NEW TOPOLOGICAL RECURSION RELATIONS

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ABSTRACT. Simple boundary expressions for the k^{th} power of the cotangent line class ψ_1 on $\overline{M}_{g,1}$ are found for $k \geq 2g$. The method is by virtual localization on the moduli space of maps to $\mathbb{P}^1$. As a consequence, nontrivial tautological classes in the kernel of the boundary push-forward map

$$\iota_* : A^*(\overline{M}_{g,2}) \to A^*(\overline{M}_{g+1})$$

are constructed. The geometry of genus $g+1$ curves then provides universal equations in genus g Gromov-Witten theory. As an application, we prove all the Gromov-Witten identities conjectured recently by K. Liu and H. Xu.

0. INTRODUCTION

0.1. **Tautological classes.** Let $\overline{M}_{g,n}$ be the moduli space of stable curves of genus g with n marked points. Let $A^*(\overline{M}_{g,n})$ denote the Chow ring with $\mathbb{Q}$-coefficients. The system of tautological rings is defined in [5] to be the set of smallest $\mathbb{Q}$-subalgebras of the Chow rings,

$$R^*(\overline{M}_{g,n}) \subset A^*(\overline{M}_{g,n}),$$

satisfying the following two properties:

(i) The system is closed under push-forward via all maps forgetting markings:

$$\pi_* : R^*(\overline{M}_{g,n}) \to R^*(\overline{M}_{g,n-1}).$$

(ii) The system is closed under push-forward via all gluing maps:

$$\iota_* : R^*(\overline{M}_{g_1,n_1\cup\{*\}}) \otimes_{\mathbb{Q}} R^*(\overline{M}_{g_2,n_2\cup\{\bullet\}}) \to R^*(\overline{M}_{g_1+g_2,n_1+n_2}),$$

$$\iota_* : R^*(\overline{M}_{g,n\cup\{*,\bullet\}}) \to R^*(\overline{M}_{g+1,n}),$$

with attachments along the markings $*$ and $\bullet$.

Natural algebraic constructions typically yield Chow classes lying in the tautological ring. For example, the standard ψ, κ, and λ classes in $A^*(\overline{M}_{g,n})$ all lie in the tautological ring. The tautological rings also possess a rich conjectural structure, see [4] for a detailed discussion.

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The moduli space $\overline{M}_{g,n}$ admits a stratification by topological type indexed by decorated graphs. The normalized stratum closures are simply quotients of products of simpler moduli spaces of pointed curves. A *descendent stratum class* in $R^*(\overline{M}_{g,n})$ is a push-forward from a stratum S of a monomial in the cotangent line classes of the special points[1] of S.

A relation in $R^*(\overline{M}_{g,n})$ among descendent stratum classes yields a universal genus g equation[2] in Gromov-Witten theory by the splitting axiom. For example, the equivalence of boundary strata in $\overline{M}_{0,4}$ implies the WDVV equation. Several other relations have since been found [2, 7, 8, 12].

Let $g \geq 1$. Boundary expressions for powers $\psi_1^k \in R^*(\overline{M}_{g,1})$ of the cotangent line class are the most basic *topological recursion relations*. For $k \geq g$, boundary expressions for ψ_1^k have been proved to exist [5, 10]. While the arguments are constructive, the method in practice is very difficult. The answers for $k = g$ appear, for low g, to be rather complicated. [3]

The results of the paper concern simple boundary expressions for ψ_1^k for $k \geq 2g$. The relations have two interesting consequences. The first is the construction of nontrivial classes in the kernel of the boundary push-forward map

$$\iota_* : A^*(\overline{M}_{g,2}) \to A^*(\overline{M}_{g+1}).$$

By the splitting axioms of Gromov-Witten theory in genus $g+1$, we obtain universal equations in genus g from linear combinations of descendent stratum classes in the kernel of ι_*. The possibility for such Gromov-Witten equations was anticipated earlier in discussions with Faber, but a nontrivial example was not found. The existence of such nontrivial equations now opens the door to new possibilities. Are there equations in Gromov-Witten theory in genus g obtained by boundary embeddings in even higher genera? Are there new equations[4] waiting to be found in genus 0 and 1?

The second consequence of our new topological recursion relations is a proof of the Gromov-Witten conjectures of K. Liu and H. Xu [13]. The conjectures are universal relations in Gromov-Witten theory related to high powers of the cotangent line classes. We prove all the conjectures made there.

0.2. **Topological recursion.** Let $g \geq 1$. Let $L_1 \to \overline{M}_{g,1}$ be the cotangent line bundle with fiber $T^*_{p_1}(C)$ at the moduli point $[C, p_1] \in \overline{M}_{g,1}$. Let

$$\psi_1 = c_1(L_1) \in A^1(\overline{M}_{g,1})$$

be the cotangent line class. For a genus splitting $g_1 + g_2 = g$, let

$$\iota : \Delta_{1,\emptyset}(g_1, g_2) \stackrel{\sim}{=} \overline{M}_{g_1,2} \times \overline{M}_{g_2,1} \to \overline{M}_{g,1}$$

denote the boundary divisor parametrizing reducible curves

$$C = C_1 \cup C_2$$

[1]The special points correspond to the n markings and the singularities of curves parametrized by the stratum.

[2]A genus g equation is allowed to involve all genera up to g.

[3]Boundary relations in codimension g for certain linear combinations of Hodge classes appear in [1].

[4]The Gromov-Witten equations obtained from relations in $R^*(\overline{M}_{0,n})$ are known by Keel's study [11]. Getzler has claimed complete knowledge of relations in $R^*(\overline{M}_{1,n})$.

satisfying $g(C_i) = g_i$ with a single meeting point,

$$C_1 \cap C_2 = p_\star,$$

and marking $p_1 \in C_1$. Let

$$\psi_{\star_1}, \psi_{\star_2} \in A^1\big(\Delta_{1,\emptyset}(g_1, g_2)\big)$$

denote the cotangent line classes at the point $p_\star$. Here, $\psi_{\star_1}$ is the cotangent line along C_1 and $\psi_{\star_2}$ is the cotangent line along C_2.

Theorem 1. *For $g \geq 1$ and $r \geq 0$,*

$$\psi_1^{2g+r} = \sum_{g_1+g_2=g,\ g_i>0} \ \sum_{a+b=2g-1+r} (-1)^a \ \frac{g_2}{g} \cdot \iota_*\Big(\psi_{\star_1}^a \psi_{\star_2}^b \cap [\Delta_{1,\emptyset}(g_1, g_2)]\Big)$$

in $A^{2g+r}(\overline{M}_{g,1})$.

