DESCENDENT BOUNDS FOR EFFECTIVE DIVISORS ON $\overline{M}_{g,1}$

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Let $g \geq 1$. Let $\overline{M}_{g,1}$ be the moduli space of genus g stable curves with 1 marking. We are interested in two basic divisors classes,

$$\psi, \delta \in A^1(\overline{M}_{g,1}) ,$$

where ψ is the cotangent line class at the marking and δ is the locus parameterizing singular curves (the full boundary divisor). We consider here a question asked by I. Smith:

For which values of $(a, b) \in \mathbb{Z}^2_{>0}$ is $a\psi - b\delta$ an effective divisor?

The divisor class ψ is well-known to be nef. Using the Weierstrass divisor, multiples of ψ can be shown to be effective. We might expect then that $a\psi - b\delta$ is effective if a is sufficiently larger than b. The method developed in [2] gives the following simple bound.

Proposition 1. *If the class $a\psi - b\delta$ is effective, then*

$$\frac{a}{b} \geq \frac{12g}{2g-1} > 6 .$$

Proof. Let C be a freely moving curve on $\overline{M}_{g,1}$. If $a\psi - b\delta$ is effective, then

$$[C] \cdot (a\psi - b\delta) \geq 0 .$$

Let δ_0 be the boundary divisor parameterizing irreducible singular curves. Since δ_0 is an effective summand of δ,

$$[C] \cdot (a\psi - b\delta_0) \geq 0 .$$

As explained in [2], we can construct (limits of) freely moving classes on $\overline{M}_{g,1}$ by pushing-forward the curve class

$$\psi_2^{3g-2} \in A_1(\overline{M}_{g,2})$$

to $\overline{M}_{g,1}$ via the map

$$\epsilon : \overline{M}_{g,2} \to \overline{M}_{g,1}$$

forgetting the second marking. Hence, we find the bound

$$\frac{a}{b} \geq \frac{\int_{\overline{M}_{g,2}} \psi_2^{3g-2} \epsilon^*(\delta_0)}{\int_{\overline{M}_{g,2}} \psi_2^{3g-2} \epsilon^*(\psi_1)} . \tag{1}$$

Date: 1 January 2011.

We now calculate the integrals appearing on the right side of (1). The numerator was essentially evaluated in [2]. The pull-back by ϵ of δ_0 is simply δ_0 on $\overline{M}_{g,2}$. Using the normalization map of the irreducible divisor,

$$\int_{\overline{M}_{g,2}} \psi_2^{3g-2}\epsilon^*(\delta_0) = \frac{1}{2}\int_{\overline{M}_{g-1,4}} \psi_2^{3g-2} = \frac{1}{2}\int_{\overline{M}_{g-1,1}} \psi_1^{3g-5}.$$

The string equation is used in the last equality. The evaluation $\int_{\overline{M}_{g-1,1}} \psi_1^{3g-5} = \frac{1}{(24)^{g-1}(g-1)!}$ is well-known. Hence

$$\int_{\overline{M}_{g,2}} \psi_2^{3g-2}\epsilon^*(\delta_0) = \frac{1}{2}\frac{1}{(24)^{g-1}(g-1)!}. \tag{2}$$

The denominator on the right side of (1) is also easily computed:

$$\begin{aligned} \int_{\overline{M}_{g,2}} \psi_2^{3g-2}\epsilon^*(\psi_1) &= \int_{\overline{M}_{g,2}} \psi_2^{3g-2}(\psi_1 - \delta_{12}) \\ &= \int_{\overline{M}_{g,2}} \psi_2^{3g-2}\psi_1 \\ &= (2g-2+1)\int_{\overline{M}_{g,*}} \psi_*^{3g-2} \\ &= \frac{2g-1}{(24)^g g!} \end{aligned}$$

Taking the ratio with (2), we find

$$\frac{a}{b} \geq \frac{\frac{1}{2}\frac{1}{(24)^{g-1}(g-1)!}}{\frac{2g-1}{(24)^g g!}} = \frac{12g}{2g-1}\ .$$

Of course, the latter approaches 6 from above as g tend to ∞. □

We now look briefly at the existence issue. To start, let g be odd. The Brill-Noether locus of curves in M_g which carry a g^1_d for $2d = g+1$ is a divisor. Taking the closure and pulling-back to $\overline{M}_{g,1}$ yields the Brill-Noether class[1]

$$\mathsf{B} = (g+3)\lambda - \left(\frac{g+1}{6}\right)\delta_0 - \sum_{i=1}^{g-1} i(g-i)\delta_i\ . \tag{3}$$

Here, λ is the first Chern class of the Hodge bundle, and δ_i is the boundary divisor parametrizing reducible curves with the marking 1 on a component of genus i. Another basic effective class is the Weierstrass divisor

$$\mathsf{W} = -\lambda + \binom{g+1}{2}\psi - \sum_{i=1}^{g-1}\binom{g-i+1}{2}\delta_i\ , \tag{4}$$

