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Fourier Analysis in Function Space Theory

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1.1 The Fourier transforms of L^1 functions

Theorem-Definition 1.1. *Let $f \in L^1(\mathbb{R}^n, \mathbb{C})$ define the Fourier transform of f as follows:*

$$\forall \xi \in \mathbb{R}^n \quad \widehat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\, dx.$$

We have that $\widehat{f} \in L^\infty(\mathbb{R}^n)$ and

$$\|\widehat{f}\|_{L^\infty(\mathbb{R}^n)} \le (2\pi)^{-\frac{n}{2}} \|f\|_{L^1(\mathbb{R}^n)}. \tag{1.1}$$

Moreover $\widehat{f} \in C^0(\mathbb{R}^n)$ and

$$\lim_{|\xi| \to +\infty} |\widehat{f}(\xi)| = 0. \tag{1.2}$$

We shall also sometimes denote the Fourier transform of f by $\mathcal{F}(f)$.

Remark 1.2. There are several possible normalizations for defining the Fourier transform of an L^1 function such as for instance

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} e^{-2i\pi x\cdot\xi} f(x)\, dx.$$

None of them give a full satisfaction. The advantages of the one we chose are the following:

i) $f \longmapsto \widehat{f}$ will define an isometry of L^2 as we will see in Proposition 1.5.

ii) With our normalization we have the convenient formula (see Lemma 1.11)

$$\forall k = 1 \ldots n \quad \partial_{\xi_k} \widehat{f} = -i\, \xi_k\, \widehat{f}$$

but the less convenient formula (see Proposition 1.32)

$$\widehat{g * f} = (2\pi)^{-n} \;\widehat{g}\, \widehat{f}.$$

Proof of Theorem 1.1. The first part of the theorem that is inequality (1.1) is straightforward. We prove now that $\widehat{f} \in C^0(\mathbb{R}^n)$. Let $f_k \in C_0^\infty(\mathbb{R}^n)$ such that

$$f_k \longrightarrow f \text{ strongly in } L^1(\mathbb{R}^n).$$

It is clear that since $f_k \in C_0^\infty(\mathbb{R}^n)$ the functions $\widehat{f_k}$ are also C^∞. Inequality (1.1) gives

$$\|\widehat{f} - \widehat{f_k}\|_{L^\infty(\mathbb{R}^n)} \le (2\pi)^{-\frac{n}{2}}\; \|f - f_k\|_{L^1(\mathbb{R}^n)}.$$

Thus $\widehat{f}$ is the uniform limit of continuous functions and, as such, it is continuous. It remains to prove that $|f|(\xi)$ uniformly converges to zero as $|\xi|$ converge to infinity. In

Proposition 1.9 we shall prove that (1.2) holds if $f \in C_0^\infty(\mathbb{R}^n)$. Let $f \in L^1(\mathbb{R}^n)$, let $\varepsilon > 0$ and let $\varphi \in C_0^\infty$ such that

$$\|f - \varphi\|_{L^1(\mathbb{R}^n)} \leq \frac{\varepsilon}{2}\,(2\pi)^{\frac{n}{2}}. \tag{1.3}$$

There exists $R > 0$ such that

$$|\xi| > R \Longrightarrow |\widehat{\varphi}(\xi)| \leq \frac{\varepsilon}{2}. \tag{1.4}$$

Combining (1.3) and (1.4) together with (1.1) applied to the difference $f - \varphi$, we obtain

$$\begin{aligned}|\xi| > R \Longrightarrow |\widehat{f}(\xi)| &\leq \|\widehat{f} - \widehat{\varphi}\|_{L^\infty} + |\widehat{\varphi}(\xi)| \\ &\leq \varepsilon.\end{aligned}$$

This implies (1.2) and Theorem 1.1 is proved. □

Exercise 1.3. Prove that for any $a \in \mathbb{R}_+^*$

$$\widehat{e^{-a|x|^2}} = \frac{1}{(2a)^{\frac{n}{2}}}\, e^{-\frac{|\xi|^2}{4a}}.$$

Prove that for any $a \in \mathbb{R}_+^*$

$$\widehat{f_a(x)} = a^n\, \widehat{f}(a\xi)$$

where $f_a(x) := f(\frac{x}{a})$ for any $x \in \mathbb{R}^n$.

It is then natural to ask among the functions which are continuous, bounded in L^∞ and converging uniformly to zero at infinity, which one is the Fourier transform of an L^1 function. Unfortunately, there seems to be no satisfactory condition characterizing the space of Fourier transforms of $L^1(\mathbb{R}^n)$. We have nevertheless the following theorem.

Theorem 1.4. *(Inverse of the Fourier transform)*
Let $f \in L^1(\mathbb{R}^n; \mathbb{C})$ such that $\widehat{f} \in L^1(\mathbb{R}^n; \mathbb{C})$ then

$$\forall x \in \mathbb{R}^n \quad f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, \widehat{f}(\xi)\, d\xi.$$

Proof of Theorem 1.4. We can of course explicitly write

$$(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, \widehat{f}(\xi)\, d\xi = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\, d\xi \int_{\mathbb{R}^n} e^{ix\cdot y}\, f(y)\, dy.$$

The problem at this stage is that we cannot a-priori reverse the order of integrations because the hypothesis for applying Fubini's theorem are not fullfilled:

$$(\xi, y) \longmapsto e^{i\xi(x-y)} f(y) \notin L^1(\mathbb{R}^n \times \mathbb{R}^n)$$

unless $f \equiv 0$.

The idea is to insert the Gaussian function $e^{-\frac{\varepsilon^2|\xi|^2}{4}}$ where ε is a positive number that we are going to take smaller and smaller. Introduce

$$I_\varepsilon(x) := (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}e^{-\frac{\varepsilon^2|\xi|^2}{4}}\,d\xi\int_{\mathbb{R}^n} e^{-i\xi\cdot y}\,f(y)\,dy.$$

Now we have

$$(\xi,y)\longmapsto e^{-\frac{\varepsilon^2|\xi|^2}{4}}\;e^{i\xi(x-y)}f(y)\in L^1(\mathbb{R}^n\times\mathbb{R}^n)$$

and we can apply Fubini's theorem.

We have in one hand

$$I_\varepsilon(x) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\;e^{-\frac{\varepsilon^2|\xi|^2}{4}}\;\widehat{f}(\xi)\,d\xi.$$

We can bound the integrand uniformly as follows:

$$\forall x,\xi\in\mathbb{R}^n\quad \left|e^{ix\cdot\xi}\;e^{-\frac{\varepsilon^2|\xi|^2}{4}}\;\widehat{f}(\xi)\right|\le|\widehat{f}*\xi)|.$$

By assumption, the right-hand side of the inequality is integrable and we have moreover, for every x and ξ

$$\lim_{\varepsilon\Rightarrow 0} e^{ix\cdot\xi}\;e^{-\frac{\varepsilon^2|\xi|^2}{4}}\;\widehat{f}(\xi) = e^{ix\cdot\xi}\;\widehat{f}(\xi).$$

Hence dominated convergence theorem implies that for any $x\in\mathbb{R}$

$$\lim_{\varepsilon\Rightarrow 0} I_\varepsilon(x) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}\widehat{f}(x)\,e^{ix\cdot\xi}\,d\xi. \tag{1.5}$$

Applying Fubini gives also

$$\begin{aligned}I_\varepsilon(x) &= (2\pi)^{-n}\int_{\mathbb{R}^n} f(y)\,dy\;\int_{\mathbb{R}_n} e^{-i(y-x)\cdot\xi}\,e^{-\frac{\varepsilon^2|\xi|^2}{4}}\;d\xi\\ &= (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} f(y)\,\mathcal{F}\big(e^{-\frac{\varepsilon^2}{4}|\xi|^2}\big)(y-x)\,dy\end{aligned}$$

using Exercise 1.3, we then obtain

$$I_\varepsilon(x) = (2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} f(y)\;e^{\frac{-|y-x|^2}{\varepsilon^2}}\;\frac{2^{\frac{n}{2}}}{\varepsilon^n}\;dy.$$

One proves without much difficulties that for any Lebesgue point $x\in\mathbb{R}$ for f the following holds

$$\lim_{\varepsilon\Rightarrow 0}(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} f(y)\;e^{-\frac{|y-x|^2}{\varepsilon^2}}\;\frac{2^{\frac{n}{2}}}{\varepsilon^n}\;dy = f(x).$$

Continuing this identity with (1.5) gives the theorem. □

The transformation

$$f\in L^1(\mathbb{R}^n)\longmapsto(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\,f(\xi)\,d\xi$$

will be denoted $\overset{\vee}{f}$ or also $\mathcal{F}^{-1}(f)$.

Proposition 1.5. *Let f and $g \in L^1(\mathbb{R}^n;\mathbb{C})$. When*

$$\int_{\mathbb{R}^n} f(x)\,\widehat{g}(x)\,dx = \int_{\mathbb{R}^n} \widehat{f}(x)\,g(x)\,dx.$$

Let $f \in L^1(\mathbb{R}^n;\mathbb{C})$ such that $\widehat{f} \in L^1(\mathbb{R}^n;\mathbb{C})$, then

$$\int_{\mathbb{R}^n} f(x)\,\overline{f(x)}\,dx = \int_{\mathbb{R}^n} \widehat{f}(\xi)\,\overline{\widehat{f}(x)}\,d\xi.$$

This last identity is called Plancherel identity.

Proof of Proposition 1.5. The proof of the first identity in Proposition 1.5 is a direct consequence of Fubini's theorem. The second identity can be deduced from the first one by taking $g := \mathcal{F}^{-1}(\overline{f})$ and by observing that

$$\mathcal{F}^{-1}(\overline{f}) = \overline{\mathcal{F}(f)}\,.$$

□

The second identity is an invitation to extend the Fourier transform as an isometry of L^2. The purpose of the present chapter is to extend the Fourier transform to an even larger class of distributions. To that aim we will first concentrate on looking at the Fourier transform in a "small" class of very smooth function with very fast decrease at infinity: the Schwartz space.

1.2 The Schwartz Space $\mathcal{S}(\mathbb{R}^n)$

The Schwartz functions are C^∞ functions whose successive derivatives decrease faster than any polynominal at infinity. We shall use below the following notations:

$$\forall \alpha = (\alpha, \ldots, \alpha_n) \in \mathbb{N}^n \qquad x^\alpha := x_1^{\alpha_1}, \ldots, x_n^{\alpha_n}$$

$$\forall \beta(\beta, \ldots, \beta_n) \in \mathbb{N}^n \qquad \partial^\beta f := \frac{\partial}{\partial n_1^{\beta_1}} \cdots \frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}}\,(f)$$

and $|\alpha| := \sum \alpha_i$.

Definition 1.6. *The space of Schwartz functions is the following subspace of $C^\infty(\mathbb{R}^n;\mathbb{C})$:*

$$\mathcal{S}(\mathbb{R}^n) := \left\{ \begin{array}{l} \varphi \in C^\infty(\mathbb{R}^n;\mathbb{C}) \quad s.t. \\ \forall p \in \mathbb{N} \quad \mathcal{N}_p(\varphi) := \sup\limits_{\substack{|\alpha| \le p \\ |\beta| \le p}} \|x^\alpha\,\partial^\beta \varphi\|_{L^\infty(\mathbb{R}^n)} < +\infty \end{array} \right\}.$$

The following obvious proposition holds

Proposition 1.7. *$\mathcal{S}(\mathbb{R}^n)$ is stable under the action of derivatives and the multiplication by polynomials in $\mathbb{C}[x_1, \ldots, x_n]$.*

We prove now the following proposition:

Proposition 1.8. *There exists $C_n > 0$ s.t. $\forall \varphi \in \mathcal{S}(\mathbb{R}^n)$*

$$\sum_{\substack{|\alpha| \leq p \\ |\beta| \leq p}} \|x^\alpha \, \partial^\beta \varphi\|_{L^1(\mathbb{R}^n)} \leq C_n \, \mathcal{N}_{p+n+1}(\varphi).$$

Proof of Proposition 1.8. We have

$$\begin{aligned} \int_{\mathbb{R}^n} |x^\alpha \, \partial^\beta \varphi(x)| \, dx &\leq \int_{\mathbb{R}^n} \frac{dx}{(1+|x|^{n+1})} \, (1+|x|^{n+1}) \, |x^\alpha| \, |\partial^\beta \varphi|(x) \\ &\leq C_n \, \mathcal{N}_{p+n+1}(\varphi). \end{aligned} \tag{1.6}$$

This concludes the proof of the proposition. □

The following proposition is fundamental in the theory of tempered distributions we are going to introduce later on.

Proposition 1.9. *Let φ be a Schwartz function on $\mathbb{R}^n$, then it's Fourier transform is also a Schwartz function. Moreover for any $p \in \mathbb{N}$ there exists $C_{n,p} > 0$ such that*

$$\mathcal{N}_p(\widehat{\varphi}) \leq C_{n,p} \, \mathcal{N}_{p+n+1}(\varphi).$$

Hence the Fourier transform is a one to one linear transformation from $\mathcal{S}(\mathbb{R}^n)$ into itself. We shall see in the next sub-chapter that it is also continuous for the topology induced by the ad-hoc Frechet structure on $\mathcal{S}(\mathbb{R})$.

Before proving Proposition 1.9, we need to establish two intermediate elementary lemmas whose proofs are left to the reader. (They are direct applications respectively of the derivation with respect to a parameter in an integral as well as integration by parts. Both operations are justified due to the smoothness of the integrands as well as the fast decrease at infinity).

We have first

Lemma 1.10. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then $\widehat{\varphi}$ is a C' function and*

$$\forall j = 1, \dots, n \qquad \partial_{\xi_j} \, \widehat{\varphi}(\xi) = F(-ix_j \, \varphi).$$

We have also the following lemma:

Lemma 1.11. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then*

$$\forall j = 1, \dots, n \qquad \widehat{\partial_{x_j} \varphi} = i \, \xi_j \, \widehat{\varphi}(\xi).$$

Observe that the two previous lemmas are illustrating the heuristic idea according to which Fourier transform exchanges derivatives or smoothness with decrease at infinity.

Proof of Proposition 1.9. By iterating Lemma 1.10 and Lemma 1.11, we obtain that $\widehat{\varphi} \in C^\infty$ and we have

$$|\xi^\alpha \, \partial_\xi^\beta \, \widehat{\varphi}(\xi)| = \big|\mathcal{F}\big(\partial_x^\alpha (x^\beta \varphi)\big)\big|.$$

Hence using inequality (1.1) we obtain

$$
\begin{aligned}
\mathcal{N}_p(\widehat{\varphi}) &= \sup_{\substack{|\alpha| \le p \\ |\beta| \le p}} \|\xi^\alpha \, \partial_\xi^\beta \, \widehat{\varphi}\|_{L^\infty(\mathbb{R}^n)} \\
&= \sup_{\substack{|\alpha| \le p \\ |\beta| \le p}} \big\|\mathcal{F}\big(\partial_x^\alpha (x^\beta \varphi)\big)\big\|_{L^\infty(\mathbb{R}^n)} \\
&\le \sup_{\substack{|\alpha| \le p \\ |\beta| \le p}} (2\pi)^{-\frac{n}{2}} \|\partial_x^\alpha (x^\beta \varphi)\|_{L^1(\mathbb{R}^n)} \\
&\le C_{n,p} \sup_{\substack{|\alpha| \le p \\ |\beta| \le p+n+1}} \|x^\beta \, \partial_x^\alpha \, \varphi\|_{L^\infty(\mathbb{R}^n)} \le C_{n,p} \, \mathcal{N}_{p+n+1}(\varphi),
\end{aligned}
$$

where we used (1.6). This concludes the proof of the proposition. □

We shall now use the Fourier transform on $\mathcal{S}(\mathbb{R}^n)$ in order to extend by duality the Fourier transform to the “dual” space to $\mathcal{S}(\mathbb{R}^n)$ as the first identity of proposition 1.5 is inviting to do. The idea behind is that $\mathcal{S}(\mathbb{R}^n)$ is a relatively small space and we expect the “dual” to be big and we would then extend Fourier to this larger space. Now the question is to give a precise meaning to the dual space to $\mathcal{S}(\mathbb{R}^n)$. The classical framework of Banach space is not sufficient since $(\mathcal{S}(\mathbb{R}^n), \mathcal{N}_p)$ is not complete. We have to build a topology out of the countable family of norms $(\mathcal{N}_p)_{p\in\mathbb{N}}$. This is the purpose of the next subsection.

1.3 Frechet Spaces

Definition 1.12. *Let V be a $\mathbb{R}$ (or $\mathbb{C}$) vector space*

$$\mathcal{N} : V \to \mathbb{R}_+$$

is a pseudo-norm if

i) $\forall \lambda \in \mathbb{R}$ *(or $\mathbb{C}$)*, $\forall x \in V \quad \mathcal{N}(\lambda x) = |\lambda| \, \mathcal{N}(x)$

ii) $\forall x, y \in V \quad \mathcal{N}(x+y) \le \mathcal{N}(x) + \mathcal{N}(y).$

In other words, a pseudo-norm is a norm without the non-degeneracy axiom.

Definition 1.13. *(Frechet Space)*

Let V be a $\mathbb{R}$ or $\mathbb{C}$ vector space equipped with an increasing sequence of pseudo-norms

$$\mathcal{N}_p \le \mathcal{N}_{p+1}$$

such that the following non-degeneracy condition is satisfied

$$\left.\begin{array}{l} \mathcal{N}_p(x) = 0 \\ \forall p \in \mathbb{N} \end{array}\right\} \Longleftrightarrow x = 0.$$

Introduce on $V \times V$ the following distance:

$$\forall x, y \in V \quad d(x,y) = \sum_{p=0}^{+\infty} 2^{-p} \min\{1, \mathcal{N}_p(x-y)\}.$$

We say that $(V, (\mathcal{N}_p)_{p\in\mathbb{N}})$ defines a Frechet space if (V,d) is a complete metric space.

Examples of Frechet Spaces (left as exercise)

i) A Banach space $(V, \|\cdot\|)$ for the constant sequence of norms $\mathcal{N}_p(\cdot) := \|\cdot\|$ is Frechet.

ii) The space of smooth functions $C^\infty(B_1^n(0))$ over the unit ball of $\mathbb{R}^n$ is a Frechet space for the sequence of C^p-norms

$$\forall p \in \mathcal{N} \quad \|f\|_{C^p} := \sup_{\substack{x \in B_1^n(0) \\ |\alpha| \le p}} |\partial^\alpha f|(x).$$

iii) The space $L^q_{\text{loc}}(\mathbb{R}^n)$ of measurable functions of $\mathbb{R}^n$ which are L^q on every compact of $\mathbb{R}^n (q \in [1,\infty])$ is Frechet for the family of pseudo-norms

$$\big(L^q(B_{2^p}(0)\big)_{p\in\mathbb{N}}.$$

iv)

$$\big(\mathcal{S}(\mathbb{R}^n), (\mathcal{N}_p)_{p\in\mathbb{N}}\big),$$

where $\mathcal{N}_p$ are the pseudo-norms defined in Definition 1.6 define a Frechet Space. □

In practice the distance d is never really used and can also be replaced by

$$d_a(x,y) := \sum_{p\in\mathbb{N}} a_p \, \min\{1, \mathcal{N}_p(x-y)\},$$

where $a = (a_p)_{p\in\mathbb{N}}$ is an arbitrary sequence of positive number such that $\sum_{p\in\mathbb{N}} a_p < +\infty$. The following proposition happens to be very useful in the context of Frechet space.

Proposition 1.14. *Let $F = (V, (\mathcal{N}_p)_{p\in\mathbb{N}})$ be a Frechet space, then the following three assertions hold true:*

i) *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of elements from V*

$$f_n \underset{n\Rightarrow+\infty}{\overset{d}{\longrightarrow}} f \Longleftrightarrow \forall p \in \mathbb{N} \quad \mathcal{N}_p(f_n - f) \underset{n\Rightarrow+\infty}{\longrightarrow} 0.$$

ii) *$(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (F,d) if and only if $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence for all the pseudo-norms $\mathcal{N}_p$.*

iii) *Each of the pseudo-norm $\mathcal{N}_p$ is continuous in (F,d).*

Proof of Proposition 1.14. First we prove the assertion i):

$$f_n \underset{n\Rightarrow+\infty}{\xrightarrow{d}} f \Longrightarrow \forall p \in \mathbb{N} \ \min\{1, \mathcal{N}_p(f_n - f)\} \underset{n\Rightarrow+\infty}{\longrightarrow} 0$$
$$\Longleftrightarrow \forall p \in \mathbb{N} \ \ \mathcal{N}_p(f_n - f) \underset{n\Rightarrow+\infty}{\longrightarrow} 0.$$

We now prove the reciprocal of i):

Let $\varepsilon > 0$ and choose $Q \in \mathbb{N}$ such that

$$\sum_{p=Q}^{+\infty} 2^{-p} < \frac{\varepsilon}{2}.$$

Since $\mathcal{N}_P(f_n - f) \underset{n\to+\infty}{\longrightarrow} 0$ for every p there exists $N \in \mathbb{N}$ such that

$$\forall p < Q \ \text{ and } \ n \geq \mathbb{N} \quad \mathcal{N}_p(f_n - f) \leq \frac{\varepsilon}{4}.$$

Thus $\forall n \geq N$:

$$\begin{aligned} &\sum_{p=0}^{+\infty} 2^{-p} \min\{1, \mathcal{N}_p(f_n - f)\} \\ &\leq \sum_{p=0}^{Q-1} 2^{-p}\, \mathcal{N}_p(f_n - f) + \sum_{p=Q}^{+\infty} 2^{-p} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This implies that $f_n \underset{n\to+\infty}{\xrightarrow{d}} f$. This proves i).

The same arguments imply ii).

The proof of iii) is straightforward. Indeed, let $p \in \mathbb{N}$

$$d(f, g) \leq 2^{-p}\, \varepsilon \Longrightarrow \mathcal{N}_p(f - g) \leq \varepsilon.$$

This concludes the proof of Proposition 1.14. $\square$

The following proposition extends a well-known fact in normed space topology.

Proposition 1.15. *Let $F = (V, (\mathcal{N}_p)_{p\in\mathbb{N}})$ and $G = (w, (\mathcal{M}_q)_{q\in\mathbb{N}})$ be two Frechet spaces and let $L : V \to W$ be a linear map. The following three assertions are equivalent:*

i) *L is continuous at 0,*

ii) *L is continuous everywhere,*

iii) $\forall q \in \mathbb{N} \quad \exists C_q > 0$ *and* $\exists p \in \mathbb{N}, \quad$ *s.t.* $\forall f \in F \quad \mathcal{M}_q(Lf) \leq C_q\, \mathcal{N}_p(f)$.

Proof of Proposition 1.15. The implication ii) $\Longrightarrow$ i) is tautological. We are now proving i) $\Longrightarrow$ iii).

Since L is continuous at 0, for any neighbors G of $0 \in W$, there exists an open neighborhood $\mathcal{U}$ of $0 \in V$ such that

$$L(\mathcal{U}) \subset G.$$

In ther words, $\mathcal{U} \subset L^{-1}(G)$. Let $q \in \mathbb{N}$ and choose $G_q = \mathcal{M}_q^{-1}([0,1))$. Since $\mathcal{M}_q$ is continuous in (w, d_G), due to Proposition 1.15, G_q is an open set containing 0. Because the topology in F is a metric topology, there exists $\alpha_q > 0$ such that

$$B_{d_q}^{d_F}(0) \subset \mathcal{U}_q \subset L^{-1}(G_q),$$

where $B_{d_q}^{d_F}(0)$ denotes the ball of center $0 \in V$ and radius α_q for the Frechet distance d_F. In other words, we have

$$\sum_{p\in\mathbb{N}} 2^{-p} \min\{1, \mathcal{N}_p(f)\} < \alpha_q \Longrightarrow \mathcal{M}_q\big(L(f)\big) < 1. \tag{1.7}$$

Let $p_0 \in \mathbb{N}$ such that

$$\sum_{p=p_0+1}^{+\infty} 2^{-p} \leq \frac{\alpha_q}{4}. \tag{1.8}$$

Since $\mathcal{N}_p$ is increasing with respect to p

$$\mathcal{N}_{p_0}(f) < \frac{\alpha_q}{4} \Longrightarrow \sum_{p\leq p_0} 2^{-p}\, \mathcal{N}_p(f) < \frac{\alpha_q}{2}. \tag{1.9}$$

Hence, combining (1.7), (1.8) and (1.9), we obtain for any $f \in V$

$$\mathcal{N}_{p_0}(f) < \frac{\alpha_q}{4} \Longrightarrow \mathcal{M}_q\big(L(f)\big) < 1$$

using the homogeneity of the two pseudo-norms $\mathcal{M}_{p_0}$ and $\mathcal{M}_q$, we have proved

$$\mathcal{M}_q\big(L(f)\big) \leq \frac{\alpha_q}{4}\, \mathcal{N}_{p_0}(f).$$

Hence we have proved the implication i) $\Longrightarrow$ iii).

In order to conclude the proof of Proposition 1.15, it suffices to establish the implication iii) $\Longrightarrow$ ii).

We assume iii) and we are going to prove that L is continuous. Since the topologies of both F and G are metric, it suffices to show that for any sequence $f_n \in V$ converging to $f \in V$ for d_F, then

$$\lim_{n\to+\infty} d_G\big(L(f_n),\, L(f)\big) = 0. \tag{1.10}$$

Because of Proposition 1.14 i) in order to establish (1.10), it suffices to prove

$$\forall q \in \mathbb{N} \quad \lim_{n\to+\infty} \mathcal{M}_q\big(L(f_n - f)\big) = 0. \tag{1.11}$$

Let $q \in \mathbb{N}$, because we are assuming iii), there exists $p_0 \in \mathbb{N}$ and $C_q > 0$ such that

$$\forall g \in V \quad \mathcal{M}_q\big(L(g)\big) \le C_q\, \mathcal{N}_{p_0}(g).$$

Let $\varepsilon > 0$. Let N be large enough such that

$$\forall n \ge N \quad \mathcal{N}_{p_0}\,(f_n - f) \le \frac{\varepsilon}{C_q},$$

then we have

$$\forall n \ge N \quad \mathcal{M}_q\,\big(L(f_n - f)\big) \le \varepsilon.$$

This implies (1.11) and L is continuous everywhere. □

The following theorem is the extension of Frechet spaces of the famous Banach-Steinhaus theorem for normed spaces.

Theorem 1.16. *(Banach-Steinhaus for Frechet Spaces)*

Let $F = (V, (\mathcal{N}_p)_{p\in\mathbb{N}}$ and $G = (W, (\mathcal{M}_q)_{q\in\mathbb{N}})$ be two Frechet spaces. Let L_n be a sequence of linear maps from V into W and assume that each L_n is continuous from F into G. Assume moreover that for any $f \in V$ the sequence $L_n f$ converges to a limit Lf in W. Then L defines a linear and continuous map.

Proof of Theorem 1.16. The linearity of L is straightforward. It remains to prove that L is continuous. For any $q \in \mathbb{N}$ and positive number A we introduce the following subset of V:

$$C^q_A := \{f \in V \text{ s.t. } \forall n \in \mathbb{N} \quad \mathcal{M}_q(L_n f) \le A\}.$$

First, we observe that C^q_A is a closed set. Indeed, it is the intersection of closed sets

$$C^q_A = \bigcap_{n\in\mathbb{N}} (\mathcal{M}_q \circ L_n)^{-1}([0, A]).$$

We now claim that

$$\bigcup_{A\in\mathbb{R}^*_+} C^q_A = V. \tag{1.12}$$

Indeed, by assumption, $d_F(L_n f, Lf) \underset{n\to+\infty}{\longrightarrow} 0$, this implies that

$$\forall q \in \mathbb{N} \quad \sup_{n\in\mathbb{N}} \mathcal{M}_q(L_n f) < +\infty.$$

Thus if one takes $A > \sup_{n\in\mathbb{N}} \mathcal{M}_q(L_n f)$, one has that $f \in C^q_A$ and this proves the claim (1.12).

Obviously $A \geq A' \Longrightarrow C^q_{A^1} \subset C^q_A$. Thus

$$V = \bigcup_{j \in \mathbb{N}} C^q_{2^j}.$$

By assumption (V, d_F) is a complete metric space to which we can apply Baire's theorem and there exists $j_0 \in \mathbb{N}$ such that $C^q_{2^{j_0}}$ has a non-empty interior:

$$\dot{C}^q_{2^{j_i}} \neq \emptyset.$$

Let $f_0 \in \dot{C}^q_{2^{j_0}}$, then there exists $\alpha > 0$ such that

$$B^{d_F}_\alpha(f_0) \subset C^q_{2^{j_0}} = \bigcap_{n \in \mathbb{N}} (\mathcal{M}_q \circ L_n)^{-1}([0, 2^{j_0}]).$$

In other words:

$$d_F(f, f_0) < \alpha \Longrightarrow \sup_{n \in \mathbb{N}} \mathcal{M}_q(L_n f) \leq 2^{j_0}. \tag{1.13}$$

Let $p_0 \in \mathbb{N}$ such that

$$\sum_{j=p_0+1}^{\infty} 2^{-j} < \frac{\alpha}{4}. \tag{1.14}$$

Since $\mathcal{N}_p$ is increasing with respect to p

$$\mathcal{N}_{p_0}(f - f_0) < \frac{\alpha}{4} \Longrightarrow \sum_{j=0}^{p_0} 2^{-j} \, \mathcal{N}_p(f - f_0) < \frac{\alpha}{2}.$$

Thus, because of (1.13) and (1.14), we deduce

$$\begin{aligned} \mathcal{N}_{p_0}(f - f_0) < \frac{\alpha}{4} &\Longrightarrow d_F\,(f, f_0) < \alpha \\ &\Longrightarrow \sup_{n \in \mathbb{N}} \mathcal{M}_q(L_n f) \leq 2^{j_0}. \end{aligned}$$

Since $\sup_{n \in \mathbb{N}} \mathcal{M}_q(L_n\, f_0) < 2^{j_0}$, we have

$$\mathcal{N}_{p_0}(h) < \frac{\alpha}{4} \Longrightarrow \sup_{n \in \mathbb{N}} \mathcal{M}_q(L_n h) \leq 2^{j_0+1}.$$

The homogeneity of the pseudo-norms gives then

$$\sup_{n \in \mathbb{N}} \mathcal{M}_q(L_n h) \leq \frac{4}{\alpha} \, 2^{j_0+1} \, \mathcal{N}_{p_0}(h).$$

Since $L_n\, h \underset{n \to +\infty}{\longrightarrow} Lh$ by continuity of $\mathcal{M}_q$, we deduce

$$\mathcal{M}_q(Lh) \leq \frac{\alpha}{4} \, 2^{j_0+1} \, \mathcal{N}_{p_0}(h).$$

This holds for arbitrary $q \in \mathbb{N}$. Then, from the characterization of continuity given by Proposition 1.15 iii), we deduce that L is continuous. □

It is now time to define the dual of the Schwartz Space in the Frechet Space theory.

1.4 The space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$

The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is from now on equipped with the Frechet topology issued by the sequence of pseudo-norms $\mathcal{N}_p$ introduced in Definition 1.6.

Definition 1.17. *The space of tempered distributions denoted $\mathcal{S}'(\mathbb{R}^n)$ is the space of continuous and linear maps from $\mathcal{S}(\mathbb{R}^n)$ into $\mathbb{C}$.*

We have the following important characterization of tempered distributions: The action of a linear form T on $\varphi \in \mathcal{S}'(\mathbb{R}^n)$ will be denoted either $T(\varphi)$ or $\langle T, \varphi\rangle$.

Proposition 1.18. *Let T be a linear map from $\mathcal{S}(\mathbb{R}^n)$ into $\mathbb{C}$. The following equivalence holds*

$$
\begin{aligned}
T \in \mathcal{S}'(\mathbb{R}^n) \Longleftrightarrow \quad & \exists C > 0 \text{ and } p \in \mathbb{N} \text{ such that} \\
& \forall \varphi \in \mathcal{S}(\mathbb{R}^n) \quad |\langle T, \varphi\rangle| \le C\, \mathcal{N}_p(\varphi).
\end{aligned}
\tag{1.15}
$$

The minimal $p \in \mathbb{N}$ for which (1.15) is called the order of the tempered distribution T.

Observe that general distributions in $\mathcal{D}'(\mathbb{R}^n)$ don't always have an order. The L^1_{loc} function on $\mathbb{R}$ given by $t \longmapsto e^t$ is an element of $\mathcal{D}'(\mathbb{R})$ but cannot be an element of $\mathcal{S}'(\mathbb{R}^n)$ for that reason: indeed, one easily proves that for any $p \in \mathbb{N}$

$$
\sup_{\varphi \in C^\infty_c(\mathbb{R})} \frac{\int_{\mathbb{R}} e^t \varphi(t)\, dt}{\mathcal{N}_p(\varphi)} = +\infty.
$$

Consider $\varphi \ge 0$ compactly supported such that $\int_{\mathbb{R}} \varphi = 1$ and take for $k \in \mathbb{N} \quad \varphi_k(t) := \varphi(t-k)$. We have $\mathcal{N}_p(\varphi_k) \le C k^p$ but

$$
\lim_{k \to +\infty} \int_{\mathbb{R}} k^{-p}\, e^t\, \varphi_k(t) = +\infty.
$$

The proof of Proposition 1.18 follows from a direct application of the characterization of continuity in Frechet space given by Proposition 1.15 iii). Indeed, $\mathbb{C}$ equipped with the modulus norm is interpreted as a Frechet space with

$$
\forall a \in \mathbb{C} \quad \mathcal{M}_q(a) := |a|.
$$

Example of elements in $\mathcal{S}'(\mathbb{R}^n)$

i) We have for any $p \in [1, +\infty]$

$$
L^p(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n).
$$

Indeed, let $f \in L^p(\mathbb{R}^n)$, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ Hölder inequality gives

$$
\begin{aligned}
\left| \int_{\mathbb{R}^n} f(x)\, \varphi(x)\, dx \right| &\le \|f\|_{L^p}\, \|\varphi\|_{L^{p'}}, \\
&\le \|f\|_{L^p} \left[\int_{\mathbb{R}^n} \frac{(1+|x|^{n+1})^{p'}}{(1+|x|^{n+1})^{p'}}\, |\varphi|^{p'}(x)\, dx \right]^{\frac{1}{p'}} \\
&\le C_{n,p}\, \|f\|_{L^p}\, \mathcal{N}_{n+1}(\varphi).
\end{aligned}
$$

ii) The space $\mathbb{C}[x_1 \dots x_n]$ of complex polynomials in $\mathbb{R}^n$ is included in $\mathcal{S}'(\mathbb{R}^n)$.

iii) Let $a \in \mathbb{R}^n$, the *Dirac Mass* $\delta_a : \varphi \in \mathcal{S}(\mathbb{R}^n) \longmapsto \varphi(a)$ is obviously a tempered distribution of order 0:

$$|\langle \delta_a, \varphi \rangle| \leq \mathcal{N}_0(\varphi).$$

Definition 1.19. *A sequence of tempered distributions $(T_k)_{k\in\mathbb{N}}$ is said to converge weakly if for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ the sequence $\langle T_k, \varphi \rangle$ converges in $\mathbb{C}$. From Banach Steinhaus theorem for Frechet spaces we deduce that there exists $T \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\lim_{k\to+\infty} \langle T_k, \varphi \rangle = \langle T, \varphi \rangle.$$

The weak convergence of a sequence $(T_k)_{k\in\mathbb{N}}$ in $\mathcal{S}'(\mathbb{R}^n)$ towards an element $T \in \mathcal{S}'(\mathbb{R}^n)$ is denoted

$$T_k \rightharpoonup T \ \ in \ \ \mathcal{S}'(\mathbb{R}^n).$$

Exercise: Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \varphi(x)\, dx = 1$ denote $\varphi_k(x) := 2^{k_n} \varphi(2^k x)$. Prove that

$$\varphi_k \rightharpoonup \delta_0 \ \text{ in } \ \mathcal{S}'(\mathbb{R}^n).$$

Definition-Proposition 1.20. *Let $T \in \mathcal{S}'(\mathbb{R}^n)$ for any $j = 1, \dots, n$ we denote by $\partial_{x_j} T$ the partial derivative of T along the direction x_j which is the following element of $\mathcal{S}'(\mathbb{R}^n)$*

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^n) \quad \langle \partial_{x_j} T, \varphi \rangle := -\langle T, \partial_{x_j} \varphi \rangle. \tag{1.16}$$

Proof of Proposition 1.20. Let $T \in \mathcal{S}'(\mathbb{R}^n)$. It is clear that the map $\partial_{x_j} T$ defined by (1.16) is linear. Let $p \in \mathbb{N}$ and $c > 0$ such that

$$|\langle T, \varphi \rangle| \leq c\, \mathcal{N}_p(\varphi).$$

By (1.16) we have

$$\begin{aligned} |\langle \partial_{x_j} T, \varphi \rangle| = |\langle T, \partial_{x_j} \varphi \rangle| &\leq c\, \mathcal{N}_p(\partial_{x_j}) \\ &\leq c\, \mathcal{N}_{p+1}(\varphi). \end{aligned}$$

Hence from the characterization of tempered distributions given by Proposition (1.9), we deduce that $\partial_{x_j} T \in \mathcal{S}'(\mathbb{R}^n)$ and this concludes the proof of Proposition 1.20. $\Box$

More generally, by iterating proposition 1.20, we deduce that for any $T \in \mathcal{S}'(\mathbb{R}^n)$ and any $\alpha = (\alpha, \dots, \alpha_n) \in \mathbb{N}^n$ the linear map on $\mathcal{S}(\mathbb{R}^n)$ given by

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^n) \quad \langle \partial^\alpha T, \varphi \rangle := (-1)^{|\alpha|} \langle T, \partial^\alpha, \varphi \rangle$$

is an element of $\mathcal{S}'(\mathbb{R}^n)$.

Definition 1.21. *The space of slowly growing functions denoted $\mathcal{O}(\mathbb{R}^n)$ is the subspace of C^∞ functions f in $\mathbb{R}^n$ such that*

$$\forall \beta = (\beta_1, \dots, \beta_n) \quad \exists\, m_\beta \in \mathbb{N} \ \ and \ \ C_\beta > 0$$

such that

$$|\partial^\beta f|(x) \leq C_\beta (1 + |x|)^{m_\beta}.$$

Exercise: Let $f \in \mathcal{O}(\mathbb{R}^n)$ prove that the map

$$\varphi \in \mathcal{S}(\mathbb{R}^n) \longmapsto \int f(x)\,\varphi(x)\,dx$$

defines a tempered distribution that we shall simply denote by f.

Observe that $\mathbb{C}[x_1, \ldots, x_n] \subset \mathcal{O}(\mathbb{R}^n)$.

