# Gromov-Witten/Donaldson-Thomas correspondence for toric 3-folds

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September 2008

## Abstract

We prove the equivariant Gromov-Witten theory of a nonsingular toric 3-fold $X$ with primary insertions is equivalent to the equivariant Donaldson-Thomas theory of $X$. As a corollary, the topological vertex calculations by Agangic, Klemm, Mariño, and Vafa of the Gromov-Witten theory of local Calabi-Yau toric 3-folds are proven to be correct in the full 3-leg setting.

# Contents

# 1 The GW/DT correspondence

## 1.1 Gromov-Witten theory

Let $X$ be a nonsingular projective 3-fold. Let $\overline{M}'_{g,r}(X,\beta)$ denote the moduli space of $r$-pointed stable maps

$$f : C \to X$$

from possibly disconnected genus $g$ curves to $X$ representing the class

$$\beta \in H_2(X,\mathbb{Z})$$

and *not* collapsing any connected components.

Denote the evaluation map corresponding to the $i^{th}$ marked point by

$$\mathrm{ev}_i : \overline{M}'_{g,r}(X,\beta) \to X.$$

Given classes $\gamma_i \in H^*(X,\mathbb{Z})$, the corresponding *primary* Gromov-Witten invariants are defined by

$$\langle \gamma_1, \ldots, \gamma_r \rangle'_{g,\beta} = \int_{\left[\overline{M}'_{g,r}(X,\beta)\right]^{\mathrm{vir}}} \prod_{i=1}^r \mathrm{ev}_i^*(\gamma_i) \,, \tag{1}$$

the virtual counts of genus $g$ degree $\beta$ curves in $X$ meeting cycles Poincaré dual to $\gamma_i$. The integration in (1) is against the *virtual fundamental class* of dimension

$$\dim \left[\overline{M}'_{g,r}(X,\beta)\right]^{\mathrm{vir}} = \int_\beta c_1(T_X) + r \,.$$

We will often call the primary fields $\gamma_i$ *insertions*. Foundational aspects of the theory are treated, for example, in [2, 3, 18].

We assemble the primary invariants into the generating function

$$\mathsf{Z}'_{GW}\left(X, u \mid \gamma_1, \dots, \gamma_r\right)_\beta = \sum_{g \in \mathbb{Z}} \langle \gamma_1, \dots, \gamma_r \rangle'_{g,\beta} \, u^{2g-2}. \tag{2}$$

Since the domain components must map nontrivially, an elementary argument shows the genus $g$ in the sum (2) is bounded from below. Following the terminology of [25], we view (2) as a *reduced* partition function.

For brevity, we will often omit the arguments of $\mathsf{Z}'_{GW}$ which are clear from context. The genus subscript, however, will follow a different convention. When the genus is not present, a summation over all genera is understood

$$\langle \gamma_1, \dots, \gamma_r \rangle'_\beta = \mathsf{Z}'_{GW}\left(X, u \mid \gamma_1, \dots, \gamma_r\right)_\beta .$$

## 1.2 Donaldson-Thomas theory

Donaldson-Thomas theory is defined via integration over the moduli space of ideal sheaves [5, 35] of $X$. An *ideal sheaf* is a torsion-free sheaf of rank 1 with trivial determinant. For an ideal sheaf $\mathcal{I}$, the canonical map

$$0 \to \mathcal{I} \to \mathcal{I}^{\vee\vee}$$

is an injection. As $\mathcal{I}^{\vee\vee}$ is reflexive of rank 1 with trivial determinant,

$$\mathcal{I}^{\vee\vee} \cong \mathcal{O}_X,$$

see [27]. Each ideal sheaf $\mathcal{I}$ determines a subscheme $Y \subset X$,

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0.$$

By the triviality of the determinant, the components of the subscheme $Y$ have dimension at most 1. The 1-dimensional components of $Y$ (weighted by their intrinsic multiplicities) determine an element,

$$[Y] \in H_2(X, \mathbb{Z}).$$

Let $I_n(X, \beta)$ denote the moduli space of ideal sheaves $\mathcal{I}$ satisfying

$$\chi(\mathcal{O}_Y) = n \quad \text{and} \quad [Y] = \beta \in H_2(X, \mathbb{Z}).$$

3

Here, $\chi$ denotes the holomorphic Euler characteristic.

The Donaldson-Thomas invariant is defined via integration against virtual class $[I_n(X, \beta)]^{\mathrm{vir}}$ of dimension

$$\dim[I_n(X, \beta)]^{\mathrm{vir}} = \int_\beta c_1(T_X).$$

Foundational aspects of the theory are treated in [25, 35].

The moduli space $I_n(X, \beta)$ is canonically isomorphic to the Hilbert scheme [25]. As the Hilbert scheme is a fine moduli space, we have the *universal ideal sheaf*

$$\mathfrak{I} \to I_n(X, \beta) \times X \,. \tag{3}$$

Let $\pi_1$ and $\pi_2$ denote the projections to the respective factors of $I_n(X, \beta) \times X$. Since $\mathfrak{I}$ is $\pi_1$-flat and $X$ is nonsingular, a finite resolution of $\mathfrak{I}$ by locally free sheaves on $I_n(X, \beta) \times X$ exists. Hence, the Chern classes of $\mathfrak{I}$ are well-defined. The second Chern class $c_2(\mathfrak{I})$ may be interpreted as the class of the universal subscheme.

For each class $\gamma \in H^*(X, \mathbb{Z})$, let $c_2(\gamma)$ denote the operator[1] on the homology $H_*(I_n(X, \beta), \mathbb{Z})$ defined by

$$c_2(\gamma)\big(\xi\big) = \pi_{1*}\big(c_2(\mathfrak{I}) \cdot \pi_2^*(\gamma) \cap \pi_1^*(\xi)\big) \,. \tag{4}$$

Since $\pi_1$ is flat, the homological pull-back $\pi_1^*$ is well-defined [6].

The primary Donaldson-Thomas invariants are defined by

$$\langle \gamma_1, \ldots, \gamma_r \rangle_{n,\beta} = \int_{[I_n(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^r c_2(\gamma_i) \,, \tag{5}$$

where the latter integral is the push-forward to a point of the class

$$c_2(\gamma_1) \, \circ \, \cdots \, \circ \, c_2(\gamma_r) \left( [I_n(X, \beta)]^{\mathrm{vir}} \right) .$$

A similar slant product construction can be found in the Donaldson theory of 4-manifolds.

As in Gromov-Witten theory, we assemble primary invariants into a generating function,

$$\mathsf{Z}_{DT}\left(X, q \mid \gamma_1, \ldots, \gamma_r\right)_\beta = \sum_{n \in \mathbb{Z}} \langle \gamma_1, \ldots, \gamma_r \rangle_{n,\beta} \, q^n. \tag{6}$$

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[1]Since $c_1(\mathfrak{I})$ vanishes, $c_2(\mathfrak{I}) = -\mathrm{ch}_2(\mathfrak{I})$. Thus, the definition of primary field insertions here agrees with the definition of [26].

An elementary argument shows $n$ in (6) is bounded from below.

The reduced partition function is obtained by formally removing the degree 0 contributions,

$$\mathsf{Z}'_{DT}\left(X, q \mid \gamma_1, \ldots, \gamma_r\right)_\beta = \frac{\mathsf{Z}_{DT}\left(X, q \mid \gamma_1, \ldots, \gamma_r\right)_\beta}{\mathsf{Z}_{DT}(X, q)_0} \, .$$

The evaluation of the degree 0 partition function was conjectured in [25, 26] and proven in [13, 16],

$$\mathsf{Z}_{DT}(X, q)_0 = M(-q)^{\int_X c_3(T_X \otimes K_X)} \, ,$$

where

$$M(q) = \prod_{n \geq 1} \frac{1}{(1 - q^n)^n} \tag{7}$$

is the McMahon function.

## 1.3 Primary GW/DT correspondence

The primary correspondence consists of two claims [25, 26]:

(i) The series $\mathsf{Z}'_{DT}$ is a rational function of $q$.

(ii) After the change of variables $e^{iu} = -q$,

$$(-iu)^{\boldsymbol{\delta}} \, \mathsf{Z}'_{GW}\left(X, u \mid \gamma_1, \ldots, \gamma_r\right)_\beta = (-q)^{-\boldsymbol{\delta}/2} \, \mathsf{Z}'_{DT}\left(X, q \mid \gamma_1, \ldots, \gamma_r\right)_\beta \, ,$$

where $\boldsymbol{\delta} = \int_\beta c_1(T_X)$ is the virtual dimension.

The main result of the paper is the proof of the primary correspondence for toric $X$. Let $T$ be the 3-dimensional torus acting on $X$.

**Theorem 1.** *The primary GW/DT correspondence holds for all nonsingular toric 3-folds $X$ in $T$-equivariant cohomology.*

In the toric case, both $Z'_{GW}$ and $Z'_{DT}$ take values in the $T$-equivariant cohomology ring of a point

$$H^*_T(\bullet, \mathbb{Q}) = \mathrm{Sym}(\mathfrak{t}^*) \, , \quad \mathfrak{t} = \mathrm{Lie}\, T \, .$$

The total codimension of the insertions in the $T$-equivariant theory is allowed to exceed the virtual dimension. Since both theories are well-defined by

residues in the noncompact case, we do not require $X$ to be projective for Theorem 1.

If $X$ is projective, the usual Gromov-Witten and Donaldson-Thomas invariants are obtained in the non-equivariant limit. Hence, Theorem 1 implies the usual GW/DT correspondence in the nonsingular projective toric case.

## 1.4 Relative GW/DT correspondence

In addition to the primary correspondence, the correspondence of *relative* theories will play an important role.

Let $D \subset X$ be a smooth divisor. Relative Gromov-Witten and Donaldson-Thomas theories enumerate curves with specified tangency to the divisor $D$. See [26] for a technical discussion of relative theories.

In Gromov-Witten theory, relative conditions are represented by a partition $\mu$ of the number

$$d = \beta \cdot [D],$$

each part $\mu_i$ of which is marked by a cohomology class $\gamma_i \in H^*(D, \mathbb{Z})$. The numbers $\mu_i$ record the multiplicities of intersection with $D$ while the cohomology labels $\gamma_i$ record where the tangency occurs. More precisely, we integrate the pull-backs of $\gamma_i$ via the evaluation maps

$$\overline{M}'_{g,r}(X/D, \beta) \to D$$

at the points of tangency. By convention, an absent cohomology label stands for $1 \in H^*(D, \mathbb{Z})$.

In Donaldson-Thomas theory, the relative moduli space admits a natural morphism to the Hilbert scheme of $d$ points in $D$. Cohomology classes on $\mathrm{Hilb}(D, d)$ may thus be pulled back to the relative moduli space. We will work in the *Nakajima basis* of $H^*(\mathrm{Hilb}(D, d), \mathbb{Q})$ indexed by a partition $\mu$ of $d$ labeled by cohomology classes of $D$. For example, the class

$$\big|\mu\big\rangle \in H^*(\mathrm{Hilb}(D, d), \mathbb{Q}),$$

with all cohomology labels equal to the identity, is $\prod \mu_i^{-1}$ times the Poincaré dual of the closure of the subvariety formed by unions of schemes of length

$$\mu_1, \ldots, \mu_{\ell(\mu)}$$

supported at $\ell(\mu)$ distinct points of $D$.

The *Fock space* associated to $D$ is the sum

$$\mathcal{F} = \bigoplus_{d \geq 0} H^*(\mathrm{Hilb}(D, d), \mathbb{Q}).$$

The Fock space is equipped with the Nakajima basis, a natural inner product from the Poincaré pairing on $H^*(\mathrm{Hilb}(D, d), \mathbb{Q})$, and standard[2] operators $\alpha_{\pm r}$ for raising and lowering.

The conjectural relative GW/DT correspondence [26] equates the generating functions

$$(-iu)^{\boldsymbol{\delta} + \ell(\mu) - |\mu|} \, \mathsf{Z}'_{GW} \left( X/D, u \mid \gamma_1, \ldots, \gamma_r \mid \mu \right)_\beta \overset{?}{=}$$
$$(-q)^{-\boldsymbol{\delta}/2} \, \mathsf{Z}'_{DT} \left( X/D, q \mid \gamma_1, \ldots, \gamma_r \mid \mu \right)_\beta , \quad (8)$$

after the change of variables $e^{iu} = -q$. Here, $\boldsymbol{\delta} = \int_\beta c_1(T_X)$ is the virtual dimension, and $\mu$ is a cohomology weighted partition with $\ell(\mu)$ parts. As before, (8) is conjectured to be a rational function of $q$.

## 1.5 Degeneration formulas

Relative theories satisfy degeneration formulas. Let

$$\mathfrak{X} \to B$$

be a nonsingular 4-fold fibered over an irreducible and nonsingular base curve $B$. Let $X$ be a nonsingular fiber and

$$X_1 \cup_D X_2$$

be a reducible special fiber consisting of two nonsingular 3-folds intersecting transversally along a nonsingular surface $D$.

