\Psi_i))\wedge d\Psi_i^l\right)\quad . \tag{VIII.3}
$$

Let δ_r^k given by

$$
\left\{
\begin{array}{l}
\Delta_{\Sigma_r}\delta_r^k=*\left(\sum_{l=1}^{2p}d(J_l^k(\Psi_i))\wedge d\Psi_i^l\right)\qquad \text{in } \Sigma_r\\[2mm]
\delta_r^k=0 \qquad \text{on } \partial\Sigma_r
\end{array}
\right.
\tag{VIII.4}
$$

From [Ge] and [To] there exists a universal constant K such that

$$
\|\delta_r\|_{L^\infty(\Sigma_r)}+\|\nabla\delta\|_{L^2(\Sigma_r)}\le K\|J\|_{C^1}\int_{\Sigma_r}|\nabla\Psi|^2\le K r_0^2\quad . \tag{VIII.5}
$$

Because of the above estimates, taking some sequence $r_n \to 0$, one can always extract a subsequence $r_{n'}$ such that $\eta_{r_{n'}}$ and $\delta_{r_{n'}}$ converge to limits η_0 and δ_0 that satisfy in particular

$$
\begin{aligned}
&\overline{\partial}(\Psi + \eta_0) = 0 \qquad \text{in } \Sigma \\
&\Delta_\Sigma(\eta_0 + \delta_0) = 0 \qquad \text{in } \Sigma \\
&\|\nabla\delta_0\|_{L^2(\Sigma)} + \|\delta_0\|_{L^\infty(\Sigma)} \le K r_0^2
\end{aligned}
\tag{VIII.6}
$$

For any $k = 1, \cdots, 2p$ we consider the harmonic function $u^k := \eta^k + \delta^k$. Using the coarea formula we have for any $r < r_0$

$$
\int_0^r ds \int_{\Gamma_s} |\nabla u^k| = \int_{\Sigma\setminus\Sigma_r} |\nabla u^k|\ |\nabla|\Psi|| \le r \left(\int_{\Sigma\setminus\Sigma_r} |\nabla u^k|^2\right)^{\frac{1}{2}} \quad . \tag{VIII.7}
$$

Therefore using a mean formula, for any $\varepsilon > 0$ there exists $s > 0$ such that

$$
\int_{\Gamma_s} |\nabla u^k| \le \varepsilon \quad . \tag{VIII.8}
$$

We have

$$
0 = \int_{\Sigma_s} \Delta_\Sigma u^k = \int_{\Gamma_s} \frac{\partial u^k}{\partial \nu} + \int_{\partial\Sigma} \frac{\partial u^k}{\partial \nu} \tag{VIII.9}
$$

By choosing ε smaller and smaller and taking the corresponding s given by (VIII.8), one gets

$$
\int_{\partial\Sigma} \frac{\partial u^k}{\partial \nu} \quad ., \tag{VIII.10}
$$

Let $m < M$ be two values such that $sup_{\partial\Sigma} u^k < m$ and consider the truncation $T_m^M u^k$ equal to m if $u^k \le m$ equal to M if $u^k \ge M$ and equal to u^k otherwise. We have

$$
0 = \int_{\Sigma_s} T_m^M u^k\ \Delta_\Sigma u^k = -\int_{\Sigma_s} |\nabla T_m^M u^k|^2 + \int_{\Gamma_s} T_m^M u^k \frac{\partial u^k}{\partial \nu} + m \int_{\partial\Sigma} \frac{\partial u^k}{\partial \nu} \quad . \tag{VIII.11}
$$

Therefore

$$
\int_{\Sigma_s} |\nabla T_m^M u^k|^2 \le M \int_{\Gamma_s} |\nabla u^k| \quad , \tag{VIII.12}
$$

and by choosing againg s tending to zero according to (VIII.8), one gets that $T_m^M u^k$ is identically equal to m and we deduce that $u^k \leq m$ in Σ. Similarly one gets that u^k is bounded from below and then we have proved that $\|u\|_{L^\infty(\Sigma)} < +\infty$. Combining this fact with (VIII.6) we have that

$$\|\eta_0\|_{L^\infty(\Sigma)} < +\infty \quad . \tag{VIII.13}$$

Being more careful above by taking eventually Σ_r instead of Σ for some $r \in [r_0/2, r_0]$, and using [Ri3] we could have shown that $\|\eta_0\|_{L^\infty(\Sigma)} < K\ r_0^2$. We claim now that $\partial(\Psi + \eta_0)_*[\Sigma] = (\Psi + \eta_0)_*[\partial\Sigma]$, that is : for any smooth 1-form ϕ equal to zero in a neighborhood of $(\Psi + \eta_0)(\partial\Sigma)$, one has

$$\int_\Sigma (\Psi + \eta_0)^* d\phi = 0 \quad . \tag{VIII.14}$$

We have

$$\left| \int_{\Sigma_s} (\Psi + \eta_0)^* d\phi \right| = \left| \int_{\Gamma_s} (\Psi + \eta_0)^* \phi \right| \leq K_\phi \int_{\Gamma_s} |\nabla\Psi + \eta_0| \quad . \tag{VIII.15}$$

