ANALYSIS ASPECTS OF WILLMORE SURFACES.

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Curvatures for surfaces in $\mathbb{R}^3$.

- S oriented closed surface in $\mathbb{R}^3$.
- g induced metric on S.
- $\vec{n}$ the unit normal to S (Gauss map).

The 2nd Fundamental form :

$$\forall X, Y \in T_xS \quad \vec{\mathbb{I}}_x(X,Y) := -\left\langle d\vec{n}_x \cdot X, Y \right\rangle_g \ \vec{n} \tag{1}$$

is bilinear symmetric from $(T_xS)^2$ into N_xS the normal direction to T_xS.

$$\vec{\mathbb{I}}_x \simeq_g \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix} \vec{n} \tag{2}$$

- Principal curvatures : κ_1 and κ_2
- Mean curvature : $H = \frac{1}{2}(\kappa_1 + \kappa_2)$.
- Mean curvature vector : $\vec{H} = H\ \vec{n}$.
- Gauss curvature : $K = \kappa_1\kappa_2$.

The Willmore energy of a closed surface in $\mathbb{R}^3$.

Willmore energy :

$$W(S) := \int_S H^2\, dvol_g = \frac{1}{4}\int_S (\kappa_1+\kappa_2)^2\, dvol_g \quad . \tag{3}$$

Gauss-Bonnet Theorem :

$$\int_S K\, dvol_g = 2\pi\chi(S) = 4\pi - 4\pi\, g(S) \quad . \tag{4}$$

Umbillic energy :

$$\tilde{W}(S) := \int_S H^2 - K\, dvol_g = \frac{1}{4}\int_S (\kappa_1-\kappa_2)^2\, dvol_g \quad . \tag{5}$$

Gauss equation :

$$2|H|^2 - K = \frac{1}{2}|\vec{\mathbb{I}}|^2_g = |\nabla\vec{n}|^2_g \quad . \tag{6}$$

Conclusion : Modulo a topological invariant $W(S)$ is comparable to the homgeneous $W^{1,2}$ norm of the Gauss map.

Willmore energy in various fields of science and technology.

- Conformal geometry (presumably the origin). Thomsen, Schadow 1923 - Blaschke 1929. -...-Willmore 1965.

- General Relativity. Hawking 1968 mass of 2 spheres :

$$m_{Haw}(S) = \frac{|S|^{\frac{1}{2}}}{(16\pi)^{\frac{3}{2}}} \left[16\pi - \int_S |H|^2 \, dvol_g \right]$$

- Cell Biology. Helfrich 1973. Spontaneous curvature model for biomembranes, vesicles and smectic A-liquid crystals.

- Mechanic-Elasticity. Non-linear plate theory. Γ-limit elastic energy $3D \longrightarrow 2D$ (Friesecke, James, Müller 2001).

- Optical design. Rubinstein 1990.

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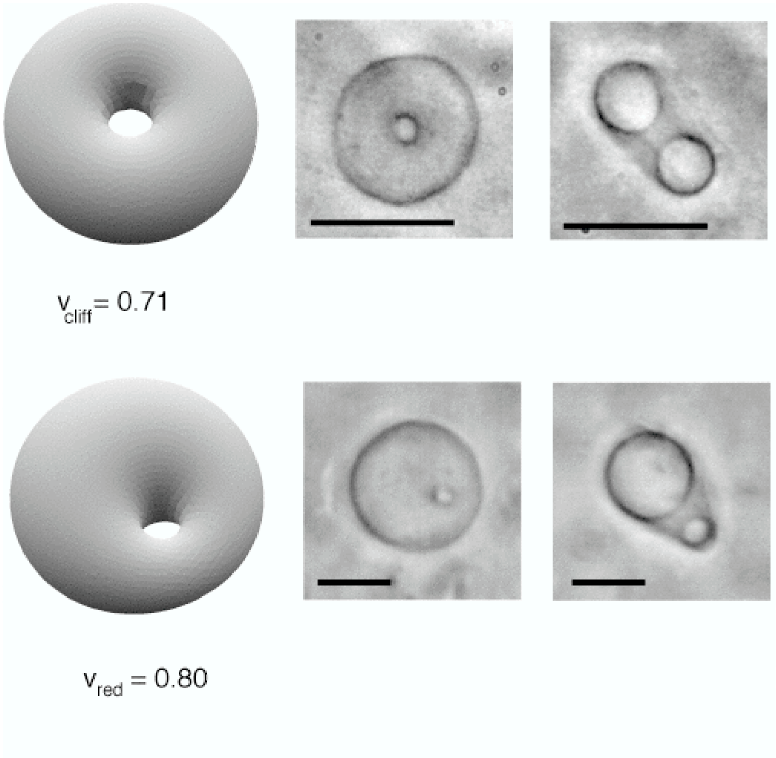
Conformal invariance of Willmore energy.