For $r > g-2$, both sides of the above relation vanish for dimension reasons. Theorem 1 is nontrivial only if $g \geq 2$ and $0 \leq r \leq g-2$. On the right side of the relation, the marking 1 carries no cotangent line classes.

Theorem 1 and several similar relations are proved in Sections 1.2-1.3 using the virtual geometry of the moduli space of stable maps $\overline{M}_{g,n}(\mathbb{P}^1, 1)$. Special intersections against the virtual class $[\overline{M}_{g,n}(\mathbb{P}^1, 1)]^{vir}$ of the moduli space, known to vanish for geometric reasons, are evaluated via virtual localization [9] and pushed-forward to $\overline{M}_{g,n}$ to obtain relations. The technique was first used in [3].

0.3. **Consequences.** Let $g \geq 1$ and $r \geq 1$. Consider the class

$$\xi_{g,r} = \sum_{a+b=2g+r} (-1)^a \psi_1^a \psi_2^b \in A^{2g+r}(\overline{M}_{g,2}). \tag{1}$$

Let $\iota : \overline{M}_{g,2} \to \overline{M}_{g+1}$ be the irreducible boundary map. As a corollary of the new topological recursion relations, we prove the following result in Section 1.4.

Theorem 2. *For $g \geq 1$ and $r \geq 1$, $\iota_*(\xi_{g,r}) = 0 \in A^{2g+r+1}(\overline{M}_{g+1})$.*

For r odd, the push-forward $\iota_*(\xi_{g,r})$ is easily seen to vanish by the antisymmetry of the sum (1). We view the class $\xi_{g,r}$ as an uninteresting element of the kernel of

$$\iota_* : R^*(\overline{M}_{g,2}) \to R^*(\overline{M}_{g+1}).$$

The universal Gromov-Witten relation obtained from $\iota_*(\xi_{g,r}) = 0$ is trivial in the r odd case.

The r even case is much more subtle. Here, $\xi_{g,r}$ is a remarkable element. For $r \leq g-2$,

$$\xi_{g,r} \neq 0 \in A^*(\overline{M}_{g,2})$$

since we can compute

$$\int_{\overline{M}_{g,2}} \xi_{g,r} \cdot \psi_2^{g-2-r} \cap [\Delta_{1,2}(1, g-1)] = \int_{\overline{M}_{1,2}} \psi_1^2 \cdot \int_{\overline{M}_{g-1,2}} \psi_2^{3g-4} = \frac{1}{24} \cdot \frac{1}{24^{g-1}(g-1)!}.$$

The vanishing of $\iota_*(\xi_{g,r})$ is nontrivial — not a consequence of any elementary symmetry. Hence, the associated Gromov-Witten relation is also nontrivial.

0.4. **Gromov-Witten theory.** Let X be a nonsingular projective variety over $\mathbb{C}$ of dimension d. Let $\{\gamma_\ell\}$ be a basis of $H^*(X,\mathbb{C})$ with Poincaré dual classes $\{\gamma^\ell\}$. The descendent Gromov-Witten invariants of X are

$$\left\langle \tau_{k_1}(\gamma_{\ell_1})\dots\tau_{k_n}(\gamma_{\ell_n})\right\rangle_{g,\beta}^X = \int_{[\overline{M}_{g,n}(X,\beta)]^{vir}} \psi_1^{k_1}\cup \mathrm{ev}_1^*(\gamma_{\ell_1})\cdots\psi_n^{k_n}\cup \mathrm{ev}_n^*(\gamma_{\ell_n})$$

where ψ_i are the cotangent line classes and

$$\mathrm{ev}_i : \overline{M}_{g,n}(X,\beta)\to X$$

are the evaluation maps associated to the markings.

Let $\{t_k^\ell\}$ be a set of variables. Let F_g^X be the generating function of the genus g descendent invariants,

$$F_g^X = \sum_{\beta\in H_2(X,\mathbb{Z})} q^\beta \sum_{n\geq 0}\frac{1}{n!}\sum_{\substack{\ell_1\dots\ell_n\\ k_1\dots k_n}} t_{k_n}^{\ell_n}\dots t_{k_1}^{\ell_1}\,\langle\tau_{k_1}(\gamma_{\ell_1})\dots\tau_{k_n}(\gamma_{\ell_n})\rangle_{g,\beta}^X.$$

Double brackets denote differentiation,

$$\left\langle\left\langle \tau_{k_1}(\gamma_{\ell_1})\dots\tau_{k_n}(\gamma_{\ell_n})\right\rangle\right\rangle_g^X = \frac{\partial}{\partial t_{k_1}^{\ell_1}}\cdots\frac{\partial}{\partial t_{k_n}^{\ell_n}}\, F_g^X.$$

The Gromov-Witten equation obtained from Theorem 2 is the following result (trivial unless r is even) conjectured by K. Liu and H. Xu.

Theorem 3. *For $g\geq 0$ and $r\geq 1$,*

$$\sum_{a+b=2g+r}\ \sum_{\ell} (-1)^a\left\langle\left\langle\tau_a(\gamma_\ell)\tau_b(\gamma^\ell)\right\rangle\right\rangle_g^X = 0\ .$$

Theorem 3 and several related Gromov-Witten equations conjectured by Liu-Xu are proved in Section 2. Proofs in case $g\leq 2$ or $r>g-2$ were obtained earlier in [16].

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1. Localization relations

1.1. $\mathbb{C}^*$**-action.** Let t be the generator of the $\mathbb{C}^*$-equivariant ring of a point,

$$A_{\mathbb{C}^*}^*(\bullet)=\mathbb{C}[t].$$

Let $\mathbb{C}^*$ act on $\mathbb{P}^1$ with tangent weights $t,-t$ at the fixed points $0,\infty\in\mathbb{P}^1$ respectively. There is an induced $\mathbb{C}^*$-action on the moduli space of maps $\overline{M}_{g,n}(\mathbb{P}^1,1)$. A $\mathbb{C}^*$-equivariant virtual class

$$[\overline{M}_{g,n}(\mathbb{P}^1,1)]^{vir}\in A_{2g+n}^{\mathbb{C}^*}(\overline{M}_{g,n}(\mathbb{P}^1,1))$$

is obtained. The $\mathbb{C}^*$-equivariant evaluation maps

$$\mathrm{ev}_i : \overline{M}_{g,n}(\mathbb{P}^1,1)\to\mathbb{P}^1$$

determine $\mathbb{C}^*$-equivariant classes

$$\mathrm{ev}_i^*([0]), \mathrm{ev}_i^*([\infty]) \in A^1_{\mathbb{C}^*}\big(\overline{M}_{g,n}(\mathbb{P}^1,1)\big).$$

