TAUTOLOGICAL PROJECTION FOR CYCLES ON THE MODULI SPACE OF ABELIAN VARIETIES

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Abstract. We define a tautological projection operator for algebraic cycle classes on the moduli space of principally polarized abelian varieties $\mathcal{A}_g$: every cycle class decomposes canonically as a sum of a tautological and a non-tautological part. The main new result required for the definition of the projection operator is the vanishing of the top Chern class of the Hodge bundle over the boundary $\overline{\mathcal{A}}_g \smallsetminus \mathcal{A}_g$ of any toroidal compactification $\overline{\mathcal{A}}_g$ of the moduli space $\mathcal{A}_g$. We prove the vanishing by a careful study of residues in the boundary geometry.

The existence of the projection operator raises many natural questions about cycles on $\mathcal{A}_g$. We calculate the projections of all product cycles $\mathcal{A}_{g_1} \times \ldots \times \mathcal{A}_{g_\ell}$ in terms of Schur determinants, discuss Faber's earlier calculations related to the Torelli locus, and state several open questions. The Appendix contains a conjecture about the projection of the locus of abelian varieties with real multiplication.

Contents

1. Introduction

1.1. Tautological rings of $\mathcal{A}_g$ and $\overline{\mathcal{A}}_g$. Let $\mathcal{A}_g \subset \overline{\mathcal{A}}_g$ be a toroidal compactification of the moduli space of principally polarized abelian varieties. The space $\mathcal{A}_g$ is a nonsingular Deligne-Mumford stack of dimension $\binom{g+1}{2}$, and the compactification $\overline{\mathcal{A}}_g$ is a reduced and irreducible (but possibly singular) proper Deligne-Mumford stack, see [FC]. The Hodge bundle

$$\mathbb{E} \to \mathcal{A}_g$$

is defined as the pullback to $\mathcal{A}_g$ via the zero section s of the relative cotangent bundle of the universal family $\mathcal{X}_g$ of abelian varieties,

$$p : \mathcal{X}_g \to \mathcal{A}_g\,, \quad s : \mathcal{A}_g \to \mathcal{X}_g\,, \quad \mathbb{E} \cong s^*\Omega_p\,.$$

There is a canonical extension of the Hodge bundle over $\overline{\mathcal{A}}_g$ by [FC, Theorems V.2.3, VI.1.1, VI.4.2],

$$\mathbb{E} \to \overline{\mathcal{A}}_g\,.$$

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By [vdG1, EV], the Chern classes λ_i of the Hodge bundle satisfy Mumford's relation[1]:

$$(1) \qquad (1+\lambda_1+\lambda_2+\ldots+\lambda_g)(1-\lambda_1+\lambda_2-\ldots+(-1)^g\lambda_g) = 1 \in \mathsf{CH}^{\mathsf{op}}(\overline{\mathcal{A}}_g)\,,$$

for all $g \geq 1$. Van der Geer [vdG1] defined the tautological rings

$$\mathsf{R}^*(\mathcal{A}_g) \subset \mathsf{CH}^*(\mathcal{A}_g)\,, \qquad \mathsf{R}^*(\overline{\mathcal{A}}_g) \subset \mathsf{CH}^{\mathsf{op}}(\overline{\mathcal{A}}_g)$$

to be the $\mathbb{Q}$-subalgebras generated by the λ-classes. Both tautological rings are calculated by a fundamental result of [vdG1].

Theorem 1 (van der Geer)**.** *The following properties hold:*

(i) *The kernel of the quotient*

$$\mathbb{Q}[\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_g] \to \mathsf{R}^*(\overline{\mathcal{A}}_g) \to 0$$

is generated as an ideal by Mumford's relation (1)*.*

(ii) $\mathsf{R}^*(\overline{\mathcal{A}}_g)$ *is a Gorenstein local ring with socle in codimension* $\binom{g+1}{2}$,

$$\mathsf{R}^{\binom{g+1}{2}}(\overline{\mathcal{A}}_g) \cong \mathbb{Q}\,.$$

The class $\lambda_1\lambda_2\lambda_3\cdots\lambda_g$ *is a generator of the socle.*

(iii) $\mathsf{R}^*(\mathcal{A}_g) \cong \mathsf{R}^*(\overline{\mathcal{A}_g})/(\lambda_g)$ *is a Gorenstein local ring with socle in codimension* $\binom{g}{2}$,

$$\mathsf{R}^{\binom{g}{2}}(\mathcal{A}_g) \cong \mathbb{Q}\,.$$

The class $\lambda_1\lambda_2\lambda_3\cdots\lambda_{g-1}$ *is a generator of the socle.*

1.2. **Tautological projection for $\overline{\mathcal{A}}_g$.** The idea of tautological projection on $\overline{\mathcal{A}}_g$ (Definition 2 below) appears in work of Faber [Fa] and of Grushevsky and Hulek [GH]. Since $\overline{\mathcal{A}}_g$ is proper of dimension $\binom{g+1}{2}$, we obtain an evaluation

$$\epsilon^{\mathsf{cpt}} : \mathsf{R}^{\binom{g+1}{2}}(\overline{\mathcal{A}}_g) \to \mathbb{Q}\,, \qquad \alpha \mapsto \int_{\overline{\mathcal{A}}_g} \alpha\,,$$

and a pairing between classes on $\overline{\mathcal{A}}_g$,

$$(2) \qquad \langle\,,\,\rangle^{\mathsf{cpt}} : \mathsf{CH}^k(\overline{\mathcal{A}}_g) \times \mathsf{R}^{\binom{g+1}{2}-k}(\overline{\mathcal{A}}_g) \to \mathbb{Q}\,, \qquad \langle\gamma,\delta\rangle^{\mathsf{cpt}} = \int_{\overline{\mathcal{A}}_g} \gamma\cdot\delta\,.$$

Here, cpt stands for compact.

By Theorem 1(ii), the socle of $\mathsf{R}^*(\overline{\mathcal{A}}_g)$ is spanned by the class $\lambda_1\lambda_2\lambda_3\cdots\lambda_g$. Equivalently,

$$(3) \qquad \gamma_g = \epsilon^{\mathsf{cpt}}(\lambda_1\lambda_2\lambda_3\cdots\lambda_g) \neq 0\,.$$

The exact evaluation[2],

$$(4) \qquad \gamma_g = \prod_{i=1}^{g} \frac{|B_{2i}|}{4i}\,,$$

[1] Since $\overline{\mathcal{A}}_g$ is possibly singular, even as a stack, some care must be taken with the Chow theories. Here, $\mathsf{CH}^{\mathsf{op}}$ is the $\mathbb{Q}$-algebra of operational Chow classes. Usual Chow cycle theory, indexed by codimension, is denoted by CH^*.

[2] Here, B_{2i} is the Bernoulli number.

is computed in [vdG1, page 9]. By the Gorenstein property of $\mathsf{R}^*(\overline{\mathcal{A}}_g)$, the pairing of tautological classes

$$\mathsf{R}^k(\overline{\mathcal{A}}_g) \times \mathsf{R}^{\binom{g+1}{2}-k}(\overline{\mathcal{A}}_g) \to \mathsf{R}^{\binom{g+1}{2}}(\overline{\mathcal{A}}_g) \cong \mathbb{Q}$$

is non-degenerate (where the last isomorphism is via ϵ^{cpt}).

Definition 2. Let $\gamma \in \mathsf{CH}^*(\overline{\mathcal{A}}_g)$. The *tautological projection* $\mathsf{taut}^{\mathsf{cpt}}(\gamma) \in \mathsf{R}^*(\overline{\mathcal{A}}_g)$ is the unique[3] tautological class which satisfies

$$\langle \mathsf{taut}^{\mathsf{cpt}}(\gamma), \delta \rangle^{\mathsf{cpt}} = \langle \gamma, \delta \rangle^{\mathsf{cpt}}$$

for all classes $\delta \in \mathsf{R}^*(\overline{\mathcal{A}}_g)$.

• If $\gamma \in \mathsf{R}^*(\overline{\mathcal{A}}_g)$, then $\gamma = \mathsf{taut}^{\mathsf{cpt}}(\gamma)$, so we have a $\mathbb{Q}$-linear projection operator:

$$\mathsf{taut}^{\mathsf{cpt}} : \mathsf{CH}^*(\overline{\mathcal{A}}_g) \to \mathsf{R}^*(\overline{\mathcal{A}}_g)\,, \quad \mathsf{taut}^{\mathsf{cpt}} \circ \mathsf{taut}^{\mathsf{cpt}} = \mathsf{taut}^{\mathsf{cpt}}\,.$$

• From the point of view of $\mathcal{A}_g$, a difficulty with the theory on $\overline{\mathcal{A}}_g$ is the dependence upon compactification. Given a subvariety

$$V \subset \mathcal{A}_g\,,$$

we can define a projection

$$\mathsf{taut}^{\mathsf{cpt}}([\overline{V}]) \in \mathsf{CH}^*(\overline{\mathcal{A}}_g) \tag{5}$$

with respect to the Zariski closure $V \subset \overline{V}$ in a toroidal compactification $\mathcal{A}_g \subset \overline{\mathcal{A}}_g$, but the projection (5) will depend upon the choice of $\overline{\mathcal{A}}_g$. In order to study cycles on the moduli of abelian varieties, we would like to construct a canonical projection operator depending just upon $\mathcal{A}_g$.

1.3. Top Chern class of the Hodge bundle.

To define a tautological projection operator on the interior $\mathcal{A}_g$, we will define a pairing similar to (2). The theory depends upon a new vanishing result for the top Chern class of Hodge bundle on $\overline{\mathcal{A}}_g$.

We recall the λ_g-pairing on tautological classes[4] on the moduli space of curves of compact type $\mathcal{M}_g^{\mathrm{ct}} \subset \overline{\mathcal{M}}_g$. On the tautological ring of $\mathcal{M}_g^{\mathrm{ct}}$, the λ_g-pairing is given by

$$\mathsf{R}^k(\mathcal{M}_g^{\mathrm{ct}}) \times \mathsf{R}^{2g-3-k}(\mathcal{M}_g^{\mathrm{ct}}) \to \mathsf{R}^{2g-3}(\mathcal{M}_g^{\mathrm{ct}}) \cong \mathbb{Q}\,, \quad (\alpha, \beta) \mapsto \int_{\overline{\mathcal{M}}_g} \overline{\alpha} \cdot \overline{\beta} \cdot \lambda_g\,,$$

where $\overline{\alpha}$ and $\overline{\beta}$ are arbitrary lifts of α and β to $\overline{\mathcal{M}}_g$. The λ_g-pairing is well-defined, independent of the lifts, because $\lambda_g \in \mathsf{R}^g(\overline{\mathcal{M}}_g)$ restricts trivially to the boundary

$$\lambda_g|_{\overline{\mathcal{M}}_g \smallsetminus \mathcal{M}_g^{\mathrm{ct}}} = 0\,,$$

see [FP2]. A parallel boundary vanishing for $\lambda_g \in \mathsf{R}^*(\overline{\mathcal{A}}_g)$ is our first result.

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[3]The existence and uniqueness of $\mathsf{taut}^{\mathsf{cpt}}(\gamma)$ follows from the Gorenstein property of $\mathsf{R}^*(\overline{\mathcal{A}}_g)$ applied to the functional $\delta \mapsto \langle \gamma, \delta \rangle^{\mathsf{cpt}}$ on $\mathsf{R}^*(\overline{\mathcal{A}}_g)$.

[4]We refer the reader to [FP3, P] for a review of the theory of tautological classes on the moduli spaces of curves. Unlike the case of $\mathcal{A}_g$, the tautological ring $\mathsf{R}^*(\mathcal{M}_g^{\mathrm{ct}})$ is *not* a Gorenstein local ring, see [CLS, Pix], and [Pet] for the pointed case.

Theorem 3. *The restriction of λ_g to $\overline{\mathcal{A}}_g \smallsetminus \mathcal{A}_g$ vanishes for every toroidal compactification $\overline{\mathcal{A}}_g$.*

In characteristic p, the Theorem 3 can be proven[5] by considering the p-rank zero locus in $\overline{\mathcal{A}}_g$. The p-rank zero locus avoids the boundary and has class in $\mathsf{R}^*(\overline{\mathcal{A}}_g)$ equal to a multiple of λ_g, [vdG1, Theorem 2.4]. In all characteristics, the vanishing of λ_g over the boundary of the partial compactification $\mathcal{A}_g^{\mathrm{part}}$ of torus rank 1 degenerations follows from the discussion of [vdG1, page 6]. The statement for the entire boundary is new.

Our proof of Theorem 3 is obtained as a consequence of the following statements about semistable degenerations of abelian varieties:

(i) The sheaf of relative log differentials has a trivial rank 1 quotient on the singularities of the fibers of the universal family. The trivial quotient statement is true for any semistable family, independently of abelian structure (also applying, for example, to families of curves).

(ii) For abelian schemes, the sheaf of relative log differentials is isomorphic to the pullback of the Hodge bundle [FC].

The full proof is presented in Section 2 after a review of log structures, semistable degenerations, and residues.

The vanishing of Theorem 3 of the top Chern class of the Hodge bundle for the moduli of abelian varieties implies the parallel vanishing for the moduli of curves,

$$\lambda_g|_{\overline{\mathcal{A}}_g \smallsetminus \mathcal{A}_g} = 0 \quad \Longrightarrow \quad \lambda_g|_{\overline{\mathcal{M}}_g \smallsetminus \mathcal{M}_g^{\mathrm{ct}}} = 0\,,$$

via the Torelli map from $\overline{\mathcal{M}}_g$ to a suitable[6] toroidal compactification $\overline{\mathcal{A}}_g$. We can hope for an even deeper connection: a lifting of Pixton's formula [HMPPS, JPPZ] for λ_g on $\overline{\mathcal{M}}_g$ to $\overline{\mathcal{A}}_g$. The natural context for such a lifting should be the logarithmic Chow ring of the moduli space of abelian varieties. A discussion of these ideas is presented in Section 2.6.

1.4. **Tautological projection for $\mathcal{A}_g$.** Given $\alpha \in \mathsf{CH}^*(\mathcal{A}_g)$, we define an evaluation,

$$\epsilon^{\mathrm{ab}} : \mathsf{CH}^{\binom{g}{2}}(\mathcal{A}_g) \to \mathbb{Q}\,, \qquad \alpha \mapsto \int_{\overline{\mathcal{A}}_g} \overline{\alpha} \cdot \lambda_g \;=\; \deg(\lambda_g \cap \overline{\alpha})\,,$$

where $\overline{\alpha}$ is a lift to the toroidal compactification $\overline{\mathcal{A}}_g$. The answer is well-defined (independent of lift) by the vanishing of Theorem 3. We also have an induced pairing between classes on $\mathcal{A}_g$,

$$\langle\, , \,\rangle : \mathsf{CH}^k(\mathcal{A}_g) \times \mathsf{R}^{\binom{g}{2}-k}(\mathcal{A}_g) \to \mathbb{Q}\,, \qquad \langle \gamma, \delta \rangle = \int_{\overline{\mathcal{A}}_g} \overline{\gamma} \cdot \overline{\delta} \cdot \lambda_g = \deg(\overline{\delta} \cdot \lambda_g \cap \overline{\gamma})\,, \tag{6}$$

which we call the λ_g-*pairing* for the moduli of abelian varieties. Here, $\overline{\delta}$ is a lift of

$$\delta \in \mathsf{R}^*(\mathcal{A}_g) = \mathsf{R}^*(\overline{\mathcal{A}}_g)/\langle \lambda_g \rangle$$

to a tautological class on $\overline{\mathcal{A}}_g$, while $\overline{\gamma}$ is an arbitrary lift of γ. The lift of δ is well-defined up to the class λ_g, while the lift of γ is well-defined up to cycles supported on the boundary. The vanishing of λ_g^2 on $\overline{\mathcal{A}}_g$ and Theorem 3 ensure that the λ_g-pairing is well-defined.