Proposition 1.22. *Let $f \in \mathcal{O}(\mathbb{R}^n)$ the multiplication by f*

$$\begin{aligned} M_f \;\; \mathcal{S}(\mathbb{R}^n) &\longrightarrow \mathcal{S}(\mathbb{R}^n) \\ \varphi &\longrightarrow f\,\varphi \end{aligned}$$

is a continuous linear map from $\mathcal{S}(\mathbb{R}^n)$ into itself.

Proof of Proposition 1.22. Let $f \in \mathcal{O}(\mathbb{R}^n)$, $q \in \mathbb{N}$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we have using mostly Leibnitz rule and triangular inequality

$$\begin{aligned} \mathcal{M}_q(f\,\varphi) &= \sup_{\substack{|\alpha| \leq q \\ |\beta| \leq q}} \|x^\alpha\,\partial^\beta(f\,\varphi)\|_{L^\infty(\mathbb{R}^n)} \\ &\leq \sup_{\substack{|\alpha| \leq q \\ |\beta| \leq q}} \sum_{\gamma \leq \beta} C_{\gamma,\beta}\|x^\alpha\;\partial^\gamma\varphi\;\partial^{\beta-\gamma} f\|_{L^\infty(\mathbb{R}^n)} \\ &\leq \sup_{\substack{|\alpha| \leq q \\ |\beta| \leq q}} \sum_{\gamma \leq \beta} C_{\gamma,\beta}\|\;|x^\alpha|\;|\partial^\gamma\varphi|(x)\;(1+|x|)^{m_{\beta-\gamma}}\|_{L^\infty(\mathbb{R}^n)} \\ &\leq C_q \sum_{|\beta| \leq q} \mathcal{N}_{m_\beta+q}(\varphi) \leq C'_q\;\; \mathcal{N}_{q\,+\,\max\limits_{|\beta|\,\leq\,q} m_\beta}(\varphi). \end{aligned}$$

This implies the proposition. $\square$

We define now the Fourier transform of a tempered distribution. This definition is motivated by the first identity in Proposition 1.5.

Definition-Proposition 1.23. *Let T be a tempered distribution. We define the Fourier transform of T that we denote by $\widehat{T}$ or $\mathcal{F}(T)$ to be the following linear map on $\mathcal{S}(\mathbb{R}^n)$*

$$\varphi \in \mathcal{S}(\mathbb{R}^n)\;\langle \widehat{T}, \varphi\rangle := \langle T, \widehat{\varphi}\rangle,$$

$\widehat{T}$ is a tempered distribution as well.

Proof of Proposition 1.23. Let $T \in \mathcal{S}'(\mathbb{R}^n)$ and let p be the order of T and $C > 0$ such that

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^n)\;\; |\langle T, \varphi\rangle| \leq c\,\mathcal{N}_p(\varphi).$$

Using Proposition 1.9 we then deduce

$$\varphi \in \mathcal{S}(\mathbb{R}^n) \ \ |\langle \widehat{T}, \varphi\rangle| = |\langle T, \widehat{\varphi}\rangle| \ \begin{aligned} &\le c\,\mathcal{N}_p(\widehat{\varphi}) \\ &\le c^1\,\mathcal{N}_{p+n+1}(\varphi). \end{aligned}$$

Using one more time the characterization of $\mathcal{S}'(\mathbb{R}^n)$ given by Proposition 1.18, we deduce that $\widehat{T}$ is a tempered distribution.

Example: Let $a \in \mathbb{R}^n$ we have

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^n) \ \ \langle \widehat{\delta_a}, \varphi\rangle = \langle \delta_a, \widehat{\varphi}\rangle = \widehat{\varphi}(a) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ia\cdot x}\,\varphi(x)\,dx.$$

Hence

$$\widehat{\delta_a} = (2\pi)^{-\frac{n}{2}}\, e^{-ia\cdot x} \in L^\infty(\mathbb{R}^n).$$

In other words, the Fourier transform exchange the "most concentrated" measure into the "most dispersed" wave function. This phenomenon is known as the *Heisenberg Uncertainty Principle* in quantum mechanics. □

Example: More generally, given $\alpha = (\alpha, \ldots, \alpha_n) \in \mathbb{N}^n$ we have, using Lemma 1.10,

$$\begin{aligned} \langle \widehat{\partial_\alpha\,\delta_a}, \varphi\rangle &= (-1)^{|\alpha|}\,\langle \delta_a, \partial_a\widehat{\varphi}\rangle \\ &= (-1)^{|\alpha|}\langle \delta_a,\ \widehat{(-i)^{|\alpha|}x^\alpha\varphi}\rangle \\ &= (i)^{|\alpha|}(2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ia\cdot x}\, x^\alpha\,\varphi(x)\,dx. \end{aligned}$$

Hence we have established

$$\widehat{\partial_\alpha\,\delta_a} = (i)^{|\alpha|}(2\pi)^{-\frac{n}{2}}\, e^{-ia\cdot x}x^\alpha \in \mathcal{O}(\mathbb{R}^n).$$

Exercise: Prove that

$$\widehat{\mathbf{1}} = (2\pi)^{\frac{n}{2}}\ \delta_0$$

and more generally

$$\forall \alpha \in \mathbb{N}^n \quad \widehat{x^\alpha} = (2\pi)^{\frac{n}{2}}\ i^{|\alpha|}\,\partial^\alpha\,\delta_0.$$

□

We shall also denote by $\check{T}$ or $\mathcal{F}^{-1}(T)$ the inverse Fourier transform of T

$$\langle \check{T}, \varphi\rangle := \langle T, \check{\varphi}\rangle$$

and obviously $\forall T \in \mathcal{S}'(\mathbb{R}^n)$, $\check{T} \in \mathcal{S}'(\mathbb{R}^n)$.

We shall now prove the following proposition:

Proposition 1.24. *Let T be a tempered distribution supported at the origin that is to say $\forall \varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\varphi \equiv 0$ in a neighborhood of 0, then $\langle T, \varphi\rangle = 0$. Then, there exists $p \in \mathbb{N}$ such that for any $\beta = (\beta_1, \ldots, \beta_n)$ satisfying $|\beta| \le p$, there exists $c_\beta \in \mathbb{C}$ such that*

$$T = \sum_{|\beta|\le p} C_\beta\,\partial^\beta\delta_0.$$

Proof of Proposition 1.24. Let p be the order of T. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we proceed to the Taylor expansion of φ to the order p at the origin: for any $\gamma \in \mathbb{N}^n$ and $|\gamma| \le p$ there exists a_γ independent of φ such that

$$\varphi(x) = \sum_{|\gamma|\le p} a_j\, \partial^\gamma\, \varphi(0)\, x^\gamma + R_p(x)$$

where

$$\lim_{|x|\to 0} \frac{|R_p(x)|}{|x|^p} = 0.$$

Moreover $\forall \gamma$, $|\gamma| \le p$

$$\lim_{|x|\to 0} \frac{|\partial^\gamma R_p(x)|}{|x|^{p-|\gamma|}} = 0. \tag{1.17}$$

Let χ b a non-negative cut-off function in $C_c^\infty(B_1(0))$ such that χ is identically equal to one on $B_{\frac{1}{2}}(0)$. By assumption

$$\begin{aligned}\langle T, \varphi\rangle &= \langle T, \chi\, \varphi\rangle + \langle T, (1-\chi)\, \varphi\rangle \\ &= \langle T, \chi\, \varphi\rangle.\end{aligned}$$

We have, using the Taylor expansion of φ,

$$\langle T, \chi\varphi\rangle = \sum_{|\gamma|\le p} a_\gamma\, \partial^\gamma \varphi(0)\, \langle T, \chi(x)\, x^\gamma\rangle + \langle T, \chi(x)\, R_p(x)\rangle.$$

Observe that the functions $\chi(x)\, x^\gamma$ are Schwartz functions and hence $\langle T, \chi(x)\, x^\gamma\rangle$ are well-defined complex numbers. We claim that

$$\langle T, \chi(x)\, R_p(x)\rangle = 0. \tag{1.18a}$$

The proof of the claim implies obviously the proposition. Let

$$\eta_\varepsilon(x) := 1 - \chi\big(\frac{x}{\varepsilon}\big)$$

where $0 < \varepsilon \ll 1$. By assumption we have

$$\begin{aligned}\langle T, \chi\, R_p\rangle &= \langle T, \chi\, R_p\, \eta_\varepsilon\rangle + \langle T, \chi\, R_p\, \chi_\varepsilon\rangle \\ &= \langle T,\ R_p\, \chi_\varepsilon\rangle,\end{aligned} \tag{1.18b}$$

where $\chi_\varepsilon(x) = \chi(x/\varepsilon)$. Since T is of order p, there exists $C > 0$ such that

$$|\langle T,\ R_p\, \chi_\varepsilon\rangle| \le C\, \mathcal{N}_p(R_p\, \chi_\varepsilon).$$

We have, using Leibnitz formula and triangular inequality,

$$
\begin{aligned}
\mathcal{N}_p(R_p\,\chi_\varepsilon) &= \sum_{\substack{|\alpha|\le p\\ |\beta|\le p}} \|x^\alpha\,\partial^\beta(R_p\,\chi_\varepsilon)\|_{L^\infty(\mathbb{R}^n)}\\
&\le \sum_{\substack{|\alpha|\le p\\ |\beta|\le p}} \sum_{\gamma\le\beta} C_\gamma\,\|x^\alpha\,\partial^{\beta-\gamma}\,R_p\,\partial^\gamma\,\chi_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \qquad (1.19)\\
&\le C_p \sum_{|\beta|\le p}\sum_{\gamma\le\beta} \|\partial^{\beta-\gamma}\,R_p\,\partial^\gamma\,\chi_\varepsilon\|_{L^\infty(\mathbb{R}^n)}
\end{aligned}
$$

We clearly have

$$
|\partial^\gamma\,\chi_\varepsilon|(x) \le \frac{C_\gamma}{\varepsilon^{|\gamma|}}\,\mathbf{1}_{\beta_\varepsilon(0)}(x) \qquad (1.20)
$$

where $\mathbf{1}_{B_2^{(x)}}(0)$ is the characteristic function of the ball centered at the origin and of radius ε. Because of (1.17) we have

$$
\|\partial^{\beta-\gamma}\,R_p(x)\,\mathbf{1}_{B_\varepsilon(0)}(x)\|_{L^\infty(\mathbb{R}^n)} = o(\varepsilon^{p-|\beta-\gamma|}).
$$

Combining this inequality with (1.19) and (1.20) we obtain

$$
\mathcal{N}_p(R_p\,\chi_\varepsilon) = o\Big(\sum_{|\beta|\le p}\sum_{\gamma\le\beta}\varepsilon^{p-|\beta-\gamma|-|\gamma|}\Big).
$$

Since $\gamma\le\beta$, we have $|\beta-\gamma|+|\gamma| = \sum\beta_i-\gamma_i+\sum\gamma_i = |\beta|$. Hence

$$
\lim_{\varepsilon\to 0}\mathcal{N}_p(R_p\,\chi_\varepsilon) = 0
$$

From (1.18b) we deduce (1.18a) and this concludes the proof of Proposition 1.24. □

We shall need the following proposition which is a direct consequence of Proposition 1.22.

Definition-Proposition 1.25. *Let $f\in\mathcal{O}(\mathbb{R}^n)$ be a slowly increasing function for any $T\in\mathcal{S}'(\mathbb{R}^n)$, we define the multiplication of T by f as follows:*

$$
\forall\varphi\in\mathcal{S}(\mathbb{R}^n)\quad \langle f\,T,\varphi\rangle := \langle T, f\,\varphi\rangle.
$$

This multiplication denoted $f\,T$ is a tempered distribution.

Proposition 1.26. *Let $T\in\mathcal{S}'(\mathbb{R}^n)$, then for any $\alpha=(\alpha_1,\dots,\alpha_n)$ and any $\beta=(\beta_1,\dots,\beta_n)$ we have respectively*

$$
\partial^\alpha\,\widehat{T} = (-i)^{|\alpha|}\,\widehat{x^\alpha\,T}
$$

and

$$
\widehat{\partial^\beta T} = i^{|\beta|}\xi^\beta\,\widehat{T},
$$

where the products $x^\alpha T$ and $\xi^\beta\widehat{T}$ have to be understood in the sense of Proposition 1.25.

Proposition 1.26 is a direct consequence of Lemma 1.10 and Lemma 1.11. We have the following theorem:

Theorem 1.27. *Let T be an harmonic tempered distribution that is an element of $\mathcal{S}'(\mathbb{R}^n)$ satisfying*

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^n) \quad \langle \Delta T, \varphi \rangle = \langle T, \Delta\varphi \rangle = 0.$$

Then T is a polynomial.

This result is a bit "confusing" since we know many more harmonic functions than polynomials. For instance in $\mathbb{R}^2$ every holomorphic function is harmonic but is not necessarily a polynomial (i.e. $f(z) = e^z$). This illustrates the difference between $\mathcal{S}'$ and $\mathcal{D}'$. $\mathcal{S}'$ being roughly the space of distributions for which one can define a Fourier transform.

Proof of Theorem 1.27. For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\begin{aligned} 0 = \langle \Delta T, \widehat{\varphi} \rangle &= \langle T, \Delta\, \widehat{\varphi} \rangle \\ &= -\langle T, \widehat{|x|^2\, \varphi} \rangle \\ &= -\langle \widehat{T}, |x|^2 \varphi \rangle. \end{aligned} \tag{1.21}$$

Let $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that ψ is identically 0 in a neighborhood of 0. Then $\psi(x)/|x|^2 = \varphi(x)$ is still an element of $\mathcal{S}(\mathbb{R}^n)$.

Then we deduce from (1.21) that for such a ψ we have $\langle \widehat{T}, \psi \rangle = 0$. In other words, the support of the Fourier transform of T is included in the origin. Applying Proposition 1.24 to $\widehat{T}$, we deduce the existence of $p \in \mathbb{N}$ and $C_\beta \in \mathbb{C}$ for any $\beta \in \mathbb{N}^n$ with $|\beta| \le p$ such that

$$\widehat{T} = \sum_{|\beta| \le p} c_\beta\, \partial^\beta\, \delta_0.$$

Using Proposition 1.26, we deduce that

$$T = \sum_{|\beta| \le p} \frac{C_\beta (-i)^{|\beta|}}{(2\pi)^{\frac{n}{2}}}\, x^\beta.$$

This implies the theorem. □

We shall now meet our first Calderón-Zygmund Kernel in this course.

The function $t \longmapsto \frac{1}{t}$ misses by "very little" to be an L^1 function. This is a measurable function which is only in the L^1-weak space (see the following chapters).

Nevertheless one can construct a tempered distribution out of $\frac{1}{t}$ that we shall denote $pv(\frac{1}{t})$ where pv stands for *principal value*. We proceed as follows. Observe that

$$\forall \varphi \in \mathcal{S}(\mathbb{R}) \quad \forall \varepsilon > 0 \quad \int_{|t|>\varepsilon} \left| \frac{\varphi(t)}{t} \right| dt < +\infty.$$

Moreover

$$\lim_{\varepsilon \to \infty} \int_{|t|>\varepsilon} \frac{\varphi(t)}{t}\, dt = \left\langle pv\left(\frac{1}{t}\right), \varphi \right\rangle \in \mathbb{C} \tag{1.22}$$

exists. Indeed, we write

$$\int_{|t|>\varepsilon} \frac{\varphi(t)}{t}\,dt = \int_{|t|>1} \frac{\varphi(t)}{t}\,dt + \int_{-1}^{-\varepsilon} \frac{\varphi(t)}{t}\,dt + \int_{\varepsilon}^{1} \frac{\varphi(t)}{t}\,dt.$$

Using the fact that $\frac{1}{t}$ is odd, we have also

$$\int_{|t|>\varepsilon} \frac{\varphi(t)}{t}\,dt = \int_{|t|>1} \frac{\varphi(t)}{t}\,dt + \int_{\varepsilon<|t|<1} \frac{\varphi(t)-\varphi(0)}{t}\,.$$

Since φ in particular is Lipschitz, we have that $\frac{\varphi(t)-\varphi(0)}{t}$ is uniformly bounded in L^∞ which justifies the passage to the limit (1.22). Moreover we obviously have

$$\begin{aligned} \Big|\Big\langle pv\big(\tfrac{1}{t}\big), \varphi\Big\rangle\Big| &\le c\big(\|t\,\varphi(t)\|_{L^\infty} + \|\varphi'\|_{L^\infty}\big) \\ &\le c\,\mathcal{N}_1(\varphi). \end{aligned}$$

This proves that $pv(\frac{1}{t}) \in \mathcal{S}'(\mathbb{R})$.

One can also without too much difficulty establish that the order of $pv(\frac{1}{t})$ is exactly 1.

We shall now compute the Fourier transform of $pv(\frac{1}{t})$. First, we claim that

$$t\,pv\big(\frac{1}{t}\big) = 1 \ \text{ in } \ \mathcal{S}'(\mathbb{R}), \tag{1.23}$$

where the product by t has to be understood in the sense given by Proposition 1.25. Indeed,

$$\begin{aligned} \forall \varphi \in \mathcal{S}(\mathbb{R}) \quad \Big\langle t pv\big(\frac{1}{t}\big), \varphi\Big\rangle &= \big\langle pv\big(\frac{1}{t}\big),\, t\varphi(t)\big\rangle \\ &= \lim_{\varepsilon\to 0} \int_{|t|>\varepsilon} \varphi(t)\,dt = \int_{\mathbb{R}} \varphi(t)\,dt. \end{aligned}$$

This proves (1.23). The computation above of the Fourier transform of 1 gives then

$$\mathcal{F}\Big(t\,pv\big(\frac{1}{t}\big)\Big) = (2\pi)^{\frac{1}{2}}\,\delta_0.$$

Using now Proposition 1.26, we have

$$\frac{d}{dt}\,\widehat{pv\big(\frac{1}{t}\big)} = -i\;\widehat{t\,pv\big(\frac{1}{t}\big)} = -i\,\sqrt{2\pi}\,\delta_0.$$

Let $H(t)$ be the *Heaviside function* equal to the characteristic function of $\mathbb{R}_+$. An elementary calculus gives

$$\frac{d}{dt}\,H(t) = \delta_0.$$

Hence

$$\frac{d}{dt}\,\Big[\,\widehat{pv\big(\frac{1}{t}\big)} + i\sqrt{2\pi}\,H(t)\Big] = 0. \tag{1.24}$$

We shall now need the following lemma:

Lemma 1.28. *Let T be an element of $\mathcal{S}'(\mathbb{R})$ such that*

$$\frac{d}{dt}\,T = 0,$$

then T is the multiplication by a constant.

Proof of Lemma 1.28. Let $\varphi \in \mathcal{S}(\mathbb{R})$. It is not difficult to prove that if $\int_{-\infty}^{+\infty} \varphi(t)\,dt = 0$, then $t \longmapsto \int_{-\infty}^{t} \varphi(s)ds$ is still a Schwartz function. Hence since $\frac{d}{dt}\int_{-\infty}^{t}\varphi(s)ds = \varphi(t)$, we have by assumption of the lemma $\forall \varphi \in \mathcal{S}(\mathbb{R})$ such that $\int_{-\infty}^{+\infty}\varphi(s)\,ds = 0$

$$\langle T, \varphi\rangle = 0.$$

Let $\varphi \in \mathcal{S}(\mathbb{R})$ arbitrary. We have then

$$\left\langle T,\, \varphi(t) - e^{t^2}\,\frac{\int_{-\infty}^{+\infty}\varphi(s)\,ds}{\int_{-\infty}^{+\infty} e^{-s^2}\,ds}\right\rangle = 0.$$

This gives

$$\langle T, \varphi\rangle = \int_{-\infty}^{+\infty} \frac{\langle T, e^{-t^2}\rangle}{\int_{-\infty}^{+\infty} e^{-s^2}\,ds}\,\varphi(t)\,dt.$$

Hence T is the multiplication by the constant $\frac{\langle T, e^{-t^2}\rangle}{\int_{-\infty}^{+\infty} e^{-s^2}\,ds}$. This concludes the proof of the lemma. $\square$

Combining (1.24) and lemma 1.19, we obtain that there exists a constant $A \in \mathbb{C}$ such that

$$\widehat{pv\big(\frac{1}{t}\big)} = -i\,\sqrt{2\pi}\,H(t) + A.$$

Observe that for any even function $\varphi(t)$, one has

$$\left\langle\, \widehat{pv\big(\frac{1}{t}\big)},\; \check{\varphi} \right\rangle = 0.$$

It is not difficult to prove that a Schwartz function is even if and only if it's Fourier transform is even too. Hence for any even function we have

$$\int_{-\infty}^{+\infty}\big(-i\sqrt{2\pi}\,H(t) + A\big)\,\varphi(t)\,dt = 0,$$

this implies that $-i\sqrt{2\pi}\,H(t) + A$ is odd and we have proved that

$$\widehat{pv\big(\frac{1}{t}\big)} = -\frac{i}{2}\,\sqrt{2\pi}\,\mathrm{sign}(t).$$

This function belongs to the family of Calderón-Zygmund multipliers that we are going to study more systematically in Chapter IV. The map

$$f \longrightarrow -\mathcal{F}^{-1}\left(\frac{i}{\sqrt{2\pi}}\,\mathrm{sign}(t)\hat{f}\right)$$

is called the *Hilbert transform* and is a first Calderon Zygmund convolution operator we are considering in this course.

1.5 Convolutions in $\mathcal{S}'(\mathbb{R}^n)$

Let φ and ψ be two Schwartz functions, we recall the classical definition of the convolution

$$\begin{aligned}\varphi * \psi(x) &:= \int_{\mathbb{R}} \varphi(x-y)\,\psi(y)\,dy \\ &= \int_{\mathbb{R}^n} \psi(x-y)\,\varphi(y) l dy.\end{aligned}$$

We have the following proposition

Proposition 1.29. *Let φ and ψ be two Schwartz Functions, then for any $p \in \mathbb{N}$*

$$\mathcal{N}_p(\varphi * \psi) \leq C_{p,n}\,\mathcal{N}_p(\varphi)\,\mathcal{N}_{p+n+1}(\psi).$$

and then $\varphi * \psi$ is also a Schwartz function.

Proof of Proposition 1.29.We have

$$\begin{aligned}\mathcal{N}_p(\varphi * \psi) = &\sup_{\substack{|\alpha| \leq p \\ |\beta| \leq p}} \left\| x^\beta \int_{\mathbb{R}^n} \varphi(x-y)\,\partial^\alpha \psi(y)\,dy \right\|_{L^\infty(\mathbb{R}^n)} \\ = &\sup_{\substack{|\alpha| \leq p \\ |\beta| \leq p}} \left\| \int_{\mathbb{R}^n} (x-y+y)^\beta \varphi(x-y)\,\partial^\alpha \psi(y)\,dy \right\|_{L^\infty(\mathbb{R}^n)}.\end{aligned}$$

Using the binomial formula, we obtain

$$\begin{aligned}\mathcal{N}_p(\varphi * \psi) \leq &\sup_{\substack{|\alpha| \leq p \\ |\beta| \leq p}} \sum_{\gamma \leq \beta} C_{\beta,\gamma} \left\| \int_{\mathbb{R}^n} |x-y|^{\beta-\gamma}\,|\varphi(x-y)|\,|y|^\gamma\,|\partial^\alpha \psi|(y)\,dy \right\|_{L^\infty(\mathbb{R}^n)} \\ &\leq C_p\,\mathcal{N}_p(\varphi) \int \sum_{\substack{|\alpha| \leq p \\ |\beta| \leq p}} |y|^\gamma\,|\partial^\alpha \psi|(y)\,dy \\ &\leq C_p\,\mathcal{N}_p(\varphi)\,\mathcal{N}_{p+n+1}(\psi).\end{aligned}$$

A classical computation gives for any φ, ψ and η in $\mathcal{S}(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \psi * \eta(x)\;\varphi(x)\;dx = \int_{\mathbb{R}} \psi(x-y)\,\eta(y)\,\varphi(x)\,dy\,dx = \int_{\mathbb{R}} \eta(y)\;\psi^\# * \varphi(y)\;dy$$

where $\psi^\#(z) := \psi(-z)$. Inspired by this elementary computation, we introduce the following definition.

Definition-Proposition 1.30. *Let T be an arbitrary element in $\mathcal{S}'(\mathbb{R}^n)$ and let ψ be an arbitrary element in $\mathcal{S}(\mathbb{R}^n)$. Define*

$$\forall\varphi \in \mathcal{S}(\mathbb{R}^n)\ \langle \psi * T, \varphi\rangle := \langle T, \psi^{\#} * \varphi\rangle,$$

$\psi * T$ is called the convolution between ψ and T. It defines an element of $\mathcal{S}'(\mathbb{R}^n)$.

Proof of Definition-Proposition 1.30. Let p be the order of T and $c > 0$ such that

$$\forall\eta \in \mathcal{S}(\mathbb{R}^n) \quad |\langle T, \eta\rangle| \le c\,\mathcal{N}_p(\eta).$$

$\psi * T$ acts obviously linearly on $\mathcal{S}(\mathbb{R}^n)$, moreover using Proposition 1.29, by definition we have

$$|\langle \psi * T, \varphi\rangle| \le c\,\mathcal{N}_p(\psi^{\#} * \varphi) \le c\, N_p(\psi)\, N_{p_{n+1}}(\varphi),$$

Using the characterization of the membership to $\mathcal{S}'(\mathbb{R}^n)$ given by Proposition 1.18, we obtain that $\psi * T \in \mathcal{S}'(\mathbb{R}^n)$ and Proposition 1.30 is proved. □

We recall from classical analysis the fact that

$$\forall\varphi, \psi \in \mathcal{S}(\mathbb{R}^n) \quad \forall\alpha \in \mathbb{N}^n\ \partial^\alpha(\varphi * \psi) = \varphi * \partial^\alpha\psi = \partial^\alpha * \psi.$$

Using this fact, one easily establishes the following proposition:

Proposition 1.31. *For any T in $\mathcal{S}'(\mathbb{R}^n)$ and any $\psi \in \mathcal{S}(\mathbb{R}^n)$, one has*

$$\forall\alpha \in \mathbb{N}^n\ \ \partial^\alpha(\psi * T) = \partial^\alpha\psi * T = \psi * \partial^\alpha T.$$

We denote

$$\mathcal{E}'(\mathbb{R}^n) = \{T \in \mathcal{S}'(\mathbb{R}^n);\ \text{supp}(T) \text{ is compact}\}.$$

In other words:

$$T \in \mathcal{E}'(\mathbb{R}^n) \Longleftrightarrow \exists\, K \text{ compact such that}$$

$\forall\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\varphi \equiv 0$ on an open set $U \supset K$, then

$$\langle T, \varphi\rangle \equiv 0.$$

We shall now prove the following proposition:

Proposition 1.32. *For any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and for any $T \in \mathcal{E}'(\mathbb{R}^n)$ the convolution $\varphi * T$ defines a Schwartz function.*

Proof of Proposition 1.32. First we prove that the tempered distribution $\varphi * T$ defines an L^∞ function. To that aim, using Riesz representation theorem, it suffices to establish the following inequality:

$$\exists\, C > 0 \ \text{ such that } \ \forall\,\psi \in \mathcal{S}(\mathbb{R}^n) \quad |\langle \varphi * T, \psi\rangle| \le C\,\|\psi\|_{L^1(\mathbb{R}^n)}. \tag{1.25a}$$

Since T is compactly supported, there exists $R \in \mathbb{R}_+^*$ such that for any ψ in $\mathcal{S}'(\mathbb{R}^n)$

$$
\begin{aligned}
|\langle \varphi * T, \psi\rangle| = |\langle T, \varphi^\# * \psi\rangle| &\leq C \sup_{\substack{|\alpha| \leq p \\ |\beta| \leq p}} \|x^\alpha\, \partial^\beta (\varphi^\# * \psi)\|_{L^\infty(B_R(0))} \\
&\leq C_p \sup_{|\beta|\leq p} \|\partial^\beta(\varphi^\#) * \psi\|_{L^\infty(B_R(0))} \\
&\leq C_p\, \mathcal{N}_p(\varphi^\#)\, \|\psi\|_{L^1(\mathbb{R}^n)},
\end{aligned}
$$

where we have used Young inequality: $L^\infty * L^1 \hookrightarrow L^\infty$. Hence (1.25) holds true and $\varphi * T$ is a measurable function. Now we write for any $\alpha \in \mathbb{N}^n$ and $\beta \in \mathbb{N}^n$

$$
\begin{aligned}
\|x^\alpha\, \partial^\beta(\varphi * T)\|_{L^\infty(\mathbb{R}^n)} &= \sup_{\substack{\psi \in \mathcal{S}(\mathbb{R}^n) \\ \|\psi\|_{L^1(\mathbb{R}^n)} \leq 1}} \left| \int \partial^\beta(x^\alpha \psi)\, \varphi * T\, dx \right| \\
&= \sup_{\substack{\psi \in \mathcal{S}(\mathbb{R}^n) \\ L^\infty(\mathbb{R}^n)\, \|\psi\|_{L^1(\mathbb{R}^n)} \leq 1}} |\langle T, (\partial^\beta \varphi^*) * x^\alpha \psi\rangle| \\
&\leq \sup_{\substack{|\gamma| \leq p \\ |\delta| \leq p}} \sup_{\|\psi\|_{L^1(\mathbb{R}^n)} \leq 1} \left\| x^\gamma \partial^\delta \int_{\mathbb{R}^n)} \partial^\beta \varphi(y-x)\, y^\alpha \psi(y)\, dy \right\|_{L^\infty(B_R(0))} \\
&\leq C_{p,R} \sup_{|\delta|\leq p} \sup_{\|\psi\|_{L^1(\mathbb{R}^n)} \leq 1} \left\| \int_{\mathbb{R}^n} \partial^{\beta+\delta} \varphi(y-x)(y-x+x)^\alpha \psi(y)\, dy \right\|_{L^\infty(B_R(0))}.
\end{aligned}
$$

Using the binomial formula, we finally obtain

$$
\|x^\alpha \partial^\beta(\varphi * T)\|_{L^\infty(\mathbb{R}^n)} \leq C_{p,R}\, \mathcal{N}_{p+|\beta|+|\alpha|}(\varphi),
$$

and this concludes the proof of Proposition 1.32. □

We shall now extend the definition of convolution on $\mathcal{E}' \times \mathcal{S}'$. We have the following definition-proposition:

Definition-Proposition 1.33. *Let $T \in \mathcal{S}'(\mathbb{R}^n)$ and $U \in \mathcal{E}'(\mathbb{R}^n)$, then we define*

$$
\forall \varphi \in \mathcal{S}(\mathbb{R}^n) \quad \langle U * T, \varphi\rangle := \langle T, U^\# * \varphi\rangle,
$$

where

$$
\forall \psi \in \mathcal{S}(\mathbb{R}^n) \quad \langle U^\#, \psi\rangle := \langle U, \psi^\#\rangle.
$$

The convolution between U and T, $U * T$, defines an element of $\mathcal{S}'(\mathbb{R}^n)$.

The proof of Proposition 1.33 follows from the proof of Proposition 1.32, where we have established that for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and any $T \in \mathcal{S}'(\mathbb{R}^n)$ of order p one has

$$\forall q \in \mathbb{N} \quad \mathcal{N}_q(\varphi * T) \leq C_{q,p}\,\mathcal{N}_{p+2q}(\varphi).$$

The convolution operation extends in fact to a larger subspace of $\mathcal{E}' \times \mathcal{S}'$ in $\mathcal{S}' \times \mathcal{S}'$. It extends to the pairs of tempered distributions with *convolutive supports* but we shall not explore this notion in this course.

Finally, we establish the following proposition:

Proposition 1.34. *Let T be a tempered distribution, then*

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^n) \quad \widehat{\varphi * T} = (2\pi)^{\frac{n}{2}}\,\widehat{\varphi}\,\widehat{T}. \tag{1.25b}$$

For any $U \in \mathcal{E}'(\mathbb{R}^n)$ we have $\widehat{U} \in \mathcal{O}(\mathbb{R}^n)$ and

$$\widehat{U * T} = (2\pi)^{\frac{n}{2}}\,\widehat{U}\;\widehat{T}. \tag{1.25c}$$

Proof of Proposition 1.34. We have

$$\begin{aligned}\forall \psi \in \mathcal{S}(\mathbb{R}^n) \quad \langle \widehat{\varphi * T}, \psi \rangle &= \langle \varphi * T, \widehat{\psi} \rangle \\ &= \langle T, \varphi^{\#} * \widehat{\psi} \rangle.\end{aligned} \tag{1.26}$$

Observe moreover that

$$\begin{aligned}\varphi^{\#} * \widehat{\psi}(x) &= \int_{\mathbb{R}^n} \varphi(y-x)\,\widehat{\psi}(y)\,dy \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \varphi(y-x) \int_{\mathbb{R}^n} e^{-iy\cdot\xi}\,\psi(\xi)\,d\xi\,dy \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-iy\cdot\xi}\,\psi(\xi) \int_{\mathbb{R}^n} \varphi(z)\,e^{-iy\cdot\xi}\,dz \\ &= \int_{\mathbb{R}^n} e^{-iy\cdot\xi}\;\psi(\xi)\;\widehat{\varphi}(\xi) = (2\pi)^{\frac{n}{2}}\,\mathcal{F}(\psi\,\widehat{\varphi}).\end{aligned} \tag{1.27}$$

Combining (1.26) and (1.27) gives

$$\langle \widehat{\varphi * T}, \psi \rangle = (2\pi)^{\frac{n}{2}}\,\langle T, \widehat{\psi\,\widehat{\varphi}} \rangle.$$

This implies (1.25b).

We claim now that for any $U \in \mathcal{E}'(\mathbb{R}^n)$ there exists $q \in \mathbb{N}$ such that

$$\frac{\widehat{U}(\xi)}{(1+|\xi|^2)^q} \in L^\infty(\mathbb{R}^n) \tag{1.28}$$

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\begin{aligned}
\left|\left\langle \frac{\widehat{U}(\xi)}{(1+|\xi|^2)^q}\,,\ \varphi(\xi)\right\rangle\right| &= \left|\left\langle U, \widehat{\frac{\varphi(\xi)}{(1+|\xi|^2)^q}}\right\rangle\right| \\
&\leq C \sup_{|\beta|\leq p} \left\|\partial^\beta\left(\widehat{\frac{\varphi(\xi)}{(1+|\xi|^2)^q}}\right)\right\|_{L^\infty(B_R(0))} \\
&\leq C \sup_{|\beta|\leq p} \left\|(i)^{|\beta|}\ \widehat{\frac{\xi^\beta\,\varphi(\xi)}{(1+|\xi|^2)^q}}\right\|_{L^\infty(\mathbb{R}^n)} \\
&\leq C \int_{\mathbb{R}^n} \frac{|\xi|^p+1}{(1+|\xi|^2)^q}\ |\varphi(\xi)|\, d\xi.
\end{aligned}$$

Hence by taking $2q > p$, we have

$$\sup_{\|\varphi\|_{L^1(\mathbb{R}^n)}} \left|\left\langle \frac{\widehat{U}(\xi)}{1+|\xi|^2)^q}\,, \varphi(\xi)\right\rangle\right| < +\infty.$$

Using Riesz representation theorem, we obtain (1.28).

Obviously, $U \in \mathcal{E}'(\mathbb{R}^n) \Longrightarrow \partial^\alpha U \in \mathcal{E}'(\mathbb{R}^n)$ for any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. Hence applying (1.28) to the successive derivative of U, we obtain that $\widehat{U} \in \mathcal{O}(\mathbb{R}^n)$.

The inequality (1.25c) is obtained by combining (1.25b) and the fact that $\check{U}^\# = \widehat{U}$. Indeed we have

$$\langle \widehat{U} * T, \varphi\rangle = \langle U * T, \widehat{\varphi}\rangle = \langle T, U^\# * \widehat{\varphi}\rangle = \langle \widehat{T}, U^* * \widehat{\varphi}\rangle = (2\pi)^{\frac{n}{2}}\, \langle \widehat{T}, \check{U}^*\,\varphi\rangle = (2\pi)^{\frac{n}{2}}\, \langle \widehat{T}\,\widehat{U}, \varphi\rangle.$$

This concludes the proof of Proposition 1.34.

2 The Hardy-Littlewood Maximal Function

2.1 Definition and elementary properties.

The Lebesgue measure on $\mathbb{R}^n$ will be denoted by μ. By *measurable function* or *measurable set* in this book we implicitly mean *measurable function with respect to* μ or *measurable set with respect to* μ unless we precise the underlying measure. Integration along a variable x in $\mathbb{R}^n$ with respect to the Lebesgue measure on $\mathbb{R}^n$ will be simply denoted by dx.

If E is a measurable set, we denote by χ_E it's characteristic function.

Definition 2.1. *For a measurable function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, we define its associated distribution function by*

$$d_f(\alpha) = \mu(\{x \in \mathbb{R}^n \ : \ |f(x)| > \alpha\}) \,,$$

with $\alpha \geq 0$.

With these notations we establish the following lemma.

Lemma 2.2. *For a measurable function f and $0 < p < \infty$, we have*

$$\|f\|^p_{L^p} = p \int_0^\infty \alpha^{p-1} d_f(\alpha)\, d\alpha \,. \tag{2.1}$$

Proof of lemma 2.2. From elementary calculus, we get

$$|f(x)|^p = p \int_0^{|f(x)|} \alpha^{p-1}\, d\alpha = p \int_0^\infty \alpha^{p-1} \chi_{\{x\,:\,\alpha<|f(x)|\}}\, d\alpha \,.$$

By integration over $\mathbb{R}^n$ and Fubini's theorem, it then follows

$$\|f\|^p_{L^p} = p \int_0^\infty \alpha^{p-1} \left(\int_{\mathbb{R}^n} \chi_{\{x\,:\,|f(x)|>\alpha\}}\, dx \right) d\alpha = p \int_0^\infty \alpha^{p-1} d_f(\alpha)\, d\alpha \,.$$

$\square$

For every x in $\mathbb{R}^n$ and every $r > 0$ we denote by $B_r(x)$ the euclidian ball of center x and radius r.

Definition 2.3. *For a locally integrable function $f \in L^1_{loc}(\mathbb{R}^n)$, we define its associated Hardy-Littlewood maximal function at the point x by*

$$Mf(x) = \sup_{r>0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y)|\, dy \in \mathbb{R}_+ \cup \{+\infty\} \quad . \tag{2.2}$$

We now prove the following elementary proposition.