Theorem. [Blaschke 1929] Let S be a closed oriented surface of $\mathbb{R}^3$, let Ψ be a conformal diffeomorphism of $\mathbb{R}^3 \cup \{\infty\}$ then the following holds

$$W(\Psi(S)) = W(S) \quad .$$



$\Psi(M) = M'$ where $AM' = AM/|AM|^2$ (inversion)

Generalization : Willmore energy of immersed surfaces in $\mathbb{R}^m$.

- Σ abstract oriented closed 2 dimensional manifold.
- $\vec{\Phi}$ smooth immersion of Σ in $\mathbb{R}^m$.
- g induced metric on Σ.
- $\vec{n}(x)$ oriented $m-2$ space normal to the tangent 2-space at x (Gauss Map).
- $\pi_{\vec{n}}$: orthonormal projection onto the normal space given by $\vec{n}$.

The 2nd Fundamental form :

$$\forall X,Y \in T_x\Sigma \quad \vec{\mathbb{I}}_x(X,Y) := \pi_{\vec{n}} \circ d^2\vec{\Phi}(X,Y) \quad . \tag{7}$$

is bilinear symmetric.

- Mean curvature vector : $\vec{H} = \frac{1}{2} tr_g \vec{\mathbb{I}}$.
- Willmore energy : $W(\vec{\Phi}) = \int_\Sigma |\vec{H}|_g^2 \, dvol_g$.

In this talk we will restrict to $m = 2$ though most of the results presented are valid in arbitrary dimension.

Willmore Immersions.

Definition : An immersion $\vec{\Phi} : \Sigma \longrightarrow \mathbb{R}^3$ is Willmore if $\forall \vec{\xi} \in C_0^\infty(\Sigma, \mathbb{R}^3)$ the following holds

$$\frac{d}{dt} W(\vec{\Phi} + t\vec{\xi})_{|t=0} = 0 \quad .$$

Examples :

- $\vec{\Phi}$ is a minimal immersion : $\vec{H} = 0$.
- $\vec{\Phi}$ is a composition of a minimal immersion and a conformal transformation.
- The round sphere S^2. Consequence of the following inequality

  $$W(S) = \int_S |H|^2 \, dvol_g \geq 4\pi$$

  holds for any closed surface S with equality iff S is a round sphere.
- The Willmore Torus T_0.

  $$W(T_0) = 2\pi^2$$

  Minimizes W among all immersions of T^2 ? (Willmore conjecture).

Geometric and Analysis questions related to Willmore immersions.

- Explicit and exhaustive description of the space of Willmore immersions of a given surface Σ.

- Is the set of Willmore surfaces below a certain level of energy strongly, weakly compact ? (modulo the action of conformal transformations).

- Does there exist minimizers of W among all immersions of a given surface Σ ?...identify these minimizers.

- Is there a notion of Weak Willmore immersions ? If so, what are the possible singularities ?

The Euler-Lagrange Equation of Thomsen and Schadow.