[5]We thank van der Geer for the characteristic p argument.

[6]The second Voronoi compactification can be taken here [A1], [N].

By Theorem 1(iii), the socle of $\mathsf{R}^*(\mathcal{A}_g)$ is spanned by the class $\lambda_1\lambda_2\lambda_3\cdots\lambda_{g-1}$. The Gorenstein property of $\mathsf{R}^*(\mathcal{A}_g)$ together with the non-vanishing (3) implies that the restriction of the λ_g-pairing to tautological classes

$$\mathsf{R}^k(\mathcal{A}_g) \times \mathsf{R}^{\binom{g}{2}-k}(\mathcal{A}_g) \to \mathsf{R}^{\binom{g}{2}}(\mathcal{A}_g) \cong \mathbb{Q}$$

is non-degenerate (where the last isomorphism is via ϵ^{ab}).

Definition 4. Let $\gamma \in \mathsf{CH}^*(\mathcal{A}_g)$. The *tautological projection* $\mathsf{taut}(\gamma) \in \mathsf{R}^*(\mathcal{A}_g)$ is the unique[7] tautological class which satisfies

$$\langle \mathsf{taut}(\gamma), \delta \rangle = \langle \gamma, \delta \rangle$$

for all classes $\delta \in \mathsf{R}^*(\mathcal{A}_g)$.

- If $\gamma \in \mathsf{R}^*(\mathcal{A}_g)$, then $\gamma = \mathsf{taut}(\gamma)$, so we have a $\mathbb{Q}$-linear projection operator:

$$\mathsf{taut} : \mathsf{CH}^*(\mathcal{A}_g) \to \mathsf{R}^*(\mathcal{A}_g)\,, \quad \mathsf{taut} \circ \mathsf{taut} = \mathsf{taut}\,.$$

- For $\gamma \in \mathsf{CH}^*(\mathcal{A}_g)$, tautological projection provides a canonical decomposition

$$\gamma = \mathsf{taut}(\gamma) + (\gamma - \mathsf{taut}(\gamma))$$

into purely tautological and purely non-tautological parts.
- Tautological projection commutes with restriction: for every toroidal compactification $\mathcal{A}_g \subset \overline{\mathcal{A}}_g$ and every class $\gamma \in \mathsf{CH}^*(\overline{\mathcal{A}}_g)$,

$$\mathsf{taut}^{\mathsf{cpt}}(\gamma)\big|_{\mathcal{A}_g} = \mathsf{taut}\left(\gamma\big|_{\mathcal{A}_g}\right)\,.$$

To prove the restriction property, consider classes

$$\gamma \in \mathsf{CH}^*(\overline{\mathcal{A}}_g) \ \text{ and } \ \delta \in \mathsf{R}^*(\mathcal{A}_g)\,.$$

Equations (2) and (6) imply the compatibility between pairings

$$\langle \gamma\big|_{\mathcal{A}_g}, \delta \rangle = \langle \gamma, \overline{\delta}\,\lambda_g \rangle^{\mathsf{cpt}}, \tag{7}$$

where $\overline{\delta}$ is any lift of δ to the compactification $\overline{\mathcal{A}}_g$. Then,

$$\begin{aligned} \langle \mathsf{taut}^{\mathsf{cpt}}(\gamma), \overline{\delta}\,\lambda_g \rangle^{\mathsf{cpt}} = \langle \gamma, \overline{\delta}\lambda_g \rangle^{\mathsf{cpt}} \quad &\Longrightarrow \quad \langle \mathsf{taut}^{\mathsf{cpt}}(\gamma)\big|_{\mathcal{A}_g}, \delta \rangle = \langle \gamma\big|_{\mathcal{A}_g}, \delta \rangle \\ &\Longrightarrow \quad \mathsf{taut}^{\mathsf{cpt}}(\gamma)\big|_{\mathcal{A}_g} = \mathsf{taut}\left(\gamma\big|_{\mathcal{A}_g}\right)\,. \end{aligned}$$

Here, we have used Definition 2, equation (7), and Definition 4 (and the argument is not possible without Theorem 3).

[7] The existence and uniqueness of $\mathsf{taut}(\gamma)$ follows from the Gorenstein property of $\mathsf{R}^*(\mathcal{A}_g)$.

1.5. **Tautological projection of product classes.** As an application of the theory, we consider the tautological projections of product loci. For $g = g_1 + g_2$ with $g_i \geq 1$, the product map

$$\mathcal{A}_{g_1} \times \mathcal{A}_{g_2} \to \mathcal{A}_g$$

defines a class $[\mathcal{A}_{g_1} \times \mathcal{A}_{g_2}] \in \mathsf{CH}^*(\mathcal{A}_g)$ by pushforward of the fundamental cycle. More generally, for every partition $g = \sum_{i=1}^{\ell} g_i$ in positive parts, we have a product map and an associated class:

$$\prod_{1=1}^{\ell} \mathcal{A}_{g_i} \to \mathcal{A}_g\,, \qquad \left[\prod_{1=1}^{\ell} \mathcal{A}_{g_i}\right] \in \mathsf{CH}^*(\mathcal{A}_g)\,. \tag{8}$$

Whether these product maps and classes (8) naturally extend to a compactification $\overline{\mathcal{A}}_g$ depends upon the choice of toroidal compactification. Toroidal compactifications of $\mathcal{A}_g$ correspond to choices of admissible fans Σ_g in $\mathrm{Sym}^2_{\mathrm{rc}}(\mathbb{R}^g)$, the rational closure of the positive-definite symmetric forms on $\mathbb{R}^g$. Following [GHT], a collection of such fans $\{\Sigma_g\}_{g\in\mathbb{N}}$ is *additive* if the sum $\sigma_1 \oplus \sigma_2$ of any two cones $\sigma_1 \in \Sigma_{g_1}$ and $\sigma_2 \in \Sigma_{g_2}$ is a cone in $\Sigma_{g_1+g_2}$. Let $\overline{\mathcal{A}}_g^{\Sigma_g}$ be a toroidal compactification corresponding to an additive collection of fans $\{\Sigma_g\}$. The *perfect cone* compactification satisfies these properties, see [SB]. In the additive case, the product maps extend,

$$\prod_{i=1}^{\ell} \overline{\mathcal{A}}_{g_i}^{\Sigma_{g_i}} \to \overline{\mathcal{A}}_g^{\Sigma_g}\,,$$

and we can therefore define cycles

$$\left[\prod_{i=1}^{\ell} \overline{\mathcal{A}}_{g_i}^{\Sigma_{g_i}}\right] \in \mathsf{CH}^*(\overline{\mathcal{A}}_g^{\Sigma_g})\,.$$

While the definition of tautological projection is independent of toroidal compactification, natural compactifications can be used for the calculation. We prove a closed formula for the tautological projection of the product cycles. The result extends calculations of [GH] for $g \leq 5$.

Theorem 5. *For $g_1 + \ldots + g_\ell = g$, the tautological projection of the product locus $\overline{\mathcal{A}}_{g_1}^{\Sigma_{g_1}} \times \cdots \times \overline{\mathcal{A}}_{g_\ell}^{\Sigma_{g_\ell}}$ in $\overline{\mathcal{A}}_g^{\Sigma_g}$ is given by a $g \times g$ determinant,*

$$\mathsf{taut}^{\mathsf{cpt}}([\overline{\mathcal{A}}_{g_1}^{\Sigma_{g_1}} \times \cdots \times \overline{\mathcal{A}}_{g_\ell}^{\Sigma_{g_\ell}}]) = \frac{\gamma_{g_1} \cdots \gamma_{g_\ell}}{\gamma_g} \begin{vmatrix} \lambda_{\alpha_1} & \lambda_{\alpha_1+1} & \cdots & \lambda_{\alpha_1+g-1} \\ \lambda_{\alpha_2-1} & \lambda_{\alpha_2} & \cdots & \lambda_{\alpha_2+g-2} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{\alpha_g-g+1} & \lambda_{\alpha_g-g+2} & \cdots & \lambda_{\alpha_g} \end{vmatrix},$$

for the vector

$$\alpha = (\underbrace{g-g_1, \ldots, g-g_1}_{g_1}, \underbrace{g-g_1-g_2, \ldots, g-g_1-g_2}_{g_2}, \ldots, \underbrace{g-g_1-\ldots-g_\ell, \ldots, g-g_1-\ldots-g_\ell}_{g_\ell}).$$

We set $\lambda_k = 0$ for $k < 0$ or $k > g$ and $\lambda_0 = 1$.

In the above determinant, α_i denotes the i^{th} component of the vector α. The last g_ℓ entries of α are 0, and contribute rows with 1's on the main diagonal and 0's below the main diagonal. These last entries do not change the determinant, but are included for a more symmetric formulation. The constants γ_g are defined in (4).

The proof of Theorem 5 in Section 3 relies on the connections between the tautological ring of $\overline{\mathcal{A}}_g^{\Sigma_g}$ and the Chow ring of the Lagrangian Grassmannian LG_g of $\mathbb{C}^{2g}$ as explained in [vdG2]. The argument combines properties of tautological projection, the Hirzebruch-Mumford proportionality principle, and the geometry of LG_g.

Using the restriction property of tautological projection and the relations in Theorem 1 (iii), we prove the following result in Section 3.4.

Theorem 6. *For $g_1 + \ldots + g_\ell = g$, the tautological projection of the product locus $\mathcal{A}_{g_1} \times \cdots \times \mathcal{A}_{g_\ell}$ in $\mathcal{A}_g$ is given by a $(g-\ell) \times (g-\ell)$ determinant,*

$$\mathsf{taut}([\mathcal{A}_{g_1} \times \cdots \times \mathcal{A}_{g_\ell}]) = \frac{\gamma_{g_1} \cdots \gamma_{g_\ell}}{\gamma_g} \cdot \lambda_{g-1} \cdots \lambda_{g-\ell+1} \cdot \begin{vmatrix} \lambda_{\beta_1} & \lambda_{\beta_1+1} & \cdots & \lambda_{\beta_1+g^*-1} \\ \lambda_{\beta_2-1} & \lambda_{\beta_2} & \cdots & \lambda_{\beta_2+g^*-2} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{\beta_{g^*}-g^*+1} & \lambda_{\beta_{g^*}-g^*+2} & \cdots & \lambda_{\beta_{g^*}} \end{vmatrix},$$

for the vector

$$\beta = (\underbrace{g^* - g_1^*, \ldots, g^* - g_1^*}_{g_1^*}, \underbrace{g^* - g_1^* - g_2^*, \ldots, g^* - g_1^* - g_2^*}_{g_2^*}, \ldots, \underbrace{g^* - g_1^* - \ldots - g_\ell^*, \ldots, g^* - g_1^* - \ldots - g_\ell^*}_{g_\ell^*}),$$

where $g^* = g - \ell$ and $g_i^* = g_i - 1$.

The tautological projections of the product loci in $\mathcal{A}_g$ from Theorem 6 in the simplest cases are:

$$\mathsf{taut}\,([\mathcal{A}_1 \times \mathcal{A}_{g-1}]) = \frac{g}{6|B_{2g}|}\lambda_{g-1}\,, \tag{9}$$

$$\mathsf{taut}\,([\mathcal{A}_2 \times \mathcal{A}_{g-2}]) = \frac{1}{360} \cdot \frac{g(g-1)}{|B_{2g}||B_{2g-2}|} \cdot \lambda_{g-1}\lambda_{g-3}\,, \tag{10}$$

$$\mathsf{taut}\,([\mathcal{A}_3 \times \mathcal{A}_{g-3}]) = \frac{1}{45360} \cdot \frac{g(g-1)(g-2)}{|B_{2g}||B_{2g-2}||B_{2g-4}|} \cdot \lambda_{g-1}(\lambda_{g-4}^2 - \lambda_{g-3}\lambda_{g-5})\,, \tag{11}$$

$$\mathsf{taut}\,([\mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_{g-3}]) = \frac{1}{90} \cdot \frac{g(g-1)(g-2)}{|B_{2g}||B_{2g-2}||B_{2g-4}|} \cdot \lambda_{g-1}\lambda_{g-2}\lambda_{g-4}\,, \tag{12}$$

$$\mathsf{taut}\left(\left[\underbrace{\mathcal{A}_1 \times \ldots \times \mathcal{A}_1}_{k} \times \mathcal{A}_{g-k}\right]\right) = \left(\prod_{i=g-k+1}^{g} \frac{i}{6|B_{2i}|}\right) \lambda_{g-1} \cdots \lambda_{g-k}\,. \tag{13}$$

In genus $g = 4$, formula (13) yields

$$\mathsf{taut}\,([\mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_2]) = 420\lambda_3\lambda_2\,, \quad \mathsf{taut}\,([\mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_1]) = 4200\lambda_3\lambda_2\lambda_1\,.$$

In fact, $[\mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_2]$ and $[\mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_1 \times \mathcal{A}_1]$ are tautological [COP].

