LOCALIZATION OF VIRTUAL CLASSES

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0. Introduction

We prove a localization formula for the virtual fundamental class in the general context of $\mathbb{C}^*$-equivariant perfect obstruction theories. Let X be an algebraic scheme with a $\mathbb{C}^*$-action and a $\mathbb{C}^*$-equivariant perfect obstruction theory. The virtual fundamental class $[X]^{vir}$ in the expected equivariant Chow group $A_*^{\mathbb{C}^*}(X)$ may be constructed by the methods of Li-Tian [LT] and Behrend-Fantechi [B], [BF]. Each connected component X_i of the fixed point scheme carries an associated $\mathbb{C}^*$-fixed perfect obstruction theory. A virtual fundamental class in $A_*(X_i)$ is thus determined. The virtual normal bundle to X_i is obtained from the moving part of the virtual tangent space determined by the obstruction theory. The localization formula is then:

$$[X]^{vir} = \iota_* \sum \frac{[X_i]^{vir}}{e(N_i^{vir})} \tag{1}$$

in $A_*^{\mathbb{C}^*}(X) \otimes \mathbb{Q}[t, \frac{1}{t}]$ where t is the generator of the $\mathbb{C}^*$ -equivariant ring of a point. This localization formula is the main result of the paper. The proof requires an additional hypothesis on X: the existence of a $\mathbb{C}^*$-equivariant embedding in a nonsingular variety Y.

In case X is nonsingular with the trivial perfect obstruction theory, equation (1) reduces immediately to the standard localization formula [Bo], [AB]. Originally, this localization was proven in equivariant cohomology. Algebraic localization in equivariant Chow theory has recently been established in [EG2]. The point of view of our paper is entirely algebraic.

The definitions and constructions related to the virtual localization formula (1) are discussed in Section 1. The simplest example of a $\mathbb{C}^*$-equivariant perfect obstruction theory is given by the following situation. Let Y be a nonsingular algebraic variety with a $\mathbb{C}^*$-action. Let V be a $\mathbb{C}^*$-equivariant bundle on Y. Let $v \in H^0(Y, V)^{\mathbb{C}^*}$ be a $\mathbb{C}^*$-fixed section. Let X be the scheme-theoretic zero locus of v. X is naturally endowed with an equivariant perfect obstruction theory which yields

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the refined Euler class (top Chern class) of V as the virtual fundamental class. The localization formula in this basic setting is proven in Section 2. The method is to deduce (1) for X from the known localization formula for the nonsingular variety Y.

The proof of (1) for general $\mathbb{C}^*$-equivariant perfect obstruction theories on an algebraic scheme X proceeds in a similar manner. Again, formula (1) is deduced from the ambient localization formula for Y. The argument here is more subtle: explicit manipulation of cones and a rational equivalence due to Vistoli [V] are necessary. This proof is given in Section 3.

There are two immediate applications of the virtual localization formula. First, a local complete intersection scheme is endowed with a canonical perfect obstruction theory obtained from the cotangent complex. A localization formula is thus obtained for these singular schemes (at least when equivariant embeddings in nonsingular varieties exist). Second, the proper Deligne-Mumford moduli stack $\overline{M}_{g,n}(V,\beta)$ of stable maps to a nonsingular projective variety V is equipped with a canonical perfect obstruction theory. If V has a $\mathbb{C}^*$-action, then a natural $\mathbb{C}^*$-action on $\overline{M}_{g,n}(V,\beta)$ is defined by translation of the map. An equivariant perfect obstruction theory on $\overline{M}_{g,n}(V,\beta)$ can be obtained. Moreover, $\overline{M}_{g,n}(V,\beta)$ admits an equivariant embedding in a nonsingular Deligne-Mumford stack. As a result, the virtual localization formula holds for $\overline{M}_{g,n}(V,\beta)$.

In the last two sections of the paper, consequences of the localization formula in Gromov-Witten theory are explored. In Section 4, an explicit graph summation formula for the Gromov-Witten invariants (for all genera) of $\mathbf{P}^r$ is presented via localization on the moduli space of maps $\overline{M}_{g,n}(\mathbf{P}^r,d)$. The invariants are expressed as a sum over graphs corresponding to the fixed point loci. For each graph, the summand is a product over vertex terms. The vertex terms are integrals over associated spaces $\overline{M}_{g',n'}$ of the Chern classes of the cotangent line bundles and the Hodge bundle. All these integrals may be calculated from Witten's conjectures (Kontsevich's theorem) by a method due to Faber [Fa]. Similar graph sum formulas exist for Gromov-Witten invariants (and their descendents) of all nonsingular projective toric varieties and all compact algebraic homogeneous spaces $\mathbf{G}/\mathbf{P}$. The localization formula also determines the full system of Gromov-Witten invariants (of all genus) in the sense of [KM]. This system includes the top dimension numerical invariants as well as higher codimension cohomology classes in $\overline{M}_{g,n}$. A corollary of localization is the following result: the full system of Gromov-Witten invariants in these cases lies in the tautological rings of the moduli spaces $\overline{M}_{g,n}$.

The positive degree Gromov-Witten invariants of $\mathbf{P}^2$ coincide with enumerative geometry: they count the number N^g_d of genus g, degree d, nodal plane curves passing through $3d+g-1$ points in $\mathbf{P}^2$. Localization presents a solution of this enumerative geometry problem via integrals of tautological classes over the moduli space of pointed curves. The numbers N^g_d have been computed via more classical degeneration methods in [R1], [CH]. The character of the solutions in [R1], [CH] is markedly different: it is by recursion over wider classes of enumerative questions.

In Section 5, localization is applied to a question suggested to us by S. Katz: the calculation of excess integrals on the moduli spaces $\overline{M}_{g,0}(\mathbf{P}^1, d)$ that arise in the study of Calabi-Yau 3-folds. Under suitable conditions, the integral

$$\int_{[\overline{M}_{g,0}(\mathbf{P}^1,d)]^{vir}} c_{\text{top}}(R^1\pi_*\mu^*N) \tag{2}$$

is the contribution to the genus g Gromov-Witten invariant of a Calabi-Yau 3-fold of multiple covers of a fixed rational curve (with normal bundle $N = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$). In [M], the integral (2) is explicitly evaluated to be $1/d^3$ in the genus $g = 0$ case via localization on the nonsingular stack $\overline{M}_{0,0}(\mathbf{P}^1, d)$. A trick of setting one of the $\mathbb{C}^*$-weights on $\mathbf{P}^1$ to be 0 is used. We evaluate the excess integral in the genus $g = 1$ case in Section 5 via virtual localization on $\overline{M}_{1,0}(\mathbf{P}^1, d)$. Manin's trick [M] and formulas for cotangent line integrals on $\overline{M}_{1,n}$ are used to handle the graph sum. The answer obtained, $1/12d$, agrees with the physics result of [BCOV]. The higher genus integrals may be explicitly evaluated in any given case by virtual localization and the algorithm implemented by Faber [Fa] to calculate the vertex integrals. The conjecture obtained from these calculations is: for $g \geq 2$,

$$\int_{[\overline{M}_{g,0}(\mathbf{P}^1,d)]^{vir}} c_{\text{top}}(R^1\pi_*\mu^*N) = \frac{|B_{2g}| \cdot d^{2g-3}}{2g \cdot (2g-2)!} = |\chi(M_g)| \frac{d^{2g-3}}{(2g-3)!} \tag{3}$$

where B_{2g} is the $2g^{th}$ Bernoulli number and $\chi(M_g) = B_{2g}/2g(2g-2)$ is the orbifold Euler characteristic of M_g. This conjecture was made jointly with C. Faber. We have not yet been able to evaluate the graph sums uniformly to establish (3).

The localization formula and graph sum formulas were first introduced in the context of stable maps by Kontsevich in [K] following related work of Ellingsrud and Strømme [ES]. Kontsevich studied the convex genus 0 case where the moduli spaces are nonsingular Deligne-Mumford stacks. Many ideas about the virtual fundamental class and localization described here are implicit in [K]. In particular, the higher genus formulas of Section 4 are identical to the genus 0 formulas of [K] except for the new Hodge bundle terms. However, the higher genus map spaces are in general nonreduced, reducible, and singular, so the virtual localization formula (1) is essential. Givental has stated a localization axiom for genus 0 Gromov-Witten invariants of toric varieties in [G] which follows from (1). Localization formulas are used in [G] to prove predictions of mirror symmetry in the case of Calabi-Yau complete intersections in toric varieties. A different approach to localization in the case of $\overline{M}_{g,n}(\mathbf{P}^r, d)$ has been pursued by Behrend via a factoring of the fixed point inclusion through Artin stacks.

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1. **The virtual localization formula**

Let X be an algebraic scheme over $\mathbb{C}$. A perfect obstruction theory consists of the following data:

(i) A two term complex of vector bundles $E^\bullet = [E^{-1} \to E^0]$ on X.

(ii) A morphism ϕ in the derived category (of quasi-coherent sheaf complexes bounded from above) from $E^\bullet$ to the cotangent complex $L^\bullet X$ of X satisfying two properties.

  (a) ϕ induces an isomorphism in cohomology in degree 0.

  (b) ϕ induces a surjection in cohomology in degree -1.

The constructions of [LT], [BF] give rise to a virtual fundamental class, $[X]^{vir}$ in $A_d(X)$ where $d = \mathrm{rk}(E^0) - \mathrm{rk}(E^{-1})$ is the expected dimension. Let $E_\bullet = [E_0 \to E_1]$ denote the dual complex of $E^\bullet$.

If X admits a global closed embedding in a nonsingular scheme (or Deligne-Mumford stack) Y, one can give a relatively straightforward construction of the virtual class as follows. In this situation, the two term cut-off of the cotangent complex can be taken to be:

$$L^\bullet X = [I/I^2 \to \Omega_Y]$$

where I is the ideal sheaf of X in Y. Since only this cut-off will be used, the cotangent complex will be identified with its cut-off throughout this section. We assume for simplicity that

$$\phi : E^\bullet \to [I/I^2 \to \Omega_Y] \tag{4}$$

is an actual map of complexes. This hypothesis is not required for the constructions of [LT], [BF]. However, if X has enough locally frees, such a representative $(E^\bullet, \phi)$ may always be chosen in the derived category.

The mapping cone associated to the morphism ϕ of complexes yields an exact sequence of sheaves:

$$E^{-1} \to E^0 \oplus I/I^2 \xrightarrow{\gamma} \Omega_Y \to 0. \tag{5}$$

In fact, ϕ satisfies (a) and (b) if and only if (5) is exact. We consider the associated exact sequence of abelian cones

$$0 \to TY \to C(I/I^2) \times_X E_0 \to C(Q) \to 0 \tag{6}$$

where $C(Q)$ is the cone associated to the kernel Q of γ. As Q is a quotient of E^{-1}, $C(Q)$ embeds in E_1. The normal cone of X in Y, $C_{X/Y}$, is naturally a closed subscheme of $C(I/I^2)$. If we define $D = C_{X/Y} \times_X E_0$, then D is a TY-cone (see [BF]), and the quotient of D by TY is a subcone of $C(Q)$ which we will denote by D^{vir}. The virtual fundamental class of X associated to this obstruction theory is then the refined intersection of D^{vir} with the zero section of the vector bundle E_1.

