Diss. ETH No. 19629

QUALITATIVE AND QUANTITATIVE PROPERTIES OF ENTROPIC SOLUTIONS TO HYPERBOLIC SCALAR CONSERVATION LAWS

A dissertation submitted to

ETH Zurich

for the degree of

Doctor of Sciences ETH Zurich

presented by

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2011

Contents

Abstract

In this thesis we consider weak solutions $u(x,t)$ of scalar conservation laws in $1+1$ dimensions

$$\left.\begin{array}{rcll} \partial_t u + \partial_x f(u) & = & 0 & \text{in} \quad \mathbb{R} \times \mathbb{R}_+ \,, \\ u(x,0) & = & u_0(x)\,, & \end{array}\right\}$$

where we assume that the flux function satisfies $f \in C^2(\mathbb{R})$.

The first part is concerned with the rate of entropy production. The Second Law of Thermodynamics asserts that the physical entropy of an adiabatic system is an increasing function in time. In this part we will study a more stringent version of this law, according to which the entropy should not only increase in time, but the rate of increase is optimal in absolute value among all possible evolutions. We will establish this property in the framework of non-linear scalar hyperbolic conservation law with strictly convex fluxes.

In the second part we present a new local Poincaré-type inequality for scalar conservation laws in $1+1$ dimensions with strictly non-linear flux, i.e. we control the oscillation of an entropy solution u in terms of the defect measure $m(x,t,a)$ given by the kinetic formulation.

Zusammenfassung

In dieser Arbeit untersuchen wir schwache Lösungen $u(x,t)$ von Erhaltungssätzen in $1+1$ Dimensionen

$$\left.\begin{array}{rcll} \partial_t u + \partial_x f(u) & = & 0 & \text{in} \quad \mathbb{R} \times \mathbb{R}_+ \,, \\ u(x,0) & = & u_0(x)\,, & \end{array}\right\}$$

wobei wir annehmen, dass $f \in C^2(\mathbb{R})$ ist.

Der erste Teil der Arbeit beschäftigt sich mit der Rate der Entropieproduktion. Das zweite Gesetz der Thermodynamik besagt, dass die Entropie eines adiabatisch abgeschlossenen Systems, in Abhängigkeit der Zeit, eine wachsende Funktion ist. In diesem Teil der Arbeit betrachten wir eine strengere Version von diesem Gesetz. Die Entropie soll nicht nur wachsend sein, sondern der Absolutbetrag der Wachstumsrate ist optimal verglichen mit allen anderen möglichen Entwicklungen. Dies werden wir im Rahmen von skalarwertigen hyperbolischen Erhaltungssätzen mit konvexem Fluss beweisen.

Im zweiten Teil präsentieren wir eine Poincaré-Ungleichung für skalarwertige Erhaltungssätze in $1+1$ Dimensionen mit streng nicht-linearem Fluss. Mit anderen Worten wir kontrollieren die Oszillation einer Entropielösung u mit dem Defektmass $m(x,t,a)$, welches durch die kinetische Formulierung des Erhaltungssatzes gegeben ist.

Acknowledgments

I want to express my deep gratitude to my supervisor Tristan Rivière for the open and upright working atmosphere he created, the lively discussions and his constant support throughout my work on this thesis. He left me all the freedom I wanted and was always available whenever I needed support. He was there to share his ideas and to guide me through the academic world. With his numerous comments and suggestions he improved this thesis. I also wish to thank Camillo De Lellis for readily accepting the task of being co-examiner.

Chapter 1

Introduction

We consider solutions to the following equation

$$\left.\begin{array}{rcll} \partial_t u + \operatorname{div}_x f(u) & = & 0 & \text{in} \quad \mathbb{R}_+ \times \mathbb{R}^n \,, \\ u(x,0) & = & u_0(x)\,, & \end{array}\right\} \tag{1.1}$$

where the flux $f \in C^2(\mathbb{R}^n)$ and the initial data $u_0 \in L^\infty(\mathbb{R}^n)$. It is well known, that, even for smooth initial data, the classical solution can cease to exist in finite time, due to the possible formation of shocks (see Chapter 4.2 in [Da]). Therefore one has to consider weak solutions of (1.1), i.e. solutions, which satisfy (1.1) in the distributional sense. However it turned out, that, for a given initial data, the space of weak solutions is huge (see Chapter 4.4 in [Da]). Therefore additional conditions have to be imposed to single out the physical relevant weak solutions in some models.

For strictly convex fluxes f and $n = 1$ Oleinik proved 1957 in [Ol] uniqueness of bounded weak solutions, which satisfy almost everywhere her '*E-condition*':

$$u(y,t) - u(x,t) \leq \frac{y-x}{ct}\,, \quad \text{for} \quad x < y\,, t > 0\,, \tag{1.2}$$

where $c = \inf f''$. A immediate consequence of this condition (1.2) is a spectacular regularization phenomena. Oleinik proved, that for bounded measurable initial data, the weak solution satisfying almost everywhere (1.2) has immediately locally bounded variation in the complement of the initial line .

A more powerful approach was given by Kruzhkov in [Kr], where he replaces condition (1.2) by a family of integral inequalities. This approach covers also cases, where f is non-convex and the space dimension is bigger than one. Moreover in the case of convex fluxes one can show that his *entropy condition* is equivalent to Oleinik's *E-condition* (see Chapter 8.5 in [Da]).

More precisely for $u_0 \in L^\infty$ he proved existence and uniqueness of weak solutions satisfying the *entropy condition*: He considers the family of convex entropy flux pairs $(\eta_a, \xi_a)_{a\in\mathbb{R}}$, where

$$\eta_a(u) = (u-a)^+ \quad \text{and} \quad \xi_a(u) = \operatorname{sign}(u-a)^+(f(u)-f(a))\,, \tag{1.3}$$

and w^+ stands for $\max\{w, 0\}$. Then an entropy solution is a bounded function u, which satisfies (1.1) in the sense of distributions and

$$\partial_t \eta_a(u) + \partial_x \xi_a(u) \leq 0\,. \tag{1.4}$$

Equivalently one can replace the one parameter family $(\eta_a, \xi_a)_{a\in\mathbb{R}}$ and assume, that (1.4) is fulfilled for all convex η with corresponding entropy flux ξ, which is defined by $\xi = \int \eta' f'$. As a consequence of this one can show, if the initial data u_0 is in BV that $u(\cdot, t)$ is locally in BV for all later times.
A different approach to scalar conservation laws is introduced by Lions, Perthame and Tadmor in [LPT]: The kinetic formulation of a scalar conservation law (1.1). A comprehensive introduction is found in [Pe]. For a weak solution $u \in L^\infty$ of (1.1) one considers the set

$$E_a = \{(x,t):\ a \leq u(x,t)\}$$

and we will denote the characteristic function of E_a by

$$\mathbb{1}_{a\leq u(x,t)}\,.$$

Then one can show

Theorem 1.1 ([LPT]). *A bounded measurable function u on $\mathbb{R}\times\mathbb{R}_+$, which satisfies*

$$\partial_t \mathbb{1}_{a\leq u} + f'(a)\operatorname{div}_x \mathbb{1}_{a\leq u} = \partial_a m(x,t,a) \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{R}) \tag{1.5}$$

for a non-negative measure $m(x,t,a)$ together with the initial condition

$$\mathbb{1}_{a\leq u(x,0)} = \mathbb{1}_{a\leq u_0(x)}\,,$$

is the admissible solution of (1.1).

One can relate the measure m in (1.5) with (1.4) as follows:

$$\partial_t|u-a| + \operatorname{div}_x[\operatorname{sign}(u-a)(f(u)-f(a))] = -2m(x,t,a) \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n\times\mathbb{R}_+) \tag{1.6}$$

or equivalently

$$\partial_t u\wedge a + \operatorname{div}_x f(u\wedge a) = m(x,t,a)\,, \tag{1.7}$$

where $u \wedge a = \min\{u, a\}$.

For non-convex respective non-concave fluxes there exist intitial data $u_0 \in L^\infty$, such that the entropy solution $u(x,t)$ is not locally BV for all later times. If f is linear, the regularity of weak solutions u won't be better than the regularity of the initial data u_0. Therefore a natural question is, if there is still a regularization effect for weak solutions u of (1.1) if f is sufficiently non-linear. One approach is given by *kinetic averaging.*
Instead of (1.5) one can consider the general Cauchy-Problem

$$\partial_t \chi(x,t,a) + b(a)\partial_x \chi(x,t,a) = g(x,t,a) \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}^n) \tag{1.8}$$

with initial condition

$$\chi(x,0,a) = \chi_0(x,a)\,. \tag{1.9}$$

In [GLPS] it was observed that that compactness and regularity results exist, not for the solution χ of (1.8), but for velocity averages of χ. For any $\phi \in C_c^\infty$, the velocity average of χ associated to ϕ is defined by

$$\rho(x,t) = \int_{\mathbb{R}} f(x,t,a)\phi(a)\,da\,. \tag{1.10}$$

In the case of (1.5), $\rho(x,t)$ is exactly the entropy solution $u(x,t)$ of (1.1), if we choose $\phi \in C_c^\infty$ such that $\phi = 1$ on $[-\|u\|_\infty, \|u\|_\infty]$. The main result in [GLPS] is then as follows: if χ, $g \in L^2(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n)$ and satisfy (1.8) with $b(a) = a$, then any average $\rho(x,t)$ of χ is in $H^{\frac{1}{2}}(\mathbb{R}^n \times \mathbb{R})$. Such results are called '*kinetic averaging lemmas*'. In [JP] is is shown via averaging lemmas that weak solutions u of (1.1) which lie in

$$\mathcal{W} := \left\{ \begin{array}{c} u \in L^\infty \text{ is a weak sol. of (1.1)} \\ \text{s.t. } m(x,t,a) = \partial_t(u \wedge a) + \operatorname{div}_x f(u \wedge a) \\ \text{is a Radon measure.} \end{array} \right\}$$

belonging to $u \in W^{\alpha,3/2}_{loc}$ for $\alpha < \frac{1}{3}$. It is important to notice that entropy solutions belong to the class $\mathcal{W}$. The result in [JP] is obtained under the following non-degeneracy assumption for f

$$\begin{array}{c} \forall R > 0 \; \exists C > 0 \text{ s.t. } \forall \xi \in \mathcal{S}^n\,, \; u \in \mathbb{R} \; \forall \varepsilon > 0 \\ \mathcal{L}^1\left(\{|a| \le R : \; |f'(a) \cdot \xi - u| < \varepsilon\}\right) \le C\varepsilon\,, \end{array} \tag{1.11}$$

which says that there is no open intervall on which f is affine. In [DW] it is then showed by De Lellis and Westdickenberg, that the result in [JP] are actually optimal with respect to the number of derivatives. They construct a weak solutions $u \in \mathcal{W}$ for Burger's equation in one space dimension, i.e. $f(u) = \frac{1}{2}u^2$, such that $u \notin W^{\alpha,p}_{loc}$ if $\alpha > \frac{1}{3}$, or $\alpha \leq \frac{1}{3}$ and $p \geq \frac{1}{\alpha}$. Hence there is only hope to improve integrability somewhat but not differentiability.

A different approach, which lies out of reach of kinetic averaging lemmas, is to investigate the structure of weak solutions of (1.1). One wonders if weak solutions of (1.1) such that its entropy production is a Radon measure have a structure similar to BV functions without being necessarily in BV. In a work [DOW] by De Lellis, Westdickenberg and Otto it is shown that the singular set of shock waves of such solutions is contained in a countable union of Lipschitz curves and $\mathcal{H}^{n-1}$ almost everywhere along these curves the solution has left and right approximate limits. The $\mathcal{H}^{n-1}$ dimensional part of the entropy production is concentrated on the shock waves and can be explicitly computed in terms of the approximate limits. The solution has vanishing mean oscillation $\mathcal{H}^n$ almost everywhere outside this union of curves. More precisely, they proved:

Theorem 1.2 ([DOW]). *Let f satisty (1.11) and let $u \in \mathcal{W}$ be a weak solution of (1.1). Then there exists a rectifiable set J_u of dimension $n-1$ such that*

a) u has vanishing means oscillation outside of J_u,

b) u has left and right traces on J_u.

The Theorem doesn't answer the questions, if points of vanishing mean oscillation are actually Lebesgue points or if the measure m is a $\mathcal{H}^{n-1}$ dimensional measure. From BV one conjectures that both questions can be positively answered. A similar Theorem but for $n = 1$ and f strictly convex is shown in [Le1]. For $n = 1$ Theorem 1.2 is improved by De Lellis and Rivière in [DR]. They deduce that for entropy solutions of (1.1) with strictly non-linear flux f, $m(x, t, a)$ is a $\mathcal{H}^1$-dimensional rectifiable measure and u is approximate continuous outside of J_u.

Besides the area of conservation laws, these questions appear also in understanding the Γ-limit of functionals arising in different areas of physics. It turns out that the Γ-limit can be properly understood in class of functions, which satisfy certain PDEs and for which the divergence of specific nonlinear quantities are Radon measures. Yet these classes of functions are strictly larger than BV and the same questions as in the case of scalar conservation laws are addressed. We consider for a bounded domain $\Omega \subset \mathbb{R}^2$ the space $\mathcal{M}_{div}(\Omega)$, which consists of unit vectorfields u such that $u = e^{i\varphi}$ for a $\phi \in L^\infty(\Omega, \mathbb{R})$ and $\operatorname{div} e^{i\varphi\wedge a}$ is a Radon measure over $\Omega \times \mathbb{R}$. This

space $\mathcal{M}_{div}(\Omega)$ was introduced by Serfaty and Rivière in [RS1] and [RS2] in connection to a problem related to micromagnetism. We give here a brief description.
Let Ω be a bounded and simply connected domain, for $u \in W^{1,2}(\Omega, \mathbb{S}^1)$ and a $\varepsilon > 0$ we consider

$$E_\varepsilon(u) = \varepsilon \int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon} \int_{\mathbb{R}^2} |H|^2 \,, \tag{1.12}$$

where $H = \nabla(G * \hat{u})$, $\hat{u} = u$ on Ω and $\hat{u} = 0$ in Ω^c and G is the kernel of the Laplacian on $\mathbb{R}^2$.
It was proved in [RS1], [RS2] that from any sequence $u_{\varepsilon_n} \in W^{1,2}(\Omega, \mathbb{S}^1)$ such that $\varepsilon \to 0$ and $E_{\varepsilon_n}(u_{\varepsilon_n}) < C$ one can extract a subsequence $u_{\varepsilon_{n'}}$ such that $\varphi_{\varepsilon_{n'}}$ converges strongly in $L^p(\Omega)$ for any $p < \infty$ to a limit φ such that $e^{i\varphi} = u \in \mathcal{M}_{div}(\Omega)$. Furthermore the authors are conjecturing that the Γ-Limit should be given by the following functional E_0 over $\mathcal{M}_{div}(\Omega)$:

$$E_0(u) := 2 \int_{a \in \mathbb{R}} \left|\operatorname{div}\left(e^{i\varphi \wedge a}\right)\right|(\Omega)\, da$$

Part of the $\Gamma-$convergence has been proved as they established in one hand the following inequality

$$E_0(u) := 2 \int_{a \in \mathbb{R}} \left|\operatorname{div}\left(e^{i\varphi \wedge a}\right)\right|(\Omega)\, da \leq \liminf E_{\varepsilon_{n'}}(u_{\varepsilon_{n'}})$$

and in the other hand that

$$\lim_{\varepsilon \to 0} \inf_{u \in W^{1,2}} E_\varepsilon(u) = 2 \inf_{u \in \mathcal{M}_{div}(\Omega)} \int_{a \in \mathbb{R}} \left|\operatorname{div}\left(e^{i\varphi \wedge a}\right)\right|(\Omega)\, da = 2|\partial\Omega| \,, \tag{1.13}$$

where $|\partial\Omega|$ is the perimeter of the set Ω. One can prove (see [RS1]), that the infimum on the right hand side is achieved by $u = -\nabla^\perp \operatorname{dist}(\cdot, \partial\Omega) \in \mathcal{M}_{div}(\Omega)$. The function $g = \nabla^\perp \operatorname{dist}(\cdot, \partial\Omega)$ is the viscosity solution of

$$\left.\begin{array}{rcll} |\nabla g| - 1 & = & 0 & \text{on} \quad \Omega \,, \\ g & = & 0 & \text{on} \quad \partial\Omega \,. \end{array}\right\} \tag{1.14}$$

It therefore natural to study elements of the space $\mathcal{M}_{div}(\Omega)$. In [RS2] it is explained, that the measure $\operatorname{div} e^{i\phi\wedge a}\, da$ detects singularities and it is shown for elements $e^{i\phi} \in \mathcal{M}_{div}(\Omega)$ such that ϕ has finite total variation, that $\operatorname{div} e^{i\phi\wedge a}$ is concentrated on the $\mathcal{H}^1$-rectifiable set J_ϕ of approximate jump points of ϕ. An immediate question is, if for elements of $\mathcal{M}_{div}(\Omega)$, which are not BV, the measure $\operatorname{div} e^{i\phi\wedge a}$ is still $\mathcal{H}^1$ dimensional and rectifiable. Connected to this question is, if elements of $\mathcal{M}_{div}(\Omega)$ have still a similar or even the same structure as BV functions. A partial answer is given in [AKLR], where it is shown that

Theorem 1.3. *Let $u = e^{i\phi} \in \mathcal{M}_{div}(\Omega)$. Then*

i) The jump set J_ϕ is countably $\mathcal{H}^1$-rectifiable and coinciedes, up to $\mathcal{H}^1$-negible sets, with

$$\Sigma := \left\{ x \in \Omega : \limsup_{r \to 0^+} r^{-1} \mu_\phi(B_r(x)) > 0 \right\} .$$

In addition

$$\operatorname{div}\left(e^{i\phi \wedge a}\right) = \mathbb{1}_{\phi^- < a < \phi^=} \left(e^{ia} - e^{i\phi^-}\right) \cdot \nu_\phi \mathcal{H}^1 \llcorner J_\phi \quad \forall a \in \mathbb{R} .$$

ii) For $\mathcal{H}^1$-a.e. $x \in \Omega \backslash J_\phi$ we have the following VMO property:

$$\lim_{r \to 0^+} \int_{B_r(x)} |\phi - \overline{\phi}| = 0 ,$$

where $\overline{\phi}$ is the average of ϕ on $B_r(x)$.

iii) The measure $\delta := \mu(\Omega \backslash J_\phi)$ is orthogonal $\mathcal{H}^1$, i.e.

$$B \text{ Borel with } \mathcal{H}^1(B) < \infty \Rightarrow \delta(B) = 0 .$$

The same result was obtained in [DO] by DeLellis and Otto, but in n dimensions.

A question addressed in scalar conservation laws, which is also related to problems in micromagnetic is the question of minimality for entropy production. In [RS2] it is shown, that a sequence $u_n = e^{i\phi_n}$ such that $\|\phi_n\|_\infty$ is uniformly bounded and $E_{\varepsilon_n}(u_n) \to 2|\partial\Omega|$ converges to an element $u = e^{i\phi}$ of $\mathcal{M}_{div}(\Omega)$ such that u is a minimizer of E_0 and satisfies $\operatorname{div} e^{i\phi\wedge a} \geq 0$. It is furthermore shown in [RS2], that any minimizer (u, ϕ) of E_0 satisfies either $\operatorname{div} e^{i\phi\wedge a} \geq 0$ or $\operatorname{div} e^{i\phi\wedge a} \leq 0$. A minimizer of E_0 is as mentioned above given by the viscosity solution of (1.14). Therefore one might wonder if the only minimizers of E_0 are the viscosity or the anti-viscosity solution of (1.14). In other words, for $(u, \phi) \in \mathcal{M}_{div}(\Omega)$ such that $\operatorname{div} e^{i\phi\wedge a} \geq 0$ is $g = \nabla^\perp u$ a viscosity solution of (1.14). This is indeed the case as it is shown by Ambrosio, Lecumberry and Rivière in [ALR].

For weak solutions $u \in \mathcal{W}$ of (1.1) one can define the absolute value of the entropy production over a set $\Omega \subset \mathbb{R} \times \mathbb{R}_+$ as being

$$EP = \int_{\mathbb{R}} |m|(\Omega, a)\, da . \tag{1.15}$$

For initial data $u_0 \in L^\infty$ with compact support the entropy production over the set $\mathbb{R} \times [0, T]$ is finite for the entropy solution. But the main concern is, if the entropy solution is a minimizer of the entropy production and if this minimizer is unique. It was positively answered by Dafermos in Chapter 9.7 of [Da] in a restricive setting. He proposes an entropy rate admissibility criterion, which says, that a solution satisfies this criterion, if its rate of entropy production is minimal. For the Riemann-Problem for scalar conservation laws with strictly convex flux he shows, that a self-similar solution satisfies the entropy rate criteria if and only if it satisfies the Oleinik-E-condition.

1.1 Main results and Outlook

In this section we present the main results of this work and give an outlook. In Chapter 1, we are going to show that the entropy rate criterion characterizes admissible L^∞ solutions for scalar conservation laws with strictly convex flux. Precisely we are going to show

Theorem 1.4. *Let $f \in C^2(\mathbb{R})$ such that $f'' \geq c > 0$ and*

$$\lim_{|x|\to\infty} f(x) = \infty \,. \tag{1.16}$$

Moreover let $u_0 \in L^\infty(\mathbb{R})$ be compactly supported. Let $u \in L^\infty(\mathbb{R} \times [0, T))$ be an arbitrary weak solution of (1.1), such that $m(x, t, a) = \partial_t(u \wedge a) + \partial_x f(u \wedge a)$ is locally a Radon measure in $\mathbb{R} \times [0, T) \times \mathbb{R}$. Assume the "entropy production" m satisfies

$$\int_{\mathbb{R}} |m| \left(\mathbb{R} \times (0, \bar{t}), a\right) \, da \leq \int_{\mathbb{R}} |q| \left(\mathbb{R} \times (0, \bar{t}), a\right) \, da \;\forall\; q \in \mathcal{W} \text{ and } \forall \bar{t} \in (0, T) \,. \tag{1.17}$$

Then u is the entropy solution, i.e. satisfies (1.2) and equivalently (1.4).

The proof of Theorem 1.4 is based on the work done in [AKLR]. One wishes to generalize Theorem 1.4 to more general fluxes f. However this is not straightforward from the work done in Chapter 1, since the proof of Theorem 1.4 is based on the connection between hyperbolic scalar conservation laws and Hamilton-Jacobi equations.
Another open question is the following: Let $u_0 \in L^\infty$ be compactly supported. Then let $u \in \mathcal{W}$ be a weak solution of (1.1) with defect measure m, such that for a fixed $T > 0$ u statisfies

$$\int_{\mathbb{R}} |m|(\mathbb{R} \times [0, T], a) \, da \leq \int_{\mathbb{R}} |q|(\mathbb{R} \times [0, T], a) \, da \tag{1.18}$$

for all $q(x,t,a) = \partial_t v \wedge a + \partial_x f(v \wedge a)$, where $v \in \mathcal{W}$. Now one might wonder if the optimality of u at time $T > 0$ implies, that u is also optimal for all times $T_1 < T$. If this would hold, Theorem 1.4 would immediately imply, that u is entropic. It is important to notice that via compensated compactness one could show the existence of a minimizer satisfying (1.18). Hence an improvement of Theorem 1.4 would immediately give that the entropy solutions is an unique minimizer of the entropy production on a fixed strip $\mathbb{R} \times [0,t]$.
Another desirable improvement would be to have a more local statement of Theorem 1.4. For $(x_0, t_0) \in \mathbb{R} \times \mathbb{R}_+$ let

$$D_{(x_0,t_0)} = \{(x,t) \in \mathbb{R} \times \mathbb{R}_+ : |x - x_0| \leq \lambda(t_0 - t),\ 0 \leq t \leq t_0\} ,$$

where λ is the maximal speed of propagation. Let u be the entropy solution of (1.1) for intial data $u_0 \in L^\infty$ and $m(x,t,a)$ its defect measure, then is it true that for all (x_0, t_0)

$$\int_{\mathbb{R}} |m|(D_{(x_0,t_0)}, a)\, da \leq \int_{\mathbb{R}} |q|(D_{(x_0,t_0)}, a)\, da \tag{1.19}$$

for all $q(x,t,a) = \partial_t v \wedge a + \partial_x f(v \wedge a)$ such that $v \in \mathcal{W}$?

The second Chapter is dedicated to the control of oscillation of entropy solutions u of (1.1). We are going to show

Theorem 1.5. *Let $f \in C^2(\mathbb{R}, \mathbb{R})$ be such that $|\{u \in \mathbb{R} : f'(u) = 0\}| < \infty$. For an entropy solution $u \in L^\infty \cap L^1(\mathbb{R} \times \mathbb{R}_+)$ of (1.1) there exist constants $C > 0$ and $\delta_0 > 0$ such that for all ε, $\delta \in (0, \delta_0)$ and for all (x_0, t_0), $r \in (0, t_0/4)$*

$$\begin{aligned} \frac{1}{\pi(\delta r)^2} \int_{B_{\delta r}(x_0,t_0)} |u(x,t) - \overline{u}^{\delta r}|\, dx\, dt \\ \leq C \left[\frac{1}{\delta r} \mu(B_r(x_0,t_0)) \right]^{\frac{1}{2}} + \frac{C}{\rho(\varepsilon)^{4/3}} \left[\frac{1}{r} \mu(B_r(x_0,t_0)) \right]^{\frac{1}{3}} \\ + C \left(\frac{\delta}{\rho(\varepsilon)} + \varepsilon \right) , \end{aligned} \tag{1.20}$$

where $\rho(\varepsilon)$ is defined as

$$\rho(\varepsilon) = \min_{a \in \{f''=0\}} \min_{\varepsilon \leq |u-a| \leq 2\varepsilon} |f''(u)| .$$

Estimate (1.20) should provide a new tool for improving regularity results like Theorem 1.2). Especially one wishes to deduce that the defect measure m is $\mathcal{H}^1$-dimensional. However so far we are unable to recover the result in [DR] completely. Nevertheless one wishes to generalize Theorem 1.5 to solutions $v \in \mathcal{W}$. From that one could also slightly improve the results in [DOW] like it is done in chapter two for entropic solutions. The approach in [DR] seems to be limited to entropy solutions, since a central point in the proof is the fact that for entropic solutions u to (1.1) the function $f'(u(\cdot,t))$ has locally bounded variation, if the flux f is strictly non-linear. However this is not true for non-entropic solutions even for Burgers equation, i.e. $f(u) = \frac{1}{2}u^2$.

In order to obtain a control of oscillation for general weak solutions in $\mathcal{W}$ one could use a comparison argument, i.e. let u be the entropy solution and $v \in \mathcal{W}$ be a weak solution. One wants to control

$$\frac{1}{\pi(\delta r)^2} \int_{B_{\delta r}(x_0,t_0)} |v(x,t) - \overline{v}^{\delta r}| \, dx \, dt$$

in terms of the defect measure $q(x,t,a) = \partial_t v \wedge a + \partial_x f(v \wedge a)$. We observe

$$\frac{1}{\pi(\delta r)^2} \int_{B_{\delta r}(x_0,t_0)} |v(x,t) - \overline{v}^{\delta r}| \, dx \, dt \leq 2 \frac{1}{\pi(\delta r)^2} \int_{B_{\delta r}(x_0,t_0)} |u(x,t) - v(x,t)| \, dx \, dt + \frac{1}{\pi(\delta r)^2} \int_{B_{\delta r}(x_0,t_0)} |u(x,t) - \overline{u}^{\delta r}| \, dx \, dt \,. \quad (1.21)$$

The first term on the right-hand side of (1.21) can be estimatet in terms of $m(x,t,a) = \partial_t u \wedge a + \partial_x f(u \wedge a)$ and $q(x,t,a)$ via kinetic averaging methods. For the second term one applies Theorem 1.20. Finally a minimality result in the spirit of (1.19) would then give a control of oscillation for v in terms of q.

Chapter 2

A Minimality property for entropic solutions to scalar conservation laws in $1 + 1$ dimensions

This chapter is joint work with T. Rivière and will appear in Comm. PDE 35, (2010), 1763 - 1801

2.1 Introduction

We consider solutions to the following equation

$$\left.\begin{aligned} \partial_t u + \partial_x f(u) &= 0 \qquad \text{in} \quad \mathbb{R}_+ \times \mathbb{R}\,, \\ u(x,0) &= u_0(x)\,, \end{aligned}\right\} \tag{2.1}$$

with strictly convex flux f ($f'' \geq c > 0$) and initial data $u_0 \in L^\infty$. It is well known that even for smooth initial data classical solutions can cease to exist in finite time, due to the possible formation of shocks (see Chapter 4.2 in [Da]). Therefore one has to consider weak solutions of (2.1), i.e. solutions, which satisfy (2.1) in the distributional sense. However it turned out that for given initial data the space of weak solutions is huge (see Chapter 4.4 in [Da]). Therefore additional conditions have to be imposed to single out the physical relevant weak solutions in some models.
In 1957 Oleinik proved in [Ol] uniqueness of bounded weak solutions, which satisfy almost everywhere her '*E-condition*'

$$u(y,t) - u(x,t) \leq \frac{y-x}{ct}\,, \quad \text{for} \quad x < y\,, t > 0\,, \tag{2.2}$$

where $c = \inf f''$. A immediate consequence of this condition (2.2) is a spectacular regularization phenomena. Oleinik proved that for bounded measurable initial data the weak solution satisfying almost everywhere (2.2) becomes immediately locally BV in the complement of the initial line .
A more powerful approach was given by Kruzhkov in [Kr], where he replaces condition (2.2) by a family of integral inequalities. This approach covers also cases, where f is non-convex and the space dimension is bigger than one. However in the case of convex fluxes one can show that his *entropy condition* is equivalent to Oleinik's *E-condition* (see Chapter 8.5 in [Da]). More precisely for $u_0 \in L^\infty$ he proved existence and uniqueness of weak solutions satisfying the *entropy condition*: an entropy solution is a bounded function u which satisfies (2.1) in the sense of distributions and

$$\partial_t \eta_a(u) + \partial_x \xi_a(u) \leq 0 \quad \mathcal{D}'\,, \tag{2.3}$$

where $(\eta_a, \xi_a)_{a\in\mathbb{R}}$ is the family of convex entropy flux pairs, such that

$$\eta_a(u) = (u-a)^+ \quad \text{and} \quad \xi_a(u) = \operatorname{sign}(u-a)^+(f(u) - f(a))\,, \tag{2.4}$$

and w^+ stands for $\max\{w, 0\}$.