Arguing like for proving (VIII.8), for any ε we can find s such that $\int_{\Gamma_s} |\nabla\Psi + \eta_0| \leq \varepsilon$ and we then deduce (VIII.14). Thus in $B^{2p}_{r_0}(x_0) \setminus \Psi + \eta_0(\partial\Sigma)$, $(\Psi + \eta_0)_*[\Sigma]$ is a integer-multiplicity rectifiable holomorphic cycle. Using the results of Harvey-Shiffman and King ([HS] and [Ki]) we have that there exists a compact Riemann surface with boundary Σ' and an holomorphic map Ψ' such that $(\Psi + \eta_0)_*[\Sigma] = \Psi'_*[\Sigma']$. $(\phi + \eta)(\Sigma)$ is therefore an holomophic curve - with boundary - in $\mathbb{C}^{2p}$. We claim that $\Psi + \eta_0$ is a holomorphic simple covering of Σ'. Indeed let ω_Σ be the pull-back by Ψ of the symplectic form ω in $\mathbb{R}^{2p}$ we have $\int_\Sigma \omega_\Sigma = \int_\Sigma \Psi^*\omega = \int_\Sigma |\nabla\Psi|^2 \geq \pi Q r_0^2$, because of the monotonicity formula ($x_0 \in \mathcal{C}_Q \setminus \mathcal{C}_{Q-1}$). Let $\omega_{\Sigma'}$ be the restriction of ω_0 to Σ'. We have $\int_\Sigma (\Psi + \eta_0)^*\omega_{\Sigma'} = \int_\Sigma (\Psi + \eta_0)^*\omega_0 = \int_\Sigma |\nabla(\Psi + \eta_0)|^2$. Because of (VIII.1), we have that the holomorphic covering $\Psi + \eta_0$ from Σ onto Σ' satisfies

$$\left| \int_\Sigma \omega_\Sigma - \int_\Sigma (\Psi + \eta_0)\omega_{\Sigma'} \right| = o_{r_0}\left(\int_\Sigma \omega_\Sigma \right) \quad . \tag{VIII.16}$$

Therfore, for r_0 small enough this covering has to be a simple one and Σ is a compact Riemann surface. Ψ is now a $J-$holomorphic map from a compact Riemann surface Σ into (B^{2p}, J), it is then smooth and $C \llcorner B^{2p}_{r_0}$ is a $J-$holomorphic curve.

A Appendix

Lemma A.1 *Let U be an open subset of $\mathbb{R}^2$, let $0<\lambda<1$ and let $(B^2_{r_i}(z_i))_{i\in I}$ a covering of U which is locally finite : there exists $n\in\mathbb{N}$ such that*

$$\forall z\in U \qquad Card\Big\{i\in I \quad s.\ t. \quad z\in B^2_{r_{z_i}}(z_i)\Big\}\le N \quad , \tag{A.1}$$

moreover one assumes that

$$\forall i,j\in I \qquad B_{r_i}(z_i)\cap B_{r_j}(z_j)\ne\emptyset \quad\Longrightarrow\quad r_i\ge\lambda r_j \quad . \tag{A.2}$$

Then there exists δ and $P\in\mathbb{N}$ depending on λ only such that

$$\forall i\in I \qquad Card\big\{j\in I \quad s.\ t.\ B_{r_j}(z_j)\cap B_{(1+\delta)r_i}(z_i)\ne\emptyset\big\}\le P \quad . \tag{A.3}$$

Proof of lemma A.1. We argue by contradiction. Assume there exist $\delta_n\to 0$, a sequences of coverings of U, $(B^2_{r_{n,i}}(z_{n,i}))$ for $i\in I$ satisfying (A.1) and (A.2) and a sequence of indices i_n such that

$$\text{Card}\left\{j\in I \quad \text{s. t. } B_{r_{j,n}}(z_{j,n})\cap B_{(1+\delta_n)r_{i_n,n}}(z_{i_n,n})\ne\emptyset\right\}\to+\infty \quad \text{as } n\to+\infty \tag{A.4}$$

After a possible rescaling of the whole covering and a translation we can assume that $r_{i_n,n}=1$ and $z_{i_n,n}=0$. After extraction of a subsequence, we can ensure that there exists $A\in\partial B_1(0)$ such that for any $r>0$

$$\text{Card}\left\{j\in I \quad \text{s. t. } B_{r_{j,n}}(z_{j,n})\cap B_r(A)\ne\emptyset\right\}\to+\infty \quad \text{as } n\to+\infty \quad . \tag{A.5}$$