An interesting case of (13) occurs when $k = g - 1$ since the class $\lambda_{g-1} \cdots \lambda_1$ generates the socle of the tautological ring $\mathsf{R}^{\binom{g}{2}}(\mathcal{A}_g)$. A speculation of [COP] is that the g-fold product

$$[\mathcal{A}_1 \times \cdots \times \mathcal{A}_1] \in \mathsf{CH}^{\binom{g}{2}}(\mathcal{A}_g)$$

also lies in the socle of the tautological ring. If the speculation is correct, then we obtain an exact evaluation

$$[\mathcal{A}_1 \times \cdots \times \mathcal{A}_1] = \left(\prod_{i=1}^{g} \frac{i}{6|B_{2i}|} \right) \lambda_{g-1} \cdots \lambda_1 \,. \tag{14}$$

Question A. *When is the non-tautological part of the product locus nonzero:*

$$\left[\prod_{1=1}^{\ell} \mathcal{A}_{g_i} \right] - \mathsf{taut} \left(\left[\prod_{1=1}^{\ell} \mathcal{A}_{g_i} \right] \right) \neq 0\,?$$

The cycles $[\mathcal{A}_1 \times \mathcal{A}_{g-1}] \in \mathsf{CH}^*(\mathcal{A}_g)$ are studied in [COP] via the Torelli map

$$\mathsf{Tor} : \mathcal{M}_g^{\mathrm{ct}} \to \mathcal{A}_g \,.$$

A central result of [COP] is that $[\mathcal{A}_1 \times \mathcal{A}_5]$ is not tautological on $\mathcal{A}_6$, so the purely non-tautological part is nonzero,

$$[\mathcal{A}_1 \times \mathcal{A}_5] - \mathsf{taut}\left([\mathcal{A}_1 \times \mathcal{A}_5]\right) \neq 0 \,.$$

Detection of the non-vanishing of the non-tautological part is subtle since the class

$$\Delta_g = \mathsf{Tor}^* \left([\mathcal{A}_1 \times \mathcal{A}_{g-1}] - \frac{g}{6|B_{2g}|} \lambda_{g-1} \right)$$

lies in the kernel of the λ_g-pairing

$$\mathsf{R}^{g-1}(\mathcal{M}_g^{\mathrm{ct}}) \times \mathsf{R}^{g-2}(\mathcal{M}_g^{\mathrm{ct}}) \to \mathsf{R}^{2g-3}(\mathcal{M}_g^{\mathrm{ct}}) \cong \mathbb{Q}$$

for all genera g by an argument of Pixton, see [COP].

The Noether-Lefschetz loci in $\mathcal{A}_g$ parametrize abelian varieties whose Picard rank jumps. The Noether-Lefschetz loci have been classified in [DL] (the products $\mathcal{A}_{g_1} \times \mathcal{A}_{g_2}$ for $g_1 + g_2 = g$ arise in the classification, but there are other loci as well). The Appendix contains a conjecture about the projection of the Noether-Lefschetz locus of abelian varieties with real multiplication. We can hope for a more general result.

Question B. *Calculate the tautological projections of all Noether-Lefschetz loci in $\mathcal{A}_g$.*

Beyond product and Noether-Lefschetz cycles, we can consider the tautological projection of the locus of Jacobians of genus g curves of compact type,

$$[\mathcal{J}_g] = \mathsf{Tor}_*[\mathcal{M}_g^{\mathrm{ct}}] \in \mathsf{CH}^*(\mathcal{A}_g) \,.$$

Faber [Fa] determined these explicitly for $g \leq 7$. For all genera, in the basis of monic square-free monomials in the $\lambda's$, the leading term is given by

$$\mathsf{taut}\left([\mathcal{J}_g]\right) = \left(\frac{1}{g-1} \prod_{i=1}^{g-2} \frac{2}{(2i+1)|B_{2i}|} \right) \lambda_1 \cdots \lambda_{g-3} + \dots \,,$$

as proposed in [Fa, Conjecture 1] and confirmed via [FP1, Theorem 4]. A more complicated formula for the coefficient of the term $\lambda_2 \dots \lambda_{g-4}\lambda_{g-2}$ was predicted by [Fa, Conjecture 2] and subsequently proven in [FP2, Section 5.2].

For each genus g, the class $\mathsf{taut}([\mathcal{J}_g]) \in \mathsf{R}^*(\mathcal{A}_g)$ can be computed algorithmically by a finite number of Hodge integral evaluations [Fa]. Finding expressions for the coefficients of the remaining terms of $\mathsf{taut}([\mathcal{J}_g])$ is an open question, but we could hope for more structure.

Question C. *Is there a simpler way to understand the tautological projection*

$$\mathsf{taut}([\mathcal{J}_g]) \in \mathsf{R}^*(\mathcal{A}_g)\,?$$

When is the non-tautological part nonzero?

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2. The top Chern class of the Hodge bundle

2.1. **Overview.** Logarithmic geometry provides a convenient tool for considering all toroidal compactifications of $\mathcal{A}_g$ simultaneously and plays an important role in the proof of Theorem 3. A quick review of the basic language of log geometry is given in Section 2.2. The proof of Theorem 3 relies on the residue map constructed in Section 2.3. An analogous residue map was constructed by Esnault-Viewheg [EV]. We give a different perspective on the construction. After a discussion of the Hodge bundle on toroidal compactifications of $\mathcal{A}_g$ in Section 2.4, the proof of Theorem 3 is presented in Section 2.5. Conjectures and future directions related to tautological classes on toroidal compactifications of $\mathcal{A}_g$ are discussed in Section 2.6.

2.2. **The logarithmic Chow ring for toroidal embeddings.** We will use the language of log geometry and assume some rudimentary familiarity with the central definitions as given in [K]. A summary of the relevant background information can be found in [MPS].

For a log scheme (S, M_S), we write

$$\epsilon : M_S \to \mathcal{O}_S$$

for the structure morphism from the monoid M_S. Let M_S^{gp} be the associated group, and let $\overline{M}_S$ be the characteristic monoid

$$\overline{M}_S = M_S/\mathcal{O}_S^* .$$

The sheaf $\overline{M}_S$ is constructible, and thus stratifies S.

For a toroidal embedding (S, D), the log structure is given by the étale sheaf of monoids

$$M_S = \{f \in \mathcal{O}_S : f \text{ is a unit on } S \smallsetminus D\} .$$

For toroidal embeddings, we will denote the log structure by either (S, D) or (S, M_S) depending upon context.

An important special case is that of a normal crossings pair (S, D): a smooth Deligne–Mumford stack S with a normal crossings divisor D. These are precisely the log smooth log Deligne–Mumford stacks with smooth underlying stack. The normal crossings condition is equivalent to

$$\overline{M}_{S,s} = \mathbb{N}^k$$

for each $s \in S$ and for some k depending on s. The integer k is the number of local branches of D passing though s.

For a morphism of log structures $f : X \to S$, let

$$\overline{M}_{X/S} = \overline{M}_X/\overline{M}_S$$

be the relative characteristic monoid. The morphism f is *strict* if $\overline{M}_{X/S} = 0$. For a log scheme (X, M_X) defined over a field K, a (global) *chart* for the log structure of X is a finitely generated, saturated monoid P and a strict map

$$X \to \operatorname{Spec}(K[P]) ,$$

where the target carries the canonical log structure (coming from the torus invariant divisor). We require all log schemes to admit charts étale locally.

A morphism $f : X \to S$ is *log smooth* if, étale locally on X, we can find a map of monoids $Q \to P$ such that $\operatorname{Ker}(Q^{\mathrm{gp}} \to P^{\mathrm{gp}})$ and the torsion part of $\operatorname{Coker}(Q^{\mathrm{gp}} \to P^{\mathrm{gp}})$ are finite groups of order invertible in K, and a diagram

$$\begin{array}{ccccc} X & \xrightarrow{\alpha_X} & S \times_{\operatorname{Spec} K[Q]} \operatorname{Spec} K[P] & \longrightarrow & \operatorname{Spec} K[P] \\ & & \big\downarrow & & \big\downarrow \\ & & S & \xrightarrow{\quad\alpha_S\quad} & \operatorname{Spec} K[Q] \end{array}$$

with α_X, α_S strict and α_X smooth. If we can find such a diagram with $Q \to P$ of finite index and α_X étale, the morphism is *log étale*. In particular, toroidal embeddings (S, D) are exactly the log smooth log schemes over $(\operatorname{Spec} K, K^*)$.

To a toroidal embedding (S, D), we can associate a cone complex $\Sigma_{(S,D)}$. We refer the reader to [MPS, Section 4.3] for an outline of the construction and further references. Each cone has an integral structure, and the cone complex is built by gluing the cones together with their integral structure. A *strata blowup* is a blowup of (S, D) along a smooth closed stratum. The result is a new toroidal embedding (S', D') with D' the total transform of D, so the procedure can be iterated indefinitely. A *log modification* of (S, D) is a proper birational map $S' \to S$ that can be dominated by an iterated strata blowup. More intrinsically, the log modifications of S are precisely the proper, representable, log étale monomorphisms $S' \to S$. Combinatorially, log modifications of S correspond exactly to subdivisions of the cone complex $\Sigma_{(S,D)}$.

Log modifications form a filtered system. Indeed, two log modifications

$$S' \to S \quad \text{and} \quad S'' \to S$$

can always be dominated by a third: the log modification corresponding to the common refinement of the subdivisions corresponding to S' and S'' together with the intersection of the integral structures. To a toroidal embedding (S, D), we can thus associate refined operational Chow groups

$$\mathsf{logCH}^*(S, D) = \varinjlim \mathsf{CH}^{\mathrm{op}}(S')\,,$$

where S' ranges over log modifications of S.

Toroidal compactifications of $\mathcal{A}_g$ correspond to admissible decompositions of the rational closure $\mathrm{Sym}^2_{\mathrm{rc}}(\mathbb{R}^g)$ of the cone of positive-definite symmetric quadratic forms on $\mathbb{R}^g$. Any two admissible decompositions can be refined by a third. Hence, $\mathsf{logCH}^*(\overline{\mathcal{A}}_g, \partial\overline{\mathcal{A}}_g)$ is independent of the choice of compactification. We define

$$\mathsf{logCH}^*(\mathcal{A}_g) = \mathsf{logCH}^*(\overline{\mathcal{A}}_g, \partial\overline{\mathcal{A}}_g)$$

for any toroidal compactification $\overline{\mathcal{A}}_g$.

2.3. **Semistable families and residues.** For suitable families of log schemes, we prove the existence of a residue map in Theorem 16 below. The residue map will be applied in Section 2.4 to the universal family over the moduli space of abelian varieties in order to prove Theorem 3.

The sheaf of relative *logarithmic differentials* $\Omega^{\log}_{X/S}$ is defined as the quotient of

$$\Omega_{X/S} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} M_X^{\mathrm{gp}})$$

by the subsheaf locally generated by sections of the form

$$(d\epsilon(m), 0) - (0, \epsilon(m) \otimes m) \quad \text{and} \quad (0, 1 \otimes n)\,,$$

where $m \in M_X$ and $n \in \mathrm{Im}(M_S^{\mathrm{gp}}) \subset M_X^{\mathrm{gp}}$, see [K]. As usual, we write

$$d \log m = (0, 1 \otimes m)$$

which we view as $d\epsilon(m)/\epsilon(m)$.

For a strict map $f : X \to S$, we have $\Omega^{\log}_{X/S} = \Omega_{X/S}$, and for a log étale map $f : X \to S$, we have $\Omega^{\log}_{X/S} = 0$.

Definition 7. The *sheaf of residues* is defined to be the quotient

$$\mathcal{R} = \Omega^{\log}_{X/S}/\Omega_{X/S} \, .$$

Definition 8. ([M, Definition 2.1.2]) A *logarithmic family* $X \to S$ is a log smooth, surjective, integral and saturated map of log schemes.

Families of stable curves and families of toroidal compactifications of semi-abelian schemes are all examples of log families. The condition that f is integral and saturated – called weak semistability in [M] – is a technical condition that, for log smooth f, implies that f is flat with reduced fibers [M, Lemma 3.1.2], [Ts, Theorem II.4.2].[8] For a thorough discussion, see [Og, Part III, Section 2.5]. Being integral and saturated is local on X and can be understood in terms of the cone complexes Σ_X and Σ_S. Integrality combined with saturatedness says, locally on X, that the associated map $\Sigma_X \to \Sigma_S$ maps cones of Σ_X surjectively onto cones of Σ_S and that the integral structure of a cone σ surjects onto the integral structure of its image cone.

Given a log scheme (S, M_S) and a finite index extension of sheaves $M_S \to M'_S$, there is a universal log DM stack (S', M'_S) with a log map to (S, M_S) whose map on log structures is given by the extension $M_S \to M'_S$. The stack S' is called the root of S along $M_S \to M'_S$. The simplest instance of this operation is taking a root along a boundary stratum of a normal crossings pair (S, D). We call a composition of logarithmic modifications and roots a *logarithmic alteration*. Log alterations of toroidal embeddings are isomorphisms on $S \smallsetminus D$, but are not necessarily representable. Logarithmic alterations are furthermore log étale. See [MW] for a lengthier discussion.

Remark 9. Because we work with $\mathbb{Q}$-coefficients, pullbacks via root maps induce isomorphisms on Chow groups. Therefore, for a toroidal embedding (S, D), the logarthmic Chow groups can be equivalently defined as

$$\mathsf{logCH}^*(S, D) = \varinjlim \mathsf{CH}^{\mathrm{op}}(S') \, ,$$

where S' ranges over logarithmic alterations of S.

Definition 10. Let $f : X \to S$ be a log map. A *logarithmic alteration of* f is a log map $f' : X' \to S'$ and a commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ S' & \longrightarrow & S \end{array}$$

such that $S' \to S$ and $X' \to X$ are logarithmic alterations.

[8]When S is also log smooth, the integrality condition is equivalent to f being flat, and the conditions that f is integral and saturated together are equivalent to f being flat with reduced fibers.

Theorem 11 (Semistable Reduction Theorem, [ALT, AK, M])**.** *Let $f : X \to S$ be a dominant log smooth morphism of logarithmic schemes. Then there is a log alteration $f' : X' \to S'$ of f which is a log family. Furthermore, if S is log smooth, then one can take X' and S' smooth.*

Definition 12. A pair (X, D) is called *simplest normal crossings* if $D \subset X$ is normal crossings in the Zariski topology and each intersection of components of D is connected.

Remark 13. In more geometric terms, a toroidal embedding (X, D) has simplest normal crossings if the following conditions are all satisfied:

(i) (X, D) is a normal crossings pair (with X smooth),
(ii) the components of D have no self-intersection,
(iii) intersections of components of D are connected.

We note that properties (ii) and (iii) are stable under logarithmic alterations, but (i) is not.

Corollary 14. *Let $f : (X, D_X) \to (S, D_S)$ be a log family with (S, D_S) toroidal. Then there is a log alteration $f' : (X', D_{X'}) \to (S', D_{S'})$ which is a log family where both S' and X' have simplest normal crossings.*

Proof. We can make S and X simplest normal crossings by suitable log modifications

$$S_1 \to S\,, \quad X_1 \to X \times_S S_1\,.$$

For example, we can take the log modifications corresponding to double barycentric subdivisions (see the discussion in [MPS, Section 5.6]). Since logarithmic alterations are log étale and surjective, the morphism $X_1 \to S_1$ remains log smooth and surjective. Moreover, as noted in Remark 13, any further log alteration of X_1 or S_1 which is smooth will have simplest normal crossings. Therefore, we may apply Theorem 11 to $X_1 \to S_1$ to get the desired $X' \to S'$. □

Let (X, M_X) be a log scheme. An *orientation* on M_X is an ordering of the sections of $\overline{M}_X(U)$ for all $U \subset X$ compatible with the restriction maps. We say M_X is *orientable* if it admits an orientation.