If all insertions $\gamma_1, \ldots, \gamma_r$ lie in the image of

$$H^*(X_1 \cup_D X_2, \mathbb{Z}) \to H^*(X, \mathbb{Z}),$$

the degeneration formula in Gromov-Witten theory takes the form [11, 14, 15]

$$\mathsf{Z}'_{GW} \left( X \mid \gamma_1, \ldots, \gamma_r \right)_\beta =$$
$$\sum \mathsf{Z}'_{GW} \left( X_1 \mid \ldots \mid \mu \right)_{\beta_1} \mathfrak{z}(\mu) \, u^{2\ell(\mu)} \, \mathsf{Z}'_{GW} \left( X_2 \mid \ldots \mid \mu^\vee \right)_{\beta_2} , \quad (9)$$

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[2]We will follow the Fock space notation of [30].

where the summation is over all curve splittings $\beta = \beta_1 + \beta_2$, all splitting of the insertions $\gamma_i$, and all relative conditions $\mu$.

In (9), the cohomological labels of $\mu^\vee$ are Poincaré duals of the labels of $\mu$. The gluing factor $\mathfrak{z}(\mu)$ is the order of the centralizer of in the symmetric group $S(|\mu|)$ of an element with cycle type $\mu$.

The degeneration formula in Donaldson-Thomas theory takes a very similar form,

$$\mathsf{Z}'_{DT}\left(X\,|\,\gamma_1, \ldots, \gamma_r\,\right)_\beta =$$
$$\sum \mathsf{Z}'_{DT}\left(X_1|\,\ldots\,\big|\mu\right)_{\beta_1} (-1)^{|\mu|-\ell(\mu)}\,\mathfrak{z}(\mu)\,q^{-|\mu|}\,\mathsf{Z}'_{DT}\left(X_2|\,\ldots\,\big|\mu^\vee\right)_{\beta_2},$$

see [26]. The sum over the relative conditions $\mu$ is interpreted as the coproduct of 1,

$$\Delta 1 = \sum_\mu (-1)^{|\mu|-\ell(\mu)}\,\mathfrak{z}(\mu)\,\big|\mu\big\rangle \otimes \big|\mu^\vee\big\rangle\ ,$$

in the tensor square of $H^*(\mathrm{Hilb}(D, \beta \cdot [D]), \mathbb{Z})$. Conjecture (8) is easily seen to be compatible with degeneration.

## 1.6 $\mathcal{A}_n$ geometries

Let $\zeta$ be a primitive $(n+1)^{th}$ root of unity, for $n \geq 0$. Let the generator of the cyclic group $\mathbb{Z}_{n+1}$ act on $\mathbb{C}^2$ by

$$(z_1, z_2) \mapsto (\zeta\,z_1, \zeta^{-1}z_2)\,.$$
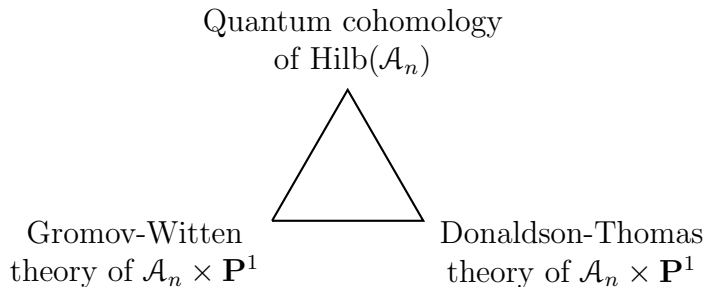
Let $\mathcal{A}_n$ be the minimal resolution of the quotient

$$\mathcal{A}_n \to \mathbb{C}^2/\mathbb{Z}_{n+1}.$$

The diagonal $(\mathbb{C}^*)^2$-action on $\mathbb{C}^2$ commutes with the action of $\mathbb{Z}_n$. As a result, the surfaces $\mathcal{A}_n$ are toric.

The starting point of our paper is the $(\mathbb{C}^*)^2$-equivariant GW/DT correspondence for the 3-folds $\mathcal{A}_n \times \mathbf{P}^1$. In fact, there exists a conjectural triangle of equivalences. The apex in that triangle is given by the $(\mathbb{C}^*)^2$-equivariant quantum cohomology of the Hilbert scheme of points of $\mathcal{A}_n$.

The triangle is established for $\mathcal{A}_0 = \mathbb{C}^2$ in the series of papers [4, 29, 30]. For the $\mathcal{A}_{n \geq 1}$ geometries, the triangle is proven[3] for a special class of operators

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[3]In fact, assuming a conjectural nondegeneracy statement for the divisorial operators, the full equivalence of the triangle follows, see [23].

8

$$\begin{array}{c} \text{Quantum cohomology} \\ \text{of Hilb}(\mathcal{A}_n) \end{array}$$

Gromov-Witten
theory of $\mathcal{A}_n \times \mathbf{P}^1$

Donaldson-Thomas
theory of $\mathcal{A}_n \times \mathbf{P}^1$

associated to divisors of $\mathcal{A}_n$ in [22, 23, 24]. The equivalence for divisorial operators is sufficient for our purposes. The Hilbert scheme vertex plays a crucial role in the establishment of the triangle.

## 1.7 Plan of the paper

We prove the GW/DT correspondence for toric 3-folds $X$ by the following method. We decompose the invariants on both sides in terms of vertex and edge contributions of the polytope associated to $X$. Such a decomposition is obtained from the localization formulas for the respective virtual classes [10]. A crucial step is to reorganize the localization data into *capped* vertex and edge contributions discussed in Section 2. The capped contributions are proven to be much better behaved — the capped contributions themselves satisfy the GW/DT correspondence. The Gromov-Witten and Donaldson-Thomas capped edges are matched in Section 3.3 by strengthening the local curves correspondence of [4, 30]. The capped vertices are matched in Sections 3.4-3.6 by exploiting the $\mathcal{A}_n \times \mathbf{P}^1$ geometries for $n \leq 2$ and using the established properties of the triangle of equivalences for divisorial operators.

Several results, conjectures, and speculations about the capped vertex are discussed in Section 4. The connection of our work to the topological vertex of [1] is explained in Section 4.1. Theorem 1 is recast symmetrically in terms of classes on the Chow variety of $X$ in Section 4.2. Finally, examples of capped vertex evaluations are given in Section 4.3.

## 1.8 Acknowledgments

# 2 Localization

## 2.1 Toric geometry

Let $X$ be a nonsingular toric 3-fold. For both Gromov-Witten and Donaldson-Thomas theory, virtual localization with respect to the action of the full 3-dimensional torus $T$ reduces all invariants of $X$ to local contributions of the vertices and edges of the associated toric polytope. However, the standard constituent pieces of the localization formula are poorly behaved with respect to the GW/DT correspondence. Instead, we will introduce a modified localization procedure with capped versions of the usual vertex and edge contributions. The capped vertex and edge terms are still building blocks for global toric calculations, but have the advantage of satisfying the GW/DT correspondence for residues.

Let $\Delta$ denote the polytope associated to $X$. The vertices of $\Delta$ are in bijection with $T$-fixed points $X^T$. The edges $e$ correspond to $T$-invariant curves

$$C_e \subset X.$$

The three edges incident to any vertex carry canonical $T$-weights — the tangent weights of the torus action.

We will consider both compact and noncompact toric varieties $X$. In the latter case, edges maybe compact or noncompact. Every compact edge is incident to two vertices.

## 2.2 Standard localization

### 2.2.1 Structure

The large-scale structure of the $T$-equivariant localization formulas in Gromov-Witten and Donaldson-Thomas theories [10] is nearly identical. The outermost sum in the localization formula runs over all assignments of partitions

$\lambda(e)$ to the compact edges $e$ of $\Delta$ satisfying

$$\beta = \sum_e |\lambda(e)| \cdot [C_e] \in H_2(X, \mathbb{Z}).$$

Such a partition assignment will be called a *marking* of $\Delta$. The weight of each marking in the localization sum for the invariants (1) and (5) equals the product of three factors:

(i) localization of the integrand,

(ii) vertex contributions,

(iii) compact edge contributions.

The complexity of the localization formula is concentrated in the vertex contributions (ii). The principal ingredients of the GW vertex are *triple Hodge integrals*.[4] The DT vertex is a weighted sum over all 3-dimensional partitions $\pi$ with fixed asymptotic cross-sections. The weight of $\pi$ is a rational function of the equivariant parameters, see [25, 26]. For the computation of the reduced invariants, the *reduced vertex*, in which the appropriate power of the McMahon function has been taken out, is more convenient.

We will avoid the fine structure of the GW and DT vertices. A precise match will be proven between *capped* vertices in the two theories. The capped vertices are $T$-equivariant residue invariants of open 1-vertex geometries defined in Section 2.3.

### 2.2.2 Gromov-Witten edges

Let $e$ be an compact edge of $\Delta$ as in Figure 1. The edge labels in Figure 1 are the weights of the torus action.

Let $e$ be marked by a partition $\lambda$. In Gromov-Witten theory, such an edge corresponds to a $T$-fixed stable map

$$f : C \to C_e \subset X,$$

where the source curve $C$ has a rational component $C_i$ for each part of $\lambda$. Restricted to $C_i$, the map $f$ has the form

$$\mathbf{P}^1 \ni z \mapsto z^{\lambda_i} \in \mathbf{P}^1.$$

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[4]An introduction to Hodge integrals in Gromov-Witten theory can be found in [7]. See [1] for the triple Hodge integrals which occur in the GW vertex in the Calabi-Yau case.
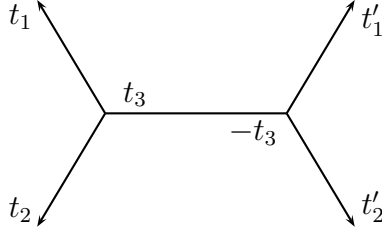
Figure 1: An edge in the toric polytope of $X$

The corresponding edge weight

$$E_{GW}(\lambda, t_1, t_2, t_3, t_1', t_2', u)$$

equals $u^{-\chi(C)}$ times the stack factor

$$\mathfrak{z}(\lambda)^{-1} = |\operatorname{Aut} f|^{-1}$$

times the product of nonzero $T$-weights on

$$H^1(f^*(TX)) \ominus H^0(f^*(TX)) \tag{10}$$

corrected by the $T$-weights of the tangent fields of $C$ vanishing at the end points. The character of the virtual $T$-module (10) may be easily calculated for each component $C_i$ to be

$$\frac{e^{t_1} + e^{t_2} + e^{t_3}}{e^{-t_3/\lambda_i} - 1} + \frac{e^{t_1'} + e^{t_2'} + e^{-t_3}}{e^{t_3/\lambda_i} - 1} + 1 \,. \tag{11}$$

Here we denote by $e^t \in \mathbb{C}[T]$ the irreducible character corresponding to a weight $t \in \mathfrak{t}^*$. The trivial character 1, which is taken out in (11), is due to the tangent fields of $C$.

### 2.2.3 Donaldson-Thomas edges

In Donaldson-Thomas theory, a marked edge corresponds to a $T$-fixed ideal sheaf $\mathfrak{I}_\lambda$ which restricts to a monomial ideal of the form

$$I_\lambda = (x^i y^j)_{i \geq \lambda_{j+1}} \subset \mathbb{C}[x, y, z] \tag{12}$$

in the neighborhood of each of the two $T$-fixed point. Here $z$ is the coordinate along the edge.

Because of the asymmetry between $x$ and $y$ in (12), the partition $\lambda$ should properly be assigned to an oriented edge, with the change in orientation resulting in the transposition of the partition. One of the many advantages of *capped localization*, which will be introduced below, is absence of orientation issues.

By definition, the edge weight

$$E_{DT}(\lambda, t_1, t_2, t_3, t'_1, t'_2, q)$$

in Donaldson-Thomas theory is $q^{\chi(\mathcal{O}/\mathcal{I}_\lambda)}$ times the product of $T$-weights on

$$\text{Ext}^2(\mathcal{I}, \mathcal{I}) \ominus \text{Ext}^1(\mathcal{I}, \mathcal{I}). \tag{13}$$

Again, the character of the virtual $T$-module (13) is easily determined, see [25, 26].

## 2.3 Capped edges and vertices

### 2.3.1 Edges

Let $e$ be a compact edge of the toric polytope corresponding to $X$. Let

$$X_e \subset X$$

denote the non-compact toric variety associated to $e$ determined by Figure 1. Let

$$F_0, F_\infty \subset X_e$$

be the $T$-invariant divisors lying over the fixed points of the base $\mathbf{P}^1 \subset X_e$.