Proposition 2.4. *Let f be a locally integrable function, then Mf is measurable function into $[0, +\infty]$. Moreover, if $f \in L^1(\mathbb{R}^n)$ then $Mf(x)$ is finite almost everywhere.*

Proof of Proposition 2.4. For any measurable function in L^1_{loc} one easily check that the map

$$(r,x) \longrightarrow A_r f(x) = \frac{1}{\mu(B_r(x))}\int_{B_r(x)} |f(x)|\, dy$$

is continuous. It implies in one hand that, for a fixed r, $A_r f(x)$ is measurable and it also implies, in the other hand, that taking the supremum at a point x among the real radii, $r \in \mathbb{R}$, coincide with the supremum among rational radii, $r \in \mathbb{Q}$. Since the supremum function of countably many measurable functions is measurable (1.1.2 in [?]), we deduce that $Mf(x)$ is measurable. The second part of the statement in proposition 2.4 is a direct consequence of Lebesgue-Besicovitch differentiation theorem (1.7.1 in [?]). It also follows from Theorem 2.5 below. □

From the Lebesgue-Besicovitch differentiation theorem (1.7.1 in [?]) we deduce the pointwise estimate $|f(x)| \le |Mf(x)|$ which holds almost everywhere for any locally integrable function. Therefore, for every $p \in [1, +\infty]$, and for every function f in $L^p(\mathbb{R}^n)$, we obtain the identity

$$\|f\|_{L^p(\mathbb{R}^n)} \le \|Mf\|_{L^p(\mathbb{R}^n)} \quad .$$

2.2 Hardy-Littlewood L^p-theorem for the Maximal Function.

The following important result gives the reverse estimate when $p > 1$ and "almost" but not quite the reverse estimate when $p = 1$.

Theorem 2.5 (Hardy-Littlewood Maximal Function Theorem)**.** *Let $1 < p \le \infty$ and $f \in L^p(\mathbb{R}^n)$. Then, we have*

$$\|Mf\|_{L^p} \le 2\left(\frac{5^n\, p}{p-1}\right)^{1/p} \|f\|_{L^p} \quad . \tag{2.3}$$

Moreover, for $f \in L^1(\mathbb{R}^n)$ and $\alpha > 0$, we have

$$\mu(\{x\ :\ Mf(x) > \alpha\}) \le \frac{5^n}{\alpha}\, \|f\|_{L^1} \quad . \tag{2.4}$$

Remark 2.1. *The last identity (2.4) is saying that the maximal function of an L^1 function is in the space L^1-weak (denoted also $L^1_w(\mathbb{R}^n)$). This space is given by the subset of measurable functions on $\mathbb{R}^n$satisfying*

$$|f|_{L^1_w} = \sup_{\alpha>0}\{\alpha\ \mu(\{x \in \mathbb{R}^n\ :\ |f(x)| > \alpha\})\} \quad . \tag{2.5}$$

L^1-weak functions do not define a-priori distributions. A typical example of a function in L^1_w is $|x|^{-n}$ in $\mathbb{R}^n$. $|\cdot|_{L^1_w}$ defines a quasi-norm on L^1_w - the triangle inequality is satisfied modulo a constant, which is 2 in the present case - and L^1_w is complete for this quasi-norm which makes L^1_w to be a quasi-Banach space by definition. However it is very important to remember that L^1_w cannot be made to be a Banach space with a norm equivalent to the quasi norm given by $|\cdot|_{L^1_w}$. If it would be the case Calderón-Zygmund theory, and this book in particular, would dramatically shrink to almost nothing ! We discuss this fact later in this chapter when we come to the Singular Integral Operators.

The proof of the Hardy-Littlewood Maximal Function Theorem that we are giving uses the following famous covering lemma.

Lemma 2.6 (Vitali's Covering Lemma)**.** *Let E be measurable subset of $\mathbb{R}^n$ and let $\mathcal{F} = \{B_j\}_{j\in J}$ be a family of euclidian balls with uniformly bounded diameter i.e.,* $\sup_j$ *diam*$(B_j) = R < \infty$*, such that $E \subset \bigcup_j B_j$. Then, there exists an at most countable subfamily $\{B_{j_k}\}_{k\in\mathbb{N}}$ of disjoint balls satisfying*

$$\mu(E) \leq 5^n \sum_{k=1}^{\infty} \mu(B_{j_k})\,. \tag{2.6}$$

Proof of lemma 2.6. For any $i \in \mathbb{N}$ we denote

$$\mathcal{F}_i = \left\{B_j \in \mathcal{F}\ ;\ 2^{-i-1}R < \text{diam } B_j \leq 2^{-i}R\right\} \quad .$$

We shall now extract our sub-covering step by step in $\mathcal{F}_i$ by induction on i.

- Denote by $\mathcal{G}_0$ a maximal disjoint collection of balls in $\mathcal{F}_0$.
- Assuming $\mathcal{G}_0, \cdots, \mathcal{G}_k$ have been selected, we choose $\mathcal{G}_{k+1}$ to be a maximal collection of balls in $\mathcal{F}_{k+1}$ such that each ball in this collection is disjoint from the balls in $\cup_{i=0}^k \mathcal{G}_i$.

We claim now that $\mathcal{G} = \cup_{i=0}^{\infty}\mathcal{G}_i$ is a suitable solution to the lemma.

It is by construction a sub-family of $\mathcal{F}$ made of disjoint balls. Let B_j be in $\mathcal{F}$. There exists $i \in \mathbb{N}$ such that $B_j \in \mathcal{F}_i$. If B_j would intersect none of the balls in $\mathcal{G}_i$ it would contradict the fact that $\mathcal{G}_i$ has been chosen to be maximal. Hence, for any $B_j \in \mathcal{F}_i$ there exist $B \in \mathcal{G}_i$ such that $B \cap B_j \neq \emptyset$. Since the ratio between the two diameters of respectively B and B_j is contained in $(2^{-1}, 2)$, the concentric ball $\widehat{B}$ to B having a radius 5 times larger than the one of B contains necessarily B_j. This proves that $E \subset \cup_{B\in\mathcal{G}}\widehat{B}$ and this finishes the proof of the lemma. □

Proof of theorem 2.5. We first consider the case $p = 1$ and prove (2.4). Let

$$E_\alpha = \{x \in \mathbb{R}^n\ ;\ Mf(x) > \alpha\} \quad .$$

By definition, for any $x \in E_\alpha$ there exists an euclidian ball B_x of center x such that

$$\int_{B_x} |f(y)|\ dy > \alpha\mu(B_x) \quad . \tag{2.7}$$

Since f is assumed to be in L^1, the size of the balls B_x is controlled as follows : $\mu(B_x) \leq \alpha^{-1}\|f\|_{L^1}$. Hence the family $\{B_x\}_{x\in E_\alpha}$ realizes a covering of E_α by balls of uniformly bounded radii. We are then in the position to apply Vitali's covering lemma 2.6. Let $(B_k)_{k\in K}$ be an at most countable sub-family to $\{B_x\}$ given by this lemma 2.6. (B_k) are disjoint balls satisfying

$$\sum_{k\in K} \mu(B_k) \geq \frac{1}{5^n}\mu(E_\alpha) \quad .$$

Combining this last inequality and (2.7) gives

$$\|f\|_{L^1(\mathbb{R}^n)} \geq \int_{\cup_{k\in K}^{+\infty} B_k} |f(y)| \; dy > \alpha \sum_{k\in K} \mu(B_k) \geq \frac{\alpha}{5^n}\mu(E_\alpha) \quad .$$

This is proves the desired inequality (2.4).

We establish now (2.3) for $1 < p < +\infty$ (the case $p = +\infty$ being straightforward). Define

$$f_1(x) := \begin{cases} f(x) & \text{if} \quad |f(x)| \geq \alpha/2 \\ 0 & \text{if} \quad |f(x)| < \alpha/2 \; . \end{cases}$$

This definition implies the following inequalities $|f(x)| \leq |f_1(x)| + \alpha/2$ and also $|Mf(x)| \leq |Mf_1(x)| + \alpha/2$ which hold for almost every $x \in \mathbb{R}^n$. Hence we have

$$E_\alpha = \{x \in \mathbb{R}^n \; ; \; Mf(x) > \alpha\} \subset \{x \in \mathbb{R}^n \; ; \; Mf_1(x) > \alpha/2\} \quad . \tag{2.8}$$

Observe that, for any $\alpha > 0$, $f_1 \in L^1(\mathbb{R}^n)$. Indeed

$$\int_{\mathbb{R}^n} |f_1(y)| \; dy \leq \left(\frac{2}{\alpha}\right)^{p-1} \int_{\mathbb{R}^n} |f(y)|^p \; dy < +\infty \quad .$$

Thus we can apply identity (2.4) to f_1 and this gives, using (2.8),

$$\begin{aligned} \mu(E_\alpha) &\leq \mu\left(\{x \in \mathbb{R}^n \; ; \; Mf_1(x) > \alpha/2\}\right) \leq \tfrac{2\cdot 5^n}{\alpha} \|f_1\|_{L^1} \\ &\leq \tfrac{2\cdot 5^n}{\alpha} \int_{\{x \; : \; |f(x)| \geq \alpha/2\}} |f(y)| \; dy \quad . \end{aligned} \tag{2.9}$$

Next, we deduce from Lemma 2.2 that

$$\begin{aligned} \|Mf\|_{L^p}^p &= p \int_0^\infty \alpha^{p-1} \mu(E_\alpha) \, d\alpha \\ &\overset{(2.9)}{\leq} p \int_0^\infty \alpha^{p-1} \left(\frac{2\cdot 5^n}{\alpha} \int_{\{x\,:\,|f(x)|\geq\alpha/2\}} |f(x)| \, dx \right) d\alpha \\ &= p \int_0^\infty \alpha^{p-1} \left(\frac{2\cdot 5^n}{\alpha} \int_{\mathbb{R}^n} \chi_{\{x\,:\,|f(x)|\geq\alpha/2\}} |f(x)| \, dx \right) d\alpha \, . \end{aligned}$$

Using Fubini's theorem it follows

$$\begin{aligned} \|Mf\|_{L^p}^p &\leq 2\cdot 5^n \, p \int_{\mathbb{R}^n} |f(x)| \left(\int_0^{2|f(x)|} \frac{\alpha^{p-1}}{\alpha} \, d\alpha \right) dx \\ &= \frac{2C\,p}{p-1} \int_{\mathbb{R}^n} |f(x)| \, 2^{p-1} |f(x)|^{p-1} \, dx \quad , \end{aligned}$$

since $p > 1$ by assumption. Thus we arrive at the desired result

$$\|Mf\|_{L^p} \leq 2 \left(\frac{5^n \, p}{p-1}\right)^{1/p} \|f\|_{L^p} \quad .$$

$\square$

Remark 2.7. *The best constant in the previous theorem, both in* (2.3) *and in* (2.4)*, is far from being known. For* $1 < p < \infty$*, a remarkable result by Stein is that the optimal constant stays bounded as n goes to infinity. Whether this holds or not for the optimal constant in* (2.4) *is still an open problem. However, one can easily replace* 5^n *with* 2^n*. Indeed, observe that the constant* 5 *in Vitali's covering theorem can be replaced with* $3+3\epsilon$ *for every* $\epsilon > 0$ *(just using* $(1+\epsilon)$ *in place of* 2 *when comparing the radii of the balls). Moreover, here we are interested in a disjoint family of balls whose dilations cover just the set of* centers *of the original family: this allows to replace* 5^n *with* $(2+2\epsilon)^n$ *for every* ϵ*.*

2.3 The limiting case $p = 1$.

It is important to emphasize that inequality (2.3) does not extend to the limiting case $p = 1$: the maximal operator M is not bounded from $L^1(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$. Assume f is a non zero integrable function on $\mathbb{R}^n$ then Mf is not integrable on $\mathbb{R}^n$. Indeed, for a non zero f there exists an euclidian ball $B_r(0)$ such that

$$\int_{B_r(0)} |f(y)|\, dy = \eta \neq 0 \quad .$$

Let x be an arbitrary point in $\mathbb{R}^n \setminus B_r(0)$. For such a point x one has $B_r(0) \subset B_{2|x|}(x)$, hence, it follows that

$$\begin{aligned} Mf(x) &= \sup_{r>0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f(y)|\, dy \\ &\geq \frac{1}{\mu(B_{2|x|}(x))} \int_{B_{2|x|}(x)} |f(y)|\, dy \\ &\geq \frac{1}{\mu(B_{2|x|}(x))} \int_{B_r(0)} |f(y)|\, dy \geq \frac{C\,\eta}{|x|^n}, \end{aligned}$$

showing that the integrability of Mf fails at infinity.

Even worth, the integrability of the function f does not ensure the local integrability of Mf. We illustrate this fact by the following example: For $n = 1$ consider the positive function

$$f(t) = \frac{1}{t(\log t)^2} \chi_{(0,1)},$$

which is integrable on $[0, 1/2]$.For $t \in (0, 1/2)$, let $B_t(t) = (0, 2t)$ and we have

$$\begin{aligned} Mf(t) &\geq \frac{1}{2t} \int_0^{2t} \frac{1}{t(\log t)^2}\, dt \\ &= \frac{1}{2t} \left(-\frac{1}{\log t} \right) \Bigg|_0^{2t} = -\frac{1}{2t(\log 2t)}. \end{aligned}$$

This directly gives that Mf is not integrable over the interval $[0, 1/2]$.

If we assume "slightly" more than the integrability of f one can reach the local integrability of Mf. Denote by $L^1\,\log L^1(\mathbb{R}^n)$ the following Orlicz space

$$L^1\,\log L^1(\mathbb{R}^n)=\left\{f\in L^1(\mathbb{R}^n)\quad;\quad\int_{\mathbb{R}^n}|f|(y)\,\log\left(e+\frac{|f(y)|}{\|f\|_{L^1}}\right)\,dy<+\infty\right\}\quad.$$

This space is of particular interest for applications due to the fact in particular that the $L^1\log L^1$ control of a non-negative integrable function f can be interpreted as an "entropy control" of the probability f - assuming it has been normalized in such a way that $\int_{\mathbb{R}^n}f=1$ -. Back to real-variable function space theory *per se*, we shall probably see in the next chapter that $L^1\log L^1$ coincide with the non-homogeneous Hardy space for non-negative functions which makes also $L^1\log L^1$ particularly interesting.

Observe that a norm can be assigned to this subspace of integrable functions by taking the Luxembourg norm :

$$\|f\|_{L^1\,\log L^1}:=\|f\|_{L^1}+\inf\left\{t>0\,;\,\int_{\mathbb{R}^n}\frac{|f(y)|}{t}\,\log^+\frac{|f(y)|}{t}\,dy\right\}\quad.$$

Theorem 2.8. *Let f be a measurable function in $L^1\log L^1(\mathbb{R}^n)$, then $Mf\in L^1_{loc}(\mathbb{R}^n)$ and for any measurable subset A of finite Lebesgue measure the following inequality holds*

$$\int_A|Mf|(y)\,dy\le C_n\int_{\mathbb{R}^n}|f|(y)\,\log\left(e+\mu(A)\,\frac{|f(y)|}{\|f\|_{L^1}}\right)\,dy\quad,\tag{2.10}$$

where $C_n>0$ only depends on n.

Proof of theorem 2.8. From lemma 2.2 we express the L^1 norm of Mf as follows

$$\int_A|Mf|(y)\,dy\le\int_0^{+\infty}\mu\left(\{x\in A;\ |Mf|(x)>\alpha\}\right)\,d\alpha\quad.$$

Denote μ^A the restriction of the Lebesgue measure to A and use again the notation $E_\alpha=\{x\in\mathbb{R}^n;\ |Mf|(x)>\alpha\}$. Let $\delta>0$ to be chosen later on. We write

$$\begin{aligned}\int_A|Mf|(y)\,dy&\le\int_0^\delta\mu^A(E_\alpha)\,d\alpha+\int_\delta^{+\infty}\mu^A(E_\alpha)\,d\alpha\\&\le\delta\,\mu(A)+2\int_{\delta/2}^{+\infty}\mu(E_{2\alpha})\,d\alpha\quad.\end{aligned}\tag{2.11}$$

Applying inequality (2.9) to (2.11) gives

$$\begin{aligned}\int_A|Mf|(y)\,dy&\le\delta\,\mu(A)+2\cdot5^n\int_{\delta/2}^{+\infty}\frac{d\alpha}{\alpha}\int_{\{x\,;\,|f(x)|>\alpha\}}|f(y)|\,dy\\&\le\delta\,\mu(A)+2\cdot5^n\int_{\mathbb{R}^n}|f(y)|\,\log^+\frac{2|f(y)|}{\delta}\,dy\quad,\end{aligned}$$

where $\log^+\cdot=\max\{0,\log\cdot\}$. Choosing $\delta=\int_{\mathbb{R}^n}|f(y)|\,dy/2\mu(A)$ gives inequality (2.10) and theorem 2.8 is proved. □

A converse of theorem 2.8 will be given in the section IV - see theorem 4.4 - once we will have at our disposal the Calderón-Zygmund decomposition.

3 Quasi-normed vector spaces

3.1 The Metrizability of quasi-normed vector spaces

In the following, $\mathbb{K}$ will denote either $\mathbb{R}$ or $\mathbb{C}$ (since the theory below works equally well for real or complex coefficients).

Definition 3.1. *A* topological vector space *over $\mathbb{K}$ is a $\mathbb{K}$-vector space V with a topology τ such that*

- *the sum, i.e. $+ : V \times V \to V$, is continuous,*
- *the multiplication by scalar, i.e. $\cdot : \mathbb{K} \times V \to V$, is continuous,*
- *the topology τ is Hausdorff.*

Example 3.2. *A normed vector space $(V, \| \ \|)$ is a topological vector space with the topology induced by the canonical distance, namely $d(x, y) := \|x - y\|$.*

Definition 3.3. *Let V be a $\mathbb{K}$-vector space. A* quasi-norm *on V is a function $| \cdot | : V \to [0, \infty)$ such that*

- *$|x| = 0$ if and only if $x = 0$,*
- *for all $\lambda \in \mathbb{K}$ and all $x \in V$ we have $|\lambda x| = |\lambda||x|$,*
- *there exists a constant $C \geq 1$ such that, for all $x, y \in V$, we have*

$$|x + y| \leq C(|x| + |y|).$$

The couple $(V, | \ |)$ is called a quasi-normed vector space.

Remark 3.4. *For $C = 1$ this is exactly the definition of a norm. In general, we use the notation $| \cdot |$ in place of $\| \cdot \|$ to recall that we are in presence of a quasi-norm. Notice that the last property in the definition, which replaces the usual triangle inequality, does not allow to say that the function $d(x, y) := |x - y|$ is a distance any longer! Nonetheless, we will see that a quasi-norm induces a canonical topology and that this topology is always metrizable (by means of a highly nontrivial construction of a true distance function d).*

Example 3.5. *Given $f : \mathbb{R}^n \to \mathbb{K}$ measurable, let $|f|_{L^{1,\infty}} := \sup_{\alpha>0} \alpha\mu\{|f| > \alpha\}$ and let $L^{1,\infty}(\mathbb{R}^n)$ be the set of all functions f such that $|f|_{L^{1,\infty}} < \infty$. Notice that, by Chebyshev–Markov inequality, $L^1(\mathbb{R}^n) \subseteq L^{1,\infty}(\mathbb{R}^n)$ and $|f|_{L^{1,\infty}} \leq \|f\|_{L^1}$. Also, $|\cdot|_{L^{1,\infty}}$ is a quasi-norm (with $C = 2$): given two functions $F, g : \mathbb{R}^n \to \mathbb{K}$, for any $\alpha > 0$ we have*

$$\mu(\{|f + g| > \alpha\}) \leq \mu\Big(\Big\{|f| > \frac{\alpha}{2}\Big\}\Big) + \mu\Big(\Big\{|g| > \frac{\alpha}{2}\Big\}\Big) \leq 2|f|_{L^{1,\infty}} + 2|g|_{L^{1,\infty}}$$

(since $\{|f + g| > \alpha\} \subseteq \{|f| > \frac{\alpha}{2}\} \cup \{|g| > \frac{\alpha}{2}\}$). Hence, $|f + g|_{L^{1,\infty}} \leq 2|f|_{L^{1,\infty}} + 2|g|_{L^{1,\infty}}$. The second requirement in the definition is satisfied since, for $\lambda \neq 0$, $\alpha\mu(\{|\lambda f| > \alpha\}) = |\lambda|\frac{\alpha}{|\lambda|}\mu(\{|f| > \frac{\alpha}{|\lambda|}\})$, while the first one is trivial.

In terms of this quasi-norm, Hardy–Littlewood maximal inequality (for $p = 1$) says that $|f|_{L^{1,\infty}} \leq 5^n\|f\|_{L^1}$.

Theorem 3.6. *A quasi-normed vector space $(V, |\ |)$ has a unique vector space topology such that*

$$B_\alpha(0) := \{x \in V : |x| < \alpha\}, \quad \alpha > 0$$

is a local basis of neighborhoods of 0.

The above requirement should be compared with the situation of a normed vector space, where $B_\alpha(0)$ is the standard ball of radius α and center 0. Notice that the theorem is not asserting that $B_\alpha(0)$ is an open set in this canonical topology (which could be false in general)!

Proof of Theorem 3.6 If such a topology τ exists, then the sets

$$B_\alpha(y) := \{x \in V : |y - x| < \alpha\}, \quad \alpha > 0$$

form a local basis of neighborhoods of y for any $y \in V$: this is because the translation by y, namely the map $x \mapsto x + y$, is continuous and has continuous inverse $x \mapsto x - y$ (with respect to τ), hence it is a homeomorphism and carries a local basis of neighborhoods of 0 into a local basis at y. So the open sets of τ must be the sets

$$U \subseteq V \text{ such that } \forall y \in U\ \exists \alpha > 0 \text{ s.t. } B_\alpha(y) \subseteq U. \tag{3.12}$$

This shows that, if τ exists, it is necessarily unique. To show existence, let us declare that the open sets are the ones satisfying (3.12). They define a topology, since the axioms for a topology are clearly satisfied. Let us check that the sets $B_\alpha(0)$ form a local basis at 0: since every open set contains one such set by definition, it suffices to check that $B_\alpha(0)$ includes an open set U containing 0. Let

$$U := \{x \in V : \exists \delta > 0 \text{ s.t. } B_\delta(x) \subseteq B_\alpha(0).$$

Clearly, $0 \in U$ and $U \subseteq B_\alpha(0)$. In order to show that U satisfies 3.12, given $x \in U$ let $\delta > 0$ such that $B_\delta(x) \subseteq B_\alpha(0)$. We claim that $B_\sigma(x) \subseteq U$, with $\sigma := \frac{\delta}{2C}$ (which will conclude the proof that U is open in τ).

Indeed, if $y \in B_\sigma(x)$ then $B_\sigma(y) \subseteq B_\delta(x) \subseteq B_\alpha(x)$, since

$$|z - x| \le C(|z - y| + |y - x|) < 2C\sigma = \delta \quad \text{for all } z \in B_\sigma(y).$$

This shows that $y \in U$ (by definition of U), i.e. that $B_\sigma(x) \subseteq U$, which is what we wanted. In order to show that τ is Hausdorff, given $x \neq y$ it suffices to observe that $B_\alpha(x) \cap B_\alpha(y) = \emptyset$, where $\alpha := \frac{|x-y|}{2} > 0$: indeed, we just proved that $B_\alpha(x)$ and $B_\alpha(y)$ are neighborhoods of x and y respectively (being τ clearly translation invariant).

Finally, we have to check that the operations are continuous. If $x + y = z$ and U is an open neighborhood of z, then there exists $\alpha > 0$ such that $B_{2C\alpha}(z) \subseteq U$. Hence, given $x' \in B_\alpha(x)$ and $y' \in B_\alpha(y)$, we have

$$|x' + y' - z| = |(x' - x) + (y' - y)| \le C\,|x' - x| + C\,|y' - y| < 2C\alpha,$$

so that the sum maps $B_\alpha(x) \times B_\alpha(y)$ to a subset of U. Since $B_\alpha(x)$ and $B_\alpha(y)$ contain open neighborhoods of x and y respectively, this shows that the sum is continuous. The continuity of the multiplication by scalar is similar and is left to the reader. $\square$

The metrizability of quasi-normed vector spaces was proved independently by Aoki and Rolewicz.

Theorem 3.7. *(Aoki-Rolewicz)*

The canonical topology of a quasi-normed vector space $(V, |\ |)$ is metrizable. In fact, it is induced by a translation-invariant distance $d(x, y) := \Lambda(x - y)$, for a suitable function $\Lambda : V \to [0, \infty)$ satisfying $\Lambda(z) = \Lambda(-z)$, $\Lambda(z + w) \le \Lambda(z) + \Lambda(w)$ and vanishing only at 0.

Remark 3.8. *In general, one cannot hope to have a distance induced by a norm (meaning that Λ is a norm, i.e. it also satisfies $\Lambda(\alpha x) = |\alpha| \Lambda(x)$ for $\alpha \in \mathbb{K}$): in this case (V, τ) would be a locally convex topological vector space, but we will see in Remark 3.17 that this fails for $L^{1,\infty}(\mathbb{R}^n)$.*

We will deduce Aoki–Rolewicz theorem from the following lemma.

Lemma 3.9. *Let $0 < p \le 1$ be defined by $2^{1/p} := 2C$. Given $x_1, \dots, x_n \in V$ we have*

$$|x_1 + \cdots + x_n|^p \le 4(|x_1|^p + \cdots + |x_n|^p).$$

Proof of Lemma 3.9. This proof illustrate the utility of decomposing dyadically a range of values. This idea will turn out to be fruitful also later in the course. Define $H : V \to [0, \infty)$ by the following formula:

$$H(x) := \begin{cases} 0 & \text{if } x = 0 \\ 2^{j/p} & \text{if } 2^{(j-1)/p} < |x| \le 2^{j/p}. \end{cases}$$

Notice that $|x| \le H(x) \le 2^{1/p}|x|$. We show, by induction on n, that

$$|x_1 + \cdots + x_n| \le 2^{1/p}(H(x_1)^p + \cdots + H(x_n)^p)^{1/p}. \tag{3.13}$$

By the observation just made, (3.13) clearly implies the statement. Also, (3.13) holds for the base case $n = 1$. We now show that it holds for a generic n, assuming it holds for $n - 1$. By symmetry, we can assume that

$$|x_1| \ge |x_2| \ge \cdots \ge |x_n|,$$

which implies that $H(x_1) \ge H(x_2) \ge \cdots \ge H(x_n)$. We distinguish two cases.

i) There exists an index $1 \le i_0 < n$ such that $H(x_{i_0}) = H(x_{i_0+1})$: let $2^{j_0/p}$ be the common value of H at x_{i_0} and x_{i_0+1} and notice that, since

$$|x_{i_0} + x_{i_0+1}| \le C(|x_{i_0}| + |x_{i_0+1}|) \le 2C \cdot 2^{j_0/p} = 2^{(j_0+1)/p},$$

we have $H(x_{i_0} + x_{i_0+1}) \leq 2^{(j_0+1)/p}$. This gives

$$H(x_{i_0} + x_{i_0+1})^p \leq 2^{j_0+1} = H(x_{i_0})^p + H(x_{i_0+1})^p$$

and so, grouping $x_1 + \cdots + x_n = x_1 + \cdots + x_{i_0-1} + (x_{i_0} + x_{i_0+1}) + x_{i_0+2} + \ldots$ and using induction,

$$\begin{aligned} |x_1 + \cdots + x_n| &\leq 2^{1/p}(H(x_1)^p + \cdots + H(x_{i_0} + x_{i_0+1})^p + \cdots + H(x_n)^p)^{1/p} \\ &\leq 2^{1/p}(H(x_1)^p + \cdots + H(x_{i_0})^p + H(x_{i_0+1})^p + \cdots + H(x_n)^p)^{1/p}. \end{aligned}$$

ii) We have a strictly decreasing sequence $H(x_1) > H(x_2) > \cdots > H(x_n)$: in this case we must have $H(x_i) \leq 2^{-(i-1)/p} H(x_1)$ for all i. Also, iterating the approximate triangle inequality we obtain

$$\begin{aligned} |x_1 + \cdots + x_n| &\leq C(|x_1| + |x_2 + \cdots + x_n|) \\ &\leq \max\{2C|x_1|, 2C|x_2 + \cdots + x_n| \\ &\leq \max\{2C|x_1|, (2C)^2 \left|x_2\right|, (2C)^2 \left|x_3 + \cdots + x_n\right| \\ &\leq \cdots \leq \max_i (2C)^i |x_i| \\ &\leq \max_i 2^{i/p} H(x_i) \\ &\leq 2^{1/p} H(x_1) \end{aligned}$$

and (3.13) trivially follows. □

Proof of Theorem 3.7. For all $x \in V$ we define

$$\Lambda(x) := \inf \sum_{i=1}^{n} |x_i|^p, \quad x = \sum_{i=1}^{n} x_i, \ n \geq 1,$$

meaning that the infimum is taken over all possible representations of x as a finite sum of elements of V. Since a possible choice is $n = 1$ and $x_1 = x$, we trivially have $\Lambda(x) \leq |x|^p$. Moreover, the previous lemma gives

$$|x|^p = |x_1 + \cdots + x_n|^p \leq 4(|x_1|^p + \cdots + |x_n|^p)$$

for all such possible representations, so $\Lambda(x) \geq \frac{1}{4}|x|^p$. In particular, this implies that Λ vanishes only at 0. From the definition it is clear that $\Lambda(-x) = \Lambda(x)$.

Also, $\Lambda(x + y) \leq \Lambda(x) + \Lambda(y)$: given $\epsilon > 0$, if $x = x_1 + \cdots + x_m$ and $y = y_1 + \cdots + y_n$ are chosen so that $\sum_{i=1}^m |x_i|^p < \Lambda(x) + \epsilon$ and $\sum_{j=1}^n |y_j|^p < \Lambda(y) + \epsilon$, then (being $x + y = \sum_i x_i + \sum_j y_j$)

$$\Lambda(x + y) \leq \sum_{i=1}^{m} |x_i|^p + \sum_{j=1}^{n} |y_j|^p < \Lambda(x) + \Lambda(y) + 2\epsilon.$$

Hence, defining $d : V \times V \to [0, \infty)$ by $d(x, y) := \Lambda(x - y)$ gives a distance on V. This induces the same topology as the quasi-metric since

$$B_{r^{1/p}}(x) \subseteq \{y \in V : d(x, y) < r\} \subseteq B_{(4r)^{1/p}}(x)$$

for all $x \in V$ and all $r > 0$. □

Remark 3.10. The space $L^p(E)$, with $0 < p < 1$, is a quasi-normed vector space, with quasi-norm

$$|f|_{L^p} := \Big(\int_E |f|^p\Big)^{1/p},$$

which yields a constant $2^{1/p-1}$ in the approximate triangle inequality. The construction Aoki–Rolewicz metric is reminiscent of the distance

$$d(f,g) := \int_E |f-g|^p$$

on $L^p(E)$ (for $0 < p < 1$), which induces the same topology as the quasi-norm but is built in a nonlinear way.

We will now see important concrete examples of quasi-normed vector spaces, namely Lorentz spaces, which refine the classical Lebesgue spaces in terms of control over the integrability of a function. Standard estimates such as Sobolev's embedding or Young's inequality can be slightly (but crucially for some applications) improved using these more refined spaces.

3.2 The Lorentz spaces $L^{p,\infty}$

Definition 3.11. *Let $E \subseteq \mathbb{R}^n$ be a set of positive measure. Given $1 \le p < \infty$ and a measurable function $f : E \to \mathbb{K}$, we let*

$$|f|_{L^{p,\infty}} := \sup_{\alpha>0} \alpha\mu(\{|f| > \alpha\})^{1/p}$$

and we define $L^{p,\infty}(E)$ to be the set of all functions $f : E \to \mathbb{K}$ with $|f|_{L^{p,\infty}} < \infty$. We also let $|f|_{L^{\infty,\infty}} := \|f\|_{L^\infty}$, so that $L^{\infty,\infty}(E) = L^\infty(E)$. The space $L^{p,\infty}$ is called weak L^p *(however, it is totally unrelated to the weak topology on the L^p space!).*

Remark 3.12. *Notice that this specializes to Example 3.5 when $p = 1$. Again, we have $L^p(E) \subset \|f\|_{L^p}$). For $p < \infty$, this inclusion is strict in general: take e.g. $E := \mathbb{R}^n$ and $f(x) := |x|^{-n/p}$, which lies in $L^{p,\infty}(\mathbb{R}^n) \setminus L^p(\mathbb{R}^n)$ (the inclusion is actually always strict for subsets of $\mathbb{R}^n$, as can be seen taking $|x - x_0|^{-n/p}$ with x_0 a density point for E).*

Remark 3.13. *Using the inequality $(\alpha + \beta)^{1/p} \le \alpha^{1/p} + \beta^{1/p}$ and arguing as in Example 3.5, we see that $L^{p,\infty}(E)$ is a quasi-normed vector space, with $C = 2$.*

Definition 3.14. *A quasi-normed vector space is called* quasi-Banach *if every $|\ |$-Cauchy sequence converges to a (necessarily unique) limit in the canonical topology, or equivalently converges with respect to the quasi-norm.*

Remark 3.15. *Notice that a sequence is Cauchy with respect to the quasi-norm if and only if it is Cauchy with respect to the Aoki-Rolewicz distance. The same holds for convergence.*

Proposition 3.16. *The space $L^{p,\infty}(E)$ is a quasi-Banach space.*

Proof. Omitted. □

Remark 3.17. *The space $L^{1,\infty}(R^n)$ is not locally convex, meaning that it does not possess a local basis of neighborhoods of* 0 *made of open convex sets. This rules out the possibility of finding a* norm *equivalent to its quasi-norm, which is the main difficulty in Calderón–Zygmund theory for singular convolution kernels (so that, as we will see, not all kernels in $L^{1,\infty}(\mathbb{R}^n)$ but only those with enough cancellation and regularity give rise to the important $L^1 \to L^{1,\infty}$ bound). Let us see this failure of convexity when $n = 1$, for simplicity.*

For all integers $m \geq 2$ and $1 \leq k \leq m$ let

$$f_{m,k}(x) := \frac{1}{\log m}|x - \frac{k}{m}|^{-1}.$$

Observe that $f_{m,k} \in L^{1,\infty}(\mathbb{R})$, with $|f_{m,k}|_{L^{1,\infty}} \leq \frac{2}{\log m}$, so that $f_{m,k} \to 0$ in $L^{1,\infty}(\mathbb{R})$ as $m \to \infty$ (uniformly in the index k). On the other hand, the arithmetic mean of $f_{m,1}, \dots, f_{m,m}$ is pointwise bounded from below on $(0,1)$:

$$F_m := \frac{f_{m,1}(x) + \cdots + f_{m,m}(x)}{m} \geq \frac{1}{\log m} \sum_{j=1}^{m} \frac{1}{j} \geq c > 0,$$

since if $\frac{k_0}{m} < x < \frac{k_0+1}{m}$ then the left-hand side is at least

$$\frac{1}{m \log m}\Big(\frac{m}{k_0} + \cdots + \frac{m}{1} + \frac{m}{1} + \cdots + \frac{m}{m - k_0}\Big)$$

(the first part being not present if $k_0 = 0$). So $|F_m|_{L^{1,\infty}} \geq c$, implying that F_m cannot converge to 0*. This however should hold if $L^{1,\infty}(\mathbb{R})$ were locally convex!*

3.3 Decreasing rearrangement

In order to define all the Lorentz spaces $L^{p,q}$ we have to introduce the notion of decreasing rearrangement.

Definition 3.18. *Given $f : E \to \mathbb{K}$ measurable, we define its* decreasing rearrangement *$f_* : [0, +\infty] \to [0, +\infty]$ as*

$$f_*(t) := \inf\{0 \leq \lambda \leq +\infty : \mu(\{|f| > \lambda\}) \leq t\}, \tag{3.14}$$

with the convention that $0 \cdot \infty = \infty \cdot 0 = 0$ (as it is customary in measure theory).

Remark 3.19. *The infimum in* (3.14) *is actually always a minimum: if $\lambda_1 \geq \lambda_2 \geq \dots$ are values such that $\mu(\{|f| > \lambda_i\}) \leq t$ and $\lambda_\infty := \lim_{i\to\infty} \lambda_i$, then we still have $\mu(\{|f| > \lambda_\infty\}) \leq t$ (since the last set is the increasing union of the sets $\{|f| > \lambda_i\}$). Hence, $\mu(\{|f| > f_*(t)\}) \leq t$.*

Remark 3.20. *Define $d_f(\lambda) := \mu(\{|f| > \lambda\})$ (as a function from $[0, +\infty]$ to itself), which is called* distribution function*, or* tail distribution *in probability theory. It is clear that*

d_f and f_* are decreasing and d_f is right-continuous. Also f_* is right-continuous: given $0 \le t_0 < +\infty$, setting $\bar\lambda := \lim_{t\to t_0^+} f_*(t)$ we have

$$\mu(\{|f| > \bar\lambda\}) = \lim_{t\to t_0^+} \mu(\{|f| > f_*(t)\}) \le \lim_{t\to t_0^+} t = t_0,$$

where the first equality holds since we have a decreasing union of sets with finite measure. Hence, $f_*(t) \le \bar\lambda$. Since the converse inequality also holds (being f_* decreasing), the claim follows. One can show that d_f and f_* are "pseudo-inverses" of each other:

- *as already said, $d_\lambda \circ f_*(t) \le t$ and, assuming $0 < t, f_*(t) < +\infty$, equality holds if and only if $f_*(t') > f_*(t)$ for all $t' < t$;*
- *similarly with f_* and d_f interchanged.*

Proposition 3.21. *The functions f and f_*, although defined on different domains, have the same distribution function (meaning that $d_f = d_{f_*}$) and the same decreasing rearrangement (meaning that $f_* = (f_*)_*$).*

Proof. Fix $0 \le \lambda \le +\infty$ and notice that, given $0 \le t < +\infty$,

$$\mu(\{|f| > \lambda\}) \le t \Leftrightarrow \lambda \ge f_*(t) \Leftrightarrow \{f_* > \lambda\} \subseteq [0, t) \Leftrightarrow \mu(\{f_* > \lambda\}) \le t.$$

The penultimate equivalence follows from the fact that f_* is decreasing, while the last one follows from the right-continuity of f_* (so that one cannot have $\{f_* > \lambda\} = [0, t]$). Both statements now follow from this chain of equivalences (observe that $f_*(+\infty) = (f_*)_*(+\infty) = 0$). $\square$

Corollary 3.22. *For any measurable $f : E \to \mathbb{K}$, we have $|f|_{L^{p,\infty}} = |f_*|_{L^{p,\infty}}$ for all $1 \le p \le \infty$. Also, we have $\|f\|_{L^p} = \|f_*\|_{L^p}$ since*

$$\|f\|_{L^p}^p = \int_0^\infty p\lambda^{p-1} d_f(\lambda)\, d\lambda = \int_0^\infty p\lambda^{p-1} d_{f_*}(\lambda)\, d\lambda = \|f_*\|_{L^p}^p$$

for $1 \le p < \infty$ and $\|f\|_{L^\infty} = \inf\{\lambda : d_f(\lambda) = 0\} = \|f_*\|_{L^\infty}$.