Equivalently one can replace the one parameter family $(\eta_a, \xi_a)_{a\in\mathbb{R}}$ and assume that (2.3) is fulfilled for all convex η with corresponding entropy flux ξ, which

is defined by $\xi = \int \eta' f'$. As a consequence one can show, if the initial data u_0 is in BV_{loc}, that u is in BV_{loc} for all later times.
Let $a \wedge b$ denote $\min\{a, b\}$. Let $u \in L^\infty(\mathbb{R} \times [0, T))$ be a weak solution of (2.1), such that

$$m(x, t, a) = \partial_t(u \wedge a) + \partial_x f(u \wedge a) \in \mathcal{M}_{loc}(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})$$

where $\mathcal{M}$ denotes the space of Radon measures. One can define the absolute value of the entropy production over a set $\Omega \subset \mathbb{R} \times \mathbb{R}_+$ as being

$$EP = \int_{\mathbb{R}} |m|(\Omega, a)\, da\,. \tag{2.5}$$

In the case of u being an entropy solution and hence in BV, the measure $m(x, t, a)$ and therefore the entropy production of u simplifies to

$$EP = \int_{\Omega} \Delta(u^+, u^-)\mathcal{H}^1 \llcorner J_u\,, \tag{2.6}$$

where J_u denotes the rectifiable set of jump points of u, u_+ and u_- are respectively the left and right approximate limits of u for some orientation of J_u and

$$\Delta(a, b) = \frac{(a-b)^2 \left[\frac{f(a)+f(b)}{2}\right] - (a-b)\int_a^b f(s)\, ds}{[(a-b)^2 + (f(a) - f(b))^2]^{\frac{1}{2}}}\,. \tag{2.7}$$

It is natural to compare the different entropic productions of the weak solutions to (2.1) - BV or not BV ! - and to ask the following questions : *does there exists a weak solution which minimizes the entropy production and, if so, what properties does a minimizer of (2.5) have.*
In this work we provide a partial answer to this question. We show a weak solution of (2.1) whose entropy production increases in time less, than any other weak solution's entropy production, has to be the entropy solution.
Precisely: Let $\mathcal{W}$ denote the set of defect measures induced by a weak solution of (2.1), i.e.

$$\mathcal{W} := \left\{ \begin{array}{c} m(x,t,a) \in \mathcal{M}_{loc} \text{ s.t. } m(x,t,a) = \partial_t u \wedge a + \partial_x f(u \wedge a), \\ \text{where } u \in L^\infty \text{ is a weak sol. of (2.1).} \end{array} \right\} \tag{2.8}$$

Our main result in the present work is the following.

Theorem 2.1. *Let $f \in C^2(\mathbb{R})$ such that $f'' \geq c > 0$ and*

$$\lim_{|x| \to \infty} f(x) = \infty\,. \tag{2.9}$$

Moreover let $u_0 \in L^\infty(\mathbb{R})$ *be compactly supported. Let* $u \in L^\infty(\mathbb{R} \times [0,T))$ *be an arbitrary weak solution of (2.1), such that* $m(x,t,a) = \partial_t u \wedge a + \partial_x f(u \wedge a)$ *is locally a Radon measure in* $\mathbb{R} \times [0,T) \times \mathbb{R}$ *. Assume the "entropy production"* m *satisfies*

$$\int_{\mathbb{R}} |m| \left(\mathbb{R} \times (0,\bar{t}), a\right) \, da \leq \int_{\mathbb{R}} |q| \left(\mathbb{R} \times (0,\bar{t}), a\right) \, da \;\forall\; q \in \mathcal{W} \text{ and } \forall \bar{t} \in (0,T) . \tag{2.10}$$

Then u *is the entropy solution, i.e. satisfies (2.2) and equivalently (2.3).*

A similar criteria in a more restrictive setting is considered by Dafermos in Chapter 9.7 of [Da]. He considers weak solutions u of (2.1) with initial data

$$u_0(x) = \begin{cases} u_l & \text{if } \quad x < 0 \,, \\ u_r & \text{if } \quad x > 0 \,. \end{cases} \tag{2.11}$$

Since the conservation law is invariant under Galilean transformations it is reasonable in this case to consider weak solutions of the form

$$u(x,t) = v\left(\frac{x}{t}\right) .$$

One can then define $\omega = \frac{x}{t}$ and consider v as a function only dependent of ω, i.e. $v = v(\omega)$. Then $v(\omega)$ satisfies the ordinary differential equation

$$\frac{d}{d\omega}\left(f(v(\omega)) - \omega v(\omega)\right) + v(\omega) = 0$$

in the sense of distributions and has prescribed end states

$$\lim_{\omega \to -\infty} v(\omega) = u_l \quad \text{and} \quad \lim_{\omega \to \infty} v(\omega) = u_r \,.$$

Furthermore it is assumed that v is in BV and J_v denotes the set of jump points ω for v. For a given entropy-entropy flux pair $(\eta(u), \xi(u))$ Dafermos defines the combined entropy of the shocks in v by

$$\mathcal{P}_v = \sum_{\omega \in J_v} \left\{ \xi(v(\omega+)) - \xi(v(\omega-)) - \omega \left[\eta(v(\omega+)) - \eta(v(\omega-)) \right] \right\} . \tag{2.12}$$

Furthermore he introduces the rate of change of the total entropy production

$$\dot{\mathcal{H}}_v = \frac{d}{dt} \int_{-\infty}^{\infty} \eta(u(x,t)) \, dx = \int_{-\infty}^{\infty} \eta(v(\omega)) \, d\omega \,,$$

for entropy-entropy flux pairs (η, ξ) such that $\eta(u_l) = \eta(u_r) = 0$.

He shows that in this simple case the rate of change of the total entropy and the entropy productions are related to each other by

$$\dot{\mathcal{H}}_v = \mathcal{P}_v + \xi(u_l) - \xi(u_r)\,.$$

We can now relate the combined entropy $\mathcal{P}_v$ to our entropy productions:

$$-\frac{1}{T}\int_{\mathbb{R}} \eta''(a) m(\mathbb{R}\times[0,T],a)\,da = \mathcal{P}_v \tag{2.13}$$

Since $T>0$ is arbitrary and $\mathcal{P}_v$ independent of T it follows from (2.13)

$$\frac{d}{dt}\int_{\mathbb{R}} \eta''(a) m(\mathbb{R}\times[0,t],a)\,da = \mathcal{P}_v \quad \text{for all} \quad t>0\,, \tag{2.14}$$

which finally relates (2.12) to (2.5).
Then a weak solution $u = v\left(\frac{x}{t}\right)$ of (2.1) with initial data (2.11) is said to satisfy the *entropy rate admissibility criterion* if it satisfies the following optimality criterion of the entropy production

$$\mathcal{P}_v \leq \mathcal{P}_{\tilde{v}}$$

or equivalently

$$\dot{\mathcal{H}}_v \leq \dot{\mathcal{H}}_{\tilde{v}}$$

holds, for any other weak solution $\tilde{u} = \tilde{v}\left(\frac{x}{t}\right)$ of (2.1) with initial condition (2.11).
Using (2.14) one can express in terms of the entropy production (2.5): A solution $u = v\left(\frac{x}{t}\right)$ with initial data (2.11) and defect measure $m(x,t,a)$ satisfies *entropy rate admissibility criterion* if

$$-\frac{d}{dt}\int_{\mathbb{R}} \eta''(a) m(\mathbb{R}\times[0,t],a)\,da \leq -\frac{d}{dt}\int_{\mathbb{R}} \eta''(a) \tilde{m}(\mathbb{R}\times[0,t],a)\,da \quad \text{for all} \quad t>0 \tag{2.15}$$

for any other weak solution $\tilde{u} = \tilde{v}\left(\frac{x}{t}\right)$ of (2.1) with initial condition (2.11) and defect measure $\tilde{m}(x,t,a)$. One can also integrate (2.15) and obtains the equivalent condition

$$-\int_{\mathbb{R}} \eta''(a) m(\mathbb{R}\times[0,t],a)\,da \leq -\int_{\mathbb{R}} \eta''(a) \tilde{m}(\mathbb{R}\times[0,t],a)\,da \quad \text{for all} \quad t>0\,. \tag{2.16}$$

Therefore (2.15) and (2.16) show, that the entropy rate admissibility criterion can be interpreted as a growth condition of the entropy production (2.5), which is similar to the growth condition (2.10) in Theorem 2.1. In Chapter 9.5 of [Da] it is proved:

Theorem. *[Da] A weak solution u of (2.1) with initial data (2.11) satisfies the entropy rate admissibility criterion for an entropy-entropy flux pair* (η, ξ) *if and only if u satisfies the E-condition (2.2).*

Again by (2.15) and (2.16) one sees, that this Theorem establishes, similar as in Theorem 2.1, a connection between growth rate of the entropy production (2.5) and entropy admissibility conditions (2.2) and (2.3). In Chapter 9.5 there is also an extension of this theorem in the case of strictly hyperbolic systems. For further references in the case of systems we refer also to Chapter 9.12 in [Da].

Another results relating an optimality criterion to entropic solution is given by A. Poliakovsky in [Po]. For $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}^k$ he considers a family of energy functionals

$$I_{\varepsilon,f}(u) = \frac{1}{2}\int_0^T \int_{\mathbb{R}^n} \left(\varepsilon|\nabla_x u|^2 + \frac{1}{\varepsilon}|\nabla_x H|^2\right) dx\, dt + \frac{1}{2}\int_{\mathbb{R}^n} |u(x,T)|^2\, dx \tag{2.17}$$

where

$$\Delta_x H_u = \partial_t u + \operatorname{div}_x f(u)\,.$$

Under certain assumptions on the flux f he shows, that there exists a minimizer to

$$\inf\{I_{\varepsilon,f}(u) :\ u(x,0) = u_0(x)\}$$

and this minimizer satisfies

$$\left.\begin{array}{rcll} \partial_t u + \operatorname{div}_x f(u) & = & \varepsilon\Delta_x H_u & \forall (x,t) \in \mathbb{R}^n \times (0,T)\,, \\ u(x,0) & = & u_0(x) & \forall x \in \mathbb{R}^n\,. \end{array}\right\}$$

In the particular case $k = 1$, he calculates the Γ-limit of (2.17) as $\varepsilon \to 0^+$ and finds an alternative variational formulation of the admissibility criterion for the particular solutions to the scalar conservation laws that can be achieved by this relaxation procedure.

The result of A.Poliakovsky has been inspired by previous works establishing a link between some variational optimality condition of a relaxed problem and the entropy condition at the limit. Among these works we can quote [RS1], [RS2] and [ALR]. Let us describe the results established in this 3 works here :

We consider for a bounded domain $\Omega \subset \mathbb{R}^2$ the space $\mathcal{M}_{div}(\Omega)$, which consists of unit vectorfields u such that $u = e^{i\varphi}$ for a $\phi \in L^\infty(\Omega, \mathbb{R})$ and $\operatorname{div} e^{i\varphi\wedge a}$ is a Radon measure over $\Omega \times \mathbb{R}$. This space $\mathcal{M}_{div}$ was introduced by S. Serfaty and the second author in [RS1] and [RS2] in connection to a problem related to micromagnetism. We give here a brief description. Let Ω be a bounded and simply connected domain, for $u \in W^{1,2}(\Omega, \mathbb{S}^1)$ and a $\varepsilon > 0$ we consider

$$E_\varepsilon(u) = \varepsilon \int_\Omega |\nabla u|^2 + \frac{1}{\varepsilon} \int_{\mathbb{R}^2} |H|^2 , \tag{2.18}$$

where $H = \nabla(G * \hat{u})$, $\hat{u} = u$ on Ω and $\hat{u} = 0$ in Ω^c and G is the kernel of the Laplacian on $\mathbb{R}^2$.
It was proved in [RS1], [RS2] that from any sequence $u_{\varepsilon_n} \in W^{1,2}(\Omega, \mathbb{S}^1)$ such that $\varepsilon \to 0$ and $E_{\varepsilon_n}(u_{\varepsilon_n}) < C$ one can extract a subsequence $u_{\varepsilon_{n'}}$ such that $\varphi_{\varepsilon_{n'}}$ converges strongly in $L^p(\Omega)$ for any $p < \infty$ to a limit φ such that $e^{i\varphi} = u \in \mathcal{M}_{div}(\Omega)$. Furthermore the authors are conjecturing that the Γ-Limit should be given by the following functional E_0 over $\mathcal{M}_{div}(\Omega)$:

$$E_0(u) := 2 \int_{a\in\mathbb{R}} \left|\text{div}\left(e^{i\varphi\wedge a}\right)\right|(\Omega)\, da$$

Part of the $\Gamma-$convergence has been proved as they established in one hand the following inequality

$$E_0(u) := 2 \int_{a\in\mathbb{R}} \left|\text{div}\left(e^{i\varphi\wedge a}\right)\right|(\Omega)\, da \leq \liminf E_{\varepsilon_{n'}}(u_{\varepsilon_{n'}})$$

and in the other hand that

$$\lim_{\varepsilon\to 0} \inf_{u\in W^{1,2}} E_\varepsilon(u) = 2 \inf_{u\in\mathcal{M}_{div}(\Omega)} \int_{a\in\mathbb{R}} \left|\text{div}\left(e^{i\varphi\wedge a}\right)\right|(\Omega)\, da = 2|\partial\Omega| , \tag{2.19}$$

where $|\partial\Omega|$ is the perimeter of the set Ω. One can prove (see [RS1]), that the infimum on the right hand side is achieved by $u = -\nabla^\perp \text{dist}(\cdot, \partial\Omega) \in \mathcal{M}_{div}(\Omega)$. The function $g = \nabla^\perp \text{dist}(\cdot, \partial\Omega)$ is the viscosity solution of

$$\left.\begin{array}{rcl} |\nabla g| - 1 &=& 0 \text{ on } \Omega\, , \\ g &=& 0 \text{ on } \partial\Omega\, . \end{array}\right\} \tag{2.20}$$

A question, which was left open in [RS1] and [RS2] was to describe the possible limits u of minimizing sequence of (2.18). It was conjectured that $u = \pm\nabla^\perp dist(\cdot, \partial\Omega)$ are the only possible limits of sequences of minimizers. A positive answer to this conjecture has been given in [ALR]. Precisely, in [RS2] it is proved that the limit u of a minimizing sequence of (2.18) satisfies the *entropy condition*

$$\text{div}\, e^{i\varphi\wedge a} \geq 0 \quad \text{for all} \quad a \in \mathbb{R} \tag{2.21}$$

or $\text{div}\, e^{i\varphi\wedge a} \leq 0$ for all $a \in \mathbb{R}$. Then in [ALR] the following result is established

Theorem. *[ALR] Let $u = -\nabla^{\perp} g$ be a divergence free unit vector-field in the space $\mathcal{M}_{div}(\Omega)$. The entropy condition (2.21) holds if and only if g is a viscosity solution of (2.20) and therefore g is locally semiconcave in Ω and $u \in BV_{loc}(\Omega, \mathbb{S}^1)$.*

Therefore, as a conclusion, one deduces the following equivalences for this particular problem

$$\text{viscosity solution to (2.20)} \quad \Longleftrightarrow \quad \text{entropy condition (2.21)}$$

$$\Longleftrightarrow \text{ minimality of the entropy production (2.19)} \quad .$$

The paper is organized as follows: First, in section 2, we establish some technical preliminary results. Then in Section 2.2.2 we will show, that the measure

$$\int_{\mathbb{R}} m(x,t,a)\, da$$

has no points with strictly negative density, outside possibly a set of 1-dimensional measure 0, i.e. we claim

$$\lim_{r\to 0^+} \frac{1}{r} \int_{\mathbb{R}} m\left(B_r(x_0,t_0),a\right)\, da \geq 0 \quad \text{ for } \mathcal{H}^1 \text{ a.e. } (x_0,t_0) \in \mathbb{R}\times(0,T). \tag{2.22}$$

In the last section, using an argument similar to the one used to prove the main result in [ALR], we deduce that the non negativity condition (2.22) implies that u is entropic.

2.2 Proof of Theorem 2.1

2.2.1 Preliminary results

In this section we define a notion of weak entropy solutions (see Definition 1) of scalar conservation laws on domain of trapezoidal shape (see (2.24). Afterward we will prove Lemma 2.1, which roughly says that for that kind of entropy solutions the same properties hold as in the classical case. We will use this results in Section 2.2 and Section 2.3.
For $0 < t_1 < t_2 < T$ and a $\delta > 0$ we define the set

$$\Gamma_{t_1}^{t_2} := \{(x,t) |\ t_2 > t > \gamma(x,t_1)\} \tag{2.23}$$

where

$$\gamma(x,t) := \begin{cases} t - \hat{\lambda}(x+\delta)) & \text{if } x \le -\delta\,, \\ t & \text{if } |x| \le \delta\,, \\ t + \hat{\lambda}(x-\delta)) & \text{if } x \ge \delta\,. \end{cases} \tag{2.24}$$

for a constant $0 < \hat{\lambda} \le 1$. Further we set

$$\Lambda_{t_1}^{t_2} := \{(x,t) |\ (x,t) = (x, \gamma(x,t_1)) \text{ and } t_1 \le t < t_2\}\,.$$

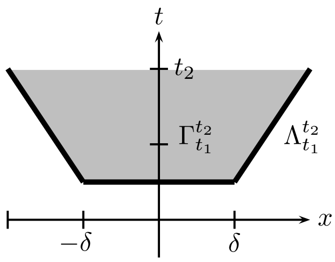

Figure 2.1: The set $\Gamma_{t_1}^{t_2}$

As for mentioned we define now a notion of weak respective entropy solution on the domain $\Gamma_{t_1}^{t_2}$

Definition 1. *For a $v_1 \in L^\infty(\Lambda_{t_1}^{t_2})$ we say that $v \in L^\infty(\Gamma_{t_1}^{t_2})$ is weak solution of*

$$\left.\begin{aligned} \partial_t v + \partial_x f(v) &= 0 \quad in \quad \Gamma_{t_1}^{t_2}\,, \\ v &= v_1 \quad on \quad \Lambda_{t_1}^{t_2}\,, \end{aligned}\right\} \tag{2.25}$$

if for all $\psi \in C_c^\infty(\mathbb{R} \times [0, t_2))$

$$\int_{\Gamma_{t_1}^{t_2}} v\partial_t\psi + f(v)\partial_x\psi \, dx \, dt = \int_{\Lambda_{t_1}^{t_2}} \psi \begin{pmatrix} f(v_1) \\ v_1 \end{pmatrix} \cdot n \, d\sigma \,. \tag{2.26}$$

holds, where n *is the outer unit normal of* $\Gamma_{t_1}^{t_2}$*. Furthermore we say that* $v \in L^\infty(\Gamma_{t_1}^{t_2})$ *is an entropy solution of (2.25), if* v *additionally satisfies*

$$q(x,t,a) := \partial_t v \wedge a + \partial_x f(v \wedge a) \in \mathcal{M}_{loc} \quad \textit{and} \quad q(x,t,a) \geq 0 \,.$$

A priory it is unclear if, for an arbitrary boundary condition $v_1 \in L^\infty(\Lambda_{t_1}^{t_2})$ the conservation law (2.25) possess a weak solution or not. We can however prove the following proposition.

Proposition 2.1. *Let* $v_1 \in L^\infty(\mathbb{R} \times [0,T))$ *be a weak solution of (2.1). Then for all* $0 < \hat\lambda < 1$ *and for almost every* $t_1 \in (0,T)$ *and all* $t_2 \in (t_1, T)$ *the problem*

$$\left.\begin{array}{rcll} \partial_t v + \partial_x f(v) & = & 0 & \text{in } \Gamma_{t_1}^{t_2} \,, \\ v & = & v_1 & \text{on } \Lambda_{t_1}^{t_2} \,, \end{array}\right\}$$

has an entropy solution in the sense of Definition 1.

The basic idea for proving Proposition 2.1 is to use the correspondence between weak solutions of (2.1) and viscosity subsolutions of

$$\left.\begin{array}{rcl} \partial_t g + f(\partial_x g) & = & 0 \,, \\ g(x,0) & = & g_0(x) \,. \end{array}\right\} \tag{2.27}$$

Before we are going to prove our assertion, we briefly repeat the definitions of viscosity sub- and supersolutions. We say that g is a viscosity solution of (2.27), if for any point $(x_0, t_0) \in \mathbb{R} \times (0,T)$ and for any $\psi \in C^1(\mathbb{R}^2)$ such that $g - \psi$ attains its maximum in (x_0, t_0) the following inequality holds

$$\partial_t\psi(x_0,t_0) + f(\partial_x\psi(x_0,t_0)) \leq 0 \,.$$

Similarly we say, that g is a viscosity supersolution of (2.27), if for any point $(x_0, t_0) \in \mathbb{R} \times (0,T)$ and for any for any $\psi \in C^1(\mathbb{R}^2)$ such that $g - \psi$ attains its minimum in (x_0, t_0) the following inequality holds

$$\partial_t\psi(x_0,t_0) + f(\partial_x\psi(x_0,t_0)) \geq 0 \,.$$

We say that g is a viscosity solution of (2.27), if g is both a sub- and supersolution. Theorem 2 in [CH] establishes a correspondence between weak solutions of (2.1) and viscosity subsolutions of (2.27).

Theorem 2.2 (Conway, Hopf)**.** *Let $u \in L^\infty(\mathbb{R} \times [0,t))$ be a weak solution of (2.1). Then there exists a $g \in W^{1,\infty}(\mathbb{R} \times [0,T))$ which satisfies (2.27) almost everywhere and is such that $u(x,t) = \partial_x g(x,t)$ and $u_0 = \partial_x g(x,0)$ for almost every $x \in \mathbb{R}$.*

Proof of Proposition 2.1. Let $v_1 \in L^\infty(\mathbb{R} \times [0,T))$ be a weak solution of (2.1); then according to Theorem 2.2 there exists $g_1 \in W^{1,\infty}(\mathbb{R} \times [0,T))$, which solves (2.27) almost everywhere. By Fubini's Theorem we can choose t_1 such that both $\partial_t g_1$ and $\partial_x g_1$ are in $L^\infty(\Lambda_{t_1}^{t_2})$ and such that

$$\int_{\Lambda_{t_1}^{t_2}} \partial_t g_1 + f(\partial_x g_1)\, d\sigma = 0 \quad \text{and} \quad v_1 = \partial_x g_1 \quad \text{a.e. on} \quad \Lambda_{t_1}^{t_2}\,. \tag{2.28}$$

For $t_1 < t_2 < T$ we want to show, that there exists a viscosity solution g of

$$\left.\begin{aligned} \partial_t g + f(\partial_x g) &= 0 \quad \text{in} \quad \Gamma_{t_1}^{t_2}\,, \\ g &= g_1 \quad \text{on} \quad \Lambda_{t_1}^{t_2}\,. \end{aligned}\right\} \tag{2.29}$$

Then we claim, that $v = \partial_x g$ is an entropy solution of (2.25), in the sense of Definition 1. The existence of such a viscosity solution g will be guaranteed by the existence result of Ishi (see Theorem 3.1 in [Is]). In order to be able to apply that theorem we must find a viscosity subsolution $\underline{g}$ and a viscosity supersolution $\overline{g}$ of (2.29), which satisfy pointwise $\underline{g} = \overline{g} = g_1$ on $\Lambda_{t_1}^{t_2}$ and $\underline{g} \le \overline{g}$ in $\Gamma_{t_1}^{t_2}$. According to Proposition 5.1 on page 77 in [BC], the fact that g_1 satisfies (2.27) almost everywhere implies, that g_1 is a viscosity subsolution of (2.27). Thus we can put $\underline{g} = g_1$ and it remains to find a viscosity supersolution $\overline{g}$ such that $\overline{g} \ge g_1$ and $\overline{g} = g_1$ on $\Lambda_{t_1}^{t_2}$. For two positive constants A, B we consider the function

$$\overline{g}_y(x,t) = g_1(y, \gamma(y,t_1)) + A|x-y| + B|t-\gamma(y)|\,.$$

We calculate for $(x,t) \in \Gamma_{t_1}^{t_2}$

$$\partial_t \overline{g}_y(x,t) + f(\partial_x \overline{g}_y(x,t)) = B\,\mathrm{sign}(t-\gamma(y)) + f(A\,\mathrm{sign}(x-y))\,.$$

By (2.9) this is positive, if we choose A large enough. Thus

$$\partial_t \overline{g}_y(x,t) + f(\partial_x \overline{g}_y(x,t)) > 0 \quad \text{for} \quad (x,t) \in \Gamma_{t_1}^{t_2}\,.$$

Proposition 5.1 on page 77 and Proposition 5.4 on page 78 in [BC] imply, that $\overline{g}$ is a viscosity supersolution. Further we notice, since $g_1 \in W^{1,\infty}(\mathbb{R} \times [0,T))$, that for all y and suitable choices of A and B

$$g_1(x,t) \le g(y,\gamma(y)) + A|x-y| + B|t-\gamma(y)|\,.$$

By Proposition 2.11 on page 302 in [BC]

$$\overline{g}(x,t) = \inf_y \overline{g}_y(x,t)$$

is still a supersolution. Furthermore $\overline{g}$ satisfies by construction $\overline{g} = g_1$ on $\Lambda_{t_1}^{t_2}$ and $\overline{g} \geq g$ in $\Gamma_{t_1}^{t_2}$. Hence all assumptions of the existence result (Theorem 3.1) in [Is] are fulfilled. Therefore there exists a viscosity solution g of (2.29) such that $g_1 \leq g \leq \overline{g}$. By Example 1 in [Is], the viscosity solution is Lipschitz continuous, i.e. $g \in W^{1,\infty}(\Gamma_{t_1}^{t_2})$. For $(x,t) \in \Gamma_{t_1}^{t_2}$ and $(y,s) \in \Lambda_{t_1}^{t_2}$ we notice that

$$g_1(x,t) - \overline{g}(y,s) \leq g(x,t) - g(y,s) \leq \overline{g}(x,t) - g_1(y,s)\,.$$

Using the fact that g_1 is Lipschitz continuous and the construction of $\overline{g}$ we deduce from the previous line

$$-\|(x,t)-(y,s)\| C_1 \leq g(x,t) - g(y,s) \leq C_2 \|(x,t)-(y,s)\|\,,$$

which means $g \in W^{1,\infty}(\Gamma_{t_1}^{t_2} \cup \Lambda_{t_1}^{t_2})$.
Next we are going to show, that $v = \partial_x g$ is a weak solution of (2.25) in $\Gamma_{t_1}^{t_2}$ in the sense of Definition 1. Since g satisfies (2.29) almost everywhere it follows for a $\psi \in C_c^\infty(\mathbb{R} \times [0,t_2))$

$$\int_{\Gamma_{t_1}^{t_2}} \partial_t\psi \partial_t g + f(\partial_x g)\partial_x \psi \, dx\, dt = 0\,. \tag{2.30}$$

We denote the outer unit normal vector of $\Gamma_{t_1}^{t_2}$ by n. Integrating the first term (2.30) twice by parts gives

$$\int_{\Gamma_{t_1}^{t_2}} \partial_x\psi \partial_t g \, dx\, dt = \int_{\partial\Gamma_{t_1}^{t_2}} g \begin{pmatrix} -\partial_t\psi \\ \partial_x\psi \end{pmatrix} \cdot n \, d\sigma + \int_{\Gamma_{t_1}^{t_2}} \partial_t\psi\partial_x g\, dx\, dt\,. \tag{2.31}$$

Rewriting the boundary term in (2.31) and using the fact that $\psi(x,t_2) = 0$ leads to

$$\begin{aligned} \int_{\partial\Gamma_{t_1}^{t_2}} g \begin{pmatrix} -\partial_t\psi \\ \partial_x\psi \end{pmatrix} \cdot n \, d\sigma &= \int_{\Lambda_{t_1}^{t_2}} g_1 \begin{pmatrix} -\partial_t\psi \\ \partial_x\psi \end{pmatrix} \cdot n \, d\sigma \\ &= \int_{s_1}^{s_2} g_1(s,\gamma(s,t_1)) \left[\partial_t\psi(s,\gamma(s,t_1))\gamma'(s,t_1) + \partial_x\psi(s,\gamma(s,t_1))\right] ds\,. \end{aligned} \tag{2.32}$$

We integrate the right-hand side of (2.32) by parts

$$\begin{aligned} &\int_{s_1}^{s_2} g_1(s,\gamma(s,t_1)) \left[\partial_t\psi(s,\gamma(s,t_1))\partial_s\gamma(s,t_1) + \partial_x\psi(s,\gamma(s,t_1))\right] ds \\ &= \int_{s_1}^{s_2} g_1(s,\gamma(s,t_1)) \frac{d}{ds}\psi(s,\gamma(s,t_1))\, ds = -\int_{s_1}^{s_2} \frac{d}{ds} g_1(s,\gamma(s,t_1)) \cdot \psi \, ds\,. \end{aligned} \tag{2.33}$$

Therefore combining (2.32) and (2.33) we can rewrite the boundary term in (2.31) as

$$\int_{\partial\Gamma_{t_1}^{t_2}} g_1 \begin{pmatrix} -\partial_t\psi \\ \partial_x\psi \end{pmatrix} \cdot n \, d\sigma = \int_{s_1}^{s_2} \left[\partial_x g_1 + \partial_t g_1 \cdot \partial_s\gamma(s,t_1)\right] \psi \, ds \tag{2.34}$$

Using (2.28) the right-hand side of (2.34) simplifies to

$$\begin{aligned} \int_{\partial\Gamma_{t_1}^{t_2}} g_1 \begin{pmatrix} -\partial_t\psi \\ \partial_x\psi \end{pmatrix} \cdot n \, d\sigma &= \int_{s_1}^{s_2} \left[\partial_x g_1 - f(\partial_x g_1) \cdot \partial_s\gamma(s,t_1)\right] \psi \, ds \\ &= \int_{\Lambda_{t_1}^{t_2}} \psi \begin{pmatrix} -f(\partial_x g_1) \\ \partial_x g_1 \end{pmatrix} \cdot n \, d\sigma \,, \end{aligned}$$

where n is the unit normal to $\Gamma_{t_1}^{t_2}$. We replace now the boundary term in (2.31) using the above identity

$$\int_{\Gamma_{t_1}^{t_2}} \partial_x\psi\partial_t g \, dx\, dt = \int_{\Lambda_{t_1}^{t_2}} \psi \begin{pmatrix} \partial_x g_1 \\ -f(\partial_x g_1) \end{pmatrix} \cdot \tau \, d\sigma + \int_{\Gamma_{t_1}^{t_2}} \partial_t\psi\partial_x g \, dx\, dt \,.$$