If $\mathbf{G}$ acts algebraically on X, then there is a theory of equivariant Chow groups $A_*^{\mathbf{G}}(X)$, [EG1], [To]. These groups are defined to be Chow groups of suitable algebraic approximations to the homotopy quotient $X \times E\mathbf{G}/\mathbf{G}$. A $\mathbf{G}$-invariant cycle naturally yields a class in $A_*^{\mathbf{G}}(X)$. Given a $\mathbf{G}$-equivariant vector bundle B over X, the standard constructions hold for equivariant Chow groups: for example, Chern classes and the refined Gysin homomorphism.

Suppose $X \subset Y$ is equipped with an equivariant $\mathbf{G}$-action together with a lifting to the complex $E^\bullet$ such that ϕ is a morphism in the derived category of $\mathbf{G}$-equivariant sheaves (with respect to the natural $\mathbf{G}$-action on $L^\bullet X$). The above construction then yields an equivariant virtual fundamental class in the equivariant Chow group $A_d^{\mathbf{G}}(X)$ since the cones used are invariant. In fact, to define the equivariant virtual class, global equivariant embeddings are not necessary.

We now assume the group $\mathbf{G}$ is the torus $\mathbb{C}^*$. We expect to be able to reduce integrals over $[X]^{vir}$ to integrals over the fixed point set. Let X^f be the maximal $\mathbb{C}^*$-fixed closed subscheme of X. X^f is the natural scheme theoretic fixed point locus. If $X = \mathrm{Spec}(A)$, then the ideal of X^f is generated by the $\mathbb{C}^*$-eigenfunctions with nontrivial characters. For nonsingular Y, Y^f is the nonsingular set theoretic fixed point locus [I]. For $X \subset Y$, the relation $X^f = X \cap Y^f$ holds. We let $Y^f = \bigcup Y_i$ be the decomposition into irreducible components. Let $X_i = X \cap Y_i$. X_i is possibly reducible.

Let S be a coherent sheaf on a fixed component X_i with a $\mathbb{C}^*$-action. S decomposes as direct sum,

$$S = \bigoplus_{k \in \mathbb{Z}} S^k,$$

of $\mathbb{C}^*$-eigensheaves of $\mathcal{O}_{X_i}$-modules. If S is locally free, each summand is also locally free. We denote the fixed subsheaf S^0 by S^f and the moving subsheaf $\oplus_{k \neq 0} S^k$ by S^m.

There is a natural isomorphism $\Omega_Y|_{Y_i}^f = \Omega_{Y_i}$ [I]. It is easy to then deduce:

$$\Omega_X|_{X_i}^f = \Omega_{X_i}$$

from the equality $X_i = X \cap Y_i$.

Let $E_i^\bullet$ denote the restriction of $E^\bullet$ to X_i. Let $E_i^{\bullet,f}$ denote the fixed part of the complex $E_i^\bullet$. $E_i^{\bullet,f}$ is a two term complex of bundles. There exists a canonical map,

$$\psi_i : E_i^{\bullet,f} \to L^\bullet X_i,$$

determined by the following construction. Let $\phi_i : E_i^\bullet \to L^\bullet X|_{X_i}$ be the pull-back of ϕ, and let $\phi_i^f : E_i^{\bullet,f} \to L^\bullet X|_{X_i}^f$ be the associated fixed map. Similarly let $\delta_i : L^\bullet X|_{X_i} \to L^\bullet X_i$ be the canonical morphism, and let δ_i^f be the associated fixed map. Then, $\psi_i = \delta_i^f \circ \phi_i^f$.

Proposition 1. *The map $\psi_i : E_i^{\bullet,f} \to L^\bullet X_i$ is a canonical perfect obstruction theory on X_i.*

Proof. To show ψ_i satisfies properties (a) and (b), it suffices to show both maps ϕ_i^f and δ_i^f satisfy these properties.

A map of complexes $\nu : A^\bullet \to B^\bullet$ satisfies (a) and (b) if and only if the sequence

$$A^{-1} \oplus B^{-2} \to A^0 \oplus B^{-1} \to B^0 \to 0$$

is exact. Since tensor product is right exact, the joint validity of (a) and (b) is preserved under pull-back. As ϕ is a perfect obstruction theory, ϕ_i satisfies (a) and (b). The fixed map ϕ_i^f also satisfies properties (a) and (b) since taking invariants is exact.

The cotangent complex of X_i can be represented by the embedding $X_i \subset Y_i$:

$$L^\bullet X_i = [I_{X_i/Y_i}/I^2_{X_i/Y_i} \to \Omega_{Y_i}|_{X_i}].$$

The zeroth cohomology of $L^\bullet X|^f_{X_i}$ is $\Omega_X|^f_{X_i} = \Omega_{X_i}$. Thus, δ^f_i satisfies property (a).

Property (b) for δ^f_i will now be established. The map δ^f_i is represented by the natural diagram:

$$\begin{array}{ccc} I_{X/Y}/I^2_{X/Y}|^f_{X_i} & \longrightarrow & \Omega_Y|^f_{X_i} \\ {\scriptstyle d^{-1}}\big\downarrow & & {\scriptstyle d^0}\big\downarrow \\ I_{X_i/Y_i}/I^2_{X_i/Y_i} & \longrightarrow & \Omega_{Y_i}|_{X_i} \end{array}$$

Since $X_i = X \cap Y_i$, the map

$$I_{X/Y}/I^2_{X/Y}|_{X_i} \to I_{X_i/Y_i}/I^2_{X_i/Y_i}$$

is surjective. Hence, d^{-1} is surjective. As d^0 is an isomorphism, δ^f_i is surjective on cohomology in degree -1. $\square$

The virtual structure on X_i is defined to be the one induced by the perfect obstruction theory $\psi_i : E^{\bullet,f}_i \to L^\bullet X_i$.

We define the virtual normal bundle, N^{vir}_i to X_i to be the moving part of $E_{\bullet,i}$. Note that $E_{\bullet,i}$ is a complex, and not a single bundle. Also note that in the non-virtual case, when the complex has just one term, this coincides with the usual normal bundle. Define the Euler class (top Chern class) of a two term complex $[B_0 \to B_1]$ to be the ratio of the Euler classes of the two bundles: $e(B_0)/e(B_1)$. We arrive at the following natural formulation of the virtual Bott residue formula for the Euler class of a bundle A of rank equal to the virtual dimension of X:

$$\int_{[X]^{vir}} e(A) = \sum \int_{[X_i]^{vir}} \frac{e(A_i)}{e(N^{vir}_i)} \tag{7}$$

where the Euler classes on the right hand side are equivariant classes. Since N^{vir}_i is a complex of bundles with nonzero $\mathbb{C}^*$-weights, the Euler class $e(N^{vir}_i)$ is invertible in the localized ring

$$A^{\mathbb{C}^*}_*(X)_t = A^{\mathbb{C}^*}_*(X) \otimes_{\mathbb{Q}[t]} \mathbb{Q}[t, \frac{1}{t}].$$

Chow groups will always be taken with $\mathbb{Q}$-coefficients. As in the case of the standard Bott residue formula, equation (7) should be a consequence of a localization formula in equivariant Chow groups. On a nonsingular variety Y, the fundamental result which immediately implies the residue formula is:

$$[Y] = \iota_* \sum \frac{[Y_i]}{e(N_i)}$$

in $A_*^{\mathbb{C}^*}(Y)_t$. The obvious generalization to the virtual setting which would just as readily imply the virtual residue formula is:

$$[X]^{vir} = \iota_* \sum \frac{[X_i]^{vir}}{e(N_i^{vir})}. \tag{8}$$

It is worth remarking that in the case of most interest to us, the moduli space of maps to projective space, the right side of (8) is directly accessible. In this case, three special properties hold. First, the fixed loci X_i for a general $\mathbb{C}^*$-action have been identified by Kontsevich in [K]: they are indexed by graphs and are essentially products of Deligne-Mumford moduli spaces of pointed curves. Second, $[X_i]^{vir} = [X_i]$. Finally, $e(N_i^{vir})$ is expressible in terms of tautological classes on X_i via the deformation theory of curves and maps. Thus (8) provides a concrete way to calculate virtual integrals on moduli spaces where it seems quite difficult to directly evaluate the virtual fundamental class.

2. **Proof in the basic case**

As a first motivational step, we prove the virtual localization formula (1) in the following situation. Let Y be a nonsingular variety equipped with a $\mathbb{C}^*$-action, a $\mathbb{C}^*$-equivariant bundle V, and an invariant section v of V. The zero scheme X of v carries a natural equivariant perfect obstruction theory. The two term complex of bundles on X,

$$E^\bullet = [V^\vee \to \Omega_Y],$$

is obtained from the the section v. The required morphism to the cotangent complex $L^\bullet X = [I/I^2 \to \Omega_Y]$ is obtained from the natural map $V^\vee \to I/I^2$ on X. The definitions show the virtual fundamental class in this case is just the refined Euler class of V. That is, if we consider the graph of the section, and take its refined intersection product with the zero section, we get a Chow homology class supported on the zero locus X. The definitions of the virtual fundamental class for general spaces are specifically designed to recover this refined Euler class from the local data of the two term complex, and to generalize this class in cases where such a geometric realization of the deformation complex does not necessarily exist.

In this basic situation, we can express all of the virtual objects in the localization formula in terms of familiar data on Y. As in Section 1, we denote the components of the fixed locus of Y by Y_i. V splits into eigenbundles on Y_i. Since v is a $\mathbb{C}^*$-invariant section, it necessarily restricts to a section of the weight 0 bundle V_i^f. Y_i is a smooth manifold with a vector bundle and a section which vanishes exactly on $X_i =$

$X \cap Y_i$. The associated $\mathbb{C}^*$-fixed obstruction theory defined in Section 1,

$$[(V_i^f)^\vee \to \Omega_{Y_i}],$$

is exactly the perfect obstruction theory obtained from the pair V_i^f and $v \in H^0(Y_i, V_i^f)$. Note the maps to the cotangent complex must be checked to agree. It follows that the virtual fundamental class of X_i is the same as the refined Euler class of V_i^f on Y_i.