For a given r and n we take the longest sequence of distinct balls of our covering $B_{r_{j_p,n}}(z_{j_p,n})$ for $p=0\cdots P_n$ satisfying

i)

$$B_{r_{j_0,n}}(z_{j_0,n})=B_1(0)$$

ii)

$$\forall p\le P_n-1 \quad B_{r_{j_p,n}}(z_{j_p,n})\cap B_{r_{j_{p+1},n}}(z_{j_{p+1},n})\ne\emptyset \quad ,$$

iii)

$$\forall p\le P_n \qquad B_{r_{j_p,n}}(z_{j_p,n})\cap B_r(A)\ne\emptyset \quad .$$

It is clear that for a given r $P_n \to +\infty$, indeed if it would not be the case, i.e. $P_n \le P_* < +\infty$ this would imply that the minimal radius for the balls of the covering intersecting $B_r(A)$ is λ^{P_*} and combining this fact with (A.5) would contradict (A.1). Therefore we can then find $r_m \to +\infty$ and $n_m \to +\infty$ as $m \to +\infty$ and sequences $(B_{r_{j_p,n_m}})$ for $1 \le p \le Q_m$ and for $m = 0 \cdots + \infty$ such that

i)

$$B_{r_{j_0,n_m}}(z_{j_0,n_m}) = B_1(0)$$

ii)

$$\forall p \le Q_{m-1} \quad B_{r_{j_p,n_m}}(z_{j_p,n_m}) \cap B_{r_{j_{p+1},n_m}}(z_{j_{p+1},n_m}) \neq \emptyset \quad , \tag{A.6}$$

iii)

$$\forall p \le Q_m \qquad B_{r_{j_p,n_m}}(z_{j_p,n_m}) \cap B_{r_m}(A) \neq \emptyset \quad . \tag{A.7}$$

iv)

$$Q_m \to +\infty \quad .$$

Since $\lambda \le r_{j_1,n_m} \le \lambda^{-1}$ and since the distance $|z_{j_1,n_m}|$ is bounded, we can extract from n_m a subsequence that we still denote n_m such that $B_{r_{j_1,n_m}}(z_{j_1,n_m})$ converges to a limiting ball $B_{r_{1,\infty}}(z_{1,\infty})$ with $\lambda \le r_{1,\infty} \le \lambda^{-1}$, $z_{1,\infty} \le 2$ and $A \in \overline{B_{r_{1,\infty}}(z_{1,\infty})}$. This procedure can be iterated and using a diagonal argument we can assume that

$$\forall p \in \mathbb{N} \quad r_{j_p,n_m} \to r_{p,\infty} \quad z_{j_p,n_m} \to z_{p,\infty}$$

such that

$$\forall p \in \mathbb{N} \quad \lambda^p \le r_{p,\infty} \le \lambda^{-p} \quad |z_{p,\infty}| \le 2 \quad \text{ and } A \in \overline{B_{r_{p,\infty}}(z_{p,\infty})}$$

Moreover because of (A.1) we have that

$$\forall z \in \mathbb{R}^2 \qquad Card\left\{p \in \mathbb{N} \quad \text{s. t.} \quad z \in B^2_{r_{p,\infty}}(z_{p,\infty})\right\} \le N \quad . \tag{A.8}$$

Because of this later fact, since $A \in \overline{B_{r_{p,\infty}}(z_{p,\infty})}$ for all p it is clear that $r_{p,\infty} \to +\infty$ as $p \to +\infty$. Because of (A.8) again, the number of open balls

$B_{r_{p,\infty}}(z_{p,\infty})$ containing A is bounded by N and we can therefore forget them while considering the sequence and assume that

$$\forall p \qquad A \in \partial B^2_{r_{p,\infty}}(z_{p,\infty}) \quad .$$

Let $\vec{t}_p(A) \in S^1$ be the unit exterior normal to $\partial B^2_{r_{p,\infty}}(z_{p,\infty})$ at A. Let $\vec{t}_\infty$ be an accumulation unit vector of the sequence $\vec{t}_p(A)$. Given a direction $\vec{t}$ and an open disk containing A in it's boundary and whose exterior normal at A is given by $\vec{t}$ any other open disk containing A in it's boundary and whose exterior unit at A is not $-\vec{t}$ as a non empty intersection with that disk. Taking $B^2_{r_{p0,\infty}}(z_{p0,\infty})$ such that $\vec{t}_{p0}(A) \neq -\vec{t}_\infty$, there exists then infinitely many disks $B^2_{r_p,\infty}(z_{p,\infty})$ having a non empty intersection with $B^2_{r_{p0},\infty}(z_{p0,\infty})$ but then, because of (A.2) that passes to the limit, all these infinitely many disks have a radius which is bounded from below by a positive number and this contradicts the fact that $r_{p,\infty} \rightarrow +\infty$ implied by (A.8). Thus lemma A.1 is proved. ■

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