Lemma 15. *Suppose (X, D) is a pair with simplest normal crossings. Then the log structure of (X, D) is orientable.*

Proof. We choose an ordering of the components D_i of the divisor D. Every stratum is the intersection

$$D_{i_1} \cap \cdots \cap D_{i_k}$$

where the components appear in the ordering we have chosen. For each x in the stratum, we have a canonical isomorphism

$$\overline{M}_{X,x} = \bigoplus \mathbb{N} m_{i_k}\,.$$

Here, m_{i_k} is the image in $\overline{M}_{X,x}$ of any element $\widetilde{m}_{i_k} \in M_{X,x}$ that maps to a local defining equation for D_{i_k}. We order the sections as $m_{i_1} < m_{i_2} \cdots < m_{i_k}$. □

Theorem 16. *Let $f : X \to S$ be a log family with X and S simplest normal crossings pairs. Every choice of orientation on $\overline{M}_X$ yields a map*

$$\mathcal{R} \to \oplus_H \mathcal{O}_H \tag{15}$$

where H ranges over the irreducible components of the stratification determined by $\overline{M}_{X/S}$ on the locus where rank $\overline{M}_{X/S} \geq 1$. *Furthermore, the projection*

$$\mathcal{R} \to \mathcal{O}_H$$

of (15) *to each summand $\mathcal{O}_H$ is surjective.*

Proof. Choose an orientation of $\overline{M}_X$ as in Lemma 15. We will construct the map (15) locally on X, and then we will prove the gluing compatibility required for the global definition.

Let x be a point of X. Choose an ordered basis

$$m_1, \dots, m_n \in \overline{M}_{X,x}$$

of $\overline{M}_X$ at x. Let $\widetilde{m}_i$ be arbitrary lifts in M_X, and write $x_i = \epsilon(\widetilde{m}_i)$ for their images in $\mathcal{O}_X$. In other words, x_i are local defining equations for the divisor D_i of X at x. Similarly, write t_i for the corresponding images in S near $f(x)$. Then, without loss of generality, we may assume that the map of characteristic monoids has the form

$$\mathbb{N}^k = \overline{M}_{S,f(x)} \to \overline{M}_{X,x} = \mathbb{N}^{n_1} \oplus \mathbb{N}^{n_2} \oplus \cdots \oplus \mathbb{N}^{n_k} \oplus \mathbb{N}^{\ell} \,,$$

with the j^{th} basis element of $\mathbb{N}^k$ mapping to the vector $(\underbrace{1, \dots, 1}_{n_j})$ of the summand $\mathbb{N}^{n_j}$ on the right.

We have equations

$$t_1 = u_1 \prod_{\alpha \in A_1} x_\alpha \,, \ \dots, \ t_k = u_k \prod_{\alpha \in A_k} x_\alpha$$

for disjoint sets $A_1, \dots, A_k \subset \{1, 2, \dots, n\}$ with $n_1, \dots, n_k$ elements respectively and units $u_i \in \mathcal{O}_{X,x}$. By the orientation assumption, the sets $A_1, \dots, A_k$ are ordered. For convenience, we write

$$A = A_1 \cup \ldots \cup A_k \,.$$

The additional ℓ parameters $y_1, \cdots, y_\ell$ have vanishing loci $V(y_i)$ representing horizontal divisors[9] over S.

The logarithmic differentials $\Omega^{\log}_{X,x}$ are generated by $\Omega_{X,x}$ and $\frac{dx_\alpha}{x_\alpha}$, $\alpha \in A$, and $\frac{dy_1}{y_1}, \dots, \frac{dy_\ell}{y_\ell}$. We have the relations

$$\frac{du_i}{u_i} + \sum_{\alpha \in A_i} \frac{dx_\alpha}{x_\alpha} = \frac{dt_i}{t_i}, \ \ 1 \leq i \leq k \,.$$

The quotient $\mathcal{R} = \Omega^{\log}_{X/S} / \Omega_{X/S}$ has a presentation as an $\mathcal{O}_X$-module with generators

$$\frac{dx_\alpha}{x_\alpha}, \frac{dy_1}{y_1}, \dots, \frac{dy_\ell}{y_\ell} \,, \tag{16}$$

[9]For example, when $X \to S$ is a family of curves, the $V(y_i)$ correspond to markings. In our study of the moduli of abelian varieties, $\ell = 0$.

where $\alpha \in A$. The relations are

$$\sum_{\alpha \in A_i} \frac{dx_\alpha}{x_\alpha} = 0\,, \quad 1 \le i \le k\,,$$

(since we are working with relative differentials and the du_i/u_i are in $\Omega_{X/S}$) and additionally

$$x_\alpha \frac{dx_\alpha}{x_\alpha} = 0\,,\ y_1 \frac{dy_1}{y_1} = 0\,, \ldots, y_\ell \frac{dy_\ell}{y_\ell} = 0\,,$$

where $\alpha \in A$. The irreducible components of the stratification of $M_{X/S}$ at x (with rk ≥ 1) are given either by

- $t_i = 0, x_\beta = 0, x_\gamma = 0$ for triples (i, β, γ) with $\beta < \gamma$ elements in A_i, or by
- $y_j = 0$ for some $1 \le j \le \ell$.

Thus, we find[10]

$$\oplus_H \mathcal{O}_H = \bigoplus_{(i,\beta,\gamma)} \mathcal{O}_X/(t_i, x_\beta, x_\gamma) \oplus \bigoplus_j \mathcal{O}_X/(y_j).$$

We define a map

$$\mathcal{R} \to \mathcal{O}_X/(t_i, x_\beta, x_\gamma)$$

by sending all the generators in (16) to 0, with the exception of

$$\frac{dx_\beta}{x_\beta} \mapsto 1, \quad \frac{dx_\gamma}{x_\gamma} \mapsto -1\,.$$

Similarly, we define a map

$$\mathcal{R} \to \mathcal{O}_X/(y_j)$$

by sending all generators in (16) to 0, with the exception of

$$\frac{dy_j}{y_j} \mapsto 1.$$

We must verify that the map is well-defined. First, for each (i, β, γ), we see that

$$\sum_{\alpha \in A_i} \frac{dx_\alpha}{x_\alpha} \mapsto 0$$

since $\frac{dx_\beta}{x_\beta}$ and $\frac{dx_\gamma}{x_\gamma}$ map to opposite elements in $\mathcal{O}_X/(t_i, x_\beta, x_\gamma)$, and the other terms map to 0. The fact that

$$x_\alpha \frac{dx_\alpha}{x_\alpha},\ y_1 \frac{dy_1}{y_1}, \ldots, y_\ell \frac{dy_\ell}{y_\ell},\ \alpha \in A$$

map to 0 in $\mathcal{O}_X/(t_i, x_\beta, x_\gamma)$ and $\mathcal{O}_X/(y_j)$ is immediate from the definitions. Surjectivity of the map to any summand $\mathcal{O}_H$ is also clear, as the generator 1 is in the image.

We now inspect how our map depended on choices; the only choices involved were the lifts $\widetilde{m}_i$ of m_i, and the choice of ordering of the m_i. A different choice of $\widetilde{m}_i'$ differs from the original one by a unit, and we have

$$\frac{d(ux)}{ux} = u^{-1} du + \frac{dx}{x}.$$

[10] Of course, $t_i \in (x_\beta, x_\gamma)$, but we have chosen to keep t_i in the notation to emphasize that β, γ belong to the same part A_i.

The term $u^{-1}du$ is an ordinary differential, and thus the residue of the logarithmic form is independent of lift. On the other hand, the map does depend on the ordering of the coordinates. Since we assume that the ordering is global, however, the local maps patch uniquely to all of X. $\square$

Remark 17. In case S is a point, (X, D) is a simplest normal crossings pair, and only the horizontal divisors $H = \mathcal{O}_X/(y_i)$ are present in our analysis. These are precisely the components D_i of the divisor D. Our residue map then reduces to the classical residue homomorphism

$$0 \longrightarrow \Omega_X \longrightarrow \Omega_X^{\log} \longrightarrow \oplus_i \mathcal{O}_{D_i} \longrightarrow 0\,,$$

see [F2, Chapter 4, Proposition 1].

The classical residue map, applied to the base and the source of a morphism of simple normal crossings pairs, is used by Esnault-Viewheg [EV, Claim 4.3] to construct a relative residue map, similar to the one of Theorem 16.

2.4. **The Hodge bundle.** Let $\overline{\mathcal{A}}_g$ be a toroidal compactification of the moduli space of principally polarized abelian varieties. The compactification $\overline{\mathcal{A}}_g$ carries a universal family of semi-abelian schemes

$$q : \mathcal{U}_g \to \overline{\mathcal{A}}_g$$

together with a zero section $s : \overline{\mathcal{A}}_g \to \mathcal{U}_g$. The Hodge bundle is the rank g vector bundle on $\overline{\mathcal{A}}_g$ defined by

$$\mathbb{E} = s^*\Omega_q\,,$$

with Chern classes $\lambda_i = c_i(\mathbb{E})$.

Definition 18. A *compactification* of $q : \mathcal{U}_g \to \overline{\mathcal{A}}_g$ is a diagram

$$\begin{array}{ccc} \mathcal{U}_g & \longrightarrow & \mathcal{X}_g \\ & \overset{q}{\searrow} & \downarrow p \\ & & \overline{\mathcal{A}}_g \end{array}$$

where p is a proper log smooth morphism, $\mathcal{U}_g$ is open and dense in $\mathcal{X}_g$, and $\mathcal{U}_g$ acts on $\mathcal{X}_g$ extending the natural action of $\mathcal{U}_g$ on itself (and commuting with p). A compactification p is a *compactified universal family* if in addition p is a log family.

The fiber $(\mathcal{X}_g)_t$ of a compactified universal family p over a point $t \in \overline{\mathcal{A}}_g$ contains the semiabelian scheme $(\mathcal{U}_g)_t$ as an open subscheme. More precisely, write

$$0 \longrightarrow T_t \longrightarrow (\mathcal{U}_g)_t \longrightarrow A_t \longrightarrow 0\,, \tag{17}$$

with T_t a torus and A_t an abelian scheme. Then $(\mathcal{X}_g)_t$ admits a fibration

$$X(T_t) \longrightarrow (\mathcal{X}_g)_t \longrightarrow A_t\,, \tag{18}$$

where $X(T_t)$ is a union of complete toric varieties with torus T_t.

An arbitrary toroidal compactification $\overline{\mathcal{A}}_g$ may not carry a compactified universal family. However, toroidal compactifications $\overline{\mathcal{A}}_g$, with compactified universal families

$$p : \mathcal{X}_g \to \overline{\mathcal{A}}_g$$

can be constructed, see [FC, Chapter VI, Section 1]. Compactifications of q correspond to $\mathrm{GL}_g \ltimes N$-admissible decompositions $\widetilde{\Sigma}_g$ of a certain subcone of $\mathrm{Sym}^2_{rc}(\mathbb{R}^g) \times \mathrm{Hom}(N, \mathbb{R})$ for a rank g lattice N. The decomposition is required to have the property that every cone in $\widetilde{\Sigma}_g$ maps into a cone of the admissible decomposition Σ_g of $\mathrm{Sym}^2_{rc}(\mathbb{R}^g)$ defining $\overline{\mathcal{A}}_g$. A compactification p is a compactified universal family if the map

$$\widetilde{\Sigma}_g \to \Sigma_g$$

satisfies the additional hypotheses of Definition 8 (the cones of $\widetilde{\Sigma}_g$ map onto cones of Σ_g, and surjectivity also holds for their integral structure).

Both notions of compactification are stable under arbitrary base change $\overline{\mathcal{A}}'_g \to \overline{\mathcal{A}}_g$. For a compactification $p : \mathcal{X}_g \to \overline{\mathcal{A}}_g$, an arbitrary log alteration $\mathcal{X}'_g \to \mathcal{X}_g$ of the *domain* of p remains a compactification. On the other hand, log alterations of $\mathcal{X}_g$ are *not* compactified families, even if the original p is a family, as the composed map $\mathcal{X}'_g \to \overline{\mathcal{A}}_g$ is rarely a log family (flatness and reducedness of fibers are typically destroyed). Nevertheless, semistable reduction by Theorem 11 ensures that there is a log alteration of the *map* $\mathcal{X}'_g \to \overline{\mathcal{A}}_g$ which is a compactified family.

The sheaf of relative logarithmic differentials of q and p are fiberwise trivial of rank g [FC, Chapter VI, Theorem 1.1]. Triviality follows from the fibration descriptions (17) and (18) since the sheaf of differentials on a semi-abelian variety and the sheaf of logarithmic differentials on a toric variety are both trivial.

When $\overline{\mathcal{A}}_g$ has a compactified family p, we can use $\Omega_p^{\log}$ to define the Hodge bundle, as

$$\mathbb{E} = s^*\Omega_q = s^*\Omega_q^{\log} = s^*\Omega_p^{\log}|_{\mathcal{U}_g} = s^*\Omega_p^{\log}$$

since the section s factors through $\mathcal{U}_g$ and the map q is strict.

A second approach to the Hodge bundle is available. The following result can be found in [FC, Chapter VI, Theorem 1.1].

Lemma 19. *Suppose $\overline{\mathcal{A}}_g$ carries a compactified universal family $p : \mathcal{X}_g \to \overline{\mathcal{A}}_g$. Then,*

$$\Omega_p^{\log} = p^*\mathbb{E} \quad \textit{and} \quad \mathbb{E} = p_*\Omega_p^{\log}\,.$$

Proof. Since $\Omega_p^{\log}$ is fiberwise trivial, cohomology and base change implies that

$$\Omega_p^{\log} = p^*p_*\Omega_p^{\log}$$

Since $s^*p^* = \mathrm{id}$, we have

$$\mathbb{E} = s^*\Omega_p^{\log} = s^*p^*p_*\Omega_p^{\log} = p_*\Omega_p^{\log}\,,$$

and the result follows.

□

Lemma 20. *The classes λ_i extend to* $\mathsf{logCH}^*(\mathcal{A}_g)$.

Proof. In light of Remark 9, we check compatibility of Hodge classes under logarithmic alterations. Suppose $\tau : \overline{\mathcal{A}}'_g \to \overline{\mathcal{A}}_g$ is a logarithmic alteration. Then we have a Cartesian diagram

$$\begin{array}{ccc} \mathcal{U}'_g & \xrightarrow{\rho} & \mathcal{U}_g \\ \big\downarrow{\scriptstyle q'} & & \big\downarrow{\scriptstyle q} \\ \overline{\mathcal{A}}'_g & \xrightarrow{\tau} & \overline{\mathcal{A}}_g \end{array}$$

Since $s \circ \tau = \rho \circ s'$ for the respective zero sections, we have

$$\tau^*\mathbb{E} = (s')^*\rho^*\Omega_q = (s')^*\Omega_{q'} = \mathbb{E}'\,,$$

which implies the required compatibility for λ_i. □

Remark 21. Toroidal compactifications of $\mathcal{A}_g$ with a compactified universal family

$$p : \mathcal{X}_g \to \overline{\mathcal{A}}_g$$

form a cofinal system among all toroidal compactifications: given an arbitrary toroidal compactification $\overline{\mathcal{A}}'_g$, we can choose a toroidal compactification $\overline{\mathcal{A}}_g$ with a compactified universal family, and then any common refinement $\overline{\mathcal{A}}''_g$ of both compactifications carries a compactified universal family.