The following two lemmas are very useful in practice, for instance when approximating a function by mollification or by simple functions.

Lemma 3.23. *If $|f_k| \to |f_\infty|$ pointwise a.e., or more generally if $|f_\infty| \le \liminf_{k\to\infty} |f_k|$ a.e., then $d_{f_\infty} \le \liminf_{k\to\infty} d_{f_k}$ and $(f_\infty)_* \le \liminf_{k\to\infty} (f_k)_*$.*

Proof. Let $N \subset E$ be a negligible subset such that $|f_\infty| \le \liminf_{k\to\infty} |f_k|$ everywhere on $E \setminus N$. Given $0 \le \lambda \le +\infty$, if $x \notin N$ has $|f_\infty(x)| > \lambda$ then $|f_k(x)| > \lambda$ eventually, so

$$\chi_{\{|f_\infty|>\lambda\}\setminus N} \le \liminf_{k\to\infty} \chi_{\{|f_k|>\lambda\}\setminus N}.$$

Integrating and applying Fatou's lemma gives the first claim. Now let $0 \le t \le +\infty$ and

$$\lambda_k := (f_k)_*(t), \qquad \bar\lambda := \liminf_{k\to\infty} \lambda_k.$$

Passing to a subsequence, we can assume that $\bar\lambda = \lim_{k\to\infty}\lambda_k$. Notice that the hypothesis still holds (in both versions). Again, if $|f_\infty(x)| > \bar\lambda$ (and $x \notin N$) then $|f_k(x)| > \lambda_k$ eventually, so as before we obtain

$$\mu(\{|f_\infty| > \bar\lambda\}) \le \liminf_{k\to\infty} \mu(\{|f_k| > \lambda_k\}) \le t$$

by Remark 3.19. By definition of decreasing rearrangement, it follows that $(f_\infty)_*(t) \le \bar\lambda = \liminf_{k\to\infty}(f_k)_*(t)$. $\square$

Lemma 3.24. *If $|f_k| \uparrow |f_\infty|$ pointwise a.e. (meaning that $|f_\infty|$ is the increasing limit of $|f_k|$), then $d_{f_k} \uparrow d_{f_\infty}$ and $(f_k)_* \uparrow (f_\infty)_*$ everywhere.*

Proof. Let $N \subset E$ be a negligible subset such that $|f_k| \uparrow |f_\infty|$ everywhere on $E \setminus N$. For every $0 \le \lambda \le +\infty$, since $\{|f_\infty| > \lambda\} \cap E$ is the increasing union of the sets $\{|f_n| > \lambda\} \cap E$, we get

$$d_{f_\infty}(\lambda) = \mu(\{|f_\infty| > \lambda\}) = \lim_{k\to\infty} \mu(\{|f_k| > \lambda\}) = \lim_{k\to\infty} d_{f_k}(\lambda).$$

Given $0 \le t \le +\infty$, we set $\lambda_k := (f_k)_*(t)$ (for $k \in \mathbb{N} \cup \{\infty\}$) and $\bar\lambda = \lim_{k\to\infty}\lambda_k$. This limit exists and is at most λ_∞ as

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_\infty.$$

We also have

$$\begin{aligned}
\mu(\{|f_\infty| > \bar\lambda\}) &= \lim_{k\to\infty} \mu(\{|f_k| > \bar\lambda\}) \\
&= \lim_{k\to\infty} \mu(\{|f_k| > \lambda_k\}) \\
&\le \liminf_{k\to\infty} \mu(\{|f_k| > \lambda_k\}) \le t,
\end{aligned}$$

so $\lambda_\infty = (f_\infty)_*(t) \le \bar\lambda$. We conclude that $\lambda_\infty = \bar\lambda$, i.e. $(f_k)_*(t) \uparrow (f_\infty)_*(t)$. $\square$

3.4 The Lorentz spaces $L^{p,q}$

Definition 3.25. *Given $1 \le p < \infty$ and $1 \le q \le \infty$, we set*

$$|f|_{L^{p,q}}^q := \int_0^\infty t^{q/p} f_*(t)^q \, \frac{dt}{t}$$

and we call $L^{p,q}(E)$ the set of all measurable functions $f : E \to \mathbb{K}$ with $|f|_{L^{p,q}} < \infty$. We also set $|f|_{L^{\infty,q}} := \|f\|_{L^\infty}$ (so that $L^{\infty,q}(E) = L^\infty(E)$).

Remark 3.26. *As we will see, even if f_* is hit by the exponent q, the first exponent p is the dominant one.*

Proposition 3.27. *The quantity $|\cdot|_{L^{p,q}}$ is a quasi-norm.*

Proof. It suffices to show that $(f+g)_*(t) \leq f_*(\frac{t}{2}) + g_*(\frac{t}{2})$. Actually, if $0 \leq s, s', t \leq +\infty$ and $s + s' \leq t$, it always holds that $(f+g)_*(t) \leq f_*(s) + g_*(s')$, since

$$\mu(\{|f+g| > f_*(s) + g_*(s')\}) \leq \mu(\{|f| > f_*(s)\}) + \mu(\{|g| > g_*(s')\}) \leq s + s' \leq t.$$

$\square$

Remark 3.28. *It follows that $L^{p,q}(E)$ is a quasi-normed vector space for all exponents $1 \leq p \leq \infty$ and $1 \leq q \leq \infty$. Again, one can show that it is always a quasi-normed vector space. For $p > 1$, as opposed to the case of $L^{1,\infty}$, we will see that the quasi-norm admits an equivalent* norm, *giving thus rise to a genuine Banach space.*

Remark 3.29. *The Lorentz quasi-norm $|\ |_{L^{p,q}}$ measures the integrability of the function, rather than the regularity. In the language of probability, it depends only on the law of f, since it is defined in terms of f_* (which in turn depends only on d_f). Rearranging the places where the values are attained, thus possibly making the function very irregular, does not alter the $L^{p,q}$-quasinorm. One can define it in the same way on general measure spaces. What we just observed can be made precise as follows: if $h : E \to E'$ is a measure-preserving map between two measure spaces, then $|f \circ h|_{L^{p,q}} = |f|_{L^{p,q}}$ for any $f : E' \to \mathbb{K}$ (since $d_f = d_{f \circ h}$ and thus $f_* = (f \circ h)_*$).*

The definition of the $L^{p,q}$-quasinorm when $q < \infty$ suggests the following equivalent definition when $q = \infty$.

Proposition 3.30. *For $1 \leq p < \infty$ we have $|f|_{L^{p,\infty}} = \sup_{0 \leq t \leq +\infty} t^{1/p} f_*(t)$.*

Proof. $(\leq)$: given $\lambda > 0$ with $d_f(\lambda) > 0$, set $t := d_f(\lambda) - \epsilon$ (where $\epsilon > 0$ is arbitrary and will tend to 0). Letting $\lambda' := f_*(t)$, being $d_f(\lambda') \leq t = d_f(\lambda) - \epsilon$ we must have $\lambda' > \lambda$. Hence,

$$\lambda (d_f(\lambda) - \epsilon)^{1/p} \leq \lambda' t^{1/p} = f_*(t) t^{1/p} \leq \sup_{0 \leq t \leq +\infty} t^{1/p} f_*(t)$$

and the inequality follows letting $\epsilon \downarrow 0$ and then taking the supremum over λ.

$(\geq)$: analogous. $\square$

Similarly, the $L^{p,q}$-quasinorm can be expressed in terms of the distribution function.

Proposition 3.31. *For all $1 \leq p < \infty$ and $1 \leq q < \infty$ we have*

$$|f|_{L^{p,q}} = p^{1/q} \Big(\int_0^\infty \lambda^{q-1} d_f(\lambda)^{q/p} \, d\lambda \Big)^{1/q}.$$

Proof. We start with the trivial observation that one has $f_*(t) > \lambda$ if and only if $d_f(\lambda) > t$, thanks to Remark 3.19. This, together with Fubini, gives

$$\begin{aligned}
|f|_{L^{p,q}}^q &= \int_0^\infty t^{q/p-1} \int_0^{f_*(t)} q \lambda^{q-1} \, d\lambda \, dt \\
&= q \int_{\{(t,\lambda) : f_*(t) > \lambda\}} t^{q/p-1} \lambda^{q-1} \, dt \, d\lambda \\
&= q \int_0^\infty \int_0^{d_f(\lambda)} t^{q/p-1} \lambda^{q-1} \, dt \, d\lambda \\
&= p \int_0^\infty d_f(\lambda)^{q/p} \lambda^{q-1} \, d\lambda.
\end{aligned}$$

$\square$

Proposition 3.32. *If $|f_k| \to |f_\infty|$ pointwise a.e., or more generally if $|f_\infty| \le \liminf_{k\to\infty} |f_k|$ a.e., then*

$$|f_\infty|_{L^{p,q}} \le \liminf_{k\to\infty} |f_k|_{L^{p,q}}.$$

If $f_k \to f_\infty$ and $|f_k| \uparrow |f_\infty|$, then

$$|f_k - f_\infty|_{L^{p,q}} \to 0,$$

provided that $f_\infty \in L^{p,q}(E)$ and $1 \le p, q < \infty$. In particular, simple functions are dense in $L^{p,q}(E)$ if $1 \le p, q < \infty$

Proof. The first part follows immediately from Lemma 3.23 and Fatou. The second part follows from the pointwise convergence $(f_k)_* \to (f_\infty)_*$ given by Lemma 3.24, together with the dominated convergence theorem. □

Proposition 3.33. *We have*

(1) $L^{p,p}(E) = L^p(E)$,

(2) $L^{p,q}(E) \subseteq L^{p,r}(E)$ if $q < r$,

(3) $L^{p,q}(E) \subseteq L^{t,u}(E)$ if $\mu(E) < \infty$ and $p > t$ (regardless of q and u).

Proof. (1) From the definition of the $L^{p,p}$-quasinorm and Corollary 3.22 we have $|f|^p_{L^{p,p}} = \|f_*\|^p_{L^p} = \|f\|^p_{L^p}$.

(2) We assume $p < \infty$ without loss of generality. We first deal with the case $r = \infty$: since f_* is decreasing, we deduce

$$t^{1/p} f_*(t) = \Big(\frac{q}{p}\int_0^t s^{q/p-1} f_*(t)^q\, ds\Big)^{1/q} \le \Big(\frac{q}{p}\int_0^t s^{q/p-1} f_*(s)^q\, ds\Big)^{1/q} \le \Big(\frac{q}{p}\Big)^{1/q} |f|_{L^{p,q}}$$

for all $0 \le t < +\infty$. Taking the supremum over t, we deduce that $|f|_{L^{p,\infty}}$ is estimated by $|f|_{L^{p,q}}$ and the inclusion follows. If $r < \infty$, notice that

$$\begin{aligned}
|f|_{L^{p,r}} &= \Big(\int_0^\infty s^{r/p} f_*(s)^r\, \frac{ds}{s}\Big)^{1/r} \\
&\le \Big(\int_0^\infty s^{q/p} f_*(s)^q\, \frac{ds}{s}\Big)^{1/r} \sup_{0\le s\le+\infty} s^{(r-q)/(pr)} f_*(s)^{(r-q)/r} \\
&= |f|^{q/r}_{L^{p,q}} |f|^{(r-q)/r}_{L^{p,\infty}} \\
&\le C(p,q,r) |f|^{q/r}_{L^{p,q}} |f|^{(r-q)/r}_{L^{p,r}}
\end{aligned}$$

by the previous case. Dividing both sides by $|f|^{(r-q)/r}_{L^{p,r}}$ and raising to the power $\frac{r}{q}$, the claim follows.

(3) From the definition of f_* it follows that $f_*(s) = 0$ for all $s \ge \mu(E)$. In view of (2), it suffices to deal with the case $u = 1$, $q = \infty$. If $p < \infty$ we have

$$|f|_{L^{t,u}} = \int_0^{\mu(E)} s^{1/t} f_*(s)\, \frac{ds}{s} \le \Big(\int_0^{\mu(E)} s^{1/t-1/p}\, \frac{ds}{s}\Big) \sup_{0\le s\le\mu(E)} s^{1/p} f_*(s).$$

Since $\frac{1}{t} - \frac{1}{p} > 0$, the first integral is a finite constant, while the supremum equals $|f|_{L^{p,\infty}}$ by Proposition 3.30. If $p = \infty$, it suffices to bound $f_*(s)$ by $\|f\|_{L^\infty}$ right after the first equality. $\square$

Remark 3.34. *The inclusion $L^{p,q}(E) \subsetneq L^{p,r}(E)$ is always strict: assuming $0 \in E$ is a density point without loss of generality, it is easy to check that*

- $|x|^{-n/p} \in L^{p,\infty}(E) \setminus \bigcup_{q<\infty} L^{p,q}(E)$,
- $|x|^{-n/p} \log(|x|^{-1})^{-\alpha} \chi_{B_{1/2}}(x) \in L^{p,q}(E)$ *if and only if* $\alpha q > 1$, *for all* $\alpha > 0$.

We now turn to the promised fact that $L^{p,q}$ is normable for $p > 1$.

Theorem 3.35 (normability of $L^{p,q}$)**.** *For all $1 < p \leq \infty$ the $L^{p,q}$-quasinorm has an equivalent norm, for all $1 \leq q \leq \infty$.*

Lemma 3.36. *Define $f_{**} : (0, +\infty) \to [0, +\infty]$ by*

$$f_{**}(t) := \frac{1}{t} \int_0^t f_*(s)\, ds.$$

This modification of the decreasing rearrangement satisfies

$$f_{**}(t) = \frac{1}{t} \sup\{ \int_F |f| \,;\, F \subseteq E, \mu(F) \leq t \}. \tag{3.15}$$

Proof. The statement holds if f is a nonnegative simple function, namely $f = \sum_{i=1}^N \lambda_i \chi_{A_i}$ with $\lambda_1 \geq \lambda_2 \geq \dots$ and $A_i \cap A_j = \emptyset$: indeed, it is easy to check that both sides of (3.15) equal

$$\frac{1}{t} \sum_{i=1}^k \lambda_i \mu(A_i) + \delta \mu(A_{i+1})$$

where k is such that $\sum_{i=1}^k \mu(A_i) \leq t < \sum_{i=1}^{k+1} \mu(A_i)$ ($k = N$ if $t \geq \sum_{i=1}^N \mu(A_i)$) and $\delta := t - \sum_{i=1}^k \mu(A_i)$. In general, we approximate $|f|$ pointwise from below with nonnegative simple functions f_k. By Lemma 3.24 and the monotone convergence theorem, both sides of (3.15) converge from below to the desired quantities. $\square$

Corollary 3.37. *We have $(f + g)_{**} \leq f_{**} + g_{**}$.*

Proof. This immediately follows from the inequality $\int_F |f + g| \leq \int_F |f| + \int_F |g|$ and the last lemma. $\square$

Lemma 3.38 (Hardy's inequality)**.** *Given $1 < p < \infty$, $1 \leq q < \infty$ and $f : (0, +\infty) \to [0, +\infty]$, it holds*

$$\Big(\int_0^\infty \Big(\frac{1}{x} \int_0^x f(t)\, dt \Big)^p dx \Big)^{1/p} \leq p' \Big(\int_0^\infty f(x)^p\, dx \Big)^{1/p}$$

and more generally

$$\Big(\int_0^\infty x^{q/p-1} \Big(\frac{1}{x} \int_0^x f(t)\, dt \Big)^q dx \Big)^{1/q} \leq p' \Big(\int_0^\infty x^{q/p-1} f(x)^q\, dx \Big)^{1/q}.$$

Proof. We argue by duality. In order to show the first inequality, let $g \geq 0$ with $\|g\|_{L^{p'}} = 1$. We get

$$\begin{aligned}\int_0^\infty \Big(\frac{1}{x}\int_0^x f(t)\,dt\Big)g(x)\,dx &= \int_0^\infty \Big(\int_0^1 f(sx)\,dt\Big)g(x)\,dx\\ &= \int_0^1 \Big(\int_0^\infty f(sx)g(x)\,dx\Big)\,ds\\ &\leq \int_0^1 \|f(s\cdot)\|_{L^p}\|g\|_{L^{p'}}\,ds\\ &= \int_0^1 s^{-1/p}\|f\|_{L^p}\,ds\\ &= p'\|f\|_{L^p}.\end{aligned}$$

The proof of the second inequality is identical, working rather with the measure space $X := ((0,\infty), x^{q/p-1}\,dx)$ and using the duality $(L^q(X))^* = L^{q'}(X)$, observing that we still have $\|f(s\cdot)\|_{L^q(X)} = s^{-1/p}\|f\|_{L^q(X)}$. □

Proof of Theorem 3.35. We assume without loss of generality that $1 < p < \infty$. We let

$$\|f\|_{L^{p,q}} := \Big(\int_0^\infty t^{q/p} f_{**}(t)^q\,\frac{dt}{t}\Big)^{1/q}$$

for $1 \leq q < \infty$ and $\|f\|_{L^{p,\infty}} := \sup_{0<t<\infty} t^{1/p} f_{**}(t)$, i.e. we are merely replacing f_* with f_{**} in the definitions. From Corollary 3.37 it follows that this is a norm (when $q < \infty$ we also use Minkowski's inequality for $L^q(X)$, where X is the same measure space as in the previous proof). Finally, since f_* is decreasing, we have $f_{**} \geq f_*$ and thus $\|f\|_{L^{p,q}} \geq |f|_{L^{p,q}}$. Conversely, by Hardy's inequality applied to f_*,

$$\|f\|_{L^{p,q}} \leq p'|f|_{L^{p,q}}.$$

This shows that the norm $\|\ \|_{L^{p,q}}$ is equivalent to the quasi-norm $|\ |_{L^{p,q}}$. □

Remark 3.39. *By Fatou's lemma, the conclusions of Lemmas 3.24 and 3.23 are still true with f_{**} in place of f_*. Hence, Proposition 3.32 still holds with $|\ |_{L^{p,q}}$ replaced with $\|\ \|_{L^{p,q}}$.*

The dual spaces of Lorentz spaces are the expected ones, for $p > 1$.

Theorem 3.40 (Dual spaces)**.** *For $1 < p < \infty$ and $1 \leq q < \infty$ we have*

$$(L^{p,q}(E))^* = L^{p',q'}(E),$$

where duality is represented by integration.

Proof. Omitted. □

3.5 Functional inequalities for Lorentz spaces

Theorem 3.41 (Hölder's inequality)**.** *Assume that $f \in L^{p_1,q_1}(E)$ and $g \in L^{p_2,q_2}(E)$ with*

$$\begin{aligned} 1 < p_1, p_2, p < \infty, &\qquad 1 \le q_1, q_2, q \le \infty, \\ \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}, &\qquad \frac{1}{q_1} + \frac{1}{q_2} \ge \frac{1}{q}. \end{aligned}$$

Then $fg \in L^{p,q}(E)$, with $\|fg\|_{L^{p,q}} \le C\|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}}$ (where C depends on p_1, p_2, q_1, q_2).

Proof. Thanks to Proposition 3.33, we can replace q_1 and q_2 with possibly higher exponents and assume, without loss of generality, that $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$. Given $0 \le t_1, t_2 \le +\infty$, notice that

$$\mu(\{|f| > f_*(t_1)\}) \le t_1, \quad \mu(\{|g| > g_*(t_2)\}) \le t_2,$$

so that, since $|fg| > f_*(t_1)g_*(t_2)$ implies either $|f| > f_*(t_1)$ or $|g| > g_*(t_2)$, we infer

$$\mu(\{|fg| > f_*(t_1)g_*(t_2)\}) \le t_1 + t_2$$

and thus

$$(fg)_*(t_1 + t_2) \le f_*(t_1)g_*(t_2).$$

This, together with the classical Hölder's inequality for Lebesgue spaces with exponents $\frac{q_1}{q}$ and $\frac{q_2}{q}$ (on the measure space $(0, +\infty)$), gives

$$\begin{aligned} |fg|_{L^{p,q}} &= \|t^{1/p-1/q}(fg)_*(t)\|_{L^q} \\ &\le \|t^{1/p-1/q} f_*\Big(\frac{t}{2}\Big) g_*\Big(\frac{t}{2}\Big)\|_{L^q} \\ &= C'\|t^{1/p_1-1/q_1} f_*(t)\ t^{1/p_2-1/q_2} g_*(t)\|_{L^q} \\ &\le C'\|t^{1/p_1-1/q_1} f_*(t)\|_{L^{q_1}}\|t^{1/p_2-1/q_2} g_*(t)\|_{L^{q_2}} \\ &= C'|f|_{L^{p_1,q_1}}|g|_{L^{p_2,q_2}}. \end{aligned}$$

□

Remark 3.42. *Of course, Hölder's inequality works also if $(p_1, q_1) = (\infty, \infty)$ (or similarly if $(p_2, q_2) = (\infty, \infty)$), since in this case it reduces to the inequality*

$$\|fg\|_{L^{p,q}} \le \|f\|_{L^\infty}\|g\|_{L^{p,q}} \le C\|f\|_{L^{\infty,\infty}}\|g\|_{L^{p_2,q_2}}.$$

Theorem 3.43 (Young's inequality)**.** *Assume that $f \in L^{p_1,q_1}(\mathbb{R}^n)$ and $g \in L^{p_2,q_2}(\mathbb{R}^n)$ with*

$$\begin{aligned} 1 < p_1, p_2, p < \infty, &\qquad 1 \le q_1, q_2, q \le \infty, \\ \frac{1}{p_1} + \frac{1}{p_2} = 1 + \frac{1}{p}, &\qquad \frac{1}{q_1} + \frac{1}{q_2} \ge \frac{1}{q}. \end{aligned}$$

Then the convolution $f * g$ is a.e. defined (meaning that the integral defining $f * g$ exists a.e.) and $f * g \in L^{p,q}(\mathbb{R}^n)$, with $\|fg\|_{L^{p,q}} \le C\|f\|_{L^{p_1,q_1}}\|g\|_{L^{p_2,q_2}}$ (where C depends on p_1, p_2, q_1, q_2).

Remark 3.44. *In some cases, this improves the classical Young's inequality for Lebesgue spaces: for instance, it gives $L^{3/2} * L^{3/2} \subseteq L^{3,1}$ rather than just $L^{3/2} * L^{3/2} \subseteq L^3$.*

The proof, due to O'Neil, is given below **(optional: it was not covered in class)**.

Lemma 3.45. *If $f, g \geq 0$ are measurable functions on $\mathbb{R}^n$ and $f \leq \alpha \chi_{E_0}$, then*

(1) $(f * g)_{**} \leq \alpha \mu(E_0) g_{**}$,

(2) $\|(f * g)_{**}\|_{L^\infty} \leq \alpha \mu(E_0) g_{**}(\mu(E_0))$.

Proof. Given $0 < t < +\infty$ and $F \subseteq \mathbb{R}^n$ with $\mu(F) \leq t$, then by (3.15)

$$\begin{aligned} t^{-1} \int_F f * g &\leq \alpha t^{-1} \int_F \int_{E_0} g(x-y)\, dy\, dx \\ &= \alpha \int_{E_0} t^{-1} \int_{F-y} g(x)\, dx\, dy \\ &\leq \alpha \int_{E_0} g_{**}(t)\, dy \\ &= \alpha \mu(E_0) g_{**}(t), \end{aligned}$$

so that taking the supremum over F and using (3.15) the first claim follows. Similarly, notice that

$$\begin{aligned} \alpha t^{-1} \int_F \int_{E_0} g(x-y)\, dy\, dx &= \alpha t^{-1} \int_F \int_{x-E_0} g(y)\, dy\, dx \\ &\leq \alpha t^{-1} \mu(F) \mu(E_0) g_{**}(\mu(E_0)) \\ &\leq \alpha \mu(E_0) g_{**}(\mu(E_0)), \end{aligned}$$

as $\mu(x - E_0) = \mu(E_0)$. This gives the second claim. □

Lemma 3.46. *For $f, g \geq 0$ and $0 < t < +\infty$, we have*

$$(f * g)_{**}(t) \leq t f_{**}(t) g_{**}(t) + \int_t^\infty f_* g_*.$$

Proof. We can assume that f is simple and finite, so we can write

$$f = \sum_{i=1}^N \alpha_i \chi_{E_i}$$

with $\alpha_i \geq 0$ and $\mathbb{R} =: E_0 \supsetneq E_1 \supsetneq \cdots \supsetneq E_N \supsetneq E_{N+1} := \emptyset$. Possibly adding artificially a set with measure t, we can assume that $t = \mu(E_{i_0})$ (with $1 \leq i_0 \leq N$). Using the previous lemma we have

$$(f * g)_{**}(t) \leq \sum_{i=1}^{i_0 - 1} \alpha_i \mu(E_i) g_{**}(\mu(E_i)) + \sum_{i=i_0}^N \alpha_i \mu(E_i) g_{**}(t). \tag{3.16}$$

Observe that f_* equals $\sum_{i=1}^{j} \alpha_i$ on the set $[\mu(E_{j+1}, \mu(E_j))$. The first sum in (3.16) equals

$$\sum_{i=1}^{i_0-1} \alpha_i \int_0^{\mu(E_i)} g_* = \sum_{i=1}^{i_0-1} \sum_{j=i}^{N} \int_{\mu(E_{j+1})}^{\mu(E_j)} \alpha_i g_* = \sum_{j=1}^{N} \sum_{i=1}^{\min\{j, i_0-1\}} \int_{\mu(E_{j+1})}^{\mu(E_j)} \alpha_i g_*$$

and the contribution for $j < i_0$ is precisely

$$\sum_{j=1}^{i_0-1} \int_{\mu(E_{j+1})}^{\mu(E_j)} \sum_{i=1}^{j} \alpha_i g_* = \sum_{j=1}^{i_0-1} \int_{\mu(E_{j+1})}^{\mu(E_j)} f_* g_* = \int_{\mu(E_{i_0})}^{\mu(E_1)} f_* g_* = \int_t^\infty f_* g_*.$$

On the other hand, the contribution for $j \geq i_0$ is

$$\begin{aligned} \sum_{j=i_0}^{N} \sum_{i=1}^{\min\{j, i_0-1\}} \int_{\mu(E_{j+1})}^{\mu(E_j)} \alpha_i g_* &= \sum_{j=i_0}^{N} \sum_{i=1}^{i_0-1} \alpha_i (\mu(E_j) g_{**}(\mu(E_j)) - \mu(E_{j+1}) g_{**}(\mu(E_{j+1}))) \\ &= \sum_{i=1}^{i_0-1} \alpha_i \mu(E_{i_0}) g_{**}(\mu(E_{i_0})) = \sum_{i=1}^{i_0-1} \alpha_i \mu(E_{i_0}) g_{**}(t), \end{aligned}$$

where $\mu(E_{N+1}) g_{**}(E_{N+1})$ has to be replaced with 0. Finally, notice that

$$\sum_{i=1}^{i_0-1} \alpha_i \mu(E_{i_0}) g_{**}(t) + \sum_{i=i_0}^{N} \alpha_i \mu(E_i) g_{**}(t) = \Big(\int_{E_{i_0}} f_* \Big) g_{**}(t) = t f_{**}(t) g_{**}(t)$$

by (3.15). $\square$

Proof of Young's inequality. We assume $q < \infty$. The case $q = \infty$ (where $q_1 = q_2 = \infty$) is far easier and left to the reader. It is clear that

$$\begin{aligned} \|t^{1/p-1/q+1} f_{**}(t) g_{**}(t)\|_{L^q} &= \|t^{1/p_1-1/q_1} f_{**}(t)\ t^{1/p_2-1/q_2} g_{**}(t)\|_{L^q} \\ &\leq \|t^{1/p_1-1/q_1} f_{**}(t)\|_{L^{q_1}} \|t^{1/p_2-1/q_2} g_{**}(t)\|_{L^{q_2}} \\ &= \|f\|_{L^{p_1,q_1}} \|g\|_{L^{p_2,q_2}} \end{aligned}$$

(assuming without loss of generality that $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$). Moreover, changing variables $t = \frac{1}{u}$, $s = \frac{1}{r}$ and using Hardy's inequality,

$$\begin{aligned} &\Big(\int_0^\infty t^{q/p-1} \Big(\int_t^\infty f_*(s) g_*(s)\, ds \Big)^q dt \Big)^{1/q} \\ &= \Big(\int_0^\infty u^{q/p'-1} \Big(\frac{1}{u} \int_0^u r^{-2} f_*(r^{-1}) g_*(r^{-1})\, dr \Big)^q du \Big)^{1/q} \\ &\leq C \Big(\int_0^\infty u^{q/p'-1} u^{-2q} f_*(u^{-1})^q g_*(u^{-1})^q\, du \Big)^{1/q} \\ &= C \Big(\int_0^\infty t^{q+q/p-1} f_*(t)^q g_*(t)^q\, dt \Big)^{1/q} \\ &= C \|t^{1/p_1-1/q_1} f_*(t)\ t^{1/p_2-1/q_2} g_*(t)\|_{L^q}, \end{aligned}$$

which can be estimated by $|f|_{L^{p_1,q_1}}|g|_{L^{p_2,q_2}}$ as before. The inequality follows from the fact that

$$\|f*g\|_{L^{p,q}} \le \|t^{1/p-1/q+1}f_{**}(t)g_{**}(t) + t^{1/p-1/q}\int_t^\infty f_*g_*\|_{L^q}$$

by the previous lemma. □

Let us now see an important consequence when $n \ge 2$.

Corollary 3.47 (improved Sobolev's embedding)**.** *We have the continuous embedding $W^{1,p}(\mathbb{R}^n) \subseteq L^{p^*,p}(\mathbb{R}^n)$ for all $1 < p < d$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$.*

Sketch of proof. By mollification and cut-off, it suffices to show that $\|f\|_{L^{p^*,p}} \le C\|f\|_{W^{1,p}}$ whenever $f \in C_c^\infty(\mathbb{R}^n)$ (since, by Lemma 3.23 and Fatou's lemma, the $L^{p,q}$-quasinorm is lower semicontinuous under pointwise convergence a.e.). We have

$$f = G * \Delta f,$$

where G is Green's function for the Laplacian. Recall that, up to a multiplicative constant, G equals $\log|x|$ if $n = 2$ and $|x|^{2-n}$ if $n \ge 3$. In all cases, commuting a derivative with the convolution, we get

$$f = \sum_{i=1}^n \frac{\partial G}{\partial x_i} * \frac{\partial f}{\partial x_i}$$

(this is legitimate since $G \in W^{1,q}_{loc}(\mathbb{R}^n)$ for any $q < \frac{n}{n-1}$) and, observing that $\frac{\partial G}{\partial x_i}$ equals $\frac{x_i}{|x|^n}$ up to a multiplicative constant, we get $|\frac{\partial G}{\partial x_i}| \in L^{n/(n-1),\infty}(\mathbb{R}^n)$. The claim follows from Young's inequality for Lorentz spaces. □

Remark 3.48. *The improved Sobolev's embedding also holds for $p = 1$, although this is not immediately clear from this proof. Instead, it can be shown using the coarea formula and the isoperimetric inequality. Here is a sketch* ***(optional: it was not covered in class)****. Assuming without loss of generality $f \in C_c^\infty$ nonnegative,*

$$\begin{aligned}
|f|_{L^{1^*,1}} &= 1^* \int_0^\infty \mu(\{f > \lambda\})^{1/1^*}\,d\lambda \\
&\le C\int_0^\infty \mathcal{H}^{n-1}(\{f = \lambda\})\,d\lambda \\
&= C\int |\nabla f|,
\end{aligned}$$

where the first equality is Proposition 3.31, the inequality is the isoperimetric inequality for the set $\{f > \lambda\}$ (which is a smooth bounded domain for a.e. λ; notice that $1/1^* = (n-1)/n$) and the last equality is the coarea formula.

Proposition 3.49. ***(optional: it was not covered in class)*** *In spite of the fact that $W^{1,n}(\mathbb{R}^n) \not\subseteq L^\infty(\mathbb{R}^n)$, a function $f \in L^1_{loc}(\mathbb{R}^n)$ with weak gradient in the Lorentz space $L^{n,1}(\mathbb{R}^n)$ has a continuous representative and satisfies*

$$\|f - c(f)\|_{L^\infty} \le C\|\nabla f\|_{L^{n,1}}$$

for a suitable constant function $c(f)$.

Proof. The main point is that, if $f \in C_c^\infty(\mathbb{R}^n)$, the same proof as Corollary 3.47 gives

$$\|f\|_{L^\infty} \le C\|\nabla f\|_{L^{n,1}}.$$

Instead of Young's inequality, we just use this version of Hölder: $L^{n,1} \cdot L^{n/(n-1),\infty} \subseteq L^1$ (same proof as Theorem 3.41). This allows to say that

$$|f(x)| \le \int \Big|\frac{\partial G}{\partial x_i}\Big|(y)\Big|\frac{\partial f}{\partial x_i}\Big|(x-y)\,d\mu(y) \le \|\nabla G\|_{L^{n/(n-1),\infty}}\|\nabla f\|_{L^{n,1}}$$

for all x. The rest of the work is to reduce to this situation.

Notice first that the convolution with a nonnegative function $\rho_\epsilon \in C_c^\infty(\mathbb{R}^n)$, with support in $B_\epsilon(0)$ and $\int \rho_\epsilon = 1$, satisfies

$$\|\nabla f - \nabla(\rho_\epsilon * f)\|_{L^{p,q}} \le \sup_{|h|\le\epsilon} \|\nabla f - \nabla f(\cdot + h)\|_{L^{p,q}} \tag{3.17}$$

for all $1 < p \le \infty$, $1 \le q \le \infty$: indeed, being $f \in W^{1,1}_{loc}(\mathbb{R}^n)$, $\rho_\epsilon * f$ is smooth and its gradient equals $\rho_\epsilon * \nabla f$, which can be thought as a pointwise limit of convex combinations of functions $\nabla f(\cdot + h)$, with $|h| \le \epsilon$ (e.g. approximating the convolution with a finite sum as for a Riemann integral). The claim follows from Remark 3.39 and the fact that $\|\ \|_{L^{p,q}}$ is a norm invariant under translations in $\mathbb{R}^n$.

As a consequence, if $1 \le p, q < \infty$ then $\nabla(\rho_\epsilon * f) \to \nabla f$: in fact, $g(\cdot + h) \to g$ as $h \to 0$ when $g = \chi_E$ is a characteristic function (with $\mu(E) < \infty$) because $\mu(E\Delta(E-h)) \to 0$ and $(\chi_E - \chi_{E-h})_* = \chi_{[0,\mu(E\Delta(E-h)))}$, so by Corollary 3.32 this holds also for a generic $g \in L^{p,q}(\mathbb{R}^n)$ and the claim follows from (3.17). So there exist smooth functions f_k such that $f_k \to f$ in $L^1_{loc}(\mathbb{R}^n)$ and $\nabla f_k \to \nabla f$ in $L^{n,1}(\mathbb{R}^n)$.

For any $R \ge 1$, the embedding $W^{n,1}(\mathbb{R}^n) \subset L^{2n}(\mathbb{R}^n)$ and Poincaré's inequality give

$$\|f_k - c_{k,R}\|_{L^{2n}(B_{2R})} \le CR^{1/2}\|\nabla f_k\|_{L^n(B_{2R})} \le CR^{1/2}\|\nabla f_k\|_{L^{n,1}(B_{2R})}$$

(with $c_{k,R} := \fint_{B_{2R}} f_k$) and thus, as the proof of Proposition 3.33(3) shows, we get

$$f_k - c_{k,R}\|_{L^{n,1}(B_{2R})} \le CR\|\nabla f_k\|_{L^{n,1}}.$$

Finally, choosing a smooth cut-off function ϕ_R with $\phi = 1$ on B_R, $\phi_R = 0$ outside B_{2R} and $|\nabla\phi_R| \le \frac{2}{R}$,

$$\|\nabla(\phi_R(f_k - c_{k,R}))\|_{L^{n,1}} \le \frac{2}{R}\|f_k - c_{k,R}\|_{L^{n,1}(B_{2R})} + \|\nabla f_k\|_{L^{n,1}} \le C\|\nabla f_k\|_{L^{n,1}}$$

and thus, by the initial part of the proof,

$$\|f_k - c_{k,R}\|_{L^\infty(B_R)} \le \|\phi_R(f_k - c_{k,R})\|_{L^\infty} \le C\|\nabla f_k\|_{L^{n,1}}.$$

The constants $c_{k,R}$ are obviously equibounded (in k, R), since this inequality gives in particular

$$|\fint_{B_1} f_k - c_{k,R}| \le C\|\nabla f_k\|_{L^{n,1}}.$$

Hence, letting $R \to \infty$ along a suitable sequence depending on k, we get $\|f_k - c_k\|_{L^\infty} \le C\|\nabla f_k\|_{L^{n,1}}$ (with $\sup_k |c_k| < \infty$). Letting $k \to \infty$, again along a subsequence, we get the statement. $\square$

3.6 Dyadic characterization of some Lorentz spaces and another proof of Lorentz–Sobolev embedding (optional)

In this part we show that, when $q \le p$, the $L^{p,q}$-norm of a function f can be measured in terms of a dyadic decomposition of f according to its values.

In the sequel, $\varphi : \mathbb{R} \to \mathbb{R}$ is a smooth function supported in the annulus $\bar{B}_2(0) \setminus B_{1/2}(0)$ and such that

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}t) = 1, \qquad \text{for all } t \in \mathbb{R}\setminus\{0\}. \tag{3.18}$$

In order to construct φ, take for instance any $\psi \in C_c^\infty(B_2)$ such that $\psi = 1$ on B_1, and set $\varphi(t) := \psi(t) - \psi(2t)$. For any $t \in \mathbb{R}\setminus\{0\}$, it holds

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}t) = \lim_{N\to\infty} \sum_{j=-N}^{N} (\psi(2^{-j}t) - \psi(2^{-(j-1)}t)) = \lim_{N\to\infty} (\psi(2^{-N}t) - \psi(2^{N+1}t)) = 1;$$

the sum is well defined and the first equality holds, since at most two terms in the sum are nonzero: if $2^k \le t \le 2^{k+1}$, then $\varphi(2^{-j}t) = 0$ for $j \ne k, k+1$ since $\varphi(2^{-j}\cdot)$ is supported in the annulus $\bar{B}_{2^{j+1}} \setminus B_{2^{j-1}}$.