By definition, the Gromov-Witten capped edge $e$,

$$\mathsf{E}_{GW}(\lambda, \mu, t_1, t_2, t_3, t'_1, t'_2, u) = \mathsf{Z}'_{GW}(X_e/F_0 \cup F_\infty, u \mid \lambda, \mu), \tag{14}$$

is the reduced $T$-equivariant partition function of $X_e$ with free[5] relative conditions $\lambda$ and $\mu$ imposed along $F_0$ and $F_\infty$ of degree

$$d = |\lambda| = |\mu|.$$

The $T$-fixed loci of the moduli space $\overline{M}'_g(X_e/F_0 \cup F_\infty, d)$ are compact. Hence, the partition function (14) is well-defined by $T$-equivariant residues, see [4].

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[5]By free a relative condition, we mean a partition with all cohomology weights 1. Hence, just a partition.

The definition of a capped edge in Donaldson-Thomas theory is identical,

$$\mathsf{E}_{DT}(\lambda, \mu, t_1, t_2, t_3, t_1', t_2', u) = \mathsf{Z}'_{DT}(X_e/F_0 \cup F_\infty, q \mid \lambda, \mu), \qquad (15)$$

again defined by $T$-equivariant residues.

The capped edge has normal bundle of type $(a, b)$ where

$$(a, b) = \left( \tfrac{t_1 - t_1'}{t_3}, \tfrac{t_2 - t_2'}{t_3} \right) \in \mathbb{Z}^2 \,.$$

We will call the geometry here an $(a, b)$-edge. The toric variety $X_e$ is isomorphic to the total space of the bundle

$$\mathcal{O}_{\mathbf{P}^1}(a) \oplus \mathcal{O}_{\mathbf{P}^1}(b) \to \mathbf{P}^1. \qquad (16)$$

### 2.3.2 Vertices

Let $U$ be the $T$-invariant 3-fold obtained by removing the three $T$-invariant lines

$$L_1, L_2, L_3 \subset \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$$

passing through the point $(\infty, \infty, \infty)$,

$$U = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \ \setminus \ \cup_{i=1}^3 L_i.$$

Let $D_i \subset U$ be the divisor with $i^{th}$ coordinate $\infty$. For $i \neq j$, the divisors $D_i$ and $D_j$ are disjoint.

In both Gromov-Witten and Donaldson-Thomas theories, the capped vertex is the *reduced* partition function of $U$ with free relative conditions imposed at the divisors $D_i$. While the relative geometry $U/ \cup_i D_i$ is noncompact, the moduli spaces of maps $\overline{M}'_g(U/ \cup_i D_i, \beta)$ and ideal sheaves $I_n(U/ \cup_i D_i, \beta)$ have compact $T$-fixed loci. The invariants of $U/ \cup_i D_i$ in both theories are well-defined by $T$-equivariant residues. In the localization formula for the reduced theories of $U/ \cup_i D_i$, nonzero degrees can occur *only* on the edges meeting the origin $(0, 0, 0) \in Y$.

We denote the capped GW vertex by

$$\mathsf{C}_{GW}(\lambda, \mu, \nu, t_1, t_2, t_3, u) = \mathsf{Z}'_{GW}(U/ \cup_i D_i, u \mid \lambda, \mu, \nu) \qquad (17)$$

where $\lambda, \mu, \nu$ denote relative conditions imposed at $D_1, D_2, D_3$ and $t_1, t_2, t_3$ are the weights of the $T$-action on the coordinate axes.

14

The definition of the capped DT vertex,

$$\mathsf{C}_{DT}(\lambda, \mu, \nu, t_1, t_2, t_3, q) = \mathsf{Z}'_{DT}(U/\cup_i D_i, q \mid \lambda, \mu, \nu),$$

is parallel. The partitions $\lambda, \mu, \nu$ here represent the Nakajima basis elements of

$$H^*(\mathrm{Hilb}(D_i, [\beta] \cdot D_i), \mathbb{Q}), \quad i = 1, \ldots, 3.$$

Unlike the uncapped DT vertex, $\mathsf{C}_{DT}$ enjoys a full $S(3)$ symmetry extending the obvious $S(3)$ action on $X$.

## 2.4 Capped localization

### 2.4.1 Overview

Capped localization expresses the primary Gromov-Witten and Donaldson-Thomas invariants of $X$ as a sum capped vertex and capped edge data.

A half-edge $h = (e, v)$ is a compact edge $e$ together with the choice of an incident vertex $v$. A partition assignment

$$h \mapsto \lambda(h)$$

to half-edges is *balanced* if the equality

$$|\lambda(e, v)| = |\lambda(e, v')|$$

always holds for the two halfs of $e$. For a balanced assignment, let

$$|e| = |\lambda(e, v)| = |\lambda(e, v')|$$

denote the *edge degree*.

The outermost sum in the capped localization formula runs over all balanced assignments of partitions $\lambda(h)$ to the half-edges $h$ of $\Delta$ satisfying

$$\beta = \sum_e |e| \cdot [C_e] \in H_2(X, \mathbb{Z}). \tag{18}$$

Such a partition assignment will be called a *capped marking* of $\Delta$. The weight of each capped marking in the localization sum for the invariants (1) and (5) equals the product of four factors:

(i) localization of the integrand,

(ii) capped vertex contributions,

(iii) capped edge contributions,

(iv) gluing terms.

The integrand terms (i) will be discussed in Section 2.4.2 below. The integrand contributions in Gromov-Witten and Donaldson-Thomas theories exactly match. Each vertex determines up to three half-edges specifying the partitions for the capped vertex. Each compact edge determines two half-edges specifying the partitions of the capped edge. The gluing terms (iv) appear in exactly the same form as in the degeneration formula. Precise formulas are written in Section 2.4.3.

The capped localization formula is easily derived from the standard localization formula. Indeed, the capped objects are obtained from the uncapped objects by rubber integral[6] factors. The rubber integrals cancel in pairs in capped localization to yield standard localization.

The GW/DT correspondence for $X$ is a direct consequence of the capped localization formula and the following results matching the capped edges and vertices.

**Proposition 1.** *Capped edges satisfy the relative GW/DT correspondence. After the substitution $q = -e^{iu}$, we have*

$$(-iu)^{d(a+b)+\sum_i \ell(\lambda^{(i)})} \mathsf{E}_{GW}(\lambda^{(1)}, \lambda^{(2)}, t_1, t_2, t_3) =$$
$$q^{-\frac{d(a+b+2)}{2}} \mathsf{E}_{DT}(\lambda^{(1)}, \lambda^{(2)}, t_1, t_2, t_3)$$

*for an $(a, b)$-edge of degree $d$. Capped edges are rational functions of $q$.*

**Proposition 2.** *Capped vertices satisfy the relative GW/DT correspondence. After the substitution $q = -e^{iu}$, we have*

$$(-iu)^{\sum_i |\lambda^{(i)}|+\ell(\lambda^{(i)})} \mathsf{C}_{GW}(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, t_1, t_2, t_3) =$$
$$q^{-\sum |\lambda^{(i)}|} \mathsf{C}_{DT}(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, t_1, t_2, t_3) \,.$$

*Capped vertices are rational functions of $q$.*

---

[6]Rubber integrals $\langle \lambda \mid \frac{1}{1-\psi_\infty} \mid \mu \rangle^\sim$ arise in the localization formulas for relative geometries. See [4, 30] for detailed discussion of the 1-leg rubber calculus. In particular, differential equations governing the rubber integrals are discussed in Section 11.2 of [30].

Proposition 1 has been proven in [4, 30] for the vertical subtorus

$$(\mathbb{C}^*)^2 \subset T.$$

The proof for the full 3-dimensional torus $T$ is given in Section 3.3. Proposition 2, proven in Sections 3.5-3.6, is really the main result of the paper. Both Propositions 1 and 2 are special cases of the GW/DT correspondence for $T$-equivariant residues conjectured in [4].

Our proof of Proposition 2 may be converted into an algorithm for the computation of the capped vertices. Capped vertex evaluations in the first few cases may be found in Section 4.3.

From the perspective of the GW/DT correspondence, the starred[7] normalizations

$$
\begin{aligned}
\mathsf{E}^*_{GW}(a,b)_{\lambda^{(1)},\lambda^{(2)}} &= (-iu)^{d(a+b)+\sum_i \ell(\lambda^{(i)})} \mathsf{E}_{GW}(\lambda^{(1)}, \lambda^{(2)}, t_1, t_2, t_3) \ , \\
\mathsf{E}^*_{DT}(a,b)_{\lambda^{(1)},\lambda^{(2)}} &= q^{-\frac{d(a+b+2)}{2}} \mathsf{E}_{DT}(\lambda^{(1)}, \lambda^{(2)}, t_1, t_2, t_3)
\end{aligned}
$$

for capped edges and

$$
\begin{aligned}
\mathsf{C}^*_{GW}(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}) &= (-iu)^{\sum_i |\lambda^{(i)}| + \ell(\lambda^{(i)})} \mathsf{C}_{GW}(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, t_1, t_2, t_3) \ , \\
\mathsf{C}^*_{DT}(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}) &= q^{-\sum |\lambda^{(i)}|} \mathsf{C}_{DT}(\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, t_1, t_2, t_3)
\end{aligned}
$$

for capped vertices will be more convenient.

### 2.4.2 Localization of the integrand

We identify here the localization contributions of the primary insertions. Let $v$ be a vertex of $\Delta$. We will denote by the corresponding $T$-fixed point by the same letter. Let $t_1, t_2, t_3$ be the tangent weights at $v$ and let $d_1, d_2, d_3$ be edge degrees determined by the capped marking.

Consider a primary insertion $\gamma \in H_T^*(X, \mathbb{Z})$. Since the fixed points form a basis of $H_T^*(X, \mathbb{Z})$ after localization, we may restrict our attention to

$$\gamma = [v] \,.$$

The pull-back of $\gamma$ to $v$ is given by

$$[v]\Big|_v = t_1 t_2 t_3 \,.$$

---

**Proposition 3.** *The localization of the incidence to* $[v]$ *in both Gromov-Witten and Donaldson-Thomas theories is*

$$t_1 t_2 d_3 + t_1 d_2 t_3 + d_1 t_2 t_3 \,.$$

Since the Donaldson-Thomas techniques are not as well-known, we derive the insertion term there. The corresponding computation of the Gromov-Witten theory is a standard application of the string equation.

We may assume that $X = \mathbb{C}^3$. Consider the universal ideal sheaf (3) and let

$$\mathcal{I} \subset \mathcal{O}_X = \mathbb{C}[x_1, x_2, x_3]$$

be a $T$-fixed ideal with degrees $d_i$ along the coordinate axes. Proposition 3 is an immediate consequence of the following result.

**Lemma 4.** *We have*

$$c_2(\mathfrak{I})\Big|_{[\mathcal{I}] \times v} = t_1 t_2 d_3 + t_1 d_2 t_3 + d_1 t_2 t_3 \,. \tag{19}$$

*Proof.* Consider a graded free resolution of length 3,

$$0 \to R_3 \to R_2 \to R_1 \to \mathcal{I} \to 0 \tag{20}$$

where

$$R_i = \bigoplus_k x^{a_{ik}} \mathcal{O}_X \,, \quad a_{ik} \in \mathbb{Z}^3 \,.$$

Since $c_1(\mathfrak{I}) = 0$, we have from (20)

$$c_2(\mathfrak{I})|_{[\mathcal{I}] \times 0} = -\mathrm{ch}_2(\mathfrak{I})|_{[\mathcal{I}] \times 0} = \frac{1}{2} \sum_{i,k} (-1)^i (t, a_{ik})^2 \,, \tag{21}$$

where $t = (t_1, t_2, t_3)$, and $(t, a)$ denotes the standard inner product.

Let $\pi$ be a 3-dimensional partition associated to $\mathcal{I}$. We may compute the character of the $T$-action on $\mathcal{O}_X/\mathcal{I}$ either directly in terms of $\pi$ or via the resolution (20). We obtain the identity

$$\prod_{i=1}^3 (1 - e^{-t_i}) \sum_{p \in \pi} e^{-(t,p)} = 1 + \sum_{i,k} (-1)^i e^{-(t,a_{ik})} \,. \tag{22}$$

The right side of (21) equals the quadratic term in (22). The sum over $p \in \pi$ of the left side of (22) may be split in 4 parts — the sums over $p$ in each of

18

the three infinite legs of $\pi$ and the finite remainder. The infinite legs yield geometric series which can be summed exactly. Since

$$\prod_{i=1}^{3}(1 - e^{-t_i}) = t_1 t_2 t_3 + O(t^4),$$

the remainder does not contribute to the quadratic term, while each infinite leg of $\pi$ contributes the corresponding term to (19). $\qquad\square$

### 2.4.3 Formulas

Consider the capped localization formula for the partition function

$$\mathsf{Z}'_{GW}\big(X, u \mid \gamma_1, \ldots, \gamma_r\big)_\beta \tag{23}$$

with equivariant primary field insertions $\gamma_i \in H_T^*(X, \mathbb{Z})$.