The virtual normal bundle is by definition the moving part of the complex $[TY \to V]$. The moving part of TY is just the normal bundle to Y_i. Hence N_i^{vir} is the complex $[N_{Y_i/Y} \to V_i^m]$. By the definition of Euler class of a complex, we obtain:

$$e(N_i^{vir}) = \frac{e(N_{Y_i/Y})}{e(V_i^m)}.$$

After substituting this expression for $e(N_i^{vir})$ into the virtual localization formula (1), we see the equality we want to prove in $A_*^{\mathbb{C}^*}(X)_t$ is:

$$e_{\mathrm{ref}}(V) = \iota_* \sum \frac{e_{\mathrm{ref}}(V_i^f) \cap e(V_i^m)}{e(N_{Y_i/Y})} \tag{9}$$

where $e_{\mathrm{ref}}(V)$ is the refined Euler class of V as a Chow homology class on X. We know by the localization formula on Y:

$$[Y] = \iota_* \sum \frac{[Y_i]}{e(N_{Y_i/Y})}.$$

Intersecting both sides with $e_{\mathrm{ref}}(V)$ yields:

$$e_{\mathrm{ref}}(V) = \iota_* \sum \frac{e_{\mathrm{ref}}(V) \cap [Y_i]}{e(N_{Y_i/Y})}.$$

Since taking refined Euler class commutes with pullback, the numerators on the right hand side are just the refined Euler classes of V_i. On each component, we have the splitting $V_i = V_i^f \oplus V_i^m$. Since the section lives entirely in V_i^f, it follows that $e_{\mathrm{ref}}(V_i) = e_{\mathrm{ref}}(V_i^f) \cap e(V_i^m)$. Formula (9) is thus obtained. The proof of (1) in the basic case is complete.

3. **Proof in the general case**

In this section, we prove the virtual localization formula for an arbitrary scheme X which admits an equivariant embedding in a nonsingular scheme Y. Recall from the construction of the virtual class in Section 1, the two cones D and D^{vir} satisfy:

$$0 \to TY \to D \to D^{vir} \to 0 \tag{10}$$

$$D = C_{X/Y} \times E_0. \tag{11}$$

D^{vir} is a embedded as a closed subcone of E_1. The virtual class is defined by $[X]^{vir} = s^*_{E_1}[D^{vir}]$. Alternatively, there is a fiber square:

$$\begin{array}{ccc} TY & \longrightarrow & D \\ \downarrow & & \downarrow \\ X & \xrightarrow{0_{E_1}} & E_1 \end{array} \tag{12}$$

where the bottom map is the zero section. Then, $[X]^{vir} = s^*_{TY} 0^!_{E_1}[D]$.

Let $X_i = X \cap Y_i$ be defined as in Section 2. X_i is a union of connected components. $\mathbb{C}^*$-fixed analogues of (10) and (11) hold for the embeddings $X_i \subset Y_i$:

$$0 \to TY_i \to D_i \to D_i^{vir} \to 0,$$

$$D_i = C_{X_i/Y_i} \times E_0^f.$$

D_i^{vir} is a embedded as a closed subcone of E_1^f, and $[X_i]^{vir} = s^*_{E_1^f}(D_i^{vir})$. Since X_i is possibly disconnected, it should be noted that the ranks of the bundles $E^f_{0,i}$ and $E^f_{1,i}$ may vary on the connected components. The Euler classes of these bundles on X_i are taken with respect to their ranks on each component. For notational convenience, the restriction subscript i will be dropped. Similarly, the pull-backs of TY and TY_i to X_i will be denoted by TY and TY_i.

The virtual localization formula for X will be deduced from localization for Y. We start with the equality

$$[Y] = \iota_* \sum \frac{[Y_i]}{e(TY^m)}$$

in $A^{\mathbb{C}^*}_*(Y)_t$. The refined intersection product with $[X]^{vir}$ yields:

$$[X]^{vir} = \iota_* \sum \frac{[X]^{vir} \cdot [Y_i]}{e(TY^m)}$$

in $A^{\mathbb{C}^*}_*(X)_t$. Comparing this equation with our desired virtual localization formula, we see that it suffices to establish:

$$\frac{[X]^{vir} \cdot [Y_i]}{e(TY^m)} = \frac{[X_i]^{vir} \cap e(E_1^m)}{e(E_0^m)} \tag{13}$$

in $A^{\mathbb{C}^*}(X_i)_t$. The refined intersection of a basic linear equivalence due to Vistoli [V], [Kr] with the zero section of a bundle will yield equation (13).

We first review Vistoli's rational equivalence. Consider the following Cartesian diagram:

$$
\begin{array}{ccc}
\iota^* C_{X/Y} & \longrightarrow & C_{X/Y} \\
\downarrow & & \downarrow \\
X_i & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y_i & \xrightarrow{\iota} & Y
\end{array} \tag{14}
$$

The cone C_{X_i/Y_i} naturally embeds in $\iota^* C_{X/Y}$. Vistoli [V] p. 641, has constructed a rational equivalence in $N_{Y_i/Y} \times \iota^* C_{X/Y}$ which implies

$$\iota^! [C_{X/Y}] = [C_{X_i/Y_i}] \tag{15}$$

in $A_*(\iota^* C_{X/Y})$ (see [BF]). Applying Vistoli's equivalence to the $\mathbb{C}^*$-homotopy quotients yields equation (15) in $A_*^{\mathbb{C}^*}(\iota^* C_{X/Y})$. We will consider the pull-back of this relation to $\iota^* D = \iota^* C_{X/Y} \times E_0$:

$$\iota^! [D] = [D_i \times E_0^m] \tag{16}$$

in $A_*^{\mathbb{C}^*}(\iota^* D)$.

Consider the pull-back of the exact sequence of abelian cones (6) to X_i:

$$0 \to TY \to \iota^* C(I/I^2) \times E_0 \to \iota^* C(Q) \to 0.$$

There is an inclusion $\iota^* C(Q) \subset E_1$. The natural inclusion $\iota^* D \subset \iota^* C(I/I^2) \times E_0$ is TY-invariant. Hence, the quotient cones

$$\iota^* D / TY_i \to \iota^* D / TY \subset \iota^* C(Q)$$

exist. We obtain a three level Cartesian diagram:

$$
\begin{array}{ccc}
TY & \longrightarrow & \iota^* D \\
\downarrow & & \downarrow \\
TY/TY_i & \longrightarrow & \iota^* D / TY_i \\
\downarrow & & \downarrow \\
X_i & \xrightarrow{0_{E_1}} & E_1
\end{array} \tag{17}
$$

Note that $TY/TY_i = TY^m$.

We now start the derivation of equation (13). The first steps are:

$$
\begin{aligned}
[X]^{vir} \cdot [Y_i] &= \iota^! s_{TY}^* 0_{E_1}^! [D] \\
&= s_{TY}^* 0_{E_1}^! \iota^! [D] \\
&= s_{TY}^* 0_{E_1}^! [D_i \times E_0^m]
\end{aligned}
$$

in $A_*^{\mathbb{C}^*}(X_i)$. The first equality is by the definition of $[X]^{vir}$. The second is obtained from the commutativity of the intersection product. The third follows from equation (16).

The TY_i-action on $\iota^* D$ leaves the cycle $D_i \times E_0^m$ invariant (since TY_i acts naturally on D_i and trivially on E_0^m). By definition,

$$D_i/TY_i = D_i^{vir}.$$

The class $[D_i \times E_0^m] \in A_*^{\mathbb{C}^*}(\iota^* D)$ is thus the pull-back of $[D_i^{vir} \times E_0^m] \in A_*^{\mathbb{C}^*}(\iota^* D/TY_i)$. Hence,

$$s_{TY}^* 0_{E_1}^! [D_i \times E_0^m] = s_{TY^m}^* 0_{E_1}^! [D_i^{vir} \times E_0^m].$$

The scheme-theoretic intersection $0_{E_1}^{-1}(D_i^{vir} \times E_0^m)$ certainly lies in TY^m. The map

$$D_i^{vir} \times E_0^m \to E_1$$

is the product of the inclusion $D_i^{vir} \subset E_1^f$ and the natural map from the obstruction theory $E_0^m \to E_1^m$. We thus observe $0_{E_1}^{-1}(D_i^{vir} \times E_0^m)$ also lies in E_0^m. We conclude the existence of the following diagram:

$$\begin{array}{ccc} 0_{E_1}^{-1}(D_i^{vir} \times E_0^m) & \longrightarrow & E_0^m \\ \downarrow & & \downarrow \\ TY^m & \longrightarrow & X_i \end{array} \tag{18}$$

To proceed, we need a relation among Gysin maps.

Lemma 1. *Let B_0 and B_1 be $\mathbb{C}^*$-equivariant bundles on X_i. Let Z be a scheme equipped with two equivariant inclusions j_0, j_1 over X_i:*

$$\begin{array}{ccc} Z & \longrightarrow & B_1 \\ \downarrow & & \downarrow \\ B_0 & \longrightarrow & X_i \end{array} \tag{19}$$

Let $\zeta \in A_*^{\mathbb{C}^*}(Z)$. Then,

$$s_{B_0}^* j_{0*}(\zeta) \cap e(B_1) = s_{B_1}^* j_{1*}(\zeta) \cap e(B_0) \quad \in A_*^{\mathbb{C}^*}(X_i).$$

Proof. Consider the family of inclusions $j_t : Z \hookrightarrow B_0 \times B_1$ defined for $t \in \mathbb{C}$ by:

$$j_t = (1-t) \cdot j_0 + t \cdot j_1.$$

The existence of this family implies:

$$s_{B_0 \times B_1}^* j_{0*}(\zeta) = s_{B_0 \times B_1}^* j_{1*}(\zeta).$$

This yields the Lemma by the excess intersection formula. $\square$

Applying Lemma 1 to diagram (18) and the class $\zeta = 0^!_{E_1}[D_i^{vir} \times E_0^m]$ yields:

$$(20) \qquad [X]^{vir} \cdot [Y_i] = s^*_{E_0^m}\left(0^!_{E_1}[D_i^{vir} \times E_0^m]\right) \cdot \frac{e(TY^m)}{e(E_0^m)}.$$

The class $0^!_{E_1}[D_i^{vir} \times E_0^m]$ is now considered to lie in $A_*^{\mathbb{C}^*}(E_0^m)$. As this class does not depend on the bundle map

$$(21) \qquad E_0^m \to E_1^m,$$

we are free to assume (21) is trivial. Then, the equality

$$(22) \qquad s^*_{E_0^m}\left(0^!_{E_1}[D_i^{vir} \times E_0^m]\right) = [X_i]^{vir} \cap e(E_1^m)$$

follows easily from the definition of $[X_i]^{vir}$ and the excess intersection formula. Equation (13) is a consequence of (20) and (22). The proof of the virtual localization formula is complete.

4. **The formula for $\mathbf{P}^r$**

We can use the virtual localization formula (1) to derive an expression for the higher genus Gromov-Witten invariants of projective space analogous to the one given for genus 0 invariants in [K]. The additional arguments needed to justify formula (1) in the category of Deligne-Mumford stacks for the moduli space of maps are given in the Appendices.

We first establish our conventions about the torus action on projective space. Let $V = \mathbb{C}^{r+1}$. Let $p_i \in \mathbf{P}(V)$ be the points determined by the basis vectors. Let $\mathbb{C}^*$ act on V with generic weights $-\lambda_0, \dots, -\lambda_r$. Then, the induced action on the tangent space to $\mathbf{P}(V)$ at p_i has weights $\lambda_i - \lambda_j$ for $j \neq i$.