Theorem. [Thomsen, Schadow 1923]
Let $\vec{\Phi}$ be a smooth immersion of a surface Σ into $\mathbb{R}^3$. $\vec{\Phi}$ is Willmore if and only if the following Euler-Lagrange equation is satisfied

$$\Delta_g H + 2H\,(|H|^2 - K) = 0 \quad , \tag{8}$$

where Δ_g is the negative Laplace Beltrami operator of the induced metric by $\vec{\Phi}$ on Σ. $\square$

Functional analysis paradox : The formulation (8) of the Euler-Lagrange Equation requires at least H to be in L^3_{loc} which is more restrictive than the condition H being in L^2 given by the finiteness of the Lagrangian $W(\vec{\Phi})$!!!

The Euler-Lagrange Equation in divergence form.

Theorem 1. [R. 2006] Let S be a smooth surface in $\mathbb{R}^3$, the following equation is satisfied

$$\Delta_g H + 2H\,(|H|^2 - K) = 0 \quad , \tag{9}$$

if and only if, in conformal coordinates, the following holds

$$div\left(2\nabla\vec{H} - 3H\,\nabla\vec{n} + \vec{H}\wedge\nabla^{\perp}\vec{n}\right) = 0 \tag{10}$$

where the operators div, ∇ and $\nabla^{\perp}$ are taken with respect to the flat metric in the conformal coordinates (x,y) : $div\,X = \partial_x X_1 + \partial_y X_2$, $\nabla = (\partial_x, \partial_y)$ and $\nabla^{\perp} = (-\partial_y, \partial_x)$. $\square$

Divergence form for Schrödinger systems with antisymmetric potentials.

Theorem 2. [R. 2006] There exists a continuous operator

$$\begin{array}{cccc} \mathfrak{G} & : & L^2(D^2, so(n)) & \longrightarrow & L^\infty \cap W^{1,2}(D^2, GL(n)) \times W^{1,2}(D^2, M(n)) \\ & & \Omega & \longrightarrow & (A_\Omega, B_\Omega) \end{array} \tag{11}$$

such that $u^{1,2}(D^2, \mathbb{R}^n)$ is a solution of the Schrödinger system

$$-\Delta u = \Omega \cdot \nabla u \qquad \text{in } D^2 \tag{12}$$

if and only if it satisfies

$$div\left(A_\Omega \cdot \nabla u + B_\Omega \cdot \nabla^\perp u\right) = 0 \quad . \tag{13}$$

□

Conformally invariant Lagrangians and Schrödinger systems with antisymmetric potentials.

Theorem 3. [R. 2006] Let L be a Lagrangian on $W^{1,2}(D^2, \mathbb{R}^n)$ of the form

$$L(u) = \int_{D^2} l(u, \nabla u)\, dx \quad , \tag{14}$$

with $C^{-1}|\xi|^2 \leq l(y, \xi) \leq C|\xi|^2$ for some constant C. Assume moreover that l is C^1 in y and C^2 in ξ. If L is conformal invariant :

$$L(u \circ \phi) = L(u)$$

for any conformal transformation ϕ in the plane, then the Euler-Lagrange equation can be written as a vectorial Schrödinger equation with antisymmetric potential. $\square$

Examples

- Harmonic maps equations into riemannian manifolds
- Prescribed mean curvature equations in riemannian manifolds

The Link between Willmore surfaces and Schrödinger equations with antisymmetric potentials.

Fact 1: This last theorem extends to harmonic map equation into Lorentzian Manifolds which can hence be written in divergence form.

Fact 2 : [Blaschke 1929] A surface S is Willmore if and only if its conformal Gauss map is harmonic into the space of oriented 2-spheres of S^3 isometric to $\simeq S^{3,1}$ the Minkowski sphere of $\mathbb{R}^{4,1}$.

Conclusion : Willmore equation can then be written in divergence form !

Good things ...and insufficiencies of the divergence form of Willmore equation.

