VERLINDE FLATNESS AND RELATIONS IN $H^*(M_g)$

CAREL, ALINA, RAHUL (*IN PROGRESS*)

1. Verlinde bundles

1.1. **Flatness constraint.** Let M_g be the moduli space of nonsingular curves of genus $g \geq 2$. Let

$$\mu : \mathcal{U}_g(r,d) \to M_g$$

be the moduli space of rank r degree d semistable bundles on nonsingular genus g curves. The space $\mathcal{U}_g(r, r(g-1))$ carries a canonical theta divisor

$$\mathbf{\Theta}_r = \{(C, E \to C) \text{ with } h^0(C,E) \neq 0\}.$$

For levels $k \geq 1$, the divisors $\mathbf{\Theta}_r^k$ are known to have no higher cohomology on the fibers of μ. The μ-pushforwards of the powers of the associated line bundle give the Verlinde vector bundles on M_g,

$$\mathcal{V}_{r,k} = \mu_\star \mathbf{\Theta}_r^k.$$

The rank of $\mathcal{V}_{r,k}$ is given by the well-known Verlinde formula [9].

For all ranks r and levels k, the Verlinde bundle $\mathcal{V}_{r,k}$ carries a projectively flat connection defined by Hitchin [1, 3, 5]. As a basic consequence, the Verlinde bundle satisifes the topological constraint

$$\text{ch}\,\mathcal{V}_{r,k} = \text{rank}\,\mathcal{V}_{r,k} \cdot \exp\left(\frac{c_1(\mathcal{V}_{r,k})}{\text{rank}\,\mathcal{V}_{r,k}}\right) \in H^*(M_g), \tag{1}$$

where $\text{ch}_i\,\mathcal{V}_{r,k}$ is the i^{th} Chern character.

1.2. **Fixed determinant.** We specialize now to the case of bundles of rank 2 and fixed determinant. We denote by

$$\mu : \mathcal{SU}_g(2, 2g-2) \to M_g$$

the moduli space of semistable rank 2 bundles over nonsingular curves C with determinant equal to the canonical bundle ω_C. In the fixed determinant situation, the push-forward

$$\mathbb{V}_{r,k} = \mu_\star \mathbf{\Theta}_2^k$$

is also projectively flat [1, 5], so the Chern character again satisfies (1) on M_g.

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We will study the moduli of semistable bundles over the moduli space $M_{g,1}$. Let

$$\mathcal{SU}_{g,1}(2, 2g-2) = \mathcal{SU}_g(2, 2g-2) \times_{M_g} M_{g,1}$$

be the fiber product, and let

$$\mu : \mathcal{SU}_{g,1}(2, 2g-2) \to M_{g,1}$$

be the projection. Similarly, we let

$$\mu : \mathcal{SU}_{g,1}(2, 2g) \to M_{g,1}$$

be the moduli space of rank 2 bundles E on nonsingular pointed curves (C, p) satisfying

$$\det E \simeq \omega_C(2p).$$

There is a canonical isomorphism,

$$\alpha : \mathcal{SU}_{g,1}(2, 2g) \longrightarrow \mathcal{SU}_{g,1}(2, 2g-2)$$

defined by

$$(C,\, p,\, E \to C) \mapsto (C,\, p,\, E(-p) \to C).$$

Certainly, we have

$$\mu_\star \alpha^\star \mathbf{\Theta}_2^k = \mu_\star \mathbf{\Theta}_2^k,$$

so the Chern classes of $\mu_\star \alpha^\star \mathbf{\Theta}_2^k$ are pulled back to $M_{g,1}$ via

$$\iota : M_{g,1} \to M_g,$$

and satisfy (1) on $M_{g,1}$.

2. The wall-crossing calculation

2.1. **Overview.** In the rank 2 case with fixed determinant, we will calculate the Chern character of the level k Verlinde bundle by geometry independent of projective flatness. The idea is to employ the wall-crossing method of Thaddeus (used to prove the Verlinde formula) uniformly over the moduli of curves. Where Thaddeus studies rank, we will require K-theory. The final result computes the Chern character of the Verlinde bundle in the tautological ring $R^*(M_{g,1})$. The projective flatness condition (1) then produces non-trivial relations.

More precisely, the pairs construction of Thaddeus [8] determines the Verlinde vector space

$$H^0(\mathcal{SU}_C(2, \Lambda), \widetilde{\mathbf{\Theta}}_2^k)$$

associated to rank 2 and level k with fixed determinant Λ on a fixed curve C. We will carry out the construction of Thaddeus canonically for the universal family of curves over $M_{g,1}$ to study

$$\mu : \mathcal{SU}_{g,1}(2, 2g) \to M_{g,1}$$

with determinant equal to $\omega_C(2p)$. The universal theta divisor $\widetilde{\mathbf{\Theta}}_2$ on $\mathcal{SU}_{g,1}(2, 2g)$ which arises from the construction of Thaddeus must be related to the divisor $\alpha^\star \mathbf{\Theta}_2$ above by a possible twist

$$\widetilde{\mathbf{\Theta}}_2 \cong \alpha^\star \mathbf{\Theta}_2 \otimes \mathcal{L} \tag{2}$$

by a line bundle $\mathcal{L}$ on the base $M_{g,1}$. Since the associated Verlinde bundle

$$\widetilde{\mathbb{V}}_{2,k} = \mu_* \widetilde{\mathbf{\Theta}}_2^k$$

is still projectively flat, the constraint (1) again holds,

$$\operatorname{ch} \widetilde{\mathbb{V}}_{2,k} = \operatorname{rank} \widetilde{\mathbb{V}}_{2,k} \cdot \exp\left(\frac{c_1(\widetilde{\mathbb{V}}_{2,k})}{\operatorname{rank} \widetilde{\mathbb{V}}_{2,k}} \right) \in H^*(M_{g,1}). \tag{3}$$

On the other hand, by the main result of the wall-crossing method of Thaddeus, we can write

$$\widetilde{\mathbb{V}}_{2,k} = \sum_{i=0}^{g-1} (-1)^i N_i \tag{4}$$

in the K-theory of $M_{g,1}$. The objects N_i and their Chern characters will be discussed below. In fact, equation (4) allows effective computation of the Chern character of $\widetilde{\mathbb{V}}_{2,k}$.

Relation (3) certainly implies the Chern character of $\widetilde{\mathbb{V}}_{2,k}$ lies in the tautological ring in cohomology

$$RH^*(M_{g,1}) \subset H^*(M_{g,1}) \ .$$

However, equation (4) together with the analysis of the N_i implies the following refined result.

Theorem 1. *The Chern characters of $\widetilde{\mathbb{V}}_{2,k}$ lie in the tautological Chow ring*

$$ch_i \, \widetilde{\mathbb{V}}_{2,k} \ \in \ R^*(M_{g,1}) \ .$$

The parallel result, $\operatorname{ch}_i \mathbb{V}_{2,k} \ \in \ R^*(M_g)$, is an easy corollary. Of course, we expect the Chern characters of the Verlinde bundles to be tautological in Chow for higher rank $r > 2$ as well.

The main point of our investigation is not Theorem 1. Our hope, rather, is to combine the constraints (3) with the calculation (4) to force new relations in the tautological ring of the moduli space of curves [2, 6]. In light of recent progress [7] in the study of $R^*(M_g)$, of particular interest is the genus 24 case.

