Gromov-Witten theory in Low dimensions

1, 2, 3

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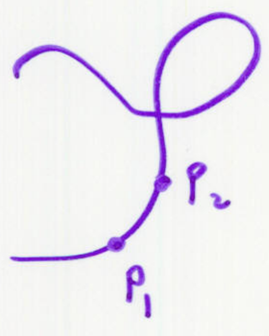
The basic object of GW theory is the moduli space of stable maps:

Let X be nonsingular, complex projective variety.

A map

$$f : (C, p_1, \ldots, p_n) \to X$$



pointed, nodal curve

is stable if there are only finitely many automorphisms

$$\sigma : C \to C$$
$$\sigma(p_i) = p_i$$
$$f \circ \sigma = f$$

Besides the genus $g = h^1(C, \mathcal{O}_C)$
and the number of markings n,

A stable map has another discrete invariant : $\beta = f_*[C] \in H_2(X, \mathbb{Z})$

Let $\overline{M}_{g,n}(X, \beta)$ be the moduli space of stable maps to X.

- $\overline{M}_{g,n}(X, \beta)$ proper D-M stack (orbischeme)
- may be reducible, non reduced, singular, and of mixed dimension
- $\overline{M}_{g,n}(\text{point}, 0) = \overline{M}_{g,n}$

While $\overline{M}_{g,n}(X,\beta)$ is badly behaved in general, the moduli space has a virtual dimension

$$\text{vdim} = \underbrace{3g-3+n}_{\substack{\text{moduli}\\ \text{of curves}}} + \chi(\underbrace{C, f^*T_X}_{\substack{\text{def of map}\\ \text{with fixed domain}}})$$

$$= 3g-3+n + \int_\beta c_1(T_X) + \dim_{\mathbb{C}} X\,(1-g)$$

In fact $\overline{M}_{g,n}(X,\beta)$ carries a virtual class

$$[\overline{M}_{g,n}(X,\beta)]^{vir} \in H_{2\cdot \text{vdim}}(\overline{M}_{g,n}(X,\beta), \mathbb{Q})$$

Li - Tian
Behrend - Fantechi

The virtual class is central to GW theory. Existence can be viewed as a shadow of the path integral

How to think about the virtual class:

(1) Deformation theory of maps $\Rightarrow$ obstruction theory

$$\begin{array}{c} E^{-1} \to E^{0} \\ \downarrow \\ \overline{M}_{g,n}(X,\beta) \end{array}$$

$$0 \to Obs^{\vee} \to E^{-1} \to E^{0} \to \Omega_{\overline{M}_{g,n}(X,\beta)} \to 0$$

Obstruction sheaf

Naively: $\left[\overline{M}_{g,n}(X,\beta)\right]^{vir}$ "=" $c_{top}(Obs)$

Valid at least when Obs is a bundle.

(2) What does $E^{-1} \to E^{0}$ look like?

If we fix complex structure of the domain, $\overline{\mathcal{M}}_{C,n}(X,\beta)$,

$$E^{\bullet} = R\pi_* f^* T_X^{\vee}$$

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ \mathcal{M}_{C,n} & X \end{tikzcd}$$

Def $H^0(C, f^*T_X)$

Obs $\to H^1(C\ f^*T_X)$

actual dim $= 3g-3+n+d$

Ex! $\overline{M}_{g,n}(X,0) = \overline{M}_{g,n} \times X_d$ $\quad$ vdim $= 3g-3+n+d(1-g)$

$(C,p) \in \overline{M}_{g,n} \times X$ $\quad$ $\mathrm{Obs}_{(C,p)} = H^1(C, \mathcal{O}_C \otimes T_p) = \mathbb{E}_C^{\vee} \otimes T_p$

$$[M_{g,n}(X,0)]^{vir} = C_{dg}(\mathbb{E}^{\vee} \otimes T_X) \cap [\overline{M}_{g,n} \times X]$$