Given $f : \mathbb{R}^n \to \mathbb{R}$, we split it according to its values: we set

$$f_j := f\,\varphi(2^{-j}|f|),$$

so that the piece f_j vanishes at x if $|f|(x)$ is not in the range $(2^{j-1}, 2^{j+1})$. Notice that, thanks to (3.18),

$$f = \sum_{j\in\mathbb{Z}} f_j$$

where the sum is actually finite at each point (since at most two terms are nonzero). *This decomposition should not be confused with the Littlewood–Paley decomposition, encountered later in the course, which involves the phase space rather than the values of f!*

Lemma 3.50. *For $1 < p < \infty$ and $1 \le q \le p$ we have*

$$C^{-1}\|f\|_{L^{p,q}} \le \Big(\sum_{j\in\mathbb{Z}} \|f_j\|_{L^p}^q\Big)^{1/q} \le C\|f\|_{L^{p,q}}$$

for some C depending on p, q.

Proof. Since $f_j \le 2^{j+1}\chi_{|f|>2^{j-1}}$, we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \|f_j\|_{L^p}^q &\le 2^q \sum_{j\in\mathbb{Z}} 2^{qj}\mu(\{|f|>2^{j-1}\})^{q/p} \\ &= 8^q \sum_{j\in\mathbb{Z}} 2^{qj}\mu(\{|f|>2^{j+1}\})^{q/p} \\ &\le 8^q \sum_{j\in\mathbb{Z}} \int_{2^j}^{2^{j+1}} \lambda^{q-1}\mu(\{|f|>\lambda\})^{q/p}\,d\lambda \\ &= 8^q \int_0^\infty \lambda^{q-1}\mu(\{|f|>\lambda\})\,d\lambda.\end{aligned}$$

Conversely, using the subadditivity of $t\mapsto t^{q/p}$ (true as $q\le p$),

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \int_{2^j}^{2^{j+1}} \lambda^{q-1}\mu(\{|f|>\lambda\})^{q/p}\,d\lambda &\le 2^{q-1}\sum_{j\in\mathbb{Z}} 2^{qj}\mu(\{|f|>2^j\})^{q/p} \\ &= 2^{q-1}\sum_{j\in\mathbb{Z}} 2^{qj}\Big(\sum_{k\ge j}\mu(\{2^k<|f|\le 2^{k+1}\})\Big)^{q/p} \\ &\le 2^{q-1}\sum_{j\in\mathbb{Z}}\sum_{k\ge j} 2^{qj}\mu(\{2^k<|f|\le 2^{k+1}\})^{q/p} \\ &\le 2^q\sum_{k\in\mathbb{Z}} 2^{qk}\mu(\{2^k<|f|\le 2^{k+1}\})^{q/p}.\end{aligned}$$

For a given $x\in E$ with $f(x)\ne 0$, if $k\in\mathbb{Z}$ is such that $2^k<|f(x)|\le 2^{k+1}$ then $2^k\le |f(x)| = |f_k(x)+f_{k+1}(x)|$, so

$$2^{pk}\mu(\{2^k<|f|\le 2^{k+1}\}) \le \int |f_k+f_{k+1}|^p \le 2^{p-1}\int |f_k|^p + 2^{p-1}\int |f_{k+1}|^p.$$

Hence, raising to the power $\frac{q}{p}$,

$$2^q\sum_{k\in\mathbb{Z}} 2^{qk}\mu(\{2^k<|f|\le 2^{k+1}\})^{q/p} \le 4^q\sum_{k\in\mathbb{Z}}\Big(\int|f_k|^p+\int|f_{k+1}|^p\Big)^{q/p} \le 2\cdot 4^q\sum_{k\in\mathbb{Z}}\|f_k\|_{L^p}^q.$$

The claim now follows from Proposition 3.31. □

We now present an alternative proof of the Lorentz–Sobolev embedding $W^{1,p}(\mathbb{R}^n)\subset L^{p^*,p}(\mathbb{R}^n)$ for $n\ge 2$ and $1\le p<n$.

Given $f\in C_c^\infty(\mathbb{R}^n)$, we apply the classical Sobolev embedding to the pieces f_j to get

$$\|f\|_{L^{p^*,p}}^p \le C\sum_{j\in\mathbb{Z}}\|f_j\|_{L^{p^*}}^p \le C\sum_{j\in\mathbb{Z}}\|\nabla f_j\|_{L^p}^p.$$

Since $\nabla f_j = \varphi(2^{-j}f)\nabla f + 2^{-j}f\varphi'(2^{-j}f)\nabla f$ is bounded by $|\nabla f|\chi_{\{2^{j-1}<|f|<2^{j+1}\}}$ up to constants (being $|2^{-j}f|\le 2$ on the support of $\varphi'(2^{-j}f)$), we finally get

$$\sum_{j\in\mathbb{Z}}\|\nabla f_j\|_{L^p}^p \le C\sum_{j\in\mathbb{Z}}\int|\nabla f|^p\chi_{\{2^{j-1}<|f|<2^{j+1}\}} \le C\int|\nabla f|^p.$$

(as it is customary, in the above estimates the value of C can change from line to line). The conclusion follows as in the previous proof.

4 The L^p-theory of Calderón-Zygmund convolution operators.

4.1 Calderón-Zygmund decompositions.

The Calderón-Zygmund decomposition of an integrable function is the key ingredient for proving the continuity of the sub-linear Maximal Operator M in L^p spaces and the continuity of Calderón-Zygmund Operators in L^p Spaces as well. The later being the starting point to the analysis of elliptic PDE in L^p and more generally in non Hilbertian Sobolev or Besov Spaces.

We adopt the following denomination : A cube of size $\delta > 0$ in $\mathbb{R}^n$ is a closed set of the form $C = \prod_{i=1}^n [a_i, a_i + \delta]$ where (a_i) is an arbitrary sequence of n real numbers.

Theorem 4.1 (Calderón-Zygmund Decomposition). *Let $f \in L^1(\mathbb{R}^n)$ with $f \geq 0$ and let $\alpha > 0$. Then there exists an at most countable family of cubes $(C_k)_{k\in K}$ having disjoint interiors such that*

(i) The average of f on all cubes is bounded from below and above by

$$\alpha < \frac{1}{\mu(C_k)} \int_{C_k} f(x)\, dx \leq 2^n \alpha\, . \tag{4.1}$$

(ii) On the complement Ω^c of the union $\Omega = \bigcup_{k\in K} C_k$, we have

$$f(x) \leq \alpha \qquad a.e.\, . \tag{4.2}$$

(iii) There exists a constant $C = C(n)$ depending only on the dimension n such that

$$\mu(\Omega) \leq \frac{C}{\alpha}\, \|f\|_{L^1}\, . \tag{4.3}$$

Remark 4.1. *An alternative way to look at the result is the following. The Calderón-Zygmund Decomposition of threshold $\alpha > 0$ is a non-linear decomposition of any function $f \in L^1$ of the form $f = g + b$ where g and b are two functions respectively in $L^1 \cap L^\infty(\mathbb{R}^n)$ and in $L^1(\mathbb{R}^n)$ satisfying*

i) $\exists\ (C_k)_{k\in K}$ *a family of disjoint cubes of $\mathbb{R}^n$ such that*

$$b = \sum_{k\in K} b_k \qquad \text{with } b_k \equiv 0 \text{ in } \mathbb{R}^n \setminus C_k \quad .$$

ii) For all $k \in K$ hold the two following conditions

$$\int_{C_k} b_k(y)\ dy = 0 \qquad \text{and} \qquad \frac{1}{\mu(C_k)} \int_{C_k} |b_k(y)|\ dy \leq\ 2^{n+1}\, \alpha \quad .$$

iii) *g satisfies the following pointwise inequalities*

$$\begin{cases} |g(x)| = |f(x)| \leq \alpha & \textit{for a.e. } x \in \mathbb{R}^n \setminus \cup_{k\in K} C_k \quad , \\ \\ |g(x)| \leq 2^n \, \alpha & \textit{for a.e. } x \in \cup_{k\in K} C_k \end{cases}$$

iv) *The L^2 norm of g is controlled as follows*

$$\|g\|^2_{L^2(\mathbb{R}^n)} \leq 2^{2n} \, \alpha \, \|f\|_{L^1(\mathbb{R}^n)} \quad .$$

v) *The Lebesgue measure of the so called "bad set" $\Omega = \cup_{k\in K} C_k$ satisfies*

$$\mu(\Omega) = \sum_{k\in K} \mu(C_k) \leq \frac{1}{\alpha} \, \|f\|_{L^1(\mathbb{R}^n)} \quad .$$

The link between our construction in the proof of theorem 4.1 (applied to $|f|$) and the decomposition $f = g + b$ satisfying $i) \cdots v)$ is made by taking

$$b_k := \left(f - \frac{1}{\mu(C_k)} \int_{C_k} f(y) \; dy \right) \; \chi_{C_k} \quad ,$$

and i)...v) follow from simple estimates. It is worth remembering that Calderón-Zygmund decomposition is not unique.

Example 4.2. *Consider the function $f = \chi_{[0,1]}$, the characteristic function of the segment $[0,1]$ in $\mathbb{R}$. A Calderón-Zygmund decomposition of f with threshold 2^{-i-1} is given by $g = 2^{-i}\chi_{[0,2^i]}$ and the set Ω is made of a unique cube : $[0,2^i]$. $b = 0$ outside $[0,2^i]$ and $b = \chi_{[0,1]} - 2^{-i}\chi_{[0,2^i]}$ has indeed average 0 on the unique cube of the decomposition.*

Proof of theorem 4.1.

We divide $\mathbb{R}^n$ into a mesh of equal cubes chosen large enough such that their volume is larger or equal than $\|f\|_{L_1}/\alpha$. Thus, for every cube C_0 in this mesh, we have

$$\frac{1}{\mu(C_0)} \int_{C_0} f(x) \, dx \leq \alpha \, . \tag{4.4}$$

Every cube C^0 from the initial mesh is decomposed into 2^n equal disjoint cubes with half of the side-length. For the resulting cubes, there are now two possibilities: Either (4.4) still holds or (4.4) is violated. Cubes of the first case are called the *good cubes*, the set of *good cubes* is denoted by $\mathcal{C}_1^g$, and the set of non *good cubes*, the *bad cubes*, is denoted by $\mathcal{C}_1^b$. In a next step, we decompose all cubes in $\mathcal{C}_1^g$ into equal disjoint cubes with half side-length and leave the cubes in $\mathcal{C}_1^b$ unchanged. The resulting cubes for which an estimate of the form (4.4) still holds are denoted by $\mathcal{C}_2^g$ - they are called *good cubes* as well - and the remaining ones by $\mathcal{C}_2^b$. Then, we proceed as before dividing the cubes in $\mathcal{C}_2^g$ and leaving the cubes in $\mathcal{C}_2^b$ unchanged. – Repeating this procedure for each cube in the initial mesh, we can define $\Omega = \bigcup_{k\in K} C_k$ as the union of all cubes which violate in some step of the decomposition process an estimate of the form (4.4). (These are precisely those cubes with an upper index b for *bad*.)

Note that for a cube C_i^b in $\mathcal{C}_i^b$ obtained in the i-th step, we have

$$\frac{1}{\mu(C_i^b)} \int_{C_i^b} f(x)\,dx > \alpha\,. \tag{4.5}$$

Since $2^n \mu(C_i^b) = \mu(C_{i-1}^g)$, where C_{i-1}^g is any cube in $\mathcal{C}_{i-1}^g$, we then deduce

$$\alpha < \frac{1}{\mu(C_i^b)} \int_{C_i^b} f(x)\,dx \leq \frac{2^n}{\mu(C_{i-1}^g)} \int_{C_{i-1}^g} f(x)\,dx \leq 2^n\,\alpha\,.$$

This shows (i) of the theorem.

In order to show (ii), we note that by Lebesgue's differentiation theorem, almost everywhere the following holds

$$f(x) = \lim_{d\to 0} \frac{1}{\mu(C_{x,d})} \int_{C_{x,d}} f(y)\,dy\,,$$

where $C_{x,d}$ denotes a cube containing $x \in \mathbb{R}^n$ with diameter d. By construction of the decomposition, there exists for every $x \in \Omega^c$ a diameter $d_x > 0$ such that all cubes $C_{x,d}$ with diameter $d < d_0$ satisfy an estimate of the form (4.4). This implies directly that $f(x) \leq \alpha$ for a.e. $x \in \Omega^c$.

The last part (iii) of the theorem can be established as follows:

$$\mu(\Omega) = \sum_{k\in K} \mu(C_k) \overset{(4.5)}{<} \frac{1}{\alpha} \int_\Omega f(x)\,dx \leq \frac{1}{\alpha}\,\|f\|_{L^1}\,.$$

□

4.2 An application of Calderón-Zygmund decomposition

The following theorem gives a statement which is close to a converse to theorem 2.8. The proof of this theorem we give is an interesting application of the Calderón-Zygmund decomposition.

Theorem 4.3. *Let f be an integrable function on $\mathbb{R}^n$ supported on an euclidian ball B. Then $Mf \in L^1(B)$ if and only if $f \in L^1 \log L^1(B)$.*

The proof of theorem 4.3 is using the following lemma.

Lemma 4.4. *Let f be a locally integrable function on $\mathbb{R}^n$. Let B be an open euclidian ball of $\mathbb{R}^n$ such that $Mf \in L^1(B)$ then $f \in (L^1 \log L^1)_{loc}(B)$.*

Proof of lemma 4.4. Let ω be an open subset strictly included in B - i.e. $\overline{\omega} \subset B$. Denote by f_ω the restriction of f to ω. It is clear that the inequality $Mf(x) \geq Mf_\omega(x)$ holds for almost every $x \in \mathbb{R}^n$. Hence, for every $\beta > 0$ the following holds

$$\mu\left(\{x\ ;\ Mf(x) > \beta\}\right) \geq \mu\left(\{x\ ;\ Mf_\omega(x) > \beta\}\right) \quad . \tag{4.6}$$

In order to show that $f_\omega \in L^1 \log L^1(\mathbb{R}^n)$, we use the following "reverse" inequality to (2.9) for the Hardy-Littlewood maximal function : there exists a constant c depending only on n such that

$$\mu(\{x \in \Omega\ :\ Mf_\omega(x) > c\,\alpha\}) \geq \frac{1}{2^n \alpha} \int_{\{x \in \mathbb{R}^n\,:\,|f_\omega(x)| > \alpha\}} |f_\omega(x)|\, dx \tag{4.7}$$

where $\Omega = \cup_{k \in K} C_k$ is the union of bad cubes for a Calderón-Zygmund decomposition of mesh α applied to f_ω on $\mathbb{R}^n$ and given by the previous theorem 4.1.

Proof of inequality (4.7). For any $\alpha > 0$ theorem 4.1 gives, for the function f_ω, a family of cubes $(C_k)_{k \in K}$ of disjoint interiors such that (see (4.1))

$$\left\{ \begin{array}{l} 2^n \alpha \geq \frac{1}{\mu(C_k)} \int_{C_k} |f_\omega(x)|\, dx \geq \alpha \quad \text{ and} \\ \\ \forall x \in \mathbb{R}^n \setminus \Omega \quad |f_\omega(x)| \leq \alpha \quad . \end{array} \right. \tag{4.8}$$

Thus, if $x \in C_k$, it follows that $Mf_\omega(x) > c\,\alpha$, where the constant $c > 0$ is an adjustment which permits to pass from cubes to balls in the definition of the maximal function. As a direct consequence, we have that

$$\mu(\{x \in \Omega\ :\ Mf_\omega(x) > c\,\alpha\}) \geq \sum_{k=1}^{\infty} \mu(C_k) \overset{(4.8)}{\geq} \frac{1}{2^n \alpha} \int_\Omega |f(x)|\, dx \quad .$$

Since $|f_\omega(x)| \leq \alpha$, for $x \in \mathbb{R}^n \setminus \Omega$, the desired inequality (4.7) is established.

Let $\delta > 0$ such that for every cube C

$$\mu(C) \leq \delta \quad \text{ and } \quad C \cap \omega \neq \emptyset \quad \Longrightarrow C \cap \mathbb{R}^n \setminus B = \emptyset \quad . \tag{4.9}$$

δ has been chosen in such a way that, for any $\alpha > \alpha_0 = \int_\omega f/\delta$, the bad set Ω is included in B - this lower bound on α ensures indeed the fact that the mesh of the starting cubes in the associated Calderón-Zygmund decomposition is less than δ . Hence we deduce using (4.6), for any $\alpha > \alpha_0$, that

$$\mu(\{x \in B\ :\ Mf(x) > c\,\alpha\}) \geq \frac{1}{2^n \alpha} \int_{\{x \in \omega\ ;\ |f(x)| > \alpha\}} |f(x)|\, dx \quad .$$

Using the previous estimate we compute

$$\begin{aligned}\|Mf\|_{L^1(B)} = \int_B Mf(x)\,dx &\overset{(2.1)}{\geq} \int_{c\alpha_0}^{\infty} \mu(\{x \in B\,:\, Mf(x) > \alpha\})\,d\alpha \\ &\overset{(4.7)}{\geq} c\int_{\alpha_0}^{\infty} \left(\frac{1}{2^n\alpha}\int_{\{x\in\omega\,:\,|f(x)|>\alpha\}} |f(x)|\,dx\right)d\alpha \\ &= c\int_\omega |f(x)| \left(\int_{\alpha_0}^{\max\{\alpha_0,|f(x)|\}} \frac{1}{\alpha}\,d\alpha\right)dx \\ &= c\int_\omega |f(x)| \log^+ \frac{|f(x)|}{\alpha_0}\,dx\,.\end{aligned}$$

This proves the lemma. □

Proof of theorem 4.3.

One direction in the equivalence has been established in theorem 2.8. It suffices then to establish that $Mf \in L^1(B)$ and f supported in B imply that $f \in L^1 \log L^1(B)$.

Let's take to simplify the presentation B to be the unit ball of center the origin $B := B_1(0)$. First we show the following statement

$$f \equiv 0 \text{ in } \mathbb{R}^n \setminus B_1(0) \quad \text{and} \quad Mf \in L^1(B_1(0)) \quad \Longrightarrow Mf \in L^1(B_2(0)) \quad . \tag{4.10}$$

Once we will have proved this implication, using the previous lemma, we will deduce that $f \in L^1 \log L^1(B)$ and this will finish the proof of theorem 4.3.

Proof of (4.10). Let x be a point in $B_2(0) \setminus B_1(0)$. Since every point in $B_1(0)$ is closer to $x/|x|^2$ than to x, for $|x| > 1$, one obtains that $B_R(x) \cap B_1(0) \subset B_R(x/|x|) \cap B_1(0)$. We then deduce

$$\int_{B_R(x)} |f(y)|\ dy \leq \int_{B_R(x/|x|^2)} |f(y)|\ dy \quad ,$$

which implies that $Mf(x) \leq Mf(x/|x|^2)$ for $|x| > 1$. Thus

$$\int_{B_2(0)\setminus B_1(0)} Mf(x)\ dx \leq 2^{2n} \int_{B_1(0)\setminus B_{1/2}(0)} Mf(y)\ dy \quad .$$

This last inequality implies (4.10) and theorem 4.3 is then proved. □

4.3 The Marcinkiewicz Interpolation Theorem - The L^p case

Definition 4.5. *Let $1 \leq p, q \leq \infty$ and let T be a mapping from $L^p(\mathbb{R}^n)$ to the space of measurable functions. For $1 \leq q \leq \infty$, we say that the mapping T is of strong type (p, q) – or simply of type (p, q) – if*

$$\|Tf\|_{L^q} \leq C\,\|f\|_{L^p}\,,$$

where the constant C is independent of $f \in L^p(\mathbb{R}^n)$. For the case of $q < \infty$, we say that T is of weak type (p, q) if

$$\mu(\{x \in \mathbb{R}^n\ :\ |Tf(x)| > \alpha\}) \leq C\left(\frac{1}{\alpha}\,\|f\|_{L^p}\right)^q\,,$$

where the constant C is independent of f and $\alpha > 0$. For $q = \infty$, we say that T is of weak type (p, ∞) if T is of type (p, ∞).

Example 4.6. *The sub-linear Maximal Operator $f \longrightarrow Mf$ is, according to the Hardy-Littlewood Maximal Function Theorem 2.5, an example of weak $(1,1)$ operator or strong (p,p) for $1 < p \leq +\infty$.*

Remark 4.2. *Observe that for $q < \infty$, and for any measurable function g we have trivialy*

$$|g|^q_{L^{q,\infty}} = \sup_{\alpha < +\infty} \alpha^q \, \mu(\{x \, : \, |g(x)| > \alpha\}) \leq \|g\|^q_{L^q} \quad . \tag{4.11}$$

Applying this inequality to $g = Tf$ we obtain the fact that T being of type (p,q) is also of weak type (p,q).

We also define $L^{p_1} + L^{p_2}(\mathbb{R}^n)$ as the space of all functions f which can be written as $f = f_1 + f_2$ with $f_1 \in L^{p_1}(\mathbb{R}^n)$ and $f_2 \in L^{p_2}(\mathbb{R}^n)$. By splitting a function in its small and large parts, one can show that $L^p(\mathbb{R}^n) \subset L^{p_1} + L^{p_2}(\mathbb{R}^n)$, for $p_1 \leq p \leq p_2$ with $p_1 < p_2$.

Theorem 4.7 (Marcinkiewicz Interpolation Theorem- The L^p case)**.** *Let $1 < r \leq \infty$ and suppose that T is a sublinear operator from $L^1 + L^r(\mathbb{R}^n)$ to the space of measurable functions, i.e., for all $f, g \in L^1 + L^r(\mathbb{R}^n)$, the following pointwise estimate holds:*

$$|T(f+g)| \leq |Tf| + |Tg| \, . \tag{4.12}$$

Moreover, assume that T is of weak type $(1,1)$ and also of weak type (r,r). Then, for $1 < p < r$, we have that T is of type (p,p) meaning that

$$\|Tf\|_{L^p} \leq C \, \|f\|_{L^p} \, ,$$

for all $f \in L^p(\mathbb{R}^n)$.

Remark 4.3. *Because of the last theorem and the fact that the Hardy-Littlewood maximal function is sublinear, we can directly deduce (2.3) in Theorem 2.5 from (2.4) – saying that the operator M is of weak type $(1,1)$ – and the obvious observation that M is of type (∞, ∞).*

Remark 4.4. *Theorem 4.7 happens to be a special case of the more general Marcinkiewicz interpolation theorem for Lorentz spaces.*

Proof of theorem 4.7.

To simplify the presentation we restrict to the case $r < +\infty$. As in the proof of theorem 2.5, for an arbitrary parameter $\alpha > 0$, we introduce the following function

$$f_1(x) := \begin{cases} f(x) & \text{if} \quad |f(x)| > \alpha \\ \\ 0 & \text{if} \quad |f(x)| \leq \alpha \end{cases} \quad ,$$

and we denote $f_2(x) := f(x) - f_1(x)$ in such a way that $|f_2(x)| \leq \alpha$. The sub-additivity of T gives then $|Tf(x)| \leq |Tf_1(x)| + |Tf_2(x)|$ and from this we deduce that

$$\{x \, ; \, |Tf(x)| > \alpha\} \subset \{x \, ; \, |Tf_1(x)| > \alpha/2\} \cup \{x \, ; \, |Tf_2(x)| > \alpha/2\} \quad .$$

Hence, using (2.3) and (2.4), we bound $d_{Tf}(\alpha) = \mu(\{x\ ;\ |Tf(x)| > \alpha\})$ as follows

$$
\begin{aligned}
d_{Tf}(\alpha) &\leq d_{Tf_1}(\alpha/2) + d_{Tf_2}(\alpha/2) \\
&\leq \tfrac{2\,C_1}{\alpha}\|f_1\|_{L^1} + \tfrac{2^r\,C_r^r}{\alpha^r}\|f_2\|_{L^r}^r \qquad (4.13)\\
&\leq \tfrac{2\,C_1}{\alpha}\int_{E_\alpha}|f(y)|\ dy + \tfrac{2^r\,C_r^r}{\alpha^r}\int_{\mathbb{R}^n\setminus E_\alpha}|f(y)|^r\ dy
\end{aligned}
$$

where E_α denotes as usual the set $\{x\ ;\ |f(x)| > \alpha\}$, $C_1 = \sup|Tf|_{L^1_w}/\|f\|_{L^1}$ and $C_r = \sup|Tf|_{L^{r,\infty}}/\|f\|_{L^r}$ and where we have also applied inequality (4.11).

Expressing now the L^p norm of Tf by the mean of lemma 2.2 and combining it with (4.13) we get, using Fubini in the third line,

$$
\begin{aligned}
&\tfrac{1}{2^r}\int_{\mathbb{R}^n}|Tf(x)|^p\ dx = \tfrac{p}{2^r}\int_0^{+\infty}\alpha^{p-1}\,d_{Tf}(\alpha)\ d\alpha\\
&\leq p\,C_1\int_0^{+\infty}\alpha^{p-2}\,d\alpha\int_{E_\alpha}|f(y)|\ dy + p\,C_r^r\int_0^{+\infty}\alpha^{p-1-r}\,d\alpha\int_{\mathbb{R}^n\setminus E_\alpha}|f(y)|^r\ dy\\
&= p\,C_1\int_{\mathbb{R}^n}|f(y)|\ dy\int_0^{|f(y)|}\alpha^{p-2}\ d\alpha + p\,C_r^r\int_{\mathbb{R}^n}|f(y)|^r\ dy\int_{|f(y)|}^{+\infty}\alpha^{p-1-r}\ d\alpha\\
&\leq 2^r\,p\left(\tfrac{C_1}{p-1} + \tfrac{C_r^r}{p-r}\right)\int_{\mathbb{R}^n}\ |f(y)|^p\ dy \quad ,
\end{aligned}
$$

which proves the theorem. □

4.4 Calderon Zygmund Convolution Operators over L^p

Convolution operators operators are special cases of Calderón-Zygmund type operators. They are the "historical" ones : the first one introduced by Calderón and Zygmund in the 50's-60's corresponding to the principal values of singular integrals. They are the key notion giving access to the L^p theory (and more generally to the non hilbertian theory) of elliptic operators. Roughly speaking a typical question relevant to the theory of Singular Integral Operators is the following : if the L^p norm of the laplacian of a function is in L^p is it true or not that every second derivatives of this function are in L^p ?

This question is answered easily in the case $p = 2$ by the mean of Fourier transform but requires a more sophisticated analysis for being considered for $p \neq 2$. Of course the interest and the use of Singular Integral Operators goes much far beyond the resolution to this question and we will see applications of them all along this book.

A singular integral operator is formally a linear mapping of the form $T\ :\ f \to K \star f$ where K is the kernel which misses to be in L^1 or even L^1_{loc} from "very little". If K would be in L^1 then the continuity of T from L^p into itself would be a simple consequence of Young's inequality on convolutions. Usually the pointwise expression of the Kernel K is only in L^1-weak :

$$\sup_{\alpha>0}\alpha\ \mu\left(\{x\ ;\ |K(x)| > \alpha\}\right) < +\infty \quad .$$

A typical example of such a convolution operator is the one which to $f = \Delta u$ assigns the second derivative of u along the i and j directions : $\partial_{x_i}\partial_{x_j}u$ (modulo harmonic functions of course). This operator is given formally for $i \neq j$ by

$$\partial_{x_i}\partial_{x_j} u = C_n \int_{\mathbb{R}^n} \frac{(x_i - y_i)\,(x_j - y_j)}{|x-y|^{n+2}}\ f(y)\ dy \quad .$$

It is a convolution type operator T of kernel $K(x) = C_n\ x_i x_j/|x|^{n+2}$. K is in L^1-weak but it is not *a priori* a distribution and this makes the use of the convolution operation and the definition of T problematic or *singular*. Calderón-Zygmund operators of the first generation share the same difficulty. The reason why the Calderón-Zygmund Kernels K can be made to be a distribution is a cancellation property. In the previous example the cancellation property happens to be (recall that we look at the case$i \neq j$)

$$\int_{S^{n-1}} \frac{x_i x_j}{|x|^{n+2}}\ dy = 0 \quad .$$

Because of this later fact, for a smooth given compactly supported function f, it is not difficult to show that

$$\lim_{\epsilon\to 0} C_n \int_{\mathbb{R}^n\setminus B_\epsilon(x)} \frac{(x_i - y_i)\,(x_j - y_j)}{|x-y|^{n+2}}\ f(y)\ dy \tag{4.14}$$

exists for every x. This singular integral is the convolution between f and the distribution called *Principal Value of* K denoted $PV(K)$.

One of the spectacular result of Calderón-Zygmund theory says the following : the limit (4.14) $PV(K) \star f(x)$ exists almost everywhere whenever f is in $L^p(\mathbb{R}^n)$ for $p \in [1, +\infty]$ and is also in $L^p(\mathbb{R}^n)$ if f is in $L^p(\mathbb{R}^n)$ for $p \in (1, +\infty)$.

Another example of Singular Integral Operator is the Hilbert Transform on $\mathbb{R}$ - which corresponds in Fourier space by multiplying $\widehat{f}(\xi)$ by the sign of ξ - that is : $f \to f * \frac{1}{i\pi\, x}$. This *singular* integral has to be understood as being the limit of the following process

$$\lim_{\epsilon\to 0} \frac{1}{i\pi} \int_{|y|>\epsilon} \frac{f(x-y)}{y}\ dy \quad . \tag{4.15}$$

at least when f is smooth and compactly supported, since x^{-1} is odd, one easily check that this limit exists everywhere. It is equal to the convolution between f and the *Principal Value of* x^{-1}, $PV(1/x)$. Here again Calderón-Zygmund theory will tell us that the limit (4.15) $PV(x^{-1}) \star f$ exists almost everywhere whenever f is in $L^p(\mathbb{R}^n)$ for $p \in [1, +\infty]$ and is also in $L^p(\mathbb{R}^n)$ if f is in $L^p(\mathbb{R}^n$ for $p \in (1, +\infty)$.

In a way which is reminiscent to the L^p-theory of the maximal operator in the previous sections, the Hilbert transform and more generally Calderón-Zygmund operator won't map L^1 functions into L^1 functions but to L^1-weak functions only. In analogy with the previous section again, Calderón-Zygmund operator will however send $L^1 \log L^1$ functions into L^1. The parallel with the results obtained for the maximal operator in the previous section has some limit since, as we will see, L^∞ functions won't be map by Calderón-Zygmund operators to L^∞ functions but to $\cap_{p<+\infty} L^p_{loc}(\mathbb{R}^n)$ functions only.

Here again the Calderón-Zygmund decomposition will be the key instrument in the proofs. This use of Calderón-Zygmund decomposition is also known under the name of the *real variable method* of Calderón and Zygmund.

Let us finish the introduction to this very important section by making the following amusing remark. If the L^1-weak would have been a Banach space for a norm $\|\cdot\|_\star$ equivalent to the quasi-norm L^1_w - (2.5) -, then the L^p theory of Calderón-Zygmund operator would be trivially true without any assumption on the Kernel K except that it is in L^1-weak and that $T\ :\ f\to K\star f$ sends L^2 into L^2. Indeed, for any finite set of k points $a_1,\cdots,a_k$ in $\mathbb{R}^n$ and any family of k reals $\lambda_1\cdots\lambda_k$ one would have using the triangular inequality

$$\left\|\sum_{i=1}^k K(x-a_i)\ \lambda_i\right\|_\star \le \|K\|_\star\sum_{i=1}^k|\lambda_i|\quad ,$$

and we would directly deduce that T sends L^1 into L^1_w. The Marcinkiewicz interpolation theorem 4.7 would then imply that T is continuous from L^p into L^p for any $p\in(1,2]$ and the continuity for $p\in[2,+\infty)$ would be obtained by a simple duality argument.

We shall see three different formulations of the continuity of a Singular Integral Operator in L^p spaces, each of these formulations are based on different assumptions on the Kernel K.

4.4.1 A "primitive" formulation

In this subsection we prove the following "primitive" formulation of the L^p-continuity of Calderón-Zygmund convolution operator. The sense we give to the adjective "primitive" here should not be interpreted as something pejorative about this formulation, which has the clear pedagogical advantage to bring us progressively to more elaborated ones in the next subsections. In this formulation the difficulties caused by the singular nature of the convolution does not appear since the kernel K is "artificially" assumed to be in L^2.

Theorem 4.8. *Let $K\in L^2(\mathbb{R}^n)$ and assume the following:*

(i) The Fourier transform $\widehat{K}$ of K is bounded in L^∞

$$\|\widehat{K}\|_{L^\infty}<+\infty\quad . \tag{4.16}$$

(ii) The function K satisfies the so-called Hörmander condition : there exists $0<B<+\infty$ such that

$$\int_{2\|y\|\le\|x\|}\big|K(x-y)-K(x)\big|\,dx\le B\,,\qquad\forall y\neq 0\quad . \tag{4.17}$$

Moreover, let T be the well-defined convolution operator on $L^1(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)$, with $1<p<\infty$, given pointwise by

$$Tf(x)=K\star f(x)=\int_{\mathbb{R}^n}K(x-y)f(y)\,dy\quad . \tag{4.18}$$

Then, there exists a constant $C_p=C(n,p,\|K\|_\infty,B)$ – independent of the L^2-norm of K – such that

$$\|Tf\|_{L^p}\le C_p\,\|f\|_{L^p}\quad . \tag{4.19}$$

Moreover there exists a constant $C_1 = C(n, \|K\|_\infty, B)$ – independent of the L^2-norm of K – such that for any $f \in L^1(\mathbb{R}^n)$

$$\sup_{\alpha>0} \alpha \ \mu(\{x \in \mathbb{R}^n \ ; \ |K \star f(x)| > \alpha\}) \leq C_1 \ \|f\|_{L^1} \tag{4.20}$$

Remark 4.5. *a) Note that T is a densely defined linear operator on $L^p(\mathbb{R}^n)$. More precisely, the operator is well-defined on the dense linear subset $L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ of $L^p(\mathbb{R}^n)$ and from (4.19) we can deduce that T can be extended to all of $L^p(\mathbb{R}^n)$ by this.*

b) In the previous theorem, the kernel K is assumed to be in $L^2(\mathbb{R}^n)$. This happens to be "artificial" in the following sense : it permits to make the convolution operator T well defined on $L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, for $1 < p < \infty$ indeed by Young's inequality we have

$$\|Tf\|_{L^2} \leq \|K\|_{L^2} \|f\|_{L^1} \quad .$$

However the final crucial estimate leading to the continuity of T from L^p into L^p is independent of the L^2 norm of K.

c) Observe that the Hörmander condition (4.17) holds, for instance, whenever K is locally Lipschitz on $\mathbb{R}^n \setminus \{0\}$ and there exists $C > 0$ such that

$$\forall x \in \mathbb{R}^n \setminus \{0\} \qquad |\nabla K|(x) \leq \frac{C}{|x|^{n+1}} \quad .$$

This comes from the following estimate : Let $y \neq 0$ and denote $v = y/|y|$, then the following holds

$$\begin{aligned}
&\int_{2\|y\| \leq \|x\|} \big|K(x-y) - K(x)\big| \, dx \\
&= \int_{2\|y\| \leq \|x\|} \left| \int_0^{|y|} \tfrac{\partial K}{\partial v}(x + t\, v) \ dt \right| \ dx \\
&\leq \int_0^{|y|} dt \int_{2\|y\| \leq \|x\|} |\nabla K|(x + tv) \ dx \\
&\leq |y| \int_{\|y\| \leq \|z\|} |\nabla K|(z) \ dz \leq C_n \tfrac{|y|}{|y|} = C_n \quad .
\end{aligned} \tag{4.21}$$

where we have proceeded to the change of variable $z = x + tv$.

Proof of theorem 4.8 The proof is divided in the following three steps: First, we show that the convolution operator T is of strong type $(2,2)$. In a second step, we establish that T is of weak type $(1,1)$ - i.e. inequality (4.20), which is the most difficult part of the proof. Finally we obtain the inequality (4.19) from Marcinkiewicz's interpolation theorem and a duality argument.

First step: Let $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then for the Fourier transform $\widehat{Tf}$ of $Tf \in L^2(\mathbb{R}^n)$, we have

$$\|\widehat{Tf}\|_{L^2} = \|\widehat{K \star f}\|_{L^2} = \|\widehat{K}\, \widehat{f}\|_{L^2} \overset{(4.16)}{\leq} \|K\|_\infty \, \|f\|_{L^2} \quad .$$

Since $\|\widehat{Tf}\|_{L^2} = \|Tf\|_{L^2}$ by Plancherel's theorem, we then obtain

$$\|Tf\|_{L^2} \leq \|K\|_\infty \, \|f\|_{L^2} \,. \tag{4.22}$$

This shows that T is of type $(2,2)$, which also implies that T is of weak type $(2,2)$ as we mentioned in remark 4.2, precisely

$$\forall \alpha > 0 \qquad \mu(\{x \,:\, |Tf(x)| > \alpha\}) \leq \frac{\|K\|_\infty^2}{\alpha^2}\,\|f\|_{L^2}^2\,. \tag{4.23}$$

Second step: Let $f \in L^1(\mathbb{R}^n)$ and $\alpha > 0$. We apply the Calderón-Zygmund Decomposition 4.1 of threshold α to f. The resulting family of disjoint "bad cubes" will be denoted by $\{C_k\}_{k\in K}$ and we write $\Omega = \bigcup_{k=1}^\infty C_k$ for their union.