Lemma 22. *The collection of simplest normal crossings compactifications $\mathcal{A}_g \subset \overline{\mathcal{A}}_g$ that carry a compactified universal family with simplest normal crossings is cofinal among the toroidal compactifications $\overline{\mathcal{A}}_g$.*

Proof. Starting with an arbitrary compactified universal family

$$p : \mathcal{X}_g \to \overline{\mathcal{A}}_g$$

we may apply Corollary 14 to p, to obtain the desired family. □

2.5. **Proof of Theorem 3.** Let $\overline{\mathcal{A}}_g$ be a toroidal compactification of $\mathcal{A}_g$. By Lemma 22, there exists a logarithmic alteration $p : \overline{\mathcal{A}}'_g \to \mathcal{A}_g$ satisfying the following conditions:

(i) $\overline{\mathcal{A}}'_g$ admits a universal family $\mathcal{X}'_g \to \overline{\mathcal{A}}'_g$ of toroidal compactifications of semi-abelian schemes,
(ii) both $\overline{\mathcal{A}}'_g$ and $\mathcal{X}'_g$ have simplest normal crossings.

By Lemma 20, λ_g defined on $\overline{\mathcal{A}}'_g$ via its own Hodge bundle agrees with the pullback of λ_g from $\overline{\mathcal{A}}_g$. Since $\overline{\mathcal{A}}'_g \to \overline{\mathcal{A}}_g$ is a log alteration, it is proper and surjective; furthermore, p is an isomorphism over $\mathcal{A}_g$, and sends the boundary of $\overline{\mathcal{A}}'_g$ to the boundary of $\overline{\mathcal{A}}_g$. Because proper surjections are Chow envelopes, it therefore suffices to show that

$$\lambda_g\big|_{\partial\overline{\mathcal{A}}'_g} = 0\,.$$

Hence, after replacing $\overline{\mathcal{A}}_g$ by $\overline{\mathcal{A}}'_g$, we may assume that $\overline{\mathcal{A}}_g$ has properties (i) and (ii) above.

Let T be a component of the boundary divisor of $\overline{\mathcal{A}}_g$ and denote by $p_T : \mathcal{X}_T \to T$ the base change to T of the universal family

$$p : \mathcal{X}_g \to \overline{\mathcal{A}}_g \,.$$

Over T, either $\mathcal{X}_T \to T$ is smooth, or we can find a nonempty component H of the singular locus $\mathcal{X}_T^{\mathrm{sing}}$ of p_T. In the first case, $\lambda_g\big|_T = 0$ because $\lambda_g\big|_{\mathcal{A}_g} = 0$.

In the second case, H is an irreducible component of the locus where rank $\overline{M}_{\mathcal{X}_g/\overline{\mathcal{A}}_g} \geq 1$. Let

$$i : H \to \mathcal{X}_T \,, \quad p_T \circ i : H \to T$$

be the inclusion and the projection. The map $p_T \circ i$ is proper and surjective because p is a log family, so it suffices to show that $i^* p_T^* \left(\lambda_g\big|_T\right) = 0$. Using Lemma 19, we have $\Omega_p^{\log} = p^*\mathbb{E}$. After base change and pullback by i, we find

$$i^* p_T^* \mathbb{E}\big|_T = i^* \Omega_{p_T}^{\log} \,.$$

It remains to check that $c_g(i^*\Omega_{p_T}^{\log}) = 0$. By Definition 7 and Theorem 16, we have surjections

$$i^*\Omega_{p_T}^{\log} \to i^*\mathcal{R} \to 0 \,, \quad i^*\mathcal{R} \to \mathcal{O}_H \to 0 \,.$$

We therefore have an exact sequence of vector bundles on H,

$$0 \longrightarrow K \longrightarrow i^*\Omega_{p_T}^{\log} \longrightarrow \mathcal{O}_H \longrightarrow 0 \,.$$

We conclude $c_g(i^*\Omega_{p_T}^{\log}) = c_{g-1}(K) c_1(\mathcal{O}_H) = 0 \,.$ □

2.6. **Log geometry and λ_g.** There is a distinguished subalgebra of classes coming from the boundary in the logarithmic Chow theory defined by the image of the algebra PP of GL_g-invariant piecewise polynomials[11] on $\mathrm{Sym}^2_{\mathrm{rc}}(\mathbb{R}^g)$,

$$\mathsf{PP} \to \mathsf{logCH}^*(\mathcal{A}_g) \,. \tag{19}$$

We refer the reader to [MPS, MR] for further details regarding the construction of the map (19). Our main conjecture concerning λ_g in the logarithmic theory is the following claim.

Conjecture D. *The class $\lambda_g \in \mathsf{logCH}^*(\mathcal{A}_g)$ lies in the image* (19) *of the algebra of piecewise polynomials.*

Our motivation for Conjecture D comes from a parallel study of λ_g in the logarithmic Chow theory of the moduli space of curves $\mathcal{M}_g$. Using Pixton's formula [HMPPS, JPPZ], the class λ_g is proven in [MPS] to lie in the image of the algebra of piecewise polynomials in $\mathsf{logCH}^*(\mathcal{M}_g)$.

Question E. *Find a lifting to $\mathsf{logCH}^*(\mathcal{A}_g)$ of Pixton's formula for $\lambda_g \in \mathsf{logCH}^*(\mathcal{M}_g)$ which is compatible with the Torelli map.*

[11] By definition, a piecewise polynomial on $\mathrm{Sym}^2_{\mathrm{rc}}(\mathbb{R}^g)$ is an admissible decomposition together with a continuous GL_g-invariant function on the decomposition that is polynomial on each cone.

The definition by van der Geer of the tautological ring is best suited for studying classes on the moduli space of abelian varieties $\mathcal{A}_g$. There is a larger tautological ring which takes the boundaries of the various compactifications into account,

$$\mathsf{logR}^*(\mathcal{A}_g) \subset \mathsf{logCH}^*(\mathcal{A}_g)\,,$$

defined to be generated by all possible piecewise polynomial and Hodge classes on all boundary strata of all toroidal compactifications $\mathcal{A}_g \subset \overline{\mathcal{A}}_g$.

The investigation of the structure of $\mathsf{logR}^*(\mathcal{A}_g)$ is an interesting future direction. For example, pushing forward powers of the polarization, we can define κ classes over $\overline{\mathcal{A}}_g$, see [MOP] and [A2] for similar constructions in the context of the moduli of K3 surfaces and over KSBA moduli respectively. In the case of abelian varieties, we expect[12] that the κ classes lie in $\mathsf{logR}^*(\mathcal{A}_g)$. Can a precise formula be found?

3. Tautological projection of product classes

3.1. Product cycles. We compute here the tautological projections of all product cycles

$$\mathcal{A}_{g_1} \times \ldots \times \mathcal{A}_{g_\ell} \to \mathcal{A}_g$$

for all g. Calculations for product cycles for genus $g \leq 5$ can be found in [GH].

Fix toroidal compactifications $\overline{\mathcal{A}}_g$ corresponding to an additive collection of fans. The product maps

$$\coprod_{g_1+\cdots+g_\ell=g} \mathcal{A}_{g_1} \times \cdots \times \mathcal{A}_{g_\ell} \to \mathcal{A}_g$$

then extend to maps

$$\coprod_{g_1+\cdots+g_\ell=g} \overline{\mathcal{A}}_{g_1} \times \cdots \times \overline{\mathcal{A}}_{g_\ell} \to \overline{\mathcal{A}}_g\,. \tag{20}$$

For example, we could take the perfect cone compactification for every g by [SB, Lemma 2.8].

The Hodge bundle splits canonically over the product (20). Indeed, the universal semiabelian variety restricts in the natural fashion over the product, and the splitting of the Hodge bundle then follows by restricting the relative cotangent bundle to the zero section.

3.2. Lagrangian Grassmannian. As remarked in [vdG1], a consequence of Theorem 1 is the $\mathbb{Q}$-algebra isomorphism

$$\mathsf{R}^*(\overline{\mathcal{A}}_g) \simeq \mathsf{CH}^*(\mathsf{LG}_g)\,,$$

where LG_g denotes the Lagrangian Grassmannian of $\mathbb{C}^{2g}$ with respect to a symplectic form ω. An overview of the cohomology of the Lagrangian Grassmannian from the point of Schubert calculus can be found, for instance, in [FPr, KT, PR].

[12] We thank V. Alexeev for a discussion about κ classes at the conference *Higher Dimensional Algebraic Geometry* in La Jolla in January 2024.

The spaces $\overline{\mathcal{A}}_g$ and LG_g are further connected by the Hirzebruch-Mumford proportionality principle [Mu, vdG3]. Let $\mathsf{S} \to \mathsf{LG}_g$ be the universal rank g subbundle, and let $x_i = c_i(\mathsf{S}^*)$. Then,

$$\int_{\overline{\mathcal{A}}_g} \lambda_I = \gamma_g \int_{\mathsf{LG}_g} x_I \tag{21}$$

for every $I \subset \{1, 2, \ldots, g\}$. Here, we use the multindex notation

$$\lambda_I = \prod_{i \in I} \lambda_i\,, \quad x_I = \prod_{i \in I} x_i\,.$$

The proportionality constant γ_g was computed in [vdG1, page 9]:

$$\gamma_g = \int_{\overline{\mathcal{A}}_g} \lambda_1 \ldots \lambda_g = \prod_{i=1}^{g} \frac{|B_{2i}|}{4i}\,.$$

For any partition

$$g_1 + \ldots + g_\ell = g\,,$$

we can consider the product

$$\mathsf{LG}_{g_1} \times \ldots \times \mathsf{LG}_{g_\ell} \to \mathsf{LG}_g\,. \tag{22}$$

Finding the class of

$$\overline{\mathcal{A}}_{g_1} \times \ldots \times \overline{\mathcal{A}}_{g_\ell} \to \overline{\mathcal{A}}_g$$

is equivalent to finding the class of the product cycle (22) in $\mathsf{CH}^*(\mathsf{LG}_g)$ in terms of the Chern classes $x_i = c_i(\mathsf{S}^*)$ of the dual subbundle. More precisely, if

$$[\mathsf{LG}_{g_1} \times \ldots \times \mathsf{LG}_{g_\ell}] = \mathsf{P}(x_1, \ldots, x_g) \in \mathsf{CH}^*(\mathsf{LG}_g)\,, \tag{23}$$

then we have

$$\mathsf{taut}^{\mathsf{cpt}}\left([\overline{\mathcal{A}}_{g_1}^{\Sigma_{g_1}} \times \ldots \times \overline{\mathcal{A}}_{g_\ell}^{\Sigma_{g_\ell}}]\right) = \frac{\gamma_{g_1} \cdots \gamma_{g_\ell}}{\gamma_g} \cdot \mathsf{P}(\lambda_1, \ldots, \lambda_g) \in \mathsf{R}^*(\overline{\mathcal{A}}_g^{\Sigma_g})\,. \tag{24}$$

To derive (24) from (23), we use the Gorenstein property of $\mathsf{R}^*(\overline{\mathcal{A}}_g^{\Sigma_g})$. We need only check that polynomials of complementary degrees in the λ classes pair equally with both sides of (24):

(i) When restricted to the product loci in $\overline{\mathcal{A}}_g^{\Sigma_g}$ and LG_g, both the Hodge bundle $\mathbb{E}$ and the dual subbundle S^* split as direct sums.

(ii) By the Hirzebruch-Mumford proportionality principle, integrals in the λ's over $\overline{\mathcal{A}}_g^{\Sigma_g}$ can be evaluated in terms of integrals in x's over LG_g. The answers are always proportional (21), with proportionality constant γ_g.

Combining (i) and (ii), we see that the constant $\gamma_{g_1} \cdots \gamma_{g_\ell}$ arises for all factors on the left hand side, while the constant γ_g arises for all terms on the right hand side, showing that (23) implies (24). The purely non-tautological part of the cycle in (24) plays no role in the argument.

3.3. **Proof of Theorem 5.** For $g_1 + \ldots + g_\ell = g$, we must show that the tautological projection of the product locus $\overline{\mathcal{A}}_{g_1} \times \ldots \times \overline{\mathcal{A}}_{g_\ell}$ in $\overline{\mathcal{A}}_g$ is given by the $g \times g$ determinant

$$\mathsf{taut}^{\mathsf{cpt}}([\overline{\mathcal{A}}_{g_1} \times \ldots \times \overline{\mathcal{A}}_{g_\ell}]) = \frac{\gamma_{g_1} \cdots \gamma_{g_\ell}}{\gamma_g} \begin{vmatrix} \lambda_{\alpha_1} & \lambda_{\alpha_1+1} & \ldots & \lambda_{\alpha_1+g-1} \\ \lambda_{\alpha_2-1} & \lambda_{\alpha_2} & \ldots & \lambda_{\alpha_2+g-2} \\ \ldots & \ldots & \ldots & \ldots \\ \lambda_{\alpha_g-g+1} & \lambda_{\alpha_g-g+2} & \ldots & \lambda_{\alpha_g} \end{vmatrix}$$

for the vector

$$\alpha = (\underbrace{g - g_1, \ldots, g - g_1}_{g_1}, \underbrace{g - g_1 - g_2, \ldots, g - g_1 - g_2}_{g_2}, \ldots, \underbrace{g - g_1 - \ldots - g_\ell, \ldots, g - g_1 - \ldots - g_\ell}_{g_\ell}) .$$

By the connection between product cycles on $\overline{\mathcal{A}}_g$ and LG_g proven in Section 3.2, it suffices to show that the class of the product $\mathsf{LG}_{g_1} \times \ldots \times \mathsf{LG}_{g_\ell}$ in LG_g is given by the determinant

$$\begin{vmatrix} x_{\alpha_1} & x_{\alpha_1+1} & \ldots & x_{\alpha_1+g-1} \\ x_{\alpha_2-1} & x_{\alpha_2} & \ldots & x_{\alpha_2+g-2} \\ \ldots & \ldots & \ldots & \ldots \\ x_{\alpha_g-g+1} & x_{\alpha_g-g+2} & \ldots & x_{\alpha_g} \end{vmatrix} .$$

We will prove this determinantal formula using the geometry of LG_g.