Finally this together with (2.30) gives

$$\int_{\Lambda_{t_1}^{t_2}} \psi \begin{pmatrix} \partial_x g_1 \\ -f(\partial_x g_1) \end{pmatrix} \cdot \tau \, d\sigma + \int_{\Gamma_{t_1}^{t_2}} \partial_t\psi\partial_x g + f(\partial_x g)\partial_x\psi \, dx\, dt = 0 \,.$$

Since $v_1 = \partial_x g_1$ and by putting $v = \partial_x g$ we see that v is a solution of (2.25) in the sense of Definition 1. It remains to show that v is an entropy solution in the sense that

$$\partial_t v \wedge a + \partial_x f(v \wedge a) \geq 0 \quad \mathcal{D}' \,.$$

By Corollary 1.7.2 in [CS] v satisfies for all (x,t), $(y,t) \in \Gamma_{t_1}^{t_2}$ such that $x < y$

$$v(y,t) - v(x,t) \leq \frac{y-x}{ct} \,.$$

This immediately implies $q(x,t,a) \geq 0$ (see Section 8.5 in [Da]). □

Proposition 2.1 being proved, we now establish some properties for entropy solutions to (2.25) analogous to those in the classical case (see [Da]). Precisely we are going to show

Lemma 2.1. *Let $v_1 \in L^\infty(\mathbb{R} \times (0,T))$ be a weak solution of (2.1). Then there exists a constant $\lambda_0 > 0$, depending on f and $\|v_1\|_\infty$, such that, for any* *domain* $\Gamma_{t_1}^{t_2}$ *satisfying* $0 < \hat{\lambda} \le \lambda_0$*, the entropy solution* $v \in L^\infty(\Gamma_{t_1}^{t_2})$ *of (2.25) with boundary condition* $v_1 \in L^\infty(\Lambda_{t_1}^{t_2})$ *satisfies*

$$\lim_{\varepsilon \to 0^+} \int_{s_1}^{s_2-\varepsilon} |v(s,\gamma(s,t_1+\varepsilon)) - v_1(s,\gamma(s,t_1))| \, ds = 0 \,, \tag{2.35}$$

where

$$s_1 = -\frac{t_2-t_1}{\hat{\lambda}} - \delta \quad \text{and} \quad s_2 = \frac{t_2-t_1}{\hat{\lambda}} + \delta \,.$$

Moreover

$$\|v\|_\infty \le \|v_1\|_\infty \tag{2.36}$$

and there exists a constant $C > 0$*, depending only on* $\|v\|_1$ *and* $\hat{\lambda}$*, such that*

$$\int_{\Gamma_{t_1}^{t_2}} q(x,t,a) \, da \, dx \, dt \le C(t_2 - t_1) \,. \tag{2.37}$$

Let now w_1*,* $w_2 \in L^\infty(\mathbb{R} \times (0,T))$ *be weak solutions of (2.1). Then there there exists a constant* $\lambda_1 > 0$ *depending on* f *and* $\max\{\|w_1\|_\infty, \|w_2\|_\infty\}$ *such that, for any domain* $\Gamma_{t_1}^{t_2}$ *satisfying* $0 < \hat{\lambda} \le \lambda_1$ *and any choice of two entropy solutions respectively* $v_1 \in L^\infty(\Gamma_{t_1}^{t_2})$ *with boundary condition* $w_1 \in L^\infty(\Lambda_{t_1}^{t_2})$ *and* $v_2 \in L^\infty(\Gamma_{t_1}^{t_2})$ *with boundary condition* $w_2 \in L^\infty(\Lambda_{t_1}^{t_2})$ *the following holds : for any* $t \in (t_1, t_2)$ *and a constant* $C > 0$ *depending on* $\Gamma_{t_1}^{t_2}$ *and* $\max\{\|w_1\|_\infty, \|w_2\|_\infty\}$*:*

$$\int_{\theta^-(t)}^{\theta^+(t)} |v_1(x,t) - v_2(x,t)| \, dx \le C \int_{\Lambda_{t_1}^{t_2}} |w_1 - w_2| \, d\sigma \,, \tag{2.38}$$

where

$$\theta^\pm(t) = \pm\frac{t-t_1}{\hat{\lambda}} \pm \delta \,.$$

Remark 1. Inequality (2.38) implies in particular the uniqueness of the entropy solution for a given initial data w on $\Lambda_{t_1}^{t_2}$ issued from a weak solution to (2.1).

Proof of Lemma 2.1. We start to prove (2.35). Let $R > 0$ such that

$$R + f(\pm R) \ge 0 \,.$$

We choose λ_0 such that

$$\lambda_0^{-1} = \max\left\{|f'(R+1+\|v_1\|_\infty)|\,, |f'(-R-1-\|v_1\|_\infty)|\right\} \,. \tag{2.39}$$

We consider now a domain $\Gamma_{t_1}^{t_2}$ such that $\hat{\lambda} \leq \lambda_0$ and an entropy solution $v \in L^\infty(\Gamma_{t_1}^{t_2})$ of (2.25) exists. From Example 1 in [Is], we know, that

$$\|v\|_\infty \leq R + 1 \,. \tag{2.40}$$

Let $\psi \in C_c^\infty(\mathbb{R} \times [0, t_2))$. From Theorem 1.3.4 in [Da] we get for all $\varepsilon > 0$ such that $t_1 + \varepsilon < t_2$

$$\int_{\Gamma_{t_1+\varepsilon}^{t_2-\varepsilon}} v\partial_t\psi + f(v)\partial_x\psi \, dx \, dt = \int_{\partial\Gamma_{t_1+\varepsilon}^{t_2-\varepsilon}} \begin{pmatrix} f(v) \\ v \end{pmatrix} \cdot n\psi \, d\sigma \,. \tag{2.41}$$

As $\varepsilon \to 0^+$ the left-hand side of (2.41) converges to

$$\int_{\Gamma_{t_1}^{t_2}} v\partial_t\psi + f(v)\partial_x\psi \, dx \, dt \,.$$

Since v is a weak solution of (2.25) and $\psi(x, t_2) = 0$ the right-hand side of (2.41) behaves like

$$\lim_{\varepsilon \to 0^+} \int_{\Lambda_{t_1+\varepsilon}^{t_2-\varepsilon}} \begin{pmatrix} f(v) \\ v \end{pmatrix} \cdot n\psi \, d\sigma = \int_{\Lambda_{t_1}^{t_2}} \begin{pmatrix} f(v_1) \\ v_1 \end{pmatrix} \cdot \tau\psi \, d\sigma \tag{2.42}$$

In order to keep the notation simple we introduce

$$\bar{\gamma}(s) = \begin{pmatrix} s \\ \gamma(s, t_1) \end{pmatrix} \quad \text{and} \quad v_\varepsilon(x, t) = v(x, t + \varepsilon)$$

and rewrite (2.42) as

$$\lim_{\varepsilon \to 0^+} \int_{s_1+\varepsilon}^{s_2-\varepsilon} \Big\{ v_1(\bar{\gamma}(s)) - v_\varepsilon(\bar{\gamma}(s)) + \hat{\lambda} \left[f(v_\varepsilon(\bar{\gamma}(s))) - f(v_1(\bar{\gamma}(s))) \right] \Big\} \, \psi \, ds = 0 \,. \tag{2.43}$$

By (2.39) we obtain the existence of some constants C, $c > 0$ for which the following holds

$$c \leq 1 \pm f(\alpha) \leq C \quad \text{for all} \quad \alpha \in (-R - 1 - \|v_1\|_\infty, R + 1 + \|v\|_\infty) \,.$$

Therefore we get from (2.43), that

$$\lim_{\varepsilon \to 0^+} v(s, \gamma(s, t_1 + \varepsilon)) = v_1(s, \gamma(s)) \quad \text{for a.e.} \quad s \in [s_1, s_2] \,. \tag{2.44}$$

By dominated convergence, we deduce the claim (2.35).

To prove the remaining claims of our lemma, we need to introduce the kinetic formulation of conservation laws, we recommend the introduction to this subject given in [Pe]. However we need here a slight modified version of this formulation. We define for any $v \in \mathbb{R}$

$$\chi(v;a) := \mathbb{1}_{a\leq v}\,,$$

where $\mathbb{1}_{a\leq v}$ is the characteristic function of the set $\{a \in \mathbb{R} : \ a \leq v\}$. Then a weak solution $v \in L^\infty(\Gamma_{t_1}^{t_2})$ of (2.25) satisfies in the distributional sense

$$\left.\begin{array}{rcll} \partial_t\chi(v(x,t);a) + f'(a)\partial_x\chi(v(x,t);a) &=& \partial_a q(x,t,a) & \text{in} \quad \Gamma_{t_1}^{t_2}\,, \\ \chi(v;a) &=& \chi(v_1;a) & \text{on} \quad \Lambda_{t_1}^{t_2}\,. \end{array}\right\} \tag{2.45}$$

In other words this means that for all $\psi \in C_c^\infty(\mathbb{R}\times[0,t_2)\times\mathbb{R})$

$$\begin{aligned} &\int_{\Gamma_{t_1}^{t_2}}\int_{\mathbb{R}} \chi(v;a)\partial_t\psi + f'(a)\chi(v;a)\partial_x\psi\,da\,dx\,dt \\ &\qquad = \int_{\Gamma_{t_1}^{t_2}} \psi\partial_a q(x,t,a) + \int_{\Lambda_{t_1}^{t_2}}\int_{\mathbb{R}} \psi \begin{pmatrix} f'(a)\chi(v_1;a) \\ \chi(v_1;a)\end{pmatrix}\cdot n\,da\,d\sigma\,. \end{aligned} \tag{2.46}$$

In order to prove (2.38), (2.36) and (2.37) we need to regularize our kinetic equation (2.45). We choose $\varphi_1(x)$, $\varphi_2(t) \in C_c^\infty(\mathbb{R})$ non-negative functions such that

$$\operatorname{supp}\varphi_1 \subset (-1,1)\,, \quad \operatorname{supp}\varphi_2 \subset [-1,0]$$

and

$$\int_{\mathbb{R}}\varphi_2\,dx = \int_{\mathbb{R}}\varphi_1\,dx = 1\,.$$

We define the kernel

$$\varphi_\varepsilon(x,t) = \frac{1}{\varepsilon^2}\varphi_1\left(\frac{x}{\varepsilon}\right)\varphi_2\left(\frac{t}{\varepsilon}\right)\,. \tag{2.47}$$

For a constant C depending only from $\hat{\lambda}$ we have

$$\operatorname{dist}((x,t),\partial\Gamma_{t_1}^{t_2}) > \varepsilon \quad \text{for all} \quad (x,t) \in \Gamma_{t_1+C\varepsilon}^{t_2-C\varepsilon}\,.$$

Consequently for $(x,t) \in \Gamma_{t_1+C\varepsilon}^{t_2-C\varepsilon}$

$$\varphi_\varepsilon(x-y,t-s) = 0 \quad \text{for} \quad (y,s) \in \partial\Gamma_{t_1}^{t_2}\,. \tag{2.48}$$

We define moreover the two mollified functions

$$\chi_\varepsilon(x,t,a) = \int_{\Gamma_{t_1}^{t_2}} \varphi_\varepsilon(x-y,t-s)\chi(v(y,s);a)\,dy\,ds$$

and

$$q_\varepsilon(x,t,a) = \int_{\Gamma_{t_1}^{t_2}} \varphi_\varepsilon(x-y,t-s)q(y,s,a)\,dy\,ds\,.$$

For q_ε and $(x,t) \in \Gamma_{t_1+C\varepsilon}^{t_2-C\varepsilon}$ we compute

$$\begin{aligned} q_\varepsilon(x,t,a) - q_\varepsilon(x,t,b) &= \int_{\Gamma_{t_1}^{t_2}} \varphi_\varepsilon(x-y,t-s)\,[dq(y,s,a) - dq(y,s,b)] \\ &= -\int_{\Gamma_{t_1}^{t_2}} \partial_t\varphi_\varepsilon(x-y,t-s)\,[v\wedge a - v\wedge b] \\ &\quad + \partial_x\varphi_\varepsilon(x-y,t-s)\,[f(v\wedge a) - f(v\wedge b)]\;dy\,ds\,, \end{aligned}$$

where we have made use of (2.48). Since

$$|v\wedge a - v\wedge b| \le \|v\|_\infty |b-a|$$

it follows from above calculation

$$\begin{aligned} |q_\varepsilon(x,t,a) - q_\varepsilon(x,t,b)| &\le \int_{\Gamma_{t_1}^{t_2}} |\partial_t\varphi_\varepsilon|\cdot\|v\|_\infty|b-a| + C\,|\partial_x\varphi_\varepsilon|\cdot\|v\|_\infty|b-a|\,dy\,ds \\ &\le C|b-a|\,. \end{aligned}$$

Therefore q_ε is Lipschitz continuous with respect to the kinetic variable a and we have for almost every $a \in \mathbb{R}$ in the classical sense

$$\partial_t\chi_\varepsilon + f'(a)\partial_x\chi_\varepsilon = \partial_a q_\varepsilon(x,t,a) \quad \text{in} \quad \Gamma_{t_1+C\varepsilon}^{t_2-C\varepsilon}\,. \tag{2.49}$$

Notice that due to the convolution with φ_ε both χ_ε and q_ε are smooth with respect to (x,t). Furthermore for $(x,t) \in \Gamma_{t_1+C\varepsilon}^{t_2-C\varepsilon}$ the function q_ε satisfies

$$q_\varepsilon(x,t,a) = 0 \quad \text{if} \quad |a| \ge \|v\|_\infty\,. \tag{2.50}$$

This follows from the classical fact, that

$$q(x,t,a) = 0 \quad \text{for} \quad |a| \ge \|v\|_\infty\,.$$

Indeed for $|a| \ge \|v\|_\infty$ and $\psi \in C_c^\infty(\Gamma_{t_1}^{t_2})$ we compute

$$\begin{aligned} \int_{\Gamma_{t_1}^{t_2}} q(x,t,a)\psi(x,t)\,dx\,dt &= \int_{\Gamma_{t_1}^{t_2}} [\partial_t v(x,t)\wedge a + \partial_x f(v(x,t)\wedge a)]\,\psi(x,t)\,dy\,ds \\ &= \int_{\Gamma_{t_1}^{t_2}} [\partial_t v(x,t) + \partial_x f(v(x,t))]\,\psi(x,t)\,dx\,dt \\ &= -\int_{\Gamma_{t_1}^{t_2}} v(x,t)\partial_t\psi(x,t) + f(v(x,t))\partial_x\psi(x,t)\,dx\,dt \\ &= 0\,. \end{aligned}$$

We consider now a convex function $\eta(a)$ in C^2 which satisfies

$$\lim_{a\to-\infty}\eta(a)=0$$

and define

$$\xi(a):=\int \eta'(a)f'(a)\,da\,.$$

We claim

Claim 1. For all $(x,t)\in\Gamma_{t_1+C\varepsilon}^{t_2-C\varepsilon}$ the following equality holds

$$\partial_t\eta_\varepsilon(x,t)+\partial_x\xi_\varepsilon(x,t)=-\int_{\Gamma_{t_1}^{t_2}}\eta''(a)q_\varepsilon(x,t,a)\,da\,, \tag{2.51}$$

where

$$\eta_\varepsilon(x,t)=\int_{\Gamma_{t_1}^{t_2}}\eta(v(y,s))\varphi_\varepsilon\left(x-y,t-s\right)\,dy\,ds$$

and

$$\xi_\varepsilon(x,t)=\int_{\Gamma_{t_1}^{t_2}}\xi(v(y,s))\varphi_\varepsilon\left(x-y,t-s\right)\,dy\,ds\,.$$

Later will make special choices of η in order to get (2.36) and (2.37).

Proof of claim 1. We multiply the regularized kinetic formulation (2.49) by $\eta'(a)$

$$\eta'(a)\partial_t\chi_\varepsilon+\eta'(a)f'(a)\partial_x\chi_\varepsilon=\partial_a q_\varepsilon(x,t,a)\,.$$

Then integrating this equation with respect to a gives

$$\int_{\mathbb{R}}\eta'(a)\partial_t\chi_\varepsilon+\eta'(a)f'(a)\partial_x\chi_\varepsilon\,da=\int_{\mathbb{R}}\eta'(a)\partial_a q_\varepsilon(x,t,a)\,da\,. \tag{2.52}$$

We observe

$$\begin{aligned}\int_{\mathbb{R}}\eta'(a)\chi_\varepsilon\,da&=\int_{\Gamma_{t_1}^{t_2}}\int_{\mathbb{R}}\eta'(a)\chi(v(y,s);a)\varphi_\varepsilon\left(x-y,t-s\right)\,da\,dy\,ds\\&=\int_{\Gamma_{t_1}^{t_2}}\eta(v(y,s))\varphi_\varepsilon\left(x-y,t-s\right)\,dy\,ds=\eta_\varepsilon(x,t)\end{aligned}$$

and

$$\int_{\mathbb{R}}\eta'(a)f'(a)\chi_\varepsilon\,da=\int_{\Gamma_{t_1}^{t_2}}\xi(v(y,s))\varphi_\varepsilon\left(x-y,t-s\right)\,dy\,ds=\xi_\varepsilon(x,t)\,.$$

Thus (2.52) reduces to

$$\partial_t \eta_\varepsilon(x,t) + \partial_x \xi_\varepsilon(x,t) = \int_{\mathbb{R}} \eta'(a)\partial_a q_\varepsilon(x,t,a)\, da\,. \tag{2.53}$$

Integrating the right-hand side by parts gives

$$\int_{\mathbb{R}} \eta'(a)\partial_a q_\varepsilon(x,t,a)\, da = -\int_{\mathbb{R}} \eta''(a) q(x,t,a)\, da\,,$$

where we have used the fact that q_ε is compactly supported in a. This gives the result (2.51) and claim 1 is proved.

Next we integrate inequality (2.51) over the set $\Gamma^{\bar t}_{t_1+C\varepsilon}$, where $\bar t \in (t_1 + C\varepsilon, t_2 - C\varepsilon)$. We will abbreviate $t_1 + C\varepsilon$ by $\bar t_1$. We have

$$\int_{\Gamma^{\bar t}_{\bar t_1}} \partial_t \eta_\varepsilon(x,t) + \partial_x \xi_\varepsilon(x,t)\, dx\, dt = -\int_{\Gamma^{\bar t}_{\bar t_1}} \int_{\mathbb{R}} \eta''(a) q_\varepsilon(x,t,a)\, da\,. \tag{2.54}$$

Using Gauss' Theorem for (2.54) gives

$$\int_{\theta^-_\varepsilon(\bar t)}^{\theta^+_\varepsilon(\bar t)} \eta_\varepsilon(x,\bar t)\, dx = -\int_{\Lambda^{\bar t}_{\bar t_1}} \begin{pmatrix} \xi_\varepsilon \\ \eta_\varepsilon \end{pmatrix} \cdot n\, d\sigma - \int_{\Gamma^{\bar t}_{\bar t_1}} \int_{\mathbb{R}} \eta''(a) q_\varepsilon(x,t,a)\, da\,, \tag{2.55}$$

where

$$\theta^\pm_\varepsilon(t) = \pm\frac{\bar t - \bar t_1}{\hat\lambda} \pm \delta\,. \tag{2.56}$$

For suitable choices of η this equality (2.55) will imply the first two claims of Lemma 2.1.

First we prove (2.36). Let a_0 be a real number being fixed later in this proof. We choose

$$\eta(a) = \begin{cases} (a-a_0) & \text{if } a - a_0 \geq 0\,, \\ 0 & \text{if } a - a_0 \leq 0 \end{cases}$$

and we aim to deduce

$$\int_{\theta^-(\bar t)}^{\theta^+(\bar t)} |v(x,\bar t) - a_0|^+\, d\sigma \leq C \int_{\Lambda^{\bar t}_{\bar t_1}} |v_0(x) - a_0|^+\, d\sigma\,, \tag{2.57}$$

from equality (2.55). The non-negativity of $\eta''(a)$ and q_ε implies

$$\int_{\Gamma^{\bar t}_{\bar t_1}} \int_{\mathbb{R}} \eta''(a) q_\varepsilon(x,t,a)\, da \geq 0\,.$$

Using this inequality in equality (2.55) we obtain the estimate

$$\int_{\theta_\varepsilon^-(\bar t)}^{\theta_\varepsilon^+(\bar t)} \eta_\varepsilon(x,\bar t)\,dx \le -\int_{\Lambda_{t_1}^{\bar t}} \begin{pmatrix}\xi_\varepsilon\\ \eta_\varepsilon\end{pmatrix}\cdot n\,d\sigma\,.$$

Letting $\varepsilon\to 0^+$ we get

$$\int_{\theta^-(\bar t)}^{\theta^+(\bar t)} \eta(x,\bar t)\,dx \le -\int_{\Lambda_{t_1}^{\bar t}} \begin{pmatrix}\xi(v_1)\\ \eta(v_1)\end{pmatrix}\cdot n\,d\sigma\,.$$

We observe

$$|\xi(a)| \le \max_{|b|\le\|v_1\|_\infty} |f'(b)|\cdot\eta(a)\,, \tag{2.58}$$

which implies

$$\int_{\theta^-(\bar t)}^{\theta^+(\bar t)} \eta(x,\bar t)\,dx \le C\int_{\Lambda_{t_1}^{\bar t}} \eta(v_1)\,dx\,.$$

This is our desired result (2.57) and choosing $a_0=\|v_1\|_\infty$ in (2.57) gives

$$\int_{\theta_-(\bar t)}^{\theta^+(\bar t)} |v(x,\bar t)-a_0|^+\,d\sigma = 0$$

and thus (2.36) follows:

$$|v(x,t)|\le\|v_1\|_\infty \quad \text{a.e. in} \quad \Gamma_{t_1}^{t_2}\,.$$

In order to prove (2.37), we choose now

$$\eta(a) := \begin{cases} 2a^2 & \text{if}\ \ a\ge -\|v\|_\infty\,,\\ (a+\|v\|_\infty)+2\|v\|_\infty^2 & \text{if}\ \ -(\|v\|_\infty+2\|v\|_\infty^2)\le a\le -\|v\|_\infty\,,\\ 0 & \text{if}\ \ a\le -(\|v\|_\infty+2\|v\|_\infty^2)\,. \end{cases}$$

Since η is non-negative, we deduce from (2.55)

$$2\int_{\Gamma_{t_1+C\varepsilon}^{t_2-C\varepsilon}}\int_{\mathbb{R}} q_\varepsilon(x,t,a)\,da\,dx\,dt \le -\int_{\Lambda_{t_1}^{\bar t}} \begin{pmatrix}\xi_\varepsilon\\ \eta_\varepsilon\end{pmatrix}\cdot n\,d\sigma \quad . \tag{2.59}$$

Since $\eta(a)=2a^2$ for $a\in[-\|v\|_\infty,\|v\|_\infty]$ we get

$$|\xi(a)| = \int_{-\|v\|_\infty}^{a} |\eta'(b)f'(b)|\,db \le f'(\|v\|_\infty)\int_{-\|v\|_\infty}^{|a|} |\eta'(b)|\,db\,.$$

Hence by letting $\varepsilon \to 0$ in (2.59), we obain

$$\int_{\Gamma_{t_1}^{t_2}} \int_{\mathbb{R}} q(x,t,a)\, da\, dx\, dt \leq C(\delta + t_2 - t_1)\,,$$

as announced in (2.37).

Finally we are going to prove (2.38). We choose the domain $\Gamma_{t_1}^{t_2}$ in such a way that

$$0 < \hat{\lambda} \leq \lambda_1\,,$$

where

$$\lambda_1 = \tfrac{1}{2}\left(\max_{a\in[-\alpha,\alpha]} |f'(a)|\right)^{-1}$$

$$\text{and} \tag{2.60}$$

$$\alpha = \max\{\|w_1\|_\infty, \|w_2\|_\infty\}\,.$$

For the two entropy solutions v_1, v_2 with boundary conditions w_1 and w_2 we consider the kinetic equations

$$\left.\begin{array}{rcll} \partial_t \chi_i + f'(a)\partial_x \chi_i &=& \partial_a q_i & \text{in} \quad \mathcal{D}'(\Gamma_{t_1}^{t_2} \times \mathbb{R}) \\ \chi_i &=& \chi(w_i; a) & \text{on} \quad \Lambda_{t_1}^{t_2} \end{array}\right\}$$

where $\chi_i = \chi(v_i(x,t); a)$ for $i = 1, 2$. Then, as before, we can regularize our kinetic equations with the kernel defined in (2.47)

$$\partial_t \chi_i^\varepsilon + f'(a)\partial_x \chi_i^\varepsilon = \partial_a q_i^\varepsilon(x,t,a) \quad \text{in} \quad \Gamma_{t_1+C\varepsilon}^{t_2-C\varepsilon}$$

where

$$\chi_i^\varepsilon(x,t,a) = \int_{\Gamma_{t_1}^{t_2}} \chi(v_i(x,t);a)\varphi_\varepsilon(x-y,t-s)\, dx\, dt \quad \text{for} \quad i = 1, 2$$

and $C > 0$ is again chosen such that for $(x,t) \in \Gamma_{t_1+C\varepsilon}^{t_2-C\varepsilon}$

$$\varphi_\varepsilon(x-y,t-s) = 0 \quad \text{for} \quad (y,s) \in \partial\Gamma_{t_1}^{t_2}\,.$$

The function $(\chi_1^\varepsilon - \chi_2^\varepsilon)^2$ satisfies for $(x,t) \in \Gamma_{t_1+C\varepsilon}^{t_2-C\varepsilon}$ and almost every $a \in \mathbb{R}$

$$\partial_t (\chi_1^\varepsilon - \chi_2^\varepsilon)^2 + f'(a)\partial_x (\chi_1^\varepsilon - \chi_2^\varepsilon)^2 = \chi_1^\varepsilon \partial_a q_1^\varepsilon + \chi_2^\varepsilon \partial_a q_2^\varepsilon - \chi_2^\varepsilon \partial_a q_1^\varepsilon - \chi_1^\varepsilon \partial_a q_2^\varepsilon\,. \tag{2.61}$$

We use again the following abbreviation: $t_1 + C\varepsilon = \bar{t}_1$. Let $\bar{t} \in (\bar{t}_1, +t_2 - C\varepsilon)$ then we integrate (2.61) in $\Gamma_{\bar{t}_1}^{\bar{t}} \times \mathbb{R}$ which leads to

$$\int_{\Gamma_{\bar{t}_1}^{\bar{t}}} \int_{\mathbb{R}} \partial_t \left(\chi_1^\varepsilon - \chi_2^\varepsilon\right)^2 + f'(a)\partial_x \left(\chi_1^\varepsilon - \chi_2^\varepsilon\right)^2 \, da\, dx\, dt$$
$$= \int_{\Gamma_{\bar{t}_1}^{\bar{t}}} \int_{\mathbb{R}} \chi_1^\varepsilon \partial_a q_1^\varepsilon + \chi_2^\varepsilon \partial_a q_2^\varepsilon - \chi_2^\varepsilon \partial_a q_1^\varepsilon - \chi_1^\varepsilon \partial_a q_2^\varepsilon \, da\, dx\, dt\,. \quad (2.62)$$

We recall, that $\chi(v; a) = \mathbb{1}_{a \leq v}$ and

$$q_1(x,t,a) = q_2(x,t,a) = 0 \quad \text{for} \quad |a| \geq \max\{\|v_1\|_\infty, \|v_2\|_\infty\}\,.$$

Therefore we can calculate for $(x,t) \in \Gamma_{\bar{t}_1}^{\bar{t}}$ and $i,\, j \in \{1,\, 2\}$

$$\int_{\mathbb{R}} \chi_i^\varepsilon \partial_a q_j^\varepsilon \, da = \int_{\mathbb{R}} \int_{\Gamma_{t_1}^{t_2}} \chi(v_i(y,s); a) \varphi_\varepsilon(x-y, t-s) q_j^\varepsilon(x,t,a)\, dy\, ds\, da$$
$$= \int_{\Gamma_{t_1}^{t_2}} q_j^\varepsilon(x,t,v_i(y,s)) \varphi_\varepsilon(x-y,t-s)\, dy\, ds\,.$$

Since φ_ε and q_ε are non-negative we obtain

$$\int_{\mathbb{R}} \chi_2^\varepsilon \partial_a q_1^\varepsilon \, da \geq 0 \quad \text{and} \quad \int_{\mathbb{R}} \chi_1^\varepsilon \partial_a q_2^\varepsilon \, da \geq 0\,,$$

which we apply in (2.62) and gives the inequality

$$\int_{\Gamma_{\bar{t}_1}^{\bar{t}}} \int_{\mathbb{R}} \partial_t \left(\chi_1^\varepsilon - \chi_2^\varepsilon\right)^2 + f'(a)\partial_x \left(\chi_1^\varepsilon - \chi_2^\varepsilon\right)^2 \, da\, dx\, dt$$
$$\leq \int_{\Gamma_{\bar{t}_1}^{\bar{t}}} \int_{\mathbb{R}} \chi_1^\varepsilon \partial_a q_1^\varepsilon + \chi_2^\varepsilon \partial_a q_2^\varepsilon \, da\, dx\, dt\,. \quad (2.63)$$

For the left hand-side of (2.63) we compute

$$\int_{\mathbb{R}} \int_{\Gamma_{\bar{t}_1}^{\bar{t}}} \partial_t \left(\chi_1^\varepsilon - \chi_2^\varepsilon\right)^2 + f'(a)\partial_x \left(\chi_1^\varepsilon - \chi_2^\varepsilon\right)^2 \, dx\, dt\, da$$
$$= \int_{\mathbb{R}} \int_{\theta_\varepsilon^-(\bar{t})}^{\theta_\varepsilon^+(\bar{t})} \left(\chi_1^\varepsilon - \chi_2^\varepsilon\right)^2 (x,\bar{t})\, dx\, da + \int_{\Lambda_{\bar{t}_1}^{\bar{t}}} \int_{\mathbb{R}} \begin{pmatrix} f'(a)\left(\chi_1^\varepsilon - \chi_2^\varepsilon\right)^2 \\ \left(\chi_1^\varepsilon - \chi_2^\varepsilon\right)^2 \end{pmatrix} \cdot n \, da\, d\sigma\,, \quad (2.64)$$

where $\theta_\varepsilon^\pm$ are defined in (2.56). Using identity (2.64) in (2.63) gives