Denote the $\mathbb{C}^*$-equivariant universal curve and universal map by

$$\pi : U \to \overline{M}_{g,n}(\mathbb{P}^1,1), \quad f : U \to \mathbb{P}^1.$$

There is a unique lifting of the $\mathbb{C}^*$-action to

$$\mathcal{O}_{\mathbb{P}^1}(-2) \to \mathbb{P}^1$$

with fiber weights to be $-t, t$ over the fixed points $0, \infty \in \mathbb{P}^1$ respectively. Let

$$B = R^1\pi_* f^*\big(\mathcal{O}_{\mathbb{P}^1}(-2)\big) \to \overline{M}_{g,n}(\mathbb{P}^1,1).$$

The sheaf B is $\mathbb{C}^*$-equivariant and locally free of rank $g+1$. Let

$$c_g(B) \in A^g_{\mathbb{C}^*}\big(\overline{M}_{g,n}(\mathbb{P}^1,1)\big)$$

be the g^{th} Chern class.

A branch morphism for stable maps to $\mathbb{P}^1$ has been defined in [6],

$$\mathrm{br} : \overline{M}_{g,n}(\mathbb{P}^1,1) \to \mathrm{Sym}^{2g}(\mathbb{P}^1).$$

The branch morphism is $\mathbb{C}^*$-equivariant. Let $H_0 \subset \mathrm{Sym}^{2g}(\mathbb{P}^1)$ denote the hyperplane of $2g$-tuples incident to $0 \in \mathbb{P}^1$. Since H_0 is $\mathbb{C}^*$-invariant,

$$\mathrm{br}^*([H_0]) \in A^1_{\mathbb{C}^*}\big(\overline{M}_{g,n}(\mathbb{P}^1,1)\big).$$

The total space of $\mathcal{O}_{\mathbb{P}^1}(-2) \to \mathbb{P}^1$ is well-known to be the resolution of the A_1 singularity $\mathbb{C}^2/\mathbb{Z}_2$ with respect to the action

$$-(z_1,z_2) \mapsto (-z_1,-z_2).$$

A localization approach to the corresponding (reduced) Gromov-Witten theory along similar lines is developed in [17].

1.2. **Proof of Theorem 1.** We obtain a boundary expression for $\psi_1^{2g+r} \in R^*(\overline{M}_{g,1})$ by localization relations on $\overline{M}_{g,1}(\mathbb{P}^1,1)$. Let

$$I_{g,r} = \mathrm{ev}_1^*([\infty]^{2+r}) \cup c_g(B) \cup \mathrm{br}^*([H_0]) \in A^{g+r+3}_{\mathbb{C}^*}\big(\overline{M}_{g,1}(\mathbb{P}^1,1)\big).$$

Since the non-equivariant limit of $[\infty]^2$ is 0, the non-equivariant limit of $I_{g,r}$ is also 0. Let

$$\epsilon : \overline{M}_{g,1}(\mathbb{P}^1,1) \to \overline{M}_{g,1}$$

be the forgetful map. The map ϵ is $\mathbb{C}^*$-equivariant with respect to the trivial $\mathbb{C}^*$-action on $\overline{M}_{g,1}$. After push-forward,

$$\epsilon_*\big(I_{g,r} \cap [\overline{M}_{g,1}(\mathbb{P}^1,1)]^{vir}\big) \in A^{2g+r}_{\mathbb{C}^*}(\overline{M}_{g,1}). \tag{2}$$

The virtual localization formula [9] gives an explicit calculation of (2) in term of tautological classes. Setting the non-equivariant limit to 0,

$$\epsilon_*\big(I_{g,r} \cap [\overline{M}_{g,1}(\mathbb{P}^1,1)]^{vir}\big)|_{t=0} = 0, \tag{3}$$

yields an equation in $R^{2g+r}(\overline{M}_{g,1})$.

The localization computation of (3) is a sum over residue contributions of the $\mathbb{C}^*$-fixed loci of $\overline{M}_{g,1}(\mathbb{P}^1,1)$. The contributing $\mathbb{C}^*$-fixed loci $\overline{M}^{\mathbb{C}^*}_{g_1,g_2}$ are indexed by genus splittings $g_1+g_2=g$. If $g_1,g_2>0$, the $\mathbb{C}^*$-fixed locus is

$$\overline{M}^{\mathbb{C}^*}_{g_1,g_2} \cong \overline{M}_{g_1,2}\times\overline{M}_{g_2,1}\subset\overline{M}_{g,1}(\mathbb{P}^1,1), \tag{4}$$

parametrizing maps with collapsed components of genus g_1,g_2 over $\infty,0\in\mathbb{P}^1$ respectively and the marking over ∞. The restriction of ϵ to the locus (4) is isomorphic to

$$\iota:\Delta_{1,\emptyset}(g_1,g_2)\to\overline{M}_{g,1}.$$

In the degenerate cases

$$(g_1,g_2)=(0,g) \text{ or } (g,0),$$

the $\mathbb{C}^*$-fixed loci are isomorphic to $\overline{M}_{g,1}$ and $\overline{M}_{g,2}$ respectively.

By the virtual localization formula, we obtain

$$\epsilon_*(I_{g,r}\cap[\overline{M}_{g,1}(\mathbb{P}^1,1)]^{vir})=\sum_{g_1+g_2=g,\ g_i\geq 0}\epsilon_*\Big(\frac{I_{g,r}}{e(\mathrm{Norm}^{vir}_{g_1,g_2})}\cap[\overline{M}^{\mathbb{C}^*}_{g_1,g_2}]\Big).$$

If $g_1,g_2>0$, the restriction of B to $\Delta_{1,\emptyset}(g_1,g_2)$ is

$$\mathbb{E}^\vee_{g_1}\otimes(-t)\oplus\mathbb{E}^\vee_{g_2}\otimes(+t)\oplus\mathbb{C}$$

where $\mathbb{E}$ denote the Hodge bundle. The class $\mathrm{br}^*(H_0)$ restricts to $2g_2t$. The Euler class of the virtual normal bundle is

$$\frac{1}{e(\mathrm{Norm}^{vir})}=\frac{c_{g_1}(\mathbb{E}^\vee\otimes(+t))c_{g_2}(\mathbb{E}^\vee\otimes(-t))}{-t^2(t+\psi_{\star_2})(-t-\psi_{\star_1})}.$$