2.2. **The bundle N_0.** We denote the universal curve over $M_{g,1}$ by

$$\pi : \mathcal{C} \to M_{g,1}.$$

Let σ_0 be the universal section, and let ω be the π-relative canonical bundle. Let

$$V = \pi_\star \left(\mathcal{O}_\mathcal{C}(2\sigma_0) \otimes 2\omega \right)$$

The bundle N_0 is a push-forward to $M_{g,1}$ from the projective bundle

$$\rho : \mathbb{P}V^\star \to M_{g,1}.$$

Specifically, letting $\mathcal{O}_\mathbb{P}(1)$ be the hyperplane bundle on $\mathbb{P}V^\star$, we define

$$N_0 = \rho_\star \left(\mathcal{O}_\mathbb{P}(kg) \right) = \operatorname{Sym}^{kg} V.$$

2.3. **The objects $N_{i>0}$.** The wall contributions $N_{i>0}$ are push-forwards from the symmetric products

$$\epsilon^{[i]} : \mathcal{C}^{[i]} \to M_{g,1},$$

where $1 \leq i \leq \left[\frac{d-1}{2}\right] = g-1$. We consider the fiber product

$$\mathcal{C}^{[i]} \times \mathcal{C}$$

over $M_{g,1}$, and let

$$\mathcal{D}_i \subset \mathcal{C}^{[i]} \times \mathcal{C}$$

be the universal divisor. We denote by π all projections from the universal curve, for instance

$$\pi : \mathcal{C} \to M_{g,1} \quad \text{and} \quad \pi : \mathcal{C}^{[i]} \times \mathcal{C} \to \mathcal{C}^{[i]}.$$

As before, σ_0 is the universal section on the second factor of the product $\mathcal{C}^{[i]} \times \mathcal{C}$.

In order to define $N_{i>0}$, we will require several vector bundles on the symmetric product $\mathcal{C}^{[i]}$. The first two arise via cohomology along the fibers of π:

$$\begin{aligned} W_i^- &= R^0\pi_\star \left(\mathcal{O}_{\mathcal{D}_i}(-\mathcal{D}_i + 2\sigma_0) \otimes \omega \right) , \\ W_i^+ &= R^1\pi_\star \left(\mathcal{O}_\mathcal{C}(2\mathcal{D}_i - 2\sigma_0) \otimes (-\omega) \right) . \end{aligned} \tag{5}$$

Let U denote the sum

$$U = W_i^- \oplus W_i^{+\star} \tag{6}$$

and define the line bundle

$$L_i = \det{}^{-1} R\pi_\star \left(\mathcal{O}_\mathcal{C}(-\mathcal{D}_i + 2\sigma_0) \otimes \omega \right) \otimes \det{}^{-1} R\pi_\star \, \mathcal{O}_\mathcal{C}(\mathcal{D}_i) \otimes \det{}^2 \mathbb{E}_g \otimes \mathbb{L}_p^\star. \tag{7}$$

The rank g Hodge bundle $\mathbb{E}_g$ and the cotangent line $\mathbb{L}_p$ at the marking

$$\mathbb{E}_g \to M_{g,1} \ , \qquad \mathbb{L}_p \to M_{g,1}$$

enter in the definition of L_i.

Finally, we define the objects N_i in the K-theory of $M_{g,1}$ by

$$N_i = R\epsilon_\star \left(L_i^k \otimes \Lambda^i W_i^- \otimes \mathrm{Sym}^{k(g-i)-i}\, U_i \right), \tag{8}$$

with the convention $N_i = 0$ when $k(g-i) - i < 0$.

2.4. **Chern classes.** We start by defining several classes on the universal product

$$\epsilon^i : \mathcal{C}^i \to M_{g,1}.$$

The most basic is the diagonal divisor class Δ_{xy} for indices $x \neq y$. Furthermore,

- Ψ_j is the cotangent line class on the j^{th} factor of $\mathcal{C}^i \to M_{g,1}$,
- $\widehat{\Psi}$ is the cotangent line class on $M_{g,1}$,
- σ_j is the class of the section of the j^{th} factor of $\mathcal{C}^i \to M_{g,1}$,
- $\Delta_j = \Delta_{1j} + \cdots + \Delta_{j-1,j},$ with $\Delta_1 = 0$.

In order the calculate the Chern character of N_0, we calculate the Chern character of V by Riemann-Roch applied to π,

$$\begin{aligned}
\mathrm{ch}\, V &= \pi_\star \left(e^{2\sigma_0 + 2\omega} \, \frac{-\omega}{1 - e^{\omega}} \right) \\
&= \pi_\star \left(\left(1 + \frac{1 - e^{-2\omega}}{\omega}\, \sigma_0 \right) e^{2\omega} \, \frac{-\omega}{1 - e^{\omega}} \right) \\
&= -\pi_\star \left(\frac{\omega\, e^{2\omega}}{1 - e^{\omega}} \right) - \frac{1 - e^{-2\widehat{\Psi}}}{1 - e^{\widehat{\Psi}}}\, e^{2\widehat{\Psi}} \\
&= -\pi_\star \left(\frac{\omega\, e^{2\omega}}{1 - e^{\omega}} \right) + 1 + e^{\widehat{\Psi}}.
\end{aligned}$$

Here, ω is the cotangent line class on the universal curve over $M_{g,1}$. The Chern character of $\mathrm{Sym}^{kg}\, V$ is then determined by the symmetric product formula.

We calculate next the Chern roots of the bundles (5)-(7) after pull-back to $\mathcal{C}^i$ via the natural map

$$\phi : \mathcal{C}^i \to \mathcal{C}^{[i]}.$$

To start, in K-theory,

$$\begin{aligned}
W_i^- &= R\pi_\star \left(\mathcal{O}_{\mathcal{D}_i}(-\mathcal{D}_i + 2\sigma_0) \otimes \omega \right) \\
&= R\pi_\star \left(\mathcal{O}_{\mathcal{C}}(-\mathcal{D}_i + 2\sigma_0) \otimes \omega \right) - R\pi_\star \left(\mathcal{O}_{\mathcal{C}}(-2\mathcal{D}_i + 2\sigma_0) \otimes \omega \right) \\
&= R\pi_\star \left(\mathcal{O}_{\mathcal{C}}(2\mathcal{D}_i - 2\sigma_0) \right)^\star - R\pi_\star \left(\mathcal{O}_{\mathcal{C}}(\mathcal{D}_i - 2\sigma_0) \right)^\star .
\end{aligned}$$

Over the point $[C, p, p_1 + \cdots + p_i] \in \mathcal{C}^{[i]}$, the virtual sheaf $R\pi_\star \left(\mathcal{O}_{\mathcal{C}}(2\mathcal{D}_i - 2\sigma_0) \right)$ is the formal difference

$$H^0(\mathcal{O}_C(2p_1 + \cdots + 2p_i - 2p)) - H^1(\mathcal{O}_C(2p_1 + \cdots + 2p_i - 2p)).$$