[1]The formulas for B and W, calculated via the moduli space of admissible covers [1] (and by earlier methods of S. Diaz), were communicated to the author by G. Farkas. In fact, the whole paragraph was communicated by Farkas.

parameterizing (C,p) where p is a Weierstrass point of C. The irreducible boundary divisor δ_0 does not appear in the formula for W. The divisor

$$\mathsf{B} + (g+3)\mathsf{W} = (g+3)\binom{g+1}{2}\psi - \frac{g+1}{6}\delta_0 - \sum_{i=1}^{g-1}\left(i(g-i) + (g+3)\binom{g-i+1}{2}\right)\delta_i$$

is then certainly effective. Since $\frac{g+1}{6}$ is the smallest coefficient of the boundary divisors, we have established existence of effective divisors $a\psi - b\delta$ for

$$a = (g+3)\binom{g+1}{2}, \quad b = \frac{g+1}{6}, \qquad \frac{a}{b} = 3g^2 + 9g.$$

In fact, the same construction is valid in the g even case since Farkas has proven the effectivity of the class B defined by (3) for g even. Formula (4) for the Weierstrass divisor holds for all g. The slope $3g^2 + 9g$ greatly exceeds the bound of Proposition 1.

Proposition 2. *If the class $a\psi - b\delta$ is effective, then $\frac{a}{b} \geq 10$.*

Proof. The idea here is to push-forward to M_g. Let D be an effective divisor of class $a\psi - b\delta$ on $\overline{M}_{g,1}$. Since ψ is nef, the push-forward

$$\epsilon_*(\psi \cdot [\mathsf{D}]) \in A^1(\overline{M}_g)$$

is the limit of classes of effective divisors. We easily compute

$$\begin{aligned} \epsilon_*(\psi \cdot [\mathsf{D}]) &= \epsilon_*(a\psi^2 - b\psi\delta) \\ &= a\kappa_1 - (2g-2)b\delta \\ &= 12a\lambda - (a + (2g-2)b)\delta \ , \end{aligned}$$

where the basic relation

$$\lambda = \frac{1}{12}\kappa_1 + \frac{1}{12}\delta$$

on $\overline{M}_g$ is used in the last equality. By the main result of [2], effectivity of the push-forward implies

$$\frac{12a}{a + (2g-2)b} \geq \frac{60}{g+4} \ .$$

Writing s for the slope a/b, we see

$$\frac{12s}{s + (2g-2)} \geq \frac{60}{g+4} \ ,$$

or equivalently,

$$12(g+4)s \geq 60s + 60(2g-2) \ .$$

The conclusion $s \geq 10$ immediately follows for $g \geq 2$.

For $g = 1$, the class ψ has degree $\frac{1}{24}$ and the the class δ has degree $\frac{1}{2}$. Hence, the stronger bound $s \geq 12$ holds for $g = 1$. □

We have gained a little in Proposition 2, but not much. Inspired by Teichmüller curve heuristics, Dawei Chen has suggested to me that the slope bound for I. Smith's question should be linear in g.

Proposition 3. *If the slope bound for the effectivity of $a\psi - b\delta$ on $\overline{M}_{g,1}$ is linear in g, then a slope bound for the effectivity of $x\lambda - y\delta$ on $\overline{M}_g$ which goes as the inverse of g can be derived.*

Proof. Suppose D is an effective divisor of class $x\lambda - y\delta$ on $\overline{M}_g$. Certainly $\epsilon^*(\mathsf{D})$ is effective. Hence, the divisors

$$\epsilon^*(\mathsf{D}) + x\mathsf{W} = x\binom{g+1}{2}\psi - y\delta_0 - \sum_{i=1}^{g-1}\left(y + x\binom{g-i+1}{2}\right)\delta_i$$

and $x\binom{g+1}{2}\psi - y\delta$ are both effective on $\overline{M}_{g,1}$. If

$$\frac{x}{y}\binom{g+1}{2} \sim g,$$

then $\frac{x}{y} \sim \frac{1}{g}$. □

Proposition 3 appears pointless since a $\frac{1}{g}$ bound for the slope question on $\overline{M}_g$ has already been proven in [2]. However, the result does suggest that Chen's prediction is very reasonable. Also, if a g^2 bound for Smith's question can be proven, a genus independent slope bound on $\overline{M}_g$ could be deduced.

References

[1] J. Harris and I. Morrison, *The slope of effective divisors on the moduli space of stable curves*, Invent. Math. **99** (1990), 321 – 355.

[2] R. Pandharipande, *Descendent bounds for effective divisors on $\overline{M}_g$*, J. Alg. Geom. (to appear).