Putting all the terms together and using Mumford's relation[5] twice, we obtain

$$\epsilon_*\Big(\frac{I_{g,r}}{e(\mathrm{Norm}^{vir}_{g_1,g_2})}\cap[\overline{M}^{\mathbb{C}^*}_{g_1,g_2}]\Big)|_{t=0}=\iota_*\Big(\sum_{a+b=2g+r-1}(-1)^g(-1)^a 2g_2\,\psi^a_{\star_1}\psi^b_{\star_2}\cap[\Delta_{1,\emptyset}(g_1,g_2)]\Big)$$

for $g_1,g_2>0$. Because of the $2g_2t$ factor, the degenerate case $(g_1,g_2)=(g,0)$ contributes 0. However,

$$\epsilon_*\Big(\frac{I_{g,r}}{e(\mathrm{Norm}^{vir}_{0,g})}\cap[\overline{M}^{\mathbb{C}^*}_{0,g}]\Big)|_{t=0}=(-1)^g(-1)2g\,\psi_1^{2g+r}.$$

By the vanishing (3), we conclude

$$(-1)^g(-1)2g\,\psi_1^{2g+r}+\sum_{g_1+g_2=g,\ g_i>0}\ \sum_{a+b=2g+r-1}\iota_*\Big((-1)^g(-1)^a 2g_2\,\psi^a_{\star_1}\psi^b_{\star_2}\cap[\Delta_{1,\emptyset}(g_1,g_2)]\Big)=0$$

which is equivalent to Theorem 1. □

[5]Mumford's relation here is $c_g(\mathbb{E}^\vee_g\otimes(+t))\cdot c_g(\mathbb{E}^\vee_g\otimes(-t))=t^g(-t)^g$

1.3. **Variations.** Let $g \geq 0$ and $n_1, n_2 \geq 2$. Consider the moduli space $\overline{M}_{g,n_1+n_2}$. Let N_1 and N_2 denote the markings sets

$$N_1 = \{1, \ldots, n_1\}, \quad N_2 = \{n_1 + 1, \ldots, n_1 + n_2\}.$$

For $g_1, g_2 \geq 0$, let

$$\iota : \Delta_{N_1,N_2}[g_1, g_2] \to \overline{M}_{g,n_1+n_2}$$

denote the boundary divisor parametrizing reducible curves

$$C = C_1 \cup C_2$$

with markings N_i on C_i satisfying $g(C_i) = g_i$ and $C_1 \cap C_2 = p_\star$. Let

$$\psi_{\star_1}, \psi_{\star_2} \in A^1\big(\Delta_{N_1,N_2}(g_1, g_2)\big)$$

denote the cotangent line classes of $p_\star$ along C_1 and C_2 as before.

Proposition 1. *For $g \geq 0$ and $n_1, n_2 \geq 2$ and $r \geq 0$,*

$$\sum_{g_1+g_2=g,\ g_i \geq 0} \ \sum_{a+b=2g+n_1+n_2-3+r} (-1)^a \iota_*\big(\psi_{\star_1}^a \psi_{\star_2}^b \cap [\Delta_{N_1,N_2}(g_1, g_2)]\big) = 0$$

in $A^{2g+n_1+n_2-2+r}(\overline{M}_{g,n_1+n_2})$.

Proof. Consider the moduli space $\overline{M}_{g,n_1+n_2}(\mathbb{P}^1, 1)$ with the $\mathbb{C}^*$-action specified in Section 1.1. Let

$$J_{g,r} = \mathrm{ev}_1^*([\infty]^{1+r}) \cup \prod_{i \in N_1} \mathrm{ev}_i^*([\infty]) \cup \prod_{i \in N_2} \mathrm{ev}_i^*([0]) \cup c_g(B) \in A^{g+n_1+n_2+r+1}\big(\overline{M}_{g,n_1+n_2}(\mathbb{P}^1, 1)\big).$$

Since the non-equivariant limit of $[\infty]^2$ is 0, the non-equivariant limit of $J_{g,r}$ is also 0. Let

$$\epsilon : \overline{M}_{g,n_1+n_2}(\mathbb{P}^1, 1) \to \overline{M}_{g,n_1+n_2}$$

be the forgetful map. After push-forward,

$$\epsilon_*\big(J_{g,r} \cap [\overline{M}_{g,n_1+n_2}(\mathbb{P}^1, 1)]^{vir}\big) \in A_{\mathbb{C}^*}^{2g+n_1+n_2-2+r}(\overline{M}_{g,n_1+n_2}). \tag{5}$$

Setting the non-equivariant limit to 0,

$$\epsilon_*\big(J_{g,r} \cap [\overline{M}_{g,n_1+n_2}(\mathbb{P}^1, 1)]^{vir}\big)|_{t=0} = 0, \tag{6}$$

yields an equation in $R^{2g+n_1+n_2-2+r}(\overline{M}_{g,n_1+n_2})$. Evaluating the virtual localization formula as in the proof of Theorem 1 precisely yields Proposition 1. $\square$

Since $n_1, n_2 \geq 2$ in the hypothesis of Proposition 1, there are no degenerate cases. There is no difficulty in handling the degenerate cases. We single out the following result with the same proof[6] as Proposition 1.

Proposition 2. *For $g \geq 1$ and $r \geq 0$,*

$$-\psi_1^{2g+r} + (-1)^r \psi_2^{2g+r} + \sum_{g_1+g_2=g,\ g_i>0} \ \sum_{a+b=2g-1+r} (-1)^a \iota_*\big(\psi_{\star_1}^a \psi_{\star_2}^b \cap [\Delta_{1,2}(g_1, g_2)]\big) = 0$$

in $A^{2g+r}(\overline{M}_{g,2})$.

[6]The proofs of Theorem 1 and Proposition 1 are almost identical. In fact, Theorem 1 can be derived from Proposition 1 using string and dilaton equations.

Proposition 2 corresponds simply to the $n_1 = n_2 = 1$ case of Proposition 1. The first two terms are the degenerate contributions.

1.4. **Proof of Theorem 2.** We start by pushing forward the relation of Proposition 2 in genus $g+1$ to $\overline{M}_{g+1}$ for odd r,

$$-2\kappa_{2(g+1)+r-2} - \sum_{g_1+g_2=g+1,\ g_i>0} \ \sum_{a+b=2(g+1)-3+r} (-1)^a \iota_*\big(\psi_{\star_1}^a \psi_{\star_2}^b \cap [\Delta_{\emptyset,\emptyset}(g_1,g_2)]\big) = 0,$$

using the definition of the κ classes and the string equation. Equivalently,

$$\kappa_{2g+r} + \frac{1}{2} \sum_{g_1+g_2=g+1,\ g_i>0} \ \sum_{a+b=2g-1+r} (-1)^a \iota_*\big(\psi_{\star_1}^a \psi_{\star_2}^b \cap [\Delta_{\emptyset,\emptyset}(g_1,g_2)]\big) = 0 \in A^{2g+r}(\overline{M}_{g+1}) \tag{7}$$

for odd r.