Let $\mathcal{V}$ be the set of vertices of $\Delta$. To each $v \in \mathcal{V}$, let $h_1^v, h_2^v, h_2^v$ be the associated half-edges[8] with tangent weights $t_1^v, t_2^v, t_3^v$ respectively. Let $\Gamma_\beta$ be the set of capped markings satisfying the degree condition (18). Each $\Gamma \in \Gamma_\beta$ associates a partition $\lambda(h)$ to every half-edge $h$. Let

$$|h| = |\lambda(h)|$$

denote the half-edge degree.

The localization contributions of the insertions is easily identified. Let $\gamma_i^v \in H_T^*(\bullet, \mathbb{Z})$ be the $T$-equivariant restriction of $\gamma_i$ to the $T$-fixed point associated to $v$. The insertion term $\mathsf{I}_\Gamma$ is determined by

$$\mathsf{I}_\Gamma = \prod_{i=1}^{r} \sum_{v \in \mathcal{V}} \gamma_i^v \left( \frac{|h_1^v|}{t_1^v} + \frac{|h_2^v|}{t_2^v} + \frac{|h_3^v|}{t_3^v} \right) \ \in H_T^*(\bullet, \mathbb{Q}) \tag{24}$$

as a consquence of Proposition 3.

For each $v \in \mathcal{V}$, the assignment $\Gamma$ determines an evaluation of the capped vertex,

$$\mathsf{C}_{GW}(v, \Gamma) = \mathsf{C}_{GW}(\lambda(h_1^v), \lambda(h_2^v), \lambda(h_3^v), t_1^v, t_2^v, t_3^v, u).$$

Let $h_1^e$ and $h_2^e$ be the half-edges associated to the edge $e$. The assignment $\Gamma$ also determines an evaluation of the capped edge,

$$\mathsf{E}_{GW}(e, \Gamma) = \mathsf{E}_{GW}(\lambda(h_1^e), \lambda(h_2^e), t_1, t_2, t_3, t_1', t_2', u)$$

---

[8]For simplicity, we assume $X$ is projective so each vertex is incident to 3 compact edges.

where the weights are determined by Figure 1. A Gromov-Witten gluing factor is specified by $\Gamma$ at each half-edge $h_i^v \in \mathcal{H}$ by

$$\mathsf{G}_{GW}(h_i^v, \Gamma) = \mathfrak{z}(\lambda(h_i^v)) \left( \frac{\prod_{j=1}^3 t_j^v}{t_i^v} \right)^{\ell(\lambda(h_i^v))} u^{2\ell(\lambda(h_i^v))}.$$

The Gromov-Witten capped localization formula can be written exactly in the form presented in Section 2.4.1,

$$\mathsf{Z}'_{GW} \Big( X, u \mid \gamma_1, \ldots, \gamma_r \Big)_\beta =$$

$$\sum_{\Gamma \in \Gamma_\beta} \prod_{v \in \mathcal{V}} \prod_{e \in \mathcal{E}} \prod_{h \in \mathcal{H}} \mathsf{I}_\Gamma \, \mathsf{C}_{GW}(v, \Gamma) \, \mathsf{E}_{GW}(e, \Gamma) \, \mathsf{G}_{GW}(h, \Gamma)$$

where the product is over the sets of vertices $\mathcal{V}$, edges $\mathcal{E}$, and half-edges $\mathcal{H}$ of the polytope $\Delta$.

The Donaldson-Thomas capped localization formula has an identical structure,

$$\mathsf{Z}'_{DT} \Big( X, q \mid \gamma_1, \ldots, \gamma_r \Big)_\beta =$$

$$\sum_{\Gamma \in \Gamma_\beta} \prod_{v \in \mathcal{V}} \prod_{e \in \mathcal{E}} \prod_{h \in \mathcal{H}} \mathsf{I}_\Gamma \, \mathsf{C}_{DT}(v, \Gamma) \, \mathsf{E}_{DT}(e, \Gamma) \, \mathsf{G}_{DT}(h, \Gamma)$$

where the evaluations $\mathsf{C}_{DT}(v, \Gamma)$ and $\mathsf{E}_{DT}(e, \Gamma)$ are defined as before. The Donaldson-Thomas gluing factors

$$\mathsf{G}_{DT}(h_i^v, \Gamma) = (-1)^{|h_i^v| - \ell(\lambda(h_i^v))} \mathfrak{z}(\lambda(h_i^v)) \left( \frac{\prod_{j=1}^3 t_j^v}{t_i^v} \right)^{\ell(\lambda(h_i^v))} q^{-|h_i^v|}$$

are slightly different.

The most basic example of capped localization occurs for the 3-fold total space of

$$\mathcal{O}(a) \oplus \mathcal{O}(b) \to \mathbf{P}^1. \tag{25}$$

The standard localization formula has vertices over $0, \infty \in \mathbf{P}^1$ and a single edge. To write the answer in terms of capped localization, we consider a $T$-equivariant degeneration of (25) to a chain

$$(0, 0) \cup (a, b) \cup (0, 0)$$

of total spaces of bundles over $\mathbf{P}^1$ denoted here by splitting degrees. The first $(0,0)$-geometry is relative over $\infty \in \mathbf{P}^1$, the central $(a,b)$-geometry is relative on both sides, and the last $(0,0)$-geometry is relative over $0 \in \mathbf{P}^1$. The degeneration formula exactly expresses the Gromov-Witten and Donaldson-Thomas theories of (25) as capped localization with 2 capped vertices and a single capped edge in the middle.

In fact, the capped localization formula for arbitrary toric $X$ in both theories can be proven by studying the example (25) — the cancelling of the rubber caps already occurs there. We leave the details to the reader.

# 3 Proofs of Propositions 1 and 2

## 3.1 Tube and cap invariants

### 3.1.1 Partition functions

Consider the 3-fold $\mathcal{A}_n \times \mathbf{P}^1$ with the full $T$-action. The torus factors

$$T = (\mathbb{C}^*)^2 \times \mathbb{C}^*$$

into a vertical subtorus acting on $\mathcal{A}_n$ and a horizontal factor acting on $\mathbf{P}^1$. There is a $T$-equivariant projection

$$\pi : \mathcal{A}_n \times \mathbf{P}^1 \to \mathbf{P}^1 .$$

Let $0, \infty \in \mathbf{P}^1$ be the $T$-fixed points.

We define partition functions in both Gromov-Witten and Donaldson-Thomas theories with relative conditions over the divisors

$$F_0, F_\infty \subset \mathcal{A}_n \times \mathbf{P}^1$$

lying over $0, \infty \in \mathbf{P}^1$. The geometry here is called the $\mathcal{A}_n$-*tube*. We set

$$\begin{aligned}
\mathsf{GW}^* \left(\gamma_1, \ldots, \gamma_r\right)_{\lambda,\mu} = \\
(-iu)^{\ell(\lambda)+\ell(\mu)} \sum_{\sigma \in H_2(\mathcal{A}_n, \mathbb{Z})} s^\sigma \, \mathsf{Z}'_{GW} \left(\mathcal{A}_n \times \mathbf{P}^1, u \,\big|\, \gamma_1, \ldots, \gamma_r \big| \lambda, \mu\right)_{d[\mathbf{P}^1]+\sigma} ,
\end{aligned} \quad (26)$$

where $\lambda$ and $\mu$ represent relative conditions — partitions of the horizontal degree $d$ labeled by elements of $H^*_T(\mathcal{A}_n, \mathbb{Z})$. The symbols $s^\sigma$ span the group

ring of $H_2(\mathcal{A}_n, \mathbb{Z})$. For brevity, we have dropped the relative divisor $F_0 \cup F_\infty$ in the notation. Similarly, let

$$
\mathsf{DT}^*\left(\gamma_1, \ldots, \gamma_r\right)_{\lambda,\mu} =
$$
$$
(-q)^d \sum_{\sigma \in H_2(\mathcal{A}_n, \mathbb{Z})} s^\sigma \, \mathsf{Z}'_{DT}\left(\mathcal{A}_n \times \mathbf{P}^1, q \,\middle|\, \gamma_1, \ldots, \gamma_r \middle| \lambda, \mu\right)_{d[\mathbf{P}^1]+\sigma},
$$

A benefit of the above notation is a simple form for the conjectural GW/DT correspondence:

$$
\mathsf{GW}^*\left(\gamma_1, \ldots, \gamma_r\right)_{\lambda,\mu} = \mathsf{DT}^*\left(\gamma_1, \ldots, \gamma_r\right)_{\lambda,\mu}
$$

after the variable change $e^{-iu} = -q$.

### 3.1.2 Tube calculation

Another advantage of the partition functions defined in Section 3.1.1 is their uniform behavior under degeneration.

Let $g_{\lambda,\mu}$ be the natural $T$-equivariant residue pairing

$$
g_{\lambda,\mu} = \int_{\mathrm{Hilb}(\mathcal{A}_n, d)} \lambda \cup \mu \tag{27}
$$

where $|\lambda| = |\mu| = d$. For example, for $S = \mathbb{C}^2$, we have

$$
g_{\lambda,\mu} = (-1)^{|\lambda|-\ell(\lambda)} \frac{\delta_{\lambda,\mu}}{\mathfrak{z}(\lambda)} (t_1 t_2)^{-\ell(\lambda)}.
$$

Define raised partition functions by

$$
\mathsf{GW}^*\left(\gamma_1, \ldots, \gamma_r\right)_\mu^{\lambda'} = \sum_\lambda \mathsf{GW}^*\left(\gamma_1, \ldots, \gamma_r\right)_{\lambda,\mu} (-1)^d g^{\lambda,\lambda'},
$$

$$
\mathsf{DT}^*\left(\gamma_1, \ldots, \gamma_r\right)_\mu^{\lambda'} = \sum_\lambda \mathsf{DT}^*\left(\gamma_1, \ldots, \gamma_r\right)_{\lambda,\mu} (-1)^d g^{\lambda,\lambda'},
$$

where $g^{\lambda,\mu}$ is the inverse matrix of the intersection pairing.

Consider the $T$-equivariant degeneration of $\mathbf{P}^1$ into a union

$$
\mathbf{P}^1 \to \mathbf{P}^1 \cup \mathbf{P}^1.
$$

The degeneration formulas for the geometry with *no insertions* [9] are

$$\mathsf{GW^*}^\lambda_\mu = \sum_{\mu'} \mathsf{GW^*}^\lambda_{\mu'} \mathsf{GW^*}^{\mu'}_\mu.$$

$$\mathsf{DT^*}^\lambda_\mu = \sum_{\mu'} \mathsf{DT^*}^\lambda_{\mu'} \mathsf{DT^*}^{\mu'}_\mu.$$

The $u = 0$ and $s = 0$ specialization (well-defined for the starred normalization) on the Gromov-Witten side shows the matrix $\mathsf{GW^*}^\lambda_\mu$ is invertible and hence equal to the identity. The same conclusion is obtained on the Donaldson-Thomas side via the specialization $q = 0$ and $s = 0$. We conclude

$$\mathsf{GW^*}^\lambda_\mu = \mathsf{DT^*}^\lambda_\mu = \delta^\lambda_\mu.$$

### 3.1.3 Cap calculation

The $\mathcal{A}_n$-*cap* geometry is obtained by imposing relative conditions on $\mathcal{A}_n \times \mathbf{P}^1$ only along $F_\infty$ *with no insertions*. Define

$$
\begin{aligned}
\mathsf{GW}^*_\lambda &= (-iu)^{d+\ell(\lambda)} \sum_{\sigma \in H_2(\mathcal{A}_n, \mathbb{Z})} s^\sigma \, \mathsf{Z}'_{GW} \left( \mathcal{A}_n \times \mathbf{P}^1, u \,\middle|\, |\lambda\rangle \right)_{d[\mathbf{P}^1]+\sigma}, \\
\mathsf{DT}^*_\lambda &= (-q)^d \sum_{\sigma \in H_2(\mathcal{A}_n, \mathbb{Z})} s^\sigma \, \mathsf{Z}'_{DT} \left( \mathcal{A}_n \times \mathbf{P}^1, q \,\middle|\, |\lambda\rangle \right)_{d[\mathbf{P}^1]+\sigma}.
\end{aligned}
$$

The formulas for raising indices are the same in the tube case.

Consider the cap geometry for $S \times \mathbf{P}^1$ where $S$ is a compact surface and $K_S \cdot \sigma \le 0$ for all curve classes $\sigma \in H_2(S, \mathbb{Z})$ of interest. Most cap invariants vanish for dimension reasons. Indeed, if $\beta = d[\mathbf{P}^1] + \sigma$, the virtual dimension

$$\boldsymbol{\delta} = 2d - \sigma \cdot K_S$$

is bounded below by the dimension $2d$ of $\mathrm{Hilb}(S, d)$, which is also the maximal codimension of relative conditions in Gromov-Witten theory. We must have $\sigma \cdot K_S = 0$ and $\lambda \in H^0(\mathrm{Hilb}(S, d), \mathbb{Q})$ for the cap invariants $\mathsf{GW}^*{}^\lambda_S$ and $\mathsf{DT}^*{}^\lambda_S$ with *no insertions* to have chance of not vanishing.