Let $\mathbf{T}$ be the full diagonal torus acting on $\mathbf{P}^r$. Following [K], we can identify the components of the fixed point locus of the $\mathbf{T}$-action on $\overline{M}_{g,n}(\mathbf{P}^r, d)$ with certain types of marked graphs. Let $f : C \to \mathbf{P}^r$ be a $\mathbf{T}$-fixed stable map. The image of C is a $\mathbf{T}$-invariant curve in $\mathbf{P}(V)$, and the images of all marked points, nodes, contracted components, and ramification points are $\mathbf{T}$-fixed points. The $\mathbf{T}$-fixed points on $\mathbf{P}^r$ are $p_0, \dots, p_r$, and the only invariant curves are the lines joining the points p_i. It follows that each non-contracted component of C must map onto one of these lines, and be ramified only over the two fixed points. This forces such a component to be rational, and the map restricted to this component is completely determined by its degree.

We are led to identify the components of the fixed locus with marked graphs. To an invariant stable map f, we associate a marked graph Γ as follows. Γ has one edge for each non-contracted component. The edge e is marked with the degree d_e of the map from that component to its image line. Γ has one vertex for each connected component of $f^{-1}(\{p_0, \ldots, p_r\})$. Define the labeling map

$$i : \text{Vertices} \to \{0, \ldots, r\}$$

by $f(v) = p_{i(v)}$. The vertices have an additional labeling $g(v)$ by the arithmetic genus of the associated component. (Note the component may be a single point, in which case its genus is 0.) Finally, Γ has n numbered legs coming from the n marked points. These legs are attached to the appropriate vertex. An edge is incident to a vertex if the two associated subschemes of C are incident.

The set of all invariant stable maps whose associated graph is Γ is naturally identified with a finite quotient of a product of moduli spaces of pointed curves. Define:

$$\overline{M}_\Gamma = \prod_{\text{vertices}} \overline{M}_{g(v),\text{val}(v)}.$$

$\overline{M}_{0,1}$ and $\overline{M}_{0,2}$ are interpreted as points in this product. Over the Deligne-Mumford stack $\overline{M}_\Gamma$, there is a canonical family

$$\pi : \mathcal{C} \to \overline{M}_\Gamma$$

of $\mathbf{T}$-fixed stable maps to $\mathbf{P}^r$ yielding a morphism

$$\gamma : \overline{M}_\Gamma \to \overline{M}_{g,n}(\mathbf{P}^r, d).$$

There is a natural automorphism group $\mathbf{A}$ acting π-equivariantly on $\mathcal{C}$ and $\overline{M}_\Gamma$. $\mathbf{A}$ is filtered by an exact sequence of groups:

$$1 \to \prod_{\text{edges}} \mathbb{Z}/d_e \to \mathbf{A} \to \text{Aut}(\Gamma) \to 1$$

where $\text{Aut}(\Gamma)$ is the automorphism group of Γ (as a marked graph). $\text{Aut}(\Gamma)$ naturally acts on $\prod_{\text{edges}} \mathbb{Z}/d_e$ and $\mathbf{A}$ is the semidirect product. The induced map:

$$\gamma/\mathbf{A} : \overline{M}_\Gamma/\mathbf{A} \to \overline{M}_{g,n}(\mathbf{P}^r, d)$$

is a closed immersion of Deligne-Mumford stacks. It should be noted that the subgroup $\prod_{\text{edges}} \mathbb{Z}/d_e$ acts trivially on $\overline{M}_\Gamma$ and that $\overline{M}_\Gamma/\mathbf{A}$ is nonsingular. A component of the $\mathbf{T}$-fixed stack of $\overline{M}_{g,n}(\mathbf{P}^r, d)$ is supported on $\overline{M}_\Gamma/\mathbf{A}$. The fixed stack will be shown to be nonsingular by analysis of the fixed perfect obstruction theory which yields the Zariski tangent space. This nonsingularity is surprising since the moduli stack $\overline{M}_{g,n}(\mathbf{P}^r, d)$ is singular. A generic $\mathbb{C}^* \subset \mathbf{T}$ will have the same fixed point loci in $\overline{M}_{g,n}(\mathbf{P}^r, d)$. Via this fixed point identification, the virtual localization formula will relate the Gromov-Witten invariants of $\mathbf{P}^r$ to integrals over moduli spaces of pointed curves.

Following [K], we define a flag F of the graph Γ to be an incident edge-vertex pair (e, v). Define $i(F) = i(v)$. The edge e is incident to one other vertex v'. Define $j(F) = i(v')$. Define:

$$\omega_F = \frac{\lambda_{i(F)} - \lambda_{j(F)}}{d_e}.$$

This is the weight of the induced action of $\mathbb{C}^*$ on the tangent space to the rational component C_e of C corresponding to F at its preimage over $p_{i(F)}$. This fact follows from the corresponding calculation on the weight of the action on the tangent space to the image line, with a factor of $\frac{1}{d_e}$ coming from the d_e-fold ramification of the map at the fixed point.

We describe the obstruction theory of $\overline{M}_{g,n}(\mathbf{P}^r, d)$ restricted to $\overline{M}_\Gamma/\mathbf{A}$. Define sheaves $\mathcal{T}^1$ and $\mathcal{T}^2$ on $\overline{M}_\Gamma/\mathbf{A}$ via the cohomology of the restriction of the canonical (dual) perfect obstruction theory $E_\bullet$ on $\overline{M}_{g,n}(\mathbf{P}^r, d)$:

$$0 \to \mathcal{T}^1 \to E_{0,\Gamma} \to E_{1,\Gamma} \to \mathcal{T}^2 \to 0. \tag{23}$$

There is a tangent-obstruction exact sequence of sheaves on the substack $\overline{M}_\Gamma/\mathbf{A}$:

$$\begin{aligned} 0 \to \mathrm{Ext}^0(\Omega_C(D), \mathcal{O}_C) \to H^0(C, f^*TX) \to \mathcal{T}^1 \to \\ \to \mathrm{Ext}^1(\Omega_C(D), \mathcal{O}_C) \to H^1(C, f^*TX) \to \mathcal{T}^2 \to 0. \end{aligned} \tag{24}$$

The marked point divisor on C is denoted by D. The 4 terms other than the sheaves $\mathcal{T}^i$ are vector bundles and are labeled by their fibers. This sequence can be viewed as filtering the deformations of the maps by those which preserve the domain curves. It arises via the pull-back to $\overline{M}_\Gamma/\mathbf{A}$ of a distinguished triangle of complexes on $\overline{M}_{g,n}(\mathbf{P}^r, d)$ (see Appendix B). These results may be found in [LT], [R2], [B].

In the remainder of this section, the fixed and moving parts of the 4 bundles in the tangent-obstruction complex are explicitly identified following [K]. It is simpler to carry out the bundle analysis on the prequotient $\overline{M}_\Gamma$ to avoid monodromy in the nodes. In fact, the final integrals over the fixed locus will be evaluated on $\overline{M}_\Gamma$ and corrected by the order of $\mathbf{A}$.

It will be seen that there are exactly 3 fixed pieces in the 4 bundles. They occur in the 1^{st}, 2^{nd}, and 4^{th} terms of the complex. The fixed piece in the 1^{st} term maps isomorphically to the fixed piece of the 2^{nd}. $\mathcal{T}^{1,f}$ is thus isomorphic to the fixed piece in the 4^{th} term. The latter is canonically the tangent bundle to $\overline{M}_\Gamma$. Also, $\mathcal{T}^{2,f} = 0$. We can conclude that the fixed stack is nonsingular and equal to $\overline{M}_\Gamma/\mathbf{A}$. The two exact sequences (23) and (24) imply:

$$e(N^{vir}) = \frac{e(B_2^m)e(B_4^m)}{e(B_1^m)e(B_5^m)}$$

where, for example, B_2^m denotes the moving part of the 2^{nd} term in (24).

We first calculate the contribution coming from the bundle

$$\mathrm{Aut}(C) = \mathrm{Ext}(\Omega_C(D), \mathcal{O}_C)$$

parameterizing infinitesimal automorphisms of the pointed domain. For each non-contracted component of C, there is a weight zero piece coming from the infinitesimal automorphism of that component fixing the two special points. This term will cancel with a similar term in $H^0(f^*T\mathbf{P}^r)$. Also, since there is no moving part, $e(B_1^m) = 1$. If it is the case that the special points are not marked or nodes, that is the associated vertex of the graph has genus 0 and valence one, there would be an extra automorphism with nontrivial weight. We will leave this case and the case of a genus 0 valence 2 vertex to the reader. No extra trivial weight pieces arise in these two cases.

Next, we consider the bundle $\mathrm{Def}(C) = \mathrm{Ext}^1(\Omega_C(D), \mathcal{O}_C)$ parameterizing deformations of the pointed domain. A deformation of the contracted components (as marked curves) is a weight zero deformation of the map which yields the tangent space of $\overline{M}_\Gamma/\mathbf{A}$ as a summand in the weight zero piece of $\mathrm{Def}(C)$. The other deformations of C come from smoothing nodes of C which join contracted components to non-contracted components. This space splits into a product of spaces corresponding to deformations which smooth each node individually. The one dimensional space associated to each node is identified as a bundle with the tensor product of the tangent spaces of the two components at the node. We see that the tangent space to the non-contracted curve forms a trivial bundle with weight ω_F while the tangent space to the contracted curve varies but has trivial weight. Let e_F denote the line bundle on $\overline{M}_\Gamma$ whose fiber over a point is the cotangent space to the component associated to F at the corresponding node. Therefore,

$$e(B_4^m) = \prod_{\text{flags}} (\omega_F - e_F).$$

To compute the contribution coming from $H^\bullet(f^*T\mathbf{P}^r)$, we consider the normalization sequence resolving all of the nodes of C which are forced by the graph type Γ.

$$0 \to \mathcal{O}_C \to \bigoplus_{\text{vertices}} \mathcal{O}_{C_v} \oplus \bigoplus_{\text{edges}} \mathcal{O}_{C_e} \to \bigoplus_{\text{flags}} \mathcal{O}_{x_F} \to 0$$

Twisting by $f^*(T\mathbf{P}^r)$ and taking cohomology yields:

$$0 \to H^0(f^*T\mathbf{P}^r) \to \bigoplus_{\text{vertices}} H^0(C_v, f^*T\mathbf{P}^r) \oplus \bigoplus_{\text{edges}} H^0(C_e, f^*T\mathbf{P}^r) \to$$

$$\to \bigoplus_{\text{flags}} T_{p_{i(F)}}\mathbf{P}^r \to H^1(f^*T\mathbf{P}^r) \to \bigoplus_{\text{vertices}} H^1(C_v, f^*T\mathbf{P}^r) \to 0$$

where we have used the fact that there will be no higher cohomology on the non-contracted components since they are rational. Also note that $H^0(C_v, f^*(T\mathbf{P}^r) = T_{p_{i(v)}}\mathbf{P}^r$ since C_v is connected and f is constant on it. Thus, we obtain:

$$H^0 - H^1 = \begin{array}{cccc} + & \bigoplus_{\text{vertices}} T_{p_{i(v)}}\mathbf{P}^r & + & \bigoplus_{\text{edges}} H^0(C_e, f^*T\mathbf{P}^r) \\ - & \bigoplus_{\text{flags}} T_{p_{i(F)}}\mathbf{P}^r & - & \bigoplus_{\text{vertices}} H^1(C_v, f^*T\mathbf{P}^r) \end{array}$$