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\theta_\varepsilon^-(\bar t)}^{\theta_\varepsilon^+(\bar t)} \left(\chi_1^\varepsilon-\chi_2^\varepsilon\right)^2(x,\bar t)\,dx\,da\\ &\le \int_{\Gamma_{\bar t_1}^{\bar t}}\int_{\mathbb{R}} \chi_1^\varepsilon\partial_a q_1^\varepsilon+\chi_2^\varepsilon\partial_a q_2^\varepsilon\,da\,dx\,dt-\int_{\Lambda_{\bar t_1}^{\bar t}}\int_{\mathbb{R}}\begin{pmatrix} f'(a)\left(\chi_1^\varepsilon-\chi_2^\varepsilon\right)^2\\ \left(\chi_1^\varepsilon-\chi_2^\varepsilon\right)^2\end{pmatrix}\cdot n\,da\,d\sigma\,.\end{aligned} \tag{2.65}$$

We claim
Claim 2. For $i\in\{1,2\}$ we have

$$\lim_{\varepsilon\to0^+}\int_{\Gamma_{\bar t_1}^{\bar t}}\int_{\mathbb{R}}\chi_i^\varepsilon\partial_a q_i^\varepsilon\,da\,dx\,dt=0\,. \tag{2.66}$$

Proof of Claim 2. We consider the function $\chi_i^\varepsilon-(\chi_i^\varepsilon)^2$ which satisfies satisfies pointwise for $(x,t)\in\Gamma_{t_1+C\varepsilon}^{t_2-C\varepsilon}$ and almost every $a\in\mathbb{R}$

$$\partial_t\left[\chi_i^\varepsilon-(\chi_i^\varepsilon)^2\right]+f'(a)\partial_x\left[\chi_i^\varepsilon-(\chi_i^\varepsilon)^2\right]=\partial_a q_i^\varepsilon+2\chi_i^\varepsilon\partial_a q_i^\varepsilon\,.$$

Integrating this in $\Gamma_{\bar t_1}^{\bar t}\times\mathbb{R}$ leads to

$$\begin{aligned}\int_{\mathbb{R}}\int_{\Gamma_{\bar t_1}^{\bar t}}\partial_t\left[\chi_i^\varepsilon-(\chi_i^\varepsilon)^2\right]+f'(a)\partial_x\left[\chi_i^\varepsilon-(\chi_i^\varepsilon)^2\right]\,dx\,dt\,da&\\ =\int_{\mathbb{R}}\int_{\Gamma_{\bar t_1}^{\bar t}}2\chi_i^\varepsilon\partial_a q_i^\varepsilon\,dx\,dt\,da\,,&\end{aligned} \tag{2.67}$$

where we made use of the fact, that q_i^ε is compactly supported in a. For the left-hand side of (2.67) one can compute with the divergence Theorem

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\Gamma_{\bar t_1}^{\bar t}}\partial_t\left[\chi_i^\varepsilon-(\chi_i^\varepsilon)^2\right]+f'(a)\partial_x\left[\chi_i^\varepsilon-(\chi_i^\varepsilon)^2\right]\,dx\,dt\,da\\ &=\int_{\mathbb{R}}\int_{\theta_\varepsilon^-(\bar t)}^{\theta_\varepsilon^+(\bar t)}\left[\chi_i^\varepsilon-(\chi_i^\varepsilon)^2\right](x,\bar t)\,dx\,da+\int_{\Lambda_{\bar t_1}^{\bar t}}\int_{\mathbb{R}}\begin{pmatrix} f'(a)[\chi_i^\varepsilon-(\chi_i^\varepsilon)^2]\\ \chi_i^\varepsilon-(\chi_i^\varepsilon)^2\end{pmatrix}\cdot n\,da\,d\sigma\,.\end{aligned} \tag{2.68}$$

For the right-hand side of (2.68) we observe

$$\begin{aligned}\lim_{\varepsilon\to0^+}\int_{\mathbb{R}}\int_{\theta_\varepsilon^-(\bar t)}^{\theta_\varepsilon^+(\bar t)}\left[\chi_i^\varepsilon-(\chi_i^\varepsilon)^2\right](x,\bar t)\,dx\,da&\\ =\int_{\mathbb{R}}\int_{\theta^-(\bar t)}^{\theta^+(\bar t)}\left[\chi_i-(\chi_i)^2\right](x,\bar t)\,dx\,da&\end{aligned} \tag{2.69}$$

and

$$\lim_{\varepsilon\to 0^+}\int_{\Lambda_{t_1}^{\bar t}}\int_{\mathbb{R}}\begin{pmatrix} f'(a)[\chi_i^\varepsilon-(\chi_i^\varepsilon)^2] \\ \chi_i^\varepsilon-(\chi_i^\varepsilon)^2\end{pmatrix}\cdot n\,da\,d\sigma = \int_{\Lambda_{t_1}^{\bar t}}\int_{\mathbb{R}}\begin{pmatrix} f'(a)[\chi_i-(\chi_i)^2] \\ \chi_i-(\chi_i)^2\end{pmatrix}\cdot n\,da\,d\sigma\,. \quad (2.70)$$

Since

$$\chi_i=(\chi_i)^2$$

the right-hand side of (2.69) and (2.70) are zero. Thus

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}}\int_{\theta_\varepsilon^-(\bar t)}^{\theta_\varepsilon^+(\bar t)}\left[\chi_i^\varepsilon-(\chi_i^\varepsilon)^2\right](x,\bar t)\,dx\,da+\int_{\Lambda_{t_1}^{\bar t}}\int_{\mathbb{R}}\begin{pmatrix} f'(a)[\chi_i^\varepsilon-(\chi_i^\varepsilon)^2] \\ \chi_i^\varepsilon-(\chi_i^\varepsilon)^2\end{pmatrix}\cdot n\,da\,d\sigma = 0\,. \quad (2.71)$$

With (2.68) one concludes

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}}\int_{\Gamma_{t_1}^{\bar t}}\partial_t\left[\chi_i^\varepsilon-(\chi_i^\varepsilon)^2\right]+f'(a)\partial_x\left[\chi_i^\varepsilon-(\chi_i^\varepsilon)^2\right]\,dx\,dt\,da=0\,.$$

Finally taking limits on both sides of (2.67) we get

$$\lim_{\varepsilon\to 0^+}\int_{\Gamma_{t_1}^{\bar t}}\int_{\mathbb{R}}\chi_i^\varepsilon\partial_a q_i^\varepsilon\,da\,dx\,dt=0\quad\text{for}\quad i\in\{1,\,2\}\,,$$

as announced.
Letting $\varepsilon\to 0^+$ in (2.65) and using (2.66) leads to

$$\int_{\mathbb{R}}\int_{\theta^-(\bar t)}^{\theta^+(\bar t)}(\chi_1-\chi_2)^2(x,\bar t)\,dx\,da\le-\int_{\Lambda_{t_1}^{\bar t}}\int_{\mathbb{R}}\begin{pmatrix} f'(a)\,(\chi_1-\chi_2)^2 \\ (\chi_1-\chi_2)^2\end{pmatrix}\cdot n\,da\,d\sigma\,. \quad (2.72)$$

We compute

$$\int_{\mathbb{R}}\left(\chi(v_1(x,t);a)-\chi(v_2(x,t);a)\right)^2\,da=|v_1(x,t)-v_2(x,t)|\,, \quad (2.73)$$

and

$$\begin{pmatrix} (\chi_1-\chi_2)^2 \\ f'(a)\,(\chi_1-\chi_2)^2\end{pmatrix}\cdot n=(\chi_1-\chi_2)^2\cdot(1\pm\hat\lambda f'(a)) \quad (2.74)$$

Applying (2.73) and (2.74) in (2.72) gives

$$\int_{\theta^-(\bar t)}^{\theta^+(\bar t)}|v_1(x,\bar t)-v_2(x,\bar t)|\,dx\le C\int_{\Lambda_{t_1}^{\bar t}}|w_1(s,\gamma(s,t_1))-w_2(s,\gamma(s))|\,d\sigma \quad (2.75)$$

as claimed. □

2.2.2 Blow up at the points of negative density.

In this section we aim to prove the following lemma

Lemma 2.2. *Let $u \in L^\infty(\mathbb{R} \times [0,T))$ be a weak solution of (2.1), which satisfies (2.10). Then for every $(x_0,t_0) \in \mathbb{R} \times (0,T)$*

$$\limsup_{r \to 0^+} \frac{1}{r} \int_{\mathbb{R}} m\left(B_r(x_0,t_0),a\right)\,da \geq 0\,.$$

A useful lemma that will be used to prove Lemma 2.2 is the following.

Lemma 2.3. *Let $u \in L^\infty(\mathbb{R} \times (0,T))$ be a weak solution of (2.1), which satisfies (2.10). Let $r_n \to 0^+$. For $(x_0,t_0) \in \mathbb{R} \times (0,T)$ define*

$$u_n(x,t) := (D_n^{-1})^*\, u(x,t)$$

and

$$\mu_n := \frac{1}{r_n} \int_{\mathbb{R}} (D_n)_*\, m\, da\,,$$

where

$$D_n(x,t) = \left(\frac{x-x_0}{r_n}, \frac{t-t_0}{r_n}\right)\,. \tag{2.76}$$

Then there exists for every $(x_0,t_0) \in \mathbb{R} \times (0,T)$ a subsequence r_k such that

$$u_k \to u_\infty \quad in \quad L^1_{loc}(\mathbb{R}^2)\,.$$

And furthermore

$$\mu_k \rightharpoonup^* \mu_\infty \quad in \quad \mathcal{M}_{loc}(\mathbb{R}^2)\,.$$

Which means in other words

$$\int_{\mathbb{R}^2} \psi\, d\mu_k \to \int_{\mathbb{R}^2} \psi\, d\mu_\infty \quad for\ all \quad \psi \in C^0_c(\mathbb{R}^2)\,.$$

Lemma 2.3 will be a consequence of of the following proposition, which is proved in Appendix A of [Le2].

Proposition 2.2. *For any constant $M \geq 0$, for any bounded set Ω, the set*

$$\left\{ u \in L^\infty(\Omega):\ \|u\|_\infty + \int_{\mathbb{R}} |m|(\Omega,a) \leq M \right\}$$

is compact in $L^1(\Omega)$ with respect to the strong topology.

Proof of Lemma 2.3. By construction we already have

$$\|u_n\|_\infty \leq \|u\|_\infty \,. \tag{2.77}$$

For this reason it remains to show that for all $R > 0$ and for every $(x_0, t_0) \in \mathbb{R} \times (0, T)$ there exists a constant $C > 0$ depending on R, such that

$$\limsup_{n\to\infty} |\mu_n|(B_R(0,0)) \leq C \,. \tag{2.78}$$

We consider a domain $\Gamma_{t_1}^{t_2}$ such that, $2R < t_2 - t_1 < 3R$, $\delta > 2R$, $\hat{\lambda} < \lambda_0$, where λ_0 is given by Lemma 2.1 and

$$\left.\begin{array}{rcll} \partial_t v_n + \partial_x f(v_n) &=& 0 & \text{in } \Gamma_{t_1}^{t_2}\,, \\ v_n &=& u_n & \text{on } \Lambda_{t_1}^{t_2} \end{array}\right\} \tag{2.79}$$

admits an entropy solution v_n in the sense of Definition 1. Then we define

$$\mathit{\Gamma}_n = D_n(\Gamma_{t_1}^{t_2})$$

and

$$\tilde{v}_n(x,t) = \begin{cases} (D_n)^* v(x,t) & (x,t) \in \mathit{\Gamma}_n \,, \\ u(x,t) & (x,t) \in \mathbb{R} \times [0, t_0 + r_n t_2) \backslash \mathit{\Gamma}_n \,. \end{cases} \tag{2.80}$$

We claim that:
Claim 1. The function $\tilde{v}_n(x,t)$ defined in (2.80) is a weak solution of (2.1).
Proof of claim 1. We observe that u_n itself solves (2.79) and compute

$$\begin{aligned} \int_{\mathit{\Gamma}_n} \tilde{v}_n \partial_t \psi + f(\tilde{v}_n) \partial_x \psi \, dx\, dt &= r^2 \int_{\Gamma_{t_1}^{t_2}} v_n \partial_t \psi + f(v_n) \partial_x \psi \, dx\, dt \\ &= -r_n^2 \int_{\Lambda_{t_1}^{t_2}} \psi \begin{pmatrix} u_n \\ -f(u_n) \end{pmatrix} \cdot \tau \, d\sigma \\ &= r_n^2 \int_{\Gamma_{t_1}^{t_2}} u_n \partial_t \psi + f(u_n) \partial_x \psi \, dx\, dt \\ &= \int_{\mathit{\Gamma}_n} u \partial_t \psi + f(u) \partial_x \psi \, dx\, dt \,. \end{aligned}$$

Using this equality we see

$$\begin{aligned}\int_{\mathbb{R}\times[0,t_0+r_nt_2]} \tilde{v}_n\partial_t\psi + f(\tilde{v}_n)\partial_x\psi\,dx\,dt &= \int_{\Gamma_n} \tilde{v}_n\partial_t\psi + f(\tilde{v}_n)\partial_x\psi\,dx\,dt \\ &\quad + \int_{\mathbb{R}\times(0,t_0+r_nt_2)\setminus\varGamma_n} u\partial_t\psi + f(u)\partial_x\psi\,dx\,dt \\ &= \int_{\varGamma_n} u\partial_t\psi + f(u)\partial_x\psi\,dx\,dt \\ &\quad + \int_{\mathbb{R}\times(0,t_0+r_nt_2)\setminus\varGamma_n} u\partial_t\psi + f(u)\partial_x\psi\,dx\,dt \\ &= \int_{\mathbb{R}\times[0,t_0+r_nt_2]} u\partial_t\psi + f(u)\partial_x\psi\,dx\,dt \\ &= \int_{\mathbb{R}} u_0(x)\psi(x,0)\,dx\,,\end{aligned}$$

which means, that $\tilde{v}_n$ is indeed a weak solution of (2.1) and the proof of claim 1 is concluded.
Let $\tilde{q}$ denote the defect measure of $\tilde{v}_n$. By construction we have

$$m(x,t,a) = \tilde{q}_n(x,t,a) \quad \text{on} \quad \mathbb{R}\times(0,t_0+r_nt_2)\setminus\bar{\varGamma}_n$$

and therefore the minimality property (2.10) of u implies

$$\int_{\mathbb{R}} |m|(\varGamma_n\cup\varLambda_n,a)\,da \le \int_{\mathbb{R}} |\tilde{q}_n|(\varGamma_n\cup\varLambda_k,a)\,da\,, \tag{2.81}$$

where $\varLambda_n = D_n(\Lambda_{t_1}^{t_2})$.
We claim now
Claim 2. For all $n\in\mathbb{N}$

$$|\tilde{q}_n|(\varLambda_n,a) = 0 \quad \text{for all} \quad n\in\mathbb{N}\,. \tag{2.82}$$

Proof of claim 2. We define the domain Λ_ε such that

$$\partial\Lambda_\varepsilon = \Lambda_{t_1+\varepsilon}^{t_2}\cup\Lambda_{t_1-\varepsilon}^{t_2}\cup I_l\cup I_r \tag{2.83}$$

and

$$\Lambda_{t_1}^{t_2}\subset\Lambda_\varepsilon\,,$$

where

$$I_l = \left[\frac{t_2-(t_1+\varepsilon)}{\hat{\lambda}}+\delta, \frac{t_2-(t_1-\varepsilon)}{\hat{\lambda}}\right]$$

and

$$I_r = \left[-\frac{t_2 - (t_1 - \varepsilon)}{\hat{\lambda}} - \delta, -\frac{t_2 - (t_1 + \varepsilon)}{\hat{\lambda}} - \delta \right]$$

Then for $\Lambda_n^\varepsilon := D_n^{-1}(\Lambda_\varepsilon)$ and $\psi \in C_c^\infty(\mathbb{R} \times (0, t_0 + t_2 r_n))$ it follows by Theorem 1.3.4 in [Da]

$$\int_{\Lambda_n^\varepsilon} \tilde{v} \wedge a \; \partial_t \psi + f(u \wedge a) \partial_x \psi \, dx \, dt = \int_{\partial \Lambda_n^\varepsilon} \begin{pmatrix} f(\tilde{v}) \\ \tilde{v} \end{pmatrix} \cdot n \psi \, d\sigma + \int_{\Lambda_n^\varepsilon} \psi \, d\tilde{q}(x, t, a) \,, \tag{2.84}$$

where n is the outer unit normal of Λ_n^ε. The boundary term can be separated in three parts

$$\begin{aligned} \int_{\partial \Lambda_n^\varepsilon} \begin{pmatrix} f(\tilde{v}) \\ \tilde{v} \end{pmatrix} \cdot n \psi \, d\sigma = & \int_{D_n^{-1}(\Lambda_{t_1-\varepsilon}^{t_2})} \begin{pmatrix} f(u) \\ u \end{pmatrix} \cdot n \psi \, d\sigma - \int_{D_n^{-1}(\Lambda_{t_1+\varepsilon}^{t_2})} \begin{pmatrix} f(\tilde{v}) \\ \tilde{v} \end{pmatrix} \cdot n \psi \, d\sigma \\ & + \int_{D_n^{-1}(I_l)} \tilde{v}(x, t_0 + r_n t_2) \, dx + \int_{D_n^{-1}(I_r)} \tilde{v}(x, t_0 + r_n t_2) \, dx \end{aligned} \tag{2.85}$$

As $\varepsilon \to 0^+$ the two last quantities in the right-hand side of in (2.85) vanish. For the first expression on the right hand side of (2.85) one concludes

$$\lim_{\varepsilon \to 0^+} \int_{D_n^{-1}(\Lambda_{t_1-\varepsilon}^{t_2})} \begin{pmatrix} f(u) \\ u \end{pmatrix} \cdot n \psi \, d\sigma = \int_{D_n^{-1}(\Lambda_{t_1}^{t_2})} \begin{pmatrix} f(u) \\ u \end{pmatrix} \cdot n \psi \, d\sigma \,. \tag{2.86}$$

With a change of variable and with Lemma 2.1 it follows

$$\begin{aligned} \lim_{\varepsilon \to 0^+} \int_{D_n^{-1}(\Lambda_{t_1+\varepsilon}^{t_2})} \begin{pmatrix} f(\tilde{v}) \\ \tilde{v} \end{pmatrix} \cdot n \psi \, d\sigma &= \lim_{\varepsilon \to 0^+} r_n \int_{\Lambda_{t_1+\varepsilon}^{t_2}} \begin{pmatrix} f(v) \\ v \end{pmatrix} \cdot n \psi \, d\sigma \\ &= r_n \int_{\Lambda_{t_1+\varepsilon}^{t_2}} \begin{pmatrix} f(u_n) \\ u_n \end{pmatrix} \cdot n \psi \, d\sigma \\ &= \int_{D_n^{-1}(\Lambda_{t_1}^{t_2})} \begin{pmatrix} f(u) \\ u \end{pmatrix} \cdot n \psi \, d\sigma \,. \end{aligned} \tag{2.87}$$

From (2.85), (2.86) and (2.87) we conclude

$$\lim_{\varepsilon \to 0^+} \int_{\partial \Lambda_n^\varepsilon} \begin{pmatrix} f(\tilde{v}) \\ \tilde{v} \end{pmatrix} \cdot n \psi \, d\sigma = 0 \,.$$

Therefore we can conclude from (2.84)

$$\lim_{\varepsilon \to 0^+} \int_{\Lambda_n^\varepsilon} \psi \; d\tilde{q}(x, t, a) = 0 \quad \text{for} \quad \psi \in C_c^\infty(\mathbb{R} \times (0, t_0 + r_n t_2)) \,.$$

From this it follows,

$$|\tilde{q}_n|(\Lambda_n, a) = 0$$

as claimed.

From (2.81) and (2.82) we conclude

$$\int_{\mathbb{R}} |m|(B_{Rr_n}(x_0,t_0),a)\,da \leq \int_{\mathbb{R}} |m|(\Gamma_n \cup \Lambda_n, a)\,da \leq \int_{\mathbb{R}} |\tilde{q}_n|(\Gamma_n, a)\,da\,.$$

Since $\tilde{q} \geq 0$ in Γ_n we get

$$\frac{1}{r_n}\int_{\mathbb{R}} m(B_{Rr_n}(x_0,t_0),a)\,da \leq \frac{1}{r_n}\int_{\mathbb{R}} \tilde{q}_n(\Gamma_n, a)\,da\,. \tag{2.88}$$

We recall, that v_n is an entropy solution of (2.79) and we denote its defect measure by q_n. Then we get from (2.88) by a change of variable and Lemma 1

$$|\mu_n|(B_R(0,0)) \leq \frac{1}{r_n}\int_{\mathbb{R}} \tilde{q}_n(\Gamma_n, a)\,da = \int_{\mathbb{R}} q_n(\Gamma_{t_1}^{t_2}, a)\,da \leq C(R)\,. \tag{2.89}$$

Therefore we proved (2.78). Since (2.77) and (2.78) hold, the assumptions of Proposition 2.2 are fulfilled and we can extract a subsequence $r_{k'}$ such that

$$u_{k'} \to u_\infty \quad \text{in} \quad L^1_{loc}(\mathbb{R}^2)\,.$$

Additionally we have by the weak*-compactness of measures (see Theorem 1.59 in [AFP]) that possibly after extracting a further subsequence r_k,

$$\mu_k \rightharpoonup \mu_\infty \quad \text{in} \quad \mathcal{M}_{loc}(\mathbb{R}^2)\,,$$

Altogether we have for the sequence r_k

$$u_k \to u_\infty \quad \text{in} \quad L^1_{loc}(\mathbb{R}^2)\,.$$

and

$$\mu_k \rightharpoonup^* \mu_\infty \quad \text{in} \quad \mathcal{M}_{loc}(\mathbb{R}^2)\,,$$

which is what we aimed to prove. □

Proof of Lemma 2.2. We argue by contradiction. Therefore we assume that there exists a point (x_0, t_0) such that

$$\limsup_{r\to 0^+} \frac{1}{r}\int_{\mathbb{R}} m(B_r(x_0,t_0),a)\,da < 0\,. \tag{2.90}$$

For a sequence $r_n \to 0^+$ we define

$$u_n(x,t) := (D_n^{-1})^* \, u(x,t)$$

and

$$\mu_n := \frac{1}{r_n} \int_{\mathbb{R}} (D_n)_* \, m \, da \, .$$

Let u_k and μ_k be the subsequences given by Lemma 2.3 with limits u_∞, μ_∞. Then we have by strong convergence that u_∞ is a weak solution of

$$\partial_t u_\infty + \partial_x f(u_\infty) = 0$$

and by the uniqueness of the distributional limit we conclude

$$\mu_\infty = \int_{\mathbb{R}} \partial_t (u_\infty \wedge a) + \partial_x f(u_\infty \wedge a) \, da \, .$$

From (2.90) we want to conclude now that

Claim 1. For all $R > 0$

$$\mu_\infty(B_R(0,0)) < 0 \, . \tag{2.91}$$

Proof of claim 1. For the sake of contradiction, we assume, that there exists a R_0 such that

$$\mu_\infty(B_{R_0}(0,0)) \geq 0 \, .$$

In [Le2] it is proved, that there exits a set K, which is either a line, or a half-line, or the empty set, such that

$$\partial_t u_\infty \wedge a + \partial_x f(u_\infty \wedge a) = \big[(X(u_\infty^+ \wedge a) - X(u_\infty^- \wedge a) \big] \cdot \omega_K \, \mathcal{H}^1 \llcorner K \, , \tag{2.92}$$

where

$$X(u) = \begin{pmatrix} f(u) \\ u \end{pmatrix} \quad \text{and} \quad \omega_K = \frac{|u_\infty^+ - u_\infty^-|}{|X(u_\infty^+) - X(u_\infty^-)|} \begin{pmatrix} 1 \\ -\frac{f(u_\infty^+) - f(u_\infty^-)}{u_\infty^+ - u_\infty^-} \end{pmatrix} . \tag{2.93}$$

Moreover therein it is proved, that u_∞ is $\mathcal{H}^1$-a.e. approximately continuous in K^c and has $\mathcal{H}^1$-a.e. constant approximate jump points $u_\infty^\pm$ on K.
A short calculation reveals

$$\int_{\mathbb{R}} \big((X(u_\infty^+ \wedge a) - X(u_\infty^- \wedge a) \big) \cdot \omega_K$$
$$= \operatorname{sign}(u_\infty^- - u_\infty^+) \int_{\min\{u_\infty^+, u_\infty^-\}}^{\max\{u_\infty^+, u_\infty^-\}} \frac{f(u_\infty^+) + f(u_\infty^-)}{2} - f(a) \, da \, . \tag{2.94}$$

The convexity of f implies that the integral on the right-hand side of (2.94) is non-negative and henceforth the sign of μ_∞ is completely determined by $\mathrm{sign}(u_\infty^- - u_\infty^+)$. Hence

$$\mu_\infty(B_{R_0}(0,0)) \geq 0$$

can only be fulfilled, if

$$u_\infty^- \geq u_\infty^+ .$$

But this implies that the measure is μ_∞ has a sign, i.e.

$$\mu_\infty \geq 0 .$$

Let $\mu_k^\pm$ be the positive respective negative part of μ_k, i.e. $\mu_k^\pm$ are non-negative measures such that

$$\mu_k = \mu_k^+ - \mu_k^- .$$

After extracting a further subsequence k'

$$\mu_{k'}^+ \rightharpoonup^* \nu^+ \quad \text{and} \quad \mu_{k'}^- \rightharpoonup^* \nu^- \quad \text{in} \quad \mathcal{M}_{loc}(\mathbb{R}^2) .$$

For $R > 0$ and non-negative $\psi \in C_c^\infty(B_R(0,0))$ we get

$$\int_{B_R(0,0)} \psi \, d\mu_\infty = \lim_{k' \to \infty} \int_{B_R(0,0)} \psi \, d\mu_{k'} = \int_{B_R(0,0)} \psi \, d\nu^+ - \int_{B_R(0,0)} \psi \, d\nu^- .$$

Since μ_∞ is non-negative we get for all non-negative $\psi \in C_c^\infty(B_R(0,0))$

$$\int_{B_{R_0}(0,0)} \psi \, d\nu^- \leq \int_{B_R(0,0)} \psi \, d\nu^+ .$$

Hence

$$\nu^-(B_R(0,0)) \leq \nu^+(B_R(0,0)) \tag{2.95}$$

By Theorem 1.2 in [Le2] (see also Theorem 1.1 in [AKLR]) we have for a rectifiable set J_u and an $\mathcal{H}^1$ measurable function $h : \; J_u \to \mathbb{R}$

$$\int_{\mathbb{R}} |m|(x,t,a) \, da = h \cdot \mathcal{H}^1 \llcorner J_u + \delta_u , \tag{2.96}$$

where δ_u satisfies

$$\forall \; B \quad \text{Borel} \quad \mathcal{H}^1(B) < \infty \Longrightarrow \delta_u(B) = 0 .$$

Therefore we can choose R_1, such that for all k'

$$\mu_{k'}^-(\partial B_{R_1}(0,0)) \leq \frac{1}{r_{k'}} \int_{D_{k'}^{-1}(\partial B_{R_1}(0,0))} h \, d\mathcal{H}^1 \llcorner J_u = 0 .$$

Hence

$$\nu^-(\partial B_{R_1}(0,0)) = \lim_{k'\to\infty} \mu^-_{k'}(\partial B_{R_1}(0,0)) = 0\,.$$

This and (2.95) imply

$$\begin{aligned}\limsup_{k'\to\infty} \mu^-_{k'}(B_{R_1}(0,0) &\le \nu^-(\bar{B}_{R_1}(0,0)) = \nu^-(B_{R_1}(0,0)) \\ &\le \nu^+(B_{R_1}(0,0)) \le \liminf_{k'\to\infty} \mu^+_{k'}(B_{R_1}(0,0))\,.\end{aligned}$$

$$\limsup_{k'\to\infty} \mu_{k'}(B_{R_1}(0,0)) \ge \liminf_{k'\to\infty} \mu^+_{k'} B_{R_1}(0,0)) - \limsup_{k'\to\infty} \mu^-_{k'}(B_{R_1}(0,0)) \ge 0\,,$$

which obviously contradicts (2.90) and we get claim 1 is proved.