We calculate the Chern roots of the pull-back to the ordered product $\mathcal{C}^i$, by taking the cohomology of the following two exact sequences on C,

$$0 \to \mathcal{O}_C(-2p + 2p_1 + \cdots + 2p_{j-1} + p_j) \to \mathcal{O}_C(-2p + 2p_1 + \cdots + 2p_{j-1} + 2p_j) \to \mathcal{O}_C(-2p + 2p_1 + \cdots + 2p_{j-1} + 2p_j)|_{p_j} \to 0,$$

$$0 \to \mathcal{O}_C(-2p + 2p_1 + \cdots + 2p_{j-1}) \to \mathcal{O}_C(-2p + 2p_1 + \cdots + 2p_{j-1} + p_j) \to \mathcal{O}_C(-2p + 2p_1 + \cdots + 2p_{j-1} + p_j)|_{p_j} \to 0,$$

for $1 \leq j \leq i$. Leaving out the contributions of the K-class of $R\pi_\star \mathcal{O}_\mathcal{C}(-2\sigma_0)$ pulled-back from $M_{g,1}$, we can therefore write the Chern roots as

$$-2\sigma_j + 2\Delta_j - 2\Psi_j, \quad -2\sigma_j + 2\Delta_j - \Psi_j, \quad 1 \leq j \leq i.$$

Just as above, excluding the contributions of $R\pi_\star \mathcal{O}_\mathcal{C}(-2\sigma_0)$, for the virtual sheaf $R\pi_\star\left(\mathcal{O}_\mathcal{C}(\mathcal{D}_i - 2\sigma_0)\right)$, we can write the Chern roots as

$$-2\sigma_j + \Delta_j - \Psi_j, \quad 1 \leq j \leq i.$$

Since the two $R\pi_\star \mathcal{O}_\mathcal{C}(-2\sigma_0)$ terms cancel, we have

$$\operatorname{ch} W_i^- = \sum_{j=1}^{i} e^{\Psi_j + 2\sigma_j - \Delta_j}\left(e^{\Psi_j - \Delta_j} + e^{-\Delta_j} - 1\right). \tag{9}$$

Over the point $[C, p, p_1 + \cdots + p_i] \in \mathcal{C}^{[i]}$, the bundle $W_i^{+\star}$ is

$$H^0(\mathcal{O}_C(2p - 2p_1 - \cdots - 2p_i) \otimes 2\omega_C).$$

We write the Chern roots of the pull-back to the ordered product $\mathcal{C}^i$ by taking the cohomology of the following two exact sequences on C,

$$0 \to \omega_C^2 \otimes \mathcal{O}_C(2p - 2p_1 - \cdots - 2p_{j-1} - 2p_j) \to \omega_C^2 \otimes \mathcal{O}_C(2p - 2p_1 - \cdots - 2p_{j-1} - p_j) \to \omega_C^2 \otimes \mathcal{O}_C(2p - 2p_1 - \cdots - 2p_{j-1} - p_j)|_{p_j} \to 0,$$

$$0 \to \omega_C^2 \otimes \mathcal{O}_C(2p - 2p_1 - \cdots - 2p_{j-1} - p_j) \to \omega_C^2 \otimes \mathcal{O}_C(2p - 2p_1 - \cdots - 2p_{j-1}) \to \omega_C^2 \otimes \mathcal{O}_C(2p - 2p_1 - \cdots - 2p_{j-1})|_{p_j} \to 0$$

for $1 \leq j \leq i$. We find

$$\operatorname{ch} W_i^{+\star} = \operatorname{ch} \pi_\star(2\omega) + e^{\widehat{\Psi}} + 1 - \sum_{j=1}^{i} e^{2\Psi_j + 2\sigma_j - 2\Delta_j} \left(e^{\Psi_j} + 1\right), \tag{10}$$

where $\widehat{\Psi}$ is the cotangent line on $M_{g,1}$. From (9) and (10), we conclude

$$\operatorname{ch} U_i = \operatorname{ch} \pi_\star(2\omega) + e^{\widehat{\Psi}} + 1 + \sum_{j=1}^{i} e^{\Psi_j + 2\sigma_j - 2\Delta_j} \left(1 - e^{2\Psi_j} - e^{\Delta_j}\right). \tag{11}$$

Finally, L_i is a line bundle with Chern class determined by Riemann-Roch. Let Δ on $\mathcal{C}^i$ be the sum of all the diagonals

$$\Delta = \sum_{x<y} \Delta_{xy} \ .$$

The following basic push-forwards are easily calculated,

$$\begin{aligned} \pi_\star(\mathcal{D}_i^2) &= -\sum_{j=1}^{i} \Psi_j + 2\Delta, \qquad \pi_\star(\mathcal{D}_i\, \omega) = \sum_{j=1}^{i} \Psi_j, \\ \pi_\star(\mathcal{D}_i\, \sigma_0) &= \sum_{j=1}^{i} \sigma_j, \qquad \pi_\star(\omega\, \sigma_0) = \widehat{\Psi}, \qquad \pi_\star(\sigma_0^2) = -\widehat{\Psi}. \end{aligned}$$

Using the above, we calculate

$$\begin{aligned} c_1(\det R\pi_\star \mathcal{O}_{\mathcal{C}}(\mathcal{D}_i)) &= \pi_\star \left(e^{\mathcal{D}_i} \frac{-\omega}{1 - e^{\omega}} \right)_{(1)} \\ &= \pi_\star \left[\left(1 + \mathcal{D}_i + \frac{\mathcal{D}_i^2}{2}\right) \left(1 - \frac{\omega}{2} + \frac{\omega^2}{12}\right) \right]_{(1)} \\ &= \pi_\star \left(\frac{\mathcal{D}_i^2}{2} - \frac{\mathcal{D}_i\, \omega}{2} + \frac{\omega^2}{12} \right) \\ &= \Delta - (\Psi_1 + \cdots + \Psi_i) + \frac{\kappa_1}{12}. \end{aligned}$$

Similarly, for $c_1\left(\det R\pi_\star\left(\mathcal{O}_{\mathcal{C}}(-\mathcal{D}_i + 2\sigma_0) \otimes \omega\right)\right)$, we find

$$\begin{aligned} &\pi_\star \left(e^{-\mathcal{D}_i + 2\sigma_0 + \omega} \frac{-\omega}{1 - e^{\omega}} \right)_{(1)} = \\ &\quad \pi_\star \left[\left(1 - \mathcal{D}_i + \frac{\mathcal{D}_i^2}{2}\right) \left(1 + 2\sigma_0 + 2\sigma_0^2\right) \left(1 + \frac{\omega}{2} + \frac{\omega^2}{12}\right) \right]_{(1)} = \\ &\quad \pi_\star \left(\frac{\mathcal{D}_i^2}{2} + 2\sigma_0^2 + \frac{\omega^2}{12} - \frac{\mathcal{D}_i\, \omega}{2} - 2\mathcal{D}_i\, \sigma_0 + \omega\, \sigma_0 \right) = \\ &\quad \Delta - (\Psi_1 + \cdots + \Psi_i) - \widehat{\Psi} - 2(\sigma_1 + \cdots + \sigma_i) + \frac{\kappa_1}{12}. \end{aligned}$$