- The Willmore Operator. For $\vec{n} \in W^{1,2}(D^2, S^2)$ and $\vec{w} \in L^2(D^2, \mathbb{R}^3)$ denote

$$\mathcal{L}_{\vec{n}}\vec{w} = div\left(-\nabla\vec{w} + 3\vec{n}\cdot\nabla\vec{w}\;\vec{n} + \vec{w}\wedge\nabla^{\perp}\vec{n}\right) \quad , \tag{15}$$

then $\mathcal{L}_{\vec{n}}$ is formally self-adjoint. (Observe that $\vec{\Phi}$ is Willmore $\Leftrightarrow \mathcal{L}_{\vec{n}}\vec{H} = 0$).

- For any $p > 2$ $\mathcal{L}_{\vec{n}}$ is continuous from L^p into $W^{-2,p}$.

- A-priori estimates : $\exists\epsilon_0 > 0$ s.t. if $\int_{D^2} |\nabla\vec{n}|^2 < \epsilon_0$ then

$$\forall\vec{\xi} \in C^{0,\infty}(D^2, \mathbb{R}^3) \qquad \|\vec{\xi}\|_{L^p} \le C\|\mathcal{L}_{\vec{n}}\vec{\xi}\|_{W^{-2,p}} \quad . \tag{16}$$

- This last 2 facts do not work anymore for $p = 2$!

Willmore surfaces as the critical case for Willmore operator : the compactness issue

Question 1 : Modulo extraction of a subsequence is the weak limit of Willmore disks still Willmore ?

Assume

$$\vec{n}_k \rightharpoonup \vec{n} \quad \text{in } W^{1,2} \quad \text{and } \vec{H}_k \rightharpoonup \vec{H} \quad \text{in } L^2$$

Then

$$\begin{array}{ccccc} div(& 2\nabla\vec{H}_k & -3H_k\,\nabla\vec{n}_k & +\vec{H}_k\wedge\nabla^{\perp}\vec{n}_k &)=0 \\ & H^{-1}\downharpoonleft & \mathcal{M}\downharpoonleft & \mathcal{M}\downharpoonleft & \\ div(& 2\nabla\vec{H} & ? & ? &) \end{array} \tag{17}$$

($\mathcal{M}$: space of Radon measures.)

Willmore surfaces as the critical case for Willmore operator : the regularity issue

Question 2 : Regularity of Willmore $W^{2,2}$ surfaces ?

Definition : A $W^{2,2}$ closed surface in $\mathbb{R}^3$ is a compact subset of $\mathbb{R}^3$ which realizes a $W^{2,2}$ graph about every point.

$$\Longrightarrow \qquad \int_S |\nabla\vec{n}|^2 \, dvol_g < +\infty \quad .$$

Theorem [Toro 1994, Müller-Sverák 1995] A $W^{2,2}-$graph in $\mathbb{R}^3$ admits locally, about every point, a bilipschitz-$W^{2,2}$ conformal parametrization $\vec{\Phi}$. $\square$

From now on we can then assume : $\vec{\Phi} \in W^{1,\infty} \cap W^{2,2}$, $\vec{n} \in W^{1,2}$ and the conformality condition $\vec{n} \wedge \partial_x\vec{\Phi} = \partial_y\vec{\Phi}$

Further conservation laws for Willmore surfaces.

Theorem 4. [R. 2006] Let $\vec{\Phi}$ be a conformal bilipschitz local parametrization of a Willmore surface. Introduce $\vec{L}$, satisfying

$$\nabla^{\perp}\vec{L} = 2\nabla\vec{H} - 3H\,\nabla\vec{n} + \vec{H}\wedge\nabla^{\perp}\vec{n} \tag{18}$$

then the following holds

$$\nabla\vec{\Phi}\cdot\nabla^{\perp}\vec{L} = 0 \quad \text{and} \quad \nabla\vec{\Phi}\wedge\nabla^{\perp}\vec{L} + 2\nabla^{\perp}H\cdot\nabla\vec{\Phi} = 0 \quad . \tag{19}$$

Let

$$\left\{\begin{array}{l}\nabla S := \vec{L}\cdot\nabla\vec{\Phi}\\[2ex] \nabla\vec{R} := \nabla\vec{\Phi}\wedge\vec{L} + 2H\nabla\vec{\Phi} \quad .\end{array}\right. \tag{20}$$

Then

$$\Delta\vec{R} = \nabla\vec{R}\wedge\nabla^{\perp}\vec{n} - \nabla S\,\nabla^{\perp}\vec{n} \quad . \tag{21}$$

$\square$

Recall notation : $\nabla a\,\nabla^{\perp}b = \partial_y a\partial_x b - \partial_x a\partial_y b$.