As non-contracted components are rigid, we see that $H^0(C_e, f^*T\mathbf{P}^r)$ is trivial as a bundle, but we need to determine its weights. We do this via the Euler sequence. On $\mathbf{P}^r$ we have:

$$0 \to \mathcal{O} \to \mathcal{O}(1) \otimes V \to T\mathbf{P}^r \to 0.$$

Pulling back to C_e and taking cohomology gives us:

$$0 \to \mathbb{C} \to H^0(\mathcal{O}(d_e)) \otimes V \to H^0(f^*T\mathbf{P}^r) \to 0$$

Here the weight on $\mathbb{C}$ is trivial, and the weights on $H^0(\mathcal{O}(d_e))$ are given by $\frac{a}{d_e}\lambda_i + \frac{b}{d_e}\lambda_j$ for $a+b=d_e$. The weights on V are $-\lambda_0, \ldots, -\lambda_r$. So the weights of the middle term are just the pairwise sums of these, $\frac{a}{d_e}\lambda_i + \frac{b}{d_e}\lambda_j - \lambda_k$. There are exactly 2 zero weight terms here coming from $a=0, k=j$ and $b=0, k=i$. These cancel the zero weight term from the $\mathbb{C}$ on the left, and the zero weight term occurring in $\mathrm{Aut}(C)$. Breaking up the remaining terms into two groups corresponding to $k=i,j$ and $k \neq i,j$, we obtain the contribution of $H^0(C_e, f^*T\mathbf{P}^r)$ to the Euler class ratio $e(B_2^m)/e(B_5^m)$:

$$(-1)^{d_e}\frac{d_e!^2}{d_e^{2d_e}}(\lambda_i - \lambda_j)^{2d_e} \cdot \prod_{\substack{a+b=d_e \\ k\neq i,j}} (\frac{a}{d_e}\lambda_i + \frac{b}{d_e}\lambda_j - \lambda_k).$$

Finally, we evaluate the contribution of $H^1(C_v, f^*T\mathbf{P}^r)$. This is simply $H^1(C_v, \mathcal{O}_{C_v}) \otimes T_{p_{i(v)}}\mathbf{P}^r$. As a bundle, $H^1(C_v, \mathcal{O}_{C_v})$ is the dual of the

Hodge bundle $E = \pi_* \omega$ on the moduli space $\overline{M}_{g(v),\mathrm{val}(v)}$. The bundle $H^1(C_v, \mathcal{O}_{C_v}) \otimes T_{p_{i(v)}} \mathbf{P}^r$ splits into r copies of $E^\vee$ twisted respectively by the r weights $\lambda_i - \lambda_j$ for $j \neq i$. Taking the equivariant top Chern class of this bundle yields:

$$\prod_{j \neq i} c_{(\lambda_i - \lambda_j)^{-1}}(E^\vee) \cdot (\lambda_i - \lambda_j)^{g(v)}$$

where for a bundle Q of rank q:

$$c_t(Q) = 1 + tc_1(Q) + \ldots t^q c_q(Q).$$

We arrive at the following form of the inverse Euler class of the virtual normal bundle to the fixed point locus corresponding to the graph Γ.

$$\begin{aligned} & \prod_{\text{flags}} \frac{1}{\omega_F - e_F} \prod_{j \neq i(F)} (\lambda_{i(F)} - \lambda_j) \\ \frac{1}{e(N^{vir})} \quad = \quad & \prod_{\text{vertices}} \prod_{j \neq i(v)} c_{(\lambda_{i(v)} - \lambda_j)^{-1}}(E^\vee) \cdot (\lambda_{i(v)} - \lambda_j)^{g(v)-1} \\ & \prod_{\text{edges}} \frac{(-1)^{d_e} d_e^{2d_e}}{(d_e!)^2 (\lambda_i - \lambda_j)^{2d_e}} \prod_{\substack{a+b=d_e \\ k \neq i,j}} \frac{1}{\frac{a}{d_e}\lambda_i + \frac{b}{d_e}\lambda_j - \lambda_k} \end{aligned}$$

In addition, the virtual fundamental class of the fixed locus must be identified. We have already seen $\mathcal{T}^{1,f}$ is the tangent bundle of $\overline{M}_\Gamma$. and $\mathcal{T}^{2,f} = 0$. It then follows from (23) that the $\mathbb{C}^*$-fixed (dual) perfect obstruction theory is equivalent on the fixed stack to the trivial perfect obstruction theory. The virtual fundamental class of the fixed stack is simply the ordinary fundamental class.

The (numerical) Gromov-Witten invariants of $\mathbf{P}^r$ are the integrals:

$$I_{g,d}^{\mathbf{P}^r}(H^{l_1}, \ldots, H^{l_n}) = \int_{[\overline{M}_{g,n}(\mathbf{P}^r,d)]^{vir}} e_1^*(H^{l_1}) \cup \cdots \cup e_n^*(H^{l_n}),$$

where H is the hyperplane class and e_i are the evaluation maps. The above expression for $\frac{1}{e(N^{vir})}$ may be used in the virtual Bott residue formula (7) to deduce formulas expressing these Gromov-Witten invariants in terms of integrals on moduli spaces of pointed curves. The numerator terms, coming from the cohomology classes of $\mathbf{P}^r$, are identical in this higher genus case to the terms appearing in [K]. In particular, they contribute only additional weights, and no cohomological terms.

Let $[n] = \{1, \ldots, n\}$ be the marking set of an n-pointed graph Γ. Let $i : [n] \to \{0, \ldots, r\}$ be defined by $f(m) = p_{i(m)}$. The final expression for the numerical Gromov-Witten invariants of $\mathbf{P}^r$ is:

$$I_{g,d}^{\mathbf{P}^r}(H^{l_1},\ldots,H^{l_n}) = \sum_{\Gamma} \frac{1}{|\mathbf{A}_\Gamma|} \int_{\overline{M}_\Gamma} \frac{\prod_{[n]} \lambda_{i(m)}^{l_m}}{e(N_\Gamma^{vir})}.$$

The sum is over all graphs Γ indexing fixed loci of $\overline{M}_{g,n}(\mathbf{P}^r,d)$. To evaluate the integral, one expands the terms of the form $\frac{1}{\omega-e}$ as formal power series, and then integrates all terms of the appropriate degree.

Each integral that is encountered will naturally split as a product of integrals over the different moduli spaces of pointed curves. We remark that the integrals over genus 0 spaces are identical to the ones which are dealt with in [K]. In particular, while the formula given above is incorrect for graphs with vertices of genus 0 and valence 1 or 2, the formulas obtained in [K] after integrating over $\overline{M}_{0,n}$ hold for these degenerate cases as well.

In higher genera, we know of no closed formulas for the integrals which occur in these calculations. However, C. Faber has constructed an algorithm in [Fa] which determines all such integrals. Thus, this formula, together with Faber's algorithm, gives a method in principle to determine all the Gromov-Witten invariants of projective space.

The full system of Gromov-Witten invariants consists of cohomology classes in $\overline{M}_{g,n}$:

$$I_{g,d}^{\mathbf{P}^r}(H^{l_1},\ldots,H^{l_n}) = \pi_*\Big([\overline{M}_{g,n}(\mathbf{P}^r,d)]^{vir} \cap e_1^*(H^{l_1}) \cap \cdots \cap e_n^*(H^{l_n})\Big),$$

where $\pi : \overline{M}_{g,n}(\mathbf{P}^r,d) \to \overline{M}_{g,n}$ is the forgetful map. Since π is $\mathbb{C}^*$-equivariant morphism (with respect to the trivial action on $\overline{M}_{g,n}$), virtual localization may be used to explicitly equate these push-forwards to push-forwards from the fixed loci of $\overline{M}_{g,n}(\mathbf{P}^r,d)$ to $\overline{M}_{g,n}$.

The tautological ring of $\overline{M}_{g,n}$ is defined to be the subring of $H^*(\overline{M}_{g,n},\mathbb{Q})$ generated by the Chern classes of the cotangent lines, the κ classes, and the push-forwards of these classes from lower moduli spaces $\overline{M}_{g',n'}$ to the boundary strata of $\overline{M}_{g,n}$ via the standard gluing maps.

Corollary 1. *The full system of Gromov-Witten invariants of* $\mathbf{P}^r$ *lies in the tautological rings of the moduli spaces* $\overline{M}_{g,n}$.

Proof. The proof relies upon two observations. First, the fixed loci map to boundary strata of $\overline{M}_{g,n}$ via the standard gluing maps (up to an automorphism factor). Second, the classes that arise on the fixed loci in the virtual localization formula are tautological. $\square$

The above analysis for the standard $\mathbf{T}$-action on $\mathbf{P}^r$ is valid in the following more general context. Let X be a nonsingular projective variety with a torus action with finitely many 0 and 1 dimensional orbits. As in the case of $\mathbf{P}^r$, the closure of each 1 dimensional orbit is necessarily a nonsingular $\mathbf{P}^1$ with 2 fixed points. The $\mathbf{T}$-fixed loci of $\overline{M}_{g,n}(X,\beta)$ are again nonsingular stacks with trivial fixed obstruction theory. The components of $\overline{M}_{g,n}(X,\beta)^{\mathbf{T}}$ are in bijective correspondence with decorated graphs and admit analogous descriptions as in the case of $\mathbf{P}^r$. In particular, explicit graph sum formulas for equivariant integrals in Gromov-Witten theory exist for X. Corollary 1 for X also holds.

We indicate the modifications needed to conclude these results for the more general $\mathbf{T}$-actions. By the finiteness assumption on the 0 and 1 dimensional orbits in X, the *reduced* structure on the fixed stack in $\overline{M}_{g,n}(X,\beta)$ is clearly the nonsingular finite quotient stack of a product of moduli spaces of pointed curves. As in the case of $\mathbf{P}^r$, an analysis of the weight 0 pieces of the obstruction theory via diagram (24) for X is required to show the actual fixed stack is reduced with trivial obstruction theory. Only two steps in the argument for $\mathbf{P}^r$ require new justification. First, a weight computation of the bundle $H^0(C, f^*T_{\mathbf{P}^r})$ in (24) showed there was 1 fixed dimension for each noncontracted component. Moreover, this fixed dimension exactly corresponded to an infinitesimal automorphism of the noncontracted component. Second, the bundle $H^1(C, f^*T_{\mathbf{P}^r})$ had no fixed part. For more general X, we will show these two facts hold: the fixed part of $H^0(C, f^*T_X)$ will again correspond exactly to the infinitesimal automorphisms of the noncontracted components, and $H^1(C, f^*T_X)$ will have no fixed part. After this is established, the analysis in the projective space case may be followed exactly to conclude that the fixed stack is nonsingular with trivial perfect obstruction theory.