Inequality (2.91) implies that the set K in (2.92) is non-empty and

$$\mu_\infty < 0\,,$$

which gives again from above considerations

$$u^-_\infty < u^+_\infty\,.$$

Moreover the convexity of f implies for every $a \in (u^-_\infty, u^+_\infty)$

$$\begin{aligned}&\partial_t u_\infty \wedge a + \partial_x f(u_\infty \wedge a) \\ &\quad = \left(\frac{f(a) - f(u^-_\infty)}{a - u^-_\infty} - \frac{f(u^+_\infty) - f(u^-_\infty)}{u^+_\infty - u^-_\infty}\right)\left(a - u^-_\infty\right)\mathcal{H}^1 \llcorner K \le 0\,.\end{aligned}$$

For $P = (x_p, t_p) \in \mathbb{R}^2$ let $K = P + \mathbb{R}\omega_K^\perp$ if K is a line or $K = P + \mathbb{R}_+\omega_K^\perp$ if K is a halfline. Define

$$H^+ := \{(x,t) :\ ((x,t) - P)\cdot\omega_K > 0\}$$

and

$$H^- := \{(x,t) :\ ((x,t) - P)\cdot\omega_K < 0\}$$

if K is a line and

$$H^+ := \{(x,t) :\ (x,t) - P)\cdot\omega_K > 0 \text{ and } x > f'(u^+_\infty)(t - t_p) + x_p\}$$

$$H^- := \{(x,t) :\ (x,t) - P)\cdot\omega_K < 0 \text{ and } x < f'(u^-_\infty)(t - t_p) + x_p\}\ ,$$

if K is a half-line. From the proof of Proposition 3.3 in [Le2] (see also Theorem 6.2 in [AKLR] for a similar proof) we get that

$$u_\infty(x,t) = u_\infty^- \quad \text{on} \quad H^- \quad \text{and} \quad u_\infty(x,t) = u_\infty^+ \quad \text{on} \quad H^+ .$$

Now we choose $\bar{t} \in \mathbb{R}$ and $\delta > 0$ in the definition of the sets $\Lambda_{\bar{t}}^{\bar{t}+1}$ and $\Gamma_{\bar{t}}^{\bar{t}+1}$ (see (2.23)), in such a way that

$$\left[-\frac{\delta}{2}, \frac{\delta}{2} \right] \times \{t\} \cap K \neq \emptyset \quad \forall\, t \in (\bar{t}, \bar{t}+1) .$$

Furthermore $\Gamma_{\bar{t}}^{\bar{t}+1}$ is defined such that the conclusions of Lemma 2.1 applies to this trapeze. In particular the strong convergence of u_k in $L^1_{loc}(\mathbb{R}^2)$ implies

$$u_k \to u_\infty \quad \text{in} \quad L^1\left(\Gamma_{\bar{t}}^{\bar{t}+1}\right) ,$$

which directly implies by a change of variable

$$\int_{\bar{t}}^{\bar{t}+1} \int_{\Lambda_{t'}^{\bar{t}+1}} |u_k - u_\infty| \, d\sigma \, dt' \to 0 .$$

Thus for almost every $t_1 \in (\bar{t}, \bar{t}+1)$ we get

$$\int_{\Lambda_{t_1}^{\bar{t}+1}} |u_k - u_\infty| \, d\sigma \to 0 \tag{2.97}$$

and moreover by (2.96)

$$\mu_k(\Lambda_{t_1}^{\bar{t}+1}) = \int_{D_k^{-1}(\Lambda_{t_1}^{\bar{t}+1})} h \, \mathcal{H}^1 \llcorner J_u = 0 . \tag{2.98}$$

We set $t_2 := \bar{t}+1$, then according to Proposition 2.1 we can choose a $t_1 \in (\bar{t}, \bar{t}+1)$ such that for all $k \in \mathbb{N}$ (2.97) and (2.98) hold and for $k \in \mathbb{N} \cup \{\infty\}$ there exists an entropy solution w_k of

$$\left. \begin{aligned} \partial_t w_k + \partial_x f(w_k) &= 0 \quad \text{in} \quad \Gamma_{t_1}^{t_2} , \\ w_k &= u_k \quad \text{on} \quad \Lambda_{t_1}^{t_2} . \end{aligned} \right\} \tag{2.99}$$

By Lemma 2.1 we have for all $t_1 \le t < t_2$

$$\int_{\theta^-(t)}^{\theta^+(t)} |w_k(x,t) - w_\infty(x,t)| \, dx \le \int_{\Lambda_{t_1}^{t_2}} |u_k - u_\infty| \, d\sigma .$$

Together with (2.97) this implies

$$w_k \to w_\infty \quad \text{in} \quad L^1\left(\Gamma_{t_1}^{t_2}\right) .$$

By our choice of t_1, we have for an $x_1 \in \left[-\frac{\delta}{2}, \frac{\delta}{2}\right]$

$$u_\infty(x,t_1) = \begin{cases} u_\infty^- & \text{if } x < x_1 \\ u_\infty^+ & \text{if } x > x_1 \, . \end{cases}$$

This structure of u_∞ at the time t_1 allows us to compute w_∞ explicitly. Since $u_\infty^- < u_\infty^+$ the two states u_∞^- and u_∞^+ are connected by a rarefaction wave

$$w_\infty(x,t) := \begin{cases} u_\infty^- & \text{if} \quad x - x_1 < f'(u_\infty^-)(t-t_1)\, , \\ (f')^{-1}\left(\frac{x-x_1}{t-t_1}\right) & \text{if} \quad f'(u_\infty^-)(t-t_1) < x - x_1 < f'(u_\infty^+)(t-t_1)\, , \\ u_\infty^+ & \text{if} \quad x - x_1 > f'(u_\infty)^+(t-t_1)\, . \end{cases}$$

We observe, that w_∞ is a Lipschitz function and this implies pointwise almost everywhere in $\Gamma_{t_1}^{t_2}$

$$\partial_t w_\infty + \partial_x f(w_\infty) = 0\, .$$

Hence

$$\begin{aligned} q_\infty(x,t,a) &= \partial_t(w_\infty \wedge a) + \partial_x f(w_\infty \wedge a) \\ &= \mathbb{1}_{w \leq a}\left[\partial_t w_\infty + f'(w_\infty \wedge a)\partial_x w_\infty\right] = 0 \quad \text{in} \quad \Gamma_{t_1}^{t_2}\, . \end{aligned}$$

Furthermore the strong convergence of w_k in $L^1(\Gamma_{t_1}^{t_2})$ implies

$$q_k \rightharpoonup q_\infty \quad \text{in} \quad \mathcal{M}_{loc}(\mathbb{R}^2)\, ,$$

where

$$q_k = \partial_t w_k \wedge a + \partial_x f(w_k \wedge a)\, .$$

To simplify notations, we define

$$\mathit{\Gamma}_k := \left\{(x,t) \in \mathbb{R} \times (0,T) : \; D_k(x,t) \in \Gamma_{t_1}^{t_2}\right\}$$

and

$$\mathit{\Lambda}_k := \left\{(x,t) \in \mathbb{R} \times (0,T) : \; D_k(x,t) \in \Lambda_{t_1}^{t_2}\right\}\, ,$$

where the map D_k is defined in (2.76). Then we define the rescaled function

$$\tilde{w}_k(x,t) = \begin{cases} (D_k)^* \, w_k & \text{if} \quad (x,t) \in \mathit{\Gamma}_k\, , \\ u & \text{if} \quad (x,t) \in \mathbb{R} \times (0, t_0 + r_k t_2) \backslash \mathit{\Gamma}_k\, . \end{cases}$$

and claim, that $w_k \in L^\infty(\mathbb{R}\times(0,t_0+r_k t_2))$ is a weak solution of (2.1) for all $k \in \mathbb{N}$. This is the same situation as in the proof of Lemma 2.3 and hence the same steps as in the proof of claim (2.80) gives, that w_k is indeed a weak solution of (2.1). Therefore the minimality condition (2.10) of u applies and we deduce

$$\int_{\mathbb{R}} |m|(\mathbb{R}\times(0,t_0+r_k t_2),a)\,da \le \int_{\mathbb{R}\times(0,t_0+r_k t_2)\times\mathbb{R}} |\tilde{q}_k|(\mathbb{R}\times(0,t_0+r_k t_2,a)|\,da\,.$$

But since

$$m(x,t,a) = \tilde{q}_k(x,t,a) \quad \text{on} \quad \mathbb{R}\times(0,t_0+r_k t_2)\backslash \bar{\mathit{\Gamma}}_k$$

we get

$$\int_{\mathbb{R}} |m|(\mathit{\Gamma}_k \cup \mathit{\Lambda}_k,a)\,da \le \int_{\mathbb{R}} |\tilde{q}_k|(\mathit{\Gamma}_k \cup \mathit{\Lambda}_k,a)\,da\,. \tag{2.100}$$

By the same arguments as in the proof of claim (2.82) we can show

$$|\tilde{q}_k|(\mathit{\Lambda}_k,a) = 0 \quad \text{for all} \quad k \in \mathbb{N}\,. \tag{2.101}$$

In a next step we show, that (2.101) induces

$$\lim_{k\to\infty} \int_{\mathbb{R}} q_k(\Gamma_{t_1}^{t_2},a)\,da = \int_{\mathbb{R}} q_\infty(\Gamma_{t_1}^{t_2},a)\,da\,. \tag{2.102}$$

Proof of (2.102). Since w_k is an entropy solution we deduce from (2.101) that $|\tilde{q}_k|(\partial\mathit{\Gamma}_k,a) = 0$ and therefore

$$\frac{1}{r_k}\int_{\mathbb{R}} (D_k)_* \, |\tilde{q}_k|(\partial\Gamma_{t_1}^{t_2},a)\,da = 0\,. \tag{2.103}$$

Lemma 2.1 and (2.103) imply for a constant $C > 0$

$$\begin{aligned}\frac{1}{r_k}\int_{\mathbb{R}} (D_k)_* \, |\tilde{q}_k|(\bar{\Gamma}_{t_1}^{t_2},a)\,da &= \frac{1}{r_k}\int_{\mathbb{R}} (D_k)_* \, |\tilde{q}_k|(\partial\Gamma_{t_1}^{t_2},a)\,da + \int_{\mathbb{R}} q_k(\Gamma_{t_1}^{t_2},a)\,da \\ &= \int_{\mathbb{R}} q_k(\Gamma_{t_1}^{t_2},a)\,da < C\,.\end{aligned}$$

Hence one gets for a positive measure $\nu \in \mathcal{M}(\bar{\Gamma}_{t_1}^{t_1})$ after possibly extracting a subsequence

$$\frac{1}{r_k}\int_{\mathbb{R}} (D_k)_* \, |\tilde{q}_k| \rightharpoonup \nu \quad \text{in} \quad \mathcal{M}(\bar{\Gamma}_{t_1}^{t_1})\,.$$

Then Proposition 1.62 in [AFP] and (2.103) imply

$$\lim_{k\to\infty}\frac{1}{r_k}\int_{\mathbb{R}}(D_k)_*\,|\tilde{q}_k|(\partial\Gamma_{t_1}^{t_2},a)\,da=\nu(\partial\Gamma_{t_1}^{t_2})=0\,. \tag{2.104}$$

But $\nu(\partial\Gamma_{t_1}^{t_2})=0$ and Proposition 1.62 in [AFP] gives

$$\lim_{k\to\infty}\frac{1}{r_k}\int_{\mathbb{R}}(D_k)_*\,|\tilde{q}_k|(\Gamma_{t_1}^{t_2},a)\,da=\lim_{k\to\infty}\int_{\mathbb{R}}q_k(\Gamma_{t_1}^{t_2},a)\,da=\int_{\mathbb{R}}q_\infty(\Gamma_{t_1}^{t_2},a)\,da\,.$$

Since (2.98) and (2.101) holds we deduce from (2.100)

$$|\mu_k|(\Gamma_{t_1}^{t_2})\le\int_{\mathbb{R}}q_k(\Gamma_{t_1}^{t_1},a)\,da\,.$$

Taking the limit on both sides and applying (2.102) gives

$$\begin{aligned}|\mu_\infty|(\Gamma_{t_1}^{t_2})&\le\liminf_{k\to+\infty}|\mu_k|(\Gamma_{t_1}^{t_2})\le\liminf_{k\to+\infty}\int_{\mathbb{R}}q_k(\Gamma_{t_1}^{t_2},a)\,da\\&=\int_{\mathbb{R}}q_\infty(\Gamma_{t_1}^{t_2},a)\,da=0\,.\end{aligned}$$

But

$$|\mu_\infty|(\Gamma_{t_1}^{t_2})=0$$

is contradiction to (2.91). Therefore

$$\limsup_{r\to0^+}\frac{1}{r}\int_{\mathbb{R}}m(B_r(x_0,t_0),a)\,da\ge0\,,$$

which is what we aimed to prove. □

2.2.3 Proving that u is entropic

In this last section we are going to prove

Lemma 2.4. *Let $u\in L^\infty(\mathbb{R}\times[0,T)$ be a weak solution of (2.1). Let $m(x,t,a)$ its entropy defect measure. If for every $(x_0,t_0)\in\mathbb{R}\times(0,T)$*

$$\limsup_{r\to0^+}\frac{1}{r}\int_{\mathbb{R}}m(B_r(x_0,t_0),a)\,da\ge0\,, \tag{2.105}$$

then u is the entropy solution of (2.1).

Proof of Lemma 2.4. We follow closely [ALR]. Without loss of generality we can assume $f(0) = 0$ and $f \geq 0$. According to Theorem 2.2 there exists a $g \in W^{1,\infty}(\mathbb{R} \times [0,T))$ such that $u = \partial_x g$ and it satisfies almost everywhere

$$\left.\begin{aligned} \partial_t g + f(\partial_x g) &= 0\,, \\ \partial_x g(x,0) &= u_0(x)\,. \end{aligned}\right\} \tag{2.106}$$

We want to show, that g is a viscosity solution of (2.106), i.e. we want to prove, that g is a sub- and supersolution of (2.106). This immediately implies by Corollary 1.7.2 in [ALR], that u is an entropy solution. We already now, that g satisfies (2.106) almost everywhere, then Proposition 5.1 in [BC] implies, that g is a subsolution. Therefore it remains to show, that g is a supersolution of (2.106). Let $\psi \in C^1(\mathbb{R} \times \mathbb{R}_+)$ such that $g - \psi$ has a local minimum in (x_0, t_0). Without loss of generality we can assume $g(x_0,t_0) = \psi(x_0,t_0)$. We want to show that

$$\partial_t \psi(x_0,t_0) + f\left(\partial_x \psi(x_0,t_0)\right) \geq 0\,.$$

We argue by contradiction, therefore we assume

$$\partial_t \psi(x_0,t_0) + f\left(\partial_x \psi(x_0,t_0)\right) < 0\,.$$

Since $f \geq 0$ this immediately implies

$$\partial_t \psi(x_0,t_0) < 0\,. \tag{2.107}$$

For a sequence $r_n \to 0^+$ we introduce

$$\begin{aligned} u_n(x,t) &= u(x_0 + r_n x, t_0 + r_n t), \\ \psi_n(x,t) &= \frac{1}{r_n}\left(\psi(x_0 + \lambda r_n x, t_0 + r_n t) - \psi(x_0,t_0)\right), \\ g_n(x,t) &= \frac{1}{r_n}\left(g(x_0 + r_n x, t_0 + r_n t) - g(x_0,t_0)\right), \end{aligned}$$

where $0 < \lambda < 1$ is a constant, which we choose later. According to Lemma 2.3 we can extract a subsequence r_k such that

$$u_k \to u_\infty \quad \text{in} \quad L^1(B_1)$$

Since $\partial_x g_k = u_k$ and $\partial_t g_k = f(u_k)$ we have by Arzela-Ascoli, that g_k converges uniformly to a Lipschitz function g_∞ such that $\partial_x u_\infty = g_\infty$ and g_∞ fulfills (2.106) almost everywhere. Furthermore we have for $\psi_\infty := \nabla\psi(x_0,t_0) \cdot (\lambda x, t)^{\mathrm{T}}$

$$\lim_{k\to\infty} \psi_k(x,t) = \psi_\infty\,.$$

We notice, that for all $0 < \lambda < 1$ and for all k the functions $g_k - \psi_k$ have a local minimum in $(0,0)$. By uniform convergence the function $g_\infty - \psi_\infty$ admits also a local minimum in $(0,0)$. Moreover

$$\mu_k = \frac{1}{r_k} \int_{\mathbb{R}} (D_k)_* \, m \, da \rightharpoonup \mu_\infty \quad \text{in} \quad \mathcal{M}(B_1) \, .$$

Similar as in Section 2.2.2 from

$$\lim_{k \to \infty} \int_{B_1(0,0)} \mu_k(B_1(0,0)) \geq 0 \, ,$$

we can conclude

$$m_\infty(x,t,a) := \partial_t u_\infty \wedge a + \partial_x f(u_\infty \wedge a) \geq 0 \, .$$

Let $\delta > 0$, then the function

$$h_\delta(x,t) := g_\infty - \psi_\infty + \frac{\delta}{2} \left[(1-\lambda) x^2 + t^2 \right]$$

is defined on B_1 and has a strict minimum in $(0,0)$. Notice that $h_\delta(0,0) = 0$ and $h \geq 0$ in B_1. We claim that

$$|\nabla h_\delta| > 0 \quad \text{a.e. in} \quad B_1 \, . \tag{2.108}$$

Proof of (2.108). Let $(x,t) \in B_1$ such that h_δ is differentiable in (x,t) and $\nabla h_\delta(x,t) = 0$. It follows since g_∞ solves (2.106)

$$\begin{aligned} 0 &= \partial_t g_\infty + f(\partial_x g_\infty) \\ &= \partial_t \psi(x_0,t_0) - \delta t + f(\lambda \partial_x \psi(x_0,t_0) + (1-\lambda)\delta x) \\ &\leq \partial_t \psi(x_0,t_0) + \lambda f(\partial_x \psi(x_0,t_0)) + ((1-\lambda) f(\delta x) - \delta t) \, . \end{aligned}$$

Since (2.107) holds, we can choose δ and λ small enough the expression

$$\partial_t \psi(x_0,t_0) + \lambda f(\partial_x \psi(x_0,t_0)) + \delta(f(\delta x) - t)$$

becomes strictly negative, which is a contradiction. Therefore the claim (2.108) is proved.
Further we choose δ and λ small enough such that

$$|\partial_t \psi(x_0,t_0)| > \lambda \, \partial_x \psi(x_0,t_0) \cdot \sup_{s \in [-\|u\|_\infty, \|u\|_\infty]} f'(s) + \delta((1-\lambda)x + t) \, . \tag{2.109}$$

By $\tau > 0$ we denote the minimum of h_δ on ∂B_1 and by $\overline{a}$ the essential supremum of u_∞ on $\{h_\delta < \tau\}$. If $\overline{a} > 0$ let $\underline{a}$ be close to $\overline{a}$ such that

$0 < \underline{a} < \overline{a}$. Let $A := \{h_\delta < \tau\} \cap \{\underline{a} < u_\infty\}$. The set A has positive Lebesgue measure. Therefore by the Coarea Formula and by $|\nabla h_\delta| > 0$ it follows for $E_s := \{h_\delta = s\}$

$$0 < \int_A |\nabla h_\delta(x,t)|\, dx\, dt = \int_0^\tau \mathcal{H}^1(A \cap E_s)\, ds\,.$$

Hence the set

$$S := \big\{s \in (0,\tau) : \mathcal{H}^1(\{\underline{a} < u_\infty\} \cap E_s) > 0,\ \mathcal{H}^1(\{u_\infty > \overline{a}\} \cap E_s) = 0\big\}$$

has positive Lebesgue measure. For a vector $v = (v_1, v_2)$ we define $v^\perp := (-v_2, v_1)$ and for a $s \in S$ the function

$$s \to l(s) := \int_{E_s} \big[X(u_\infty \wedge a) - \nabla^\perp \psi_\infty + \delta((1-\lambda)x,t)^\perp\big] \cdot \nu$$

where $\nu = \frac{\nabla h_\delta}{|\nabla h_\delta|}$ and the X is the vectorfield from (2.93). We choose $s \in S$ such that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{s-\varepsilon}^{s} l(s')ds' = l(s)\,.$$

We define $\zeta_\varepsilon(x,t) := 1 \wedge (s - h_\delta)^+/\varepsilon$ and calculate

$$\nabla \zeta_\varepsilon = \begin{cases} 0 & \text{if } h_\delta > s \text{ or } h_\delta < s - \varepsilon \\ -\frac{1}{\varepsilon}\nabla h_\delta & \text{if } s - \varepsilon < h_\delta < s\,. \end{cases}$$

The choice of $s \in S$ an the Coarea Formula implies

$$\lim_{\varepsilon \to 0} \int_{B_1} \big[X(u_\infty \wedge \underline{a}) - \nabla^\perp \psi_\infty + \delta((1-\lambda)x,t)^\perp\big] \cdot \nabla \zeta_\varepsilon = -\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{s-\varepsilon}^{s} l(s')ds' = l(s)\,.$$

The sign of m_∞ gives

$$0 \le -\int_{B_1} \big[X(u_\infty \wedge \underline{a}) - \nabla^\perp \psi_\infty + \delta((1-\lambda)x,t)^\perp\big] \cdot \nabla \zeta_\varepsilon\,.$$

As $\varepsilon \to 0$ this implies

$$0 \le \int_{E_s} \big[X(u_\infty \wedge \underline{a}) - \nabla^\perp \psi_\infty + \delta((1-\lambda)x,t)^\perp\big] \cdot \nu\,.$$

Now define $E_s^+ := E_s \cap \{u_\infty > \underline{a}\}$ and $E_s^- := E_s \cap \{u_\infty \le \underline{a}\}$. For $(x,t) \in E_s^-$ we notice

$$\big[X(u_\infty \wedge a) - \nabla^\perp \psi_\infty + \delta((1-\lambda)x,t)^\perp\big] \cdot \nu = \nabla^\perp h_\delta \cdot \nabla h_\delta = 0\,.$$

Therefore it follows

$$0 \leq \int_{E_s^+} \left[X(\underline{a}) - \nabla^\perp \psi_\infty + \delta((1-\lambda)x, t)^\perp\right] \cdot \nabla h_\delta \,.$$

In order to get a contradiction we claim

$$\left(X(\underline{a}) - \nabla^\perp \psi_\infty + \delta((1-\lambda)x, t)^\perp\right) \cdot \nabla h_\delta < 0\,. \tag{2.110}$$

We rearrange terms

$$\begin{aligned}&\left[X(\underline{a}) - \nabla^\perp \psi_\infty + \delta((1-\lambda)x, t)^\perp\right] \cdot \nabla h_\delta \\ &\qquad = X(\underline{a}) \cdot \nabla g_\infty + (\nabla \psi_\infty - \delta((1-\lambda)x, t)) \left(\nabla^\perp g_\infty - X(\underline{a})\right)\,.\end{aligned} \tag{2.111}$$

We show (2.110), by proving that each term on the right hand side of (2.111) is negative respectively strictly negative. Firstly we treat the first term and claim

$$X(\underline{a}) \cdot \nabla g_\infty < 0\,. \tag{2.112}$$

A short calculation reveals

$$\begin{aligned}X(\underline{a}) \cdot \nabla g_\infty &= f(\underline{a}) u_\infty - f(u_\infty)\underline{a} \\ &= f(\underline{a})(u_\infty - \underline{a}) + (f(a) - f(u_\infty))\underline{a} \\ &= \underline{a}(u_\infty - \underline{a}) \left(\frac{f(\underline{a}) - f(0)}{\underline{a}} - \frac{f(u_\infty) - f(\underline{a})}{u_\infty - \underline{a}}\right)\,.\end{aligned}$$

By convexity of f we have in the case $\underline{a} < u_\infty < \overline{a} < 0$

$$\frac{f(u_\infty) - f(\underline{a})}{u_\infty - \underline{a}} < \frac{f(\overline{a}) - f(\underline{a})}{\overline{a} - \underline{a}} < \frac{f(\underline{a}) - f(0)}{\underline{a}}\,.$$

This implies

$$\underline{a} \left(\frac{f(\underline{a}) - f(0)}{\underline{a}} - \frac{f(u_\infty) - f(\underline{a})}{u_\infty - \underline{a}}\right) \leq 0$$

and henceforth (2.112), if $\overline{a} \leq 0$. On the other hand if $0 < \underline{a} < \overline{a}$, we get for $\xi \in (0, \underline{a})$, $\alpha \in (\underline{a}, u_\infty)$

$$\frac{f(\underline{a}) - f(0)}{a} = f'(\xi) < f'(\underline{a}) < f'(\alpha) = \frac{f(u_\infty) - f(\underline{a})}{u_\infty - \underline{a}}\,,$$

which implies (2.112). Hence the first term of (2.111) is non-positive and it remains to treat the second term. A short calculation gives

$$\begin{aligned}&(\nabla \psi_\infty - \delta((1-\lambda)x, t)) \left(\nabla^\perp g_\infty - X(\underline{a})\right) \\ &= (u_\infty - \underline{a}) \left[\partial_t \psi(x_0, t_0) + \lambda \partial_x \psi(x_0, t_0) \frac{f(u_\infty) - f(\underline{a})}{u_\infty - \underline{a}} + \delta((1-\lambda)x + t)\right]\,.\end{aligned}$$

Our choice of δ and λ (see (2.109)) imply, that

$$(\nabla\psi_\infty - \delta((1-\lambda)x,t))\left(\nabla^\perp g_\infty - X(\underline{a})\right) < 0$$

and thus (2.110). Finally (2.110) implies

$$\int_{E_s^+} \left(X(\underline{a}) - \lambda\nabla^\perp\psi(x_0,t_0) + (1-\lambda)\delta(x,t)^\perp\right)\cdot\nabla h_\delta = 0\,.$$

Since

$$\left(X(\underline{a}) - \lambda\nabla^\perp\psi(x_0,t_0) + (1-\lambda)\delta(x,t)^\perp\right)\cdot\nabla h_\delta < 0$$

it follows $\mathcal{H}^1(E_s^+) = 0$, which is a contradiction to our choice of $s \in S$. Thus

$$\partial_t\psi(x_0,t_0) + f\left(\partial_x\psi(x_0,t_0)\right) \geq 0$$

as claimed. Henceforth g is the viscosity solution of (2.106) and $u = \partial_x g$ the entropy solution of (2.1) as claimed. □

Proof of Theorem 1 Thanks to Lemma 2.2 and Lemma 2.4 we can conclude the proof of Theorem 2.1. Indeed, we see that a weak solution $u \in L^\infty(\mathbb{R}\times[0,T))$ satisfying the assumptions of Theorem 2.1, has by Lemma 2.2 only points of positive density, i.e.

$$\limsup_{r\to 0^+}\frac{1}{r}\int_{\mathbb{R}} m(B_r(x_0,t_0),a)\,da\,dx\,dt \geq 0 \quad \text{for all} \quad (x_0,t_0)\in\mathbb{R}\times(0,T)\,.$$

By Lemma 2.4 we know then, that u has to be entropic.

Chapter 3

Control of Oscillation of Entropic Solutions to Scalar Conservation Laws in $1+1$ Dimensions

This chapter is submitted to Calc. Var. and PDE

3.1 Introduction

For $f \in C^2(\mathbb{R})$ we consider solutions to the scalar conservation law

$$\left.\begin{array}{rcll} \partial_t u + \partial_x f(u) &=& 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}\,, \\ u(x,0) &=& u_0(x)\,, & \end{array}\right\} \tag{3.1}$$

for initial data $u_0 \in L^\infty(\mathbb{R})$. It is well known that even for smooth initial data, the classical solution can cease to exist in finite time, due to the possible formation of shocks (see Chapter 4.2 in [Da]). Therefore one has to consider weak solutions of (3.1), i.e. solutions, which satisfy (3.1) in the distributional sense. However it turned out that for given initial data, the space of weak solutions is large (see Chapter 4.4 in [Da]). Therefore additional conditions have to be imposed to single out the physical relevant weak solutions in some models.

An approach was given by Kruzhkov in [Kr], where he introduces a family of integral inequalities. More precisely, for $u_0 \in L^\infty$ he proved existence and uniqueness of weak solutions satisfying the *entropy condition*: He considers the family of convex entropy flux pairs $(\eta_a, \xi_a)_{a\in\mathbb{R}}$, where

$$\eta_a(u) = |u-a| \quad \text{and} \quad \xi_a(u) = \operatorname{sign}(u-a)(f(u)-f(a))\,. \tag{3.2}$$

Then an entropy solution is a bounded function u, which satisfies (3.1) in the sense of distributions and

$$\partial_t \eta_a(u) + \partial_x \xi_a(u) \leq 0\,. \tag{3.3}$$

One can also replace the one parameter family $(\eta_a, \xi_a)_{a\in\mathbb{R}}$ and assume, that (3.3) is fulfilled for all convex η with corresponding entropy flux ξ, which satisfies

$$\xi = \int \eta' f'\,. \tag{3.4}$$

As a consequence of this, one can show if the initial data u_0 is in BV u is in BV for all later times.

A different approach to scalar conservation laws is introduced by Lions, Perthame and Tadmor in [LPT]: The kinetic formulation of a scalar conservation law (3.1). A comprehensive introduction is found in [Pe]. For a weak solution $u \in L^\infty$ of (3.1) one considers the set

$$E_a = \{(x,t):\ a \leq u(x,t)\}$$

and we will denote the characteristic function of E_a by

$$\mathbb{1}_{a\leq u(x,t)}\,.$$

Then one can show

Theorem 3.1 ([LPT]). *A bounded measurable function u on $\mathbb{R}\times\mathbb{R}_+$, which satisfies*

$$\partial_t \mathbb{1}_{a\le u} + f'(a)\partial_x \mathbb{1}_{a\le u} = \partial_a m(x,t,a) \quad in \quad \mathcal{D}'(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}) \tag{3.5}$$

for a non-negative measure $m(x,t,a)$ together with initial condition

$$\mathbb{1}_{a\le u(x,0)} = \mathbb{1}_{a\le u_0(x)}\,,$$

is the admissible solution of (3.1).