(3) Obs is never a bundle in most interesting cases

Then we need full machinery

$$\begin{array}{ccc} E^{-1} & \to & E^{0} \\ \downarrow \phi^{1} & & \downarrow \phi^{0} \\ L^{-1} & \to & L^{0} \end{array}$$

$L^{\bullet}$ cotangent complex $\overline{M}_{g,n}(X,\beta)$

virtual class obtained from a cone in $E_1 = (E^{-1})^{\vee}$ determined by $E^{\bullet} \to L^{\bullet}$

(4) Symplectic approach to the subject :

(X, ω, J) symplectic manifold

Consider $\overline{M}_{g,n}(X,\beta)$ moduli of stable pseudo-holomorphic maps

Perturb ω, J, equations $\Rightarrow$ expected dim moduli space

Tian, Ruan, Seibert

Gromov-Witten theory concerns integration against $[\overline{M}_{g,n}(X,\beta)]^{vir}$

Integrand?

(1) Cotangent line classes $\psi_i = c_1(L_i)$ as before

$$\begin{tikzcd} L_i \arrow[d] & T_{p_i}^*C \arrow[l] \arrow[d] \\ & \end{tikzcd}$$

$\overline{M}_{g,n}(X,\beta) \ni \Big[$ 

$\to X \Big]$

(2) evaluation classes

$$\overline{M}_{g,n}(X,\beta) \xrightarrow{ev_i} X$$

$$ev_i\Big(\big[f:(C,p_1,\dots,p_n)\to X\big]\Big) = f(p_i)$$

$\forall \gamma \in H^*(X,\mathbb{Q})$

$ev_i^*(\gamma) \in H^*(\overline{M}_{g,n}(X,\beta),\mathbb{Q})$

Gromov-Witten invariant

$$\left\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \right\rangle_{g,\beta}^{X}$$

$$= \int_{[\overline{M}_{g,n}(X,\beta)]^{vir}} \prod_{i=1}^{n} ev_i^*(\gamma_i) \cup \psi_i^{a_i}$$

$$= \left(\prod_{i=1}^{n} ev_i^*(\gamma_i) \cup \psi_i^{a_i} \right) \cap [\overline{M}_{g,n}(X,\beta)]^{vir}$$

$$= \# \in \mathbb{Q}.$$

We discussed the GW theory of a point previously.

The next case is $\mathbb{P}^1$. (Okounkov - P)

$$\left\langle \tau_{\ell_1}(\omega) \cdots \tau_{\ell_n}(\omega) \right\rangle_d^{\mathbb{P}^1}$$

$$\shortparallel$$

disconnected

$\omega = [\text{point}] \in H^2(\mathbb{P}^1, \mathbb{Z})$

$$\sum_g \left\langle \tau_{\ell_1}(\omega) \cdots \tau_{\ell_n}(\omega) \right\rangle_{g,d}^{\bullet\, \mathbb{P}^1} z^{2g}$$

Solution is related to

$\Lambda^{\frac{\infty}{2}} V$ infinite wedge representation of $Gl(\infty)$

Let $\mathbb{Z}+\frac{1}{2}$ be $\{ \dots \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \dots \}$

Let $V = \bigoplus_{k \in \mathbb{Z}+\frac{1}{2}} \mathbb{C}\, \underline{k}$

∞-dim vector space with basis $\{ \underline{k} \}_{k \in \mathbb{Z}+\frac{1}{2}}$