Lemma 2. *Let L be a line bundle on $\mathbf{P}^1$. Let $\mathbf{T}$ act equivariantly on L and $\mathbf{P}^1$. Let $p \in \mathbf{P}^1$ be a fixed point. If the weights of the torus representations L_p and $T_{\mathbf{P}^1,p}$ are linearly independent, then the $\mathbf{T}$-invariant subspaces of $H^0(\mathbf{P}^1, L)$ and $H^1(\mathbf{P}^1, L)$ are 0.*

Proof. Let the weights of the representation L_p and $T_{\mathbf{P}^1,p}$ be λ and ω respectively. First consider $H^0(\mathbf{P}^1, L)$. Let d be the degree of L. If $d < 0$, then $H^0(\mathbf{P}^1, L) = 0$. If $d \geq 0$, the weight decomposition of $H^0(\mathbf{P}^1, L)$ is:

$$\oplus_{i=0}^{d} \, (\lambda - i\omega).$$

Hence, $H^0(\mathbf{P}^1, L)$ has no fixed part by the independence assumption. We conclude the result for H^1 by applying Serre duality, and the result for $H^0(K\otimes L^\vee)$. The linear independence assumption holds for $K\otimes L^\vee$ since the weight of the representation $(K\otimes L^\vee)_p$ is is $-\omega-\lambda$. $\square$

Lemma 3. *Let X be a nonsingular variety with a $\mathbf{T}$-action with finitely many 0 and 1 dimensional orbits. Let $f:\mathbf{P}^1\to P\subset X$ be a $\mathbf{T}$-fixed map onto a 1 dimensional orbit closure P. Then, the $\mathbf{T}$-invariant subspace of $H^0(\mathbf{P}^1, f^*T_X)$ is 1 dimensional (obtained from the infinitesimal automorphism of the domain) and the $\mathbf{T}$-invariant subspace of $H^1(\mathbf{P}^1, f^*T_X)$ is 0.*

Proof. Let $p\in P\subset X$ be a fixed point. The $\mathbf{T}$-action on X is isomorphic étale locally at p to a linear representation of $\mathbf{T}$ (see [Bi]). The finiteness assumption on 1 dimensional orbits then implies the weights of the $\mathbf{T}$-action on $T_{X,p}$ are linearly independent. The pull-back f^*T_X splits equivariantly as a direct sum of line bundles with respect to the induced (fractional) action on $\mathbf{P}^1$. As the weights of $T_{X,p}$ are distinct, there is a canonical equivariant decomposition of f^*T_X. One of these factors is f^*T_P. The representation $H^0(\mathbf{P}^1, f^*T_P)$ has a 1 dimensional fixed piece (obtained from the infinitesimal automorphism of the domain) and $H^1(\mathbf{P}^1, f^*T_P)=0$. It remains to show the other line bundle factors yield no invariant sections or first cohomology on $\mathbf{P}^1$. Let $p_1\in\mathbf{P}^1$ satisfy $f(p_1)=p$. As the weights of the torus action on the p_1 fiber of each of the other factors are linearly independent from the weight of $T_{\mathbf{P}^1,p_1}$, the proof is complete by Lemma 2. $\square$

Lemma 3 implies the two required facts about the bundles $H^0(C, f^*T_X)$ and $H^1(C, f^*T_X)$ on the fixed loci of $\overline{M}_{g,n}(X,\beta)$ via the standard normalization sequence.

Both nonsingular projective toric varieties and projective algebraic homogeneous spaces (with the natural maximal torus action) satisfy the finiteness condition on 0 and 1 dimensional orbits. These examples cover all the deformation classes of nonsingular projective rational surfaces.

5. **Multiple cover calculations**

Let $C\subset X$ be a nonsingular rational curve with balanced normal bundle $N\cong\mathcal{O}(-1)\oplus\mathcal{O}(-1)$ in a nonsingular Calabi-Yau 3-fold X. Let $[C]\in H_2(X,\mathbb{Z})$ be the homology class of C. The space of stable elliptic maps to X representing the curve class $d[C]$ contains a component Y_d consisting of maps which factor through a d-fold cover of C. Y_d is naturally isomorphic to $\overline{M}_{1,0}(C,d)$, the space of unpointed, genus 1 stable maps. The contribution of Y_d to the elliptic Gromov-Witten invariant $I^X_{1,d[C]}$ has been computed in physics [BCOV]. The answer obtained is $\frac{1}{12d}$ (accounting for the differing treatment of the elliptic involution).

Mathematically, the excess contribution of Y_d is expressed as an integral over $\overline{M}_{1,0}(C,d)$. The integral is computed here for all d via localization. Localization reduces the contribution to a graph sum which can be explicitly evaluated by Manin's trick [M] and a formula for intersections of cotangent lines on $\overline{M}_{1,n}$.

Let $\pi : U \to \overline{M}_{1,0}(C,d)$ be the universal family over the moduli space. Let $\mu : U \to C$ be the universal evaluation map. The expected dimension of $\overline{M}_{1,0}(C,d)$ is $2d$. By the cohomology and base change theorems, $R^1\pi_*\mu^*N$ is a vector bundle of rank $2d$ on $\overline{M}_{1,0}(\mathbf{P}^1,d)$. The contribution of Y_d to the elliptic Gromov-Witten invariant of curve class $d[C]$ is:

$$\int_{[\overline{M}_{1,0}(C,d)]^{vir}} c_{2d}(R^1\pi_*\mu^*N). \tag{25}$$

Natural lifts of $\mathbb{C}^*$-actions on C to $\overline{M}_{1,0}(C,d)$, N, and $R^1\pi_*\mu^*N$ exist. The localization formula can therefore be applied to compute (25). The answer obtained agrees with the physics calculation.

Proposition 2.

$$\int_{[\overline{M}_{1,0}(C,d)]^{vir}} c_{2d}(R^1\pi_*\mu^*N) = \frac{1}{12d}.$$

Let $V \cong \mathbb{C}^2$. Let $C = \mathbf{P}(V)$. Let $\mathbb{C}^*$ act by weights 0 and -1 on V. Let x_0 and x_{-1} be the respective fixed points in C. The $\mathbb{C}^*$-action lifts naturally to the tautological line $\mathcal{O}(-1)$ and thus to N. Consider the graph sum obtained by the localization formula for the integral (25). The 0 weight leads to a drastic collapse of the sum. This was observed by Manin in [M] for an analogous excess integral over a space of genus 0 maps. In fact, the only graphs which contribute are comb graphs where the backbone is an elliptic curve contracted over x_{-1} and the teeth are rational curves multiple covering $\mathbf{P}(V)$. The degree d is distributed over the teeth by $\sum_1^k m_i = d$. The denominator terms in the localization formula are determined by the results of Section 4. The numerator is given by the bundle $R^1\pi_*\mu^*N$ which is decomposed on each fixed point locus via the natural normalization sequence. The formula

$$\sum_{m \vdash d} \frac{(-1)^{d-L(m)}}{\mathrm{Aut}(m)\ \Pi_1^{L(m)} m_i} \int_{\overline{M}_{1,L(m)}} \frac{1+\lambda}{\Pi_1^{L(m)}(1-m_i e_i)} \tag{26}$$

is obtained for the degree d contribution. The sum is over all positive partitions:

$$m = (m_1, \ldots, m_k), \quad m_i > 0, \quad \sum_1^k m_i = d.$$

$L(m)$ denotes the length of m. $\mathrm{Aut}(m)$ is the order of the stabilizer of the symmetric group S_k-action on the string $(m_1, \ldots, m_k)$. The class λ in the numerator is the first Chern class of the Hodge bundle on $\overline{M}_{1,n}$. As before, e_i is the i^{th} cotangent line bundle on $\overline{M}_{1,n}$.

The integral (26) is calculated in two parts to prove Proposition 2.

Lemma 4.

$$\sum_{m \vdash d} \frac{(-1)^{d-L(m)}}{\mathrm{Aut}(m)\ \Pi_1^{L(m)} m_i} \int_{\overline{M}_{1,L(m)}} \frac{\lambda}{\Pi_1^{L(m)}(1 - m_i e_i)} = \frac{1}{24d}$$

Lemma 5.

$$\sum_{m \vdash d} \frac{(-1)^{d-L(m)}}{\mathrm{Aut}(m)\ \Pi_1^{L(m)} m_i} \int_{\overline{M}_{1,L(m)}} \frac{1}{\Pi_1^{L(m)}(1 - m_i e_i)} = \frac{1}{24d}$$

We start with Lemma 4. The first step is to use the boundary expression for λ to reduce to an integral over genus 0 pointed moduli spaces. On $\overline{M}_{1,1}$, the equation:

$$\lambda = \frac{\Delta_0}{12} \tag{27}$$

holds where Δ_0 is the irreducible boundary divisor. Since λ on $\overline{M}_{1,n}$ is a pull-back from a one pointed space, (27) is valid on $\overline{M}_{1,n}$. Using the standard identification of Δ_0 with the $\mathbb{Z}/2\mathbb{Z}$-quotient of $\overline{M}_{0,n+2}$, the equality:

$$\int_{\overline{M}_{1,L(m)}} \frac{\lambda}{\Pi_1^{L(m)}(1 - m_i e_i)} = \frac{1}{24} \int_{\overline{M}_{0,L(m)+2}} \frac{1}{\Pi_1^{L(m)}(1 - m_i e_i)}$$

is obtained. Next, using the well-known formula for intersection numbers on the genus 0 spaces, we see:

$$\int_{\overline{M}_{0,L(m)+2}} \frac{1}{\Pi_1^{L(m)}(1 - m_i e_i)} = (\sum_1^{L(m)} m_i)^{L(m)-1} = d^{L(m)-1}.$$

After substituting these equalities, the sum of Lemma 4 is transformed to:

$$\frac{(-1)^d}{24d} \sum_{m \vdash d} \frac{(-1)^{-L(m)}}{\mathrm{Aut}(m)\ \Pi_1^{L(m)} m_i} d^{L(m)}. \tag{28}$$

The summation term in (28) was encountered by Manin in [M]. It evaluates explicitly to $(-1)^d$ via a generating function argument (see [M] p.416). The value of (28) is thus $\frac{1}{24d}$. Lemma 4 is established.