Let $V = \mathbb{C}^{2g}$ with symplectic form ω. We consider a splitting

$$(V, \omega) \simeq (V_1, \omega_1) \oplus \ldots \oplus (V_\ell, \omega_\ell) , \tag{25}$$

where $V_1, \ldots, V_\ell$ are symplectic subspaces of V with $\dim V_i = 2g_i$. The splitting (25) defines an embedding

$$j : \mathsf{LG}_{g_1} \times \ldots \times \mathsf{LG}_{g_\ell} \to \mathsf{LG}_g , \quad (P_1, \ldots, P_\ell) \mapsto P = P_1 \oplus \ldots \oplus P_\ell .$$

Consider the embedding of LG_g into the usual Grassmannian $\mathsf{G} = \mathsf{G}(g, 2g)$:

$$\iota : \mathsf{LG}_g \to \mathsf{G} .$$

Let $\mathsf{S} \to \mathsf{G}$ be the universal subbundle (which restricts to the universal subbundle $\mathsf{S} \to \mathsf{LG}_g$ via the embedding ι). Similarly, let x_i be the Chern classes of S^* on G (which restrict to the classes x_i on LG_g). Let Σ be the Schubert cycle in the Grassmannian G associated to the partition α with respect to any complete flag $F_1 \subset F_2 \subset \ldots \subset F_{2g} = \mathbb{C}^{2g}$ satisfying the property

$$F_{2(g_1+\ldots+g_i)} = V_1 \oplus \ldots \oplus V_i , \quad 1 \le i \le \ell .$$

By definition, $P \in \Sigma$ provided

$$\dim(P \cap F_{g+j-\alpha_j}) \ge j , \quad 1 \le j \le g .$$

For $1 \le i \le \ell$, let $j = g_1 + \ldots + g_i$, so that $\alpha_j = g - (g_1 + \ldots + g_i)$. We see that for $P \in \Sigma$ we have

$$\dim \left(P \cap F_{2(g_1+\ldots+g_i)}\right) \ge g_1 + \ldots + g_i , \quad 1 \le i \le \ell . \tag{26}$$

The converse is also true. While there are additional requirements about dimensions of intersections with other members of the flag, these are automatically fulfilled by elementary linear algebra considerations.

In $\mathsf{CH}^*(\mathsf{G})$, we have the standard expression [F1, Chapter 14]:

$$[\Sigma] = \begin{vmatrix} x_{\alpha_1} & x_{\alpha_1+1} & \dots & x_{\alpha_1+g-1} \\ x_{\alpha_2-1} & x_{\alpha_2} & \dots & x_{\alpha_2+g-2} \\ \dots & \dots & \dots & \dots \\ x_{\alpha_g-g+1} & x_{\alpha_g-g+2} & \dots & x_{\alpha_g} \end{vmatrix}.$$

Moreover, we have

$$\operatorname{codim}(\Sigma, \mathsf{G}) = |\alpha| = \sum_{i=1}^{\ell} (g - g_1 - \ldots - g_i) g_i = \sum_{i>j} g_i g_j \, ,$$

which agrees with

$$\operatorname{codim}(\mathsf{LG}_1 \times \ldots \times \mathsf{LG}_{g_\ell}, \mathsf{LG}_g) = \binom{g+1}{2} - \sum_{i=1}^{\ell} \binom{g_i+1}{2} = \sum_{i>j} g_i g_j \, .$$

The scheme-theoretic claim

$$\mathsf{LG}_{g_1} \times \ldots \times \mathsf{LG}_{g_\ell} = i^{-1}\Sigma = \Sigma \cap \mathsf{LG}_g \, , \tag{27}$$

then implies

$$[\mathsf{LG}_{g_1} \times \ldots \times \mathsf{LG}_{g_\ell}] = \iota^*[\Sigma] = \begin{vmatrix} x_{\alpha_1} & x_{\alpha_1+1} & \dots & x_{\alpha_1+g-1} \\ x_{\alpha_2-1} & x_{\alpha_2} & \dots & x_{\alpha_2+g-2} \\ \dots & \dots & \dots & \dots \\ x_{\alpha_g-g+1} & x_{\alpha_g-g+2} & \dots & x_{\alpha_g} \end{vmatrix}, \tag{28}$$

as required.

We first establish (27) set-theoretically. The left to right containment is clear for split subspaces $P = P_1 \oplus \ldots \oplus P_\ell$, so we show the converse. Let $P \in \Sigma \cap \mathsf{LG}_g$. For convenience, write

$$h_i = g_1 + \ldots + g_i \, .$$

We set $P_i = P \cap V_i$. Note that $P \cap F_{2h_i}$ is isotropic in F_{2h_i}, hence $\dim(P \cap F_{2h_i}) \leq h_i$. By the Schubert condition (26), we must have

$$\dim(P \cap F_{2h_i}) = h_i \, . \tag{29}$$

We will prove that $\dim P_i = g_i$ for all $1 \leq i \leq \ell$.

The case $i = 1$ is clear by (29) since $V_1 = F_{2h_1}$. For the general case, we induct on i. We assume that

$$\dim(P \cap V_1) = g_1, \ \ldots, \ \dim(P \cap V_i) = g_i \, ,$$

and show that

$$\dim(P \cap V_{i+1}) = g_{i+1} \, .$$

To this end, let $Q = P \cap F_{2h_{i+1}}$, so that

$$\dim Q = h_{i+1} \, , \quad \dim(Q \cap F_{2h_i}) = h_i$$

by (29). Furthermore, Q is isotropic hence Lagrangian in $(F_{2h_{i+1}}, \eta)$ where η is the restriction of the symplectic form ω. To show

$$\dim(P \cap V_{i+1}) = \dim(Q \cap V_{i+1}) = g_{i+1}\,,$$

we compute

$$\dim(Q \cap V_{i+1}) = \dim Q + \dim V_{i+1} - \dim(Q + V_{i+1}) = h_{i+1} + 2g_{i+1} - \dim(Q + V_{i+1})\,.$$

It suffices then to show that $\dim(Q + V_{i+1}) = h_{i+1} + g_{i+1}$, or equivalently,

$$\dim(Q + V_{i+1})^{\eta} = 2h_{i+1} - (h_{i+1} + g_{i+1}) = h_i\,. \tag{30}$$

Here, the complement is taken in $F_{2h_{i+1}}$. Since Q is Lagrangian, $Q^{\eta} = Q$. By construction, $V_{i+1}^{\eta} = F_{2h_i}$. We can therefore rewrite (30) as

$$\dim(Q \cap F_{2h_i}) = h_i\,,$$

which is correct by the Schubert condition (29). The inductive step is proven.

Since $P_1 \oplus \ldots \oplus P_\ell \subset P$, equality must hold for dimension reasons. Therefore, $P \in \mathsf{LG}_{g_1} \times \ldots \times \mathsf{LG}_{g_\ell}$, and the proof of the set-theoretic equality (27) is complete.

To show (27) holds scheme-theoretically, it suffices to prove that the scheme-theoretic intersection $\Sigma \cap \mathsf{LG}_g$ is nonsingular at all points $P \in \Sigma \cap \mathsf{LG}_g$. Equivalently, we will show

$$\dim T_P\,(\Sigma \cap \mathsf{LG}_g)\; = \dim(T_P\,\Sigma\; \cap\; T_P\,\mathsf{LG}_g)\; \leq \dim \mathsf{LG}_{g_1} \times \ldots \mathsf{LG}_{g_\ell} = \sum_{i=1}^{\ell} \binom{g_i + 1}{2}\,. \tag{31}$$

We claim first that all $P \in \Sigma \cap \mathsf{LG}_g$ are nonsingular points of the Schubert variety $\Sigma \subset \mathsf{G}$. We use here a result due to [LS], [C, Corollary 2.5]: singular points of Σ must lie in Schubert varieties for singular partitions associated to α, see [C, Definition 2.1] for the terminology. In our case, nonsingularity at $P \in \Sigma \cap \mathsf{LG}_g$ is due to the fact that equality holds in (29). Equality (29) prevents P from satisfying the Schubert conditions for any of the singular partitions associated to α.

The tangent space of Σ at nonsingular points is computed in [EH, Theorem 4.1]: $T_P\Sigma$ is identified with a subspace of the space of linear maps

$$\Phi : P \to \mathbb{C}^{2g}/P$$

satisfying the property

$$\Phi : (P \cap F_{2h_i}) \to (P + F_{2h_i})/P\,.$$

For tangent space $T_P\mathsf{LG}_g$, we require

$$\Phi : P \to P^*$$

to be symmetric, where we identify $\mathbb{C}^{2g}/P \simeq P^*$ using the symplectic form.

Assume $\Phi \in T_P\Sigma \cap T_P\mathsf{LG}_g$. A straighforward check shows that

$$(P + F_{2h_i})/P \simeq F_{2h_i}/(P \cap F_{2h_i})$$

gets identified with $(P\cap F_{2h_i})^*$, so that

$$\Phi : (P\cap F_{2h_i}) \to (P\cap F_{2h_i})^* .$$

We have shown above that

$$(P\cap F_{2h_i}) = (P\cap V_1)\oplus \ldots \oplus (P\cap V_i) .$$

Therefore, Φ must be symmetric block diagonal with blocks of size $g_1,\ldots,g_\ell$. Equation (31) then follows. ◻

Example 23. For $g_1+g_2=g$, the tautological projection of the product locus $\overline{\mathcal{A}}_{g_1}\times\overline{\mathcal{A}}_{g_2}$ is given by the $g_1\times g_1$ determinant

$$\mathsf{taut}^{\mathsf{cpt}}([\overline{\mathcal{A}}_{g_1}\times\overline{\mathcal{A}}_{g_2}]) = \frac{\gamma_{g_1}\gamma_{g_2}}{\gamma_g}\begin{vmatrix} \lambda_{g_2} & \lambda_{g_2+1} & \cdots & \lambda_{g-1}\\ \lambda_{g_2-1} & \lambda_{g_2} & \cdots & \lambda_{g-2}\\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{g_2-g_1+1} & \lambda_{g_2-g_1+2} & \cdots & \lambda_{g_2}\end{vmatrix} . \tag{32}$$

The right hand side is the Schur determinant associated to the partition $(\underbrace{g_2,\ldots,g_2}_{g_1})$. The Schur determinant is in general not preserved by exchanging g_1 and g_2 (which amounts to transposing the partition), but it is so in the presence of Mumford's relation by precisely [F1, Lemma A.9.2].

- In case $g_1=1$, we obtain

$$\mathsf{taut}^{\mathsf{cpt}}([\overline{\mathcal{A}}_1\times\overline{\mathcal{A}}_{g-1}]) = \frac{g}{6|B_{2g}|}\lambda_{g-1} .$$

- In case $g=2$, we obtain

$$\mathsf{taut}^{\mathsf{cpt}}([\overline{\mathcal{A}}_2\times\overline{\mathcal{A}}_{g-2}]) = \frac{1}{360}\cdot\frac{g(g-1)}{|B_{2g}||B_{2g-2}|}\cdot(\lambda_{g-2}^2-\lambda_{g-1}\lambda_{g-3}) .$$

Example 24. For $g_1=\ldots=g_k=1$, $g_{k+1}=g-k$, Theorem 5 yields

$$\mathsf{taut}^{\mathsf{cpt}}\left(\left[\underbrace{\overline{\mathcal{A}}_1\times\ldots\times\overline{\mathcal{A}}_1}_{k}\times\overline{\mathcal{A}}_{g-k}\right]\right) = \frac{\gamma_1^k\gamma_{g-k}}{\gamma_g}\begin{vmatrix} \lambda_{g-1} & \lambda_g & 0 & \cdots & 0\\ \lambda_{g-3} & \lambda_{g-2} & \lambda_{g-1} & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ \lambda_{g_2-2k+1} & \lambda_{g-2k+2} & \lambda_{g-2k+3} & \cdots & \lambda_{g-k}\end{vmatrix} . \tag{33}$$

For example, we have

$$\mathsf{taut}^{\mathsf{cpt}}\left([\overline{\mathcal{A}}_1\times\overline{\mathcal{A}}_1\times\overline{\mathcal{A}}_{g-2}]\right) = \frac{1}{36}\cdot\frac{g(g-1)}{|B_{2g}||B_{2g-2}|}\cdot(\lambda_{g-1}\lambda_{g-2}-\lambda_g\lambda_{g-3}) .$$

3.4. **Proof of Theorem 6.** Our goal is to prove that after restriction to $\mathcal{A}_g$, the tautological projections of the product cycles admit further factorization:

$$\mathsf{taut}([\mathcal{A}_{g_1}\times\cdots\times\mathcal{A}_{g_\ell}]) = \frac{\gamma_{g_1}\cdots\gamma_{g_\ell}}{\gamma_g}\cdot\lambda_{g-1}\cdots\lambda_{g-\ell+1}\cdot\begin{vmatrix} \lambda_{\beta_1} & \lambda_{\beta_1+1} & \cdots & \lambda_{\beta_1+g^*-1}\\ \lambda_{\beta_2-1} & \lambda_{\beta_2} & \cdots & \lambda_{\beta_2+g^*-2}\\ \cdots & \cdots & \cdots & \cdots\\ \lambda_{\beta_{g^*}-g^*+1} & \lambda_{\beta_{g^*}-g^*+2} & \cdots & \lambda_{\beta_{g^*}}\end{vmatrix},$$

for the vector

$$\beta = (\underbrace{g^* - g_1^*, \ldots, g^* - g_1^*}_{g_1^*}, \underbrace{g^* - g_1^* - g_2^*, \ldots, g^* - g_1^* - g_2^*}_{g_2^*}, \ldots, \underbrace{g^* - g_1^* - \ldots - g_\ell^*}_{g_\ell^*}),$$

where $g^* = g - \ell$ and $g_i^* = g_i - 1$.

The term $\lambda_{g-1} \cdots \lambda_{g-\ell+1}$ is expected to appear in the formula of Theorem 6 by the following reasoning. First,

$$\lambda_{g-m} \cdot [\mathcal{A}_{g_1} \times \ldots \times \mathcal{A}_{g_\ell}] = 0\,, \quad 1 \leq m \leq \ell - 1\,.$$

Indeed, the splitting of the Hodge bundle distributes a top Hodge class on at least one of the ℓ factors $\mathcal{A}_{g_i}$, yielding the vanishing by Theorem 1(iii). Second, we compute the annihilator ideal

$$\operatorname{Ann}\langle \lambda_{g-1}, \ldots, \lambda_{g-\ell+1}\rangle = \langle \lambda_{g-1} \ldots \lambda_{g-\ell+1}\rangle\,.$$

The right to left containment follows from the relations

$$\lambda_j^2 \lambda_{j+1} \ldots \lambda_{g-1} = 0\,, \quad 1 \leq j \leq g-1 \tag{34}$$

on $\mathcal{A}_g$ noted in [vdG1, page 4]. The left to right inclusion can be justified by expressing an arbitrary element z of the annihilator in terms of the square-free monomial basis in the λ's. Using (34), in particular $\lambda_{g-1}^2 = 0$, it follows that all monomials that appear in z must contain λ_{g-1}. If not, $z \cdot \lambda_{g-1}$ would contain nonzero terms in the square-free monomial basis, corresponding to the monomials of z not containing λ_{g-1}. This contradicts that z is in the annihilator ideal. Successively, we see that $\lambda_{g-2}, \ldots, \lambda_{g-\ell+1}$ must also appear in each of the monomials of z, proving the claim.