Wente estimates and integrability by compensation.

Theorem [Wente 1969, Tartar 1983, Coifman-Lions-Meyer-Semmes 1989]

Let a and b be two functions in $W^{1,2}(D^2,\mathbb{R})$. Let φ be the unique solution of

$$\left\{\begin{array}{l} -\Delta\varphi = \nabla a\cdot\nabla^{\perp} b = \partial_y a\partial_x b - \partial_x a\partial_y b \quad \text{in } D^2 \\ \varphi = 0 \qquad \text{on } \partial D^2 \quad . \end{array}\right. \tag{22}$$

Then the following estimates hold true

$$\|\varphi\|_{L^\infty} + \|\nabla\varphi\|_{L^{2,1}} + \|\nabla^2\varphi\|_{L^1} \le C_0\,\|\nabla a\|_{L^2}\,\|\nabla b\|_{L^2} \tag{23}$$

$\square$

Recall that f, measurable, is in $L^{2,1}$ if and only if

$$\int_0^\infty \left|\left\{x\in D^2 \quad \text{s. t.} \quad |f|(x)\ge\lambda\right\}\right|^{\frac{1}{2}}\, d\lambda < +\infty$$

Regularity of $W^{2,2}$ Willmore surfaces.

Theorem 5 [R. 2006] $W^{2,2}$ Willmore surfaces are analytic. $\square$

Proof :

- Recall $\vec{\Phi} \in W^{1,\infty} \cap W^{2,2}$, $\vec{n} \in W^{1,2}$ and $H \in L^2$ and

$$\nabla^\perp \vec{L} = 2\nabla \vec{H} - 3H\, \nabla\vec{n} + \vec{H} \wedge \nabla^\perp \vec{n} \quad .$$

- Hence $\vec{L} \in L^{2,\infty}$ therefore

$$\nabla \vec{R} \in L^{2\infty} \quad \text{and} \quad \nabla S \in L^{2,\infty} \quad .$$

Recall that f, measurable, is in $L^{2,\infty}$ if and only if

$$\sup_{\lambda \in \mathbb{R}_+} \lambda^2 \left|\left\{x \in D^2 \quad \text{s. t.} \quad |f|(x) \geq \lambda\right\}\right| < +\infty$$

Further integrability by compensation.

Corollary [Bethuel 1992]

Let $a \in W^{1,2}(D^2,\mathbb{R})$ and $\nabla b \in L^{2,\infty}(D^2,\mathbb{R}^2)$. Let φ solving of

$$\begin{cases} -\Delta\varphi = \nabla a\cdot\nabla^{\perp} b = \partial_y a\partial_x b - \partial_x a \partial_y b \quad \text{in } D^2 \\ \varphi = 0 \qquad \text{on } \partial D^2 \quad . \end{cases} \tag{24}$$

Then

$$\|\nabla\varphi\|_{L^2(D^2)} \le C_1\ \|\nabla a\|_{L^2(D^2)}\ \|\nabla b\|_{L^{2,\infty}(D^2)} \tag{25}$$

□

Conclusion of the proof of the regularity : Applying this result to

$$\Delta\vec{R} = \nabla\vec{R}\wedge\nabla^{\perp}\vec{n} - \nabla S\ \nabla^{\perp}\vec{n}$$

gives $\nabla\vec{R}\in L^2$...$\Rightarrow \nabla S \in L^2$.