We now prove Lemma 5. A generating function approach is taken. For $d \geq 1$, let

$$g_d = \sum_{m \vdash d} \frac{(-1)^{d-L(m)}}{\mathrm{Aut}(m)\ \Pi_1^{L(m)} m_i} \int_{\overline{M}_{1,L(m)}} \frac{1}{\Pi_1^{L(m)}(1-m_i e_i)}.$$

Define $\gamma(t)$ by:

$$\gamma(t) = \sum_{\alpha \geq 1} (-1)^\alpha g_\alpha t^\alpha.$$

An important observation is that $\gamma(t)$ can be rewritten in the following form:

$$\gamma(t) =< \exp\ (-\sum_{\alpha\geq 1}\sum_{i\geq 0} \alpha^{i-1} t^\alpha \sigma_i) >_1 . \tag{29}$$

Here, Witten's notation,

$$< \sigma_0^{r_0} \sigma_1^{r_1} \cdots \sigma_k^{r_k} >_1, \tag{30}$$

is used to denote the integral:

$$\int_{\overline{M}_{1,r}} \underbrace{e_{r_0+1} \ldots e_{r_0+r_1}}_{r_1} \cdot \underbrace{e^2_{r_0+r_1+1} \ldots e^2_{r_0+r_1+r_2}}_{r_2} \cdots \underbrace{e^k_{r-r_k+1} \ldots e^k_r}_{r_k}$$

where $r = \sum_0^k r_i$. Equality (29) is a simply a rewriting of terms. The genus 1 integrals (30) are determined from genus 0 integrals by a beautiful formula in the formal variables $\{z_i\}_{i\geq 0}$:

$$< \exp \sum_{i\geq 0} z_i \sigma_i >_1 = \frac{1}{24} \log < \sigma_0^3 \exp \sum_{i \geq 0} z_i \sigma_i >_0 . \tag{31}$$

Formula (31) can be found, for example, in [D]. Let $z_i = -\sum_{\alpha \geq 1} \alpha^{i-1} t^\alpha$. Using (29) and (31), $\gamma(t)$ may be expressed as:

$$\gamma(t) = \frac{1}{24} \log < \sigma_0^3 \exp(-\sum_{\alpha\geq 1}\sum_{i\geq 0} \alpha^{i-1} t^\alpha \sigma_i) >_0 \tag{32}$$

Equation (32) will be used to determine $\gamma(t)$. First, define another generating function $\psi(t)$ by:

$$\psi(t) = 1 + \sum_\beta s_\beta t^\beta$$

where the coefficients s_β are:

$$(33)\qquad s_\beta = \sum_{m \vdash \beta} \frac{(-1)^{-L(m)}}{\mathrm{Aut}(m)\ \Pi_1^{L(m)} m_i} \int_{\overline{M}_{0,L(m)+3}} \frac{1}{\Pi_1^{L(m)}(1 - m_i e_i)}.$$

As before, the equality:

$$\psi(t) = < \sigma_0^3 \exp(-\sum_{\alpha \geq 1} \sum_{i \geq 0} \alpha^{i-1} t^\alpha \sigma_i) >_0$$

is a rewriting of terms. However, the expression (33) may be explicitly evaluated by the genus 0 intersection formulas and Manin's summation argument to yield:

$$s_\beta = (-1)^\beta.$$

Hence, $\psi(t)$ is simply $1/(1+t)$, and

$$\gamma(t) = -\frac{\log(1+t)}{24} = \frac{1}{24}(-t + \frac{t^2}{2} - \frac{t^3}{3} + \ldots).$$

Thus, $g_d = \frac{1}{24d}$. Lemma 5 is proven. Proposition 2 follows from (26) and the two Lemmas.

Localization may be applied to the analogous excess integrals for arbitrary genus g. The resulting formula is:

$$(34)\qquad \int_{[\overline{M}_{g,0}(\mathbf{P}^1,d)]^{vir}} c_{\mathrm{top}}(R^1\pi_*\mu^* N) =$$

$$\sum_{m \vdash d} \frac{(-1)^{d-L(m)}}{\mathrm{Aut}(m)\ \Pi_1^{L(m)} m_i} \int_{\overline{M}_{g,L(m)}} \frac{1 + c_1(E) + \ldots + c_g(E)}{\Pi_1^{L(m)}(1 - m_i e_i)}$$

where E is the Hodge bundle. For $g \geq 2$, we have conjectured with C. Faber the above integral sum is equal to:

$$\frac{|B_{2g}| \cdot d^{2g-3}}{2g \cdot (2g-2)!} = \frac{|\chi(M_g)| \cdot d^{2g-3}}{(2g-3)!}.$$

This equality has been verified in case $g + d \leq 7$.

Appendix A. Global nonsingular embeddings

Let V be an nonsingular projective algebraic variety with a $\mathbb{C}^*$-action. There exists a $\mathbb{C}^*$-equivariant polarization $\mathcal{L}$ on V (see [MFK]). $\overline{M}_{g,n}(V,\beta)$ is a $\mathbb{C}^*$-equivariant closed substack of $\overline{M}_{g,n}(\mathbf{P}^r,\beta)$ via the equivariant embedding determined by $\mathcal{L}$. It will be shown that the

Deligne-Mumford stack $\overline{M}_{g,n}(\mathbf{P}^r, \beta)$ admits a global closed equivariant embedding into a nonsingular Deligne-Mumford stack.

In [FP], the moduli space of maps to $\mathbf{P}^r$ is expressed as a quotient of a locally closed scheme J of an associated (product) Hilbert scheme $\mathcal{H}$ by a reductive group $\mathbf{G} = \mathbf{PGL}$. Four properties of this quotient will be needed here.

(i) The stack quotient $[J/\mathbf{G}]$ is the Deligne-Mumford moduli stack of maps $\overline{M}_{g,n}(\mathbf{P}^r, \beta)$.
(ii) $\mathbf{G}$ acts with finite stabilizers on J (in fact, the $\mathbf{G}$-action on J is proper).
(iii) There is a $\mathbb{C}^* \times \mathbf{G}$-action on J which descends to the given $\mathbb{C}^*$-action on $\overline{M}_{g,n}(\mathbf{P}^r, \beta)$.
(iv) There is a $\mathbb{C}^* \times \mathbf{G}$-equivariant linearized embedding of $J \subset \mathcal{H}$ in a nonsingular Grassmannian $\mathbb{G}$.

All these properties are obtained directly from the construction in [FP].

The $\mathbf{G}$-equivariant open set $U \subset \mathbb{G}$ on which the $\mathbf{G}$-action has finite stabilizers contains J and is $\mathbb{C}^*$-equivariant. Note that $\Delta = \overline{J} \setminus J$ is closed in $\mathbb{G}$ and is $\mathbb{C}^* \times \mathbf{G}$-equivariant. After discarding $\Delta \cap U$, it may be assumed that J is closed in U. Let Y be the nonsingular Deligne-Mumford stack $[U/\mathbf{G}]$. The moduli space of maps embeds $\mathbb{C}^*$-equivariantly in Y. It should be noted that while Y is a (quasi-separated) Deligne-Mumford stack of finite type, Y need not be separated.

Appendix B. The obstruction theory on $\overline{M}_{g,n}(V, \beta)$

In [B] and [LT], a canonical obstruction theory on $\overline{M}_{g,n}(V, \beta)$ is defined which is locally a two term complex of vector bundles. To obtain a global two term complex, a polarization is required. Since the constructions are $\mathbb{C}^*$-equivariant, a $\mathbb{C}^*$-equivariant perfect obstruction theory on $\overline{M}_{g,n}(V, \beta)$ may be defined using $\mathcal{L}$.

We sketch here the method of [B] to obtain the equivariant perfect obstruction theory on $\overline{M}_{g,n}(V, \beta)$. The relative deformation problem is considered for the canonical morphism

$$\tau : \overline{M}_{g,n}(V, \beta) \to \mathfrak{M}_{g,n}$$

where $\mathfrak{M}_{g,n}$ is the nonsingular Artin stack of prestable curves. The theory of the cotangent complex for Artin stacks has been developed in [LM-B]. The morphism τ yields a distinguished triangle of cotangent complexes on $\overline{M}_{g,n}(V, \beta)$:

$$\tau^* L^\bullet_{\mathfrak{M}} \to L^\bullet_{\overline{M}} \to L^\bullet_{\tau} \to \tau^* L^\bullet_{\mathfrak{M}}[1].$$

The complex $\tau^* L^\bullet_{\mathfrak{M}}$ has a two term bundle resolution of amplitude $[0,1]$, $A^\bullet = A^0 \to A^1$, in the $\mathbb{C}^*$-equivariant derived category (obtained from $\mathcal{L}$). There is a relative obstruction theory (see [B])

$$R^\bullet\pi_*(f^*T_V)^\vee \to L^\bullet_\tau \tag{35}$$

with a natural map to $\tau^* L^\bullet_{\mathfrak{M}}[1]$. Here, f is the map to V from the universal curve and π is the projection from the universal curve to $\overline{M}_{g,n}(V,\beta)$. Moreover, $R^\bullet\pi_*(f^*T_V)^\vee$ has a two term equivariant bundle resolution $B^\bullet = B^{-1} \to B^0$. Representatives of $L^\bullet_\tau$ and $L^\bullet_{\overline{M}}$ in the equivariant derived category may be found from the global nonsingular embeddings constructed above. The diagram below of distinguished triangles may be canonically completed in the equivariant derived category to obtain an equivariant obstruction theory of amplitude [-1,1]:

$$E^\bullet = B^{-1} \to B^0 \oplus A^0 \to A^1,$$

$$\begin{tikzcd}
B^\bullet \arrow[r] \arrow[d] & A^\bullet[1] \arrow[r] \arrow[d] & E^\bullet[1] \arrow[r] \arrow[d] & B^\bullet[1] \arrow[d] \\
L^\bullet_\tau \arrow[r] & \tau^* L^\bullet_{\mathfrak{M}}[1] \arrow[r] & L^\bullet_{\overline{M}}[1] \arrow[r] & L^\bullet_\tau[1].
\end{tikzcd} \tag{36}$$

The stability condition implies the cohomology of $E^\bullet$ at grade 1 vanishes. Hence, $E^\bullet$ may be represented by a two term equivariant complex $E^{-1} \to E^0$. The morphism constructed in diagram (36),

$$\phi : E^\bullet \to L^\bullet_{\overline{M}}$$

can be seen to be a equivariant perfect obstruction theory.

Property (iv) of Appendix A implies that the moduli stack $\overline{M}_{g,n}(V,\beta)$ has enough $\mathbb{C}^*$-equivariant locally frees. Hence, representatives $(E^\bullet, \phi)$ in the equivariant derived category may be chosen to map to the two term cutoff $[I/I^2 \to \Omega_Y]$ of $L^\bullet_{\overline{M}}$ determined by the nonsingular embedding (see Section 1). The distinguished triangle (36) is used in the computations of Section 4.

Appendix C. **Localization for Deligne-Mumford stacks**

In this appendix, we extend the virtual localization formula to the case of a Deligne-Mumford stack X with a $\mathbb{C}^*$-equivariant perfect obstruction theory under the additional assumption that X admits an equivariant global embedding in a nonsingular Deligne-Mumford stack. This condition is satisfied for the moduli space of maps to a nonsingular projective variety V with a $\mathbb{C}^*$-action by the existence of the quotient construction reviewed in Appendix A. We assume Deligne-Mumford stacks are of finite type (but not necessarily separated). In fact, for the application to the moduli space of maps, only quotient stacks $[U/\mathbf{G}]$ (where U is a quasi-projective scheme and $\mathbf{G}$ acts with finite stabilizers) need be considered. The $\mathbb{C}^*$-actions on $[U/\mathbf{G}]$ which arise descend from $\mathbb{C}^* \times \mathbf{G}$-actions on U.