Now, we define

$$g(x) = \begin{cases} f(x) & \text{for } \; x \in \Omega^c \\ \dfrac{1}{\mu(C_k)} \displaystyle\int_{C_k} f(y)\,dy & \text{for } \; x \in C_k\,. \end{cases} \tag{4.24}$$

Following remark 4.1 C-Z Decomposition permits to write f as sum of a "good" and a "bad" function, namely $f = g + b$ - "good" and "bad" stand for the fact that there is a better control, namely L^∞, on g than on b - where

$$b = \sum_{k\in K} b_k \quad , \tag{4.25}$$

with

$$b_k(x) = \left(f(x) - \frac{1}{\mu(C_k)} \int_{C_k} f(y)\,dy \right) \chi_{C_k}(x) \quad .$$

From the linearity of the convolution operator T and the triangular inequality we have for all $x \in \mathbb{R}^n$

$$|Tf(x)| \leq |Tg(x)| + |Tb(x)| \quad . \tag{4.26}$$

Hence we deduce

$$\begin{aligned} \mu(\{x \,:\, |Tf(x)| > \alpha\}) \;&\leq\; \mu(\{x \,:\, |Tg(x)| > \alpha/2\}) \\ &\quad + \mu(\{x \,:\, |Tb(x)| > \alpha/2\}) \quad . \end{aligned} \tag{4.27}$$

In order to get an estimate for the first term on the right-hand side of (4.27), we first use the fact that g is an element of $L^2(\mathbb{R}^n)$- see remark 4.1 iv) - with the following control

$$\|g\|_{L^2(\mathbb{R}^n)}^2 \leq 2^{2n}\; \alpha\; \|f\|_{L^1(\mathbb{R}^n)} \quad .$$

As a consequence, we can apply (4.23) to $g \in L^2(\mathbb{R}^n)$ in order to get the following estimate for the first term on the right-hand side of (4.27):

$$\begin{aligned} \mu(\{x \,:\, |Tg(x)| > \alpha/2\}) \;&\leq\; \frac{4\|K\|_\infty^2}{\alpha^2}\|g\|_{L^2}^2 \\ &\leq\; 2^{2n+2}\; \frac{\|K\|_\infty^2}{\alpha}\; \|f\|_{L^1(\mathbb{R}^n)} \quad . \end{aligned} \tag{4.28}$$

Next, we estimate the second term on the right hand-side of (4.27). – For this purpose, we expand each cube C_k in the Calderón-Zygmund decomposition by the factor $2\sqrt{n}$ leaving its center c_k fixed. The new bigger cubes are denoted by $\tilde{C}_k$ and its union by $\tilde{\Omega} = \bigcup_{k\in K} \tilde{C}_k$. It is easy to see that $\Omega \subset \tilde{\Omega}$, $\tilde{\Omega}^c \subset \Omega^c$ and $\mu(\tilde{\Omega}) \leq (2\sqrt{n})^n \, \mu(\Omega)$. Moreover, for $x \notin \tilde{C}_k$, we have

$$\|x - c_k\| \geq 2\,\|y - c_k\|\,, \qquad \text{for all } \; y \in C_k \quad . \tag{4.29}$$

Now, let c_k denote the center of the cube C_k. Then, we can write

$$\begin{aligned} Tb(x) = \sum_{k\in K} Tb_k(x) \;&=\; \sum_{k\in K} \int_{C_k} K(x-y) b_k(y)\,dy \\ &=\; \sum_{k\in K} \int_{C_k} \big(K(x-y) - K(x-c_k)\big) b_k(y)\,dy\,, \end{aligned}$$

being a direct consequence of the fact that for all C_k

$$\int_{C_k} b_k(y)\,dy = \int_{C_k} \left(f(y) - \frac{1}{\mu(C_k)} \int_{C_k} f(z)\,dz \right) dy = 0 \quad ,$$

- condition ii) in remark 4.1 -. This then leads to

$$\begin{aligned} \int_{\tilde{\Omega}^c} |Tb(x)|\,dx \;&\leq\; \sum_{k\in K} \int_{\tilde{\Omega}^c} \left(\int_{C_k} \big|K(x-y) - K(x-c_k)\big| \, \big|b_k(y)\big|\,dy \right) dx \\ &\leq\; \sum_{k\in K} \int_{\tilde{C}_k^c} \left(\int_{C_k} \big|K(x-y) - K(x-c_k)\big| \, \big|b_k(y)\big|\,dy \right) dx \\ &=\; \sum_{k\in K} \int_{C_k} \left(\int_{\tilde{C}_k^c} \big|K(x-y) - K(x-c_k)\big|\,dx \right) \big|b_k(y)\big|\,dy\,. \end{aligned}$$

Setting $\bar{x} = x - c_k$, $\bar{y} = y - c_k$ and using (4.29), the integral in parenthesis ca be bounded this way

$$\int_{\tilde{C}_k^c} \big|K(x-y) - K(x-c_k)\big|\,dx \leq \int_{2\|\bar{y}\| \leq \|\bar{x}\|} \big|K(\bar{x}-\bar{y}) - K(\bar{x})\big|\,d\bar{x} \quad .$$

The assumption (4.17) of the theorem hence implies that

$$\int_{\tilde{\Omega}^c} |Tb(x)|\,dx \leq B \sum_{k\in K} \int_{C_k} |b_k(y)|\,dy \leq C\,\|f\|_{L^1} \quad . \tag{4.30}$$

At this stage, we are ready to give the following estimate for the second term in (4.27):

$$\begin{aligned} \mu(\{x \in \mathbb{R}^n \,:\, |Tb(x)| > \frac{\alpha}{2}\}) \;&\leq\; \mu(\{x \in \tilde{\Omega}^c \,:\, |Tb(x)| > \alpha/2\}) + \mu(\tilde{\Omega}) \\ &\overset{(4.30)}{\leq}\; \frac{2C}{\alpha}\,\|f\|_{L^1} + (2\sqrt{n})^n\,\mu(\Omega) \\ &\overset{(4.3)}{\leq}\; \frac{2C}{\alpha}\,\|f\|_{L^1} + \frac{C}{\alpha}\,\|f\|_{L^1} \leq \frac{C}{\alpha}\,\|f\|_{L^1}\,. \end{aligned}$$

(4.31)

Where C only depends on n, $\|K\|_\infty$ and B. Combining (4.28) with (4.31), we end up with the existence of a constant $C_1 > 0$ such that

$$\mu(\{x \,:\, |Tf(x)| > \alpha\}) \le \frac{C_1}{\alpha}\, \|f\|_{L^1}\,, \tag{4.32}$$

showing (4.20) and hence that the convolution operator T is of weak type $(1,1)$.

Third step: Note that we have already shown the inequality (4.19) in the case of $p = 2$ in (4.22). – Putting $r = 2$ in Marcinkiewicz Interpolation Theorem 4.7 and using the fact that T is of weak type $(1,1)$, respectively $(2,2)$, by (4.23), respectively (4.32), we conclude that

$$\|Tf\|_{L^p} \le C\, \|f\|_{L^p}\,, \tag{4.33}$$

for $1 < p < 2$ and where C only depends on n, p, $\|K\|_\infty$ and B the constant in the Hörmander condition.

For the case $2 < p < \infty$, we will use a *duality argument*. – Consider the dual space $L^{p'}(\mathbb{R}^n)$ of $L^p(\mathbb{R}^n)$ with $1/p + 1/p' = 1$. We easily see that $1 < q < 2$. Consider now $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. Since L^p is itself the dual space to $L^{p'}$ and since $L^1 \cap L^{p'}$ is dense in $L^{p'}$, the L^p-norm of Tf is given by the following expression:

$$\|Tf\|_{L^p} = \sup_{\substack{g \in L^1 \cap L^{p'} \\ \|g\|_{L^{p'}} \le 1}} \left| \int_{\mathbb{R}^n} Tf(x) g(x)\, dx \right|. \tag{4.34}$$

We calculate

$$\begin{aligned} \left| \int_{\mathbb{R}^n} Tf(x) g(x)\, dx \right| &= \left| \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} K(x-y) f(y)\, dy \right) g(x)\, dx \right| \\ &= \left| \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} K(x-y) g(x)\, dx \right) f(y)\, dy \right|, \end{aligned}$$

where Fubini's theorem was applied because of $K \in L^2(\mathbb{R}^n)$ and the assumptions on g and f. For the first integral, we conclude from (4.33) that it is an element of $L^{p'}(\mathbb{R}^n)$. Using Hölder's inequality, we end up with

$$\begin{aligned} \sup_{\substack{g \in L^1 \cap L^{p'} \\ \|g\|_{L^{p'}} \le 1}} \left| \int_{\mathbb{R}^n} Tf(x) g(x)\, dx \right| &\le \int_{\mathbb{R}^n} \left| \left(\int_{\mathbb{R}^n} K(x-y) g(x)\, dx \right) f(y) \right| dy \\ &\overset{(4.33)}{\le} C\, \|g\|_{L^{p'}} \|f\|_{L^p} \le C\, \|f\|_{L^p}\,. \end{aligned}$$

This establishes the theorem. □

4.4.2 A singular integral type formulation

In the present formulation of the L^p continuity for convolution type Calderón-Zygmund Operator we will skip the too strong assumption that the kernel K is in L^2 and will assume only a L^1-weak type pointwise control of K + a cancellation property together, still with the Hörmander condition. We will be then facing the heart of the matter : how can we deal with the singular integral $K \star f$ when f is only assumed to be in L^p ?

Theorem 4.9. *Let $K : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a measurable function such that there exists $A, B > 0$ for which the following holds*

$$|K(x)| \leq \frac{A}{\|x\|^n} \quad , \qquad \forall\ x \neq 0 \quad . \tag{4.35a}$$

$$\int_{2\|y\|\leq\|x\|} \big|K(x-y) - K(x)\big|\, dx \leq B \quad , \qquad \forall\ x \neq 0 \quad . \tag{4.35b}$$

$$\int_{\partial B_r(0)} K(x)\, dx = 0 \quad , \qquad \textit{for a. e. } r > 0 \quad . \tag{4.35c}$$

For $\varepsilon > 0$ and $f \in L^p(\mathbb{R}^n)$ with $1 \leq p < \infty$, we set

$$T_\varepsilon f(x) = \int_{\|y\|\geq\varepsilon} f(x-y)K(y)\, dy \quad . \tag{4.36}$$

Then, for any $1 < p < +\infty$ there exists a positive constant C such that for any $\varepsilon > 0$ and any $f \in L^p(\mathbb{R}^n)$,

$$\|T_\varepsilon f\|_{L^p} \leq C\, \|f\|_{L^p}\,, \tag{4.37}$$

where the constant $C = C(p, n, A, B)$ is independent of ε and f. Moreover, there exists $Tf \in L^p(\mathbb{R}^n)$ such that

$$T_\varepsilon f \longrightarrow Tf \quad \textit{in } L^p \qquad (\varepsilon \longrightarrow 0) \quad . \tag{4.38}$$

For any $f \in L^1(\mathbb{R}^n)$ there exists a measurable function Tf in L^1-weak such that

$$T_\varepsilon f \longrightarrow Tf \quad \textit{in } L^1_w \tag{4.39}$$

and there exists a constant positive $C(n, A, B)$ independent of f and ϵ such that

$$\sup_{\alpha>0} \alpha\ \mu(\{x \in \mathbb{R}^n\ ;\ |Tf(x)| > \alpha\}) \leq C(n, A, B)\ \|f\|_{L^1} \tag{4.40}$$

Remark 4.6. *The singular integral defined in (4.36) is, for a fixed ϵ, absolutely convergent. To see this, note that due to (4.35a) we have that $K \in L^{p'}(\mathbb{R}^n \setminus B_\varepsilon)$, where $1 < p'$ is the Hölder conjugate exponent of p. From Young's inequality, it then follows that $\|T_\varepsilon f\|_\infty \leq \|f\|_{L^p}\|K\|_{L^{p'}}$.*

A substantial part of the proof of theorem 4.9 will be to derive from the assumptions (4.35a), (4.35b) and (4.35c) an L^∞ bound for the Fourier transform of $K_\varepsilon(y) := K(y)\ \chi_{\mathbb{R}^n\setminus B_\varepsilon(0)}$ independent of ε. This estimate will permit us to invoke theorem 4.8 at some point in our proof. Precisely the following lemma holds.

Lemma 4.10. *Let $K : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a measurable function such that*

$$|K(x)| \leq \frac{A}{\|x\|^n} \quad , \qquad \textit{for } x \neq 0 \quad . \tag{4.41a}$$

$$\int_{2\|y\|\leq\|x\|} \big|K(x-y) - K(x)\big|\, dx \leq B \quad , \qquad \textit{for } y \neq 0 \quad . \tag{4.41b}$$

$$\int_{\partial B_r(0)} K(x)\, dx = 0 \quad , \qquad \textit{for a. e. } \ r > 0 \quad .$$

(4.41c)

Moreover, for every $\varepsilon > 0$, we define

$$K_\varepsilon(x) = \begin{cases} K(x) & \text{if } \|x\| \geq \varepsilon \\ 0 & \text{if } \|x\| < \varepsilon\,. \end{cases} \tag{4.42}$$

Then, there exists a constant $C = C(n, A, B)$, independent of ε, such that

$$\|\widehat{K}_\varepsilon\|_\infty \leq C\,. \tag{4.43}$$

Before to prove this L^∞ bound we would like to show first how the hypothesis relative to the cancellation property (4.35c) is essential. How cancellation property can lead to decisive improvements in the estimates will be a leitmotiv in this book - see in particular the chapter on Hardy spaces and the integrability by compensation phenomenon.

Example 4.11. *Consider the function on $\mathbb{R}$ given by $K(t) = \frac{1}{|t|}$ It is not difficult to check that K satisfies hypothesis (4.41a) and (4.41b) but the cancellation assumption (4.41c) is violated. we now prove that for this function K the conclusion of lemma 4.10 fails. We have*

$$\begin{aligned} \widehat{K_\varepsilon}(\xi) := &\ \lim_{r\to 0} \int_{\varepsilon<|t|<r} e^{2\pi i t\,\xi}\ \frac{dt}{|t|} = \lim_{r\to 0} \int_{\varepsilon<|t|<r} cos(2\pi t\,\xi)\ \frac{dt}{|t|} \quad , \\ = &\ 2sgn(\xi)\ \int_{\varepsilon|\xi|}^{+\infty} \frac{\cos 2\pi s}{s}\ ds \quad , \end{aligned}$$

where we have used the parity and the imparity respectively of $cos(2\pi t\,\xi)/|t|$ and $sin(2\pi t\,\xi)/|t|$. Now, since $\int_0^{+1} \cos s/s\ ds = +\infty$ we deduce that $\widehat{K_\varepsilon}(\xi)$ goes to $+\infty$ as ε goes to zero for non zero ξ.

Observe that a change of sign for K that would ensure the cancellation property (4.41c) - by taking $1/t$ instead of $1/|t|$ - would lead to the integral $\int_0^{+\infty} \sin s/s$, which converges, instead of the previous integral $\int_0^{+\infty} \cos s/s$ which diverges. This illustrate the importance of the cancellation assumption (4.41c)

Proof of lemma 4.10.

For any $0 < \varepsilon < R$ Denote $K_{\varepsilon,R} := K(x)\ \chi_{B_R(0)\setminus B_\varepsilon(0)}$. For a fixed ξ such that $\varepsilon < |\xi|^{-1} < R$, we write

$$\begin{aligned} \widehat{K_{\varepsilon,R}}(\xi) &= \int_{\varepsilon<|x|<R} e^{2\pi i\,x\cdot\xi}\ K(x)\,dx \\ &= \int_{\varepsilon<|x|<|\xi|^{-1}} e^{2\pi i\,x\cdot\xi}\ K(x)\,dx + \int_{|\xi|^{-1}<|x|<R} e^{2\pi i\,x\cdot\xi}\ K(x)\,dx \\ &= I_1 + I_2 \quad . \end{aligned}$$

We bound I_1 first. Using the cancellation assumption (4.41c), we have

$$I_1 = \int_{\varepsilon<|x|<|\xi|^{-1}} (e^{2\pi i\,x\cdot\xi} - 1)\ K(x)\,dx \quad .$$

Hence we deduce the following bound, using this time assumption (4.41a)

$$|I_1| \leq 2\pi\,|\xi|\ \int_{\varepsilon<|x|<|\xi|^{-1}} |x|\ |K(x)|\ dx \leq C_n\ A \quad .$$

In order to bound I_2 we introduce $z = \xi/2|\xi|^2$. Observe that the choice of z has been made in such a way that $\exp(2\pi\, iz \cdot \xi) = -1$, hence a change of variable $x \to x + z$ will generate a minus sign in front of the integral and formally we would have

$$\int_{\mathbb{R}^n} e^{2\pi i\, x\cdot\xi}\; K(x)\, dx = \frac{1}{2}\int_{\mathbb{R}^n} e^{2\pi i\, x\cdot\xi}\; K(x) - K(x-z)\, dx$$

which would put us in position to make use of the Hörmander condition (4.41b). The only difficulty is to keep track of the domains of integrations that we precise now.

$$2I_2 = \int_{|\xi|^{-1}<|x|<R} e^{2\pi i\, x\cdot\xi}\; K(x)\, dx - \int_{|\xi|^{-1}<|x-z|<R} e^{2\pi i\, x\cdot\xi}\; K(x-z)\; dx$$

We write

$$\begin{aligned}\int_{|\xi|^{-1}<|x-z|<R} e^{2\pi i\, x\cdot\xi}\; K(x-z)\; dx &= \int_{|\xi|^{-1}<|x|<R} \cdots\; dx \\ - \int_{|x-z|<|\xi|^{-1}<|x|} \cdots\; dx &- \int_{|x|<R<|x-z|} \cdots\; dx \\ + \int_{|x|<|\xi|^{-1}<|x-z|} \cdots\; dx &+ \int_{|x-z|<R<|x|} \cdots\; dx \quad .\end{aligned}$$

The following elementary inclusions are longer to state than to prove...

$$\{x\;;\;|x-z|<|\xi|^{-1}<|x|\} \subset \{x\;;\;|x-z|<|\xi|^{-1}<|x-z|+|z|\}$$
$$\{x\;;\;|x|<R<|x-z|\} \subset \{x\;;\;|x-z|-|z|<R<|x-z|\}$$
$$\{x\;;\;|x|<|\xi|^{-1}<|x-z|\} \subset \{x\;;\;|x-z|-|z|<|\xi|^{-1}<|x-z|\}$$
$$\{x\;;\;|x-z|<R<|x|\} \subset \{x\;;\;|x-z|<R<|x-z|+|z|\} \quad .$$

Using these inclusions and the fact that $|z| = 1/2|\xi|$, we can bound I_2 in the following way

$$\begin{aligned} 2|I_2| \le &\int_{|\xi|^{-1}<|x|<R} |K(x)-K(x-z)|\, dx \\ &+ \int_{\frac{1}{2}|\xi|^{-1}<|x|<\frac{3}{2}|\xi|^{-1}} |K(x)|\; dx + \int_{R-\frac{1}{2}|\xi|^{-1}<|x|<R+\frac{1}{2}|\xi|^{-1}} |K(x)|\; dx \end{aligned} \tag{4.44}$$

Since $|z| = \frac{1}{2}|\xi|^{-1}$ we can invoke the Hörmander condition (4.41b) and bound the first integral in the right-hand-side of (4.44) by B. For the second integral we use (4.41a) and bound it by a constant $C_n\, A$ and the third integral is treated in the same way using the fact that $|\xi|^{-1} < R$ which implies that the quotient of $R+\frac{1}{2}|\xi|^{-1}$ by $R-\frac{1}{2}|\xi|^{-1}$ is bounded by 3. Hence I_2 is bounded by $B + 4C_n\, A$. So we have proved that $|\widehat{K_{\varepsilon,R}}(\xi)|$ is uniformly bounded by a constant depending only on n, A and B, which is the desired result. $\square$

Proof of theorem 4.9.

Combining lemma 4.10 and theorem 4.8 we obtain (4.37) and (4.40) where Tf is replaced by $T_\varepsilon f$. It remains to show the L^p convergence (4.38), the L^1_w convergence (4.39) and inequality (4.40) for Tf itself.

We consider first a smooth function $f \in C_0^\infty(\mathbb{R}^n)$ and using the cancellation property (4.35c) we write

$$
\begin{aligned}
T_\varepsilon f(x) &= \int_{1\leq \|y\|} f(x-y)K(y)\,dy + \int_{\varepsilon\leq\|y\|\leq 1} f(x-y)K(y)\,dy \\
&= \int_{\mathbb{R}^n} f(x-y)K_1(y)\,dy + \int_{\varepsilon\leq\|y\|\leq 1} \big(f(x-y)-f(x)\big)K(y)\,dy\,.
\end{aligned}
\tag{4.45}
$$

Because of the regularity of f, using assumption (4.35a), we have the following bound which holds for every x in $\mathbb{R}^n$ and $y\neq 0$

$$
\left|\big(f(x-y)-f(x)\big)K(y)\right| \leq \|\nabla f\|_\infty\,\|y\|\,|K(y)| \overset{(4.35a)}{\leq} \|\nabla f\|_\infty\,\frac{A}{\|y\|^{n-1}}\quad . \tag{4.46}
$$

Hence, inserting the bound (4.46) in (4.45) we can define for every x the limit

$$
Tf(x) := \lim_{\varepsilon\to 0} T_\varepsilon f(x) = \int_{\mathbb{R}^n} f(x-y)K(y)\,dy \quad . \tag{4.47}
$$

Observe that at this stage Tf is a distribution obtained by the convolution between a smooth compactly supported function and the principal value of K, $p.v.K$, which is an order 1 distribution. However using (4.46) again we have

$$
\begin{aligned}
\forall x\in\mathbb{R}^n \qquad |Tf(x)-T_\varepsilon f(x)| &\leq \int_{B_\varepsilon(0)} |f(x-y)-f(x)|\;|K(y)|\;dy \\
&\leq C_n\|\nabla f\|_\infty\;A\;\varepsilon \quad .
\end{aligned}
\tag{4.48}
$$

Thus $T_\varepsilon f$ converges uniformly to Tf and hence in $L^p_{loc}(\mathbb{R}^n)$ for any $p\geq 1$. Let $R>1$ such that $f\equiv 0$ in $\mathbb{R}^n\setminus B_R(0)$. For $|x|>4R$

$$
K(y)[f(x-y)-f(x)] = K(y)\;f(x-y)
$$

is supported in $B_R(x)$ and one has $|K(y)\;f(x-y)|\leq 2^n\|f\|_\infty A/|x|^n$. Hence the bound (4.48) can be completed by a behavior at infinity as follows :

$$
\begin{aligned}
\forall x\in\mathbb{R}^n \qquad |Tf(x)-T_\varepsilon f(x)| &\leq \int_{B_\varepsilon(0)} |f(x-y)|\;|K(y)|\;dy \\
&\leq C_n\;A\;\frac{\varepsilon^n}{|x|^n}\|f\|_\infty \quad .
\end{aligned}
\tag{4.49}
$$

This later inequality implies that $T_\varepsilon f\to Tf$ in $L^p(\mathbb{R}^n)$ for any $p>1$ and that $|T_\varepsilon f - Tf|_{L^1_w}$ converges to zero.

Let us take now $f\in L^p(\mathbb{R}^n)$ for $p\geq 1$. Since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, using inequalities (4.19) and (4.20) for $K_\varepsilon\star g$ - where g is a difference between f and a finer and finer approximation of it in C_0^∞ for the L^p norm - a classical diagonal argument implies that, for $p>1$, $T_\varepsilon f$ converges strongly in L^p and that, for $p=1$, T_ε is Cauchy for the quasi-norm L^1_w. This concludes the proof of theorem 4.9. $\square$

Remark 4.7. *The exact cancellation assumption (4.41c) can be relaxed in the statement of theorem 4.9 by requiring only the existence of a constant $C > 0$ such that for any $0 < r < R < +\infty$*

$$\left|\int_{B_R(0)\setminus B_r(0)} K(x)\ dx\right| \leq C \quad . \tag{4.50}$$

*Under this weakened assumption however the convergence of $T_\varepsilon f$ to Tf does not necessarily hold in L^p or even almost everywhere but in the distributional sense only (see a counterexample in [**?**]). The nature of this convergence nevertheless is not a main point in the theory the most important one being given by the inequalities (4.37) and (4.40) which still hold under the weakest assumption (4.50).*

4.4.3 The case of homogeneous kernels

It is interesting to look at the case of homogeneous kernels which correspond to operators of special geometric interest - such as Hilbert Transform for instance. The following result is obtained as a corollary of theorem 4.9 and has the advantage to provide a "translation", in the special case of homogeneous Kernels, of general assumptions on K that imply (4.35a), (4.35b) and (4.35c). Precisely we consider kernels K of the form

$$K(x) = \frac{\Omega(x)}{\|x\|^n} \quad , \tag{4.51}$$

where Ω is an homogeneous function of degree 0, i.e., $\Omega(\delta x) = \Omega(x)$, for $\delta > 0$. In other words, the function Ω is radially constant and therefore completely determined by its values on the sphere S^{n-1}. Note also that K is homogeneous of degree $-n$, i.e., $K(\delta x) = \delta^{-n} K(x)$.

Proposition 4.12. *Let $K : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a measurable function given by $K(x) = \Omega(x)/\|x\|^n$ where Ω is an homogeneous function of degree 0 satisfying*

i)

$$\int_{S^{n-1}} \Omega(x)\, d\sigma(x) = 0 \quad . \tag{4.52}$$

ii) If we set

$$\omega(\delta) = \sup_{\substack{\|x-y\|\leq\delta \\ x,y\in S^{n-1}}} \left|\Omega(x) - \Omega(y)\right| \quad ,$$

the following integral is finite:

$$\int_0^1 \frac{\omega(\delta)}{\delta}\, d\delta < \infty\,. \tag{4.53}$$

Then K satisfies the conditions (4.35a)–(4.35c) and theorem 4.9 can be applied to K.

Remark 4.8. *Observe that the so called Dini condition ii) implies that Ω is continuous on S^{n-1}. Moreover observe that if Ω is assumed to be Hölder continuous, $C^{0,\alpha}(S^{n-1})$, for some exponent $1 > \alpha > 0$ then the Dini condition ii) is automatically satisfied.*

Proof of proposition 4.12.

The conditions (4.35a), respectively (4.35c), follow directly from (4.53), respectively (4.52) and integration in polar coordinates. In order to establish (4.35b), we first observe that

$$\begin{aligned}\int_{2\|y\|\leq\|x\|} \big|K(x-y)-K(x)\big|\,dx \;&\leq\; \int\limits_{2\|y\|\leq\|x\|} \frac{\big|\Omega(x-y)-\Omega(x)\big|}{\|x-y\|^n}\,dx \\ &\quad+ \int\limits_{2\|y\|\leq\|x\|} |\Omega(x)|\left|\frac{1}{\|x-y\|^n}-\frac{1}{\|x\|^n}\right| dx\,.\end{aligned} \tag{4.54}$$

Since Ω is bounded due to (4.53) and as a consequence of the mean value theorem

$$\left|\frac{1}{\|x-y\|^n}-\frac{1}{\|x\|^n}\right| \leq \frac{C\|y\|}{\|x\|^{n+1}}\,,$$

we conclude by integration in polar coordinates that the second integral on the right-hand side of (4.54) is finite. Note also that

$$\begin{aligned}\big|\Omega(x-y)-\Omega(x)\big| \;&=\; \left|\Omega\left(\frac{x-y}{\|x-y\|}\right)-\Omega\left(\frac{x}{\|x\|}\right)\right| \\ &\leq\; \omega\left(\left\|\frac{x-y}{\|x-y\|}-\frac{x}{\|x\|}\right\|\right)\end{aligned}$$

by definition of the function ω. Moreover, if $2\|y\|\leq\|x\|$, then $1/\|x-y\|^n\leq C/\|x\|^n$ and also

$$\left\|\frac{x-y}{\|x-y\|}-\frac{x}{\|x\|}\right\| \leq C\,\frac{\|y\|}{\|x\|} \quad .$$

Inserting these estimates in the first integral on the right-hand side of (4.54), we obtain

$$\begin{aligned}\int\limits_{2\|y\|\leq\|x\|} \frac{\big|\Omega(x-y)-\Omega(x)\big|}{\|x-y\|^n}\,dx \;&\leq\; C\int\limits_{2\|y\|\leq\|x\|} \frac{\omega\left(C\frac{\|y\|}{\|x\|}\right)}{\|x\|^n}\,dx \\ &\leq\; C\int_{2\|y\|}^{\infty} \frac{\omega\left(C\frac{\|y\|}{r}\right)}{r}\,dr \quad .\end{aligned}$$

Changing coordinates $\delta = C\|y\|/r$ and using (4.53), we deduce that the last integral is finite showing that (4.35b) holds and proposition 4.12 is proved. □

4.4.4 A multiplier type formulation

It is useful to explicit sufficient conditions on $\widehat{K}$ only that implies the strong type (p,p) (for $1<p<+\infty$) and the weak type $(1,1)$ properties of the corresponding convolution operator T. Such results are called *multiplier theorems* - $m(\xi) := \widehat{K}(\xi)$ is the *multiplier* associated to T. We shall give more and more sophisticated multiplier theorem in this book that will play a crucial role in characterizing real-variable function spaces using the Fourier transform. Multiplier theorems are moreover the basic tools in the analysis of pseudo-differential operators. Here is maybe the most elementary one that we will deduce from the previous sections.

Theorem 4.13. *Let m be a C^∞ function on $\mathbb{R}^n$ satisfying :*

$$\begin{aligned} &\forall\, l \in \mathbb{N} \quad \exists C_l > 0 \quad s.\ t. \quad \forall \xi \in \mathbb{R}^n \\ &|\nabla^l m|(\xi) \le C_l\ |\xi|^{-l} \quad . \end{aligned} \tag{4.55}$$

Let $p \in [1,+\infty)$. Define T_m on $L^p \cap L^2$ by

$$\forall f \in L^p \cap L^2(\mathbb{R}^n) \qquad \forall \xi \in \mathbb{R}^n \qquad \widehat{T_m f}(\xi) := m(\xi)\ \widehat{f}(\xi) \quad .$$

Then for $p \in (1,+\infty)$ there exists $C_{p,m} > 0$ such that for any $f \in L^p \cap L^2$

$$\|T_m f\|_{L^p} \le C_{p,m}\, \|f\|_{L^p}\, , \tag{4.56}$$

and there exists $C_{1,m} > 0$ such that for any $f \in L^1 \cap L^2$

$$\sup_{\alpha>0} \alpha\ \mu(\{x \in \mathbb{R}^n\ ;\ |T_m f(x)| > \alpha\}) \le C_{1,m}\ \|f\|_{L^1} \quad . \tag{4.57}$$

Hence T_m extends continuously as a linear operator of strong type (p,p) - $1 < p < +\infty$ - and weak type $(1,1)$.

Remark 4.9. *It is important to compare at this stage already, before to proceed to the proof of theorem 4.13 itself, the difference between the assumption (4.55) and the assumptions we made on K in the previous subsections. Take for instance the condition $|\nabla K|(x) \le C/|x|^{n+1}$ that implies the Hörmander condition (4.17) - as it is established in remark 4.5 c) - would hold if, for instance, we would assume $\nabla^{n+1} m$ to be in L^1. Observe that this later condition is just "at the border" to be implied, but is not implied, by our assumption (4.55). As it will be seen later in the book, assumption (4.55) is however very relevant to the theory.*

Proof of theorem 4.13. Theorem 4.13 will a direct consequence of theorem 4.8 once we will have proved that assumption (4.55) implies the Hörmander condition (4.17) for $K := \widehat{m}$ - Observe that (4.55) contains (4.16) already.

In order to establish the Hörmander condition we cannot afford to be as little cautious as we were in establishing the bound (4.21). We shall use a more refined argument based on dyadic decomposition in the Fourier variable ξ - the phase space. This techniques is making use of the Littlewood-Paley decomposition presented in chapter 5. Precisely let $\psi \in C_0^\infty(B_2(0))$ be a smooth non negative function with compact support in the ball $B_2(0)$ such that ψ equals identically 1 on $B_1(0)$ and let $\phi(\xi) := \psi(\xi) - \psi(2\xi)$. It follows from this definition that $\phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and that

$$1 \equiv \lim_{N\to+\infty} \sum_{k=-N}^{k=+N} \phi(2^{-k}\xi) = \sum_{k\in\mathbb{Z}} \phi(2^{-k}\xi) \qquad \text{on } \mathbb{R}^n \quad .$$

For $k \in \mathbb{Z}$ we denote

$$m_k(\xi) := \phi(\xi)\ m(2^{-k}\xi) \quad .$$

Observe that with this notation

$$m(\xi) = \sum_{k\in\mathbb{Z}} m_k(2^k\xi) \quad .$$

Denoting $K_k(x) := \widehat{m_k}(x)$, we have :

$$K(x) := \widehat{m}(\xi) = \sum_{k\in\mathbb{Z}} 2^{-k\,n} K_k(2^{-k}x) \quad . \tag{4.58}$$

Using now the assumption (4.55) on m and the definition of m_k, it is not difficult to see that

$$\forall l \in \mathbb{N} \qquad \exists\, C_l > 0 \quad \text{s.t.} \quad \forall k \in \mathbb{Z} \qquad \|\nabla^l m_k\|_{L^\infty(\mathbb{R}^n)} \le C_l \quad . \tag{4.59}$$

Moreover, since the m_k are supported in the fixed compact set $B_2(0) \setminus B_{1/2}(0)$, we deduce that every H^s norm of m_k is bounded independently of k.

Take $s > n$, we then have the existence of C, independent of k such that

$$\int_{\mathbb{R}^n} (1+|x|^2)^{s/2} |K_k(x)|^2\ dx = \|m_k\|_{H^s}^2 \le C \quad .$$

Hence, using Cauchy-Schwarz, we deduce the following bound

$$\begin{aligned} &\int_{|x|>|y|} |K_k(x)|\ dx \\ &\le \left[\int_{|x|>|y|} \frac{1}{(1+|x|^2)^{s/2}}\ dx\right]^{\frac{1}{2}} \left[\int_{\mathbb{R}^n} (1+|x|^2)^{s/2} |K_k(x)|^2\ dx\right]^{\frac{1}{2}} \\ &\le \frac{C}{(1+|y|)^{-n/2+s/2}} \end{aligned} \tag{4.60}$$

where C is possibly a new constant but again independent of k.

Similarly as before, $\xi\, m_k(\xi)$ is a function supported in the fixed compact set $B_2(0) \setminus B_{1/2}(0)$ and, hence, (4.59) implies that

$$\forall\, l \in \mathbb{N} \qquad \exists\, C_l > 0 \quad \text{s.t.} \quad \forall k \in \mathbb{Z} \qquad \|\nabla^l(\xi\, m_k(\xi))\|_{L^\infty(\mathbb{R}^n)} \le C_l \quad . \tag{4.61}$$

Hence for the same reasons as above we obtain a uniform bound, independent of k, for ∇K_k. Precisely there exists $C>0$ such that for every $k\in\mathbb{Z}$

$$\int_{\mathbb{R}^n} |\nabla K_k(x)|\ dx \le C < +\infty \quad . \tag{4.62}$$

Let now $y\in\mathbb{R}^n$ and denote $v=y/|y|$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \big|K_k(x-y)-K_k(x)\big|\,dx\ &= \int_{\mathbb{R}^n}\left|\int_0^{|y|}\frac{\partial K_k}{\partial v}(x+t\,v)\ dt\right|\ dx\\ &\le |y|\int_{\mathbb{R}^n}|\nabla K_k|(z)\ dz\le C\ |y| \quad .\end{aligned} \tag{4.63}$$

Consider again $y\in\mathbb{R}^n\setminus\{0\}$ and let k_0 be the largest integer less than $\log_2|y|$: $k_0=[\log_2|y|]$. Using (4.60), we obtain

$$\begin{aligned}&\int_{|x|>2|y|}\left|\sum_{k\le k_0}2^{-n\,k}\left[K_k(2^{-k}(x+y)-K_k(2^{-k}x)\right]\right|\ dx\\ &\le 2\sum_{k\le k_0}\int_{|z|>2^{-k}|y|}|K_k(z)|\ dz\le\sum_{k\le k_0}\frac{C}{(1+2^{-k}|y|)^\alpha}\end{aligned}$$

where $\alpha=-n/2+s/2>0$. Hence, we have in one hand

$$\begin{aligned}\int_{|x|>2|y|}\left|\sum_{k\le k_0}2^{-n\,k}\left[K_k(2^{-k}(x+y)-K_k(2^{-k}x)\right]\right|\ &\le C\sum_{k\le k_0}2^{\alpha(k-k_0)}\\ &\le\frac{C}{1-2^\alpha} \quad .\end{aligned} \tag{4.64}$$

In the other hand, using (4.63), we have

$$\begin{aligned}&\int_{|x|>2|y|}\left|\sum_{k\ge k_0}2^{-n\,k}\left[K_k(2^{-k}(x+y)-K_k(2^{-k}x)\right]\right|\\ &\le\sum_{k\ge k_0}\int_{|z|>2^{-k}|y|}\left|K_k(z+2^{-k}y)-K_k(z)\right|\ dz\\ &\le C\sum_{k\ge k_0}2^{-k}|y|\le 2C\sum_{k\ge k_0}2^{k_0-k}\le 4\,C\end{aligned} \tag{4.65}$$

Combining (4.64) and (4.65) gives

$$\int_{|x|>2|y|}|K(x+y)-K(x)|\ dx\le B<+\infty \quad , \tag{4.66}$$

where B is independent of $y\in\mathbb{R}^n\setminus\{0\}$. This is the Hörmander condition (4.17) and theorem 4.13 is proved. □

4.4.5 Applications: The L^p theory of the Riesz Transform and the Laplace and Bessel Operators

In this subsection we apply to the Riesz Transform and the Laplace Operator the L^p continuity of the convolution type Calderón-Zygmund operators that we proved above.

For $j = 1, \ldots, n$, we now consider the kernels $K_j(x) = \Omega_j(x)/\|x\|^n$ with

$$\Omega_j(x) = c_n \frac{x_j}{\|x\|} \, , \tag{4.67}$$

where

$$c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}} \quad .$$

Observe first that $\Omega_j = c_n x_j$ is smooth on S^{n-1} and moreover, since Ω_j is an odd function the cancellation property

$$\int_{S^{n-1}} \Omega_j(x) \, d\sigma(x) = 0$$

also holds. Hence proposition 4.12 can be applied to the kernels K_j. For any $1 \le p < \infty$, any $j = 1 \cdots n$ and any $f \in L^p(\mathbb{R}^n)$ the following limit exists (in L^p or L^1_w when $p = 1$)

$$R_j f(x) = \lim_{\varepsilon \to 0} R_{j,\varepsilon} f(x) \quad , \tag{4.68}$$

where

$$\begin{aligned} R_{j,\varepsilon} f(x) &= \int_{\varepsilon \le \|y\|} f(x-y) K_j(y) \, dy \\ &= c_n \int_{\varepsilon \le \|y\|} f(x-y) \frac{y_j}{\|y\|^{n+1}} \, dy \, . \end{aligned}$$

Definition 4.14. Riesz Transform *For any function $f \in L^p(\mathbb{R}^n)$, $1 \le p < +\infty$, the $\mathbb{R}^n$ valued measurable map given almost everywhere by*

$$Rf(x) := (R_1 f(x), \cdots, R_n f(x)) \quad ,$$

is called the Riesz transform of f.

Theorem 4.9 implies the following proposition

Proposition 4.15. *For any $1 < p < +\infty$ and any $f \in L^p(\mathbb{R}^n)$*

$$\|Rf\|_{L^p} \le C_{n,p} \, \|f\|_{L^p} \quad . \tag{4.69}$$

Moreover, for any $f \in L^1(\mathbb{R}^n)$

$$\sup_{\alpha > 0} \alpha \; \mu(\{x \in \mathbb{R}^n \; ; \; |Rf(x)| > \alpha\}) \le C_n \, \|f\|_{L^1} \quad . \tag{4.70}$$

We now derive the multiplier $m(\xi) = (m_1(\xi), \cdots, m_n(\xi))$ corresponding to the Riesz transform. Precisely we establish the following result.