Let $\mathcal{A}_n \subset S$ be a $T$-equivariant embedding in a compact toric surface $S$. On the Gromov-Witten side, the moduli of maps to $\mathcal{A}_n \times \mathbf{P}^1$ occurs as an

---

[9]By convention, $\mathsf{GW^*}^\lambda_\mu = \mathsf{GW}^*(\ )^\lambda_\mu$ and similarly for Donaldson-Thomas theory.

open and closed subset of the moduli of maps to $S \times \mathbf{P}^1$. On the Donaldson-Thomas side, the dimension 0 components of the subschemes can wander off of $\mathcal{A}_n \times \mathbf{P}^1$, but a localization argument shows the reduced theory[10] of $\mathcal{A}_n \times \mathbf{P}^1$ is determined by an open and closed locus of the moduli of ideal sheaves of $S \times \mathbf{P}^1$.

By considering the inclusion $\mathcal{A}_n \subset S$, we draw two conclusions. First, the cap invariants

$$\mathsf{GW}^{*\lambda} = \mathsf{DT}^{*\lambda} = 0$$

vanish unless $\lambda$ is the unique partition $1^d$ with each part weighted by the identity in $H^0(\mathcal{A}_n, \mathbb{Z})$. Let us denote the latter partition by $\mathbf{1}^d$. Second, the cap invariants

$$\mathsf{GW}^{*\mathbf{1}^d}, \quad \mathsf{DT}^{*\mathbf{1}^d} \tag{28}$$

are *non-equivariant scalars*. Hence, the $T$-equivariant integrals (28) can be calculated via a restricted torus. For the vertical subtorus $(\mathbb{C}^*)^2$, the calculations of [22, 24] yield

$$\mathsf{GW}_S^{*\mathbf{1}^d} = \mathsf{DT}_S^{*\mathbf{1}^d} = 1.$$

## 3.2 Differential equations

### 3.2.1 Equation I

Let $t_3$ be the weight of the $T$-action on the trivial normal bundle of

$$F_0 \subset \mathcal{A}_n \times \mathbf{P}^1.$$

By localization, every class $\gamma \in H_T^*(\mathcal{A}_n \times \mathbf{P}^1, \mathbb{Z})$ satisfies

$$t_3 \gamma = \gamma_0 - \gamma_\infty \ \in H_T^*(\mathcal{A}_n \times \mathbf{P}^1, \mathbb{Z}) \tag{29}$$

where $\gamma_0, \gamma_\infty$ are the restrictions of $\gamma$ to $F_0$ and $F_\infty$ respectively.

We may express the insertion of the class $\gamma_0$ in terms of rubber integrals via the following gluing formula. For the Gromov-Witten theory of $\mathcal{A}_n \times \mathbf{P}^1$,

$$\left\langle \lambda \big| \gamma_0 \big| \mu \right\rangle_\beta' = \sum_{\nu,\, \sigma_1+\sigma_2=\sigma} \left\langle \lambda \big| \gamma_0 \big| \nu \right\rangle_{\beta_1}^{\sim}{}' \mathfrak{z}(\nu)\, u^{2\ell(\nu)} \left\langle \nu^\vee \big| \, \big| \mu \right\rangle_{\beta_2}' \tag{30}$$

---

[10]The division by the degree 0 Donaldson-Thomas theory removes contributions of the complement of $\mathcal{A}_n$ in $S$.

where
$$\beta_i = d[\mathbf{P}^1] + \sigma_i \,, \quad \sigma_i \in H_2(\mathcal{A}_n, \mathbb{Z}) \,,$$

is the decomposition of the curve class into an $\mathcal{A}_n$-component and a $\mathbf{P}^1$-component. The sum in (30) is over all intermediate relative conditions $\nu$ and all splittings of the $\mathcal{A}_n$-component $\sigma \in H_2(\mathcal{A}_n, \mathbb{Z})$ of the degree $\beta$. The bracket on the left denotes the theory of $\mathcal{A}_n \times \mathbf{P}^1$ relative to $F_0 \cup F_\infty$ with relative conditions $\lambda, \mu$ and insertion $\gamma_0$. The first set of brackets on the right with a tilde superscript denotes the rubber theory[11] of $\mathcal{A}_n \times \mathbf{P}^1$ with the insertion $\gamma_0$ pulled back from $\mathcal{A}_n$. The second bracket on the right is the $\mathcal{A}_n$-tube.

The proof of (30) is obtained from basic geometry of the relative theory. Let $\mathcal{D}$ be the $T$-equivariant Artin stack of degenerations of the 1-pointed relative geometry $\mathcal{A}_n \times \mathbf{P}^1/F_0 \cup F_\infty$. The moduli space $\mathcal{D}$ parameterizes accordian destabilizations $Y$ of $\mathcal{A}_n \times \mathbf{P}^1$ together with a point $p \in Y$ not on the boundary or the singular locus. There is a $T$-equivariant evaluation map

$$\mathrm{ev}_p : \mathcal{D} \to \mathcal{A}_n \times \mathbf{P}^1.$$

There are two divisors on $\mathcal{D}$ related to $0 \in \mathbf{P}^1$. The first is $\mathrm{ev}_p^*([F_0])$. The second is the boundary divisor $\mathcal{D}_0 \subset \mathcal{D}$ where $p$ is on a destabilization over $0 \in \mathbf{P}^1$. The following result can be easily seen by comparing the divisors on smooth charts for $\mathcal{D}$.

**Lemma 5.** $\mathrm{ev}_p^*([F_0]) = \mathcal{D}_0$ in $\mathrm{Pic}(\mathcal{D})$.

Equation (30) is proven by factoring

$$\gamma_0 = \gamma \cdot [F_0],$$

pulling-back Lemma 5 to the moduli space of stable maps to $\mathcal{A}_n \times \mathbf{P}^1/F_0 \cup F_\infty$, and using the splitting formulas [15].

The parallel result in Donaldson-Thomas theory holds with DT gluing factors,

$$\left\langle \lambda \middle| \gamma_0 \middle| \mu \right\rangle'_\beta = \sum_{\nu, \, \sigma_1 + \sigma_2 = \sigma} \left\langle \lambda \middle| \gamma_0 \middle| \nu \right\rangle^{\sim}_{\beta_1} {}' (-1)^{|\nu| - \ell(\nu)} \mathfrak{z}(\nu) \, q^{-|\nu|} \left\langle \nu^\vee \middle| \, \middle| \mu \right\rangle'_{\beta_2} \qquad (31)$$

The proof is identical. The universal relative space over the moduli space of relative ideal sheaves discussed in [30] is used.

---

[11]See [24, 30] for a foundational discussion of rubber theory in the $\mathcal{A}_n$ context.

### 3.2.2 Equation II

Let $\gamma \in H_T^*(\mathcal{A}_n \times \mathbf{P}^1, \mathbb{Z})$ be the pull-back of a divisor $\Gamma \subset \mathcal{A}_n$. By the divisor equation, the insertion of $\gamma$ simply multiplies an invariant in class $d[\mathbf{P}^1] + \sigma$ by

$$\gamma \cdot \beta = \Gamma \cdot \sigma \,.$$

The insertion of $\gamma$ may thus be interpreted as the action of a linear differential operator $\partial_\Gamma$ on the generating function (26) over all possible vertical curve classes $\sigma$.

The formulas (30)-(31) may be interpreted as left multiplication by the matrix corresponding to

$$\left\langle \lambda \middle| \gamma_0 \middle| \nu \right\rangle_\beta^{\sim}{}' = \left\langle \lambda \middle| \Gamma_F \middle| \nu \right\rangle_\beta' \,. \tag{32}$$

Here $\Gamma_F = \gamma \cdot F$, where

$$F \in H^2_{(\mathbb{C}^*)^2}(\mathcal{A}_n \times \mathbf{P}^1) \,,$$

is the fiber class — the pull-back of the generator of the nonequivariant group $H^2(\mathbf{P}^1, \mathbb{Z})$. The nonequivariance here is because equation (32) arises from rigidification of the rubber. The brackets on the right side of (32) take values in the fiberwise $(\mathbb{C}^*)^2$-equivariant cohomology.[12]

We can write the formulas (30)-(31) as

$$\mathsf{GW}^*(\gamma_0)_\mu^\lambda = \sum_\nu \mathsf{GW}^*(\Gamma_F)_\nu^\lambda \, \mathsf{GW}^*{}_\mu^\nu \,,$$

$$\mathsf{DT}^*(\gamma_0)_\mu^\lambda = \sum_\nu \mathsf{DT}^*(\Gamma_F)_\nu^\lambda \, \mathsf{DT}^*{}_\mu^\nu \,.$$

By (29), we obtain the following linear differential equations in the Gromov-Witten theory of $\mathcal{A}_n \times \mathbf{P}^1$,

$$t_3 \, \partial_\Gamma \mathsf{O}_{GW} = \left[ \mathsf{O}_{GW}(\Gamma_F), \mathsf{O}_{GW} \right], \tag{33}$$

Here, we consider the invariants $\mathsf{GW}^*{}_\bullet^\bullet$ as defining an operator $\mathsf{O}_{GW}$ on the Fock space associated to $\mathcal{A}_n$,

$$\mathsf{O}_{GW}|\mu\rangle = \sum_\lambda \mathsf{GW}^*{}_\mu^\lambda |\lambda\rangle, \tag{34}$$

---

[12]There is no possibility for confusion in equation (32) because even if $\left\langle \lambda \middle| \Gamma_F \middle| \nu \right\rangle_\beta'$ is interpreted as a $T$-equivariant bracket, the result is independent of the $t_3$ along the $\mathbf{P}^1$ direction. The proof is left to the reader.

or equivalently

$$\langle\lambda|\mathsf{O}_{GW}|\mu\rangle = (-1)^d \mathsf{GW}^*{}_{\lambda,\mu}$$

where the inner product on Fock space defined by (27) occurs on the left side. The operator $\mathsf{O}_{GW}(\Gamma_F)$ is defined similarly. The identical equation

$$t_3\,\partial_\Gamma \mathsf{O}_{DT} = \big[\mathsf{O}_{DT}(\Gamma_F), \mathsf{O}_{DT}\big], \tag{35}$$

holds in Donaldson-Thomas theory.

Unfortunately, equations (33) and (35) are trivial here. We have already seen $\mathsf{O}_{GW}$ is the identity matrix. In particular, only $\sigma = 0$ terms occur. Hence, the left sides of (33) and (35) vanish. The right sides of (33) and (35) are also 0 since commutators with the identity matrix always vanish. However, we will make nontrivial use of the differential equations in more complicated geometries in the remainder of the paper.

The calculations of $\mathsf{O}_{GW}(\Gamma_F)$ and $\mathsf{O}_{DT}(\Gamma_F)$ are certainly not formal. The equality we need is Corollary 8.5 of [24].

**Theorem 2.** *For $\mathcal{A}_n \times \mathbf{P}^1$,*

$$\mathsf{GW}^*(\Gamma_F)^\mu_\lambda = \mathsf{DT}^*(\Gamma_F)^\mu_\lambda$$

*after the variable change $e^{iu} = -q$,*

Theorem 2 is a special case of the equivariant relative GW/DT correspondence for toric varieties.[13] The result will play a crucial role for us.

## 3.3   Proof of Proposition 1

### 3.3.1   Fiberwise $(\mathbb{C}^*)^2$-action

Since capped edges are reduced partition functions in the Gromov-Witten and Donaldson-Thomas theories of local curves, the results of [4, 29, 30] establish the GW/DT correspondence for the fiberwise $(\mathbb{C}^*)^2$-equivariant cohomology. Our goal now is to strengthen the correspondence to include the full 3-dimensional $T$-action.

---

[13]More of the correspondence is proven in [22, 23, 24], but Theorem 2 is all we will require.

### 3.3.2 $(0,0)$ and $(0,-1)$-edges

The $(0,0)$-edge is the theory of $\mathbb{C}^2 \times \mathbf{P}^1$ relative to fibers over $0, \infty \in \mathbf{P}^1$ with respect to the full 3-dimensional $T$-action. The geometry here is just the $\mathcal{A}_0$-tube, so we have already proven the required GW/DT correspondence in Section 3.1.2.

By standard $T$-equivariant degeneration arguments, the correspondence for all $(a,b)$-edges follows from the $(0,0),(0,-1)$, and $(-1,0)$ cases. By symmetry, we need only consider the $(0,-1)$-edge, the total space of

$$\mathcal{O} \oplus \mathcal{O}(-1) \to \mathbf{P}^1$$

with respect to the full $T$-action.