The Chern characters of the Hodge bundle $\mathrm{ch}_{2l-1}(\mathbb{E}_{g+1})$ on $\overline{M}_{g+1}$ vanish for $l > g+1$, see [3]. Hence, by Mumford's GRR calculation,

$$\begin{aligned}\mathrm{ch}_{2g+r}(\mathbb{E}_{g+1}) \left(\frac{B_{2g+r+1}}{(2g+r+1)!}\right)^{-1} &= \\ \kappa_{2g+r} + \frac{1}{2}\iota_*(\xi_{g,r-1}) + \frac{1}{2} \sum_{g_1+g_2=g+1,\ g_i>0} \ \sum_{a+b=2g-1+r} & (-1)^a \iota_*\big(\psi_{\star_1}^a \psi_{\star_2}^b \cap [\Delta_{\emptyset,\emptyset}(g_1,g_2)]\big) = 0\end{aligned}$$

for $r \geq 3$ odd. Using the vanishing (7), we conclude

$$\iota_*(\xi_{g,r-1}) = 0 \in A^{2g+r}(\overline{M}_{g+1})$$

for $r \geq 3$ odd, which are the only nontrivial cases of Theorem 2. □

2. Gromov-Witten equations

2.1. **Liu-Xu conjecture.** Let X be a nonsingular projective variety. We prove here the following result constraining the Gromov-Witten theory of X conjectured by K. Liu and H. Xu in [13].

Theorem 4. *Let $g \geq 0$ and $x_i, y_j \in H^*(X, \mathbb{C})$. For all $p_i, q_j, r, s \geq 0$ and $m \geq 2g-3+r+s$,*

$$\sum_{k\in\mathbb{Z}} \ \sum_{g_1+g_2=g,\ g_i\geq 0} (-1)^k \left\langle\!\!\left\langle \tau_k(\gamma_\ell) \prod_{i=1}^{r} \tau_{p_i}(x_i) \right\rangle\!\!\right\rangle_{g_1} \left\langle\!\!\left\langle \tau_{m-k}(\gamma^\ell) \prod_{j=1}^{s} \tau_{q_j}(y_j) \right\rangle\!\!\right\rangle_{g_2} = 0.$$

Here, k is allowed to be an arbitrary integer. To interpret Theorem 4 correctly, the following convention is used[7]:

$$\langle \tau_{-2}(\gamma_1) \rangle_{0,0} = 1 \qquad \text{and} \qquad \langle \tau_m(\gamma_\alpha)\tau_{-1-m}(\gamma_\beta) \rangle_{0,0} = (-1)^{\max(m,-1-m)}\eta_{\alpha\beta} \tag{8}$$

for $m \in \mathbb{Z}$. All other negative descendents vanish. The sum over ℓ in Theorem 4 is implicit.

Since the genus 0 case of Theorem 4 has been proved[8] in [16], we will only consider the case $g \geq 1$. By Theorem 0.2 of [16], Theorem 3 follows from the $r = s = 0$ case of Theorem 4.

[7] γ_1 is the identity of the cohomology ring of X.

[8] The $m \geq 3g-3+r+s$ case is also proved in [16], but the result will not be used here.

2.2. **Conventions.** We will not use convention (8). Instead, we set $\tau_n(\gamma_\alpha) = 0$ for $n < 0$ and separate the negative terms in the summation of Theorem 4.

The *big phase space* is the infinite dimensional vector space with coordinate $t = (t_n^\alpha)$. It can be interpreted as an infinite product of the cohomology space $H^*(X, \mathbb{C})$. The Gromov-Witten potential F_g^X is a function on the big phase space. We will interpret the symbol $\tau_n(\gamma_\alpha)$ as the coordinate vector field $\frac{\partial}{\partial t_n^\alpha}$. Moreover, we also extend the meaning of $\langle\langle\, \mathcal{W}_1 \cdots \mathcal{W}_k \,\rangle\rangle_g$ from partial derivatives of F_g^X to covariant derivatives of F_g^X with respect to arbitrary vector fields $\mathcal{W}_1, \ldots, \mathcal{W}_k$ on the big phase space. Here, the covariant differentiation is with respect to the trivial connection ∇ for which the coordinate vector fields $\tau_n(\gamma_\alpha)$ are parallel. More precisely, if $\mathcal{W}_i = \sum_{n,\alpha} f_{n,\alpha}^i \tau_n(\gamma_\alpha)$ where $f_{n,\alpha}^i$ are functions of $t = (t_m^\beta)$, then we define

$$\langle\langle\, \mathcal{W}_1 \cdots \mathcal{W}_k \,\rangle\rangle_g = \nabla^k_{\mathcal{W}_1, \cdots, \mathcal{W}_k} F_g^X = \sum_{\substack{n_1, \cdots, n_k \\ \alpha_1, \cdots, \alpha_k}} \left(\prod_{i=1}^k f_{n_i, \alpha_i}^i \right) \langle\langle\, \tau_{n_1}(\alpha_1) \cdots \tau_{n_k}(\alpha_k) \,\rangle\rangle_g .$$

For a vector field of type $\tau_n(\gamma_\alpha)$, the integer n is called the *level of the descendent*. A vector field is *primary* if the level of the descendent is 0. The total level of descendents for a set of vector fields is defined to be the sum of the levels of descendents for all vector fields in the set. For convenience, we define the operators τ_+ and τ_- on the space of vector fields by the following formulas:

$$\tau_\pm(\mathcal{W}) = \sum_{n,\alpha} f_{n,\alpha} \tau_{n\pm 1}(\gamma_\alpha) \qquad \text{if} \qquad \mathcal{W} = \sum_{n,\alpha} f_{n,\alpha} \tau_n(\gamma_\alpha).$$

Moreover, we define $\tau_k(\mathcal{W}) = \tau_+^k(\mathcal{W})$ for any vector field $\mathcal{W}$.

2.3. **Lower cases.** We first prove a result about relations among different cases of Theorem 4.