Then [Wente, Tartar, CLMS] applied to the red equation gives $\vec{R}\in W^{2,1}$....bootstraping gives $\vec{n}\in C^0$....

Point removability result for Willmore surfaces.

Theorem 6 [Kuwert-Schätzle ($m = 3$) 2003, R. ($m \geq 3$)] Let $\vec{\Phi} \ : \ D^2 \longrightarrow \mathbb{R}^m$, $\vec{\Phi} \in C^0$. Assume

i) $\vec{\Phi}$ is a Willmore immersion on $D^2 \setminus \{0\}$,

ii)

$$\int_{D^2} |\nabla \vec{n}|^2 \, dvol_g < +\infty \quad ,$$

iii)

$$\liminf_{r \rightarrow 0} \frac{\mu(B_r(0))}{\pi \, r^2} < 2 \quad ,$$

where $\mu = N \, \mathcal{H}^2 \llcorner \vec{\Phi}(D^2)$ and $N_x = Card \left\{ p \in D^2 \quad \text{s. t.} \quad \vec{\Phi}(p) = x \right\}$

Then $\vec{\Phi}(D^2)$ is a $C^{1,\alpha}$ submanifold of $\mathbb{R}^m$ $\forall \alpha < 1$. Moreover $\exists \vec{H}_0 \in \mathbb{R}^m$ s. t.

$$\vec{H}(x) - \vec{H}_0 \, \log |x| \quad \in C^{0,\alpha}$$

and $\vec{H}_0 = 0 \Leftrightarrow \vec{\Phi}(D^2)$ is an analytic Willmore surface. $\square$

Remark : < 2 in iii) is optimal !

The Li-Yau 8π condition.

Theorem. [P.Li-S.T.Yau 1982]

Let Σ be a closed 2-manifold. Let $\vec{\Phi} \ : \ D^2 \longrightarrow \mathbb{R}^m$ be an immersion.

Assume $\exists p \in \mathbb{R}^m$ s.t.

$$\vec{\Phi}^{-1}(\{p\}) = \{x_1, \cdots, x_n\} \quad ,$$

x_i distinct.

Then

$$\int_\Sigma |\vec{H}|^2 \, dvol_g \geq 4\pi \ n \quad .$$

In particular if

$$\int_\Sigma |\vec{H}|^2 \, dvol_g < 8\pi$$

$\vec{\Phi}$ is an embedding. □

Weak compactness of Willmore surfaces below 8π.

Theorem. [Kuwert-Schätzle ($m = 3$) 2003, R. ($m \geq 3$)]

Let S_k be a sequence of Willmore embeddings of closed surfaces in B_1^m. Assume

i)

$$\text{Area}(S_k) \quad \text{unif. bounded,}$$

ii)

$$\text{genus}(S_k) \quad \text{unif. bounded,}$$

iii)

$$W(S_k) = \int_{S_k} |\vec{H}_k|^2 \, dvol_{g_k} < 8\pi$$

Then

$$\exists S_{k'} \qquad \text{s. t.} \qquad [S_{k'}] \rightharpoonup T \quad \text{as current}$$

If $T \neq 0$, $T = [S]$ is the integration along a smooth Willmore embedding S. $\square$

Strong compactness, modulo the Möbius group action, of Willmore torii below 8π.

Theorem. [Kuwert-Schätzle ($m = 3$) 2003, R. ($m = 3, 4$) 2006]

Let $\delta > 0$. Then

$$\mathfrak{M}_\delta = \left\{ \begin{array}{c} \text{embedded Willmore torii in } \mathbb{R}^m \\ \text{with } W(T^2) \le 8\pi - \delta \end{array} \right\}$$

is strongly compact modulo translations and conformal transformations. □

The Proof is based on the point removability result and the following energy lower bound obtained for <u>non umbilic</u> Willmore S^2 by the mean of geometrico-algebraic methods ([Bryant (m=3) 1984, Montiel (m=4) 2000])

$$\int_{S^2} |\vec{H}|^2 \, dvol_g \ge 8\pi \qquad .$$