The two calculations together with (7) yield

$$
\begin{aligned}
(12)\quad c_1(L_i) &= -2\Delta + 2(\Psi_1 + \cdots + \Psi_i) + 2(\sigma_1 + \cdots + \sigma_i) + \widehat{\Psi} - \frac{\kappa_1}{6} + 2\lambda_1 - \widehat{\Psi} \\
&= -2\Delta + 2(\Psi_1 + \cdots + \Psi_i) + 2(\sigma_1 + \cdots + \sigma_i).
\end{aligned}
$$

We have used here

$$\det \mathbb{E}_g = \lambda_1 = \frac{\kappa_1}{12}, \qquad c_i(\mathbb{L}_p) = \widehat{\Psi} \ .$$

2.5. **Riemann-Roch.** The Chern character of N_i is given by Riemann-Roch

$$\operatorname{ch} N_i = \epsilon_\star^{[i]} \left(\operatorname{ch} L_i^k \cdot \operatorname{ch} \Lambda^i W_i^- \cdot \operatorname{ch} \operatorname{Sym}^{k(g-i)-i} U_i \cdot \operatorname{td} T_{\epsilon^{[i]}} \right)$$

for the morphism

$$\epsilon^{[i]} : \mathcal{C}^{[i]} \to M_{g,1}.$$

Here, td is the Todd class.

We prefer to calculate the push-forward via $\epsilon^{[i]}$ after pull-back via ϕ to $\mathcal{C}^i$. Since we have already determined the Chern characters of L_i, W_i^- and U_i after pull-back via ϕ, the only term left to discuss is the Todd class class $\phi^* T_{\epsilon^{[i]}}$. The bundle $\phi^* T_{\epsilon^{[i]}}$ has fiber $H^0(\mathcal{O}_D(D))$ over the divisor

$$D = p_1 + \ldots + p_i.$$

The Chern roots have been calculated in [4] to be

$$\Delta_j - \Psi_j, \quad 1 \le j \le i.$$

We can then write a formula for the Chern character of N_i,

$$\operatorname{ch} N_i = \frac{1}{i!}\ \epsilon_\star^i \left(\operatorname{ch} L_i^k \cdot \operatorname{ch} \Lambda^i W_i^- \cdot \operatorname{ch} \operatorname{Sym}^{k(g-i)-i} U_i \cdot \prod_{j=1}^{i} \frac{\Delta_j - \Psi_j}{1 - e^{-\Delta_j + \Psi_j}} \right), \tag{13}$$

with respect to the push-forward via

$$\epsilon^i : \mathcal{C}^i \to M_{g,1} \ .$$

Every term of the right side of (13) is determined, so the push-forward can be calculated explicitly in terms of tautologcal classes on $M_{g,1}$.

Together with the flatness relation (3), we obtain relations in the tautological ring $R^*(M_{g,1})$ which can be pushed-down to yield relations in $R^*(M_g)$.

3. Construction of Thaddeus

3.1. **Comparison.** The objects W_i^-, $W_i^{+\star}$, U, and L_i all appear in the study of pairs moduli spaces by Thaddeus [8]. Since he considers only a fixed curve C, the factors $\det^2 \mathbb{E}_g \otimes \mathbb{L}_p$ are absent in his definition of L_i. However for us, the additional twisting of L_i over $M_{g,1}$ plays a crucial role. The treatment by Thaddeus of W_i^- and $W_i^{+\star}$ is sufficiently canonical to be valid over $M_{g,1}$.

We record here the difference in the calculation of our L_i and the line bundle L_i of Thaddeus. For ease of comparison, we follow here the terminology of [8]. Of course for us,

$$d = 2g \quad \text{and} \quad \Lambda = \omega(2\sigma_0)\ .$$

Also, we write $\pi_!$ for $R\pi_\star$.

In Section 5.4 of [8], Thaddeus selects a point of the curve C. Since we are working over $M_{g,1}$, a marking is always there for us. However, the equations of 5.4 must be corrected for twists over $M_{g,1}$. We have

$$\begin{aligned} \wedge^2(\mathbf{E}_0)_p &= \mathcal{O}_0(-1,0)\otimes \mathbb{L}_p^\star, \\ \wedge^2(\mathbf{E}_1)_p &= \mathcal{O}_1(0,-1)\otimes \mathbb{L}_p^\star \end{aligned}$$

where we follow the notation of [8] for the universal sheaves $\mathbf{E}_0$ and $\mathbf{E}_1$ and the line bundles $\mathcal{O}_i(m,n)$. A more important correction appears in the calculation of $\det \pi_! \mathbf{E}_1$ in the middle of 5.4,

$$\det \pi_! \mathbf{E}_1 = \mathcal{O}_1(-1, g-d)\otimes \det^2 \mathbb{E}_g \otimes \mathbb{L}_p^\star. \tag{14}$$

A factor $\det \mathbb{E}_g$ comes from $\det \pi_! \mathcal{O}(E_1^+)$ in the computation of Thaddeus, and a factor $\det \mathbb{E}_g \otimes \mathbb{L}_p^\star$ comes from $\det \pi_! \Lambda(-1)(E_1^+)$. Putting the above together, we find

$$\mathcal{O}_1(m,n) = \det^{-m} \pi_! \mathbf{E}_1 \otimes \otimes \left(\det^2 \mathbb{E}_g \otimes \mathbb{L}_p^\star\right)^m \otimes \left(\wedge^2(\mathbf{E}_1)_p\right)^{(d-g)m-n} \otimes \mathbb{L}_p^{(d-g)m-n}\ .$$

We turn now to Section 3.3 of [8] and consider the restrictions of

$$\det \pi_! \mathbf{E}_i \quad \text{and} \quad \wedge^2 (\mathbf{E}_i)_p$$

to $\mathbb{P}W_i^-$. Following Thaddeus, we drop the subscript i in the notation for $\mathbf{E}_i$. From the main extension equation at the end of the proof, we see

$$\det \pi_! \mathbf{E} = \det \pi_! \Lambda(-\mathcal{D}_i) \otimes H^{d-i-g+1} \otimes \det \pi_! \mathcal{O}_{\mathcal{C}}(\mathcal{D}_i)$$

where H is $\mathcal{O}(1)$ on the projective bundle $\mathbb{P}W_i^-$. Using the same extension, we also find

$$\wedge^2(\mathbf{E})_p = H \otimes \mathbb{L}_p^\star\ .$$

The changes in 3.3 imply corrections for Section 6.5,

$$\begin{aligned}\mathcal{O}_{i-1}(m,n) &= \det{}^{-m}\pi_!\mathbf{E}\otimes\left(\det{}^2\mathbb{E}_g\otimes\mathbb{L}_p^\star\right)^m\otimes\left(\wedge^2(\mathbf{E})_p\right)^{(d-g)m-n}\otimes\mathbb{L}_p^{(d-g)m-n}\\ &= \left(\det{}^{-1}\pi_!\Lambda(-\mathcal{D}_i)\otimes\det{}^{-1}\pi_!\mathcal{O}_{\mathcal{C}}(\mathcal{D}_i)\otimes\det{}^2\mathbb{E}_g\otimes\mathbb{L}_p^\star\right)^m\\ &\qquad\otimes H^{-m(d-i-g+1)}\otimes H^{(d-g)m-n}\otimes(\mathbb{L}_p^\star)^{(d-g)m-n}\otimes\mathbb{L}_p{}^{(d-g)m-n}\\ &= L_i^m\otimes H^{m(i-1)-n}\end{aligned}$$

where we must take now

$$L_i=\det{}^{-1}\pi_!\Lambda(-\mathcal{D}_i)\otimes\det{}^{-1}\pi_!\mathcal{O}_{\mathcal{C}}(\mathcal{D}_i)\otimes\det{}^2\mathbb{E}_g\otimes\mathbb{L}_p^\star\ .$$

While the above modifications are somewhat subtle, the main construction of Thaddeus is very natural for $M_{g,1}$ and goes through beautifully.