Proof. We only indicate the proof of Theorem 6 when $\ell = 2$. The general case is an ℓ-fold iteration of the same argument. To start, we restrict to $\mathcal{A}_g$ the expression provided by Theorem 5, see (32). Then, we must prove

$$\begin{vmatrix} \lambda_{g_2} & \lambda_{g_2+1} & \cdots & \lambda_{g-1} \\ \lambda_{g_2-1} & \lambda_{g_2} & \cdots & \lambda_{g-2} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{g_2-g_1+1} & \lambda_{g_2-g_1+2} & \cdots & \lambda_{g_2} \end{vmatrix} = \lambda_{g-1} \cdot \begin{vmatrix} \lambda_{g_2-1} & \lambda_{g_2} & \cdots & \lambda_{g-3} \\ \lambda_{g_2-2} & \lambda_{g_2-1} & \cdots & \lambda_{g-4} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{g_2-g_1+1} & \lambda_{g_2-g_1+2} & \cdots & \lambda_{g_2-1} \end{vmatrix}$$

after setting $\lambda_g = 0$. The parallel identity for the Lagrangian Grassmannian is equivalent:

$$\begin{vmatrix} x_{g_2} & x_{g_2+1} & \cdots & x_{g-1} \\ x_{g_2-1} & x_{g_2} & \cdots & x_{g-2} \\ \cdots & \cdots & \cdots & \cdots \\ x_{g_2-g_1+1} & x_{g_2-g_1+2} & \cdots & x_{g_2} \end{vmatrix} = x_{g-1} \cdot \begin{vmatrix} x_{g_2-1} & x_{g_2} & \cdots & x_{g-3} \\ x_{g_2-2} & x_{g_2-1} & \cdots & x_{g-4} \\ \cdots & \cdots & \cdots & \cdots \\ x_{g_2-g_1+1} & x_{g_2-g_1+2} & \cdots & x_{g_2-1} \end{vmatrix} \mod x_g\,. \tag{35}$$

The identity does not hold in the absence of the Mumford relations.

We will derive identity (35) geometrically via an excess intersection calculation on LG_g. Fix a symplectic splitting

$$V = W_1 \oplus U_1 \oplus W_2 \oplus U_2\,, \quad \dim W_i = 2(g_i - 1)\,, \quad \dim U_i = 2\,.$$

In addition, fix Lagrangian subspaces $P_1 \subset U_1$ and $P_2 \subset U_2$. Let

$$\iota : \mathsf{LG}_{g-1} \to \mathsf{LG}_g\,, \quad P \to P \oplus P_2\,.$$

Here LG_{g-1} is the Lagrangian Grassmannian of $W_1 \oplus U_1 \oplus W_2$. For this embedding, we have

$$\iota_* \left[\mathsf{LG}_{g-1}\right] = x_g \cap \left[\mathsf{LG}_g\right], \tag{36}$$

as can be seen by a normal bundle calculation. The reader can verify that

$$\iota^{-1}(\mathsf{LG}_{g_1} \times \mathsf{LG}_{g_2}) = \mathsf{LG}_{g_1} \times \mathsf{LG}_{g_2-1}.$$

Here, LG_{g_1}, LG_{g_2} and LG_{g_2-1} correspond to $W_1 \oplus U_1$, $W_2 \oplus U_2$ and W_2 respectively. The left hand side has codimension $g_1 g_2$ in LG_g, while the right hand side has codimension $g_1(g_2-1)$ in LG_{g-1}. Write

$$j : \mathsf{LG}_{g_1} \times \mathsf{LG}_{g_2-1} \to \mathsf{LG}_{g-1}$$

for the natural map determined by the pair $(W_1 \oplus U_1, W_2)$. The class $\iota^*(\mathsf{LG}_{g_1} \times \mathsf{LG}_{g_2})$ can be computed via excess intersection. The excess bundle is the dual tautological subbundle $\mathsf{S}^*_{g_1}$. Therefore,

$$\iota^*([\mathsf{LG}_{g_1} \times \mathsf{LG}_{g_2}]) = j_*((x_{g_1} \times 1) \cap [\mathsf{LG}_{g_1} \times \mathsf{LG}_{g_2-1}]) = j_* k_*([\mathsf{LG}_{g_1-1} \times \mathsf{LG}_{g_2-1}]), \tag{37}$$

after using (36) again. The embedding

$$k : \mathsf{LG}_{g_1-1} \times \mathsf{LG}_{g_2-1} \to \mathsf{LG}_{g_1} \times \mathsf{LG}_{g_2-1}$$

is defined by taking sum with P_1 on the first factor. Consider

$$u : \mathsf{LG}_{g_1-1} \times \mathsf{LG}_{g_2-1} \to \mathsf{LG}_{g-2}, \quad v : \mathsf{LG}_{g-2} \to \mathsf{LG}_{g-1},$$

where the first map is determined by the pair (W_1, W_2) and the second map is determined by taking sum with P_1. The equality $j \circ k = v \circ u$ follows from the definitions. By (28) in the proof of Theorem 5, we find

$$u_*([\mathsf{LG}_{g_1-1} \times \mathsf{LG}_{g_2-1}]) = v^* \begin{vmatrix} x_{g_2-1} & x_{g_2} & \cdots & x_{g-3} \\ x_{g_2-2} & x_{g_2-1} & \cdots & x_{g-4} \\ \cdots & \cdots & \cdots & \cdots \\ x_{g_2-g_1+1} & x_{g_2-g_1+2} & \cdots & x_{g_2-1} \end{vmatrix}. \tag{38}$$

Then, using (37) and (38), we have

$$\begin{aligned} \iota^*([\mathsf{LG}_{g_1} \times \mathsf{LG}_{g_2}]) &= j_* k_*([\mathsf{LG}_{g_1-1} \times \mathsf{LG}_{g_2-1}]) = v_* u_*([\mathsf{LG}_{g_1-1} \times \mathsf{LG}_{g_2-1}]) \\ &= v_* v^* \begin{vmatrix} x_{g_2-1} & x_{g_2} & \cdots & x_{g-3} \\ x_{g_2-2} & x_{g_2-1} & \cdots & x_{g-4} \\ \cdots & \cdots & \cdots & \cdots \\ x_{g_2-g_1+1} & x_{g_2-g_1+2} & \cdots & x_{g_2-1} \end{vmatrix} = x_{g-1} \begin{vmatrix} x_{g_2-1} & x_{g_2} & \cdots & x_{g-3} \\ x_{g_2-2} & x_{g_2-1} & \cdots & x_{g-4} \\ \cdots & \cdots & \cdots & \cdots \\ x_{g_2-g_1+1} & x_{g_2-g_1+2} & \cdots & x_{g_2-1} \end{vmatrix}, \end{aligned}$$

which recovers the right hand side of (35). On the other hand, by (28), the class on the left hand side equals

$$\iota^* \begin{vmatrix} x_{g_2} & x_{g_2+1} & \cdots & x_{g-1} \\ x_{g_2-1} & x_{g_2} & \cdots & x_{g-2} \\ \cdots & \cdots & \cdots & \cdots \\ x_{g_2-g_1+1} & x_{g_2-g_1+2} & \cdots & x_{g_2} \end{vmatrix},$$

while the pullback $\iota^* : \mathsf{CH}^*(\mathsf{LG}_g) \to \mathsf{CH}^*(\mathsf{LG}_{g-1})$ has the effect of setting $x_g = 0$. $\square$

Appendix A. Abelian varieties with real multiplication

Shimura-Hilbert-Blumental varietes parametrize abelian varieties with real multiplication and arise as Noether-Leschetz loci in $\mathcal{A}_g$, see [DL]. We propose here a conjecture for the tautological projections of the classes of the canonical components of Shimura-Hilbert-Blumenthal varieties in $\mathcal{A}_g$.

A.1. **Real multiplication.** Fix a totally real number field F with $[F:\mathbb{Q}]=e$. A polarized abelian variety X admits *real multiplication* by F provided that

$$F\subset \mathrm{End}_{\mathbb{Q}}(X)$$

as unital $\mathbb{Q}$-algebras. In the context of the Noether-Lefschetz loci, the embedding of the totally real field arises from the additional Néron-Severi class[13], this is explained in [DL]. By [BL, Proposition 5.5.7], we must have

$$g=me$$

for an integer m, and the Picard rank of X must be least e.

We will use the following notation for objects related to the totally real field F:

- $\sigma_1,\ldots,\sigma_e : F\to\mathbb{R}$ are the real embeddings of F.
- $\mathcal{O}_F$ denotes the ring of integers in F.
- $\mathfrak{d}^{-1}$ is the codifferent ideal of F given by

  $$\mathfrak{d}^{-1}=\{x\in F : \mathrm{Tr}_{F/\mathbb{Q}}(xy)\in\mathbb{Z}\quad \text{for all } y\in\mathcal{O}_F\}\,.$$

  $\mathfrak{d}^{-1}$ is a fractional ideal whose inverse is the different ideal $\mathfrak{d}\subset\mathcal{O}_F$.
- The Dedekind zeta function is given by

  $$\zeta_F(s)=\sum_{\mathfrak{a}\subset\mathcal{O}_F}\frac{1}{|\mathcal{O}_F/\mathfrak{a}|^s},\quad \mathrm{Re}\, s>1\,,$$

  where the sum is taken over ideals $\mathfrak{a}\subset\mathcal{O}_F$. The function ζ_F can be analytically continued to $\mathbb{C}\smallsetminus\{1\}$.

A.2. **Shimura-Hilbert-Blumenthal varieties.**

A.2.1. *Component $\mathcal{A}_F$.* If X admits real multiplication by F, then $F\cap\mathrm{End}(X)\subset F$ is an order in F. We will consider the canonical component of the Shimura-Hilbert-Blumenthal variety of abelian varieties with real multiplication by F: the component $\mathcal{A}_F$ defined by the condition that the intersection is a maximal order

$$F\cap\mathrm{End}(X)=\mathcal{O}_F\subset F\,.$$

[13] If L is the polarization, and M is the additional Néron-Severi class, the field F is spanned by $\phi_L^{-1}\phi_M\in\mathrm{End}_{\mathbb{Q}}(X)$ where $\phi_L,\phi_M : X\to\widehat{X}$ are the usual morphisms [BL, Section 2.4].

The geometric construction of $\mathcal{A}_F$ is standard [BL, G, S]. As in the construction of $\mathcal{A}_g$, the Siegel upper half space,

$$\mathfrak{H}_m^+ = \{\tau \in \mathrm{Mat}_{m\times m}(\mathbb{C}), \quad \tau = \tau^t, \quad \mathrm{Im}\,\tau > 0\},$$

plays a central role.

A.2.2. *Case* $g = e$. Consider first the case $g = e$ and $m = 1$ which corresponds to the classical Hilbert-Blumenthal varieties. For each tuple

$$\tau = (\tau_1, \ldots, \tau_g) \in \mathfrak{H}_1^+ \times \ldots \times \mathfrak{H}_1^+,$$

we define

$$j_\tau : F \times F \to \mathbb{C}^g, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \sigma_1(x)\tau_1 + \sigma_1(y) \\ \ldots \\ \sigma_g(x)\tau_g + \sigma_g(y) \end{pmatrix}.$$

Consider the lattice

$$\Gamma'_\tau = j_\tau(\mathcal{O}_F \times \mathcal{O}_F) \subset \mathbb{C}^g.$$

The quotient

$$X'_\tau = \mathbb{C}^g/\Gamma'_\tau$$

is an abelian variety via the polarization

$$H_\tau(z, w) = \sum_{j=1}^g \frac{z_j \overline{w}_j}{\mathrm{Im}\,\tau_j}.$$

The abelian variety (X'_τ, H_τ) may not be principally polarized. To construct a principally polarized abelian variety, we consider the lattice

$$\Gamma_\tau = j_\tau(\mathcal{O}_F \times \mathfrak{d}^{-1}).$$

The complex torus

$$X_\tau = \mathbb{C}^g/\Gamma_\tau$$

is then principally polarized via H_τ. The lattices Γ_τ and Γ'_τ in $\mathbb{C}^g$ are preserved by the action of $\mathcal{O}_F$ defined by

$$\mathcal{O}_F \to \mathrm{Mat}_{g\times g}(\mathbb{C}), \quad x \mapsto \mathrm{diag}(\sigma_1(x), \ldots, \sigma_g(x)).$$

Therefore,

$$\mathcal{O}_F \hookrightarrow \mathrm{End}(X_\tau) \text{ and so } F \subset \mathrm{End}(X_\tau)_\mathbb{Q}.$$

There is a moduli space

$$\mathcal{A}_F = \mathrm{SL}_2(\mathcal{O}_F \oplus \mathfrak{d}^{-1})\backslash \mathfrak{H}_1^+ \times \ldots \times \mathfrak{H}_1^+.$$

Here, we set

$$\mathrm{SL}_2(\mathcal{O}_F \oplus \mathfrak{d}^{-1}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, d \in \mathcal{O}_F, \quad b \in \mathfrak{d}, \quad c \in \mathfrak{d}^{-1}, \quad ad - bc = 1 \right\}.$$

These are matrices that preserve $\mathcal{O}_F \times \mathfrak{d}^{-1}$. The action of $\mathrm{SL}_2(\mathcal{O}_F \oplus \mathfrak{d}^{-1})$ on the product $\mathfrak{H}_1^+ \times \ldots \times \mathfrak{H}_1^+$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\tau_1, \ldots, \tau_g) = \left(\frac{\sigma_1(a)\tau_1 + \sigma_1(b)}{\sigma_1(c)\tau_1 + \sigma_1(d)}, \ldots, \frac{\sigma_g(a)\tau_1 + \sigma_g(b)}{\sigma_g(c)\tau_g + \sigma_g(d)} \right).$$

From the above discussion, it follows that there is a morphism $\mathcal{A}_F \to \mathcal{A}_g$.

The moduli space $\mathcal{A}_F$ differs slightly from the construction in [BL], but it agrees with [G, S]. The precise modular interpretation is given in [G, Theorem 2.17], for instance.

A.2.3. *Case $g = me$.* We consider tuples

$$\tau = (\tau_1, \ldots, \tau_e) \in \mathfrak{H}_m^+ \times \ldots \times \mathfrak{H}_m^+.$$

The construction of Section A.2.2 goes through with minor modifications. We set

$$j_\tau : F^m \times F^m \to \mathbb{C}^g, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \tau_1 \sigma_1(x) + \sigma_1(y) \\ \ldots \\ \tau_g \sigma_g(x) + \sigma_g(y) \end{pmatrix},$$

where $x, y \in F^m$, and we can define $\sigma_j : F^m \to \mathbb{R}^m$ by applying σ_j component by component. We consider the lattice $\Gamma_\tau = j_\tau \left(\mathcal{O}_F^m \times \left(\mathfrak{d}^{-1} \right)^m \right)$. The abelian variety $X_\tau = \mathbb{C}^g / \Gamma_\tau$ is polarized in the same fashion as above. There is a moduli space

$$\mathcal{A}_F = \mathrm{Sp}_{2m}(\mathcal{O}_F \oplus \mathfrak{d}^{-1}) \backslash \mathfrak{H}_m^+ \times \ldots \times \mathfrak{H}_m^+$$

of dimension

$$\dim \mathcal{A}_F = e \cdot \frac{m(m+1)}{2}.$$

The group $\mathrm{Sp}_{2m}(\mathcal{O}_F \oplus \mathfrak{d}^{-1})$ preserves $\mathcal{O}_F^m \times \left(\mathfrak{d}^{-1} \right)^m \subset F^{2m}$ as well as the standard symplectic form

$$E(x, y) = \sum_{i=1}^{2m} \mathrm{Tr}_{F/\mathbb{Q}} (x_i y_{i+m} - y_i x_{i+m}), \quad x, y \in F^{2m}.$$

A.3. **Tautological projection.** We conjecture a formula for the tautological projection of $[\mathcal{A}_F]$. We provide evidence for the formula via the Hirzebruch-Mumford proportionality principle and Theorem 5.