Proposition 3. *Let $g \geq 0$ be fixed. If Theorem 4 holds for $r = \hat{r}$ and $s = \hat{s}$, then Theorem 4 holds for all $r \leq \hat{r}$ and $s \leq \hat{s}$.*

Proof: We first rewrite Theorem 4 without using the special convention (8). Define

$$\tilde{t}_n^\alpha = t_n^\alpha - \delta_{\alpha,1}\delta_{n,1}.$$

Let $\mathcal{W}_i$, $\mathcal{V}_j$ be arbitrary coordinate vector fields on the big phase space of the form $\tau_n(\gamma_\alpha)$. For $r, s, g, m \geq 0$, define

$$\begin{aligned}
(9) \quad & \Psi_{r,s,g,m}(\mathcal{W}_1, \cdots, \mathcal{W}_r \mid \mathcal{V}_1, \cdots, \mathcal{V}_s) = \\
& \sum_{k=0}^{m} \sum_{g_1+g_2=g,\ g_i\geq 0} (-1)^k \langle\langle\, \tau_k(\gamma_\alpha)\, \mathcal{W}_1 \cdots \mathcal{W}_r \,\rangle\rangle_{g_1} \langle\langle\, \tau_{m-k}(\gamma^\alpha)\, \mathcal{V}_1 \cdots \mathcal{V}_s \,\rangle\rangle_{g_2} \\
& - \delta_{r,0} \sum_{n,\alpha} \tilde{t}_n^\alpha \langle\langle\, \tau_{n+m+1}(\gamma_\alpha)\, \mathcal{V}_1 \cdots \mathcal{V}_s \,\rangle\rangle_g - \delta_{r,1} \langle\langle\, \tau_{m+1}(\mathcal{W}_1)\, \mathcal{V}_1 \cdots \mathcal{V}_s) \,\rangle\rangle_g \\
& + \delta_{s,0}(-1)^{m+1} \sum_{n,\alpha} \tilde{t}_n^\alpha \langle\langle\, \tau_{n+m+1}(\gamma_\alpha)\, \mathcal{W}_1 \cdots \mathcal{W}_r \,\rangle\rangle_g + \delta_{s,1}(-1)^{m+1} \langle\langle\, \mathcal{W}_1 \cdots \mathcal{W}_r\, \tau_{m+1}(\mathcal{V}_1) \,\rangle\rangle_g .
\end{aligned}$$

The function satisfies the symmetry

$$\Psi_{r,s,g,m}(\mathcal{W}_1 \cdots \mathcal{W}_r \mid \mathcal{V}_1 \cdots \mathcal{V}_s) = (-1)^m \Psi_{s,r,g,m}(\mathcal{V}_1 \cdots \mathcal{V}_s \mid \mathcal{W}_1 \cdots \mathcal{W}_r). \tag{10}$$

Moreover, $\Psi_{0,0,g,m}$ is identically equal to 0 if m is odd.

Theorem 4 can be restated as

$$\Psi_{r,s,g,m}(\mathcal{W}_1 \cdots \mathcal{W}_r \mid \mathcal{V}_1 \cdots \mathcal{V}_s) = 0 \tag{11}$$

if $m \geq 2g + r + s - 3$, see [14].

Suppose for fixed integers $r > 0$ and $s \geq 0$, equation (11) holds for all integers $m \geq 2g+r+s-3$. Then, we must prove that equation (11) holds if r is replaced by $r-1$ for all $m \geq 2g+r+s-4$. By an inverse induction on r, if Theorem 4 holds for $r = \hat{r}$ and $s = \hat{s}$, then Theorem 4 holds for $r \leq \hat{r}$ and $s = \hat{s}$. By equation (10), we can switch the role of r and s. Hence, the Proposition will be proved.

Consider the *string vector field*,

$$\mathcal{S} = -\sum_{n,\alpha} \tilde{t}_n^\alpha \tau_{n-1}(\gamma_\alpha).$$

The *string equation* for Gromov-Witten invariants can be written as

$$\langle\!\langle \mathcal{S} \rangle\!\rangle_g = \frac{1}{2}\delta_{g,0}\eta_{\alpha\beta}t_0^\alpha t_0^\beta$$

where $\eta_{\alpha\beta} = \int_X \gamma_\alpha \cup \gamma_\beta$ is the usual pairing. Taking derivatives of the string equation, we obtain

$$\langle\!\langle \mathcal{S}\,\mathcal{W}_1 \, \cdots \, \mathcal{W}_k \rangle\!\rangle_g = \sum_{i=1}^k \langle\!\langle \mathcal{W}_1 \, \cdots \, \{\tau_-(\mathcal{W}_i)\} \, \cdots \, \mathcal{W}_k \rangle\!\rangle_g + \delta_{g,0}\nabla^k_{\mathcal{W}_1,\cdots,\mathcal{W}_k}\left(\frac{1}{2}\eta_{\alpha\beta}t_0^\alpha t_0^\beta\right). \tag{12}$$

Note that

$$\nabla^k_{\mathcal{W}_1,\cdots,\mathcal{W}_k}\left(\frac{1}{2}\eta_{\alpha\beta}t_0^\alpha t_0^\beta\right) = 0$$

if $k > 2$ or if at least one of the vector fields $\mathcal{W}_1, \cdots, \mathcal{W}_k$ has a positive descendent level.

Since equation (11) is linear with respect to each $\mathcal{W}_i$ and $\mathcal{V}_j$, we can replace them by any vector fields on the big phase space. Assume $r > 0$. We consider what happens if $\mathcal{W}_r = \mathcal{S}$ in $\Psi_{r,s,g,m}(\mathcal{W}_1, \cdots, \mathcal{W}_r \mid \mathcal{V}_1, \cdots, \mathcal{V}_s)$.

Lemma 2.4. *For $r > 0$,*

$$\begin{aligned} \Psi_{r,s,g,m}(\mathcal{W}_1 \cdots \mathcal{W}_{r-1}\mathcal{S} \mid \mathcal{V}_1 \cdots \mathcal{V}_s) = \\ - \Psi_{r-1,s,g,m-1}(\mathcal{W}_1 \cdots \mathcal{W}_{r-1} \mid \mathcal{V}_1 \cdots \mathcal{V}_s) \\ + \sum_{i=1}^{r-1} \Psi_{r-1,s,g,m}(\mathcal{W}_1 \cdots \tau_-(\mathcal{W}_i) \cdots \mathcal{W}_{r-1} \mid \mathcal{V}_1 \cdots \mathcal{V}_s). \end{aligned} \tag{13}$$

for all vector fields $\mathcal{W}_i$ and $\mathcal{V}_j$.