Proposition 4.16. *The following holds*

$$\widehat{R_j f}(\xi) = \frac{i\,\xi_j}{|\xi|}\,\widehat{f}(\xi) = m_j(\xi)\,\widehat{f}(\xi) \tag{4.71}$$

i.e. the multiplier corresponding to R_j is

$$m_j(\xi) = i\frac{\xi_j}{|\xi|} \quad .$$

Remark 4.10. *Observe that the multipliers $m_j(\xi)$ of the components R_j of the Riesz transform R satisfy the main assumption (4.55) of theorem 4.13. Hence combining the previous proposition together with the theorem 4.13 provides a new proof of proposition 4.15.*

Proof of proposition 4.16. For a C^∞_0 function f we have that

$$K_j \star f = c_n\ PV\left(\frac{x_j}{|x|^{n+1}}\right) \star f = -\frac{c_n}{n-1}\frac{\partial}{\partial x_j}|x|^{-n+1} \star f \quad .$$

Hence

$$m_j(\xi) = 2i\pi \frac{c_n}{n-1}\ \xi_j\ \widehat{|x|^{-n+1}} \quad . \tag{4.72}$$

In order to identify m_j it remains to compute the Fourier transform of $|x|^{-n+1}$. Denoting $d\sigma^{n-1}$ the canonical volume form on the $n-1$ sphere, one has for $\xi \neq 0$:

$$\begin{aligned}
\widehat{|x|^{-n+1}}(\xi) &= \lim_{\delta\to 0} \frac{\widehat{e^{-\pi\,\delta\,|x|^2}}}{|x|^{n-1}}(\xi)\\
&= \int_0^{+\infty}\int_{S^{n-1}} e^{-\pi\,\delta\,\rho^2}\ e^{2\pi\,i\ \rho\ \zeta\cdot\xi}\ d\sigma^{n-1}(\zeta)\ d\rho \quad .
\end{aligned}$$

Denote $S^{n-1}_\xi := \{\zeta \in S^{n-1}\ ;\ \zeta\cdot\xi \geq 0\}$. Using this notation, the previous identity becomes

$$\begin{aligned}
\widehat{|x|^{-n+1}}(\xi) &= \int_0^{+\infty}\int_{S^{n-1}_\xi} e^{-\pi\,\delta\,\rho^2}\ e^{2\pi\,i\,\rho\,\zeta\cdot\xi}\ d\sigma^{n-1}(\zeta)\ d\rho\\
&\quad + \int_0^{+\infty}\int_{S^{n-1}\setminus S^{n-1}_\xi} e^{-\pi\,\delta\,\rho^2}\ e^{2\pi\,i\ \rho\ \zeta\cdot\xi}\ d\sigma^{n-1}(\zeta)\ d\rho\\
&= \int_0^{+\infty}\int_{S^{n-1}_\xi} e^{-\pi\,\delta\,\rho^2}\ \left[e^{2\pi\,i\ \rho\ \zeta\cdot\xi} - e^{-2\pi\,i\ \rho\ \zeta\cdot\xi}\right]\ d\sigma^{n-1}(\zeta)\ d\rho\\
&= \int_{S^{n-1}_\xi} d\sigma^{n-1}(\zeta)\int_{\mathbb{R}} e^{-\pi\,\delta\,\rho^2}\ e^{2\pi\,i\,\rho\,|\xi|\,\alpha}\ d\rho \quad ,
\end{aligned}$$

where $\alpha := \zeta\cdot\xi/|\xi|$. Using the fact that the Fourier transform of $e^{-\pi\,\delta\,t^2}$ is equal at the point τ to $\delta^{-1/2}\ e^{-\pi\,(\tau/\sqrt{\delta})^2}$, we obtain

$$\widehat{|x|^{-n+1}}(\xi) = \int_{S^{n-1}_\xi} \frac{1}{\sqrt{\delta}}\ e^{-\pi\ \left(\frac{|\xi|\,\alpha}{\sqrt{\delta}}\right)^2}\ d\sigma^{n-1}(\zeta) \quad . \tag{4.73}$$

We interpret $\alpha = z_1$ as being the first coordinate of a positive orthonormal basis containing the unit vector $\xi/|\xi|$ as first vector. We have

$$d\sigma^{n-1} = \sum_{i=1}^{n} (-1)^{i-1}\ z_i\ dz_1 \cdots dz_{i-1} \wedge dz_{i+1} \cdots dz_n \quad .$$

We decompose $d\sigma^{n-1}$ is the following way : $d\sigma^{n-1} = dz_1 \wedge d\sigma^{n-2} + z_1\ dz_2 \cdots dz_n$

$$\begin{aligned}\int_{S_\xi^{n-1}} \frac{1}{\sqrt{\delta}}\ e^{-\pi\ \left(\frac{|\xi|\,\alpha}{\sqrt{\delta}}\right)^2}\ d\sigma^{n-1}(\zeta) &= \int_{S^{n-2}} d\sigma^{n-2} \int_0^1 \frac{1}{\sqrt{\delta}} e^{-\pi\ \left(\frac{|\xi|\,z_1}{\sqrt{\delta}}\right)^2}\ dz_1 \\ &+ \int_{S_\xi^{n-1}} \frac{z_1}{\sqrt{\delta}}\ e^{-\pi\ \left(\frac{|\xi|\,z_1}{\sqrt{\delta}}\right)^2}\ dz_2 \cdots dz_n \quad .\end{aligned} \tag{4.74}$$

Since , as δ goes to zero, $\frac{z_1}{\sqrt{\delta}}\ e^{-\pi\ \left(\frac{|\xi|\,z_1}{\sqrt{\delta}}\right)^2}$ is converging to zero uniformly on any compact subset of $S_\xi^{n-1} \setminus \{\xi/|\xi|\}$, we obtain that

$$\int_{S_\xi^{n-1}} \frac{1}{\sqrt{\delta}}\ e^{-\pi\ \left(\frac{|\xi|\,\alpha}{\sqrt{\delta}}\right)^2}\ d\sigma^{n-1}(\zeta) = |S^{n-2}|\ |\xi|^{-1} \int_0^{\frac{|\xi|}{\sqrt{\delta}}} e^{-\pi t^2}\ dt + o_\delta(1) \quad , \tag{4.75}$$

where $|S^{n-2}|$ denotes the volume of the $n-2$ unit sphere which is equal to $2\ \pi^{(n-1)/2}/\Gamma((n-1)/2)$ - Γ is the Euler Gamma Function. Recall that

$$\int_0^{+\infty} e^{-\pi t^2}\ dt = \frac{1}{2} \quad .$$

Hence, combining (4.72) and (4.75) we obtain that

$$m_j(\xi) = 2i\, \frac{\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)} \frac{c_n}{n-1}\ \frac{\xi_j}{|\xi|} = \frac{\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}\ c_n \frac{\xi_j}{|\xi|} = i\frac{\xi_j}{|\xi|} \quad .$$

where we have used that $\Gamma(z+1) = z\,\Gamma(z)$. We have proved proposition 4.16. $\square$

Let $f \in C_0^2(\mathbb{R}^n)$ and note that the Fourier transform of its second order partial derivatives are given by

$$\widehat{\partial_k \partial_j f}(\xi) = (i\,\xi_k)(i\,\xi_j)\,\widehat{f}(\xi) = -\xi_k \xi_j\,\widehat{f}(\xi)\,.$$

In particular, we have for the Fourier transform of the Laplace operator $\widehat{\Delta f}(\xi) = -\|\xi\|^2\,\widehat{f}(\xi)$. This enables us to write the following:

$$\begin{aligned}\widehat{\partial_k \partial_j f}(\xi) \quad &= \quad -\xi_k \xi_j\,\widehat{f}(\xi) = \frac{i\,\xi_k}{\|\xi\|} \frac{i\,\xi_j}{\|\xi\|}\,\widehat{\Delta f}(\xi) \\ &\overset{(4.71)}{=} \quad \frac{i\,\xi_k}{\|\xi\|} \widehat{R_j(\Delta f)}(\xi) \overset{(4.71)}{=} \big(R_i(\widehat{R_j(\Delta f)})\big)(\xi)\,.\end{aligned}$$

Thus, we get

$$\partial_i \partial_j f = R_i\big(R_j(\Delta f)\big)\,. \tag{4.76}$$

From (4.69), it then follows for $1 < p < +\infty$ that

$$\begin{aligned}\|\partial_i\partial_j f\|_{L^p} &= \big\|R_i\big(R_j(\Delta f)\big)\big\|_{L^p}\\ &\le C_{n,p}\,\|R_j(\Delta f)\|_{L^p} \le C_{n,p}^2\,\|\Delta f\|_{L^p}\,,\end{aligned}$$

Using the density of $C_0^\infty(\mathbb{R}^n)$ in the Sobolev space $W^{2,p}(\mathbb{R}^n)$, we have proved the following result.

Proposition 4.17. *Let $1 < p < +\infty$. There exists a positive constant $C_p > 0$ such that, for any function f in the Sobolev Space $W^{2,p}(\mathbb{R}^n)$ the following identity holds*

$$\|\nabla^2 f\|_{L^p(\mathbb{R}^n)} \le C_p\,\|\Delta f\|_{L^p(\mathbb{R}^n)} \quad .$$

where $\nabla^2 f$ denotes the Hessian matrix of f.

The previous result can be improved when the operator Δ is made inhomogeneous and more coercive by adding $-id$ to it. Precisely, the following result which says that the inverse of the *Bessel Operator*, given by $(\Delta - id)^{-1}$, is continuous from $L^p(\mathbb{R}^n)$ into $W^{2,p}(\mathbb{R}^n)$ is a direct application of theorem 4.13.

Proposition 4.18. *Let $1 < p < +\infty$. Let f be an L^p function on $\mathbb{R}^n$. Then there exists a unique tempered Distribution u in $\mathcal{S}'(\mathbb{R}^n)$ such that $(\Delta - id)u = f$ moreover u belongs to the Sobolev Space $W^{2,p}(\mathbb{R}^n)$ and the following inequality holds*

$$\|u\|_{W^{2,p}} \le C_p\,\|f\|_{L^p(\mathbb{R}^n)} \quad .$$

Proof of Proposition 4.18. A tempered Distribution f being given and $\widehat{f}$ being its Fourier transform, $-(1+|\xi|^2)^{-1}\widehat{f}(\xi)$ is the Fourier transform of the only tempered Distribution solution to

$$\Delta u - u = f \qquad \text{in } \mathcal{S}'(\mathbb{R}^n) \quad .$$

It is straightforward to check that the multipliers $-(1+|\xi|^2)^{-1}$, $-i\xi_j\,(1+|\xi|^2)^{-1}$ and $\xi_k\,\xi_j\,(1+|\xi|^2)^{-1}$ satisfy the assumption (4.55) of theorem 4.13 and hence proposition 4.18 follows. □

4.4.6 The limiting case $p = 1$

As for the sub-linear maximal operator, Calderón-Zygmund convolution operators are usually not bounded from L^1 into L^1. The following proposition illustrates this fact.

Proposition 4.19. *Let R be the Riesz Transform and let $f \in L^1(\mathbb{R}^n)$ such that $f \ge 0$ on $\mathbb{R}^n$ and $f \not\equiv 0$ then the measurable function Rf is not in $L^1(\mathbb{R}^n)$.*

Proof of proposition 4.19. Since f is in $L^1(\mathbb{R}^n)$, $\widehat{f}$ is a continuous function and moreover $\widehat{f}(0) = \int_{\mathbb{R}^n} f(x)\,dx > 0$.

$m_j(\xi) = \xi_j/|\xi|$ is discontinuous at the origin and hence, since $\widehat{f}$ is continuous at the origin and since $\widehat{f}(0) \ne 0$, $m_j(\xi)\,\widehat{f}(\xi)$ is also discontinuous at the origin.

Assuming $Rf \in L^1(\mathbb{R}^n)$ this implies that $\widehat{Rf}$ is continuous too on $\mathbb{R}^n$ and in particular at 0, which contradicts the previous assertion. □

Lemma 4.20. *There exists $f \in L^1(\mathbb{R}^2)$ such that, for any $u \in \mathcal{S}'(\mathbb{R}^2)$ satisfying*

$$\Delta u = f \quad \textit{in} \ \ \mathcal{S}'(\mathbb{R}^2), \tag{4.77}$$

then $\nabla^2 u \notin L^1_{\text{Loc}}(\mathbb{R}^2)$. □

Proof of Lemma 4.20. We choose

$$f(x) := \frac{\mathbf{1}_{D^2_{1/2}(x)}}{|x|^2 \operatorname{Log}^2|x|} ,$$

where $\mathbf{1}_{D^2_{1/2}}(x)$ is the characteristic function of the disc of radius $1/2$ and centered at the origin. One easily verifies that $f \in L^1(\mathbb{R}^2)$. We are now looking for an axially symmetric solution of (4.77) in $\mathcal{S}'(\mathbb{R}^2)$. That is, we look for $u(x) = v(|x|)$ and we use the conventional notation $r = |x|$. V should then satisfy

$$\ddot{v} + \frac{\dot{v}}{r} = \frac{\mathbf{1}_{[0,1/2](r)}}{r^2 \operatorname{Log}^2 r} \ \text{ in } \ \mathbb{R}^*_+$$

where $\mathbf{1}_{[0,1/2](r)}$ is the characteristic function of the segment $[0, 1/2]$. In other words,

$$\frac{d}{dr}(r\,\dot{v}) = \frac{\mathbf{1}_{[0,1/2](r)}}{r \operatorname{Log}^2 r} .$$

For this to be satisfied, it suffices

$$\dot{v}(r) = \begin{cases} -\dfrac{1}{r \operatorname{Log} r} & \text{for } \ r \in \left(0, \frac{1}{2}\right] \\[2ex] -\dfrac{1}{r \operatorname{Log} 1/2} & \text{for } \ r > \frac{1}{2} \end{cases} .$$

This holds in particular if

$$v(r) = \begin{cases} + \operatorname{Log}\Big[\dfrac{1}{\operatorname{Log} r^{-1}}\Big] \text{ for } \ r \in \left(0, \frac{1}{2}\right] \\[2ex] 1 + \dfrac{\operatorname{Log} r}{\operatorname{Log} 2} - \operatorname{Log}\operatorname{Log} 2 \text{ for } \ r > \frac{1}{2} \end{cases} .$$

Observe that $u(x) := v(|x|) \in \mathcal{S}'(\mathbb{R}^2)$ because it can obviously be decomposed as the sum of an element in $L^1(\mathbb{R}^2)$ and an element in $\mathcal{O}(\mathbb{R}^2)$. By construction, we have

$$\Delta u(x) = f(x) \ \text{ in } \ \mathcal{S}'(\mathbb{R}^2 \setminus \{0\}) .$$

Let $\chi(x)$ be a cut-off function in $C^\infty_c(B_1(0))$ with $\chi \equiv 1$ on $B_{1/2}(0)$. Denote $\chi_\varepsilon(x) = \chi(\frac{x}{\varepsilon})$. For any $\varphi \in \mathcal{S}(\mathbb{R}^2)$ one has

$$\int_{\mathbb{R}^2} \varphi[\Delta u - f(x)]\, dx + \int_{\mathbb{R}^2} \chi_\varepsilon \varphi[\Delta u - f(x)]\, dx.$$

Since $f \in L^1(\mathbb{R}^2)$

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^2} \chi_\varepsilon \varphi \; f(x)\, dx = 0. \tag{4.78}$$

We write

$$\begin{aligned} \int_{\mathbb{R}^2} \chi_\varepsilon \varphi(x)\, \Delta u(x)\, dx = &- \int_{\mathbb{R}^2} \nabla \chi_\varepsilon \; \nabla u \; \varphi(x)\, dx \\ &+ \int_{\mathbb{R}^2} \chi_\varepsilon \; \nabla_u \cdot \nabla \varphi(x)\, dx. \end{aligned} \tag{4.79}$$

Oberseve that for $|x| < \frac{1}{2}$

$$\nabla u = \dot{v}(r) \; \frac{\partial}{\partial r} = \frac{1}{r \operatorname{Log} r^{-1}} \; \frac{\partial}{\partial r} \; .$$

Since

$$\int_{B_\varepsilon(0)} |\nabla u|^2 \, dx = 2\pi \int_0^\varepsilon \frac{dr}{r (\operatorname{Log} r^{-1})^2} = \frac{2\pi}{\operatorname{Log} \varepsilon^{-1}},$$

we have

$$\lim_{\varepsilon \to 0} \int_{B_\varepsilon} |\nabla u|^2 \, dx = 0.$$

Hence this last fact implies

$$\lim_{\varepsilon \to 0} \left| \int_{\mathbb{R}^2} \chi_\varepsilon \, \nabla u \cdot \nabla \varphi(x)\, dx \right| \leq \lim_{\varepsilon \to 0} \; \|\nabla \varphi\|_\infty \; \|\chi\|_\infty \, \varepsilon \left[\int_{B_\varepsilon(0)} |\nabla u|^2 \, dx \right]^{\frac{1}{2}} = 0.$$

Moreover we have also

$$\lim_{\varepsilon \to 0} \left| \int_{\mathbb{R}^n} \nabla \chi_\varepsilon \cdot \nabla u \, \varphi \right| \leq \lim_{\varepsilon \to 0} \; \left[\int_{B_\varepsilon(0)} |\nabla u|^2 \, dx \right]^{\frac{1}{2}} = 0.$$

Hence we have proved

$$\Delta u = f \;\text{ in }\; \mathcal{S}'(\mathbb{R}^2).$$

A classical computation gives for $|x| < \frac{1}{2}$

$$\sum_{i,j=1}^{2} \frac{x_i \, x_j}{r^2} \; \frac{\partial^2 u}{\partial x_i \, \partial x_j} = \frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2 \operatorname{Log} r^{-1}} + \frac{1}{r^2 (\operatorname{Log} r)^2}.$$

Hence

$$\int_{B_{1/2}} \left| \sum_{i,j=1}^{2} \frac{x_i \, x_j}{r^2} \; \frac{\partial^2 u}{\partial x_i \, \partial x_j} \right| dx = +\infty$$

and we cannot have that $\Delta^2 u \in L^1_{\text{loc}}(\mathbb{R}^2)$. This concludes the proof of the Lemma. $\square$

This being established, if we make a slightly stronger integrability assumption on the function f such as $f \in L^1 \, \log L^1(\mathbb{R}^n)$, then, in the similar way to the case of the maximal sub-linear operator, Tf is in L^1_{loc}.

Theorem 4.21. *Let T be a convolution operator satisfying the assumptions of either theorem 4.8, theorem 4.9, theorem 4.13 or proposition 4.12. Let f be a measurable function in $L^1\log L^1(\mathbb{R}^n)$, then $Tf\in L^1_{loc}(\mathbb{R}^n)$ and for any measurable subset A of finite Lebesgue measure the following inequality holds*

$$\int_A |Tf(y)|\ dy \le C_T\ \int_{\mathbb{R}^n} |f(y)|\ \log\left(e+\mu(A)\ \frac{|f(y)|}{\|f\|_{L^1}}\right)\ dy\,, \tag{4.80}$$

where $C_T>0$ only depends on T.

Proof of theorem 4.21. We use the notations from the proof of theorem 4.8. For any positive number α we proceed to the Calderón-Zygmund decomposition of f : $f=g_\alpha+b_\alpha$. -we add the subscript α in order to insists on the fact that the result of the decomposition depends on α. Let $\delta>0$ to be fixed later and write

$$\begin{aligned}\int_A |Tf|(x)\ dx\ &= \int_0^\delta \mu\left(\{x\in A\ ;\ |Tf(x)|>\alpha\}\right)\ d\alpha\\ &+\int_\delta^{+\infty} \mu\left(\{x\in A\ ;\ |Tf(x)|>\alpha\}\right)\ d\alpha \quad .\end{aligned} \tag{4.81}$$

We use the decomposition $f=g_\alpha+b_\alpha$ in order to deduce :

$$\begin{aligned}\mu(\{x\ :\ |Tf(x)|>\alpha\})\ &\le \mu(\{x\ :\ |Tg_\alpha(x)|>\alpha/2\})\\ &+\mu(\{x\ :\ |Tb_\alpha(x)|>\alpha/2\}) \quad .\end{aligned} \tag{4.82}$$

We have, using the embedding $L^2(\mathbb{R}^n)\hookrightarrow L^{2,\infty}(\mathbb{R}^n)$

$$\int_\delta^{+\infty} \mu\Big(\Big\{x\in A:\ |Tg_\alpha(x)|>\frac{\alpha}{2}\Big\}\Big)\,d\alpha\le c_n\int_\delta^{+\infty} \|g_\alpha\|^2_{L^2(\mathbb{R}^n)}\ \frac{d\alpha}{\alpha^2}. \tag{4.83}$$

We decompose

$$\int_{\mathbb{R}^n} |g_\alpha|(x)^2 dx=\int_{\mathbb{R}^n\setminus\Omega_\alpha} |g_\alpha|^2(x)\,dx+\int_{\Omega_\alpha} |g_\alpha|^2(x)\,dx,$$

where Ω_α is the "bad set" away from which $g_\alpha\equiv f$. Recall moreover that

$$\sup_{x\in\Omega_\alpha} |g_\alpha|(x)\le 2^n\alpha$$

and

$$|f(x)|\le\alpha\ \text{ in }\ \mathbb{R}^n\setminus\Omega_\alpha.$$

Combining these facts with (4.83) give:

$$\begin{aligned}\int_\delta^{+\infty} \mu\Big(\Big\{x\in A:|Tg_\alpha(x)|<\frac{\alpha}{2}\Big\}\Big)\,d\alpha &\le C_n\int_\delta^{+\infty}\frac{d\alpha}{\alpha^2}\ \int_{|f|\le\alpha} |f|^2(x)\,dx\\ &+\int_\delta^{+\infty} 2^{2n}\ \frac{\alpha^2}{\alpha^2}\ \mu(\Omega_\alpha)\,d\alpha.\end{aligned} \tag{4.84}$$

Using Fubini, we have in one hand

$$\int_{\delta}^{+\infty} \frac{d\alpha}{\alpha^2} \int_{|f|\leq\alpha} |f|^2(x)\,dx = \int_{\mathbb{R}^n} |f|^2(x)\,dx \int_{\max\{\delta,|f|(x)\}}^{+\infty} \frac{d\alpha}{\alpha^2}$$
$$\leq \int_{\mathbb{R}^n} \frac{|f|^2(x)}{\max\{\delta,|f|(x)\}}\,dx \leq \|f\|_{L^1(\mathbb{R}^n)}. \tag{4.85}$$

In the other hand, we recall that the bad set Ω_α is the union of disjoint cubes $(C_k)_{k\in\mathbb{N}}$ and on each of these cubes the average of $|f|$ is larger than α. Hence we have

$$\mu(\Omega_\alpha) = \sum_{k\in\mathbb{N}} \mu(C_k) \leq \alpha^{-1} \sum_{k\in\mathbb{N}} \int_{C_k} |f|(x)\,dx$$
$$= \alpha^{-1} \int_{\Omega_\alpha} |f|(x)\,dx.$$

We write then

$$\mu(\Omega_\alpha) \leq \alpha^{-1} \int_{\Omega_\alpha} |f|(x)\,dx = \alpha^{-1} \int_{\Omega_\alpha\cap\{x;|f|(x)>\frac{\alpha}{2}\}} |f|(x)\,dx$$
$$+ \alpha^{-1} \int_{\Omega_\alpha\cap\{x;|f|(x)<\frac{\alpha}{2}\}} |f|(x)\,dx$$
$$\leq \frac{\mu(\Omega_\alpha)}{2} + \alpha^{-1} \int_{|f|>\frac{\alpha}{2}} |f|(x)\,dx.$$

Thus we just proved

$$\frac{\mu(\Omega_\alpha)}{2} \leq \alpha^{-1} \int_{|f|>\frac{\alpha}{2}} |f|(x)\,dx. \tag{4.86}$$

Combining (4.84), (4.85) and (4.86), we finally obtain

$$\int_{\delta}^{+\infty} \mu\Big(\Big\{x \in A : |Tg_\alpha(x)| > \frac{\alpha}{2}\Big\}\Big)\,d\alpha \leq C_n\Big[\|f\|_{L^1} + \int_{\delta}^{+\infty} \frac{d\alpha}{\alpha} \int_{|f|>\frac{\alpha}{2}} |f|(x)\,dx\Big]$$
$$\leq C_n\Big[\|f\|_{L^1} + \int |f|(x)\,\mathrm{Log}+\Big(\frac{2|f|(x)}{\delta}\Big)\Big]. \tag{4.87}$$

Now we bound the contribution of the action of T on the bad part. We have seen in the proof of the primitive formulation of L^p theorem for convolution Calderón-Zygmund kernels that the following inequality holds

$$\int_{\mathbb{R}^n\setminus\widetilde{\Omega}_\alpha} |T\,b_\alpha| \leq C(T) \int_{\widetilde{\Omega}_\alpha} |f|(x)\,dx,$$

where $\widetilde{\Omega}_\alpha = \bigcup_{k\in\mathbb{N}} \widetilde{C}_k$ and $\widetilde{C}_k$ are the cubes obtained from the C_k by dilating by the factor $2\sqrt{n}$ leaving the cube centers fixed.

For any $\beta > 0$, we bound

$$\begin{aligned}\mu(\{x \in A:\ |Tb_\alpha|(x) \geq \beta\}) &\leq \mu(\widetilde{\Omega}_\alpha) + \mu(\{x \in \mathbb{R}^n\backslash\widetilde{\Omega}_\alpha; |T\, b_\alpha(x)| > \beta\}\\ &\leq (2\sqrt{n})^n\, \mu(\Omega_\alpha) + \frac{C_n}{\beta} \int_{\mathbb{R}^n\backslash\widetilde{\Omega}_\alpha} |T\, b_\alpha|(x)\, dx\\ &\leq (2\sqrt{n})^n\, \mu(\Omega_\alpha) + \frac{C_n}{\beta} \int_{\widetilde{\Omega}_\alpha} |f|(x)\, dx.\end{aligned}$$

We apply this inequality to $\beta = \frac{\alpha}{2}$ and we integrate between δ and $+\infty$. We obtain

$$\begin{aligned}\int_\delta^{+\infty} \mu\Big(\Big\{x \in A:\ |Tb_\alpha|(x) \geq \frac{\alpha}{2}\Big\}\Big)\, d\alpha &\leq C_n \int_\delta^{+\infty} \mu(\Omega_\alpha)\, d\alpha\\ &\quad + C_n \int_\delta^{+\infty} \frac{d\alpha}{\alpha} \int_{\widetilde{\Omega}_\alpha} |f|(x)\, dx.\end{aligned}$$

We decompose again

$$\begin{aligned}\frac{1}{\alpha} \int_{\widetilde{\Omega}_\alpha} |f|(x)\, dx &\leq \frac{1}{\alpha} \int_{\{x\in\widetilde{\Omega}_\alpha;|f|(x)<\frac{\alpha}{2}\}} |f|(x)\, dx + \frac{1}{\alpha} \int_{|f|>\frac{\alpha}{2}} |f|(x)\, dx\\ &\leq C_n\, \mu(\Omega_\alpha) + \frac{1}{\alpha} \int_{|f|>\frac{\alpha}{2}} |f|(x)\, dx.\end{aligned}$$

Hence we have proved

$$\begin{aligned}\int_\delta^{+\infty} \mu\Big(\Big\{x \in A: |Tg_\alpha(x)| \geq \frac{\alpha}{2}\Big\}\Big)\, d\alpha &\leq c \int_\delta^{+\infty} \mu(\Omega_\alpha)\, d\alpha\\ &\quad + c \int_\delta^{+\infty} \frac{d\alpha}{\alpha} \int_{|f|>\frac{d\alpha}{\alpha}} |f|(x)\, dx.\end{aligned}$$

Using (4.86) again, we then have

$$\int_\delta^{+\infty} \mu\Big(\Big\{x \in A:\ |Tb_\alpha|(x) \geq \frac{\alpha}{2}\Big\}\Big) \leq c \int_{\mathbb{R}^n} |f|(x)\, \mathrm{Log} + \Big(\frac{2|f|(x)}{\delta}\Big)\, dx, \tag{4.88}$$

where c depends on T. Combining (4.87) and (4.88) together with (4.88), we obtain

$$\int_A |T\, f|(x) \leq \delta\, \mu(A) + c \int_{\mathbb{R}^n} |f|(x)\, \mathrm{Log}\Big[e + \frac{2|f|(x)}{\delta}\Big].$$

The inequality (4.81) follows by taking $\delta = 2\, \|f\|_{L^1/\mu(A)}$.

5 The L^p-Theorem for Littlewood Paley Decompositions

5.1 Bernstein and Nikolsky inequalities

Theorem 5.1. *(Bernstein inequality)*

Let $p \in [1, +\infty]$. There exists a constant $C_{n,p} > 1$ such that, for any $k \in \mathbb{N}^*$ and any $f \in L^p(\mathbb{R}^n)$ satisfying

$$\mathrm{supp}\ \widehat{f} \subset B_{2^k}(0) \setminus B_{2^{k-1}}(0),$$

then $\nabla f \in L^p(\mathbb{R}^n)$ and we have

$$C_{n,p}^{-1}\ \|f\|_{L^p(\mathbb{R}^n)} \le 2^{-k}\ \|\nabla f\|_{L^p(\mathbb{R}^n)} \le C_{n,p}\ \|f\|_{L^p(\mathbb{R}^n)}. \tag{5.1}$$

□

Proof of Theorem 5.1. Let χ be a cut of function in $C_c^\infty(\mathbb{R}^n)$ such that

$$\begin{cases} \chi \equiv 0 & \text{in } B_{\frac{1}{4}}(0) \cup \left(\mathbb{R}^n \setminus B_4(0)\right) \\ \chi \equiv 1 & \text{in } B_1(0) \setminus B_{\frac{1}{2}}(0). \end{cases}$$

By assumption we have

$$\widehat{f}(\xi) = \chi(2^{-k}\xi)\ \widehat{f}(\xi) \ \text{ in } \ \mathcal{S}'(\mathbb{R}^n).$$

Using Proposition 1.34, we deduce

$$f(x) = (2\pi)^{-\frac{n}{2}}\ \check{\chi}(2^k x)\, 2^{kn} * f(x).$$

This implies for any $j = 1, \ldots, n$ (using Proposition 1.32)

$$\partial_{x_j} f = (2\pi)^{-\frac{n}{2}}\ 2^{k(n+1)}\, \partial_{x_j} \check{\chi}(2^k x) * f \ \text{ in } \ \mathcal{S}'(\mathbb{R}^n).$$

Since $\chi \in C_c^\infty(\mathbb{R}^n)$, $\partial_{x_j} \check{\chi} \in \mathcal{S}(\mathbb{R}^n)$ and then in particular $\partial_{x_j} \check{\chi} \in L^1(\mathbb{R}^n)$. Using Young inequality, we deduce that

$$\|\nabla f\|_{L^p(\mathbb{R}^n)} \le C_n\ 2^k\, \|\nabla \check{\chi}\|_{L^1(\mathbb{R}^n)}\, \|f\|_{L^p(\mathbb{R}^n)}.$$

This implies the second inequality in (5.1).

We shall now present the proof of the first inequality in (5.1) in the particular case where $p \in (1, +\infty)$.

For the limiting cases respectively $p = 1$ and $p = +\infty$, we shall need a multiplier theorem that takes into account the support of the Fourier transform and that we shall prove in Chapter 7 only. Recall from Chapter 1 that for any $j \in \{1, \ldots, n\}$

$$\widehat{\partial_{x_j} f} = -i\, \xi_j\, \widehat{f}.$$

Multiplying the identity by $i\xi_j$ and summing out j gives[1]

$$\chi\big(2^{-k}\xi\big)\ \sum_{j=1}^n \frac{i\,\xi_j}{|\xi|^2}\ \widehat{\partial_{x_j} f} = \widehat{f}\ \text{ in }\ \mathcal{S}'(\mathbb{R}^n).$$

Denote

$$\begin{aligned} m_{j,k(\xi)} &:= i\,2^k\ \frac{\chi(2^{-k}\xi)}{|\xi|^2}\ \xi_j \\ &= i\,\chi(2^{-k}\xi)\ \frac{2^{-k}\,\xi_j}{|2^{-k}\xi|^2}\ . \end{aligned}$$

We have $m_{j,k(\xi)} = m_j(2^{-k}\xi)$ where

$$m_j(\eta) := i\ \frac{\chi(\eta)\,\eta_j}{|\eta|^2} \in C^\infty_c(\mathbb{R}^n).$$

Hence, it is straightforward to prove that

$$\forall \ell\in\mathbb{N}^n\ \ \exists\, C_\ell>0\ \text{ s.t. }\ \sup_j\, |\partial^\ell m_{j,k}(\xi)| \le \frac{C_\ell}{|\xi|^{|\ell|}}.$$

We can use the multiplyer Theorem 4.13 to deduce

$$2^k\, \|f\|_{L^p(\mathbb{R}^n)} \le C_{p,n}\, \|\nabla f\|_{L^p(\mathbb{R}^n)}.$$

This is the first inequality in (5.1) and this concludes the proof of Theorem 5.1 in the case $p\in(1,+\infty)$. The general case is postponed to Chapter 7.

While the second inequality in (5.1) looks a bit like a “reverse Poincaré inequality”, the following theorem could be interpreted as some sort of “reverse Hölder inequality”.

Theorem 5.2. *There exists $C_n>0$ such that for any $1\le p\le q\le+\infty$, for any $k\in\mathbb{N}$ and any $f\in L^p(\mathbb{R}^n)$ satisfying*

$$\operatorname{supp}\widehat{f}\subset B_{2^k}(0),$$

then $f\in L^q(\mathbb{R}^n)$ and the following inequality holds

$$\|f\|_{L^q(\mathbb{R}^n)} \le C_n\ \big(2^k\big)^{\frac{n}{p}-\frac{n}{q}}\ \|f\|_{L^p(\mathbb{R}^n)}. \tag{5.2}$$

□

Proof of Theorem 5.2. Let $f_k(x) := 2^{kn}\, f(2^k x)$. We have then

$$\widehat{f_k}(\xi) = \widehat{f}(2^{-k}\xi)$$

which gives $\operatorname{supp}\widehat{f_k}\subset B_1(0)$.

[1] We are using here the fact that $\widehat{f}$ is supported away from the origin.

Let χ now be a function in $C_c^\infty(\mathbb{R}^n)$ such that

$$\begin{cases} \chi \equiv 1 & \text{on } B_1(0) \\ \chi \equiv 0 & \text{on } R^n \backslash B_2(0). \end{cases}$$

Because of the choice of χ we have

$$\widehat{f_k}(\xi) = \chi(\xi)\, \widehat{f_k}(\xi).$$

Using Proposition 1.34, we deduce

$$f_k = (2\pi)^{-\frac{n}{2}}\ \check{\chi} * f_k \quad .$$

Since we only consider the case $p < q$ and since $p \geq 1$, we have

$$0 < \frac{1}{p} - \frac{1}{q} < 1\ .$$

Hence there exists $r \in (1, \infty)$ such that

$$1 - \frac{1}{r} = \frac{1}{p} - \frac{1}{q}\ .$$

Since $\check{\chi} \in \mathcal{S}(\mathbb{R}^n)$, we have in particular $\check{\chi} \in L^1(\mathbb{R}^n)$ and Young inequality gives then

$$\|f_k\|_{L^q(\mathbb{R}^n)} \leq (2\pi)^{-\frac{n}{2}}\, \|\check{\chi}\|_{L^1(\mathbb{R}^n)}\, \|f_k\|_{L^p(\mathbb{R}^n)}.$$

Hölder inequality gives

$$\|\check{\chi}\|_{L^1(\mathbb{R}^n)} \leq \|\check{\chi}\|_{L^1(\mathbb{R}^n)}^{1-\beta}\, \|\check{\chi}\|_{L^\infty(\mathbb{R}^n)}^{\theta},$$

where $\theta = 1 - \frac{1}{r}$. Choose $c_n = \max\{\|\check{\chi}\|_{L^1}, \|\check{\chi}\|_{L^\infty}\}$ and we have proved

$$\|f_k\|_{L^q(\mathbb{R}^n)} \leq C_n\, \|f_k\|_{L^p(\mathbb{R}^n)}.$$

(5.2) follows by substituting $f_k(x) = 2^{-nk} f(2^{-k}x)$. This concludes the proof of Theorem 5.2. $\square$

5.2 Littlewood Paley projections

In the proof of Theorem 4.13 we introduced a partition of unity over the phase space with each function $\varphi_k = \varphi(2^{-k}\xi)$ being supported in the dyadic annuli $B_{2^{k+1}}(0) \backslash B_{2^{k-1}}(0)$. We shall consider the same partition of unity of the phase space but truncated at 0. Precisely, let $\psi \in C_c^\infty(\mathbb{R}^n)$ such that

$$\begin{cases} \psi(\xi) \equiv 1 & \text{in } B_1(0) \\ \psi(\xi) \equiv 0 & \text{in } \mathbb{R}^n \backslash B_2(0), \end{cases}$$

and denote $\varphi(\xi) := \psi(\xi) - \psi(2\xi)$. We have clearly

$$\operatorname{supp} \varphi \subset B_2(0) \backslash B_{\frac{1}{2}}(0).$$

For $k > 0$ we take $\varphi_k(\xi) := \varphi(2^{-k}\xi)$ while for $k = 0$ we take $\varphi_0(\xi) = \psi(\xi)$. This gives

$$\sum_{k=0}^{N} \varphi_k(\xi) = \psi(2^{-N}\xi).$$

This implies that

$$\sum_{k\in\mathbb{N}} \varphi_k(\xi) = \lim_{N\to+\infty} \sum_{k=0}^{N} \varphi_k(\xi) \equiv 1 \text{ in } \mathbb{R}^n.$$

Definition 5.3. *Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $k \in \mathbb{N}$. We define the k-th Littlewood-Paley projection of f associated to the partition of unity $(\varphi_k)_{k\in\mathbb{N}}$ to be $f_k := \mathcal{F}^{-1}(\varphi_k \hat{f})$.*

Because of Bernstein theorem 5.1, we have in particular, by iterating (5.1):

$$\forall p \in [1,\infty] \quad \forall k \in \mathbb{N} \quad \forall q \in \mathbb{N} \quad \sup_{|\ell|=q} \|\partial^\ell f_k\|_{L^p(\mathbb{R}^n)} \sim 2^{kq} \|f_k\|_{L^p(\mathbb{R}^n)}.$$

We have for $k > 0$ (using Proposition 1.34)

$$f_k = 2^{kn}\, \check{\varphi}\,(2^k x) * f\ (2\pi)^{-\frac{n}{2}}. \tag{5.3}$$

Hence we deduce that for any $p \in [1,\infty]$

$$\sup_{k\in\mathbb{N}} \|f_k\|_{L^p(\mathbb{R}^n)} \leq C_{n,\varphi} \|f\|_{L^p(\mathbb{R}^n)}. \tag{5.4}$$

By the triangular inequality we also have trivially

$$\|f\|_{L^p(\mathbb{R}^n)} \leq \sum_{k\in\mathbb{N}} \|f_k\|_{L^p(\mathbb{R}^n)}. \tag{5.5}$$

The goal of the present chapter is to prove that for any $p \in (1,+\infty)$

$$\|f\|_{L^p(\mathbb{R}^n)} \sim \left\| \Big(\sum_{k\in\mathbb{N}} |f_k|^2 \Big)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)}.$$

To that aim we have to present briefly the L^p-spaces for families.