The starred partition functions for the $(0,-1)$-edge are

$$
\begin{aligned}
\mathsf{E}^*_{GW}(0,-1)_{\lambda,\mu} &= (-iu)^{\ell(\lambda)+\ell(\mu)-d}\ \mathsf{Z}'_{GW}\left((0,-1), u \mid \lambda,\mu\right), \\
\mathsf{E}^*_{DT}(0,-1)_{\lambda,\mu} &= (-q)^{-\frac{d}{2}}\ \mathsf{Z}'_{DT}\left((0,-1), q \mid \lambda,\mu\right).
\end{aligned}
$$

The GW/DT correspondence we require then takes the form

$$\mathsf{E}^*_{GW}(0,-1)_{\lambda,\mu} = \mathsf{E}^*_{DT}(0,-1)_{\lambda,\mu}$$

after the variable change $e^{-iu} = -q$.

### 3.3.3 DT descendents

Primary insertions in Donaldson-Thomas theory were defined in (4) via the the Künneth components of $c_2(\mathfrak{J})$. Of course, we may also consider the other characteristic classes of the universal ideal sheaf $\mathfrak{J}$.

Let $X$ be a nonsingular 3-fold. Following the terminology of (4), we define the *descendent* insertion $\sigma_k(\gamma)$ by the operation

$$\sigma_k(\gamma)(\xi) = \pi_{1*}\left(\mathrm{ch}_{k+2}(\mathfrak{J}) \cdot \pi_2^*(\gamma) \cap \pi_1^*(\xi)\right).$$

The insertion $\sigma_k(\gamma)$ lowers the (real) homological degree by $2k + \deg\gamma - 2$. In particular, $\sigma_1(1)$ preserves the degree. By Riemann-Roch,

$$\sigma_1(1) = -\chi + \frac{1}{2}\int_\beta c_1(X),$$

where $\chi = \chi(\mathcal{O}/\mathfrak{J})$ and $\beta$ is the curve class.

Specializing now to the $(0, -1)$-edge geometry, we see

$$\mathsf{Z}_{DT}((0, -1), q \mid \sigma_1(1) \mid \lambda, \mu) = (-q\frac{d}{dq} + \frac{d}{2}) \, \mathsf{Z}_{DT}((0, -1), q \mid \lambda, \mu) \quad (36)$$

where the partition functions are *unprimed*. The degree 0 series [30] is

$$\mathsf{Z}_{DT}((0, -1), q \mid \emptyset, \emptyset) = M(-q)^{-\frac{t_1 + t_2}{t_3}} \cdot M(-q)^{\frac{t_1' + t_2'}{t_3}}$$

where the weights $t_i$ are specified by Figure 1 and

$$M(-q) = \prod_{n=1}^{\infty} \frac{1}{(1 - (-q)^n)^n}$$

is the McMahon function (7)

After transforming (36), we obtain the following equation

$$- q\frac{d}{dq} \, \mathsf{E}_{DT}^*(0, -1)_\mu^\lambda =$$

$$\mathsf{E}_{DT}^*(0, -1 \mid \sigma_1(1))_\mu^\lambda - \left( \frac{t_1 + t_2}{t_3} - \frac{t_1' + t_2'}{t_3} \right) \Phi(q) \mathsf{E}_{DT}^*(0, -1)_\mu^\lambda \quad (37)$$

where

$$\mathsf{E}_{DT}^*(0, -1 \mid \sigma_1(1))_{\lambda, \mu} = (-q)^{-\frac{d}{2}} \frac{\mathsf{Z}_{DT}((0, -1), q \mid \sigma_1(1) \mid \lambda, \mu)}{\mathsf{Z}_{DT}((0, -1), q \mid \emptyset, \emptyset)}$$

and

$$\Phi(q) = q\frac{d}{dq} M(-q).$$

Next, we use the relation

$$t_3 \sigma_1(1) = \sigma_1(F_0) - \sigma_1(F_\infty)$$

and the differential equation of Section 3.2 to conclude

$$t_3 \mathsf{E}_{DT}^*(0, -1 \mid \sigma_1(1))_\mu^\lambda = \sum_\nu \mathsf{E}_{DT, t_1, t_2}^*(0, 0 \mid \sigma_1(F))_\nu^\lambda \, \mathsf{E}_{DT}^*(0, -1)_\mu^\nu$$

$$- \sum_\nu \mathsf{E}_{DT}^*(0, -1)_\nu^\lambda \, \mathsf{E}_{DT, t_1', t_2'}^*(0, 0 \mid \sigma_1(F))_\mu^\nu.$$

where

$$\mathsf{E}^*_{DT,w_1,w_2}(0,0 \mid \sigma_1(F))_{\lambda,\mu} = (-q)^{-d} \frac{\mathsf{Z}_{DT}((0,0),q \mid \sigma_1(F) \mid \lambda,\mu)}{\mathsf{Z}_{DT}((0,0),q \mid \emptyset,\emptyset)}$$

and the subscripted weights specify the fiberwise $T$-action.

Written as operators on the Fock space of $\mathcal{A}_0$, we obtain

$$
\begin{aligned}
-t_3 q \frac{d}{dq} \mathsf{O}_{DT}(0,-1) \;=\; & \quad \mathsf{O}_{DT}^{t_1;t_2}((0,0) \mid \sigma_1(F))\, \mathsf{O}_{DT}(0,-1) \\
& -\mathsf{O}_{DT}(0,-1)\, \mathsf{O}_{DT}^{t_1';t_2'}((0,0) \mid \sigma_1(F)) \\
& -(t_1 + t_2 - t_1' - t_2')\, \Phi(q)\mathsf{O}_{DT}(0,-1).
\end{aligned}
$$

By Proposition 22 of [30],

$$\mathsf{O}_{DT}^{w_1,w_2}((0,0) \mid \sigma_1(F)) = -\mathsf{M}(w_1,w_2) + (w_1 + w_2)\Phi(q) \cdot \mathrm{Id},$$

where $\mathsf{M}$ is the fundamental operator[14] on Fock space defined by

$$
\begin{aligned}
\mathsf{M}(w_1,w_2) = (w_1 + w_2) \sum_{k>0} \frac{k}{2} \frac{(-q)^k + 1}{(-q)^k - 1}\, \alpha_{-k}\, \alpha_k + \\
\frac{1}{2} \sum_{k,l>0} \Big[ w_1 w_2\, \alpha_{k+l}\, \alpha_{-k}\, \alpha_{-l} - \alpha_{-k-l}\, \alpha_k\, \alpha_l \Big].
\end{aligned}
$$

We conclude

$$-t_3 q \frac{d}{dq} \mathsf{O}_{DT}(0,-1) = -\mathsf{M}(t_1,t_2)\, \mathsf{O}_{DT}(0,-1) + \mathsf{O}_{DT}(0,-1)\mathsf{M}(t_1',t_2') \quad (38)$$

holds.

### 3.3.4  GW descendents

The dilaton equation for the descendent insertion $\tau_1(1)$ yields

$$\left( u\frac{d}{du} + d \right)\, \mathsf{E}^*_{GW}(0,-1)^\lambda_\mu = \mathsf{E}^*_{GW}((0,-1) \mid \tau_1(1))^\lambda_\mu$$

---

[14]We follow here the terminology of [30] for the raising and lowering operators $\alpha_{\pm r}$ on Fock space.

30

where
$$\mathsf{E}^*_{GW}(0,-1)_{\lambda,\mu} = (-iu)^{\ell(\lambda)+\ell(\mu)-d}\mathsf{Z}'_{GW}(0,-1)_{\lambda,\mu},$$
$$\mathsf{E}^*_{GW}((0,-1) \mid \tau_1(1))_{\lambda,\mu} = (-iu)^{\ell(\lambda)+\ell(\mu)-d}\mathsf{Z}'_{GW}((0,-1) \mid \tau_1(1))_{\lambda,\mu}.$$

Next, we use the relation
$$t_3\tau_1(1) = \tau_1(F_0) - \tau_1(F_\infty)$$

and the differential equation of Section 3.2 to conclude

$$t_3\mathsf{E}^*_{GW}(0,-1 \mid \tau_1(1))^\lambda_\mu = \sum_\nu \mathsf{E}^*_{GW,t_1,t_2}(0,0 \mid \tau_1(F))^\lambda_\nu \, \mathsf{E}^*_{GW}(0,-1)^\nu_\mu$$
$$- \sum_\nu \mathsf{E}^*_{GW}(0,-1)^\lambda_\nu \, \mathsf{E}^*_{GW,t'_1,t'_2}(0,0 \mid \tau_1(F))^\nu_\mu ,$$

where

$$\mathsf{E}^*_{GW,w_1,w_2}(0,0 \mid \tau_1(F))_{\lambda,\mu} = (-iu)^{\ell(\lambda)+\ell(\mu)}\mathsf{Z}'_{GW}((0,0),q \mid \tau_1(F) \mid \lambda,\mu) \quad (39)$$

and the subscripted weights specify the fiberwise $T$-action.

Putting the results together, we obtain the main equation,

$$t_3\left(u\frac{d}{du} + d\right)\mathsf{E}^*_{GW}(0,-1)^\lambda_\mu = \sum_\nu \mathsf{E}^*_{GW,t_1,t_2}(0,0 \mid \tau_1(F))^\lambda_\nu \, \mathsf{E}^*_{GW}(0,-1)^\nu_\mu$$
$$- \sum_\nu \mathsf{E}^*_{GW}(0,-1)^\lambda_\nu \, \mathsf{E}^*_{GW,t'_1,t'_2}(0,0 \mid \tau_1(F))^\nu_\mu .$$

The change of variables $e^{iu} = -q$ implies

$$-t_3q\frac{d}{dq} = -\frac{1}{iu}\left(t_3u\frac{d}{du}\right).$$

By a straightforward[15] evaluation of (39), we find $-\frac{1}{iu}$ times the main equation can be written in $q$ as

$$-t_3q\frac{d}{dq}\mathsf{O}_{GW}(0,-1) = -\mathsf{M}(t_1,t_2)\,\mathsf{O}_{GW}(0,-1) + \mathsf{O}_{GW}(0,-1)\mathsf{M}(t'_1,t'_2) . \quad (42)$$

---

[15]The evaluation of the fiberwise $(\mathbb{C}^*)^2$-equivariant Gromov-Witten integral

$$\langle \lambda \mid -\tau_1(F) \mid \mu \rangle^{(0,0)}, \quad (40)$$

determining (39) proceeds in several well-known steps. First, the series (40) is related to

$$\langle \lambda \mid -(2,1^{d-2}) \mid \mu \rangle^{(0,0)} \quad (41)$$

The term on the left

$$-\frac{1}{iu} \cdot t_3 d\mathsf{O}_{GW}(0,-1)$$

is exactly cancelled by the differences between (39) and the two instances of $\mathsf{M}$ on the right side. Hence, we have an exact match with (38).

### 3.3.5 Matching

Let $\mathcal{V} \subset \mathcal{F}$ be the linear subspace of the Fock space of $\mathcal{A}_0$ consisting of vectors $|v\rangle$ for which the $(0,-1)$-edge matrix element

$$\langle \lambda | \mathsf{O}_{DT}(0,-1) | v \rangle \in \mathbb{Q}(t_1,t_2,t_3)((q))$$

is a rational function of $q$ satisfying the relative GW/DT correspondence (8) for all $\lambda$.

We first prove $\mathcal{V}$ is nonempty by showing

$$\left| 1^d \right\rangle \in \mathcal{V}$$

for all $d$. Indeed,

$$\langle \lambda | \mathsf{O}_{DT}(0,-1) | 1^d \rangle = t_1^{-\ell(\lambda)} \, \langle \lambda[L_{-1}] | \mathsf{O}_{DT}(0,-1) | 1^d \rangle' \tag{43}$$

where the cohomology label $[L_{-1}]$ is the Poincaré dual of the $\mathcal{O}(-1)$-axis in the fiber over $0 \in \mathbf{P}^1$. The relative conditions imply the bracket on the right is an integral of the correct dimension over a proper space (modulo point contributions removed in the reduced invariant, see Section 10.3 of [30]).

---

by degeneration — the parallel step in Donaldson-Thomas theory is done in Section 9.1 of [30]. The evaluation of (41) is a central result of [4]. The difference between (40) and (41) is very easily evaluated. The only new integral which must be computed is

$$\int_{[\overline{M}_{g,1}(\mathbf{P}^1,1)]^{vir}} \lambda_g \lambda_{g-1} \tau_1(p),$$

where $p \in H^2(\mathbf{P}^1, \mathbb{Z})$ is the point class. By localization, the integral is immediately reduced to the Hodge integral series

$$\sum_{g \geq 1} 2g \cdot u^{2g} \int_{\overline{M}_{g,1}} \lambda_g \lambda_{g-1} \frac{\sum_{i=0}^g (-1)^i \lambda_i}{1 - \psi_1} = u \frac{d}{du} \log \frac{u/2}{\sin(u/2)}$$

calculated in [32]. We leave the details to the reader.

The bracket on the right is thus *independent* of the equivariant parameters. The same conclusion holds in Gromov-Witten theory.

The GW/DT correspondence for (43) is the same for $T$-equivariant cohomology and $(\mathbb{C}^*)^2$-equivariant cohomology since the answer is weight independent. Since the $(\mathbb{C}^*)^2$-equivariant statement has been proven, $T$-equivariant statement also holds.