Assuming the validity of Lemma 2.4, we can prove the Proposition by induction. Indeed, assume

$$\Psi_{r,s,g,m}(\mathcal{W}_1 \cdots \mathcal{W}_{r-1}\mathcal{S} \mid \mathcal{V}_1 \cdots \mathcal{V}_s) = 0,$$

for all vector fields $\mathcal{W}_i$, $\mathcal{V}_j$, and all integers $m \geq 2g-3+r+s$. By linearity, we may assume that all vector fields $\mathcal{W}_i$ are coordinate vector fields of type $\tau_n(\gamma_\alpha)$. Note that $\tau_-(\mathcal{W}_i) = 0$ if $\mathcal{W}_i$ is a primary vector field. Hence equation (13) implies

$$\Psi_{r-1,s,g,m-1}(\mathcal{W}_1 \cdots \mathcal{W}_{r-1} \mid \mathcal{V}_1 \cdots \mathcal{V}_s) = 0 \tag{14}$$

for all integers $m \geq 2g-3+r+s$ if $\mathcal{W}_1, \cdots, \mathcal{W}_{r-1}$ are all primary vector fields.

Since the total level of descendents for vector fields in the second term on the right hand side of equation (13) is strictly less than that in the first term, an induction on the total level of descendents for $\mathcal{W}_1, \cdots, \mathcal{W}_{r-1}$ shows that equation (14) also hold for all (not necessarily primary) vector fields $\mathcal{W}_1, \cdots, \mathcal{W}_{r-1}$. Hence, if Theorem 4 holds for $r > 0$ and $s \geq 0$, then Theorem 4 holds if r is replaced by $r-1$. The Proposition thus follows from Lemma 2.4. □

2.5. **Proof of Lemma 2.4.** Using equation (12), the result is straightforward for $r > 2$. The cases $r \leq 2$ are more subtle because of the last term in equation (12).

We consider the case $r = 2$ first. If $\mathcal{W}$ is a primary vector field, then

$$\nabla^2_{\mathcal{W},\tau_k(\gamma_\alpha)}\left(\frac{1}{2}\eta_{\beta\mu}t_0^\beta t_0^\mu\right) \langle\langle\, \tau_{m-k}(\gamma^\alpha)\, \mathcal{V}_1 \,\cdots\, \mathcal{V}_s \,\rangle\rangle_g \;=\; \delta_{k,0}\, \langle\langle\, \tau_m(\mathcal{W})\, \mathcal{V}_1 \,\cdots\, \mathcal{V}_s \,\rangle\rangle_g\,.$$

This will produce the extra term in $\Psi_{1,s,g,m-1}(\mathcal{W} \mid \mathcal{V}_1, \cdots, \mathcal{V}_s)$. Therefore by equation (12),

$$\Psi_{2,s,g,m}(\mathcal{W}\mathcal{S} \mid \mathcal{V}_1 \cdots \mathcal{V}_s) \;=\; -\Psi_{1,s,g,m-1}(\mathcal{W} \mid \mathcal{V}_1 \cdots \mathcal{V}_s)$$

when $\mathcal{W}$ is a primary vector field.

If $\mathcal{W}$ has a positive descendent level, then

$$\nabla^2_{\mathcal{W},\tau_k(\gamma_\alpha)}\left(\frac{1}{2}\eta_{\beta\mu}t_0^\beta t_0^\mu\right) = 0$$

for all $k \geq 0$. Equation (12) again implies

$$\Psi_{2,s,g,m}(\mathcal{W}\mathcal{S} \mid \mathcal{V}_1 \cdots \mathcal{V}_s) = - \Psi_{1,s,g,m-1}(\mathcal{W} \mid \mathcal{V}_1 \cdots \mathcal{V}_s) + \Psi_{1,s,g,m}(\tau_-(\mathcal{W}) \mid \mathcal{V}_1 \cdots \mathcal{V}_s). \tag{15}$$

When passing from $\Psi_{2,s,g,m}$ to $\Psi_{1,s,g,m}$, an extra term will emerge. The summations which we obtain from applying equation (12) to $\Psi_{2,s,g,m}(\mathcal{W}, \mathcal{S} \mid \mathcal{V}_1, \cdots, \mathcal{V}_s)$ have some missing terms when compared to the definition of $\Psi_{1,s,g,m-1}(\mathcal{W} \mid \mathcal{V}_1 \cdots \mathcal{V}_s)$ and $\Psi_{1,s,g,m}(\tau_-(\mathcal{W}) \mid \mathcal{V}_1 \cdots \mathcal{V}_s)$. The missing term for $\Psi_{1,s,g,m-1}(\mathcal{W} \mid \mathcal{V}_1 \cdots \mathcal{V}_s)$ is $-\langle\langle\, \tau_m(\mathcal{W})\, \mathcal{V}_1 \cdots \mathcal{V}_s \,\rangle\rangle_g$ while the missing term for $\Psi_{1,s,g,m}(\tau_-(\mathcal{W}) \mid \mathcal{V}_1 \cdots \mathcal{V}_s)$ is $-\langle\langle\, \tau_{m+1}(\tau_-(\mathcal{W}))\, \mathcal{V}_1 \cdots \mathcal{V}_s \,\rangle\rangle_g$. The missing terms cancel in (15) when $\mathcal{W}$ has a positive descendent level.

Since we have checked that equation (15) holds for all primary and descendent vector fields $\mathcal{W}$, Lemma 2.4 is true for $r = 2$.

Consider next the case $r=1$ and $s>0$. We have

$$\Psi_{1,s,g,m}(\mathcal{S} \mid \mathcal{V}_1 \cdots \mathcal{V}_s) = \\ - \langle\langle \tau_{m+1}(\mathcal{S})\, \mathcal{V}_1 \cdots \mathcal{V}_s \rangle\rangle_g + \delta_{s,1}(-1)^{m+1} \langle\langle \mathcal{S}\ \tau_{m+1}(\mathcal{V}_1) \rangle\rangle_g \\ + \sum_{k=0}^{m} \sum_{g_1+g_2=g,\ g_i\geq 0} (-1)^k \langle\langle \tau_k(\gamma_\alpha)\, \mathcal{S} \rangle\rangle_{g_1} \langle\langle \tau_{m-k}(\gamma^\alpha)\, \mathcal{V}_1 \cdots \mathcal{V}_s \rangle\rangle_{g_2}.$$