5.3 The spaces $L^p(\mathbb{R}^n, \ell_q)$

We recall the classical notation for any sequence $(a_k)_{k\in\mathbb{N}}$ and any $q \in [1,\infty)$

$$\|a_k\|_{\ell^q} := \Big(\sum_{k\in\mathbb{N}} |a_k|^q \Big)^{\frac{1}{q}}$$

and

$$\|a_k\|_{\ell^\infty} := \sup_{k\in\mathbb{N}} |a_k|\,.$$

It is a well-known fact that $\mathbb{R}^{\mathbb{N}}$ or $\mathbb{C}^{\mathbb{N}}$ equipped with lack of these norms is complete and then define a Banach space.

We now define

$$L^p(\mathbb{R}^n,\ell^q):=\Big\{(f_k)_{k\in\mathbb{N}}\ \text{ s.t. }\ f_k\in L^p(\mathbb{R}^n)\ \Big(\sum_{k\in\mathbb{N}}|f_k|^q\Big)^{\frac{1}{q}}(x)<+\infty$$
$$\text{for almost every } x\in\mathbb{R}^n \text{ and } \Big\|\Big(\sum_{k\in\mathbb{N}}|f_k|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\mathbb{R}^n)}<+\infty\Big\}.$$

We have the following proposition:

Proposition 5.4. *For any $p\in[1,\infty]$ and $q\in[1,\infty]$ the space $L^p(\mathbb{R}^n,\ell^q)$ defines a Banach space. Moreover for $p\in(1,\infty)$ and $q\in(1,\infty)$*

$$\big(L^p(\mathbb{R}^n,\ell^q)\big)'=L^{p'}(\mathbb{R}^n,\ell^{q'}).$$

Proof of Proposition 5.4. We first prove that $L^p(\mathbb{R}^n,\ell^q)$ is complete. Let $(f_k^j)_{k\in\mathbb{N}}$ be a Cauchy sequence in $L^p(\mathbb{R}^n,\ell^q)$. Then for each $k\in\mathbb{N}$ $(f_k^j)_{j\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\mathbb{R}^n)$. Since $L^p(\mathbb{R}^n)$ defines a Banach space, there exists $(f_k^\infty)_{k\in\mathbb{N}}$ such that

$$\forall k\in\mathbb{N}\quad f_k^j\longrightarrow f_k^\infty\ \text{ strongly in } L^p(\mathbb{R}^n).$$

This implies in particular that for any $N\in\mathbb{N}$

$$\Big(\sum_{k=0}^N|f_k^j|^q\Big)^{\frac{1}{q}}\longrightarrow\Big(\sum_{k=0}^N|f_k^\infty|^q\Big)^{\frac{1}{q}}\ \text{ strongly in } L^p(\mathbb{R}^n).$$

Let $F_N(x):=(\sum_{k=0}^N|f_k^\infty|^q)^{\frac{1}{q}}$. Because of the previous strong convergence we have

$$\begin{aligned}\|F_N\|_{L^p(\mathbb{R}^n)}&\le\limsup_{j\to+\infty}\Big\|\Big(\sum_{k=0}^N|f_k^j|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\mathbb{R}^n)}\\&\le\limsup_{j\to+\infty}\|(f_k^j)\|_{L^p(\mathbb{R}^n,\ell^q)}<+\infty.\end{aligned}$$

$(F_N)_{N\in\mathbb{N}}$ is a monotone sequence of L^p functions whose L^p norm is uniformly bounded. By using Beppo Levi monotone convergence theorem, we deduce that F_N strongly converges in L^p to a limit which is obvioulsly equal to $(\sum_{k=0}|f_k^\infty|^q)^{\frac{1}{q}}$. It implies that $(f_k^\infty)_{k\in\mathbb{N}}\in L^p(\mathbb{R}^n,\ell^q)$. It remains to prove

$$(f_k^j)_{k\in\mathbb{N}}\longrightarrow(f_k^\infty)_{k\in\mathbb{N}}\ \text{ strongly in } L^p(\mathbb{R}^n,\ell^q).$$

Let $\varepsilon>0$ and let respectively $j_0\in\mathbb{N}$ and $N_0\in\mathbb{N}$ such that

$$\text{i)}\qquad\sup_{j,\ell\ge j_0}\Big\|\Big(\sum_{k\in\mathbb{N}}|f_k^j-f_k^\ell|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\mathbb{R}^n)}<\frac{\varepsilon}{3}\ .$$

$$\text{ii)}\qquad\Big\|\Big(\sum_{k>N_0}|f_k^\infty|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\mathbb{R}^n)}<\frac{\varepsilon}{3}\ .$$

$$\text{iii)}\qquad\Big\|\Big(\sum_{k>N_0}|f_k^{j_0}|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\mathbb{R}^n)}<\frac{\varepsilon}{3}\ .$$

We then deduce that
$$\sup_{j\geq j_0}\Big\|\Big(\sum_{k>N_0}|f_k^j|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\mathbb{R}^n)}<\frac{2\varepsilon}{3}.$$
Hence we have
$$\limsup_{j\to+\infty}\Big\|\Big(\sum_{k>\mathbb{N}_0}|f_k^j-f_k^\infty|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\mathbb{R}^n)}<\varepsilon.$$
Since for each $k\in\mathbb{N}$ $f_k^j\to f_k^\infty$ strongly in $L^p(\mathbb{R}^n)$, we have
$$\lim_{j\to+\infty}\Big\|\Big(\sum_{k=0}^{N_0}|f_k^j-f_k^\infty|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\mathbb{R}^n)}=0.$$

Hence for j large enough, we have
$$\Big\|\Big(\sum_{k=0}^{+\infty}|f_k^j-f_k^\infty|^q\Big)^{\frac{1}{q}}\Big\|<\varepsilon$$
which implies the convergence of $(f_k^j)_{k\in\mathbb{N}}$ towards $(f_k^\infty)_{k\in\mathbb{N}}$ in $L^p(\mathbb{R}^n,\ell^q)$.

We prove now the second part of Proposition 5.4. Let p and q in $(1,+\infty]$. Let $T\in(L^p(\mathbb{R}^n,\ell^q))'$. Let $k_0\in\mathbb{N}$ and denote $L^p_{k_0}(\mathbb{R}^n,\ell^q)$ the subspace of $(f_k)_k\in L^p(\mathbb{R}^n,\ell^q)$ such that $f_k\equiv 0$ for $k\neq k_0=0$. $L^p_{k_0}(\mathbb{R}^n,\ell^q)$ is obviously isomorphic to $L^p(\mathbb{R}^n)$ and, using Riesz representation theorem, we define the existence of $g_{k_0}\in L^{p'}(\mathbb{R}^n)$ such that
$$T|_{L^p_{k_0}(\mathbb{R}^n,\ell^q)}\big((f_k)\big)=\int_{\mathbb{R}^n}f_{k_0}(x)\,g_{k_0}(x)\,dx.$$
Let
$$L^p_{\leq k_0}(\mathbb{R}^n,\ell^q):=\big\{(f_k)_{k\in\mathbb{N}}\in L^p(\mathbb{R}^n,\ell^q)\ \text{ such that } f_k\equiv 0 \text{ for } k>k_0\big\}.$$
By linearity we have
$$T|_{L^p_{\leq k_0}(\mathbb{R}^n,\ell^q)}\big((f_k)\big)=\sum_{k\leq k_0}\int_{\mathbb{R}^n}f_k(x)\,g_k(x)\,dx.$$
Let
$$\begin{array}{rccc}\Pi_{k_0}: & L^p(\mathbb{R}^n,\ell^q) & \longrightarrow & L^p_{\leq k_0}(\mathbb{R}^n,\ell^q)\\ & (f_k)_{k\in\mathbb{N}} & \longrightarrow & (f_k)_{k\leq k_0}.\end{array}$$
It is not difficult to prove that for any $(f_k)_{k\in\mathbb{N}}$
$$\lim_{k_0\to+\infty}\Pi_{k_0}\big((f_k)_{k\in\mathbb{N}}\big)=(f_k)_{k\in\mathbb{N}}\ \text{ in }\ L^p(\mathbb{R}^n,\ell^q).$$
Hence, by continuity of T, we deduce that
$$\forall (f_k)_{k\in\mathbb{N}}\in L^p(\mathbb{R}^n,\ell^q)\quad T\big((f_k)_{k\in\mathbb{N}}\big)=\sum_{k\in\mathbb{N}}\int_{\mathbb{R}^n}f_k(x)\,g_k(x)\,dx.$$

It remains to prove that $(g_k)_{k\in\mathbb{N}} \in L^{p'}(\mathbb{R}^n, \ell^{q'})$.

Let $k_0 \in \mathbb{N}$ and denote $f_k^{k_0} \in L^p_{\leq k_0}(\mathbb{R}^n, \ell^q)$, the element defined by

$$\forall k \leq k_0 \quad f_k^{k_0} := \frac{\dfrac{g_k}{|g_k|^{2-q'}}}{\Big(\sum\limits_{k=0}^{k_0} |g_k|^{q'}\Big)^{1-\frac{p'}{q'}}} \,.$$

We have that $\forall k \leq k_0$

$$|f_k^{k_0}|(x) \leq \frac{|g_k|^{q'-1}}{|g_k|^{q'-p'}} = |g_k|^{\frac{p'}{p}} \in L^p(\mathbb{R}^n).$$

Because of the continuity of T we have in one hand

$$\big|T\big((f_k^{k_0})\big)\big| \leq C_T \,\|(f_k^{k_0})\|_{L^p(\mathbb{R}^n,\ell^q)} = C_T\Big[\int_{\mathbb{R}^n}\Big(\sum_{k=0}^{k_0}|g_k|^{q'}\Big)^{\frac{p'}{q'}}\Big]^{\frac{1}{p}}.$$

In the other hand, a direct computation gives

$$T\big((f_k^{k_0})\big) = \int_{\mathbb{R}^n}\Big(\sum_{k=0}^{k_0}|g_k|^{q'}(x')\Big)^{\frac{p'}{q'}}\,dx.$$

Since $p > 1$, we have proved

$$\int_{\mathbb{R}^n}\Big(\sum_{k=0}^{k_0}|g_k|^{q'}(x)\Big)^{\frac{p'}{q'}}\,dx \leq C_{T,p}\,.$$

The constant in the right-hand side of the inequality is independent of k_0. Hence $(g_k) \in L^{p'}(\mathbb{R}^n, \ell^{q'})$ and this concludes the proof of Proposition 5.4. $\square$

5.4 The L^p-theorem for Littlewood-Paley decompositions

The goal of the present subsection is to give a proof of the following theorem which is the main achievement of the course.

Theorem 5.5. *Let $(\varphi_k)_{k\in\mathbb{N}}$ be a dyadic partition of unity of the phase space and let $p \in (1,\infty)$, there exists $1 < C$ such that for any $f \in L^p(\mathbb{R}^n)$:*

$$C^{-1}\,\|f\|_{L^p(\mathbb{R}^n)} \leq \|(f_k)_{k\in\mathbb{N}}\|_{L^p(\mathbb{R}^n,\ell^2)} \leq C\,\|f\|_{L^p}, \tag{5.6}$$

where $(f_k)_{k\in\mathbb{N}}$ is the Littlewood-Paley decomposition of f relative to the partition of unity $(\varphi_k)_{k\in\mathbb{N}}$. $\square$

Proof of Theorem 5.5. For any $f \in \mathcal{S}(\mathbb{R}^n)$, we denote $\forall x \in \mathbb{R}^n$

$$S(f)(x) := \Big(\sum_{k\in\mathbb{N}} |f_k|^2(x) \Big)^{\frac{1}{2}}.$$

By Minkowski inequality, we deduce that S is a sub-additive map.

We first prove that S is strong $(2,2)$. Indeed, using Plancherel theorem, we have

$$\begin{aligned} \int_{\mathbb{R}^n} |S(f)|^2(x)\,dx &= \sum_{k\in\mathbb{N}} \int_{\mathbb{R}^n} |f_k|^2(x)\,dx \\ &= \sum_{k\in\mathbb{N}} \int_{\mathbb{R}^n} \widehat{f}_k(\xi)\,\overline{\widehat{f}}_k(\xi)\,d\xi \qquad (5.7)\\ &= \sum_{k\in\mathbb{N}} \int_{\mathbb{R}^n} \varphi_k^2(\xi)\,|\widehat{f}|^2(\xi)\,d\xi\;. \end{aligned}$$

Since $\operatorname{supp}\varphi_k \subset B_{2^{k+1}}(0)\backslash B_{2^{k-1}}$, each $\xi \in \mathbb{R}^n$ is contained in the support of at most 3 different φ_k. Hence we have the bound

$$\forall \xi \in \mathbb{R}^n \quad \sum_{k\in\mathbb{N}} \varphi_k^2(\xi) \le 3\,\|\varphi\|^2_{L^\infty(\mathbb{R}^n)}. \tag{5.8}$$

Combining (5.7) and (5.8), we obtain that S is indeed strong $(2,2)$.

We claim now that S is weak $(1,1)$. Let $K_k(x) := \check{\varphi}_k(x)$. In particular for $k > 1$, we have $K_k(x) = 2^{kn}\,\check{\varphi}(2^k x)$ and

$$f_k = (2\pi)^{-\frac{n}{2}}\;2^{kn}\,\check{\varphi}(2^k x) * f.$$

In order to prove the claim, we shall be using the following lemma which is the Hörmander condition for families:

Lemma 5.6. *(Hörmander condition for families)*

Under the notations above, we have the existence of $B > 0$ such that

$$\forall y \in \mathbb{R}^n \quad \int_{|x|>2|y|} \|K_k(x-y) - K_k(x)\|_{\ell^2}\,dx \le B < +\infty. \tag{5.9}$$

□

Proof of Lemma 5.6. Let $y \neq 0$ and denote $v := \frac{y}{|y|}$. For any $x \in \mathbb{R}^n$, one has

$$|K_k(x-y) - K_k(x)| \le \int_0^{|y|} \Big|\frac{\partial K_k}{\partial v}\Big|(x - tv)\,dt.$$

Using *Minkowski integral inequality*, one has

$$\begin{aligned} \int_{|x|>2|y|} \|K_k(x-y) - K_k(x)\|_{\ell^2}\,dx &\le \int_{|x|>2|y|} \Big(\sum_{k\in\mathbb{N}} \Big| \int_0^{|y|} \Big|\frac{\partial K_k}{\partial v}\Big|(x-tv)\,dt \Big|^2 \Big)^{\frac{1}{2}} dx \\ &\le \int_{|x|>2|y|} \int_0^{|y|} \|\nabla K_k\|_{\ell^2}(x - tv)\,dt\,dx. \end{aligned}$$

We interchange the order of integration and we proceed to the change of variable $z := x - tv$. This gives

$$\int_{|x|>2|y|} \|K_k(x-y) - K_k(x)\|_{\ell^2}\, dx \le |y| \int_{|z|>|y|} \|\nabla K_k\|_{\ell^2}(z)\, dz. \tag{5.10a}$$

We have for each $k \in \mathbb{N}^*$

$$|\nabla K_k|(z) = 2^{k(n+1)} |\nabla \check{\varphi}|(2^k z).$$

Since $\varphi \in C_c^\infty(\mathbb{R}^n)$, we have that $\check{\varphi} \in \mathcal{S}(\mathbb{R}^n)$ and hence, obviously, we have in particular

$$|\nabla \check{\varphi}|(x) \le C\ \min\{1; |x|^{-n-2}\}.$$

This implies then

$$|\nabla K_k|(z) \le C\ \min\{2^{k(n+1)}; 2^{-k}\, |z|^{-n-2}\}.$$

For each z we denote by k_z the integer part of

$$\log_2 |z|^{-1} \quad \big(\text{i.e. } k_z := [\log_2\ |z|^{-1}]\big).$$

We write

$$\begin{aligned}
\Big(\sum_{k\in\mathbb{N}} |\nabla K_k|^2(z)\Big)^{\frac{1}{2}} &\le \Big(\sum_{k\le k_z} |\nabla K_k|^2(z)\Big)^{\frac{1}{2}} + \Big(\sum_{k=k_z+1}^{\infty} |\nabla K_k|^2(z)\Big)^{\frac{1}{2}} \\
&\le C\ \Big(\sum_{k\le k_z} 2^{2k(n+1)}\Big)^{\frac{1}{2}} + C\, |z|^{-n-2} \Big(\sum_{k=k_z+1}^{\infty} 2^{-2k}\Big)^{\frac{1}{2}} \\
&\le \sqrt{2}\ C\ 2^{k_z(n+1)} + \frac{C}{\sqrt{2}}\ |z|^{-n-2}\, 2^{-k_z}.
\end{aligned}$$

Using the fact that $2^{k_z} \sim \frac{1}{|z|}$, we deduce

$$\Big(\sum_{k\in\mathbb{N}} |\nabla K_k|^2(z)\Big)^{\frac{1}{2}} \le \frac{C'}{|z|^{n+1}}.$$

Inserting this last inequality in (5.10) gives then

$$\begin{aligned}
\int_{|x|>2|y|} \|K_k(x-y) - K_k(x)\|_{\ell^2}\, dx &\le C'\, |y| \int_{|z|>|y|} \frac{dz}{|z|^{n+1}} \\
&\le B\ |y| \int_{|y|}^{+\infty} \frac{d\rho}{\rho^2} \\
&\le B < +\infty.
\end{aligned}$$

This concludes the proof of Lemma 5.6. $\square$

Continuation of the proof of Theorem 5.5. Let $\alpha > 0$, we proceed to a Calderón-Zygmund decomposition of f for the threshold α. We write $f = g_\alpha + b_\alpha$ where g_α and b_α

are respectively the good and bad parts of the decomposition. Using the subadditivity of S, we have

$$\mu\big(\{x; S(f)(x) > \alpha\}\big) \le \mu\Big(\Big\{x; S(g_\alpha)(x) > \frac{\alpha}{2}\Big\}\Big) + \mu\Big(\Big\{x; S(b_\alpha)(x) > \frac{\alpha}{2}\Big\}\Big). \tag{5.10b}$$

Using the fact that S is strong $(2,2)$, we deduce

$$\begin{aligned} \frac{\alpha^2}{4}\,\mu\Big(\Big\{x \in \mathbb{R}^n; S(g_\alpha)(x) > \frac{\alpha}{2}\Big\}\Big) &\le C \int_{\mathbb{R}^n} |g_\alpha|^2(x)\,dx \\ &\le C\,2^{n+1}\alpha \int_{\mathbb{R}^n} |f|(x)\,dx. \end{aligned} \tag{5.11}$$

We recall the notations from Chapter 4:
The bad part of $\mathbb{R}^n$ for the decomposition is a union of disjoint cubes with faces parallel to the canonical hyperplanes: $\Omega = \bigcup_{\ell\in\mathbb{N}} C_\ell$ and $\widetilde{C}_\ell$ are the dilations of these cubes by the factor $2\sqrt{n}$ leaving each center c_ℓ fixed. This dilation factor is chosen in such a way that

$$\forall x \in \mathbb{R}^n\backslash\widetilde{C}_\ell \quad \forall y \in C_\ell \quad |x - c_\ell| \ge 2|y - c_\ell|.$$

Denote as usual $\widetilde{\Omega} = \bigcup_{\ell\in\mathbb{N}} \widetilde{C}_\ell$.

We estimate

$$\int_{\mathbb{R}^n\backslash\widetilde{\Omega}} |S(b_\alpha)|(x)\,dx = \int_{\mathbb{R}^n\backslash\widetilde{\Omega}} \Big|\sum_{k\in\mathbb{N}} |K_k * b_\alpha|^2(x)\Big|^{\frac{1}{2}} dx.$$

We write $b_\alpha = \sum_{\ell\in\mathbb{N}} b_\ell$ where $b_\ell = b\,\mathbf{1}_{c_\ell}$ and we use Minkowski inequality to obtain

$$\int_{\mathbb{R}^n\backslash\widetilde{\Omega}} |S(b_\alpha)|(x)\,dx \le \sum_{\ell\in\mathbb{N}} \int_{\mathbb{R}^n\backslash\widetilde{\Omega}} \|K_k * b_\ell\|_{\ell^2}(x)\,dx. \tag{5.12}$$

Using the fact that $\int_{C_\ell} b_\ell(y)\,dy = 0$, we write

$$\begin{aligned} \|K_k * b_\ell\|_{\ell^2}(x) &= \Big\| \int_{y\in C_\ell} K_k(x-y)\, b_\ell(y)\,dy \Big\|_{\ell^2} \\ &= \Big\| \int_{y\in c_\ell} \big[K_k\big(x - c_\ell - (y - c_\ell)\big) - K_k(x - c_\ell)\big]\, b_\ell(y)\,dy \Big\|_{\ell^2}. \end{aligned} \tag{5.13}$$

Using again *Minkowski integral inequality* and continuing (5.12) and (5.13), we obtain by the mean of Lemma 5.6

$$\begin{aligned} \int_{\mathbb{R}^n\backslash\widetilde{\Omega}} |S(b_\alpha)|(x)\,dx &\le \sum_{\ell\in\mathbb{N}} \int_{\mathbb{R}^n\backslash\widetilde{\Omega}} dx \int_{C_\ell} |b_\ell(y)|\, \big\|K_k\big(x - c_\ell - (y - c_\ell) - K_k(x - c_\ell)\big)\big\|_{\ell^2}\,dy \\ &\le \sum_{\ell\in\mathbb{N}} \int_{C_\ell} |b_\ell(y)|\,dy \int_{|x-c_\ell|>2|y-c_\ell|} \big\|K_k(x - c_\ell - (y - c_\ell)\big) - \\ &\qquad\qquad K_k(x - c_\ell)\big\|_{\ell^2}\,dx \\ &\le B \sum_{\ell\in\mathbb{N}} \int_{C_\ell} |b_\ell(y)|\,dy \le B \int_{\mathbb{R}^n} |b_\alpha(y)|\,dz \le 2B \int_{\mathbb{R}^n} |f(y)|\,dy. \end{aligned}$$

This implies that

$$\sup_{\beta>0} \beta\,\mu\bigl(\{x\in\mathbb{R}^n\backslash\widetilde{\Omega};\ S(b_\alpha)(x)>\beta\}\bigr)\leq 2B\int_{\mathbb{R}^n}|f(y)|\,dy.$$

Applying this inequality to $\beta := \frac{\alpha}{2}$ and recalling that $|\widetilde{\Omega}| \leq C_n\,\alpha^{-1}\int_{\mathbb{R}^n}|f(x)|\,dx$, we deduce

$$\alpha\,\mu\Bigl(\Bigl\{x\in\mathbb{R}^n;\ S(b_\alpha)(x)>\frac{\alpha}{2}\Bigr\}\Bigr)\leq C\int_{\mathbb{R}^n}|f(x)|\,dx.$$

Combining this inequality with (5.10b) and (5.11) gives that S is weak $(1,1)$ and the claim is proved. Using now Marcinkiewicz interpolation theorem 4.7, we deduce that S is strong (p,p) for $p\in(1,2]$.

We claim now that S is strong (p,p) for $p\in(2,+\infty)$. We shall use a duality argument. Thanks to Proposition 5.4, using Hahn Banach theorem, we have

$$\begin{aligned}\Bigl[\int |S(f)|^p\,(x)\,dx\Bigr]^{\frac{1}{p}} = \|(f_k)\|_{L^p(\mathbb{R}^n,\ell^2)} &= \sup_{\|(h_k)\|_{L^{p'}(\mathbb{R}^n,\ell^2)}\leq 1}\ \sum_{k\in\mathbb{N}}\int f_k(x)\,h_k(x)\,dx\\ &= \sup_{\|(h_k)\|_{L^{p'}(\mathbb{R}^n,\ell^2)}\leq 1}\ \sum_{k\in\mathbb{N}}\int_{\mathbb{R}^n} K_k * f(x)\,h_k(x)\,dx\\ &= \sup_{\|(h_k)\|_{L^{p'}(\mathbb{R}^n,\ell^2)}\leq 1}\ \int_{\mathbb{R}^n} f(x)\sum_{k\in\mathbb{N}} K_k^{\#} * h_k(x)\,dx.\end{aligned}$$

Therefore, in order to prove that S is strong (p,p) for $p>2$, it suffices to prove that the operator S^* defined by

$$S^*(h_k)_{k\in\mathbb{N}} := \sum_{k\in\mathbb{N}} K_k^{\#} * h_k$$

maps continuously $L^{p'}(\mathbb{R}^n,\ell^2)$ into $L^{p'}(\mathbb{R}^n)$. Precisely, we are proving the following lemma:

Lemma 5.7. *Under the above notations, for any $p'\in(1,2]$, there exists $C>0$ such that $\forall (h_k)\in L^{p'}(\mathbb{R}^n,\ell^2)$, we have*

$$\Bigl\|\sum_{k\in\mathbb{N}} K_k^{\#} * h_k\Bigr\|_{L^{p'}(\mathbb{R}^n)} \leq C\,\|(h_k)\|_{L^{p'}(\mathbb{R}^n,\ell^2)} \tag{5.14}$$

Proof of Lemma 5.7. We use a natural extension of Marcinkiewicz interpolation theorem 4.7 to the framework of mappings from $L^{p'}(\mathbb{R}^n,\ell^2)$ into $L^{p'}(\mathbb{R}^n)$ whose proof is left to the reader in order to infer that the lemma is proved if it holds for $p'=2$ and if there exists $C>0$ such that

$$|S^*(h_k)_{k\in\mathbb{N}}|_{L^{1,\infty}(\mathbb{R}^n)} \leq C\,\|(h_k)\|_{L^1(\mathbb{R}^n,\ell^2)}. \tag{5.15}$$

We then first consider the case $p'=2$.

To justify all steps in the computations below, we can of course restrict to elements $(h_k)_{k\in\mathbb{N}}\in L^2(\mathbb{R}^n,\ell^2)$ such that $h_k\in\mathcal{S}(\mathbb{R}^n)$ and $h_k\equiv 0$ for k large enough. It is not difficult to prove that this class is dense in $L^2(\mathbb{R}^n,\ell^2)$.

We have by using Plancherel theorem:

$$\int_{\mathbb{R}^n} |S^*(h_k)|^2 = (2\pi)^{-n} \int_{\mathbb{R}^n} \Big(\sum_{k\in\mathbb{N}} \widehat{K_k^{\#}}\, \widehat{h}_k \Big) \overline{\Big(\sum_{\ell\in\mathbb{N}} \widehat{K_\ell^{\#}}\, \widehat{h}_\ell \Big)}$$
$$= (2\pi)^{-n} \int_{\mathbb{R}^n} \sum_{k,\ell\in\mathbb{N}} \varphi_k^{\#}(\xi)\, \overline{\varphi_\ell^{\#}(\xi)}\, \widehat{h}_k(\xi)\, \overline{\widehat{h}}_\ell(\xi)\, d\xi.$$

Recall that $\operatorname{supp} \varphi_k(\xi) \subset B_{2^{k+1}}(0) \backslash B_{2^{k-1}}(0)$, hence

$$\varphi_k^{\#}(\xi)\, \varphi_\ell^{\#}(\xi) \not\equiv 0 \Longrightarrow |k-\ell| \le 3.$$

This implies that

$$\int_{\mathbb{R}^n} |S^*(h_k)|^2(x)\, dx = (2\pi)^{-n} \int_{\mathbb{R}^n} \sum_{|k-\ell|<4} \varphi_k^{\#}(\xi)\, \varphi_\ell^{\#}(\xi)\, \widehat{h}_k(\xi)\, \overline{\widehat{h}_\ell}(\xi)\, d\xi$$
$$\le C\, (2\pi)^{-n} \int_{\mathbb{R}^n} 7 \sum_{k\in\mathbb{N}} |\varphi_k^{\#}(\xi)|^2\, |\widehat{h}_k|\, (\xi)\, d\xi$$
$$\le 7\, (2\pi)^{-n}\, \|\varphi\|^2_{L^\infty(\mathbb{R}^n)} \sum_{k\in\mathbb{N}} \int_{\mathbb{R}^n} |\widehat{h}_k|^2(\xi)\, d\xi$$
$$\le C\, \|(h_k)\|_{L^2(\mathbb{R}^n,\ell^2)}.$$

Hence we have proved (5.14) for $p'=2$.

We establish now (5.15). Let

$$H(x) := \Big(\sum_{k\in\mathbb{N}} |h_k|^2(x) \Big)^{\frac{1}{2}}.$$

We fix $\alpha>0$ and we proceed to a Calderón-Zygmund decomposiiton for H. As usual, we denote by $\Omega = \bigcup_{\ell\in\mathbb{N}} C_\ell$ the union of the bad cubes relative to this decomposition. For each $k\in\mathbb{N}$, we write $h_k = g_k + b_k$, where

$$g_k(x) = \begin{cases} h_k(x) & \text{for } x \in \mathbb{R}^n \backslash \Omega \\ \displaystyle\fint_{C_\ell} h_k(y)\, dy & \text{for } x \in C_\ell \ (\ell\in\mathbb{N}). \end{cases}$$

Since $H(x) \le \alpha$ on $\mathbb{R}^n\backslash\Omega$ and $\fint_{C_\ell} (H(y)\, dy \le 2^n \alpha$ for any ℓ, we deduce, using Minkowski inequality, that

$$\|(g_k)\|_{L^\infty(\mathbb{R}^n,\ell^2)} \le 2^n \alpha. \tag{5.16}$$

For any $k\in\mathbb{N}$ and $\ell\in\mathbb{N}$, we denote

$$b_{k,\ell} := b_k\, \mathbf{1}_{C_\ell},$$

where $\mathbf{1}_{C_\ell}$ denotes the characteristic function of the bad cube C_ℓ. Observe that we have fixed

$$\forall k,\ell\in\mathbb{N}\quad \fint_{C_\ell} b_{k,\ell}(y)\,dy=0. \tag{5.17}$$

Moreover, using Minkowski inequality, we have also for any $\ell\in\mathbb{N}$

$$\begin{aligned}\fint_{C_\ell}\Big(\sum_{k\in\mathbb{N}}|b_{k,\ell}|^2(y)\Big)^{\frac12}dy &\le \fint_{C_\ell}\Big(\sum_{k\in\mathbb{N}}|h_k-\fint_{C_\ell}h_k|^2(y)\Big)^{\frac12}dy\\ &\le \fint_{C_\ell}\|h_k\|_{\ell^2}(y)\,dy+\Big\|\fint_{C_\ell}h_k\Big\|_{\ell^2}.\end{aligned}$$

Using Minkowski integral inequality, we then deduce

$$\forall \ell\in\mathbb{N}\quad \fint_{C_\ell}\|b_{k,\ell}\|_{\ell^2}(y)\,dy\le 2\fint_{C_\ell}\|h_k\|_{\ell^2}(y)\,dy. \tag{5.18}$$

Finally, recall that from the fundamental properties of the Calderón-Zygmund decomposition one has

$$\mu(\Omega)=\sum_{\ell\in\mathbb{N}}\mu(C_\ell)\le\frac{\displaystyle\int_{\mathbb{R}^n}\|h_k\|_{\ell^2}(y)\,dy}{\alpha}. \tag{5.19}$$

Using the strong $(2,2)$ property, we have

$$\alpha^2\mu\Big(\Big\{x\in\mathbb{R}^n;\ |S^*(g_k)|(x)>\frac{\alpha}{2}\Big\}\Big)\le C\,\|(g_k)\|^2_{L^2(\mathbb{R}^n,\ell^2)}.$$

Combining this inequality with (5.16) gives then

$$\begin{aligned}\alpha\,\mu\Big(\Big\{x\in\mathbb{R}^n;\ |S^*(g_k)|(x)>\frac{\alpha}{2}\Big\}\Big)&\le C\int_{\mathbb{R}^n}\|(g_k)\|_{\ell^2}(y)\,dy\\ &\le C\int_{\mathbb{R}^n}\|(h_k)\|_{\ell^2}(y)\,dy\,,\end{aligned} \tag{5.20}$$

where we used again Minkowski integral inequality.

Denote as usual $\widetilde{C}_\ell$ the dilated cubes by the factor $2\sqrt{n}$ and $\widetilde{\Omega}=\bigcup_{\ell\in\mathbb{N}}\widetilde{C}_\ell$ with respect to the center c_l of C_l. We estimate now

$$\int_{\mathbb{R}^n\setminus\widetilde{\Omega}}|S^*(b_k)|(x)\,dx\le\int_{\mathbb{R}^n\setminus\widetilde{\Omega}}\sum_{\ell\in\mathbb{N}}\Big(\sum_{k\in\mathbb{N}}|K_k^**b_{k,\ell}|(x)\Big)\,dx.$$

As usual we write

$$\begin{aligned}|K_k^{\#}*b_{k,\ell}|(x)&=\Big|\int_{C_\ell}K_k^{\#}\big(x-c_\ell-(y-c_\ell)\big)\,b_{k,\ell}(y)\,dy\Big|\\ &=\Big|\int_{C_\ell}\big[K_k^{\#}\big(x-c_\ell-(y-c_\ell)\big)-K_k^{\#}(x-c_\ell)\big]\,b_{k,\ell}(y)\,dy\Big|.\end{aligned}$$

We then bound using Cauchy-Schwarz inequality

$$\sum_{k\in\mathbb{N}} |K_k^{\#} * b_{k,\ell}|(x) \le \int_{c_\ell} \sum_{k\in\mathbb{N}} \left|K_k^{\#}(x-c_\ell-(y-c_\ell)-K_k^*(x-c_\ell)\right| |b_{k,\ell}(x)|\,dy$$
$$\le \int_{c_\ell} \|K_k^{\#}\big(x-c_\ell-(y-c_\ell)\big)-K_k^{\#}(x-c_\ell)\|_{\ell_2}\,\|b_{k,\ell}(y)\|_{\ell^2}\,dy.$$

This gives

$$\int_{\mathbb{R}^n\setminus\widetilde{\Omega}} |S^*(b_k)|(x)\,dx \le$$
$$\sum_{\ell\in\mathbb{N}} \int_{C_\ell} \|b_{k,\ell}\|_{\ell^2}(y)\,dy \int_{|x-c_\ell|>2|y-c_\ell|} \|K_k^{\#}\big(x-c_\ell-(y-c_\ell)\big)-K_k(x-c_\ell)\|_{\ell^2}\,dy.$$

Using Lemma 5.6 (i.e. Hörmander property for families), we then deduce

$$\begin{aligned}\int_{\mathbb{R}^n\setminus\widetilde{\Omega}} |S^*(b_k)|(x)\,dx &\le C \sum_{\ell\in\mathbb{N}} \int_{C_\ell} \|b_{k,\ell}\|_{\ell^2}(y)\,dy\\ &= C\int_{\mathbb{R}^n} \|b_k\|_{\ell^2}(y)\,dy \\ &\le 2\,C\int_{\mathbb{R}^n} \|h_k\|_{\ell^2}(y)\,dy.\end{aligned} \tag{5.21}$$

Combining (5.19), (5.20) and (5.21) gives

$$\alpha\,\mu\big(\{x\in\mathbb{R}^n;\ |S^*(h_k)|(x)>\alpha\}\big) \le C\int_{\mathbb{R}^n} \|h_k\|_{\ell^2}(y)\,dy,$$

which is the weak $(1,1)$ property for S^* (5.15). We then deduce Lemma 5.7. $\square$

End of the proof of Theorem 5.5. We recall the identity

$$\left[\int_{\mathbb{R}^n} |S(f)|^p(x)\,dx\right]^{\frac{1}{p}} = \sup_{\|(h_k)\|_{L^{p'}(\mathbb{R}^n,\ell^2)}\le 1} \int_{\mathbb{R}^n} f(x)\ S^*(h_k)(x)\,dx.$$

Since by Lemma 5.7 S^* is continuously mapping $L^{p'}(\mathbb{R}^n,\ell^2)$ into $L^{p'}(\mathbb{R}^n)$ for any $p'\in(1,2]$, we deduce then $\forall p\in[2,\infty)$ $\exists C_p>0$ such that

$$\left[\int_{\mathbb{R}^n} |S(f)|^p(x)\,dx\right]^{\frac{1}{p}} \le C_p\,\|f\|_{L^p(\mathbb{R}^n)}.$$

Hence we have proved the second inequality in (5.6). It remains to prove the first one in order to conclude the proof of the theorem.

We use the following duality argument

$$\begin{aligned}\|f\|_{L^p(\mathbb{R}^n)} &= \sup_{\|g\|_{L^{p'}(\mathbb{R}^n)}\le 1} \int_{\mathbb{R}^n} f(x)\,g(x)\,dx\\ &= \sup_{\|g\|_{L^{p'}(\mathbb{R}^n)}\le 1} \int_{\mathbb{R}^n} \sum_{k,\ell\in\mathbb{N}} f_k(x)\ g_\ell(x)\,dx.\end{aligned}$$

Since $\operatorname{supp} \widehat{f}_k \cap \operatorname{supp} \widehat{g}_\ell = \emptyset$ for $|k-\ell| \geq 4$, we deduce

$$\begin{aligned}
\|f\|_{L^p(\mathbb{R}^n)} &= \sup_{\|g\|_{L^{p'}(\mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^n} \sum_{|k-\ell|<4} f_k(x)\, g_\ell(x)\, dx \\
&\leq \sup_{\|g\|_{L^{p'}(\mathbb{R}^n)} \leq 1} 7 \int_{\mathbb{R}^n} \|f_k\|_{\ell^2}(x)\ \|g_k\|_{\ell^2}(x)\, dx \\
&\leq \sup_{\|g\|_{L^{p'}(\mathbb{R}^n)} \leq 1} 7\, \|(f_k)\|_{L^p(\mathbb{R}^n,\ell^2)} \|(g_k)\|_{L^{p'}(\mathbb{R}^n,\ell^2)} \\
&\leq C\ \|(f_k)\|_{L^p(\mathbb{R}^n,\ell^2)},
\end{aligned}$$

where we used

$$\|(g_k)\|_{L^p(\mathbb{R}^n,\ell^2)} \leq C\ \|g\|_{L^{p'}(\mathbb{R}^n)}.$$

This concludes the proof of the Theorem 5.5. $\square$