First of all, a good theory of rational Chow groups on Deligne-Mumford stacks has been constructed in [V]. A finite covering result of [LM-B] (Theorem 10.1) is required for Vistoli's theory to apply to general Deligne-Mumford stacks. Essentially all properties of Chow groups for schemes hold for Deligne-Mumford stacks. In particular, one obtains flat pullbacks and proper pushforwards. Flat pullback gives an isomorphism in Chow groups between any stack and any vector bundle over the stack. Refined Gysin maps exist for regular embeddings, giving rise to an intersection product on the Chow groups of smooth Deligne-Mumford stacks. Finally, Chern and Segre classes for vector bundles and cones exist and satisfy the same properties as they do for schemes. Also, because these groups are defined in terms of closed substacks, it is immediate that the Chow groups are non-zero only in dimensions between zero and the dimension of the stack. (This last condition would not be possible if one required a theory with integral coefficients satisfying the hypothesis that flat pullback to a vector bundle gave isomorphisms in Chow groups.)

We can define a theory of $\mathbb{C}^*$-equivariant Chow groups on Deligne-Mumford stacks by the following construction. One simply defines $A_*^{\mathbb{C}^*}(X)$ to be the Chow groups of appropriate approximations to the homotopy quotient as in [To], [EG1]. The only difference is that the homotopy quotient is now taken to be the stack quotient, $[X \times E\mathbb{C}^*/\mathbb{C}^*]$. Since this is a free group action, the quotient is also a Deligne-Mumford stack. Therefore, $\mathbb{C}^*$-equivariant Chow groups are well defined for Deligne-Mumford stacks:

$$A_*^{\mathbb{C}^*}(X) = A_*[X \times E\mathbb{C}^*/\mathbb{C}^*].$$

The proof in Section 3 proves the virtual localization formula on Deligne-Mumford stacks satisfying the embedding assumption provided we know that the standard localization formula holds for nonsingular Deligne-Mumford stacks.

The key step in the proof of the localization formula for nonsingular Deligne-Mumford stacks, as in proofs of localization in other categories, is the following lemma:

Lemma 6. *If U is a Deligne-Mumford stack on which $\mathbb{C}^*$ acts without fixed points, then the equivariant Chow groups $A_*^{\mathbb{C}^*}(U)$ vanish after localization.*

Proof. We consider the stack quotient, $[U/\mathbb{C}^*]$. Because the $\mathbb{C}^*$-action has no fixed points on U, the quotient is again Deligne-Mumford. Furthermore, we claim that the Chow groups of $[U/\mathbb{C}^*]$ are naturally isomorphic to the equivariant Chow groups of U. In the diagram below, both horizontal arrows are open sets of vector bundles.

$$\begin{array}{ccc} U \times E\mathbb{C}^*_k & \longrightarrow & U \\ \downarrow & & \downarrow \\ [U \times E\mathbb{C}^*_k/\mathbb{C}^*] & \longrightarrow & [U/\mathbb{C}^*] \end{array}$$

$E\mathbb{C}^*_k$ is an approximation to $E\mathbb{C}^*$ determined by an open set of a $\mathbb{C}^*$-representation ([To], [EG1]). As these approximations to the homotopy quotient are realized as open sets of vector bundles over $[U/\mathbb{C}^*]$, the isomorphism in Chow groups follows. We have already observed that $A_*[U/\mathbb{C}^*]$ has only finitely many graded components. Hence, $A^{\mathbb{C}^*}_*(U)$ has only finitely many graded components and thus is trivial after localization. $\square$

Let Y be a nonsingular Deligne-Mumford stack with a $\mathbb{C}^*$-action. The fixed substack Y^f is defined as the stack theoretic zero locus of the canonical vector field determined in TY by the flow. Equivalently, this is the maximal substack on which each element of $\mathbb{C}^*$ acts trivially. In the category of stacks, this does not imply that the action of $\mathbb{C}^*$ on Y^f is trivial. However, the action becomes trivial after a finite cover $\mathbb{C}^* \to \mathbb{C}^*$.

If we consider the pushforward ι_* from the equivariant Chow groups of the fixed locus, Y^f to Y, we obtain the exact sequence in equivariant Chow groups:

$$A^{\mathbb{C}^*}_*(Y^f) \xrightarrow{\iota_*} A^{\mathbb{C}^*}_*(Y) \to A^{\mathbb{C}^*}_*(U) \to 0.$$

By the Lemma, ι_* is surjective after localization.

To prove ι_* is injective after localization, the nonsingularity of Y^f will be used. Some care must to be taken to establish this nonsingularity in the category of Deligne-Mumford stacks. Let ζ be a $\mathbb{C}^*$-fixed geometric point. Although it doesn't make sense to talk about the local ring of a point in a stack, the completion of the local ring, $\widehat{\mathcal{O}}_{\zeta,Y}$, is well defined, because it is invariant under étale covers. $\mathbb{C}^*$ (or rather, the finite cover of $\mathbb{C}^*$ mentioned above) acts on this completion. The fixed scheme of the induced action on the spectrum of this ring is an étale cover of a formal neighborhood of ζ in Y^f. The smoothness then follows from the complete reducibility of the action of $\mathbb{C}^*$ on $\widehat{\mathcal{O}}_{\zeta,Y}$ as in [T].

After the finite cover, $\mathbb{C}^*$-equivariant sheaves on Y^f split into eigensheaves as before. The normal bundle to the fixed locus is seen to have the standard identification with the moving part of the restriction of the tangent bundle of Y. The representations of the covering $\mathbb{C}^*$ may be described by fractional weights of the original $\mathbb{C}^*$.

The nonsingularity of Y and Y^f implies that there is a pullback from $A_*^{\mathbb{C}^*}(Y)$ to $A_*^{\mathbb{C}^*}(Y^f)$ and that, on each component of $Y^f = \cup Y_i$, the usual self-intersection formula $\iota^*\iota_*\alpha = e(N_i)\cdot\alpha$ holds.

Suppose $\alpha = \sum \alpha_i$ in $A_*^{\mathbb{C}^*}(Y^f)_t$ pushes forward to zero. Then,

$$0 = \iota^*\iota_*\alpha = \sum e(N_i)\cdot\alpha_i$$

since pushing forward from one component of Y^f and restricting to another necessarily gives zero. Hence, for each i, $e(N_i)\cdot\alpha_i = 0$. However, since $e(N_i)$ is invertible in the localized ring, each α_i is zero. We have proven:

Proposition 3. *The map $\iota_* : A_*^{\mathbb{C}^*}(Y^f) \to A_*^{\mathbb{C}^*}(Y)$ is an isomorphism after localization.*

The self intersection formula also quickly implies the explicit localization formula:

$$[Y] = \sum_i \frac{[Y_i]}{e(N_i)}.$$

By Proposition 3, there exists a unique class α satisfying $\iota_*(\alpha) = [Y]$. The condition $\iota^*\iota_*(\alpha) = [Y^f]$ then determines α.

If $X \subset Y$ is a $\mathbb{C}^*$-equivariant embedding, the fixed substack X^f may be *defined* by $X^f = X \cap Y^f$. It follows from this definition that

$$\Omega_X|_{X^f}^f = \Omega_{X^f}.$$

It is not difficult to show the substack structure X^f is independent of the choice of nonsingular equivariant embedding. The constructions and arguments for the virtual localization formula for equivariant perfect obstruction theories on X now go through unchanged.

References

[AB] M. Atiyah and R. Bott, *The moment map and equivariant cohomology*, Topology **23** (1984), 1-28.

[B] K. Behrend, *Gromov-Witten invariants in algebraic geometry*, Invent. Math. **127** (1997), 601-617.

[BF] K. Behrend and B. Fantechi, *The intrinsic normal cone*, Invent. Math. **128** (1997), 45-88.

[BCOV] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, *Holomorphic anomalies in topological field theories* (with an appendix by S. Katz), Nucl.Phys. **B405** (1993), 279-304.

[Bi] A. Bialynicki-Birula, *Some theorems on actions of algebraic groups,* Ann. of Math. (2) **98** (1973), 480-497.

[Bo] R. Bott, *A residue formula for holomorphic vector fields*, JDG **4** (1967), 311-332.

[CH] L. Caporaso and J. Harris, *Counting plane curves of any genus*, preprint.

[D] R. Dijkgraaf, *Intersection theory, integrable hierarchies, and topological field theory*, Lectures given at the Cargese Summer School on 'New Symmetry Principles in Quantum Field Theory,' July 16-27, 1991, hep-th/9201003.

[EG1] D. Edidin and W. Graham, *Equivariant intersection theory*, Invent. Math. (to appear).

[EG2] D. Edidin and W. Graham, *Localization in equivariant intersection theory and the Bott residue formula*, preprint.

[ES] G. Ellingsrud and S. Strømme, *Bott's formula and enumerative geometry*, J. Amer. Math. Soc. **9** (1996), no. 1, 175-193.

[Fa] C. Faber, *Algorithms for computing intersection numbers of curves, with an application to the class of the locus of Jacobians*, Mittag-Leffler preprint no. 18, 1997.

[F] W. Fulton, *Intersection Theory*, Springer-Verlag, 1980.

[FP] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, Proc. AMS (to appear).

[I] B. Iverson, *A fixed point formula for action of tori on algebraic varieties*, Invent. Math. **16** (1972), 229-236.

[KR] T. Kambayashi and P. Russell, *On linearizing algebraic torus actions*, J. Pure Appl. Algebra **23** (1982).

[K] M. Kontsevich, *Enumeration of rational curves via torus actions*, in *The moduli space of curves*, R. Dijkgraaf, C. Faber, and G. van der Geer, eds., Birkhauser, 1995, pp 335-368.

[KM] M. Kontsevich and Yu. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, Commun. Math. Phys. **164** (1994), 525-562.

[Kr] A. Kresch*Canonical rational equivalence of intersections of divisors*, preprint.

[LM-B] G. Laumon and L. Moret-Baily, *Champs algébriques*, prépublications mathématiques de l'Université de Paris-Sud no. 88-33, 1995.

[LT] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties*, preprint 1996.

[M] Yu. Manin, *Generating functions in algebraic geometry and sums over trees*, in *The moduli space of curves*, R. Dijkgraaf, C. Faber, and G. van der Geer, eds., Birkhauser, 1995, pp 401-417.

[MFK] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric Invariant Theory*, Springer-Verlag: Berlin, 1994.

[R1] Z. Ran, *The degree of a Severi variety*, Bull. Amer. Math. Soc. (N.S.) **17** (1987), 125–128.

[R2] Z. Ran, *Deformation of maps*, in *Algebraic curves and projective geometry (Trento 1988)*, E. Ballico and C. Ciliberto, eds., LNM 1389, Springer 1989, pp 246-253.

[T] R. Thomason, *Une formule de Lefschetz*, Duke J. **68** (1992), 447-462.

[To] B. Totaro, Chow ring of the symmetric group, preprint.
[V] A. Vistoli, Intersection theory on algebraic stacks and their moduli, Invent. Math. **97** (1989), 613-670.

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