3.2. **Twisting the Verlinde bundle.** Following the terminology of Section 2.1, the Verlinde bundles $\widetilde{\mathbb{V}}_{2,k}$ and $\iota^\star\mathbb{V}_{2,k}$ differ by a twist

$$\widetilde{\mathbb{V}}_{2,k}\cong\iota^\star\mathbb{V}_{2,k}\otimes\mathcal{L}\tag{15}$$

by a line bundle $\mathcal{L}$ on $M_{g,1}$.

Proposition 1. *We have* $\widetilde{\mathbb{V}}_{2,k}\cong\iota^*\mathbb{V}_{2,k}\otimes(\wedge^g\mathbb{E}_g)^2$.

Proof. Since both $\widetilde{\Theta}_2$ and $\alpha^\star\Theta_2$ restrict to the positive generator of the Picard group of each fiber of

$$\mathcal{SU}_{g,1}(2,2g)\to M_{g,1},$$

a line bundle $\mathcal{L}$ on $M_{g,1}$ satisfying

$$\widetilde{\Theta}_2\cong\alpha^\star\Theta_2$$

and thus (15) must exist.

Following the notation of [8], let $\mathcal{M}_w$ be the last space of rank 2 stable pairs of fixed determinant $\omega_C(2p)$ over $M_{g,1}$. Let

$$\gamma:\mathcal{M}_w\to\mathcal{SU}_{g,1}(2,2g)$$

be the contraction. By Section 5.9 of [8], the universal theta divisor $\widetilde{\Theta}_2$ which arises from the construction of Thaddeus satisfies

$$\gamma^\star\widetilde{\Theta}_2\cong\mathcal{O}_w(1,g-1)\ .\tag{16}$$

By definition, the canonical theta divisor $\gamma^\star\alpha^\star\Theta_2$ arises from the determinant of cohomology,

$$\gamma^\star\alpha^\star\Theta_2=\det{}^{-1}\left(\pi_!\mathbf{E}_w(-\sigma_0)\right),\tag{17}$$

where $\mathbf{E}_w$ is the univeral sheaf.

Our calculation of the difference between (16) and (17) can be carried out on any of the stable pairs moduli spaces. We choose to work on $\mathcal{M}_1$ which is the simplest. Then,

$$\begin{aligned}\gamma^\star\alpha^\star\Theta_2 &= \det{}^{-1}\left(\pi_!\mathbf{E}_1(-\sigma_0)\right)\\ &= \det{}^{-1}(\pi_!\mathbf{E}_1)\otimes\wedge^2(\mathbf{E}_1)_p\end{aligned}$$

Using the identification of $\det(\pi_!\mathbf{E}_1)$ and $(\mathbf{E}_1)_p$ from Section 3.1, we find

$$\begin{aligned}\gamma^\star\alpha^\star\Theta_2 &= \mathcal{O}_1(1,g-1)\otimes(\wedge^g\mathbb{E}_g)^{-2}\\ &= \gamma^\star\widetilde{\Theta}_2\otimes(\wedge^g\mathbb{E}_g)^{-2}\end{aligned}$$

which is equivalent to the claim of the Proposition. □

As a direct consequence, we conclude a result which is not at all obvious from the formulas for the Chern character of $\widetilde{\mathbb{V}}_{2,k}$.

Proposition 2. *The first Chern class of* $\widetilde{\mathbb{V}}_{2,k}$ *on* $M_{g,1}$ *is proportional to* κ_1.

The class κ_1, pulled-back from M_g via ι, is the generator of $H^2(M_g)$. Let us now find a formula for $\mathrm{ch}_1\,\widetilde{\mathbb{V}}_{2,k}$ ...

4. Genus 2

4.1. **Level 1.** The genus 2 case is not of much interest to us since $R^{>0}(M_2)$ and $R^{>1}(M_{2,1})$ vanish. There is no room for any further non-trivial relations. Nevertheless, we can calculate the Chern character of the Verlinde bundle in level 1. Since

$$(g-i)-i=2-2i$$

is non-negative only for $i=1$, we see

$$\widetilde{\mathbb{V}}_{2,1}=N_0-N_1$$

in the K-theory of $M_{2,1}$ by (4).

We use the formulas of Section 2.4 to find the nonvanishing Chern characters of V,

$$\begin{aligned}\mathrm{ch}_0\,V &= 5,\\ \mathrm{ch}_1\,V &= \frac{13}{12}\kappa_1+\widehat{\Psi}.\end{aligned}$$

Since N_0 is the second symmetric power of V,

$$\begin{aligned}\mathrm{ch}_0\,N_0 &= 15,\\ \mathrm{ch}_1\,N_0 &= \frac{13}{2}\kappa_1+6\widehat{\Psi}.\end{aligned}$$

To calculate the Chern character of N_1, we use formula (13),

$$\begin{aligned}\operatorname{ch} N_1 &= \epsilon^1_\star\left(\operatorname{ch} L_1 \cdot \operatorname{ch} \Lambda^1 W_1^- \cdot \operatorname{ch}\operatorname{Sym}^0 U_1 \cdot \frac{-\Psi_1}{1-e^{\Psi_1}}\right)\\ &= \epsilon^1_\star\left(\operatorname{ch} L_1 \cdot \operatorname{ch} W_1^- \cdot \frac{-\Psi_1}{1-e^{\Psi_1}}\right)\ .\end{aligned}$$

By equations (9) and (12), we have

$$\operatorname{ch}(L_1) = e^{2\Psi_1+2\sigma_1} \quad \text{and} \quad \operatorname{ch} W_1^- = e^{2\Psi_1+2\sigma_1}.$$

After calculating the push-forward, we find

$$\begin{aligned}\operatorname{ch}_0 N_1 &= 11,\\ \operatorname{ch}_1 N_1 &= \frac{73}{12}\kappa_1 + 6\widehat{\Psi}.\end{aligned}$$

Putting the above equations together yields

$$\begin{aligned}\operatorname{ch}_0 \widetilde{\mathbb{V}}_{2,1} &= 4,\\ \operatorname{ch}_1 \widetilde{\mathbb{V}}_{2,1} &= \frac{5}{12}\kappa_1.\end{aligned}$$

Since the ch_0 is the rank, we recover the Verlinde rank calculation by Thaddeus. The Verlinde formula here is

$$\operatorname{rank} \widetilde{\mathbb{V}}_{2,1} = \left(\frac{3}{2\sin^2(\frac{\pi}{3})}\right) + \left(\frac{3}{2\sin^2(\frac{2\pi}{3})}\right) = 4.$$

By the first Chern class calculation, the line bundle $\mathcal{L}$ of equation (2) is pulled-back from M_2. Hence our Verlinde bundle is also pulled-back from M_2.