A set $S \subset \mathbb{Z}+\frac{1}{2}$ is <u>admissible</u>

if (i) S has finitely many positive elements

(ii) S omits finitely many negative elements

$$S = \{ s_1 > s_2 > s_3 > \cdots \} \subset \mathbb{Z}+\tfrac{1}{2}$$

$$v_S = \underline{s_1} \wedge \underline{s_2} \wedge \underline{s_3} \wedge \cdots$$

$$\Lambda^{\frac{\infty}{2}} V = \bigoplus_{S \text{ admissible}} v_S$$ half inf wedge

orthonormal basis

Creation operators $k \in \mathbb{Z} + 1/2$

$$\psi_k : \Lambda^{\infty/2} V \to \Lambda^{\infty/2} V$$

$$\psi_k (v_S) = \underline{k} \wedge \underline{s_1} \wedge \underline{s_2} \wedge \ldots$$

Annihilation ψ_k^* ← adjoint

Canonical anti commutation relations

$$\psi_i \psi_j^* + \psi_j^* \psi_i = \delta_{ij}$$

$$E_{ij} = :\psi_i \psi_j^*: \begin{cases} \psi_i \psi_j^* & j > 0 \\ -\psi_j^* \psi_i & j < 0 \end{cases}$$

E_{ij} define (projective) rep $gl(\infty)$

$$v_\phi = \underline{-\tfrac{1}{2}} \wedge \underline{-\tfrac{3}{2}} \wedge \underline{-\tfrac{5}{2}} \wedge \cdots$$

$$v = \underbrace{\underline{\tfrac{5}{2}} \wedge \underline{\tfrac{1}{2}}}_{\text{particles}} \wedge \underline{-\tfrac{3}{2}} \wedge \cdots$$

hole at $-\frac{1}{2}$

Creation : adds particle or removes holes

$$\psi_{\frac{3}{2}}\, v = -\left(\underline{\tfrac{5}{2}} \wedge \underline{\tfrac{3}{2}} \wedge \underline{\tfrac{1}{2}} \wedge \underline{-\tfrac{3}{2}} \wedge \cdots \right)$$

$$\psi^*_{\frac{5}{2}}\, v = \underline{\tfrac{1}{2}} \wedge \underline{-\tfrac{3}{2}} \wedge \cdots$$

Annihilator : remove particl or adds holes.

$$E_{ij} = \;:\psi_i \psi_j^*:$$

basis of $\mathfrak{gl}(V)$

define :

$$\mathcal{E}(z) = \sum_{k \in \mathbb{Z}+1/2} e^{zk} E_{kk} + \frac{1}{e^{z/2} - e^{-z/2}}$$

for integral l,

$$T_l = \text{Coeff } [z^l] \text{ in } \mathcal{E}(z)$$

define :

$$\alpha_1 = \sum_{k \in \mathbb{Z}+1/2} E_{k-1,k}$$

$$\alpha_{-1} = \sum_{k \in \mathbb{Z}+1/2} E_{k+1,k}$$

$$T_l \, , \; \alpha_{\pm 1} \; : \; \Lambda^{\infty/2} V \to \Lambda^{\infty/2} V$$

Evaluation of GW theory of $\mathbb{P}^1$

$$\left\langle \tau_{l_1}(\omega) \cdots \tau_{l_n}(\omega) \right\rangle_{g,d}^{\mathbb{P}^1} =$$

(Okounkov-P)

$$\left\langle v_\emptyset \,\middle|\, \frac{\alpha_{+1}^d}{d!} \, T_{l_1+1} \cdots T_{l_n+1} \, \frac{\alpha_{-1}^d}{d!} \,\middle|\, v_\emptyset \right\rangle$$

GW theory point solved by

Kontsevich's Comb model

$\Rightarrow$ KdV

GW theory $\mathbb{P}^1$ solved by vacuum

expectation in $\Lambda^{\frac{\infty}{2}} V$

$\Rightarrow$ Toda

Conjecture Eguchi Hori Yang

GW(point) $\to$ KdV

GW($\mathbb{P}^1$) $\to$ Toda

GW(X) $\to$? Integrable systems.