One can relate the measure m in (3.5) with (3.3) as follows:

$$\partial_t|u-a| + \partial_x[\mathrm{sign}(u-a)(f(u)-f(a))] = -m(x,t,a) \quad \text{in} \quad \mathcal{D}'(\mathbb{R}\times\mathbb{R}_+) \tag{3.6}$$

or equivalently

$$\partial u\wedge a + \partial_x f(u\wedge a) = m(x,t,a)\,, \tag{3.7}$$

where $u\wedge a = \min\{u,a\}$.
Instead of (3.5) one can consider the general Cauchy-Problem

$$\partial_t\chi(x,t,a) + b(a)\partial_x\chi(x,t,a) = g(x,t,a) \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{R}^n) \tag{3.8}$$

with initial condition

$$\chi(x,0,a) = \chi_0(x,a)\,. \tag{3.9}$$

In [GLPS] it was observed that that compactness and regularity results exist, not for the solution χ of (3.8), but for velocity averages of χ. For any $\phi\in C_c^\infty$, the velocity average of χ associated to ϕ is defined by

$$\rho(x,t) = \int_{\mathbb{R}} f(x,t,a)\phi(a)\,da\,. \tag{3.10}$$

In the case of (3.5), $\rho(x,t)$ is exactly the entropy solution $u(x,t)$ of (3.1), if we choose $\phi\in C_c^\infty$ such that $\phi=1$ on $[-\|u\|_\infty,\|u\|_\infty]$. The main result in [GLPS] is then as follows: if χ, $g\in L^2(\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n)$ and satisfy (3.8) with $b(a)=a$, then any average $\rho(x,t)$ of χ is in $H^{\frac{1}{2}}(\mathbb{R}^n\times\mathbb{R})$. Such results are called 'kinetic averaging lemmas'. For a survey in this topic we refer to [BGP]. Usually one assumes some regularity for χ and g to prove kinetic averaging lemmas and there are only a few results concerning χ only to be in L^1_{loc}. This is however the case if one deals with scalar conservation laws. In [Le1] a result in this direction is shown for the case $n=1$ and the following non-degeneracy condition for $b(a)$

$$\begin{gathered}\forall M>0\ \exists C>0 \text{ s.t. } \forall\xi\,,\ u\in\mathbb{R}\ \forall\varepsilon>0\\ \mathcal{L}^1\left(\{a\in[-M,M]:\ |b(a)\xi-u|<\varepsilon\}\right)\le C\varepsilon\,.\end{gathered} \tag{3.11}$$

Then the main theorem in [Le1] reads as follows

Theorem 3.2 ([Le1]). *Let $\chi \in L^1_{loc}(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R})$ satisfy (3.8) with zero initial condition. If $b(a) \in C^\infty(\mathbb{R},\mathbb{R})$ satisfies (3.11) and $g \in L^1(\mathbb{R}_x\times\mathbb{R}_t^+, BV(\mathbb{R}_a))$, then any velocity average ρ is in $L^{2,\infty}$ and satisfies*

$$\|\rho\|_{L^{2,\infty}} \le C\|g\|_{L^1_{x,t}BV_a} \,. \tag{3.12}$$

A motivation for regularity results like Theorem 3.2 or compactness results comes from studying blow up limits of solutions u of (3.1). Considering sequences $u(x_0+rx, t_0+rx)$ and $\frac{1}{r}m(B_r(x_0,t_0)\times\mathbb{R})$ as $r\to 0^+$, one would expect, that the singular set of entropy solutions u of (3.1) coincides with the set of points of $\mathbb{R}\times\mathbb{R}_+$, where the upper 1-dimensional density of m is strictly larger than zero (see [Le2] and [DOW]).
However, so far there were no estimates available which compare the local behavior of $u(x,t)$ to the behavior of $m(x,t,a)$. In this work we present a new Poincaré-type inequality, where we control the local oscillation of an entropic solution u of (3.1) in terms of the defect measure $m(x,t,a)$ given by (3.7). Before we can state our main result, we need to introduce the following two definitions: For $(x_0,t_0)\in\mathbb{R}\times\mathbb{R}_+$ and $r>0$ we denote the average of u over the ball $B_r(x_0,y_0)$ by

$$\overline{u}^r = \frac{1}{\pi r^2}\int_{B_r(x_0,t_0)} u(y,s)\,dy\,ds\,. \tag{3.13}$$

Furthermore

Definition 2. *For an entropy solution u of (3.1) let $m(x,t,a)$ be the defect measure given by the kinetic formulation (3.7). Then we define the (x,t)-marginal as*

$$\mu(A) = m(A\times\mathbb{R}_a) \quad \forall\ A \text{ Borel.} \tag{3.14}$$

We can now state our main result:

Theorem 3.3. *Let $f\in C^2(\mathbb{R},\mathbb{R})$ be such that $|\{u\in\mathbb{R} : f''(u)=0\}|<\infty$. For an entropy solution $u\in L^\infty\cap L^1(\mathbb{R}\times\mathbb{R}_+)$ of (3.1) there exist constants $C>0$ and $\delta_0>0$ such that for all ε, $\delta\in(0,\delta_0)$ and for all (x_0,t_0), $r\in(0,t_0/4)$*

$$\begin{aligned}\frac{1}{\pi(\delta r)^2}&\int_{B_{\delta r}(x_0,t_0)}|u(x,t)-\overline{u}^{\delta r}|\,dx\,dt\\ &\le C\left[\frac{1}{\delta r}\mu(B_r(x_0,t_0))\right]^{\frac12} + \frac{C}{\rho(\varepsilon)^{4/3}}\left[\frac{1}{r}\mu(B_r(x_0,t_0))\right]^{\frac13}\\ &\qquad + C\left(\frac{\delta}{\rho(\varepsilon)}+\varepsilon\right)\,,\end{aligned} \tag{3.15}$$

where $\rho(\varepsilon)$ is defined as

$$\rho(\varepsilon) = \min_{a \in \{f''=0\}} \min_{\varepsilon \le |u-a| \le 2\varepsilon} |f''(u)| \,.$$

We want to emphasize the fact that the inequality (3.15) is local and there is no further regularity assumption needed for u. Moreover, one can see that (3.15) immediately implies that $u(x,t)$ has vanishing mean oscillation at points where the upper 1-dimensional density of μ is zero, as expected from BV theory. Naturally one expects that Theorem 3.3 is not restricted to entropy solutions, it should be considered as a first step towards a similar estimate for weak solutions $v \in L^1$ of (3.1) such that $\partial_t v \wedge a + \partial_x f(v \wedge a)$ is a measure. Also it is reasonable to conjecture that (3.15) holds for more general fluxes f, i.e. fluxes $f \in C^1$ such that, f' satisfies (3.11).

From BV-theory one expects, that μ is $\mathcal{H}^1$-dimensional and rectifiable. In [DOW] it is shown for scalar conservation laws in n space dimensions, that the $\mathcal{H}^n$-dimensional part of μ is rectifiable. The control of oscillation (3.15) should then provide a tool for dealing with **0-density points** of the measure μ, in order to establish that μ has no higher dimensional parts. For scalar conservation laws in $1+1$-dimensions with strictly non-linear flux, this is shown in [DR] by De Lellis and Rivière, i.e. they prove that for entropy solutions u of (3.1) the measure μ is $\mathcal{H}^1$-dimensional and rectifiable. Unfortunately we are not able to deduce the main result in [DR] completely. Nevertheless we will give in Section 3.3 some applications of Theorem 3.3.

Estimate (3.15) is also desirable in the case of Γ-limits in micromagnetism (see [RS1], [RS2] and [DO] and references therein). One would like to show, that entropy-measures of limit configurations are $\mathcal{H}^1$-dimensional and rectifiable. In [AKLR] and [DO] it is shown that the $\mathcal{H}^1$-dimensional part of such entropy-measures is rectifiable. Then the control of oscillation might provide a tool for dealing with 0-density-points.

This work is organized as follows. In Section 3.2.1 we proof a version of Theorem 3.3 in the case of convex respective concave fluxes f. In Section 3.2.2 we are going to prove the general Theorem 3.3, which builds upon the work done in the previous sections. In the last section we will give applications of Theorem 3.3.

3.2 Control of Oscillation

3.2.1 The Strictly Convex and Concave Case

We consider the case where the flux f in (3.1) is strictly convex. Our aim in this section is to prove:

Theorem 3.4. *Let f be a strictly convex flux, i.e there exists a $\rho > 0$ such that $f''(u) \geq \rho$ for all $u \in \mathbb{R}$. Then for an entropy solution u of (3.1) with initial data $u_0 \in L^\infty(\mathbb{R})$ there exists constants $C > 0$ and $\delta_0 > 0$ depending on u_0 and f, such that for all $(x_0, t_0) \in \mathbb{R} \times \mathbb{R}_+$ and all $0 < \delta \leq \delta_0$*

$$\frac{1}{\pi(\delta r)^2} \int_{B_{\delta r}(x_0,t_0)} |u(x,t) - \overline{u}^{\delta r}| \, dx \, dt \leq \frac{C}{\rho^{4/3}} \left[\frac{1}{r} \mu(B_r(x_0,t_0)) \right]^{\frac{1}{3}} + C \frac{\delta}{\rho} \tag{3.16}$$

for all $0 < r < \frac{t_0}{4}$.

Before we proof Theorem 3.4, we need a few preliminary results about entropy solutions u of (3.1) in the particular case of convex fluxes.

An entropy solution u of (3.1) with initial data $u_0 \in L^\infty$ has in this case striking features. According to Theorem 11.2.2 in [Da] u is in $BV_{loc}(\mathbb{R} \times \mathbb{R}_+)$. This yields the following structure of an entropy solution u: One can decompose $\mathbb{R} \times \mathbb{R}_+$ in three disjoint sets C_u, J_u and I_u, such that

- for any open set $\Omega \subset C_u$ the solution u is continuous on Ω (see Theorem 11.3.2 in [Da]),
- the set J_u is $\mathcal{H}^1$ rectifiable and u has strong traces on J_u (see Theorem 11.3.3 in [Da]),
- the set of irregular points I_u is at most countable (see Theorem 11.3.4 in [Da]).

Therefore one can apply the method of characteristics. For every $(x,t) \in \mathbb{R} \times \mathbb{R}_+$ one can define the extremal backward characteristics as follows:

Definition 3. *A generalized characteristics for (3.1) associated with a weak solution u of (3.1) on the time interval $[\sigma, \tau] \subset [0, \infty)$ is a Lipschitz function $\xi : [\sigma, \tau] \to (-\infty, \infty)$, which satisfies the differential inclusion*

$$\dot{\xi}(t) \in \Lambda(\xi(t), t) \quad a.e. \ on \quad [\sigma, \tau] \,, \tag{3.17}$$

where

$$\Lambda(\overline{x}, \overline{t}) := \bigcap_{\epsilon > 0} [\operatorname{essinf}_{[\overline{x}-\epsilon, \overline{x}+\epsilon]} f'(u(x, \overline{t})), \operatorname{esssup}_{[\overline{x}-\epsilon, \overline{x}+\epsilon]} f'(u(x, \overline{t}))] \,. \tag{3.18}$$

We recall the following Theorem out of [Da]:

Theorem 3.5 (Theorem 10.2.2 in [Da]). *Through any fixed point* $(\overline{x}, \overline{t}) \in \mathbb{R} \times \mathbb{R}_+$ *pass two (not necessarily distinct) generalized characteristics associated with* u *and defined on* $[0, \infty)$, *namely the minimal* ξ^- *and the maximal* ξ^+, *with* $\xi^-(t) \leq \xi^+(t)$ *for* $t \in [0, \infty)$.

Furthermore one can show that generalized characteristics have the following properties:

Theorem 3.6 (Theorem 10.2.3 in [Da]). *Let* ξ *be a generalized characteristic associated with an entropy solution* u *of (3.1) and defined on* $[\sigma, \tau]$. *The following holds for almost all* $t \in [\sigma, \tau]$: *When* $(\xi(t), t) \in C_u$, *then* $\dot{\xi}(t) = f'(u(\xi(t)\pm, t))$. *When* $(\xi(t), t) \in J_u$, *then*

$$\dot{\xi}(t) = \frac{f(u(\xi(t)+, t)) - f(u(\xi(t)-, t))}{f(u(\xi(t)+, t)) - f(u(\xi(t)-, t))}.$$

Finally one can show that the minimal and maximal backward characteristics don't intersect.

Theorem 3.7 (Theorem 11.1.3 in [Da]). *Let* ξ^- *and* ξ^+ *denote the minimal and maximal backward characteristics associated with some entropy solution* u *of (3.1) emanating from any point* $(\overline{x}, \overline{t}) \in \mathbb{R} \times \mathbb{R}_+$. *Then*

$$\begin{aligned} u(\xi^-(t)-, t) &= u(\overline{x}-, \overline{t}) = u(\xi^-(t)+, t) \\ u(\xi^+(t)-, t) &= u(\overline{x}+, \overline{t}) = u(\xi^+(t)+, t) \end{aligned} \tag{3.19}$$

for all $t \in (0, \overline{t})$. *Moreover*

$$\begin{aligned} u_0(\xi^-(0)) &\leq u(\overline{x}-, \overline{t}) \leq u_0(\xi^-(0)+)\,, \\ u_0(\xi^+(0)) &\leq u(\overline{x}+, \overline{t}) \leq u_0(\xi^+(0)+)\,. \end{aligned} \tag{3.20}$$

From Theorem 3.5 and Theorem 3.7 one can conclude that

Corollary 3.1. *For any point* $(\overline{x}, \overline{t}) \in \mathbb{R} \times \mathbb{R}_+$ *the extremal backward characteristics* $\xi^\pm$ *are straight lines with constant speed* $\dot{\xi}^\pm = f'(u(\overline{x}\pm, \overline{t}))$. *Furthermore different backward characteristics* $\xi_1^\pm$ *and* $\xi_2^\pm$, *which originate from different points, don't intersect.*

The fact that an entropy solution u is in BV has also a useful consequence for the defect measure $m(x, t, a)$, which we summarize in the following lemma.

Lemma 3.1. *Let $m(x,t,a)$ be the defect measure of an entropy solution $u \in L^\infty$ of (3.1) with strictly convex flux f, $f'' \geq \rho > 0$. Then there exists a constant $C > 0$ such that*

$$\mu(A) = \int_A \Delta(u^+,u^-)\, d\mathcal{H}^1 \llcorner J_u \geq C\rho \int_A |u^+ - u^-|^3\, d\mathcal{H}^1 \llcorner J_u\,, \tag{3.21}$$

for all A Borel, where

$$\Delta(a,b) = \frac{\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(s)\, ds}{\sqrt{1+[(f(b)-f(a))/(b-a)]^2}}\,.$$

Proof of Lemma 3.1. The explicit formula for the measure μ follows from Vol'perts chain-rule (see Theorem 3.96 in [AFP]). The error estimate

$$\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(s)\, d \geq \frac{1}{12} \min_{\xi\in[a,b]} |f''(\xi)|(b-a)^3 \tag{3.22}$$

is given in equation 5.1.7 in [At]. Since $f'' \geq \rho$ (3.21) follows. □

After these preliminaries, we are able to show Theorem (3.16).

Proof of Theorem 3.4. We fix a point $(x_0,t_0) \in \mathbb{R} \times \mathbb{R}_+$ and define for any $s > 0$

$$Q_s(x_0,t_0) = \{(x,t) :\ (x-x_0, t-t_0) \in [-r,r] \times [-r,r]\}\,. \tag{3.23}$$

Let $Q_{\delta r}(x_0,t_0)$, where

$$\delta \leq \delta_0 := \min\left\{ \left(\max_{|\alpha|\leq \|u\|_\infty} |f'(\alpha)| \right)^{-1}, \frac{1}{3} \right\} \tag{3.24}$$

and $0 < r < \frac{t_0}{4}$. From the choice of δ we deduce that there exists for almost all $(x,t) \in Q_{\delta r}(x_0,t_0)$ a $(\overline{x},\overline{t}) \in Q_r((x_0,t_0))$ such that for an extremal backward characteristic ξ emanating at $(\overline{x},\overline{t})$ we have $(\xi(t),t) = (x,t)$.

For two different points (x,t) and (y,s) in $Q_{\delta r}(x_0,t_0)$ let ξ denote an extremal backward characteristic starting at $(\overline{x},\overline{t}) \in J_u$ and going through (x,t) and let ζ denote the extremal backward characteristic starting at $(\overline{y},\overline{s}) \in J_u$ and going through (y,s). Then we claim

Claim 1. If $\overline{t} - t_0 \geq \frac{r}{2}$ and $\overline{s} - t_0 \geq \frac{r}{2}$ we have

$$|u(x,t) - u(y,s)| \leq C\frac{\delta}{\rho}\,. \tag{3.25}$$

Proof of Claim 1. Without loss of generality, we can assume $y < x$. By Corollary 3.1 backward characteristics don't intersect, which implies

$$0 < \xi(\tau) - \zeta(\tau) \quad \forall 0 < \tau < \min\{\bar{t}, \bar{s}\}\,. \tag{3.26}$$

Moreover by Corollary 3.1 we know that backward characteristics are straight lines and that u is constant along such characteristics. Thus for $t_m := \min\{\bar{t}, \bar{s}\}$ we find

$$\begin{gathered} \xi(\tau) = f'(u(x,t))(\tau - t_m) + \xi(t_m) \\ \text{and} \\ \zeta(\tau) = f'(u(y,s))(\tau - \bar{t}) + \zeta(t_m)\,. \end{gathered} \tag{3.27}$$

From (3.26) and (3.27) we deduce

$$\begin{aligned} \xi(t) - \zeta(t) &= f'(u(x,t))(t - t_m) + \xi(t_m) - f'(u(y,s))(t - t_m) - \zeta(t_m) \\ &\geq f'(u(x,t))(t - t_m) - f'(u(y,s))(t - t_m) \end{aligned}$$

and hence

$$f'(u(y,s)) - f'(u(x,t)) \leq \frac{\xi(t) - \zeta(t)}{t_m - t}\,. \tag{3.28}$$

Since $(x,t)\,,\ (y,s) \in Q_{\delta r}(x_0, t_0)$ we observe

$$0 < x - y = \xi(t) - \zeta(s) \leq \delta r \quad \text{and} \quad |t - s| \leq \delta r\,. \tag{3.29}$$

Thus (3.27) and (3.29) yields

$$\xi(t) - \zeta(t) \leq (1 + 2\lambda)\delta r\,, \tag{3.30}$$

where

$$\lambda := \max_{|\alpha| \leq \|u\|_\infty} |f'(\alpha)|\,.$$

Combining (3.28) and (3.30) results in

$$f'(u(y,s)) - f'(u(x,t)) \leq \frac{(1 + 2\lambda)\delta r}{t_m - t}\,,$$

which implies

$$f'(u(y,s)) - f'(u(x,t)) \leq \frac{1 + 2\lambda}{\frac{1}{2} - \delta_0}\delta\,, \tag{3.31}$$

since $t_m - t \geq (\frac{1}{2} - \delta_0)r$. But on the other hand from (3.26) we have for $\tilde{t} = t_0 - r$

$$\begin{aligned}\xi(t) - \zeta(t) &= f'(u(x,t))\big(t-\tilde{t}\big) + \xi(\tilde{t}) - f'(u(y,s))\big(t-\tilde{t}\big) - \zeta(\tilde{t})\\ &\geq [f'(u(x,t)) - f'(u(y,s))](t-\tilde{t})\end{aligned}$$

and thus from (3.30)

$$\begin{aligned}f'(u(x,t)) - f'(u(y,s)) &\leq \frac{\xi(t)-\zeta(t)}{t-\tilde{t}} \leq \frac{(1+2\lambda)\delta r}{t-\tilde{t}}\\ &\leq \frac{1+2\lambda}{1-\delta_0}\delta\,.\end{aligned} \tag{3.32}$$

Putting (3.31) together with (3.32) gives

$$-\frac{1+2\lambda}{\frac{1}{2}-\delta_0}\delta \leq f'(u(x,t)) - f'(u(ys)) \leq \frac{1+2\lambda}{1-\delta_0}\delta\,.$$

Hence

$$|f'(u(x,t)) - f'(u(y,s))| \leq C\delta\,. \tag{3.33}$$

From (3.33) and strict convexity of f we obtain

$$|u(x,t) - u(y,s)| \leq C\frac{\delta}{\rho}\,, \tag{3.34}$$

which is exactly what what we claimed in (3.25).
Let $(x,t) \in Q_{\delta r}(x_0,t_0)$ and $(\overline{x},\overline{t}) \in J_u$ be such that $\overline{t} \leq t_0 + \frac{r}{2}$ and (x,t) lies on an extremal backward characteristics originating at $(\overline{x},\overline{t})$. Since $(\overline{x},\overline{t}) \in J_u$ according to Theorem 11.1.5 in [Da] there exists a Lipschitz curve $(\chi(\tau),\tau)$ such that $(\chi(\tau),\tau) \in J_u$ for all $\tau \geq \overline{t}$. We fix a point $(\chi(t_1),t_1)$ such that $t_1 \geq t_0 + \frac{3}{4}r$ and consider the minimal and maximal backward characteristics ξ^-, ξ^+ starting from $(\chi(t_1),t_1)$. Similar let ζ^+, ζ^- be the extremal characteristics emanating from $(\overline{x},\overline{t})$.
Claim 2. For a $C>0$ depending on f' and $\|u\|_\infty$ we have

$$|u(x,t) - u(\chi(t_1)-,t_1)| \leq \frac{C}{\rho}\left|u(\chi(t_1)+,t_1) - u(\chi(t_1)-,t_1)\right|\,. \tag{3.35}$$

Proof of Claim 2. By Corollary 3.1 the backward characteristics don't intersect each other. Therefore

$$\xi^-(\tau) \leq \zeta^\pm(\tau) \leq \xi^+(\tau) \quad \forall \tau \leq \overline{t}\,. \tag{3.36}$$

The fact that backward characteristics are straight lines implies

$$\dot{\zeta}^- = \frac{\zeta^-(\tau) - \overline{x}}{\tau - \overline{t}} \quad \forall \tau < \overline{t}\,. \tag{3.37}$$

From (3.37) we conclude

$$\dot{\zeta}^- - \dot{\xi}^- = \frac{\zeta^-(t_0 - r) - \overline{x}}{t_0 - r - \overline{t}} - \dot{\xi}^-\,. \tag{3.38}$$

Then (3.36) implies

$$\xi^-(t_0 - r) - \xi^+(\overline{t}) \leq \zeta^-(t_0 - r) - \zeta^+(\overline{t}) = \zeta^-(t_0 - r) - \overline{x}\,. \tag{3.39}$$

Since $t_0 - r - \overline{t}$ is strictly negative (3.39) applied to (3.38) implies

$$\dot{\zeta}^- - \dot{\xi}^- \leq \frac{\xi^-(t_0 - r) - \xi^+(\overline{t})}{t_0 - r - \overline{t}} - \dot{\xi}^-\,. \tag{3.40}$$

A quick calculation reveals

$$\begin{aligned} \xi^-(t_0 - r) - \xi^+(\overline{t}) &= \dot{\xi}^-(t_0 - r - \overline{t}) + \xi^-(\overline{t}) - \xi^+(\overline{t}) \\ &= \dot{\xi}^-(t_0 - r - \overline{t}) + \left[\dot{\xi}^- - \dot{\xi}^+\right](\overline{t} - t_1)\,. \end{aligned} \tag{3.41}$$

Using (3.41) in (3.40) gives

$$\dot{\zeta}^- - \dot{\xi}^- \leq \frac{\overline{t} - t_1}{t_0 - r - \overline{t}} \left(\dot{\xi}^- - \dot{\xi}^+\right)\,. \tag{3.42}$$

Since

$$0 \leq t_1 - \overline{t} \leq r \quad \text{and} \quad (1 - \delta) r < \overline{t} - (t_0 - r)\,,$$

the previous inequality (3.42) implies

$$\xi^-(t_0 - r) - \xi^+(\overline{t}) \leq \frac{1}{1 - \delta} \left(\dot{\xi}^- - \dot{\xi}^+\right)\,. \tag{3.43}$$

On the other hand we have also by (3.36) and (3.37)

$$\begin{aligned} \dot{\zeta}^- - \dot{\xi}^- &= \frac{\zeta^-(t_0 - r) - \overline{x}}{t_0 - r - \overline{t}} - \dot{\xi}^- \geq \frac{\xi^+(t_0 - r) - \xi^-(\overline{t})}{t_0 - r - \overline{t}} - \dot{\xi}^- \\ &= \left(1 + \frac{\overline{t} - t_1}{t_0 - r - \overline{t}}\right)\left(\dot{\xi}^+ - \dot{\xi}^-\right) \end{aligned}$$

and since $t_1 - \overline{t} \geq \frac{1}{2}r$ we get

$$\dot{\zeta}^- - \dot{\xi}^- \geq \frac{3}{2}\left(\dot{\xi}^+ - \dot{\xi}^-\right) \tag{3.44}$$

Hence (3.38) and (3.44) together result in

$$\left|\dot{\zeta}^- - \dot{\xi}^-\right| \leq C\left|\dot{\xi}^+ - \dot{\xi}^-\right|. \tag{3.45}$$

By Theorem 3.6 inequality (3.45) is equivalent to

$$|f'(u(x,t)) - f'(u(\chi(t_1)-,t_1))| \leq C\,|f'(u(\chi(t_1)+,t_1)) - f'(u(\chi(t_1)-,t_1))|\,.$$

Using that f is strictly convex gives

$$|u(x,t) - u(\chi(t_1)-,t_1)| \leq \frac{C}{\rho}\,|u(\chi(t_1)+,t_1) - u(\chi(t_1)-,t_1)| \tag{3.46}$$

which is what we claimed (3.35).

From Claim 1 and Claim 2 we want now to conclude the Theorem 3.4. We claim

Claim 3. Let $(x,t),\ (y,s) \in Q_{\delta r}(x_0,t_0)$, then

$$|u(x,t) - u(y,s)|^3 \leq \frac{C}{\rho^4}\frac{1}{r}\mu(Q_r(x_0,t_0)) + C\frac{\delta^3}{\rho^3}\,. \tag{3.47}$$

Proof of Claim 3. For $(x,t),\ (y,s) \in Q_{\delta r}(x_0,t_0)$ we assume without loss of generality $x < y$. Again let $(\overline{x},\overline{t}),\ (\overline{y},\overline{s}) \in J_u$ such that (x,t) lies on an extremal backward characteristic emanating from $(\overline{x},\overline{t})$ and (y,s) lies on a backward characteristic of $(\overline{y},\overline{s})$. If $\overline{t} - t_0 \geq \frac{r}{2}$ and $\overline{s} - t_0 \geq \frac{r}{2}$, we can apply (3.25) from Claim 1 and (3.47) immediately follows.

We consider the case where $\overline{t} - t_0 \leq \frac{r}{2}$ and $\overline{s} - t_0 \leq \frac{r}{2}$. Then let $(\alpha(\tau),\tau)$ be the shock-curve starting at $(\overline{x},\overline{t})$ and $(\beta(\tau),\tau)$ the shock-curve starting at $(\overline{y},\overline{s})$. We deduce with (3.35) from Claim 2 for all $t_1 \geq t_0 + 3/4r$

$$\begin{aligned}
|u(x,t) - u(y,s)| &\leq |u(x,t) - u(\alpha(t_1)-,t_1)| + |u(\alpha(t_1)+,t_1) - u(\alpha(t_1)-,t_1)| \\
&\quad + |u(\alpha(t_1)+,t_1) - u(\beta(t_1)-,t_1)| \\
&\quad + |u(y,s) - u(\beta(t_1)-,t_1)| \\
&\leq \frac{C}{\rho}\big\{|u(\alpha(t_1)+,t_1) - u(\alpha(t_1)-,t_1)| \\
&\quad + |u(\beta(t_1)+,t_1) - u(\beta(t_1)-,t_1)|\big\} \\
&\quad + |u(\alpha(t_1)+,t_1) - u(\beta(t_1)-,t_1)|\,.
\end{aligned} \tag{3.48}$$

It remains to handle the term $|u(\alpha(t_1)+,t_1) - u(\beta(t_1)-,t_1)|$. Let ξ^+ be the maximal backward characteristic starting at $(\alpha(t_1),t_1)$ and ζ^- be the minimal backward characteristic starting at $(\beta(t_1),t_1)$. Necessarily we have

$$x < \xi^+(t) < \zeta^-(t) \quad \text{and} \quad \xi^+(s) < \zeta^-(s) < y\,. \tag{3.49}$$

Therefore there exits t_3, $t_4 \in [t_0 - \delta r, t_0 + \delta r]$ such that

$$(\xi^+(t_3),t_3),\ (\zeta^-(t_4),t_4) \in Q_{\delta r}(x_0,t_0)\,.$$

From (3.25) in Claim 1 we get

$$|u(\xi^+(t_3),t_3) - u(\zeta^-(t_4),t_4)| \le C\frac{\delta}{\rho}$$

and since u is constant along extremal backward characteristics we discover

$$|u(\alpha(t_1)+,t_1) - u(\beta(t_1)-,t_1)| = |u(\xi^+(t_3),t_3) - u(\zeta^-(t_4),t_4)| \le C\frac{\delta}{\rho}\,. \tag{3.50}$$

Applying (3.50) in (3.48) gives for $t_1 \in [t_0 + 3/4r, t_0 + r]$

$$\begin{aligned}|u(x,t) - u(y,s)| \le \frac{C}{\rho}&\left\{|u(\alpha(t_1)+,t_1) - u(\alpha(t_1)-,t_1)|\right\}\\ &+ \frac{C}{\rho}\left\{|u(\beta(t_1)+,t_1) - u(\beta(t_1)-,t_1)| + \delta\right\}\,.\end{aligned} \tag{3.51}$$

From (3.51) and lemma 3.1 we deduce by integrating in t_1 over $[t_0{+}3/4r, t_0{+}r]$

$$\begin{aligned}|u(x,t) - u(y,s)|^3 &\le \frac{C}{\rho^3}\frac{1}{r}\int_{t_0+\frac{3}{4}r}^{t_0+r} |u(\chi(t_1)-,t_1) - u(\chi(t_1)-,t_1)|^3\, dt_1 + C\frac{\delta^3}{\rho^3}\\ &\le \frac{C}{\rho^4}\frac{1}{r}\mu(Q_r(x_0,t_0)) + C\frac{\delta^3}{\rho^3}\,,\end{aligned}$$

which is our claim (3.47).

Finally from Claim 3 the conclusion of Theorem 3.4 is straightforward. ☐

Corollary 3.2. *Let f be a strictly concave flux, i.e there exists a $\rho > 0$ such that $f''(u) \le -\rho$ for all $u \in \mathbb{R}$. Then for an entropy solution u of (3.1) with initial condition $u_0 \in L^\infty(\mathbb{R})$ there exist constants $C > 0$ and $\delta_0 > 0$ depending on u_0 and f and such that for all $(x_0,t_0) \in \mathbb{R}\times\mathbb{R}_+$ and all $0 < \delta \le \delta_0$*

$$\frac{1}{\pi r^2}\int_{B_{\delta r}(x_0,t_0)} |u(x,t) - \overline{u}^{\delta r}|\, dx\, dt \le \frac{C}{\rho^{4/3}}\left[\frac{1}{r}\mu(B_r(x_0,t_0))\right]^{\frac{1}{3}} + \frac{C\delta}{\rho} \tag{3.52}$$

for all $0 < r < \frac{t_0}{4}$.