4.2. **Level 2.** For the Verlinde bundle in level 2,

$$2(g-i) - i = 4 - 3i$$

is non-negative only for $i = 1$. Again, we have

$$\widetilde{\mathbb{V}}_{2,1} = N_0 - N_1$$

in the K-theory of $M_{2,1}$ by (4).

In level 2, N_0 is the fourth symmetric power of V. Hence

$$\begin{aligned}\operatorname{ch}_0 N_0 &= 70,\\ \operatorname{ch}_1 N_0 &= \frac{182}{3}\kappa_1 + 56\widehat{\Psi}.\end{aligned}$$

To calculate the Chern character of N_1, we use formula (13),

$$\begin{aligned}\operatorname{ch} N_1 &= \epsilon^1_\star\left(\operatorname{ch} L_1^2 \cdot \operatorname{ch} \Lambda^1 W_1^- \cdot \operatorname{ch}\operatorname{Sym}^1 U_1 \cdot \frac{-\Psi_1}{1-e^{\Psi_1}}\right)\\ &= \epsilon^1_\star\left(\operatorname{ch} L_1^2 \cdot \operatorname{ch} W_1^- \cdot \operatorname{ch} U_1 \cdot \frac{-\Psi_1}{1-e^{\Psi_1}}\right)\ .\end{aligned}$$

By equations (9) and (12), we have

$$\mathrm{ch}\,(L_1^2) = e^{4\Psi_1+4\sigma_1} \quad \text{and} \quad \mathrm{ch}\,W_1^- = e^{2\Psi_1+2\sigma_1}.$$

The Chern character of the bundle U_1 is determined in (11) by

$$\mathrm{ch}\,U_1 = \mathrm{ch}\,\pi_\star(2\omega) + e^{\widehat{\Psi}} + 1 - e^{3\Psi_1+2\sigma_1}.$$

After calculating the push-forward, we find

$$\begin{aligned} \mathrm{ch}_0\,N_1 &= 60, \\ \mathrm{ch}_1\,N_1 &= \frac{231}{4}\kappa_1 + 56\widehat{\Psi}. \end{aligned}$$

Putting the above equations together yields

$$\begin{aligned} \mathrm{ch}_0\,\widetilde{\mathbb{V}}_{2,1} &= 10, \\ \mathrm{ch}_1\,\widetilde{\mathbb{V}}_{2,1} &= \frac{35}{12}\kappa_1. \end{aligned}$$

We have agreement here with the Verlinde formula,

$$\mathrm{rank}\,\widetilde{\mathbb{V}}_{2,1} = \left(\frac{4}{2\sin^2(\frac{\pi}{4})}\right) + \left(\frac{4}{2\sin^2(\frac{2\pi}{4})}\right) + \left(\frac{4}{2\sin^2(\frac{3\pi}{4})}\right) = 10.$$

5. Example in genus 3

5.1. **Flatness constraint.** We study here the rank 2 and level 1 Verlinde bundle on $R^*(M_{3,1})$. By the Verlinde formula,

$$\mathrm{rank}\,\widetilde{\mathbb{V}}_{2,1} = \left(\frac{3}{2\sin^2(\frac{\pi}{3})}\right)^2 + \left(\frac{3}{2\sin^2(\frac{2\pi}{3})}\right)^2 = 8.$$

Genus 3 is still too low for the flatness constraint to be of much interest. Nevertheless, the calculation will not go unrewarded.

5.2. **The bundle N_0.** We turn now to the geometric calculation of the Chern characters of $\widetilde{\mathbb{V}}_{2,1}$. Since

$$(g-i) - i = 3 - 2i$$

is non-negative only for $i = 1$, we see

$$\widetilde{\mathbb{V}}_{2,1} = N_0 - N_1$$

in the K-theory of $M_{4,1}$ by (4).

In order to calculate the first few Chern characters of N_0, we use the formulas of Section 2.4 to find the nonvanishing Chern characters of V,

$$\begin{aligned}
\mathrm{ch}_0\, V &= 8,\\
\mathrm{ch}_1\, V &= \frac{13}{12}\kappa_1 + \widehat{\Psi},\\
\mathrm{ch}_2\, V &= \frac{1}{2}\widehat{\Psi}^2,
\end{aligned}$$

where we have used the well-known vanishing of κ_2 in $R^*(M_3)$ and thus in $R^*(M_{3,1})$. We will impose the vanishing of $R^2(M_3)$ and $R^3(M_{3,1})$ in all our calculations. Since N_0 is the third symmetric power of V,

$$\begin{aligned}
\mathrm{ch}_0\, N_0 &= 120,\\
\mathrm{ch}_1\, N_0 &= \frac{195}{4}\kappa_1 + 45\widehat{\Psi},\\
\mathrm{ch}_2\, N_0 &= \frac{65}{6}\kappa_1\widehat{\Psi} + \frac{65}{2}\widehat{\Psi}^2.
\end{aligned}$$

5.3. **The object** N_1**.** Our next task is to calculate the Chern character of N_1. By formula (13), we see

$$\begin{aligned}
\mathrm{ch}\, N_1 &= \epsilon^1_\star\left(\mathrm{ch}\, L_1 \cdot \mathrm{ch}\,\Lambda^1 W_1^- \cdot \mathrm{ch}\,\mathrm{Sym}^1\, U_1 \cdot \frac{-\Psi_1}{1-e^{\Psi_1}}\right)\\
&= \epsilon^1_\star\left(\mathrm{ch}\, L_1 \cdot \mathrm{ch}\, W_1^- \cdot \mathrm{ch}\, U_1 \cdot \frac{-\Psi_1}{1-e^{\Psi_1}}\right)\ .
\end{aligned}$$

By equations (9) and (12), we have

$$\mathrm{ch}\,(L_1) = e^{2\Psi_1+2\sigma_1} \quad \text{and} \quad \mathrm{ch}\, W_1^- = e^{2\Psi_1+2\sigma_1}.$$

The bundle U_1 is determined as a K-theoretic difference in (11),

$$\mathrm{ch}\, U_1 = \mathrm{ch}\,\pi_\star(2\omega) + e^{\widehat{\Psi}} + 1 - e^{3\Psi_1+2\sigma_1}.$$

Putting the above equations together yields

$$\begin{aligned}
\mathrm{ch}_0\, N_1 &= 112,\\
\mathrm{ch}_1\, N_1 &= \frac{565}{12}\kappa_1 + 45\widehat{\Psi},\\
\mathrm{ch}_2\, N_1 &= \frac{151}{12}\kappa_1\widehat{\Psi} + \frac{51}{2}\widehat{\Psi}^2.
\end{aligned}$$

5.4. **Flatness constraint.** We now have enough information to calculate the Chern characters of the Verlinde bundle,

$$\begin{aligned} \mathrm{ch}_0\, \widetilde{\mathbb{V}}_{2,1} &= 8, \\ \mathrm{ch}_1\, \widetilde{\mathbb{V}}_{2,1} &= \frac{5}{3}\kappa_1, \\ \mathrm{ch}_2\, \widetilde{\mathbb{V}}_{2,1} &= -\frac{7}{4}\kappa_1 \widehat{\Psi} + 7\widehat{\Psi}^2. \end{aligned}$$