In the definition of $\mathcal{S}$, $\tilde{t}_0^\alpha$ is not included since $\tau_{-1}(\gamma_\alpha)=0$. Hence

$$\langle\langle \tau_{m+1}(\mathcal{S})\, \mathcal{V}_1 \cdots \mathcal{V}_s \rangle\rangle_g = -\sum_{n=1}^{\infty} \sum_{\alpha} \tilde{t}_n^\alpha \langle\langle \tau_{n+m}(\gamma_\alpha)\, \mathcal{V}_1 \cdots \mathcal{V}_s \rangle\rangle_g. \tag{16}$$

By equation (12), $\langle\langle \mathcal{S}\ \tau_{m+1}(\mathcal{V}_1) \rangle\rangle_g = \langle\langle \tau_m(\mathcal{V}_1) \rangle\rangle_g$ and

$$\langle\langle \tau_k(\gamma_\alpha)\, \mathcal{S} \rangle\rangle_{g_1} = \langle\langle \tau_{k-1}(\gamma_\alpha) \rangle\rangle_{g_1} + \delta_{g_1,0}\delta_{k,0}\eta_{\alpha\beta}t_0^\beta.$$

The effect of the second term on the right hand side of this equation is just to compensate for the missing case $n=0$ in the summation for n in equation (16) when computing

$$\Psi_{1,s,g,m}(\mathcal{S} \mid \mathcal{V}_1, \cdots, \mathcal{V}_s).$$

Therefore we have

$$\Psi_{1,s,g,m}(\mathcal{S} \mid \mathcal{V}_1 \cdots \mathcal{V}_s) = -\Psi_{0,s,g,m-1}(\mathcal{V}_1 \cdots \mathcal{V}_s).$$

Hence, Lemma 2.4 is true for $r=1$ and $s>0$.

Now only the case $r=1$ and $s=0$ is left. By definition,

$$\begin{aligned}\Psi_{1,0,g,m}(\mathcal{S}) &= -\langle\langle \tau_{m+1}(\mathcal{S}) \rangle\rangle_g + (-1)^{m+1} \sum_{n,\alpha} \tilde{t}_n^\alpha \langle\langle \tau_{n+m+1}(\gamma_\alpha)\ \mathcal{S} \rangle\rangle_g \\ &\quad + \sum_{k=0}^{m} \sum_{g_1+g_2=g,\ g_i\geq 0} (-1)^k \langle\langle \tau_k(\gamma_\alpha)\, \mathcal{S} \rangle\rangle_{g_1} \langle\langle \tau_{m-k}(\gamma^\alpha) \rangle\rangle_{g_2}.\end{aligned}$$

By equation (12), we have

$$\begin{aligned}\Psi_{1,0,g,m}(\mathcal{S}) &= \left\{1+(-1)^{m+1}\right\} \sum_{n,\alpha} \tilde{t}_n^\alpha \langle\langle \tau_{n+m}(\gamma_\alpha) \rangle\rangle_g \\ &\quad - \sum_{k=0}^{m-1} \sum_{g_1+g_2=g,\ g_i\geq 0} (-1)^k \langle\langle \tau_k(\gamma_\alpha) \rangle\rangle_{g_1} \langle\langle \tau_{m-1-k}(\gamma^\alpha) \rangle\rangle_{g_2} \\ &= -\Psi_{0,0,g,m-1}.\end{aligned}$$

The proof for Lemma 2.4 is complete. □

2.6. **Proof of Theorem 4.** Relations in $R^*(\overline{M}_{g,n})$ can be translated into universal equations for Gromov-Witten invariants by the splitting axiom and cotangent line comparison equations. Define the operator T on the space of vector fields by

$$T(\mathcal{W}) = \tau_+(\mathcal{W}) - \langle\langle \mathcal{W}\, \gamma^\alpha \rangle\rangle_0\, \gamma_\alpha$$

for any vector field $\mathcal{W}$. Properties of T have been studied in [15]. The operator is very useful for the translation into universal equations. In the process, each marked point corresponds to a vector field, and the cotangent line class corresponds to the operator T. Each node is translated into a pair of primary vector fields γ_ℓ and γ^ℓ. In particular, the relation of Proposition 1 is translated into the following universal equation

$$\sum_{k=0}^{m} \sum_{g_1+g_2=g,\ g_i\geq 0} (-1)^k \left\langle\!\left\langle \mathcal{W}_1 \cdots \mathcal{W}_{n_1} T^k(\gamma_\ell) \right\rangle\!\right\rangle_{g_1} \left\langle\!\left\langle T^{m-k}(\gamma^\ell) \mathcal{V}_1 \cdots \mathcal{V}_{n_2} \right\rangle\!\right\rangle_{g_2} = 0 \tag{17}$$

for all vector fields $\mathcal{W}_i$ and $\mathcal{V}_j$ if $n_1, n_2 \geq 2$ and $m \geq 2g + n_1 + n_2 - 3$.

Let P and Q be two arbitrary contravariant tensors on the big phase space. The following formula was proved in [16, Proposition 3.2]:

$$\sum_{k=0}^{m} (-1)^k P(T^k(\gamma_\ell))\ Q(T^{m-k}(\gamma^\ell)) = \sum_{k=0}^{m} (-1)^k P(\tau_k(\gamma_\ell))\ Q(\tau_{m-k}(\gamma^\ell))$$

for $m \geq 0$. In particular, if we take $P(\mathcal{U}) = \langle\!\langle \mathcal{W}_1 \cdots \mathcal{W}_{n_1} \mathcal{U} \rangle\!\rangle_{g_1}$ and $Q(\mathcal{U}) = \langle\!\langle \mathcal{U}\ \mathcal{V}_1 \cdots \mathcal{V}_{n_2} \rangle\!\rangle_{g_2}$, then the left hand side of equation (17) is equal to

$$\sum_{k=0}^{m} \sum_{g_1+g_2=g,\ g_i\geq 0} (-1)^k \left\langle\!\left\langle \mathcal{W}_1 \cdots \mathcal{W}_{n_1} \tau_k(\gamma_\ell) \right\rangle\!\right\rangle_{g_1} \left\langle\!\left\langle \tau_{m-k}(\gamma^\ell) \mathcal{V}_1 \cdots \mathcal{V}_{n_2} \right\rangle\!\right\rangle_{g_2} =$$
$$\Psi_{n_1,n_2,g,m}(\mathcal{W}_1 \cdots \mathcal{W}_{n_1} \mid \mathcal{V}_1 \cdots \mathcal{V}_{n_2}).$$

Therefore equation (17) implies that Theorem 4 is true for $r = n_1 \geq 2$ and $s = n_2 \geq 2$. By Proposition 3, all other cases of Theorem 4 follow. $\square$

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