$\Rightarrow$ line of thought leads to very interesting directions in the subject

- Dubrovin's theory of Frobenius Manifold

- E H X - K (Eguchi, Hori, Xiong, S. Katz) Virasoro Constraints

One of the basic mysteries of the subject

GW theory of surfaces S

Two basic cases

(1) $p_g(S) = 0$: $\mathbb{P}^2$, toric surfaces

Primary GW invariants

$$\left\langle \tau_0(\text{point}) \cdots \tau_0(\text{point}) \right\rangle_{g,\beta}^{S}$$

often $\parallel$ enumerative geometry.

Structure?

Göttsche conjecture :

$\beta >> 0$ ... express answer in terms of topology β^2, $c_1 \cdot \beta$, c_1^2, c_2

Bryan-Leung, Ai-Ko Liu

Case (2) S general type

Let S be a min surface of gen type.

Let $C \subset S$ be a nonsingular canonical divisor. (Assumption for simplicity).

By adjunction $\mathcal{O}_S(C+C)/_C = K_C$

$$2C/_C = K_C$$

$\Rightarrow$ Normal bundle of C is a theta characteristic

The theta char $\mathcal{C}|_C$ is

either <u>odd</u> or <u>even</u>

$h^0(C, \mathcal{C}|_C)$ mod 2.

def invariant of S.

How can we tell what the

parity of the $\mathcal{C}|_C$ is?

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_C(C) \to 0$$

$$0 \to H^0(S, \mathcal{O}_S) \to H^0(S, K) \to H^0(C, \mathcal{C}|_C)$$

$$\to H^1(S, \mathcal{O}_S) \xrightarrow{\phi} H^1(S, K)$$

rewrite dim

$$0 \to 1 \to p_g \to x$$

$$\to \operatorname{Ker} \phi \to 0$$

$$\chi = 1 + p_g + \dim \operatorname{Ker} \phi$$

$$\phi : H^1(S, \mathcal{O}_S) \to H^1(S, k)$$

$\operatorname{Im} \phi$ has <u>even</u> rank

(exercise, because of skew-sym form)

Conclusion

$$\chi = 1 + p_g + q \mod 2.$$

$$p_g = h^0(S, K_S)$$

$$q = h^1(S, \mathcal{O}_S)$$

S surface

Exp dim of gen g curv in clas β?

$$\text{Vir dim} = +\int_\beta c_1(S) + g - 1$$

Most basic GW invariant of S:

class C (canonical class)

genus g_C (adj genus, smooth genus)

$$g_C = \frac{1}{2}(2C\cdot C + 2)$$

$$= C^2 + 1$$

$$\text{Vir dim} = -C^2 + C^2 + 1 - 1 = 0$$

$$\langle 1 \rangle^S_{g_C, C} \qquad C = \text{canonical class.}$$

S min surface of gen type

Taubes : $G = SW$

$$\langle 1 \rangle^{S}_{g_c, C} = SW(S) = (-1)^{1+p_g+q}$$

C = canonical

What about the rest of the GW theory of S?

Conj framework (with D. Maulik, conversations with T. Parker)

- All GW invariants of S vanish unless $\beta = rC$ (multiple of canonical)
- GW invariants in class rC depend only upon r, g_C ← Taubes genus of S

$$\left\langle \tau_{d_1}(c)\cdots\tau_{d_n}(c)\right\rangle^{S}_{rC} = f_{r,g_C}(d_1,\dots d_n)$$

What are these universal functions?

Conj known $r=1$

$r=2$

$\vdots$ work in progress

(ask Davesh Maulik)

$$f_{2,g_C}(d_1,\dots,d_n) =$$

$$(-1)^{1+p_g+L}\,(g-1)^n\,2^g\cdot 2^{n-1}\prod_{i=1}^{n}\frac{d_i!}{(2d_i+1)!}(-2)^{d_i}$$

Étale case leads to a new Hurwitz problem.

f étale $\Rightarrow$ $f^*\Theta$ theta char on D

Problem: Count étale covers of C with sign determined by parity of $f^*\Theta$.