Using the identical equations (38) and (42) for the two theories, we conclude $\mathcal{V} \otimes \mathbb{Q}(q, t_1, t_2, t_3)$ is closed under the action of the operator

$$\nabla = t_3 \, q \frac{d}{dq} - \mathsf{M}(t_1', t_2') \, .$$

Let $p(d)$ denote the number of partitions of $d$. We claim the vectors

$$\nabla^k \left| 1^d \right\rangle , \quad k = 0, \ldots, p(d) - 1$$

are linearly independent over $\mathbb{Q}(q, t_1, t_2, t_3)$ and therefore span the entire subspace $\mathcal{F}_d \subset \mathcal{F}$ of vectors of energy $d$. In fact, the $t_3$-constant terms of these vectors are already linearly independent, see [4, 29]. Hence, $\mathcal{V}$ is the entire Fock space $\mathcal{F}$, and the proof of Proposition 1 is complete. $\qquad\square$

## 3.4 Capped rubber

We start with the tube,
$$\pi : \mathcal{A}_n \times \mathbf{P}^1 \to \mathbf{P}^1$$
relative to the fibers over $0, \infty \in \mathbf{P}^1$. In the moduli space of stable maps, we define a $T$-equivariant open subset

$$U_{g,\beta}^{CR} \subset \overline{M}_g'(\mathcal{A}_n \times \mathbf{P}^1 / F_0 \cup F_\infty, \beta)$$

consisting of map with *no positive degree components* in the destabilization of the fiber over $0 \in \mathbf{P}^1$. The open set

$$V_{n,\beta}^{CR} \subset I_n(\mathcal{A}_n \times \mathbf{P}^1 / F_0 \cup F_\infty, \beta)$$

is defined in exactly the same way. The $T$-equivariant residue theories of $U^{CR}$ and $V^{CR}$ are well-defined since the $T$-fixed loci are compact. The geometry here is called the *capped $\mathcal{A}_n$-rubber*.[16]

---

[16] Another approach to capped rubber in Gromov-Witten and Donaldson-Thomas theory is to start with $A_n$-rubber [22, 24] and then add $n$ degree 0 caps. The result is precisely what is obtained by localization on $U^{CR}$ and $V^{CR}$.

We define partition functions with relative conditions over the divisors $F_0, F_\infty$ lying over $0, \infty \in \mathbf{P}^1$. We set[17]

$$\mathsf{GW}^{CR*}_{\lambda,\mu} = (-iu)^{\ell(\lambda)+\ell(\mu)} \sum_{\sigma \in H_2(\mathcal{A}_n, \mathbb{Z})} s^\sigma \, \mathsf{Z}'_{GW} \left( U^{CR}, u \,\middle|\, \lambda, \mu \right)_{d[\mathbf{P}^1]+\sigma},$$

$$\mathsf{DT}^{CR*}_{\lambda,\mu} = (-q)^d \sum_{\sigma \in H_2(\mathcal{A}_n, \mathbb{Z})} s^\sigma \, \mathsf{Z}'_{DT} \left( V^{CR}, q \,\middle|\, \lambda, \mu \right)_{d[\mathbf{P}^1]+\sigma}. \qquad (44)$$

following the conventions of Section 3.1.1. The GW/DT correspondence takes the form

$$\mathsf{GW}^{CR*}_{\lambda,\mu} = \mathsf{DT}^{CR*}_{\lambda,\mu}$$

after the variable change $e^{-iu} = -q$.

**Lemma 6.** *The capped $\mathcal{A}_n$-rubber invariants are rational functions of $q$ and satisfy the GW/DT correspondence.*

*Proof.* The capped rubber invariants (44) determine operators

$$\mathsf{O}_{GW}(\mathsf{CR}), \ \mathsf{O}_{DT}(\mathsf{CR}) : \mathcal{F} \to \mathcal{F}$$

acting on the the Fock space associated to $\mathcal{A}_n$ following Section 3.2.2.

Equations (30)-(31) remain valid in the residue theory with the restriction $\sigma_1 = 0$ by the definition of $U_{CR}$ and $V_{CR}$. As a result, we obtain

$$t_3 \, \partial_\Gamma \mathsf{O}_{GW}(\mathsf{CR}) = \mathsf{O}_{GW}(\Gamma_F)_0 \cdot \mathsf{O}_{GW}(\mathsf{CR}) - \mathsf{O}_{GW}(\mathsf{CR}) \cdot \mathsf{O}_{GW}(\Gamma_F), \qquad (45)$$

where $\mathsf{O}_{GW}(\Gamma_F)_0$ denotes the contribution of curves with degree 0 in the $\mathcal{A}_n$-direction. An identical equation holds in Donaldson-Thomas theory.

The horizontal part of the capped rubber operator is simply equal to the tube,

$$\mathsf{O}_{GW}(\mathsf{CR})_0 = \mathrm{Id}, \quad \mathsf{O}_{DT}(\mathsf{CR})_0 = \mathrm{Id}.$$

We will use (45) to uniquely reconstruct the capped rubber operators from their horizontal parts.

Let $\mathsf{O}_{GW}(\mathsf{CR})_\sigma$ denote the contribution of curves with vertical degree $\sigma$. Equation (45) may be written as

$$t_3 \, (\Gamma \cdot \sigma) \, \mathsf{O}_{GW}(\mathsf{CR})_\sigma = \left[ \mathsf{O}_{GW}(\Gamma_F)_0, \mathsf{O}_{GW}(\mathsf{CR})_\sigma \right] + \dots, \qquad (46)$$

---

[17]The symbols $U^{CR}$ and $V^{CR}$ here indicates not the targets but rather the open sets of the moduli spaces associated to $\mathcal{A}_n \times \mathbf{P}^1$.

where the dots stand for terms involving $O_{GW}(CR)_{\sigma'}$ with $\sigma - \sigma'$ nonzero and effective. The latter terms may be assumed known by induction.

As long as $\sigma \neq 0$, the left side of (46) can be made nonzero by a suitable choice of $\Gamma$. Since the operator $O_{GW}(\Gamma_F)_0$ does not depend on $t_3$, neither do the eigenvalues of the commutation action of $O_{GW}(\Gamma_F)_0$. Hence, the linear equation (46) has a unique solution for $O_{GW}(CR)_\sigma$ in the the field of rational functions of $q$ and $t_i$,

An identical discussion is valid in Donaldson-Thomas theory. In fact, since

$$O_{GW}(\Gamma_F) = O_{DT}(\Gamma_F)$$

by Theorem 2, the reconstruction is the same in the two theories. The GW/DT correspondence is therefore proven. □

## 3.5   Correspondence for 2-leg vertices

We first prove Proposition 2 for the capped vertices of the form $C^*(\lambda, \nu, \emptyset)$. Since the last partition is trivial, such vertices are said to have *2-legs*. Our constructions will be parallel for Gromov-Witten and Donalson-Thomas theory, so we omit the subscript in the capped vertex notation.

**Lemma 7.** *2-leg capped vertices satisfy the GW/DT correspondence.*

*Proof.* The cap geomeetry

$$\pi : \mathcal{A}_1 \times \mathbf{P}^1 \to \mathbf{P}^1$$

with relative conditions along $F_\infty$ has already been proven to satisfy the GW/DT correspondence in Section 3.1.3.

On the other hand, the $\mathcal{A}_1$-cap may be computed by relative capped localization. The capped vertices and edges over $0 \in \mathbf{P}^1$ occur exactly as explained in Section 2. Over $\infty \in \mathbf{P}^1$, a single vertex occurs given by capped $\mathcal{A}_1$-rubber. Again, relative capped localization is easily seen to be equivalent to usual relative localization. In the starred normalization, the relative capped localization formulas for Gromov-Witten and Donaldson-Thomas theory are exactly parallel — the former is obtained from the latter by replacing all occurances of $DT$ by $GW$.

A schematic representation of the localization procedure is depicted in Figure 2. The lines in Figure 2 represent the edges of the toric polyhedron of $X$. The squiggly lines belong to the relative divisor. Partitions $\lambda'$ and
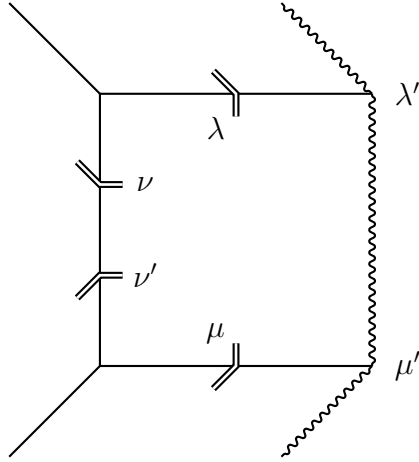
Figure 2: Capped localization for the $\mathcal{A}_1$-cap

$\mu'$ represent relative conditions in the fixed point basis of $H_T^*(\mathcal{A}_1, \mathbb{Z})$. Other partitions represent intermediate relative conditions to be summed over.

Horizontal edges in the $\mathbf{P}^1$-direction have a trivial normal bundle. Hence, the corresponding edge operators are identity operators — which is why there is only one intermediate partition on those edges. The other compact edge carries two partitions satisfying

$$|\nu| = |\nu'|$$

connected by an capped $(0, -2)$-edge. Figure 2 also represents two 2-legged capped vertices $\mathsf{C}^*(\lambda, \nu, \emptyset)$ and $\mathsf{C}^*(\nu', \mu, \emptyset)$, capped $\mathcal{A}_1$-rubber connecting $(\lambda, \mu)$ to $(\lambda', \mu')$, and 4 gluing factors indicated by double lines.

By Lemma 6, the capped $\mathcal{A}_1$-rubber invariants are rational functions of $q$ and satisfy the GW/DT correspondence. Since

$$\mathsf{O}_{GW}(\mathsf{CR}) = \mathsf{O}_{DT}(\mathsf{CR}) = \mathrm{Id} + o\left(s^E\right),$$

where $E$ is the exceptional divisor, the operator $\mathsf{CR}$ is invertible.

We conclude the combination of two 2-legged capped vertices and one capped edge illustrated in Figure 3 satisfies the GW/DT correspondence for any value of $|\nu| = |\nu'|$.

We will now prove the GW/DT correspondence for 2-legged capped vertices $\mathsf{C}^*(\lambda, \nu, \emptyset)$ by induction on
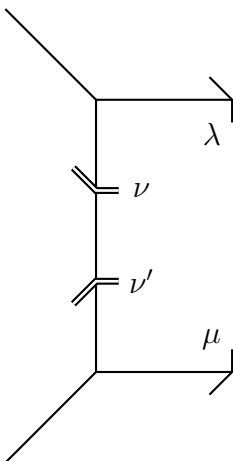
$$n = \min(|\lambda|, |\nu|).$$

36

Figure 3: $\mathcal{A}_1$-cap minus the capped rubber

The base case $n = 0$ of the induction is provided by the correspondence for 1-legged vertices. The 1-legged vertex is simply the $\mathcal{A}_0$-cap geometry treated in Section 3.1.3.

We assume the partitions in Figure 3 satisfy

$$|\lambda| \geq |\nu| = |\nu'| > |\mu|\,.$$

Then all capped vertices $\mathsf{C}^*(\nu', \mu, \emptyset)$ are known by induction. Figure 3 may be interpreted as a system of linear equations for the unknown vertices $\mathsf{C}^*(\lambda, \nu, \emptyset)$ in terms of the known vertices $\mathsf{C}^*(\nu', \mu, \emptyset)$. The linear system has more equations than unknowns. Indeed, for fixed $|\lambda|$, the number of unknowns equals $p(n)$, the number of partitions of $n$, while the number of equations equals the number of possibilities for $\mu$,

$$p(n-1) + \cdots + p(1) + p(0) \geq p(n)\,, \quad n \geq 1\,.$$

Since the $(a, b)$-edge operator is invertible[18], the unique solubility of the linear system is guaranteed by the Lemma 8 below. $\qquad\qquad\qquad\qquad\square$

**Lemma 8.** *The matrix of capped vertices*

$$\left[\mathsf{C}^*(\lambda, \mu, \emptyset)\right]\,, \quad |\lambda| = n\,, |\mu| < n\,,$$

---

[18]The invertibility of the $(a, b)$-edge operator is easily proven by consider the degeneration of a $(0, 0)$-edge to an $(a, b)$-edge and a $(-a, -b)$-edge.