Proof of Corollary 3.2. We define $g(s) = -f(-s)$ and for an entropy solution u of (3.1) we define $v = -u$. Then v is a weak solution of

$$\left.\begin{array}{rcll} \partial_t v + \partial_x g(v) & = & 0 & \text{in } \mathbb{R}\times\mathbb{R}_+ \,, \\ v(x,0) & = & -u_0(x)\,. & \end{array}\right\} \tag{3.53}$$

We claim that

Claim 1. The weak solution v is an entropy solution of (3.53).
Proof of Claim 1. For $\psi \in C_c^\infty(\mathbb{R}\times\mathbb{R}_+)$ and $a \in \mathbb{R}$ we compute

$$\begin{aligned} &\int_{\mathbb{R}\times\mathbb{R}} v \wedge (-a)\partial_t\psi + g(v\wedge(-a))\partial_x\psi \, dx\, dt \\ &\qquad\qquad = -\int_{\mathbb{R}\times\mathbb{R}} u \vee a\partial_t\psi + f(u\vee a)\partial_x\psi\, dx\, dt\,, \end{aligned}$$

where $v \vee a = \max\{v, a\}$. This implies

$$\partial_t v \wedge (-a) + \partial_x g(v\wedge(-a)) = -[\partial_t u \vee a + \partial_x f(u\vee a)] \quad \text{in} \quad \mathcal{D}'\,. \tag{3.54}$$

Since $u \in BV_{loc}$ we compute with Vol'pert's chain rule (see Theorem 3.96 in [AFP])

$$\begin{aligned} &\partial_t u\vee a + \partial_x f(u\vee a) \\ &\qquad = f(u^+\vee a) - f(u^-\vee a) - \sigma[u^+\vee a - u^-\vee a]\, d\mathcal{H}^1 \llcorner J_u\,, \end{aligned} \tag{3.55}$$

where σ is the jump-speed given by the Rankine-Hugoniot condition

$$\sigma = \frac{f(u^+) - f(u^-)}{u^+ - u^-}\,.$$

Since u is an entropy solution it satisfies the Lax E-condition: $f'(u^-) > f'(u^+)$. Thus the concavity of f gives $u^+ > u^-$. For $a \notin [u^-, u^+]$ we compute

$$f(u^+\vee a) - f(u^-\vee a) - \sigma[u^+\vee a - u^-\vee a] = 0$$

and for $a \in [u^-, u^+]$

$$\begin{aligned} &f(u^+) - f(a) - \sigma(u^+ - a) \\ &\qquad = (u^+ - a)\left(\frac{f(u^+) - f(a)}{u^+ - a} - \frac{f(u^+) - f(u^-)}{u^+ - u^-}\right)\,. \end{aligned} \tag{3.56}$$

Since f is concave, the right-hand side of (3.56) is non-positive and therefore also right-hand side of (3.55) is non-positive. Thus by (3.54) we get that

$$\partial_t v \wedge (-a) + \partial_x f(v \wedge (-a))$$

is a non-negative measure and hereby v is indeed an entropy solution of (3.53) and Claim 1 is proven.

Since g is strictly convex, we can apply Theorem 3.4 for v and get

$$\frac{\pi}{r^2} \int_{B_{\delta r}(x_0,t_0)} |v(x,t) - \overline{v}^{\delta r}| \, dx \, dt \leq \frac{C}{\rho^{4/3}} \left(\frac{1}{r} \nu(B_r(x_0,t_0)) \right)^{\frac{1}{3}} + \frac{C\delta}{\rho} \tag{3.57}$$

for all $0 < r < \frac{t_0}{4}$ and $\nu = \int_{\mathbb{R}} \partial_t v \wedge a + g(v \wedge a) \, da$. For the measure ν we compute

$$\begin{aligned} \nu &= \int_{\mathbb{R}} \partial_t v \wedge a + \partial_x g(v \wedge a) \, da = - \int_{\mathbb{R}} \partial_t u \vee a + \partial_x f(u \vee a) \, da \\ &= - \left(\int_{u^-}^{u^+} f(u^+) - f(a) - \sigma[u^+ - a] \, da \right) d\mathcal{H}^1 \llcorner J_u \\ &= \int_{u^-}^{u^+} f(a) \, da - \frac{f(u^+) + f(u^-)}{2} (u^+ - u^-) d\mathcal{H}^1 \llcorner J_u \, . \end{aligned} \tag{3.58}$$

In a similar way we compute

$$\mu = \left\{ \int_{u^-}^{u^+} f(a) \, da - \frac{f(u^+) + f(u^-)}{2} (u^+ - u^-) \right\} d\mathcal{H}^1 \llcorner J_u \, . \tag{3.59}$$

Hence combining (3.58) and (3.59) gives

$$\nu = \mu \, . \tag{3.60}$$

Therefore using (3.60) and $v = -u$ in (3.57) concludes the proof. □

3.2.2 The Strictly Non-linear Case

In this section we are going to show a similar estimate as in Theorem 3.4, but for more general conservation laws. We consider the scalar conservation law (3.1), where $f \in C^2(\mathbb{R})$ is strictly non-linear, i.e.

$$|\{u \in \mathbb{R} : \ f''(u) = 0\}| < \infty \, . \tag{3.61}$$

We define

$$\rho(\varepsilon) = \min_{a \in \{f''=0\}} \min_{\varepsilon \leq |u-a| \leq 2\varepsilon} |f''(u)| \, . \tag{3.62}$$

Our main result is, as already mentioned in Section, 3.1:

Theorem. *Let $f \in C^2(\mathbb{R},\mathbb{R})$ be such that $|\{u \in \mathbb{R} : f'(u) = 0\}| < \infty$. For an entropy solution $u \in L^\infty \cap L^1(\mathbb{R}\times\mathbb{R}_+)$ of (3.1) there exist constants $C > 0$ and $\delta_0 > 0$ such that for all ε, $\delta \in (0, \delta_0)$ and for all (x_0, t_0), $r \in (0, t_0/4)$*

$$[15]\frac{1}{\pi(\delta r)^2}\int_{B_{\delta r}(x_0,t_0)} |u(x,t) - \overline{u}^{\delta r}|\, dx\, dt \leq C\left[\frac{1}{\delta r}\mu(B_r(x_0,t_0))\right]^{\frac{1}{2}} + \frac{C}{\rho(\varepsilon)^{4/3}}\left[\frac{1}{r}\mu(B_r(x_0,t_0))\right]^{\frac{1}{3}} + C\left(\frac{\delta}{\rho(\varepsilon)} + \varepsilon\right),$$

where $\rho(\varepsilon)$ is defined as

$$\rho(\varepsilon) = \min_{a\in\{f''=0\}} \min_{\varepsilon\leq|u-a|\leq 2\varepsilon} |f''(u)|\,.$$

The main idea for the proof of Theorem 3.3 is to compare the entropy solution u with solutions v of (3.1) which take only values in the convex resp. concave parts of f.

Proof of Theorem 3.3

For $a_0 \in \mathbb{R}$ we consider the problem

$$\left.\begin{array}{rcll} \partial_t v + \partial_x f(v) & = & 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}\,, \\ v(x,0) & = & u_0(x) \wedge a_0\,. & \end{array}\right\} \tag{3.63}$$

Our goal is to compare entropy solutions v of (3.63) to entropy solutions u of (3.1)

Lemma 3.2. *For initial data $u_0 \in L^\infty(\mathbb{R})$, let v be an entropy solution of (3.63) with defect measure $q(x,t,a)$ and let u be an entropy solution of (3.1) with defect measure $m(x,t,a)$. We define for any $(x_0,t_0) \in \mathbb{R}\times\mathbb{R}_+$*

$$\Gamma^r_{(x_0,t_0)} = \{(x,t):\ 0 \leq t \leq t_0\,,\ |x - x_0| \leq r + \lambda(t_0 - t)\}\,, \tag{3.64}$$

where $\lambda = \max_{|\alpha|\leq\|u_0\|_\infty} |f'(\alpha)|$. Then

$$q(\Gamma^r_{(x_0,t_0)} \times \{a\}) \leq m(\Gamma^r_{(x_0,t_0)} \times \{a\}) \quad \forall a \leq a_0 \tag{3.65}$$

and

$$\int_{|x-x_0|\leq r} |u(x,t_0)\wedge a - v(x,t_0)\wedge a|^2\, dx \leq \mu(\Gamma^r_{(x_0,t_0)}) \quad \forall a \leq a_0\,, \tag{3.66}$$

where μ is the marignal of $m(x,t,a)$.

Proof of Lemma 3.2. By definition of the defect measure we have for all $\psi \in C_c^\infty(\mathbb{R}\times[0,\infty))$ and all $a \in \mathbb{R}$

$$\begin{aligned}\int_{\mathbb{R}\times[0,\infty)} (u\wedge a - v\wedge a)\partial_t\psi + (f(u\wedge a) - f(v\wedge a))\partial_x\psi\,dx\,dt \\ + \int_{\mathbb{R}} (u(x,0)\wedge a - v(x,0)\wedge a)\psi(x,0)\,dx \\ = -\int_{\mathbb{R}\times[0,\infty)} \psi\,dm(x,t,a) + \int_{\mathbb{R}\times[0,\infty)} \psi\,dq(x,t,a)\,.\end{aligned} \tag{3.67}$$

For $a \le a_0$ it follows since v is a solution of (3.63)

$$u(x,0)\wedge a - v(x,0)\wedge a = u_0(x)\wedge a - u_0(x)\wedge a\wedge a_0 = 0\,. \tag{3.68}$$

Hence in the case $a \le a_0$ equation (3.67) simplifies to

$$\begin{aligned}\int_{\mathbb{R}\times[0,\infty)} (u\wedge a - v\wedge a)\partial_t\psi + (f(u\wedge a) - f(v\wedge a))\partial_x\psi\,dx\,dt \\ = -\int_{\mathbb{R}\times[0,\infty)} \psi\,dm(x,t,a) + \int_{\mathbb{R}\times[0,\infty)} \psi\,dq(x,t,a)\,.\end{aligned} \tag{3.69}$$

We introduce

$$\omega_\varepsilon(t) = \begin{cases} 1 & 0 \le t \le t_0\,, \\ \varepsilon^{-1}(t_0 - t) + 1 & t_0 \le t \le t_0 + \varepsilon\,, \\ 0 & t_0 + \varepsilon \le t < \infty \end{cases} \tag{3.70}$$

and

$$\kappa_\varepsilon(x,t) = \begin{cases} 1 & |x - x_0| - r - \lambda(t_0 - t) < 0\,, \\ \varepsilon^{-1}[r + \lambda(t_0 - t) - |x - x_0|] + 1 & 0 \le |x - x_0| - r - \lambda(t_0 - t) < \varepsilon\,, \\ 0 & |x - x_0| - r - \lambda(t_0 - t) \ge \varepsilon\,. \end{cases} \tag{3.71}$$

Then putting $\psi_\varepsilon(x,t) = \kappa_\varepsilon(x,t)\omega_\varepsilon(t)$ in (3.69) leads to

$$\begin{aligned}-\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\int_{|x-x_0|\le r} (u\wedge a - v\wedge a)\,dx\,dt \\ -\frac{1}{\varepsilon}\int_{\Gamma^{r+\varepsilon}_{(x_0,t_0)}} \lambda(u\wedge a - v\wedge a) + [f(u\wedge a) - f(v\wedge a)]\operatorname{sign}(x - x_0)\,dx\,dt \\ = -\int_{\mathbb{R}\times[0,\infty)} \psi\,dm + \int_{\mathbb{R}\times[0,\infty)} \psi\,dq(x,t,a)\,.\end{aligned} \tag{3.72}$$

From Theorem 6.2.3 in [Da] we know that

$$u_0 \geq v_0 = u_0 \wedge a_0$$

implies

$$u(x,t) \geq v(x,t) \quad \text{for a.e.} \quad (x,t) \in \mathbb{R} \times [0,\infty)$$

and hence

$$u(x,t) \wedge a \geq v(x,t) \wedge a \quad \forall a \leq a_0 \quad \text{and for a.e.} \quad (x,t) \in \mathbb{R} \times [0,\infty)\,. \tag{3.73}$$

Therefore by (3.73) we get

$$u \wedge a - v \wedge a = |u \wedge a - v \wedge a|\,. \tag{3.74}$$

We observe that (3.74) and the choice of λ imply

$$0 \leq \lambda(u \wedge a - v \wedge a) + [f(u \wedge a) - f(v \wedge a)]\,\mathrm{sign}(x - x_0)\,. \tag{3.75}$$

Using (3.75) in (3.72) gives the estimate

$$-\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\int_{|x-x_0|\leq r}(u \wedge a - v \wedge a)\,dx\,dt \\ \geq -\int_{\mathbb{R}\times[0,\infty)}\psi_\varepsilon\,dm_a + \int_{\mathbb{R}\times[0,\infty)}\psi_\varepsilon\,dq_a\,.$$

and thus

$$\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\int_{|x-x_0|\leq r}|u \wedge a - v \wedge a|\,dx\,dt \leq \int_{\mathbb{R}\times[0,\infty)}\psi_\varepsilon\,dm_a - \int_{\mathbb{R}\times[0,\infty)}\psi_\varepsilon\,dq_a\,. \tag{3.76}$$

Letting $\varepsilon \to 0^+$ in (3.76) gives for all $a \leq a_0$

$$\int_{|x-x_0|\leq r}|u(x,t_0) \wedge a - v(x,t_0) \wedge a|\,dx \\ \leq m(\Gamma^r_{(x_0,t_0)} \times \{a\}) - q(\Gamma^r_{(x_0,t_0)} \times \{a\})\,. \tag{3.77}$$

Since the left-hand side of (3.77) is non-negative (3.65) follows immediately. It remains to show (3.66). Integrating (3.77) with respect to a over $(-\infty, a_1)$ for $a_1 \leq a_0$ gives

$$\int_{|x-x_0|\leq r}|u(x,t_0) \wedge a_1 - v(x,t_0) \wedge a_1|^2\,dx \leq C\mu(\Gamma^r_{(x_0,t_0)})\,,$$

which is the desired result (3.66). $\square$

We now consider a similar case as in Lemma 3.2. For a $a_0 \in \mathbb{R}$, let v be an entropy solution of

$$\left.\begin{array}{rcll} \partial_t v + \partial_x f(v) & = & 0 & \text{in} \quad \mathbb{R}_+ \times \mathbb{R}\,, \\ v(x,0) & = & u_0(x) \vee a_0\,. & \end{array}\right\} \tag{3.78}$$

As before we compare solutions of (3.78) to an entropy solution of (3.1).

Lemma 3.3. *For initial data $u_0 \in L^\infty(\mathbb{R})$, let v be an entropy solution of (3.78) with defect measure $q(x,t,a)$ and let u be an entropy solution of (3.1) with defect measure $m(x,t,a)$. We define for any $(x_0,t_0) \in \mathbb{R} \times \mathbb{R}_+$*

$$\Gamma^r_{(x_0,t_0)} = \{(x,t) :\ 0 \le t \le t_0\,,\ |x-x_0| \le r + \lambda(t_0-t)\}\,, \tag{3.79}$$

where $\lambda = \max_{|\alpha|\le\|u_0\|_\infty} |f'(\alpha)|$. Then

$$q(\Gamma^r_{(x_0,t_0)} \times \{a\}) \le m(\Gamma^r_{(x_0,t_0)} \times \{a\}) \quad \forall a \ge a_0 \tag{3.80}$$

and

$$\int_{|x-x_0|\le r} |u(x,t_0) \vee a - v(x,t_0) \vee a|^2\, dx \le \mu(\Gamma^r_{(x_0,t_0)}) \quad \forall a \ge a_0\,, \tag{3.81}$$

where μ is the marignal of $m(x,t,a)$.

Proof of Lemma 3.3. Let $\varphi(x,t) = \varphi_1(x)\varphi_2(t)$, where $\varphi_1,\ \varphi_2 \in C_c^\infty(\mathbb{R})$ are such that $\operatorname{supp}\varphi_1 \subset (0,1)$, $\operatorname{supp}\varphi_2 \subset (-1,0]$ and $\int_R \varphi_1(x)\,dx = \int_{\mathbb{R}} \varphi_2(t)\,dt = 1$. Define the convolution kernel $\varphi_\varepsilon(x,t) = \frac{1}{\varepsilon}\varphi(\frac{(x,t)}{\varepsilon})$. Then we introduce the regularized functions:

$$\chi_\varepsilon(x,t,a) = \mathbb{1}_{a\le u} * \varphi_\varepsilon(x,t) \quad \text{and} \quad \theta_\varepsilon(x,t,a) = \mathbb{1}_{a\le v} * \varphi_\varepsilon(x,t)$$

and the regularized defect measures

$$m_\varepsilon(x,t,a) = \int_{\mathbb{R}\times\mathbb{R}_+} \varphi_\varepsilon(x-y,t-s)\, dm(y,s,a) \tag{3.82}$$

and

$$q_\varepsilon(x,t,a) = \int_{\mathbb{R}\times\mathbb{R}_+} \varphi_\varepsilon(x-y,t-s)\, dq(y,s,a)\,. \tag{3.83}$$

Let $\psi \in C_c^\infty(\mathbb{R}\times\mathbb{R}_+)$. Then by the kinetic formulation (3.5) we get pointwise in (x,t) and for almost every a

$$[\chi_\varepsilon - \theta_\varepsilon]\partial_t\psi + f'(a)[\chi_\varepsilon - \theta_\varepsilon]\partial_x\psi = -\psi\partial_a[m_\varepsilon - q_\varepsilon]\,. \tag{3.84}$$

For $\delta > 0$ and $\overline{a} \geq a_0$ we define

$$S'(a) = \begin{cases} 1 & \overline{a} + \delta \leq a\,, \\ \dfrac{1}{\delta}(a - \overline{a}) & \overline{a} \leq a \leq \overline{a} + \delta\,, \\ 0 & a \leq \overline{a}\,. \end{cases} \tag{3.85}$$

Multiplying (3.84) with $S'(a)$ and integrating with respect of a gives

$$\int_{\mathbb{R}} S'(a)[\chi_\varepsilon - \theta_\varepsilon]\partial_t \psi + S'(a) f'(a)[\chi_\varepsilon - \theta_\varepsilon]\partial_x \psi \, da \\ = - \int_{\mathbb{R}} S'(a)\psi \partial_a [m_\varepsilon - q_\varepsilon]\, da\,. \tag{3.86}$$

We integrate the right-hand side of (3.86) by parts, which gives

$$\int_{\mathbb{R}} S'(a)\psi\partial_a[m_\varepsilon - q_\varepsilon]\, da = -\frac{1}{\delta}\int_{\overline{a}}^{\overline{a}+\delta} m_\varepsilon(x,t,a) - q_\varepsilon(x,t,a)\, da\,. \tag{3.87}$$

As m_ε and q_ε are uniformly Lipschitz continuous in a, we notice that

$$\frac{1}{\delta}\int_{\overline{a}}^{\overline{a}+\delta} m_\varepsilon(x,t,a) - q_\varepsilon(x,t,a)\, da \to m(x,t,\overline{a}) - q_\varepsilon(x,t,\overline{a}) \quad \text{as} \quad \delta \to 0^+\,. \tag{3.88}$$

For a function $g(a) \in W^{1,\infty}(\mathbb{R})$ we compute

$$\int_{\mathbb{R}} g'(a)S'(a)\chi_\varepsilon(x,t,a)\, da = \int_{\mathbb{R}\times\mathbb{R}_+}\int_{\mathbb{R}} g'(a)S'(a)\mathbb{1}_{a\leq u(y,s)}\varphi_\varepsilon(x-y,t-s)\, da\, dx\, dt \\ = \int_{\mathbb{R}\times\mathbb{R}_+} S_\delta^g(u(y,s))\varphi_\varepsilon(x-y,t-s)\, dx\, dt\,, \tag{3.89}$$

where

$$S_\delta^g(a) = \begin{cases} 0 & a \leq \overline{a}\,, \\ g(a) - \dfrac{1}{\delta}\displaystyle\int_{\overline{a}}^{a} g'(\alpha)\, d\alpha & \overline{a} \leq a \leq \overline{a}+\delta\,, \\ g(\overline{a}) - \dfrac{1}{\delta}\displaystyle\int_{\overline{a}}^{\overline{a}+\delta} g(\alpha)\, d\alpha + g(a) - g(\overline{a}) & \overline{a}+\delta \leq a\,. \end{cases} \tag{3.90}$$

We notice that

$$S_\delta^g(a) \to g(a \vee \overline{a}) - g(\overline{a}) \quad \text{in} \quad L^\infty_{loc}(\mathbb{R}) \quad \text{as} \quad \delta \to 0^+\,. \tag{3.91}$$

Applying identities (3.87) and (3.89) in (3.86) results in

$$\left[S_\delta^{Id}(u) - S_\delta^{Id}\right] * \varphi_\varepsilon \partial_t \psi + \left[S_\delta^f(u) - S_\delta^f\right] * \varphi_\varepsilon \partial_x \psi = \frac{1}{\delta} \int_{\overline{a}}^{\overline{a}+\delta} m_\varepsilon(x,t,a) - q_\varepsilon(x,t,a)\, da\,. \quad (3.92)$$

Recalling (3.88) and (3.91) gives as we let $\delta \to 0^+$ in (3.92)

$$[u \vee \overline{a} - v \vee \overline{a}] * \varphi_\varepsilon \partial_t \psi + [f(u \vee \overline{a}) - f(v \vee \overline{a})] * \varphi_\varepsilon \partial_x \psi = [m_\varepsilon(x,t,\overline{a}) - q_\varepsilon(x,t,\overline{a})]\psi\,. \quad (3.93)$$

Integrating (3.93) over $\mathbb{R} \times \mathbb{R}_+$ and letting $\varepsilon \to 0^+$ gives

$$\int_{\mathbb{R}\times\mathbb{R}_+} [u \vee \overline{a} - v \vee \overline{a}]\partial_t \psi + [f(u \vee \overline{a}) - f(v \vee \overline{a})]\, \partial_x \psi\, dx\, dt = \int_{\mathbb{R}\times\mathbb{R}_+} \psi\, dm_{\overline{a}} - \int_{\mathbb{R}\times\mathbb{R}_+} \psi\, dq_{\overline{a}}\,. \quad (3.94)$$

If we choose $\psi(x,t) = \psi_\varepsilon := \omega_\varepsilon(t)\kappa_\varepsilon(x,t)$ in (3.94) where $\omega_\varepsilon(t)$ is as in (3.70) and $\kappa_\varepsilon(x,t)$ as in (3.71), we compute similar to (3.72)

$$\begin{aligned} &-\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon} \int_{|x-x_0|\le r} u \vee \overline{a} - v \vee \overline{a}\, dx\, dt \\ &\quad - \frac{1}{\varepsilon}\int_{\Gamma_{(x_0,t_0)}^{r+\varepsilon}} \lambda(u \vee a - v \vee a) + [f(u \vee a) - f(v \vee a)]\, \mathrm{sign}(x)\, dx\, dt \\ &= \int_{\mathbb{R}\times\mathbb{R}_+} \psi_\varepsilon\, dm_{\overline{a}} - \int_{\mathbb{R}\times\mathbb{R}_+} \psi_\varepsilon\, dq_{\overline{a}}\,. \end{aligned} \quad (3.95)$$

By Theorem 6.2.3 in [Da] we get that $v_0 = u_0 \vee \overline{a}_0 \ge u_0$ implies

$$v(x,t) \ge u(x,t) \quad \text{a.e. in} \quad \mathbb{R} \times \mathbb{R}_+$$

and thus

$$v(x,t) \vee \overline{a} \ge u(x,t) \vee \overline{a} \quad \text{a.e. in} \quad \mathbb{R} \times \mathbb{R}_+\,. \quad (3.96)$$

Therefore (3.96) implies

$$-(u \vee \overline{a} - v \vee \overline{a}) = |u \vee \overline{a} - u \vee \overline{a}|\,. \quad (3.97)$$

Applying (3.97) in (3.95) gives

$$\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\int_{|x-x_0|\le r} |u\vee\overline{a} - v\vee\overline{a}|\,dx\,dt$$
$$\frac{1}{\varepsilon}\int_{\Gamma^{r+\varepsilon}_{(x_0,t_0)}} \lambda|u\vee a - v\vee a| - [f(u\vee a) - f(v\vee a)]\operatorname{sign}(x-x_0)\,dx\,dt$$
$$= \int_{\mathbb{R}\times\mathbb{R}_+}\psi_\varepsilon\,dm_{\overline{a}} - \int_{\mathbb{R}\times\mathbb{R}_+}\psi_\varepsilon\,dq_{\overline{a}}\,. \tag{3.98}$$

By choice of λ we deduce

$$\lambda|u\vee a - v\vee a| - [f(u\vee a) - f(v\vee a)]\operatorname{sign}(x-x_0) \ge 0 \quad \text{a. e. in} \quad \mathbb{R}\times\mathbb{R}_+\,. \tag{3.99}$$

We use (3.99) in (3.98) and receive

$$\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\int_{|x-x_0|\le r} |u\vee\overline{a} - v\vee\overline{a}|\,dx\,dt$$
$$\le \int_{\mathbb{R}\times\mathbb{R}_+}\psi_\varepsilon\,dm(x,t,\overline{a}) - \int_{\mathbb{R}\times\mathbb{R}_+}\psi_\varepsilon\,dq(x,t,\overline{a})\,. \tag{3.100}$$

Letting $\varepsilon \to 0^+$ in (3.100) gives

$$\int_{|x-x_0|\le r} |u(x,t_0)\vee\overline{a} - v(x,t_0)\vee\overline{a}|\,dx \le m(\Gamma^r_{(x_0,t_0)},\overline{a}) - q(\Gamma^r_{(x_0,t_0)},\overline{a})\,. \tag{3.101}$$

Since the left-hand side of (3.101) is non-negative (3.81) follows. Integrating (3.101) with respect to $\overline{a}$ over (a,∞) for $a \ge a_0$ concludes the claim (3.81)

$$\int_{|x-x_0|\le r} |u(x,t_0)\vee\overline{a} - v(x,t_0)\vee\overline{a}|^2\,dx \le \mu(\Gamma^r_{(x_0,t_0)})\,.$$

□

After these preliminaries we are no able to show Theorem 3.3.

Proof of Theorem 3.3. Let

$$N := |\,\{u\in\mathbb{R} : f''(u) = 0\}\,|$$

and

$$a_1,\ldots,a_N \in \{u\in\mathbb{R} : f''(u) = 0\}$$

such that

$$a_1 < a_2 < \ldots < a_N .$$

Further we define

$$\rho_j(\varepsilon) = \min_{x\in[a_{j-1}+\varepsilon, a_j-\varepsilon]} |f''(s)|$$

and

$$\rho(\varepsilon) = \min_{1\le j\le N} \rho_j . \tag{3.102}$$

For $j \in \{1, \ldots, N\}$ and

$$0 < \varepsilon \le \frac{1}{4} \min_{1\le j\le N-1} |a_{j+1} - a_j| .$$

Let v_j be the entropy solution of

$$\left.\begin{array}{rcl} \partial_t v_j + \partial_x f(v_j) &=& 0 \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R} , \\ v_j(x,0) &=& u(x, t_0 - 2r) \wedge (a_j - \varepsilon) , \end{array}\right\} \tag{3.103}$$

with defect measure

$$m_j(x,t,a) = \partial_t v_j \wedge a + \partial_x f(v_j \wedge a)$$

and (x,t)-marginal

$$\mu_j = m_j(\cdot \times \mathbb{R}) .$$

Similarly for $j \in \{1, \ldots, N-1\}$ let w_j be the entropy solution of

$$\left.\begin{array}{rcl} \partial_t w_j + \partial_x f(w_j) &=& 0 \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R} , \\ w_j(x,0) &=& u(x, t_0 - 2r) \vee (a_j + \varepsilon) \wedge (a_{j+1} - \varepsilon) , \end{array}\right\} \tag{3.104}$$

with defect measure

$$q_i(x,t,a) = \partial_t w_i \wedge a + \partial_x f(w_i \wedge a)$$

and (x,t)-marginal ν_i. Furthermore let w_N be the entropy solution of

$$\left.\begin{array}{rcl} \partial_t w_N + \partial_x f(w_N) &=& 0 \quad \text{in} \quad \mathbb{R}_+ \times \mathbb{R} , \\ w_N(x,0) &=& u(x, t_0 - 2r) \vee (a_N + \varepsilon) \end{array}\right\} \tag{3.105}$$

and we denote by $q_N(x,t,a)$ its defect measure and by ν_N the (x,t)-marginal.

Claim 1. There exist constants $C > 0$ and $\delta_0 > 0$ such that for all $1 \leq j \leq N$ the entropy solution v_j of (3.103) satisfies for $(x_0, t_0) \in \mathbb{R} \times \mathbb{R}_+$

$$\frac{1}{(\delta r)^4} \int_{B_{\delta r}(x_0,2r)} |v_j(x,t) - \overline{v}_j^{\delta r}|\, dx\, dt \leq Cj \left[\frac{1}{\delta r}\mu_j(\Gamma^r_{(x_0,3r)})\right]^{\frac{1}{2}} + \frac{2Cj}{\rho(\varepsilon)^{4/3}} \left[\frac{1}{r}\mu_j(\Gamma^r_{(x_0,3r)})\right]^{\frac{1}{3}} + Cj\left(\frac{\delta}{\rho(\varepsilon)} + \varepsilon\right). \quad (3.106)$$

for all $0 < \delta < \delta_0$ and $0 < r < t_0/2$.

Proof of Claim 1. We are going to show (3.106) by induction.
Base case. For the entropy solution v_2 of (3.103) we claim that there exist constants $C > 0$ and $\delta_0 > 0$ such that for all $(x_0, t_0) \in \mathbb{R} \times \mathbb{R}_+$

$$\frac{1}{\pi(\delta r)^2} \int_{B_{\delta r}(x_0,2r)} |v_2(x,t) - \overline{v}_2^{\delta r}|\, dx\, dt \leq 2 \left[\frac{1}{\delta r}\mu_2(\Gamma^r_{(x_0,3r)})\right]^{\frac{1}{2}} + \frac{2C}{\rho(\varepsilon)^{4/3}} \left[\frac{1}{r}\mu_2(\Gamma^r_{(x_0,3r)})\right]^{\frac{1}{3}} + \frac{C\delta}{\rho} + 4\varepsilon \quad (3.107)$$

for all $0 < \delta < \delta_0$ and $0 < r < t_0/2$.