The flatness constraint (3) requires

$$\mathrm{ch}_2\, \widetilde{\mathbb{V}}_{2,1} = \frac{1}{16}\mathrm{ch}_1^2\, \widetilde{\mathbb{V}}_{2,1} = \frac{25}{144}\kappa_1^2 = 0 \in H^4(M_{3,1})\ .$$

Since $R^2(M_{3,1}) \cong \mathbb{Q}$, we can check the vanishing after push-forward to M_3,

$$\iota : M_{3,1} \to M_3\ .$$

We easily calculate

$$\iota_*(\mathrm{ch}_2\, \widetilde{\mathbb{V}}_{2,1}) = -7\kappa_1 + 7\kappa_1 = 0\ \in R^1(M_3)\ .$$

6. Example in genus 4

6.1. **Flatness constraint.** We compute here the Chern character of the Verlinde bundle $\widetilde{\mathbb{V}}_{2,1}$ on $R^*(M_{4,1})$. By the Verlinde formula,

$$\mathrm{rank}\, \widetilde{\mathbb{V}}_{2,1} = \left(\frac{3}{2\sin^2(\frac{\pi}{3})}\right)^3 + \left(\frac{3}{2\sin^2(\frac{2\pi}{3})}\right)^3 = 16.$$

The flattness constraint (3) takes the following form:

$$\begin{aligned} \mathrm{ch}\, \widetilde{\mathbb{V}}_{2,1} &= 16\exp\left(\frac{\mathrm{ch}_1(\widetilde{\mathbb{V}}_{2,1})}{16}\right) \\ &= 16 + \mathrm{ch}_1(\widetilde{\mathbb{V}}_{2,1}) + \frac{1}{32}\mathrm{ch}_1(\widetilde{\mathbb{V}}_{2,1})^2 + \frac{1}{1536}\mathrm{ch}_1(\widetilde{\mathbb{V}}_{2,1})^3 + \dots\ . \end{aligned} \tag{18}$$

6.2. **The bundle N_0.** We now consider the geometric calculation of the Chern characters of $\widetilde{\mathbb{V}}_{2,1}$. Since

$$(g-i) - i = 4 - 2i$$

is non-negative only for $i = 1$ and 2, we see

$$\mathbb{V}_{2,1} = N_0 - N_1 + N_2$$

in the K-theory of $M_{4,1}$ by (4).

In order to calculate the first few Chern characters of N_0, we use the formulas of Section 2.4 to find the nonvanishing Chern characters of V,

$$\begin{aligned}
\mathrm{ch}_0\, V &= 11,\\
\mathrm{ch}_1\, V &= \frac{13}{12}\kappa_1 + \widehat{\Psi},\\
\mathrm{ch}_2\, V &= \frac{1}{2}\kappa_2 + \frac{1}{2}\widehat{\Psi}^2,\\
\mathrm{ch}_3\, V &= \frac{1}{6}\widehat{\Psi}^3,
\end{aligned}$$

where we have used the well-known vanishing of κ_3 in $R^*(M_4)$ and thus in $R^*(M_{4,1})$. We will impose the vanishing of $R^3(M_4)$ and $R^4(M_{4,1})$ in all our calculations. Since N_0 is the fourth symmetric power of V,

$$\begin{aligned}
\mathrm{ch}_0\, N_0 &= 1001,\\
\mathrm{ch}_1\, N_0 &= \frac{1183}{3}\kappa_1 + 364\widehat{\Psi},\\
\mathrm{ch}_2\, N_0 &= \frac{455}{2}\kappa_2 + \frac{15379}{288}\kappa_1^2 + \frac{1183}{12}\kappa_1\widehat{\Psi} + 273\widehat{\Psi}^2,\\
\mathrm{ch}_3\, N_0 &= \frac{105}{2}\kappa_2\widehat{\Psi} + \frac{1183}{144}\kappa_1^2\widehat{\Psi} + \frac{1547}{24}\kappa_1\widehat{\Psi}^2 + \frac{497}{3}\widehat{\Psi}^3.
\end{aligned}$$

6.3. **On Sym and Λ.** Suppose C is a bundle written in K-theory as a virtual difference

$$C = A - B$$

of two bundles. We can calculate the Chern character of Sym^*C and Λ^*C in terms of the Chern characters of A and B. For example, we have in K-theory

$$\mathrm{Sym}^2 A = \mathrm{Sym}^2 C + \mathrm{Sym}^2 B + C \otimes B\ .$$

After rewriting, we find

$$\mathrm{Sym}^2 C = \mathrm{Sym}^2 A - \mathrm{Sym}^2 B - A \otimes B + B \otimes B$$

which easily leads to the desired Chern character formulas. Similarly,

$$\Lambda^2 C = \Lambda^2 A - \Lambda^2 B - A \otimes B + B \otimes B\ .$$

Formulas for the higher symmetric and wedge products are obtain in the same manner.

Since the bundles W_i, W_i^+, and U_i have been determined in K-theory in Section 2.4 as virtual differences, we will require such formulas in the computations below.

6.4. **The object N_1.** Our next task is to calculate the Chern character of N_1. By formula (13), we see

$$\mathrm{ch}\, N_1 = \epsilon^1_\star \left(\mathrm{ch}\, L_1 \cdot \mathrm{ch}\, W_1^- \cdot \mathrm{ch}\,\mathrm{Sym}^2\, U_1 \cdot \frac{-\Psi_1}{1-e^{\Psi_1}} \right).$$

By equations (9) and (12), we have

$$\mathrm{ch}\,(L_1) = e^{2\Psi_1+2\sigma_1} \quad \text{and} \quad \mathrm{ch}\, W_1^- = e^{2\Psi_1+2\sigma_1}.$$

The bundle U_1 is determined as a K-theoretic difference in (11),

$$\mathrm{ch}\, U_1 = \mathrm{ch}\, \pi_\star(2\omega) + e^{\widehat{\Psi}} + 1 - e^{3\Psi_1+2\sigma_1}.$$

We can write $U_1 = A - B$ where A is rank 11, B is rank 1, and

$$\mathrm{ch}\, A = \mathrm{ch}\, \pi_\star(2\omega) + e^{\widehat{\Psi}} + 1, \quad \mathrm{ch}\, B = e^{3\Psi_1+2\sigma_1}.$$

More explicitly, the Chern characters of A are

$$\begin{aligned}
\mathrm{ch}_0\, A &= 11,\\
\mathrm{ch}_1\, A &= \frac{13}{12}\kappa_1 + \widehat{\Psi},\\
\mathrm{ch}_2\, A &= \frac{1}{2}\kappa_2 + \frac{1}{2}\widehat{\Psi}^2,\\
\mathrm{ch}_3\, A &= \frac{1}{6}\widehat{\Psi}^3,
\end{aligned}$$

with higher Chern characters vanishing in $R^*(M_{4,1})$ and thus in $R^*(\mathcal{C}^1)$. We find

$$\begin{aligned}
\mathrm{ch}_0\, \mathrm{Sym}^2 A &= 66,\\
\mathrm{ch}_1\, \mathrm{Sym}^2 A &= 13\kappa_1 + 12\widehat{\Psi},\\
\mathrm{ch}_2\, \mathrm{Sym}^2 A &= \frac{13}{2}\kappa_2 + \frac{169}{288}\kappa_1^2 + \frac{13}{12}\kappa_1\widehat{\Psi} + 7\widehat{\Psi}^2,\\
\mathrm{ch}_3\, \mathrm{Sym}^2 A &= \frac{1}{2}\kappa_2\widehat{\Psi} + \frac{13}{24}\kappa_1\widehat{\Psi}^2 + 3\widehat{\Psi}^3.
\end{aligned}$$