*has maximal rank.*

*Proof.* It is enough to consider the special *topological vertex* case

$$t_1 + t_2 + t_3 = 0\,,$$

in which case great simplifications occur. The capped DT vertex is related to the standard uncapped DT vertex by invertible capped rubbers. The uncapped DT vertex may be evaluated directly, see [25, 31]. Up to further invertible factors, the matrix to consider becomes

$$\sum_\eta s_{\lambda^t/\eta}(q^\rho)\, s_{\mu/\eta}(q^\rho)$$

where $s_{\lambda/\eta}$ are skew Schur functions evaluated at

$$q^\rho = (q^{-1/2}, q^{-3/2}, \dots)\,.$$

As $\mu$ and $\eta$ vary over all partitions of size at most $n-1$, the matrix of skew Schur functions $s_{\mu/\eta}$ is invertible and upper-triangular. We are thus reduced to proving the matrix

$$\left[ s_{\lambda/\eta}(q^\rho) \right]\,, \quad |\lambda| = n\,, |\eta| < n\,,$$

has maximal rank.

For a symmetric function $f$ of degree $r$, let

$$f^\perp : \Lambda_k \to \Lambda_{k-r}$$

denote the linear map adjoint to multiplication by $f$ under the standard inner product. By a basic property of skew Schur functions,

$$s_\rho^\perp s_\lambda = s_{\lambda/\rho}$$

and also

$$p_k^\perp g = k \frac{\partial}{\partial p_k}\, g,$$

where $p_k$ is the power-sum symmetric function $\sum x_i^k$ and the derivative is obtained by expressing $g$ as a polynomial expression of the functions $p_k$.

Using these two facts, the full rank statement on skew Schur functions is equivalent to the following claim. For any symmetric function $g$ of degree $n$, there exists a partition $\mu$, $|\mu| < n$, for which

$$\frac{\partial}{\partial p_{\mu_1}} \cdots \frac{\partial}{\partial p_{\mu_l}} g \bigg|_{q^\rho} \neq 0 \,. \tag{47}$$

Indeed, we can arrange the partial derivative (47) to be a multiple of a single $p_k$ by focusing on the leading lexicographic term of $g$ with respect to the ordering

$$p_1 > p_2 > p_3 > \dots \,.$$

The proof of Lemma 8 and thus Lemma 7 is complete. $\qquad\square$

## 3.6 Proof of Proposition 2

We now prove the GW/DT correspondence for 3-legged capped vertices $\mathsf{C}^*(\lambda, \mu, \nu)$. The argument follows the proof for the 2-legged vertices. For the 3-leg case, we use capped localization for the $\mathcal{A}_2$-cap. A schematic view of the localization is illustrated in Figure 4.
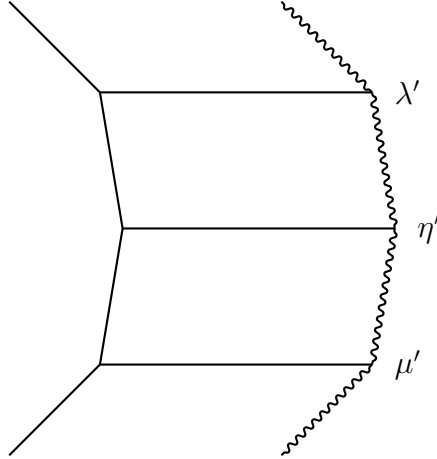


Figure 4: Capped localization for the $\mathcal{A}_2$-cap

Partitions $\lambda'$, $\eta'$, and $\mu'$ represent arbitrary relative conditions. Since the 3-leg vertex occurs in Figure 4 once, the capped localization may be viewed as a system of linear equations for the 3-leg vertex. By Proposition 8, the 3-leg vertex is uniquely determined.

In fact, the nondegeneracy of the square matrix

$$\left[ \mathsf{C}^*(\lambda, \mu, \emptyset) \right], \quad |\lambda|, |\mu| \le n \,,$$

is sufficient to conclude. $\qquad \square$

Since the GW/DT correspondence for toric varieties is a consequence of Proposition 1 and 2, the proof of Theorem 1 is complete.

# 4  Properties of the vertex

## 4.1  Connections

### 4.1.1  Topological vertex

Let $X$ be a local Calabi-Yau toric 3-fold with a 2-dimensional subtorus

$$T_{CY} \subset T$$

preserving the Calabi-Yau form. A procedure for computing the Gromov-Witten series via the topological vertex was proposed by Aganagic, Klemm, Mariño, and Vafa [1]. In [25, 31], the procedure of [1] was shown to exactly compute the Donaldson-Thomas series of $X$. By the GW/DT correspondence of Theorem 1, the two theories coincide.

The 1-leg version of the topological vertex appeared first in [21] and was proven in [19, 28]. The 2-leg case was proven in [20]. Finally, an evaluation of the Gromov-Witten theory of a local Calabi-Yau toric 3-fold $X$ in a form closely resembling the topological vertex was obtained in [17].

### 4.1.2  Stable pairs

The virtual enumeration of stable pairs in the derived category of $X$ was conjectured in [33, 34] to be equivalent to both Gromov-Witten and Donaldson-Thomas theory. After the triangle of equivalences of Section 1.6 is extended to include the stable pairs theory of $\mathcal{A}_n \times \mathbf{P}^1$, our proof of Theorem 1 will also apply to the stable pairs theory of nonsingular toric 3-folds.

## 4.2 Conjectures

### 4.2.1 Chow varieties

Let $X$ be a nonsingular projective 3-fold with very ample line bundle $L$. The Chow variety of curves parameterizes cycles of class $\beta \in H_2(X, \mathbb{Z})$ in $X$. More precisely, we define $\mathrm{Chow}(X, \beta)$ to be the seminormalization of the subvariety of Chow forms associated to the embedding $X \subset \mathbf{P}(H^0(X, L)^\vee)$. The variety $\mathrm{Chow}(X, \beta)$ is independent of $L$. See [12] for a detailed treatment.

For both the moduli space of stable maps $\overline{M}'_g(X, \beta)$ and the Hilbert scheme of curves $I_n(X, \beta)$, the associated seminormalized varieties admit maps to $\mathrm{Chow}(X, \beta)$ for all $g$ and $n$,

$$\overline{M}'_g(X, \beta)_{\mathrm{sn}} \qquad\qquad\qquad I_n(X, \beta)_{\mathrm{sn}} \ .$$

$$\rho_{GW} \searrow \qquad \swarrow \rho_{DT}$$

$$\mathrm{Chow}(X, \beta)$$

As a corollary of our proof of Theorem 1, we immediately obtain the following result.

**Corollary 1.** *Let $X$ be a nonsingular projective toric 3-fold. We have*

$$\sum_g u^{2g-2} \, \rho_{GW*}\left(\left[\overline{M}'_{g,n}(X, \beta)\right]^{\mathrm{vir}}\right) =$$

$$\frac{1}{\mathsf{Z}_{DT}(X, q)_0} \sum_n q^n \, \rho_{DT*}\left([I_n(X, \beta)]^{\mathrm{vir}}\right) \quad (48)$$

*in $H_*(\mathrm{Chow}(X, \beta), \mathbb{Q}) \otimes \mathbb{Q}(q)$, after the variable change $e^{iu} = -q$.*

*Proof.* Define the Chow variety with respect to a $T$-equivariant very ample bundle $L$ on $X$. Then, both $\rho_{GW}$ and $\rho_{DT}$ are also $T$-equivariant, and the respective push-forwards can be calculated by $T$-equivariant localization. The Chow classes of $T$-fixed points in both Gromov-Witten and Donaldson-Thomas theory are determined by the edge degrees. By the GW/DT correspondence for capped vertices and edges, the contributions in both theories are matched when the edge degrees are fixed. $\qquad\square$

Since Chow and homology groups are preserved by seminormalization, the above homological push-forwards $\rho_*$ are well-defined. As the primary

Gromov-Witten and Donaldson-Thomas invariants can be computed in terms of the push-forwards $\rho_*$, Corollary 1 may be viewed as a refinement of Theorem 1.

**Conjecture 1.** *The equivalence of Corollary 1 holds for all nonsingular projective 3-folds $X$.*

Unfortunately, the Chow statement of Corollary 1 does not capture the full GW/DT correspondence as descendent insertions are not pulled-back from the Chow variety.

### 4.2.2 Polynomiality

We have proven the capped vertex is a rational function

$$\mathsf{C}_{DT}(\lambda, \mu, \nu, t_1, t_2, t_3) \in \mathbb{Q}(q, t_1, t_2, t_3).$$

Using a combination of geometry and box counting, we can further prove the following result.[19]

**Theorem 3.** *We have*

$$\mathsf{C}_{DT}(\lambda, \mu, \nu, t_1, t_2, t_3) \in \mathbb{Q}(q) \otimes \mathbb{Q}(t_1, t_2, t_3)$$

*with possible poles in $q$ occuring only at roots of unity and 0.*

An identical statement is obtained for the capped edge by studying the differential equations of the rubber calculus. An analysis of the singularities of the differential equations proves a property encountered earlier by J. Bryan: the capped $(a, b)$-edge $\mathsf{E}_{DT}(\lambda, \mu, t_1, t_2, t_3, t'_1, t'_2)$ is a Laurent polynomial in $q$ if $a, b \geq 0$. We expect polynomiality to also be true for the capped vertex.

**Conjecture 2.** $\mathsf{C}_{DT}(\lambda, \mu, \nu, t_1, t_2, t_3) \in \mathbb{Q}(t_1, t_2, t_3)[q, \frac{1}{q}].$

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[19]The proof together with further properties of the capped vertex will be presented in a future paper.

## 4.3 Calculations

We denote by $\mathsf{C}^{\circ}_{DT}(\lambda, \mu, \nu, t_1, t_2, t_3)$ the connected version of the capped vertex. Omitting the weights $t_i$ from the notation,

$$\sum_{\lambda,\mu,\nu} \mathsf{C}_{DT}(\lambda, \mu, \nu) x^\lambda y^\mu z^\nu = \exp(\sum_{\lambda,\mu,\nu} \mathsf{C}^{\circ}_{DT}(\lambda, \mu, \nu) x^\lambda y^\mu z^\nu),$$

$$x^\lambda = \prod_i x_{\lambda_i}, \quad y^\mu = \prod_i y_{\mu_i}, \quad z^\nu = \prod_i z_{\nu_i}.$$

Let $P_{\lambda,\mu,\nu}$ be the polynomial

$$t_1^{\ell(\mu)+\ell(\nu)} t_2^{\ell(\lambda)+\ell(\nu)} t_3^{\ell(\lambda)+\ell(\mu)} \prod_{i=1}^{|\mu|}\prod_{j=1}^{|\lambda|}(it_1 + jt_2) \prod_{j=1}^{|\nu|}\prod_{k=1}^{|\mu|}(jt_2 + kt_3) \prod_{i=1}^{|\nu|}\prod_{k=1}^{|\lambda|}(it_1 + kt_3).$$

Define $\mathsf{R}(\lambda, \mu, \nu)$ by the formula

$$\mathsf{C}^{\circ}_{DT}(\lambda, \mu, \nu) = \mathsf{R}(\lambda, \mu, \nu) q^{1-|\lambda|-|\mu|+|\nu|}(1 + q)^{|\lambda|+|\mu|+|\nu|-2} \Pi_\lambda \Pi_\mu \Pi_\nu / P_{\lambda,\mu,\nu},$$

$$\Pi_\lambda = \prod_i (1 - (-q)^{\lambda_i})/\mathfrak{z}(\lambda).$$

Our calculations suggest $\mathsf{R}(\lambda, \mu, \nu)$ is a Laurent polynomial of $q$.

The formula for $\mathsf{R}$ is known in the 1-leg case,

$$\mathsf{R}(\lambda, \emptyset, \emptyset) = 1, \text{ if } |\lambda| \le 1, \text{ and } \mathsf{R}(\lambda, \emptyset, \emptyset) = 0 \text{ otherwise.}$$

We can also prove

$$\mathsf{R}(\lambda, [1], \emptyset) = t_3^{\ell(\lambda)}, \text{ if } |\lambda| > 0.$$

Below we give some further values of $\mathsf{R}(\lambda, \mu, \nu)$ with small partitions.

| $\lambda, \mu, \nu$ | $\mathsf{R}(\lambda, \mu, \nu)$ |
|---|---|
| $1^2, 1^2, \emptyset$ | $((t_1 + t_2 - t_3)(q + q^{-1}) + (-10t_2 - 10t_1 - 2t_3))t_3^3$ |
| $1^2, 2, \emptyset$ | $((t_1 + t_2 - t_3)(q + q^{-1}) + (-6t_1 - 2t_3 - 8t_2))t_3^2$ |
| $2, 2, \emptyset$ | $((t_1 + t_2 - t_3)(q + q^{-1}) + (-2t_3 - 4t_2 - 4t_1))t_3$ |
| $1, 1, 1$ | $(t_1 + t_2)(t_2 + t_3)(t_1 + t_3)$ |
| $2, 1, 1$ | $(t_1 + t_2 + t_3)(t_1 + 2t_2)(t_2 + t_3)(t_1 + 2t_3)$ |
| $1^2, 1, 1$ | $(t_3^2 + t_1 t_3 + t_2^2 + t_2 t_3 + t_1 t_2)(t_1 + 2t_2)(t_2 + t_3)(t_1 + 2t_3)$ |

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