Our considerations rely on a few assumptions on toroidal compactifications which we now state. Consider the moduli stack

$$\mathcal{A}'_F = \mathrm{Sp}_{2m}(\mathcal{O}_F) \backslash \mathfrak{H}_m^+ \times \ldots \times \mathfrak{H}_m^+.$$

Let $\overline{\mathcal{A}}'_F$ be a smooth toroidal compatification with simple normal crossings boundary. We assume

- the Hodge bundle $\mathbb{E}$ extends to $\overline{\mathcal{A}}'_F$ and splits as direct sums of bundles of rank m

$$\mathbb{E} = \mathbb{E}_1 \oplus \ldots \oplus \mathbb{E}_e,$$

- each one of the bundles $\mathbb{E}_1, \ldots, \mathbb{E}_e$ satisfies the Mumford relations,
- over $\overline{\mathcal{A}}'_F$, we have

$$\Omega^{\log} = \mathrm{Sym}^2 \mathbb{E}_1 \oplus \cdots \oplus \mathrm{Sym}^2 \mathbb{E}_e.$$

In fact, we assume a bit more, in particular that there is a compactification $\overline{\mathcal{A}}_F$ compatible with the choice of $\overline{\mathcal{A}}'_F$, as well as a morphism $\overline{\mathcal{A}}_F \to \overline{\mathcal{A}}_g$ extending $\mathcal{A}_F \to \mathcal{A}_g$.

The compact dual of $\mathfrak{H}_m^+ \times \ldots \times \mathfrak{H}_m^+$ is the product of e Lagrangian Grassmannians

$$\mathsf{LG}_m \times \ldots \times \mathsf{LG}_m \,.$$

By the Hirzebruch-Mumford proportionality principle [Mu], we must have

$$\int_{\overline{\mathcal{A}}'_F} \mathsf{P}(\lambda_1, \ldots, \lambda_g) = \gamma'_F \int_{\mathsf{LG}_m \times \ldots \times \mathsf{LG}_m} \mathsf{P}(x_1, \ldots, x_g) \tag{39}$$

for some constant γ'_F. Here, $x_1, \ldots, x_g$ are the Chern classes of the dual bundle

$$\mathsf{S}^* = \mathsf{S}_1^* \oplus \ldots \oplus \mathsf{S}_e^* \to \mathsf{LG}_m \times \ldots \times \mathsf{LG}_m \,,$$

where S_i is the tautological subbundle on the i^{th} Lagrangian Grassmannian.

To find the constant γ'_F, we use the polynomial

$$\mathsf{P}(\lambda_1, \ldots, \lambda_g) = \lambda_e \lambda_{2e} \cdots \lambda_{me} \,,$$

where $\deg \mathsf{P} = \dim \mathcal{A}'_F$.

For each bundle $\mathcal{V}$ that satisfies the Mumford relation, we have

$$\mathsf{e}(\mathrm{Sym}^2 \mathcal{V}) = 2^g c_1(\mathcal{V}) \cdots c_g(\mathcal{V}) \,. \tag{40}$$

The identity can be found in [vdG3, Section 2].

We have a logarithmic analogue of Gauss-Bonnet computing the orbifold Euler characteristic[14]

$$\chi_{\mathsf{orb}}(\mathcal{A}'_F) = (-1)^{\dim \mathcal{A}'_F} \int_{\overline{\mathcal{A}}'_F} \mathsf{e}(\Omega^{\log}) \,.$$

Thus, using our assumptions, we find

$$\chi_{\mathsf{orb}}(\mathcal{A}'_F) = (-1)^{\dim \mathcal{A}'_F} \int_{\overline{\mathcal{A}}'_F} \mathsf{e}(\mathrm{Sym}^2 \mathbb{E}_1) \cdots \mathsf{e}(\mathrm{Sym}^2 \mathbb{E}_e) \,. \tag{41}$$

Invoking (40), we have

$$\mathsf{e}(\mathrm{Sym}^2 \mathbb{E}_1) \cdots \mathsf{e}(\mathrm{Sym}^2 \mathbb{E}_e) = \left(2^m \lambda_1^{(1)} \cdots \lambda_m^{(1)}\right) \cdots \left(2^m \lambda_1^{(e)} \cdots \lambda_m^{(e)}\right) .$$

Here $\lambda_1^{(i)}, \ldots, \lambda_m^{(i)}$ are the Hodge classes for the summand $\mathbb{E}_i$. Using the Mumford relations for each of the summands $\mathbb{E}_i$, the above expression can be rewritten as

$$\left(\lambda_1^{(1)} \cdots \lambda_m^{(1)}\right) \cdots \left(\lambda_1^{(e)} \cdots \lambda_m^{(e)}\right) = \lambda_e \lambda_{2e} \cdots \lambda_{me} \,.$$

Indeed, the last term λ_{me} on the right hand side splits as $\lambda_m^{(1)} \cdots \lambda_m^{(e)}$. The next term $\lambda_{(m-1)e}$ is forced to split as $\lambda_{m-1}^{(1)} \cdots \lambda_{m-1}^{(e)}$, since all $\lambda_m^{(i)}$ contributions to this term will cancel when paired with the term $\lambda_{me} = \lambda_m^{(1)} \cdots \lambda_m^{(e)}$, via Mumford's relations, and so on. Therefore,

$$\mathsf{e}(\mathrm{Sym}^2 \mathbb{E}_1) \cdots \mathsf{e}(\mathrm{Sym}^2 \mathbb{E}_e) = 2^g \lambda_e \lambda_{2e} \cdots \lambda_{me} \,. \tag{42}$$

[14] The result is well-known. See, for example, [Sil]. A proof for Deligne-Mumford stacks appears in [CMZ, Proposition 2.1].

The same argument shows

$$\mathsf{e}(\mathrm{Sym}^2\,\mathsf{S}_1^*)\cdots\mathsf{e}(\mathrm{Sym}^2\,\mathsf{S}_e^*) = 2^g x_e x_{2e}\cdots x_{me}\,. \tag{43}$$

Using (39), (41), (42), and (43), we find

$$\begin{aligned}
\chi_{\mathsf{orb}}(\mathcal{A}'_F) &= (-1)^{\dim \mathcal{A}'_F}\int_{\overline{\mathcal{A}}'_F}\mathsf{e}(\mathrm{Sym}^2\,\mathbb{E}_1)\cdots\mathsf{e}(\mathrm{Sym}^2\,\mathbb{E}_e)\\
&= (-1)^{\dim \mathcal{A}'_F}\int_{\overline{\mathcal{A}}'_F} 2^g\lambda_e\lambda_{2e}\cdots\lambda_{me}\\
&= (-1)^{\dim \mathcal{A}'_F}\gamma'_F\int_{\mathsf{LG}_m\times\ldots\times\mathsf{LG}_m} 2^g x_e x_{2e}\cdots x_{me}\\
&= (-1)^{\dim \mathcal{A}'_F}\gamma'_F\int_{\mathsf{LG}_m\times\ldots\times\mathsf{LG}_m}\mathsf{e}(\mathrm{Sym}^2\,\mathsf{S}_1^*)\cdots\mathsf{e}(\mathrm{Sym}^2\,\mathsf{S}_e^*)\\
&= (-1)^{\dim \mathcal{A}'_F}\gamma'_F\int_{\mathsf{LG}_m\times\ldots\times\mathsf{LG}_m}\mathsf{e}(\mathrm{Tan}_{\mathsf{LG}_1})\cdots\mathsf{e}(\mathrm{Tan}_{\mathsf{LG}_m})\\
&= (-1)^{\dim \mathcal{A}'_F}\gamma'_F\cdot\mathsf{e}(\mathsf{LG}_m)^e = (-1)^{e\cdot\frac{m(m+1)}{2}}\cdot\gamma'_F\cdot(2^m)^e\,.
\end{aligned}$$

Using the Harder-Siegel formula [H, page 493], [Si], we have

$$\chi_{\mathsf{orb}}(\mathcal{A}'_F) = \chi_{\mathsf{orb}}(\mathrm{Sp}_{2m}(\mathcal{O}_F)) = \zeta_F(-1)\cdots\zeta_F(1-2m)\,,$$

which yields

$$\gamma'_F = (-1)^{e\cdot\frac{m(m+1)}{2}}\cdot\frac{1}{2^g}\cdot\zeta_F(-1)\cdots\zeta_F(1-2m)\,.$$

We can compare integrals of Hodge classes on $\overline{\mathcal{A}}_F$ and $\overline{\mathcal{A}}'_F$ using a common finite cover corresponding to the quotient of $\mathfrak{H}_m^+\times\ldots\times\mathfrak{H}_m^+$ by $\mathrm{Sp}_{2m}(\mathcal{O}_F\oplus\mathfrak{d}^{-1})\cap\mathrm{Sp}_{2m}(\mathcal{O}_F)\subset\mathrm{Sp}_{2m}(F)$. To this end, we need to be able to make compatible choices of the compactification data. Assuming such choices, we find

$$\int_{\overline{\mathcal{A}}_F}\mathsf{P}(\lambda_1,\ldots,\lambda_g) = \gamma_F\int_{\mathsf{LG}_m\times\ldots\times\mathsf{LG}_m}\mathsf{P}(x_1,\ldots,x_g) \tag{44}$$

where

$$\gamma_F = \gamma'_F\cdot[\mathrm{Sp}_{2m}(\mathcal{O}_F):\mathrm{Sp}_{2m}(\mathcal{O}_F\oplus\mathfrak{d}^{-1})]\,.$$

Here, for two commensurable subgroups G_1, G_2 of $\mathrm{Sp}_{2m}(F)$, we write

$$[G_1:G_2] = \frac{[G_1:(G_1\cap G_2)]}{[G_2:(G_1\cap G_2)]}\,.$$

Using the definition of the tautological projection and (44), we obtain

$$\begin{aligned}
\int_{\overline{\mathcal{A}}_g}\mathsf{taut}^{\mathsf{cpt}}\left([\overline{\mathcal{A}}_F]\right)\cdot\mathsf{P}(\lambda_1,\ldots,\lambda_g) = \int_{\overline{\mathcal{A}}_F}\mathsf{P}(\lambda_1,\ldots,\lambda_g) &= \gamma_F\int_{\mathsf{LG}_m\times\ldots\times\mathsf{LG}_m}\mathsf{P}(x_1,\ldots,x_g)\\
&= \gamma_F\int_{\mathsf{LG}_g}[\mathsf{LG}_m\times\ldots\times\mathsf{LG}_m]\cdot\mathsf{P}(x_1,\ldots,x_g)\\
&= \gamma_F\int_{\mathsf{LG}_g}\mathrm{Schur}_{m,e}(x_1,\ldots,x_g)\cdot\mathsf{P}(x_1,\ldots,x_g)\\
&= \frac{\gamma_F}{\gamma_g}\int_{\overline{\mathcal{A}}_g}\mathrm{Schur}_{m,e}(\lambda_1,\ldots,\lambda_g)\cdot\mathsf{P}(\lambda_1,\ldots,\lambda_g)\,,
\end{aligned}$$

which implies

$$\mathsf{taut}^{\mathsf{cpt}}\left(\left[\overline{\mathcal{A}}_F\right]\right) = \frac{\gamma_F}{\gamma_g} \cdot \mathrm{Schur}_{m,e}(\lambda_1, \ldots, \lambda_g)\,.$$

On the last line of the above derivation, we have used the Hirzebruch-Mumford principle again. Here,

$$\gamma_g = (-1)^{\frac{g(g+1)}{2}} \cdot \frac{1}{2^g} \cdot \zeta(-1) \cdots \zeta(1-2g)$$

is the Hirzebruch-Mumford proportionality constant[15] for $\overline{\mathcal{A}}_g$ found in [vdG1, Theorem 1.13]. The polynomial

$$\mathrm{Schur}_{m,e}(\lambda_1, \ldots, \lambda_g) = \begin{vmatrix} \lambda_{\alpha_1} & \lambda_{\alpha_1+1} & \cdots & \lambda_{\alpha_1+g-1} \\ \lambda_{\alpha_2-1} & \lambda_{\alpha_2} & \cdots & \lambda_{\alpha_2+g-2} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{\alpha_g-g+1} & \lambda_{\alpha_g-g+2} & \cdots & \lambda_{\alpha_g} \end{vmatrix}$$

corresponds to the partition

$$\alpha = (\underbrace{m(e-1), \ldots, m(e-1)}_{m}, \underbrace{m(e-2), \ldots, m(e-2)}_{m} \ldots, \underbrace{m, \ldots, m}_{m})$$

via Theorem 5.

Conjecture F. *The projection of* $\left[\overline{\mathcal{A}}_F\right]$ *is computed by*

$$\mathsf{taut}^{\mathsf{cpt}}\left(\left[\overline{\mathcal{A}}_F\right]\right) = \mathsf{const} \cdot \mathrm{Schur}_{m,e}(\lambda_1, \ldots, \lambda_g)$$

for the constant

$$\mathsf{const} = (-1)^{\frac{g(g-m)}{2}} \cdot \frac{\zeta_F(-1) \cdots \zeta_F(1-2m)}{\zeta(-1) \cdots \zeta(1-2g)} \cdot [\mathrm{Sp}_{2m}(\mathcal{O}_F) : \mathrm{Sp}_{2m}(\mathcal{O}_F \oplus \mathfrak{d}^{-1})]\,.$$

The restriction to $\mathcal{A}_g$ can be slightly simplified using Theorem 6. For instance, when $m = 1$ and $g = e$, on $\mathcal{A}_g$ we have

$$\mathrm{Schur}_{1,g}(\lambda_1, \ldots, \lambda_g) = \lambda_{g-1} \cdots \lambda_1\,,$$

yielding in this case

$$\mathsf{taut}\left([\mathcal{A}_F]\right) = (-1)^{\frac{g(g-1)}{2}} \frac{\zeta_F(-1)}{\zeta(-1) \cdots \zeta(1-2g)} \cdot [\mathrm{SL}_2(\mathcal{O}_F) : \mathrm{SL}_2(\mathcal{O}_F \oplus \mathfrak{d}^{-1})] \cdot \lambda_{g-1} \cdots \lambda_1\,.$$

Aitor Iribar López has informed us that a proof of Conjecture F together with further results about the projections of Shimura-Hilbert-Blumenthal varieties will appear in his upcoming paper [IL].

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