Firstly we notice for any v, $a \in \mathbb{R}$

$$v = v \wedge a_1 + v \vee a_1 - a_1\,. \quad (3.108)$$

We write

$$\mathbf{B_{\delta r}} = B_{\delta r}(x_0, 2r) \times B_{\delta r}(x_0, 2r)$$

and we deduce from (3.108)

$$\begin{aligned} \frac{1}{(\delta r)^4} \int_{\mathbf{B_{\delta r}}} &|v_2(x,t) - v_2(y,s)|\, dx\, dt\, dy\, ds \\ &\leq \frac{1}{(\delta r)^4} \int_{\mathbf{B_{\delta r}}} |v_2(x,t) \wedge a_1 - v_2(y,s) \wedge a_1|\, dx\, dt\, dy\, ds \qquad (3.109) \\ &\quad + \frac{1}{(\delta r)^4} \int_{\mathbf{B_{\delta r}}} |v_2(x,t) \vee a_1 - v_2(y,s) \vee a_1|\, dx\, dt\, dy\, ds\,. \end{aligned}$$

With

$$|v_2(x,t) \wedge a_1 - v_2(y,s) \wedge a_1| \\ \leq |v_2(x,t) \wedge (a_1 - \varepsilon) - v_2(y,s) \wedge (a_1 - \varepsilon)| + 2\varepsilon$$

and

$$|v_2(x,t) \vee a_1 - v_2(y,s) \vee a_1| \\ \leq |v_2(x,t) \vee (a_1 + \varepsilon) - v_2(y,s) \vee (a_1 + \varepsilon)| + 2\varepsilon$$

we obtain from (3.109)

$$\begin{aligned} &\frac{1}{\pi^2(\delta r)^4} \int_{\mathbf{B}_{\delta \mathbf{r}}} |v_2(x,t) - v_2(y,s)| \, dx\, dt\, dy\, ds \\ &\quad \leq \frac{1}{\pi^2(\delta r)^4} \int_{\mathbf{B}_{\delta \mathbf{r}}} |v_2(x,t) \wedge (a_1 - \varepsilon) - v_2(y,s) \wedge (a_1 - \varepsilon)| \, dx\, dt\, dy\, ds \\ &\quad + \frac{1}{\pi^2(\delta r)^4} \int_{\mathbf{B}_{\delta \mathbf{r}}} |v_2(x,t) \vee (a_1 + \varepsilon) - v_2(y,s) \vee (a_1 + \varepsilon)| \, dx\, dt\, dy\, ds \\ &\qquad + 4\varepsilon\,. \end{aligned} \tag{3.110}$$

Foremost we deal with the first term on the right-hand side of (3.110). We consider the entropy solution v_1 of (3.104) Since

$$v_1(x,0) - u(x, t_0 - 2r) \wedge (a_1 - \varepsilon) \leq a_1 - \varepsilon$$

we get by Theorem 6.2.3 in [Da] that

$$v(x,t) \leq a_1 - \varepsilon \quad \text{a.e.}$$

and therefore

$$v_1(x,t) \wedge (a_1 - \varepsilon) = v_1(x,t) \quad \text{a. e. in} \quad \mathbb{R} \times \mathbb{R}_+\,. \tag{3.111}$$

We deduce from (3.111)

$$\begin{aligned} &\frac{1}{(\delta r)^4} \int_{\mathbf{B}_{\delta \mathbf{r}}} |v_2(x,t) \wedge (a_1 - \varepsilon) - v_2(y,s) \wedge (a_1 - \varepsilon)| \, dx\, dt\, dy\, ds \\ &\quad \leq 2 \frac{1}{(\delta r)^2} \int_{B_{\delta r}(x_0, 2r)} |v_2(x,t) \wedge (a_1 - \varepsilon) - v_1(x,t) \wedge (a_1 - \varepsilon)| \, dx\, dt \\ &\qquad + \frac{1}{(\delta r)^4} \int_{\mathbf{B}_{\delta \mathbf{r}}} |v_1(x,t) - v_1(y,s)| \, dx\, dt\, dy\, ds\,. \end{aligned} \tag{3.112}$$

Since v_1 is an entropy solution and $|f''(a)| \geq \rho(\varepsilon)$ for $a \in [-\|v_1\|_\infty, \|v_1\|_\infty]$ the assumptions of Theorem 3.4 (resp. Corollary 3.2) are fulfilled and we get

$$\frac{1}{\pi^2(\delta r)^4} \int_{\mathbf{B}_{\delta \mathbf{r}}} |v_1(x,t) - v_1(y,s)|\, dx\, dt\, dy\, ds \leq \frac{C}{\rho(\varepsilon)^{4/3}} \left[\frac{1}{r}\mu_1(B_r(x_0, 2r))\right]^{\frac{1}{3}} + C\frac{\delta}{\rho(\varepsilon)} \tag{3.113}$$

for all $0 < \delta < \delta_0$. Applying (3.113) in (3.112) leads to

$$\begin{aligned} \frac{1}{(\delta r)^4} \int_{\mathbf{B}_{\delta \mathbf{r}}} &|v_2(x,t) \wedge (a_1 - \varepsilon) - v_2(y,s) \wedge (a_1 - \varepsilon)|\, dx\, dt\, dy\, ds \\ &\leq 2\frac{1}{(\delta r)^2} \int_{B_{\delta r}(x_0, 2r)} |v_2(x,t) \wedge (a_1 - \varepsilon) - v_1(x,t) \wedge (a_1 - \varepsilon)|\, dx\, dt \\ &\quad + \frac{C}{\rho(\varepsilon)^{4/3}} \left[\frac{1}{r}\mu_1(B_r(x_0, 2r))\right]^{\frac{1}{3}} + C\frac{\delta}{\rho(\varepsilon)} \end{aligned} \tag{3.114}$$

Lemma 3.2 aloud us to estimate the right-hand side of (3.114) in terms of μ_2

$$\begin{aligned} \frac{1}{(\delta r)^4} \int_{\mathbf{B}_{\delta \mathbf{r}}} &|v_2(x,t) \wedge (a_1 - \varepsilon) - v_2(y,s) \wedge (a_1 - \varepsilon)|\, dx\, dt\, dy\, ds \\ &\leq 2\left[\frac{1}{\delta r}\mu_2(\Gamma^r_{x_0,3r})\right]^{\frac{1}{2}} + \frac{C}{\rho(\varepsilon)^{4/3}} \left[\frac{1}{r}\mu_2(B_r(x_0, 2r))\right]^{\frac{1}{3}} + C\frac{\delta}{\rho(\varepsilon)} \end{aligned} \tag{3.115}$$

Next we consider the second term in (3.110). As (3.112) we deduce for w_1 instead of v_1

$$\begin{aligned} \frac{1}{(\delta r)^4} \int_{\mathbf{B}_{\delta \mathbf{r}}} &|v_2(x,t) \vee (a_1 + \varepsilon) - v_2(y,s) \vee (a_1 + \varepsilon)|\, dx\, dt\, dy\, ds \\ &\leq 2\frac{1}{(\delta r)^2} \int_{B_{\delta r}(x_0, 2r)} |v_2(x,t) \vee (a_1 + \varepsilon) - w_1(x,t) \wedge (a_1 + \varepsilon)|\, dx\, dt \\ &\quad + \frac{1}{(\delta r)^4} \int_{\mathbf{B}_{\delta \mathbf{r}}} |w_1(x,t) - w_1(y,s)|\, dx\, dt\, dy\, ds\, . \end{aligned} \tag{3.116}$$

Since w_1 is an entropy solution and $|f''(a)| \geq \rho(\varepsilon)$ for $a \in [-\|w_1\|_\infty, \|w_1\|_\infty]$ the assumptions of Theorem 3.4 (resp. Corollary 3.2) are fulfilled and therefore

$$\frac{1}{\pi^2(\delta r)^4}\int_{\mathbf{B}_{\delta\mathbf{r}}} |w_1(x,t)-w_1(y,s)|\,dx\,dt\,dy\,ds \\ \le \frac{C}{\rho(\varepsilon)^{4/3}}\left[\frac{1}{r}\nu_1(B_r(x_0,2r))\right]^{\frac{1}{3}} + C\frac{\delta}{\rho(\varepsilon)} \quad (3.117)$$

Then we use (3.113) and Lemma 3.3 in (3.116) and achieve similar to (3.115)

$$\frac{1}{(\delta r)^4}\int_{\mathbf{B}_{\delta\mathbf{r}}} |v_2(x,t)\vee(a_1+\varepsilon)-v_2(y,s)\vee(a_1+\varepsilon)|\,dx\,dt\,dy\,ds \\ \le 2\left[\frac{1}{\delta r}\mu_2(\Gamma^r_{x_0,3r})\right]^{\frac{1}{2}} + \frac{C}{\rho(\varepsilon)^{4/3}}\left[\frac{1}{r}\mu_2(B_r(x_0,2r))\right]^{\frac{1}{3}} + C\frac{\delta}{\rho(\varepsilon)} \quad (3.118)$$

Bringing (3.115) and (3.118) together in (3.110) concludes the base step.

For the inductive step we assume

$$\frac{1}{(\delta r)^4}\int_{\mathbf{B}_{\delta\mathbf{r}}} |v_{j-1}(x,t)-v_{j-1}(y,s)|\,dx\,dt\,dy\,ds \\ \le C(j-1)\left[\frac{1}{\delta r}\mu_{j-1}(\Gamma^r_{(x_0,3r)})\right]^{\frac{1}{2}} + \frac{C(j-1)}{\rho(\varepsilon)^{4/3}}\left[\frac{1}{r}\mu_{j-1}(\Gamma^r_{(x_0,3r)})\right]^{\frac{1}{3}} \\ + C(j-1)\left(\frac{\delta}{\rho(\varepsilon)}+\varepsilon\right). \quad (3.119)$$

We compute for the entropy solution v_j of (3.103)

$$\frac{1}{\pi^2(\delta r)^4}\int_{\mathbf{B}_{\delta\mathbf{r}}} |v_j(x,t)-v_j(y,s)|\,dx\,dt\,dy\,ds \\ \le \frac{1}{\pi^2(\delta r)^4}\int_{\mathbf{B}_{\delta\mathbf{r}}} |v_j(x,t)\wedge(a_{j-1}-\varepsilon)-v_j(y,s)\wedge(a_{j-1}-\varepsilon)|\,dx\,dt\,dy\,ds \\ + \frac{1}{\pi^2(\delta r)^4}\int_{\mathbf{B}_{\delta\mathbf{r}}} |v_j(x,t)\vee(a_{j-1}+\varepsilon)-v_j(y,s)\vee(a_{j-1}+\varepsilon)|\,dx\,dt\,dy\,ds \\ + 4\varepsilon. \quad (3.120)$$

The first term on the right-hand side of (3.120) can be estimated by

$$\begin{aligned}\frac{1}{(\delta r)^4}\int_{\mathbf{B_{\delta r}}} &|v_j(x,t)\wedge(a_{j-1}-\varepsilon)-v_j(y,s)\wedge(a_{j-1}-\varepsilon)|\,dx\,dt\,dy\,ds\\ &\leq 2\frac{1}{(\delta r)^2}\int_{B_{\delta r}(x_0,2r)}|v_j(x,t)\wedge(a_{j-1}-\varepsilon)-v_{j-1}(x,t)\wedge(a_{j-1}-\varepsilon)|\,dx\,dt\\ &\qquad+\frac{1}{(\delta r)^4}\int_{\mathbf{B_{\delta r}}}|v_{j-1}(x,t)-v_{j-1}(y,s)|\,dx\,dt\,dy\,ds\,. \qquad (3.121)\end{aligned}$$

Then we use Lemma 3.2 for the first term on the right-hand side of (3.121) and we use assumption (3.119) for the second terms

$$\begin{aligned}\frac{1}{(\delta r)^4}\int_{\mathbf{B_{\delta r}}} &|v_j(x,t)\wedge(a_{j-1}-\varepsilon)-v_j(y,s)\wedge(a_{j-1}-\varepsilon)|\,dx\,dt\,dy\,ds\\ &\leq Cj\left[\frac{1}{\delta r}\mu_j(\Gamma^r_{(x_0,3r)})\right]^{\frac{1}{2}}+\frac{Cj}{\rho(\varepsilon)^{4/3}}\left[\frac{1}{r}\mu_j(\Gamma^r_{(x_0,3r)})\right]^{\frac{1}{3}}\\ &\qquad+Cj\left(\frac{\delta}{\rho(\varepsilon)}+\varepsilon\right)\,. \qquad (3.122)\end{aligned}$$

For the second term in (3.120) we can argue step by step as in (3.116). Thus Claim 1 follows.

We want now to conclude the proof of Theorem 3.3. Let $\tilde{u}$ be the entropy solution of

$$\left.\begin{aligned}\partial_t\tilde{u}+\partial_x f(\tilde{u}) &= 0 \qquad \text{in } \mathbb{R}_+\times\mathbb{R}\,,\\ \tilde{u}(x,0) &= u(x,t_0-2r)\,,\end{aligned}\right\} \qquad (3.123)$$

with defect measure $\tilde{m}(x,t,a)=\partial_t\tilde{u}\wedge a+\partial_x f(\tilde{u}\wedge a)$. With Claim 1 for $j=N$ we conclude in exact the same as in the proof of (3.106):

$$\begin{aligned}\frac{1}{\pi^2(\delta r)^4}\int_{B_{\delta r}(x_0,2r)\times B_{\delta r}(x_0,2r)} &|\tilde{u}(x,t)-\tilde{u}(y,s)|\,dx\,dt\,dy\,ds\\ &\leq C\left[\frac{1}{\delta r}\tilde{\mu}(\Gamma^r_{(x_0,3r)})\right]^{\frac{1}{2}}+\frac{C}{\rho(\varepsilon)^{4/3}}\left[\frac{1}{r}\tilde{\mu}(\Gamma^r_{(x_0,3r)})\right]^{\frac{1}{3}}\\ &\qquad+C\left(\frac{\delta}{\rho(\varepsilon)}+\varepsilon\right)\,. \qquad (3.124)\end{aligned}$$

Since $\tilde{u}(x,t) = u(x, t + t_0 - 2r)$ (3.124) gives for a $c > 0$

$$\frac{1}{\pi(\delta r)^2}\int_{B_{\delta r}(x_0,t_0)} |u(x,t) - \overline{u}^{\delta r}|\,dx\,dt$$
$$\leq C\left[\frac{1}{\delta r}\mu(B_{cr}(x_0,t_0))\right]^{\frac{1}{2}} + \frac{C}{\rho(\varepsilon)^{4/3}}\left[\frac{1}{r}\mu(B_{cr}(x_0,t_0))\right]^{\frac{1}{3}} + C\left(\frac{\delta}{\rho(\varepsilon)} + \varepsilon\right) \tag{3.125}$$

and therefore Theorem 3.3 follows. □

3.3 Applications of Theorem 3.3

Our goal in this section is to deduce b) and c) of the following proposition from Theorem 3.3.

Proposition 3.1. *Let $f \in C^2(\mathbb{R})$ such that $|\{u \in \mathbb{R} : f'(u) = 0\}| < \infty$ and for a $p > 0$ $\rho(\varepsilon) = \mathcal{O}(\varepsilon^p)$ for ρ defined as in (3.62). We consider an entropy solution $u \in L^1 \cap L^\infty(\mathbb{R})$ of (3.1). Then there exists a rectifiable set J_u and a $\mathcal{H}^1$-dim. set $V_0 \subset \mathbb{R} \times \mathbb{R}_+ \backslash J_u$ such that*

a) u has approx. jump-points on J_u,

b) u is approximate continuous on $\mathbb{R} \times \mathbb{R}_+ \backslash (J_u \cup V_0)$ and $\mu(V_0) = 0$,

c) for $B_R(x_0,t_0)$ such that $\mu(B_R(x_0,t_0)) = 0$ one has

$$u \in C^{0,1/(2+p)}(B_{R/2}(x_0,t_0))\,.$$

Before we show Proposition 3.1 we prove a reversed estimate as in Theorem 3.3, i.e. the entropy production is controlled by the oscillation

Proposition 3.2. *For f Lipschitz, let u be a solution of (3.1) with initial condition $u_0 \in L^\infty$. Then there exist constants C, $c > 0$ depending on u_0 and f such that for any $(x_0,t_0) \in \mathbb{R} \times \mathbb{R}$ and $0 < r < t_0/2$ the following estimate holds*

$$\mu(B_{cr}(x_0,t_0)) \leq C\int_{B_r(x_0,t_0)} |u(x,t) - \overline{u}^r|\,dx\,dt\,. \tag{3.126}$$

Proof of Proposition 3.2. Let

$$\lambda := \sup_{|\alpha| \leq \|u_0\|_\infty} |f'(\alpha)|\,.$$

For (x_0, t_0) and $0 < r < t_0/(2\lambda)$ and $a_0 \in \mathbb{R}$ let

$$\tilde{u}_0(x) = \begin{cases} u(x, t_0 - r/2) & \text{if} \quad |x - x_0| \le \lambda r\,, \\ a_0 & \text{else.} \end{cases} \tag{3.127}$$

Then let $\tilde{u}$ be the entropy solution of

$$\left.\begin{array}{rcll} \partial_t \tilde{u} + \partial_x f(\tilde{u}) & = & 0 & \text{in} \quad \mathbb{R} \times \mathbb{R}_+\,, \\ \tilde{u}(x,0) & = & \tilde{u}_0 & \end{array}\right\} \tag{3.128}$$

with defect measure $\tilde{m}(x,t,a)$. We notice that by the contraction principle (see Theorem 6.2.3 in [Da]) we have on

$$\Gamma := \{(x,t) :\ 0 \le t \le r\,,\ |x - x_0| \le \lambda(r - t)\}$$

that

$$\tilde{u}(x,t) = u(x, t + t_0 - r/2)\,. \tag{3.129}$$

As in the proof of Lemma 3.3 we consider the regularized kinetic formulation for the solution $\tilde{u}$

$$\partial_t \chi_\varepsilon(x,t,a) + f'(a)\partial_x \chi_\varepsilon(x,t,a) = \partial_a \tilde{m}_\varepsilon(x,t,a) \tag{3.130}$$

Since one has $\tilde{u}(x,t) = 0$ for $|x - x_0| \ge r + \lambda t$, we get

$$\chi_\varepsilon(x,t,a) = \mathbb{1}_{a \le a_0} \quad \text{for} \quad |x - x_0| \ge \lambda(r+t) + \varepsilon\,. \tag{3.131}$$

For $a_0 \in \mathbb{R}$ we consider $S'(a) = 2(a - a_0)$. Multiplying (3.130) with $S'(a)$ and integrating in $\mathbb{R} \times [0,r]$ gives with (3.131)

$$\int_{\mathbb{R}} S'(a)\chi_\varepsilon(x,r,a) - S'(a)\chi_\varepsilon(x,0,a)\,dx = \int_{\mathbb{R}\times[0,r]} S'(a)\partial_a \tilde{m}_\varepsilon(x,t,a)\,dx\,dt\,. \tag{3.132}$$

Integrating (3.132) also with respect to a over $\mathbb{R}$ leads to

$$\int_{\mathbb{R}} (\tilde{u} - a_0)^2 * \varphi_\varepsilon(x,r) - (\tilde{u} - a_0)^2 * \varphi_\varepsilon(x,0)\,dx = \int_{\mathbb{R}\times[0,r]\times\mathbb{R}} S'(a)\partial_a \tilde{m}_\varepsilon(x,t,a)\,dx\,dt\,da\,. \tag{3.133}$$

We integrate the right-hand side of (3.133) by parts

$$\int_{\mathbb{R}} (\tilde{u} - a_0)^2 * \varphi_\varepsilon(x, r) - (\tilde{u} - a_0)^2 * \varphi_\varepsilon(x, 0)\, dx$$
$$= -2 \int_{\mathbb{R}\times[0,r/\lambda]\times\mathbb{R}} \tilde{m}_\varepsilon(x, t, a)\, dx\, dt\, da\,. \quad (3.134)$$

From (3.134) we get the inequality

$$\int_{\mathbb{R}\times[0,r]\times\mathbb{R}} \tilde{m}_\varepsilon(x, t, a)\, dx\, dt\, da \le 2 \int_{\mathbb{R}} (\tilde{u} - a_0)^2 * \varphi_\varepsilon(x, 0)\, dx \quad (3.135)$$

We let $\varepsilon \to 0^+$ in (3.134) and obtain

$$\tilde{\mu}(\mathbb{R} \times [0, r]) \le \int_{\mathbb{R}} |\tilde{u}(x, 0) - a_0|^2\, dx\,. \quad (3.136)$$

We recall the definition of $\tilde{u}$ (3.127) and obtain from (3.136)

$$\tilde{\mu}(\mathbb{R} \times [0, r]) \le \int_{|x-x_0|\le\lambda r} |u(x, t_0 - r/2) - a_0|^2\, dx\,. \quad (3.137)$$

Therefore

$$\tilde{\mu}(\mathbb{R} \times [0, r]) \le (\|u\|_\infty + a_0) \int_{|x-x_0|\le\lambda r} |u(x, t_0 - r/2) - a_0|\, dx\,. \quad (3.138)$$

We observe that (3.129) implies

$$\tilde{\mu}(\mathbb{R} \times [0, r]) \ge \tilde{\mu}(\Gamma) = \mu(\Gamma)\,. \quad (3.139)$$

Thus from (3.138) and (3.139) we conclude

$$\mu(\Gamma) \le (\|u\|_\infty + a_0) \int_{|x-x_0|\le\lambda r} |u(x, t_0 - r/2) - a_0|\, dx\,. \quad (3.140)$$

By the contraction principle for scalar conservation laws (see Theorem 6.2.3 in [Da]) we get

$$\int_{|x-x_0|\le\lambda r} |u(x, t_0 - r/2) - a_0|\, dx \le \int_{|x-x_0|\le\lambda(r+s} |u(x, t_0 - r/2 - s) - a_0|\, dx$$

for all $0 < s \le t_0 - r/2$ and thus

$$\frac{1}{\lambda r} \int_{|x-x_0|\le\lambda r} |u(x, t_0 - r/2) - a_0|\, dx$$
$$\le \frac{2}{\lambda r^2} \int_0^{r/2} \int_{|x-x_0|\le\lambda(r+s)} |u(x, t_0 - r/2 - s) - a_0|\, dx\, ds\,. \quad (3.141)$$

We apply (3.141) in (3.140) and receive

$$\frac{1}{r}\mu(\Gamma) \leq \frac{2}{\lambda r^2}\int_0^{r/2}\int_{|x-x_0|\leq\lambda(r+s)} |u(x,t_0-r/2-s)-a_0|\,dx\,ds\,. \qquad (3.142)$$

We can find a $\tilde{c} \in \mathbb{R}$, depending on λ such that $B_{\tilde{c}r}(x_0,t_0) \subset \Gamma$. Hence we get from (3.142)

$$\frac{1}{r}\mu(B_{\tilde{c}r}(x_0,t_0)) \leq (\|u\|_\infty + a_0)\frac{C}{r^2}\int_{B_{3/2\lambda r}(x_0,t_0)} |u(x,t)-a_0|\,dx\,dt\,. \qquad (3.143)$$

We put $a_0 = \overline{u}^{3/2\lambda r}$ in (3.143) and our claim (3.126) follows. □

We prove now Proposition 3.1:

Proof of Proposition 3.1. First we define the set

$$J_u := \left\{(x,t) \in \mathbb{R}\times\mathbb{R}_+ :\ \limsup_{r\to 0^+} r^{-1}\mu(B_r(x,t)) > 0\right\}. \qquad (3.144)$$

By Theorem 2.4 in [DOW] we know that J_u is rectifiable and also that a) is satisfied. In a next step our aim is to show b). For $\alpha \in (0,1)$ we define the set

$$V_\alpha = \left\{(x,t) \in \mathbb{R}\times\mathbb{R}_+\backslash J_u : \alpha = \sup_{\gamma\in(0,1)}\{\limsup_{r\to 0^+} r^{-1-\gamma}\mu(B_r(x,t)) < \infty\}\right\}. \qquad (3.145)$$

By construction we get, that

$$\mathcal{H}^{1+\alpha}(V_\alpha) < \infty\,. \qquad (3.146)$$

Choosing $\varepsilon > 0$ such that $\rho(\varepsilon) = \delta^{\frac{1}{2}}$ we get from Theorem 3.3

$$\frac{1}{(\delta r)^2}\int_{B_{\delta r}(x,t)} |u(y,s)-\overline{u}^{\delta r}|\,dy\,ds \leq \frac{C}{\delta^{\frac{2}{3}}}\left[\frac{1}{r}\mu(B_r(x,t))\right]^{\frac{1}{3}} + C\delta^{\frac{1}{2}} \qquad (3.147)$$

For $(x,t) \in V_\alpha$ we choose $\delta = r^{\alpha/2}$ and $\varepsilon = r^{\alpha/(8p)}$ and obtain from Theorem 3.3

$$\frac{1}{r^{2(1+\alpha/2)}}\int_{B_{r^{1+\alpha/2}}(x,t)} |u(y,s)-\overline{u}^{r^{1+\alpha/2}}|\,dy\,ds \leq Cr^{\gamma\alpha}\,, \qquad (3.148)$$

where

$$\gamma := \min\{1/6, 1/(8p)\}\,.$$

Thus after rescaling we deduce from (3.148)

$$\frac{1}{r^2}\int_{B_r(x,t)} |u(y,s)-\overline{u}^r|\,dy\,ds \le Cr^\theta\,, \tag{3.149}$$

where

$$\theta = 2\gamma\alpha/(2+\alpha)\,. \tag{3.150}$$

We claim now

Claim 1. For $(x_0,t_0)\in V_\alpha$ and $0<r<R$ we have

$$|\overline{u}^R-\overline{u}^r| \le CR^\theta\,, \tag{3.151}$$

where θ is defined in (3.150).

Proof of Claim 1. for $R>0$ we set $R_k = 2^{-k}R$. Then we obtain from (3.149)

$$\begin{aligned} |\overline{u}^R-\overline{u}^{R_k}| &\le \sum_{j=0}^{k-1}|\overline{u}^{R_j}-\overline{u}^{R_{j+1}}| \\ &\le \sum_{j=0}^{k-1}\frac{4}{R_j^2}\int_{B_{R_j}(x_0,t_0)}|u(y,s)-\overline{u}^{R_j}|\,dy\,ds \\ &\le 4R^\theta\sum_{j=0}^{k-1}2^{-j\theta} \le CR^\theta\,, \end{aligned} \tag{3.152}$$

and the claim follows.

From Claim 1 one easily gets that for any $(x,t)\in V_\alpha$ and $(r_n)_{n\in\mathbb{N}}$ such that $r_n\to 0^+$ the sequence $\overline{u}^{r_n}$ is a Cauchy-sequence and therefore converges. Hence u is approximate continuous for $(x,t)\in V_a$ and it remains to consider points (x,t) such that

$$(x,t)\in V_0 = \mathbb{R}\times\mathbb{R}_+\backslash(J_u\cup\bigcup_{0<\alpha<1}V_\alpha)\,.$$

This set can be characterized as

$$V_0 = \left\{(x,t)\in\mathbb{R}\times\mathbb{R}_+ :\ 0 = \sup_{\gamma\in(0,1)}\{\limsup_{r\to 0^+} r^{-1-\gamma}\mu(B_r(x,t))<\infty\}\right\} \tag{3.153}$$

From (3.153) we deduce that $\mathcal{H}^1(V_0) < \infty$. Therefore we have to show $\mu(V_0) = 0$ to complete the proof b). Let $\varepsilon > 0$. For $k \in \mathbb{N}$ we define the set

$$E_k = \left\{ (x,t) \in V_0 : \frac{c^2}{\pi r^2} \int_{B_{rc}(x_0,t_0)} |u(y,s) - \overline{u}^r| \, dy \, ds \le \varepsilon \quad \forall r \in (0, 1/k) \right\} . \tag{3.154}$$

For every $(x,t) \in E_k$ we have by Proposition 3.2

$$\mu(B_r(x,t)) \le \varepsilon r \quad \forall 0 < r < 1/k \, . \tag{3.155}$$

Let $(C_i)_{i \in \mathbb{N}}$ be a covering of E_k such that $\operatorname{diam} C_i =: r_i < 1/k$ and $C_i \cap E_k$ contains at least one point (x_i, t_i) and

$$\sum_i r_i \le \mathcal{H}^1_{1/k}(E_k) + \frac{1}{k} \, . \tag{3.156}$$

Then $B_{2r_i}(x_i, t_i)$ is still a covering of E_k and we get together with (3.155) and (3.156)

$$\mu(E_k) \le \sum_i \mu(B_{2r_i}(x_i,t_i)) \le \varepsilon \sum_i 2r_i \le 2\varepsilon \left(\mathcal{H}^1(V_0) + \frac{1}{k} \right) \, . \tag{3.157}$$

Since E_k is an increasing sequence and its union is V_0 we get by letting $k \to \infty$ in (3.157)

$$\mu(V_0) \le 2\varepsilon \mathcal{H}^1(E) \, .$$

Since ε was arbitrary we get

$$\mu(V_0) = 0$$

as claimed and therefore b) is shown.

Let $B_R(x_0,t_0)$ be an open set such that $\mu(B_R(x_0,t_0)) = 0$, then for every $(x,t) \in B_R(x_0,t_0)$ and $r > 0$ such that $B_r(x,t) \subset B_R(x_0,t_0)$ we obtain from Theorem 3.3

$$\frac{1}{(\delta r)^2} \int_{B_{\delta r}(x,t)} |u(y,s) - \overline{u}^{\delta r}| \, dy \, ds \le C(\frac{\delta}{\varepsilon^p} + \varepsilon) \, . \tag{3.158}$$

From (3.158) we obtain for suitable choices of δ and ε that for all $(x,t) \in B_R(x_0,t_0)$ and $r > 0$ such that $B_r(x,t) \subset B_R(x_0,t_0)$ we have

$$\frac{1}{r^2} \int_{B_r(x,t)} |u(y,s) - \overline{u}^r| \, dy \, ds \le C r^{2+p} \, . \tag{3.159}$$

Similar as Claim 1 we deduce for any $(x,t) \in B_R(x_0,t_0)$ and $0 < r < r_1 < \mathrm{diam}((x,t), \partial B_R(x_0,t_0))$

$$|\overline{u}^{r}_{(x,t)} - \overline{u}^{r_1}_{(x,t)}| \leq CR^{1/(2+p)} \tag{3.160}$$

Especially we obtain by letting $r \to 0^+$ in (3.160)

$$|\overline{u}^{r_1}_{(x,t)} - u(x,t)| \leq Cr_1^{1/(2+p)}\,. \tag{3.161}$$

Let (x,t), $(y,s) \in B_{R/2}(x_0,t_0)$, then for $r_1 = |(x,t)-(y,s)|$ it follows with (3.161)

$$\begin{aligned} |u(x,t)-u(y,s)| &\leq |u(x,t) - \overline{u}^{2r_1}_{(x,t)}| + |u(y,s) - \overline{u}^{2r_1}_{(y,s)}| + |\overline{u}^{2r_1}_{(x,t)} - \overline{u}^{2r_1}_{(y,s)}| \\ &\leq Cr_1^{1/(2+p)} = |(x,t)-(y,s)|^{1/(2+p)} \end{aligned} \tag{3.162}$$

and therefore our claim follows. □

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