By the discussion in Section 6.3,

$$\mathrm{ch}\,\mathrm{Sym}^2 U_1 = \mathrm{ch}\,\mathrm{Sym}^2 A - \mathrm{ch}\, A \cdot \mathrm{ch}\, B\ .$$

Putting the above equations together yields

$$\begin{aligned}
\mathrm{ch}_0\, N_1 &= 1155,\\
\mathrm{ch}_1\, N_1 &= \frac{2675}{6}\kappa_1 + 420\widehat{\Psi},\\
\mathrm{ch}_2\, N_1 &= \frac{203}{2}\kappa_2 + \frac{20423}{288}\kappa_1^2 + \frac{537}{4}\kappa_1\widehat{\Psi} + 165\widehat{\Psi}^2,\\
\mathrm{ch}_3\, N_1 &= \frac{159}{2}\kappa_2\widehat{\Psi} + \frac{91}{9}\kappa_1^2\widehat{\Psi} + \frac{2251}{24}\kappa_1\widehat{\Psi}^2 - 275\widehat{\Psi}^3.
\end{aligned}$$

6.5. **The object N_2.** In order to calculate the Chern character of N_2, we use the equation

$$\begin{aligned}\operatorname{ch} N_2 &= \frac{1}{2}\epsilon_\star^2\left(\operatorname{ch} L_2\cdot\operatorname{ch}\Lambda^2 W_2^-\cdot\operatorname{ch}\operatorname{Sym}^0 U_2\cdot\frac{-\Psi_1}{1-e^{\Psi_1}}\cdot\frac{\Delta-\Psi_2}{1-e^{-\Delta+\Psi_2}}\right)\\ &= \frac{1}{2}\epsilon_\star^2\left(\operatorname{ch} L_2\cdot\operatorname{ch}\det W_2^-\cdot\frac{-\Psi_1}{1-e^{\Psi_1}}\cdot\frac{\Delta-\Psi_2}{1-e^{-\Delta+\Psi_2}}\right).\end{aligned}$$

By equations (9) and (12), we have

$$\begin{aligned}\operatorname{ch}(L_2) &= e^{-2\Delta+2\Psi_1+2\Psi_2+2\sigma_1+2\sigma_2},\\ \operatorname{ch}\det W_2^- &= e^{-3\Delta+2\Psi_1+2\Psi_2+2\sigma_1+2\sigma_2}.\end{aligned}$$

Putting the above equations together yields

$$\begin{aligned}\operatorname{ch}_0 N_2 &= 170,\\ \operatorname{ch}_1 N_2 &= \frac{329}{6}\kappa_1+56\widehat{\Psi},\\ \operatorname{ch}_2 N_2 &= -\frac{267}{2}\kappa_2+\frac{5329}{144}\kappa_1^2+\frac{73}{2}\kappa_1\widehat{\Psi}-113\widehat{\Psi}^2,\\ \operatorname{ch}_3 N_2 &= 42\kappa_2\widehat{\Psi}+\frac{511}{12}\kappa_1\widehat{\Psi}^2-\frac{1652}{3}\widehat{\Psi}^3.\end{aligned}$$

6.6. **Chern characters.** We now have enough information to calculate the Chern characters of the Verlinde bundle,

$$\begin{aligned}\operatorname{ch}_0\widetilde{\mathbb{V}}_{2,1} &= 16,\\ \operatorname{ch}_1\widetilde{\mathbb{V}}_{2,1} &= \frac{10}{3}\kappa_1,\\ \operatorname{ch}_2\widetilde{\mathbb{V}}_{2,1} &= -\frac{15}{2}\kappa_2+\frac{95}{96}\kappa_1^2+\frac{5}{6}\kappa_1\widehat{\Psi}-5\widehat{\Psi}^2,\\ \operatorname{ch}_3\widetilde{\mathbb{V}}_{2,1} &= 15\kappa_2\widehat{\Psi}-\frac{91}{48}\kappa_1^2\widehat{\Psi}+\frac{53}{4}\kappa_1\widehat{\Psi}^2-110\widehat{\Psi}^3.\end{aligned}$$

The flatness relation occuring in degree 2 is

$$\operatorname{ch}_2\widetilde{\mathbb{V}}_{2,1}-\frac{1}{32}\operatorname{ch}_1^2\widetilde{\mathbb{V}}_{2,1}=0\ \in H^4(M_{4,1}).$$

After expanding the left side, we find the relation

$$-\frac{15}{2}\kappa_2+\frac{185}{288}\kappa_1^2+\frac{5}{6}\kappa_1\widehat{\Psi}-5\widehat{\Psi}^2=0\in H^4(M_{4,1})$$

which can be checked to hold. The flatness relation in degree 3 is

$$\operatorname{ch}_3\widetilde{\mathbb{V}}_{2,1}=\frac{1}{1536}\operatorname{ch}_1^2\widetilde{\mathbb{V}}_{2,1}=\frac{25}{3456}\kappa_1^3=0\in H^6(M_{4,1})$$

which is also true.

7. Example in genus 5

The results for the Verlinde bundle of rank 2 and level 1 on the moduli space $M_{5,1}$ are given below:

$$
\begin{aligned}
\operatorname{ch}_0 \widetilde{\mathbb{V}}_{2,1} &= 32, \\
\operatorname{ch}_1 \widetilde{\mathbb{V}}_{2,1} &= \frac{20}{3}\kappa_1, \\
\operatorname{ch}_2 \widetilde{\mathbb{V}}_{2,1} &= \frac{5}{2}\kappa_2 + \frac{25}{48}\kappa_1^2, \\
\operatorname{ch}_3 \widetilde{\mathbb{V}}_{2,1} &= \frac{17303}{72}\kappa_3 - \frac{601}{24}\kappa_2\kappa_1 + \frac{9763}{10368}\kappa_1^3 \\
&\quad - \frac{47}{2}\kappa_2\widehat{\Psi} + \frac{701}{288}\kappa_1^2\widehat{\Psi} - \frac{121}{6}\kappa_1\widehat{\Psi}^2 + 198\widehat{\Psi}^3, \\
\operatorname{ch}_4 \widetilde{\mathbb{V}}_{2,1} &= -\frac{9445}{48}\kappa_3\widehat{\Psi} + \frac{589}{24}\kappa_2\kappa_1\widetilde{\Psi} - \frac{1183}{1152}\kappa_1^3\widetilde{\Psi} \\
&\quad - \frac{363}{4}\kappa_2\widehat{\Psi}^2 + \frac{6343}{576}\kappa_1^2\widehat{\Psi}^2 - \frac{707}{4}\kappa_1\widehat{\Psi}^3 + 2192\widehat{\Psi}^4.
\end{aligned}
$$

The associated flatness relations can be verified to hold in $H^*(M_{5,1})$ by known results governing the tautological ring in genus 5.

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